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$(T, \mathbf{V})$-**Cat** IS EXTENSIVE

Dedicated to Bob Rosebrugh

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Abstract. For a complete and cocomplete monoidal-closed category $\mathbf{V}$ and a **Set**-monad T suitably extended to **V-Rel**, we show that the category of $(T, \mathbf{V})$-categories and $(T, \mathbf{V})$-functors is infinitary extensive.

1. Introduction

The introduction of $(T, \mathbf{V})$-categories in [7] – as both a generalization of Eilenberg-Moore algebras and enriched categories – led mostly to the study of topological aspects of these structures in the particular case when $\mathbf{V}$ is a thin category (see e.g. the monograph [10]). Much less is known in the case of a general monoidal-closed category $\mathbf{V}$, although it includes as examples of $(T, \mathbf{V})$-categories Lambek's multicategories, Burroni's T-categories [3], and Hermida's generalized multicategories when $\mathbf{V} = \mathbf{Set}$ [8, 9] (as a bridge between the quantalic and the categorical examples see also [5]).

In this note, generalizing Mahmoudi-Schubert-Tholen's proof [12], we show that, for a complete, cocomplete, symmetric monoidal-closed category $\mathbf{V}$, the category $(T, \mathbf{V})$-**Cat** of $(T, \mathbf{V})$-categories and $(T, \mathbf{V})$-functors is infinitary extensive, proving that $(T, \mathbf{V})$-**Cat** has coproducts and pullbacks along coproduct injections, and that coproducts are universal and disjoint.

2. The setting

Throughout this paper we use essentially the setting of [7]; that is,

(1) $\mathbf{V}$ *is a (non-degenerate) complete, cocomplete, symmetric monoidal-closed category, with tensor product* $\otimes$ *and unit* I.

We make use of the bicategory **V-Rel** (or $\mathbf{Mat}(\mathbf{V})$: see [2], [13]), whose objects are sets, arrows (=1-cells) $r \colon X \relbar\joinrel\mapsto Y$ are families of $\mathbf{V}$-objects $r(x, y)$, for $x \in X$, $y \in Y$,

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i.e. functors $r: X \times Y \longrightarrow \mathbf{V}$ (where X and Y are considered as discrete categories), and 2-cells $\varphi: r \Longrightarrow r'$ are families of $\mathbf{V}$-morphisms $\varphi_{x,y}: r(x,y) \longrightarrow r'(x,y)$, i.e. natural transformations $\varphi: r \Longrightarrow r'$.

Transposition of $\mathbf{V}$-relations defines a pseudo-involution: the transpose $r^\circ: Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$ of $r: X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ is defined by $r^\circ(y,x) = r(x,y)$. The category **Set** of sets embeds naturally into $\mathbf{V}$-**Rel**: if $f: X \longrightarrow Y$ is a map, as a $\mathbf{V}$-relation $f: X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ is defined by $f(x,y) = I$ if $x = y$ and $f(x,y) = 0$ otherwise. Its transpose f° is a right adjoint to f; we will denote the unit and the counit of this adjunction by $\lambda_f: 1_X \Longrightarrow f^\circ f$ and $\rho_f: f f^\circ \Longrightarrow 1_Y$, respectively.

(2) (T, m, e) *is a* **Set**-*monad with an extension* (T, m, e) *to* $\mathbf{V}$-**Rel**.

More precisely, $T: \mathbf{V}\text{-}\mathbf{Rel} \longrightarrow \mathbf{V}\text{-}\mathbf{Rel}$ is a lax functor which extends the given **Set**-functor, with given natural and coherent 2-cells $\kappa_{s,r}: Ts \cdot Tr \longrightarrow T(s \cdot r)$, for $\mathbf{V}$-relations $r: X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $s: Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$; the 2-cells $\kappa_{s,r}$ are isomorphisms whenever r is a **Set**-map (and therefore also when s° is a **Set**-map), and $(Tf)^\circ = T(f^\circ)$. The functor T extends to 2-cells functorially, and m and e become oplax natural transformations, with given α_r and β_r, for $r: X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ a $\mathbf{V}$-relation, as in the diagrams:

$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[d, "r"', "|" marking] \arrow[dr, phantom, "\overset{\alpha_r}{\Longrightarrow}"] & TX \arrow[d, "Tr", "|" marking] \\
Y \arrow[r, "e_Y"'] & TY
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
TTX \arrow[r, "m_X"] \arrow[d, "TTr"', "|" marking] \arrow[dr, phantom, "\overset{\beta_r}{\Longrightarrow}"] & TX \arrow[d, "Tr", "|" marking] \\
TTY \arrow[r, "m_Y"'] & TY
\end{tikzcd}$$

such that

$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[d, "r"', "|" marking] \arrow[dr, phantom, "\overset{\alpha_r}{\Longrightarrow}"] & TX \arrow[d, "Tr", "|" marking] \arrow[dd, bend left=60, "T(s\cdot r)", "|" marking] \\
Y \arrow[r, "e_Y"] \arrow[d, "s"', "|" marking] \arrow[dr, phantom, "\overset{\alpha_s}{\Longrightarrow}"] & TY \arrow[d, "Ts", "|" marking] \arrow[r, phantom, "\overset{\kappa_{s,r}}{\Longrightarrow}"] & {} \\
Z \arrow[r, "e_Z"'] & TZ
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[d, "s\cdot r"', "|" marking] \arrow[dr, phantom, "\overset{\alpha_{s\cdot r}}{\Longrightarrow}"] & TX \arrow[d, "T(s\cdot r)", "|" marking] \\
Z \arrow[r, "e_Z"'] & TZ
\end{tikzcd}$$

$$\begin{tikzcd}
T^2X \arrow[r, "m_X"] \arrow[d, "T^2r"', "|" marking] \arrow[dr, phantom, "\overset{\beta_r}{\Longrightarrow}"] & TX \arrow[d, "Tr", "|" marking] \arrow[dd, bend left=60, "T(s\cdot r)", "|" marking] \\
T^2Y \arrow[r, "m_Y"] \arrow[d, "T^2s"', "|" marking] \arrow[dr, phantom, "\overset{\beta_s}{\Longrightarrow}"] & TY \arrow[d, "Ts", "|" marking] \arrow[r, phantom, "\overset{\kappa_{s,r}}{\Longrightarrow}"] & {} \\
T^2Z \arrow[r, "m_Z"'] & TZ
\end{tikzcd}
=
\begin{tikzcd}
T^2X \arrow[r, "m_X"] \arrow[d, "T^2(s\cdot r)", "|" marking] \arrow[d, bend right=60, "T^2s\cdot T^2r"', "|" marking] \arrow[d, phantom, shift right=4ex, "\overset{\kappa}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\beta_{s\cdot r}}{\Longrightarrow}"] & TX \arrow[d, "T(s\cdot r)", "|" marking] \\
T^2Z \arrow[r, "m_Z"'] & TZ
\end{tikzcd}$$

$$\begin{tikzcd}
TX \arrow[r, "e_{TX}"] \arrow[d, "Tr"', "|" marking] \arrow[dr, phantom, "\overset{\alpha_{Tr}}{\Longrightarrow}"] & T^2X \arrow[r, "m_X"] \arrow[d, "T^2r", "|" marking] \arrow[dr, phantom, "\overset{\beta_r}{\Longrightarrow}"] & TX \arrow[d, "Tr", "|" marking] \\
TY \arrow[r, "e_{TY}"'] & T^2Y \arrow[r, "m_Y"'] & TY
\end{tikzcd}
=
\begin{tikzcd}
TX \arrow[r, "1"] \arrow[d, "Tr"', "|" marking] \arrow[dr, phantom, "="] & TX \arrow[d, "Tr", "|" marking] \\
TY \arrow[r, "1"'] & TY
\end{tikzcd}
=
\begin{tikzcd}
TX \arrow[r, "Te_X"] \arrow[d, "Tr"', "|" marking] \arrow[dr, phantom, "\overset{T\alpha_r}{\Longrightarrow}"] & T^2X \arrow[r, "m_X"] \arrow[d, "T^2r", "|" marking] \arrow[dr, phantom, "\overset{\beta_r}{\Longrightarrow}"] & TX \arrow[d, "Tr", "|" marking] \\
TY \arrow[r, "Te_Y"'] & T^2Y \arrow[r, "m_Y"'] & TY
\end{tikzcd}$$

$$\begin{tikzcd}
T^3X \arrow[r, "m_{TX}"] \arrow[d, "T^3r"', "|" marking] \arrow[dr, phantom, "\overset{\beta_{Tr}}{\Longrightarrow}"] & T^2X \arrow[r, "m_X"] \arrow[d, "T^2r"', "|" marking] \arrow[dr, phantom, "\overset{\beta_r}{\Longrightarrow}"] & TX \arrow[d, "Tr", "|" marking] \\
T^3Y \arrow[r, "m_{TY}"'] & T^2Y \arrow[r, "m_Y"'] & TY
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
T^3X \arrow[r, "Tm_X"] \arrow[d, "T^3r"', "|" marking] \arrow[dr, phantom, "\overset{T\beta_r}{\Longrightarrow}"] & T^2X \arrow[r, "m_X"] \arrow[d, "T^2r"', "|" marking] \arrow[dr, phantom, "\overset{\beta_r}{\Longrightarrow}"] & TX \arrow[d, "Tr", "|" marking] \\
T^3Y \arrow[r, "Tm_Y"'] & T^2Y \arrow[r, "m_Y"'] & TY
\end{tikzcd}$$

(For the pointwise version of these conditions see [7, Section 3].) We point out that here, as well as in the remaining text, no coherence issues occur since in each composition of $\mathbf{V}$-relations at most two of them are not maps.

In addition to the conditions of [7], we assume throughout that:

(3) *the initial object* 0 *of* $\mathbf{V}$ *is strict*;

(4) *the* **Set**-*functor* T *is taut*, that is, it preserves pullbacks along monomorphisms (which is in fact weaker than the Beck-Chevalley condition usually assumed in this context);

(5) $\kappa_{s,r}\colon Ts\cdot Tr \longrightarrow T(s\cdot r)$ *is an isomorphism when* s *is a* **Set**-*map.*

3. The category $(T,\mathbf{V})$-**Gph**

A $(T,\mathbf{V})$-*graph* is a pair (X,a) where X is a set and $a\colon TX \relbar\joinrel\mapstochar\joinrel\rightarrow X$ a $\mathbf{V}$-relation. (For $\mathfrak{x}\in TX$ and $x\in X$, we will sometimes denote $a(\mathfrak{x},x)$ by $X(\mathfrak{x},x)$ *à la* Lawvere [11]). A *morphism* between two $(T,\mathbf{V})$-graphs (X,a), (Y,b) is given by a map $f\colon X\longrightarrow Y$ and a 2-cell $\varphi_f\colon f\cdot a\Longrightarrow b\cdot Tf$:

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "a"', "|" marking] \arrow[dr, phantom, "\overset{\varphi_f}{\Longrightarrow}"] & TY \arrow[d, "b", "|" marking] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Given a map $f\colon X\longrightarrow Y$, there are several different ways of defining the morphism structure on f; indeed, any of the 2-cells

(Φ1) $\varphi_f\colon f\cdot a\Longrightarrow b\cdot Tf$,

(Φ2) $\widetilde{\varphi}_f\colon a\Longrightarrow f^\circ\cdot b\cdot Tf$,

(Φ3) $\widehat{\varphi}_f\colon a\cdot (Tf)^\circ\Longrightarrow f^\circ\cdot b$,

defines a morphism (f,φ_f). Each of these three descriptions can be stated pointwise. We present the one we will use mostly:

(Φ4) $\forall\mathfrak{x}\in TX,\ x\in X \qquad X(\mathfrak{x},x)\xrightarrow{\widetilde{\varphi}_f} Y(Tf(\mathfrak{x}),f(x))$.

The following diagrams show how $\widetilde{\varphi}_f$ and $\widehat{\varphi}_f$ may be obtained from φ_f.

$$\begin{tikzcd}
 & & & & TY \arrow[d, "(Tf)^\circ"'] \arrow[dr, "1"] \arrow[dr, phantom, shift right=3, "\overset{\rho_f}{\Rightarrow}"] & \\
 & TX \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\overset{\varphi_f}{\Longrightarrow}"] & TY \arrow[d, "b"] & & TX \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\overset{\varphi_f}{\Longrightarrow}"] & TY \arrow[d, "b"] \\
\widetilde{\varphi}_f\colon & X \arrow[r, "f"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3, "\overset{\lambda_f}{\Rightarrow}"] & Y \arrow[d, "f^\circ"] & \widehat{\varphi}_f\colon & X \arrow[r, "f"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3, "\overset{\lambda_f}{\Rightarrow}"] & Y \arrow[d, "f^\circ"] \\
 & & X & & & X
\end{tikzcd}$$

3.1. Definition. *A morphism* $(f,\varphi_f)\colon (X,a) \longrightarrow (Y,b)$ *is said to be*

1. fully faithful *if* $\widetilde{\varphi}_f$ *is pointwise an isomorphism;*
2. *an* embedding *if* f *is injective and fully faithful;*
3. open *if* $\widehat{\varphi}_f$ *is pointwise an isomorphism.*

(Although not used throughout, we mention that f is said to be *proper* if φ_f is pointwise an isomorphism.)

The lax functor $T\colon \mathbf{V}\text{-}\mathbf{Rel} \longrightarrow \mathbf{V}\text{-}\mathbf{Rel}$ induces an endofunctor

$$\overline{T}\colon (T,\mathbf{V})\text{-}\mathbf{Gph} \longrightarrow (T,\mathbf{V})\text{-}\mathbf{Gph},$$

with $\overline{T}(\ (X,a) \xrightarrow{(f,\varphi_f)} (Y,b)\ ) = \ ((TX,Ta) \xrightarrow{(Tf,\overline{T}(\varphi_f))} (TY,Tb)\ )$, where

$$\overline{T}(\varphi_f)\colon\ Tf\cdot Ta \xrightarrow[\cong]{\kappa_{f,a}} T(f\cdot a) \xrightarrow{T\varphi_f} T(b\cdot Tf) \xrightarrow[\cong]{\kappa^{-1}_{b,Tf}} Tb\cdot T^2f.$$

3.2. Lemma. *The functor* $\overline{T}\colon (T,\mathbf{V})\text{-}\mathbf{Gph} \longrightarrow (T,\mathbf{V})\text{-}\mathbf{Gph}$ *preserves fully faithful morphisms, embeddings, open and proper morphisms.*

Proof. Straightforward. ∎

The following result was essentially proved in [6].

3.3. Theorem. *The category* $(T,\mathbf{V})$-**Gph** *is complete and cocomplete.* ∎

We point out that in [6] by $(T,\mathbf{V})$-graph we meant *reflexive* $(T,\mathbf{V})$-graph. Here we do not assume reflexivity *a priori*. It is important to recall that in $(T,\mathbf{V})$-**Gph** limits and colimits are built as in **Set**, with a $(T,\mathbf{V})$-structure built pointwise as a limit in $\mathbf{V}$. That is, given a functor $J\colon \mathbf{D} \longrightarrow (T,\mathbf{V})\text{-}\mathbf{Gph}$ (with $\mathbf{D}$ small), where $J(\ D \xrightarrow{f} D'\ ) = (\ (X_D,a_D) \xrightarrow{(\check{f},\varphi_{\check{f}})} (X_{D'},a_{D'})\ )$, one equips the limit in **Set** $(\ L \xrightarrow{\pi_D} X_D\ )_D$ with the structure defined, for each $\mathfrak{x} \in TL$, $x \in L$, by the limit in $\mathbf{V}$ of $J_{\mathfrak{x},x}\colon \mathbf{D} \longrightarrow \mathbf{V}$, where

$$J_{\mathfrak{x},x}(\ D \xrightarrow{f} D'\ ) = (\ (X_D(T\pi_D(\mathfrak{x}),\pi_D(x)) \xrightarrow{\widetilde{\varphi}_{\check{f}}} X_{D'}(T\pi_{D'}(\mathfrak{x}),\pi_{D'}(x))\ ).$$

Colimits are constructed analogously.

We recall the infinitary version of Proposition 2.14 of [4]:

3.4. Proposition. *A category with coproducts and pullbacks along coproduct injections is infinitary extensive if, and only if, coproducts are universal and disjoint.*

We recall that a coproduct $(\sigma_D\colon X \longrightarrow X_D)_{D\in\mathbf{D}}$ is said to be *universal* if, when pulling back along any morphism $f\colon Y \longrightarrow X$, the diagram

$$\begin{tikzcd}
Y_D \arrow[r, "\sigma'_D"] \arrow[d, "f_D"'] & Y \arrow[d, "f"] \\
X_D \arrow[r, "\sigma_D"'] & X
\end{tikzcd} \tag{3.i}$$

is a coproduct diagram, i.e. $(Y_D \xrightarrow{\sigma_D} Y)_D$ is a coproduct, for every $D \in \mathbf{D}$; the coproduct $(X \xrightarrow{\sigma_D} X_D)_D$ is *disjoint* if, for every $D, D' \in \mathbf{D}$ with $D \neq D'$, the pullback of $X_D \xrightarrow{\sigma_D} X \xleftarrow{\sigma_{D'}} X_{D'}$ is the initial object.

In order to show that $(T,\mathbf{V})$-**Gph** is infinitary extensive, we revisit in particular the construction of coproducts and pullbacks.

The *coproduct* of a family $(X_D, a_D)_{D\in\mathbf{D}}$ is given by (X, a) with X the disjoint union of the sets X_D, with inclusions $\sigma_D\colon X_D \longrightarrow X$, and

$$X(\mathfrak{x}, x) = \begin{cases} X_D(\mathfrak{x}, x) & \text{if } \mathfrak{x} \in TX_D,\ x \in X_D \\ 0 & \text{otherwise} \end{cases}$$

(where, for simplicity, we consider that the injective map $T\sigma_D$ is an inclusion). With $\varphi_D = \mathrm{id}\colon \sigma_D \cdot a_D \Longrightarrow a \cdot T\sigma_D$, $(\sigma_D, \varphi_D)\colon (X_D, a_D) \longrightarrow (X, a)$ are morphisms of $(T,\mathbf{V})$-graphs, and it is easily checked that they have the coproduct universal property. The coproduct of the empty family, that is, the *initial object* in $(T,\mathbf{V})$-**Gph** is the empty set with the trivial $(T,\mathbf{V})$-graph structure.

The description of the $(T,\mathbf{V})$-graph structure of the coproduct gives us immediately the following result:

3.5. Proposition. *Let $(X_D, a_D)_D$ be a family of $(T,\mathbf{V})$-graphs and $(\sigma_D\colon X_D \longrightarrow X)_D$ a coproduct in* **Set**. *For a $(T,\mathbf{V})$-graph (X, a), the following assertions are equivalent.*

*(i) $(\sigma_D, \varphi_D)\colon (X_D, a_D) \longrightarrow (X, a)$ is a coproduct in $(T,\mathbf{V})$-**Gph**.*

(ii) Each (σ_D, φ_D) is an open embedding. ∎

Given morphisms $(X, a) \xrightarrow{f} (Y, b) \xleftarrow{g} (Z, c)$ of $(T,\mathbf{V})$-graphs, their *pullback* is the pullback in **Set**

$$\begin{tikzcd}
X \times_Y Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd} \tag{3.ii}$$

and, for each $\mathfrak{w} \in T(X \times_Y Z)$, $(x,z) \in X \times_Y Z$, $(X \times_Y Z)(\mathfrak{w},(x,z))$ and $\widetilde{\varphi}_{\pi_1}$ and $\widetilde{\varphi}_{\pi_2}$ are given by the pullback in $\mathbf{V}$

$$\begin{tikzcd}
(X \times_Y Z)(\mathfrak{w},(x,z)) \arrow[r, "\widetilde{\varphi}_{\pi_2}"] \arrow[d, "\widetilde{\varphi}_{\pi_1}"'] & Z(T\pi_2(\mathfrak{w}),z) \arrow[d, "\widetilde{\varphi}_g"] \\
X(T\pi_1(\mathfrak{w}),x) \arrow[r, "\widetilde{\varphi}_f"'] & Y(T(f\cdot\pi_1)(\mathfrak{w}),f(x)).
\end{tikzcd} \tag{3.iii}$$

3.6. Lemma.

1. *Both fully faithful morphisms and embeddings are stable under pullback.*
2. *Open embeddings are pullback-stable.*

Proof. 1. In diagrams (3.ii) and (3.iii) above, assume that f is fully faithful. If f is injective, then π_2 is injective; pointwise $\widetilde{\varphi}_{\pi_2}$ is defined as the pullback of an isomorphism, therefore both fully faithful morphisms and embeddings are stable under pullback.

2. Now let $f\colon (X,a) \longrightarrow (Y,b)$ be an open embedding. Then, for each $\mathfrak{y} \in TY$, $x \in X$, $(\, a\cdot(Tf)^\circ)(\mathfrak{y},x) \xrightarrow{\cong} (f^\circ \cdot b)(\mathfrak{y},x)$, that is,

$$\textstyle\sum_{Tf(\mathfrak{x})=\mathfrak{y}} X(\mathfrak{x},x) \xrightarrow{\widehat{\varphi}_f} Y(\mathfrak{y},f(x)) \text{ is an isomorphism.}$$

With f also Tf is injective, and therefore this isomorphism translates to

$$Y(\mathfrak{y},f(x)) = \begin{cases} X(\mathfrak{x},f(x)) & \text{if } \mathfrak{y} = Tf(\mathfrak{x}) \\ 0 & \text{otherwise.} \end{cases}$$

To show that π_2 is an open embedding, let $\mathfrak{z} \in TZ$ and $(x,z) \in X \times_Y Z$. If $\mathfrak{z} = T\pi_2(\mathfrak{w})$ for some $\mathfrak{w} \in T(X \times_Y Z)$, then we already know that $(X \times_Y Z)(\mathfrak{w},(x,z)) \cong Z(\mathfrak{z},z)$; otherwise, since T preserves the pullback (3.ii), $Tg(\mathfrak{z})$ is not in the image of Tf, and therefore $Y(Tg(\mathfrak{z}),g(z)) = 0$. Since 0 is a strict initial object of $\mathbf{V}$, we may conclude that $Z(\mathfrak{z},z) = 0$. ■

3.7. Theorem. *The category* $(T,\mathbf{V})$-**Gph** *is infinitary extensive.*

Proof. $(T,\mathbf{V})$-**Gph** is complete, and so in particular it has finite limits.

Let $(\, (X_D,a_D) \xrightarrow{\sigma_D} (X,a)\,)_D$ be a coproduct in $(T,\mathbf{V})$-**Gph**. Given a morphism $f\colon (Y,b) \longrightarrow (X,a)$ in $(T,\mathbf{V})$-**Gph**, form the pullback of σ_D along f:

$$\begin{tikzcd}
(Y_D,b_D) \arrow[r, "\sigma'_D"] \arrow[d, "f'_D"'] & (Y,b) \arrow[d, "f"] \\
(X_D,a_D) \arrow[r, "\sigma_D"'] & (X,a).
\end{tikzcd}$$

Then, due to extensivity of **Set**, $(\, Y_D \xrightarrow{\sigma'_D} Y\,)_D$ is the coproduct in **Set**; together with pullback stability of open embeddings, using Proposition 3.5 one concludes that $(\,(Y_D, b_D) \xrightarrow{\sigma'_D} (Y, b)\,)_D$ is a coproduct in $(T, \mathbf{V})$-**Gph**, that is, coproducts are universal.

To check that they are also disjoint, let $X_D \xrightarrow{\sigma_D} X \xleftarrow{\sigma_{D'}} X_{D'}$ be distinct coproduct injections. Since coproducts in **Set** are disjoint, their pullback is the empty set with the only possible $(T, \mathbf{V})$-graph structure, that is, it is the initial object of $(T, \mathbf{V})$-**Gph**. ■

4. $(T, \mathbf{V})$-**Cat** is infinitary extensive

A $(T, \mathbf{V})$-*category* is a $(T, \mathbf{V})$-graph (X, a) equipped with two additional natural transformations

$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[dr, "1"'] & TX \arrow[d, "a", "\shortmid"{marking}] \arrow[dl, phantom, "\overset{\eta_a}{\Rightarrow}" near start] \\
& X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
TX \arrow[d, "a"', "\shortmid"{marking}] & T^2X \arrow[l, "Ta"', "\shortmid"{marking}] \arrow[d, "m_X"] \arrow[dl, phantom, "\overset{\mu_a}{\Longrightarrow}"] \\
X & TX \arrow[l, "a", "\shortmid"{marking}]
\end{tikzcd}$$

providing a *generalized monad structure on* a; that is,

$$\begin{tikzcd}
TX \arrow[r, "e_{TX}"] \arrow[d, "a"', "\shortmid"{marking}] \arrow[dr, phantom, "\overset{\alpha_a}{\Longrightarrow}"] & T^2X \arrow[d, "Ta", "\shortmid"{marking}] \arrow[dr, "m_X"] & \\
X \arrow[r, "e_X"] \arrow[dr, "1"'] & TX \arrow[d, "a", "\shortmid"{marking}] \arrow[r, phantom, "\overset{\mu_a}{\Longrightarrow}"] \arrow[dl, phantom, "\overset{\eta_a}{\Rightarrow}" near start] & TX \arrow[dl, "a", "\shortmid"{marking}] \\
& X &
\end{tikzcd}
=
\begin{tikzcd}
TX \arrow[r, "1"] \arrow[dd, "a"', "\shortmid"{marking}] & TX \arrow[dd, "a", "\shortmid"{marking}] \\
\arrow[r, phantom, "="] & {} \\
X \arrow[r, "1"'] & X
\end{tikzcd}
=
\begin{tikzcd}
TX \arrow[d, bend right=60, "1"', "\shortmid"{marking}] \arrow[d, "T(ae_X)", "\shortmid"{marking}] \arrow[dr, "Te_X"] \arrow[d, phantom, "\overset{T\eta}{\Rightarrow}", shift right=6] \arrow[dr, phantom, "\cong"'] & \\
TX \arrow[d, "a"', "\shortmid"{marking}] & T^2X \arrow[l, "Ta", "\shortmid"{marking}] \arrow[d, "m_X"] \arrow[dl, phantom, "\overset{\mu_a}{\Longrightarrow}"] \\
X & TX \arrow[l, "a", "\shortmid"{marking}]
\end{tikzcd}$$

(4.i)

and

$$\begin{tikzcd}
T^3X \arrow[r, "m_{TX}"] \arrow[d, "T^2a"', "\shortmid"{marking}] \arrow[dr, phantom, "\overset{\beta_a}{\Longrightarrow}"] & T^2X \arrow[d, "Ta", "\shortmid"{marking}] \arrow[dr, "m_X"] & \\
T^2X \arrow[r, "m_X"] \arrow[d, "Ta"', "\shortmid"{marking}] \arrow[dr, phantom, "\overset{\mu_a}{\Longrightarrow}"] & TX \arrow[d, "a", "\shortmid"{marking}] \arrow[r, phantom, "\overset{\mu_a}{\Longrightarrow}"] & TX \arrow[dl, "a", "\shortmid"{marking}] \\
TX \arrow[r, "a"', "\shortmid"{marking}] & X &
\end{tikzcd}
=
\begin{tikzcd}
& & T^3X \arrow[dll, "T^2a"', "\shortmid"{marking}] \arrow[drr, "Tm_X"] \arrow[dd, bend right=30, "T(a\cdot Ta)"', "\shortmid"{marking}] \arrow[dd, bend left=30, "T(a\cdot m_X)", "\shortmid"{marking}] & & & \\
T^2X \arrow[drr, "Ta"', "\shortmid"{marking}] \arrow[r, phantom, "\overset{\kappa}{\Longrightarrow}"] & {} & \overset{T\mu}{\Rightarrow} & \cong & T^2X \arrow[r, "m_X"] \arrow[dll, "Ta", "\shortmid"{marking}] & TX \arrow[d, "a", "\shortmid"{marking}] \arrow[dl, phantom, "\overset{\mu_a}{\Longrightarrow}"] \\
& & TX \arrow[rrr, "a"', "\shortmid"{marking}] & & & X
\end{tikzcd}$$

(4.ii)

Given two $(T, \mathbf{V})$-categories (X, a), (Y, b), a $(T, \mathbf{V})$-*functor* $(f, \varphi_f)\colon (X, a) \longrightarrow (Y, b)$ is a map $f\colon X \longrightarrow Y$ together with a natural transformation

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "a"', "\shortmid"{marking}] \arrow[dr, phantom, "\overset{\varphi_f}{\Longrightarrow}"] & TY \arrow[d, "b", "\shortmid"{marking}] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

– i.e. it is a morphism in $(T,\mathbf{V})$-**Gph** – *preserving the generalized monad structures on a and b*:

$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[dr, "1"'] \arrow[dr, phantom, "\overset{\eta_a}{\Rightarrow}", shift left=3] & TX \arrow[r, "Tf"] \arrow[d, "a", "\shortmid" marking] \arrow[dr, phantom, "\overset{\varphi_f}{\Longrightarrow}"] & TY \arrow[d, "b", "\shortmid" marking] & = & X \arrow[r, "f"] & Y \arrow[r, "e_Y"] \arrow[dr, "1"'] \arrow[dr, phantom, "\overset{\eta_b}{\Rightarrow}", shift left=3] & TY \arrow[d, "b", "\shortmid" marking] \\
& X \arrow[r, "f"'] & Y & & & & Y
\end{tikzcd} \tag{4.iii}$$

and

$$\begin{tikzcd}
& T^2X \arrow[rr, "T^2f"] \arrow[dl, "Ta"', "\shortmid" marking] \arrow[dr, "m_X"] \arrow[dr, phantom, "="', shift left=8] & & T^2Y \arrow[dr, "m_Y"] & & & & T^2X \arrow[rr, "T^2f"] \arrow[dl, "Ta"', "\shortmid" marking] \arrow[dr, phantom, "\overset{\overline{T}\varphi_f}{\Longrightarrow}"] & & T^2Y \arrow[dl, "Tb"', "\shortmid" marking] \arrow[dr, "m_Y"] & \\
TX \arrow[dr, "a"', "\shortmid" marking] \arrow[rr, phantom, "\overset{\mu_a}{\Longrightarrow}"] & & TX \arrow[rr, "Tf"] \arrow[dl, "a", "\shortmid" marking] \arrow[dr, phantom, "\overset{\varphi_f}{\Longrightarrow}", shift left=4] & & TY \arrow[dl, "b", "\shortmid" marking] & = & TX \arrow[rr, "Tf"] \arrow[dr, "a"', "\shortmid" marking] \arrow[dr, phantom, "\overset{\varphi_f}{\Longrightarrow}", shift left=6] & & TY \arrow[dr, "b"', "\shortmid" marking] \arrow[rr, phantom, "\overset{\mu_b}{\Longrightarrow}"] & & TY \arrow[dl, "b"', "\shortmid" marking] \\
& X \arrow[rr, "f"'] & & Y & & & & X \arrow[rr, "f"'] & & Y &
\end{tikzcd} \tag{4.iv}$$

(For the pointwise version of these equalities see [7, Section 4].)

4.1. Examples. As shown in [7], when $\mathbf{V} = \mathbf{Set}$ and T is the free-monoid **Set**-monad naturally extended to **Set**-**Rel**, $(T,\mathbf{V})$-**Cat** is the category of multicategories. Furthermore, when T is the ultrafilter monad on **Set** and $\mathbf{V} = \{0 < 1\}$ or $\mathbf{V}$ is the half-real line *à la* Lawvere, then $(T,\mathbf{V})$-**Cat** is, respectively, the category of topological spaces (Barr [1]) and the category of Lowen's approach spaces. (For more examples see [7].)

4.2. Proposition. $(T,\mathbf{V})$-**Cat** *has coproducts and they are preserved by the forgetful functor* $(T,\mathbf{V})\text{-}\mathbf{Cat} \longrightarrow (T,\mathbf{V})\text{-}\mathbf{Gph}$.

Proof. Let $(X_D, a_D)_{D\in\mathbf{D}}$ be a family of $(T,\mathbf{V})$-categories, and (X,a) their coproduct in $(T,\mathbf{V})$-**Gph** as built in Section 3; that is, X is the disjoint union of the sets X_D, with inclusions $\sigma_D\colon X_D \longrightarrow X$, and, for each $\mathfrak{x} \in TX$ and $x \in X_D$, $X(\mathfrak{x}, x) = X_D(\mathfrak{x}_D, x)$ if there is $\mathfrak{x}_D \in TX_D$ such that $T\sigma_D(\mathfrak{x}_D) = \mathfrak{x}$, and $X(\mathfrak{x}, x) = 0$ otherwise. Hence, we can define η_a, for each $x \in X_D$, as:

$$\eta_a(x,x)\colon I \xrightarrow{\eta_{a_D}} X_D(e_{X_D}(x), x) = X(e_X(x), x)\ .$$

In order to define, for each $\mathfrak{X} \in T^2X$, $\mathfrak{x} \in TX$, $x \in X_D$,

$$\mu_a\colon TX(\mathfrak{X}, \mathfrak{x}) \otimes X(\mathfrak{x}, x) \longrightarrow X(m_X(\mathfrak{X}), x)\ ,$$

we observe that $T\sigma_D$ is also an open embedding. If $\mathfrak{x} = T\sigma_D(\mathfrak{x}_D)$ and $\mathfrak{X} = T^2\sigma_D(\mathfrak{X}_D)$, then

$$\mu_a\colon TX(\mathfrak{X}, \mathfrak{x}) \otimes X(\mathfrak{x}, x) = TX_D(\mathfrak{X}_D, \mathfrak{x}_D) \otimes X_D(\mathfrak{x}_D, x) \xrightarrow{\mu_{a_D}} a_D(m_{X_D}(\mathfrak{X}_D), x) = a(m_X(\mathfrak{X}), x).$$

Otherwise, $TX(\mathfrak{X}, \mathfrak{x}) \otimes X(\mathfrak{x}, x) = 0$ and μ_a is trivial. From the way η_a and μ_a were defined we conclude that:

- the equalities of diagrams (4.i) and (4.ii) follow from the corresponding equalities for η_{a_D} and μ_{a_D};
- this way σ_D becomes a $(T,\mathbf{V})$-functor for every D, and, moreover, this is the only $(T,\mathbf{V})$-category structure on the $(T,\mathbf{V})$-graph (X,a) that makes σ_D a $(T,\mathbf{V})$-functor;
- $(\sigma_D\colon (X_D,a_D)\longrightarrow (X,a))_D$ is a coproduct in $(T,\mathbf{V})$-**Cat**, and, as in $(T,\mathbf{V})$-**Gph**, the coproduct injections are open embeddings.

■

4.3. Lemma. *If (Y,b) is a $(T,\mathbf{V})$-category and $(f,\varphi_f)\colon (X,a)\longrightarrow (Y,b)$ is an embedding in $(T,\mathbf{V})$-**Gph**, then (X,a) has a $(T,\mathbf{V})$-category structure so that (f,φ_f) is a $(T,\mathbf{V})$-functor.*

Proof. With f, also Tf is an embedding in $(T,\mathbf{V})$-**Gph**. Hence we may consider that both f and Tf are inclusions, and the isomorphisms of $(\Phi 4)$ read, for every $\mathfrak{X}\in T^2X$, $\mathfrak{x}\in TX$, $x\in X$, as

$$TX(\mathfrak{X},\mathfrak{x})\cong TY(\mathfrak{X},\mathfrak{x}),\ \text{and}\ X(\mathfrak{x},x)\cong Y(\mathfrak{x},x).$$

Defining η_a and μ_a as (co)restrictions of η_b and μ_b, the equalities of diagrams (4.i) and (4.ii) for (X,a) follow immediately from the corresponding equalities for (Y,b).

The equalities of diagrams (4.iii) and (4.iv) follow by similar arguments, taking into account that both f and Tf are inclusions, and therefore (f,φ_f) is a morphism in $(T,\mathbf{V})$-**Cat** as claimed. ■

4.4. Proposition. $(T,\mathbf{V})$-**Cat** *has pullbacks along embeddings and they are preserved by the forgetful functor* $(T,\mathbf{V})$-**Cat** $\longrightarrow (T,\mathbf{V})$-**Gph**.

Proof. Let (X,a), (Y,b), (Z,c) be $(T,\mathbf{V})$-categories, and $(f,\varphi_f)\colon (X,a)\longrightarrow (Y,b)$ and $g\colon (Z,c)\longrightarrow (Y,b)$ be $(T,\mathbf{V})$-functors, with f an embedding. Form their pullback (3.ii)-(3.iii) in $(T,\mathbf{V})$-**Gph**. Since π_2 is an embedding, by the lemma above $X\times_Y Z$ has a $(T,\mathbf{V})$-category structure induced by the one of (Z,c) which makes π_2 a $(T,\mathbf{V})$-functor. Moreover, π_1 is nothing but a restriction and a corestriction of the $(T,\mathbf{V})$-functor g, hence it is also a $(T,\mathbf{V})$-functor. The universal property of the pullback follows easily from the universal property of the diagram when considered in $(T,\mathbf{V})$-**Gph** and the fact that π_2 is an embedding. ■

4.5. Theorem. *The category* $(T,\mathbf{V})$-**Cat** *is infinitary extensive.*

Proof. We make use again of Proposition 3.4. Propositions 4.2 and 4.4 assure that $(T,\mathbf{V})$-**Cat** has coproducts and pullbacks along coproduct injections. Given diagrams (3.i) in $(T,\mathbf{V})$-**Cat**, we know that $(\ Y_D \xrightarrow{\sigma'_D} Y\ )$ is a coproduct in $(T,\mathbf{V})$-**Gph** and that each σ'_D is an open embedding in $(T,\mathbf{V})$-**Cat**. Hence, from Proposition 4.2 (and its proof) we conclude that Y, as a $(T,\mathbf{V})$-category, must have the structure that makes $(\, Y_D \xrightarrow{\sigma'_D} Y \,)$ a coproduct in $(T,\mathbf{V})$-**Cat**.

Finally, from Proposition 4.4 it follows that coproducts in $(T,\mathbf{V})$-**Cat** are disjoint. ■

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