

# SPATIAL REALIZATION OF A LIE ALGEBRA AND THE BAR CONSTRUCTION

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**ABSTRACT.** We prove that the spatial realization of a rational complete Lie algebra  $L$ , concentrated in degree 0, is isomorphic to the simplicial bar construction on the group, obtained from the Baker-Campbell-Hausdorff product on  $L$ .

## Introduction

In [5], we construct a cosimplicial differential graded complete Lie algebra (henceforth  $\mathbf{cdgl}$ )  $(\mathfrak{L}_\bullet, d)$ , in which  $(\mathfrak{L}_1, d)$  is the Lawrence-Sullivan model of the interval introduced in [8]. As in the work of Sullivan ([10]) for differential commutative graded algebras, the existence of this cosimplicial object gives an adjoint pair of functors between the category  $\mathbf{cdgl}$  of  $\mathbf{cdgl}$ 's and the category  $\mathbf{Sset}$  of simplicial sets, see [5] or [3]. In this work, we focus on one of them, the spatial realization functor,

$$\langle - \rangle : \mathbf{cdgl} \rightarrow \mathbf{Sset},$$

defined by  $\langle L \rangle = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L)$  for  $L \in \mathbf{cdgl}$ . (Let us also notice that  $\langle L \rangle$  is isomorphic to the nerve of  $L$ , a deformation retract of the Getzler-Hinich realization, see [2], [9].)

More precisely, we are interested in the realization  $\langle L \rangle$  of a complete Lie algebra,  $L$ , concentrated in degree 0 and (thus) with the differential 0. In this case, a group structure can be defined on the set  $L$  from the Baker-Campbell-Hausdorff formula. We denote by  $\exp L$  this group. The realization  $\langle L \rangle$  is an Eilenberg-MacLane space of type  $K(\pi, 1)$ , see [6]. The purpose of this work is the determination of  $\langle L \rangle$  up to isomorphism.

**0.1. MAIN THEOREM.** *Let  $L$  be a complete differential graded Lie algebra, concentrated in degree 0. Then, its spatial realization  $\langle L \rangle$  is isomorphic to the simplicial bar construction on  $\exp L$ .*

Let  $K(G, 1)$  be an Eilenberg-MacLane space and  $L$  the Lie algebra structure on its fundamental group. We can consider the realizations of the  $A_{PL}(K(G, 1))$  of D. Sullivan ([10]) and  $\mathrm{MC}_*(L)$  of E. Getzler ([7]). If  $L$  is of finite type, A. Berglund proves in [1,

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Proposition 1.6] that these two realizations are isomorphic. In [5, Theorem 8.1], we show the existence of a weak equivalence between them and the previous spatial realization  $\langle L \rangle$ . In our setting, we prove that the simplicial set  $\langle L \rangle = \text{Hom}_{\mathbf{cdgl}}(\mathcal{L}_\bullet, L)$  is isomorphic to a simplicial bar resolution. With [3], we know that spaces more general than a  $K(G, 1)$  admit a  $\mathbf{cdgl}$  model  $L$ . For them, the simplicial set  $\text{Hom}_{\mathbf{cdgl}}(\mathcal{L}_\bullet, L)$  appears as a natural extension of the bar construction. We will come back on this point in a forthcoming work.

In Section 1, we recall basic background on  $\mathbf{cdgl}$  and our construction  $\mathfrak{L}_\bullet$ . Section 2 consists of the proof of the Main theorem.

## 1. Some reminders

We first recall the construction of the cosimplicial  $\mathbf{cdgl}$   $\mathfrak{L}_\bullet$ . Let  $V$  be a finite dimensional graded vector space. The completion of the free graded Lie algebra on  $V$ ,  $\mathbb{L}(V)$ , is the inverse limit,

$$\widehat{\mathbb{L}}(V) = \varprojlim_n \mathbb{L}(V)/\mathbb{L}^{\geq n}(V),$$

where  $\mathbb{L}^{\geq n}(V)$  is the ideal generated by the Lie brackets of length  $\geq n$ . We call  $\widehat{\mathbb{L}}(V)$  the free complete graded Lie algebra on  $V$ .

As a graded Lie algebra,  $\mathfrak{L}_n$  is the free complete graded Lie algebra on the rational vector space generated by the elements  $a_{i_0 \dots i_k}$  of degree  $|a_{i_0 \dots i_k}| = k - 1$ , with  $0 \leq i_0 < \dots < i_k \leq n$ . We denote by  $\text{ad}_{i_0 \dots i_k}$  the Lie derivation  $[a_{i_0 \dots i_k}, -]$ . The  $\mathbf{cdgl}$   $\mathfrak{L}_n$  satisfies the following properties.

–  $\mathfrak{L}_0$  is the free  $\mathbf{cdgl}$  on a Maurer-Cartan element  $a_0$ , that is:

$$\mathfrak{L}_0 = (\mathbb{L}(a_0), d), \quad da_0 = -\frac{1}{2}[a_0, a_0].$$

–  $\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), d)$  is the Lawrence-Sullivan interval (see [8]), where  $a_0$  and  $a_1$  are Maurer-Cartan elements and

$$da_{01} = [a_{01}, a_1] + \frac{\text{ad}_{01}}{e^{\text{ad}_{01}} - 1}(a_1 - a_0).$$

– A model  $\mathfrak{L}_2$  for the triangle has been described in [5] (see also [4]):

$$\mathfrak{L}_2 = (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{02}, a_{12}, a_{012}), d) \text{ with } d(a_{012}) = a_{01} * a_{12} * a_{02}^{-1} - [a_0, a_{012}].$$

Here  $*$  denotes the Baker-Campbell-Hausdorff product defined for any pair of elements  $a, b$  of degree 0 by  $a * b = \log(\exp^a \exp^b)$ .

– Moreover these structures appear in each  $\mathfrak{L}_n$ : each vertex  $a_r$  is a Maurer-Cartan element, each triple  $(a_r, a_s, a_{rs})$  is a Lawrence-Sullivan interval and each family  $(a_r, a_s, a_t, a_{rs}, a_{rt}, a_{st}, a_{rst})$  is a triangle as above.

The family  $(\mathfrak{L}_n)_{n \geq 0}$  forms a cosimplicial cdgl which allows the definition of the spatial realization of  $L \in \mathbf{cdgl}$  by,

$$\langle L \rangle := \text{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L).$$

The cofaces  $\delta^i$  and the codegeneracies  $\sigma^i$  of the cosimplicial cdgl  $\mathfrak{L}_\bullet$  are defined by

$$\delta^i a_{i_0 \dots i_p} = a_{j_0 \dots j_p} \quad \text{with} \quad j_k = \begin{cases} i_k & \text{if } i_k < i, \\ i_k + 1 & \text{if } i_k \geq i, \end{cases} \quad (1)$$

$$\sigma^i a_{i_0 \dots i_p} = \begin{cases} 0 & \text{if } \{i, i+1\} \subset \{i_0, \dots, i_p\}, \\ a_{j_0 \dots j_p} & \text{otherwise, with } j_k = \begin{cases} i_k & \text{if } i_k \leq i, \\ i_k - 1 & \text{if } i_k > i. \end{cases} \end{cases} \quad (2)$$

## 2. Proof of the main theorem

The simplicial bar construction on a group  $G$ , is the simplicial set  $B_\bullet G$  with set of  $n$ -simplices  $B_n G = G^n$ . Its elements are denoted  $[g_1 | \dots | g_n]$ , with  $g_i \in G$ . The faces  $d_i$  and degeneracies  $s_i$  of  $B_\bullet G$  are defined as follows:

$$\begin{aligned} d_0 [g_1 | \dots | g_n] &= [g_2 | \dots | g_n], \\ d_i [g_1 | \dots | g_n] &= [g_1 | \dots | g_i g_{i+1} | \dots | g_n], \quad \text{for } 0 < i < n, \\ d_n [g_1 | \dots | g_n] &= [g_1 | \dots | g_{n-1}]. \end{aligned} \quad (3)$$

The degeneracy  $s_i$  inserts the identity  $e$  of  $G$  in position  $i$ .

Let  $L \in \mathbf{cdgl}$  be generated in degree 0 and  $f: \mathfrak{L}_n \rightarrow L$  a morphism in  $\mathbf{cdgl}$ . For degree reasons, we have  $f(a_{i_0 \dots i_k}) = 0$  if  $k \neq 1$ . Moreover, since  $f$  commutes the differential, from the definition of the differential in  $\mathfrak{L}_2$ , we get

$$0 = df(a_{0rs}) = f(a_{0r}) * f(a_{rs}) * f(a_{0s})^{-1}.$$

Therefore, for any  $r, s > 0$ , we have

$$f(a_{rs}) = f(a_{0r})^{-1} * f(a_{0s}).$$

The map  $f$  being entirely defined by its values on the  $a_{0i}$ , we have a bijection

$$\Phi: \text{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_n, L) \rightarrow L^n, \quad \text{defined by } f \mapsto (f(a_{01}), f(a_{02}), \dots, f(a_{0n})).$$

We now determine the image of the faces and degeneracies on  $\text{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L)$ , induced from (1) and (2). For the face operators, as only the elements  $(a_{0r})$  play a role, it suffices to consider,

$$\delta^i(a_{0r}) = \begin{cases} a_{0r} & \text{if } r < i \\ a_{0(r+1)} & \text{if } r \geq i \end{cases} \quad \text{for } i > 0, \quad \text{and} \quad \delta^0(a_{0r}) = a_{1(r+1)}.$$

Let  $f: \mathfrak{L}_n \rightarrow L$  be specified by  $(f(a_{01}), \dots, f(a_{0n}))$ . Then  $d_0 f = f \circ \delta^0: \mathfrak{L}_{n-1} \rightarrow L$  is described by

$$\begin{aligned} d_0(f(a_{01}), \dots, f(a_{0(n-1)})) &= (f \circ \delta^0(a_{01}), \dots, f \circ \delta^0(a_{0(n-1)})) \\ &= (f(a_{12}), \dots, f(a_{1n})) \\ &= (f(a_{01})^{-1} * f(a_{02}), \dots, f(a_{01})^{-1} * f(a_{0n})). \end{aligned}$$

Therefore, the face operator  $d_0$  on  $L^\bullet$ , induced from  $\Phi$ , is

$$d_0(x_1, \dots, x_n) = (x_1^{-1} * x_2, \dots, x_1^{-1} * x_n).$$

Similar arguments give, for  $i > 0$ ,

$$d_i(x_1, \dots, x_n) = (x_1, \dots, \hat{x}_i, \dots, x_n).$$

As for the degeneracies, starting from

$$\begin{aligned} \sigma^i(a_{0r}) &= \begin{cases} a_{0r} & \text{if } r \leq i \\ a_{0(r-1)} & \text{if } r > i \end{cases} \quad \text{for } i > 0, \\ \sigma^0(a_{0r}) &= a_{0(r-1)} \quad \text{if } r > 1, \quad \text{and } \sigma^0(a_{01}) = 0, \end{aligned}$$

we get

$$s_0(x_1, \dots, x_n) = (0, x_1, \dots, x_n)$$

and for  $i > 0$ ,

$$s_i(x_1, \dots, x_n) = (x_1, \dots, x_i, x_i, \dots, x_n).$$

Now a straightforward and easy computation shows that the morphism

$$\Psi: \text{Hom}_{\text{cdgl}}(\mathfrak{L}_\bullet, L) \rightarrow B_\bullet(\exp L)$$

defined by

$$\Psi(f) = [f(a_{01}) | f(a_{01})^{-1} f(a_{02}) | f(a_{02})^{-1} f(a_{03}) | \dots | f(a_{0(n-1)})^{-1} f(a_{0n})]$$

is an isomorphism of simplicial sets.

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