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SPATIAL REALIZATION OF A LIE ALGEBRA AND THE BAR CONSTRUCTION

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ABSTRACT. We prove that the spatial realization of a rational complete Lie algebra L, concentrated in degree 0, is isomorphic to the simplicial bar construction on the group, obtained from the Baker-Campbell-Hausdorff product on L.

Introduction

In [5], we construct a cosimplicial differential graded complete Lie algebra (henceforth cdgl) $(\mathfrak{L}_\bullet, d)$, in which $(\mathfrak{L}_1, d)$ is the Lawrence-Sullivan model of the interval introduced in [8]. As in the work of Sullivan ([10]) for differential commutative graded algebras, the existence of this cosimplicial object gives an adjoint pair of functors between the category **cdgl** of cdgl's and the category **Sset** of simplicial sets, see [5] or [3]. In this work, we focus on one of them, the spatial realization functor,

$$\langle - \rangle \colon \mathbf{cdgl} \to \mathbf{Sset},$$

defined by $\langle L \rangle = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L)$ for $L \in \mathbf{cdgl}$. (Let us also notice that $\langle L \rangle$ is isomorphic to the nerve of L, a deformation retract of the Getzler-Hinich realization, see [2], [9].)

More precisely, we are interested in the realization $\langle L \rangle$ of a complete Lie algebra, L, concentrated in degree 0 and (thus) with the differential 0. In this case, a group structure can be defined on the set L from the Baker-Campbell-Hausdorff formula. We denote by $\exp\ L$ this group. The realization $\langle L \rangle$ is an Eilenberg-MacLane space of type $K(\pi, 1)$, see [6]. The purpose of this work is the determination of $\langle L \rangle$ up to isomorphism.

0.1. MAIN THEOREM. *Let L be a complete differential graded Lie algebra, concentrated in degree 0. Then, its spatial realization $\langle L \rangle$ is isomorphic to the simplicial bar construction on* $\exp\ L$.

Let $K(G, 1)$ be an Eilenberg-Maclane space and L the Lie algebra structure on its fundamental group. We can consider the realizations of the $A_{PL}(K(G, 1))$ of D. Sullivan ([10]) and $\mathrm{MC}_*(L)$ of E. Getzler ([7]). If L is of finite type, A. Berglund proves in [1,

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Proposition 1.6] that these two realizations are isomorphic. In [5, Theorem 8.1], we show the existence of a weak equivalence between them and the previous spatial realization $\langle L\rangle$. In our setting, we prove that the simplicial set $\langle L\rangle = \mathrm{Hom}_{\mathbf{cdgl}}(\mathcal{L}_\bullet, L)$ is isomorphic to a simplicial bar resolution. With [3], we know that spaces more general than a $K(G,1)$ admit a cdgl model L. For them, the simplicial set $\mathrm{Hom}_{\mathbf{cdgl}}(\mathcal{L}_\bullet, L)$ appears as a natural extension of the bar construction. We will come back on this point in a forthcoming work.

In Section 1, we recall basic background on **cdgl** and our construction $\mathfrak{L}_\bullet$. Section 2 consists of the proof of the Main theorem.

1. Some reminders

We first recall the construction of the cosimplicial cdgl $\mathfrak{L}_\bullet$. Let V be a finite dimensional graded vector space. The completion of the free graded Lie algebra on V, $\mathbb{L}(V)$, is the inverse limit,

$$\widehat{\mathbb{L}}(V) = \varprojlim_n \mathbb{L}(V)/\mathbb{L}^{\geq n}(V),$$

where $\mathbb{L}^{\geq n}(V)$ is the ideal generated by the Lie brackets of length $\geq n$. We call $\widehat{\mathbb{L}}(V)$ the free complete graded Lie algebra on V.

As a graded Lie algebra, $\mathfrak{L}_n$ is the free complete graded Lie algebra on the rational vector space generated by the elements $a_{i_0\dots i_k}$ of degree $|a_{i_0\dots i_k}| = k-1$, with $0 \leq i_0 < \dots < i_k \leq n$. We denote by $\mathrm{ad}_{i_0\dots i_k}$ the Lie derivation $[a_{i_0\dots i_k}, -]$. The cdgl $\mathfrak{L}_n$ satisfies the following properties.
– $\mathfrak{L}_0$ is the free cdgl on a Maurer-Cartan element a_0, that is:

$$\mathfrak{L}_0 = (\mathbb{L}(a_0), d)\,, \quad da_0 = -\frac{1}{2}[a_0, a_0].$$

– $\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), d)$ is the Lawrence-Sullivan interval (see [8]), where a_0 and a_1 are Maurer-Cartan elements and

$$da_{01} = [a_{01}, a_1] + \frac{\mathrm{ad}_{01}}{e^{\mathrm{ad}_{01}} - 1}(a_1 - a_0).$$

– A model $\mathfrak{L}_2$ for the triangle has been described in [5] (see also [4]):

$$\mathfrak{L}_2 = (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{02}, a_{12}, a_{012}), d) \text{ with } d(a_{012}) = a_{01} * a_{12} * a_{02}^{-1} - [a_0, a_{012}].$$

Here $*$ denotes the Baker-Campbell-Hausdorff product defined for any pair of elements a, b of degree 0 by $a * b = \log(\exp^a \exp^b)$.
– Moreover these structures appear in each $\mathfrak{L}_n$: each vertex a_r is a Maurer-Cartan element, each triple (a_r, a_s, a_{rs}) is a Lawrence-Sullivan interval and each family $(a_r, a_s, a_t, a_{rs}, a_{rt}, a_{st}, a_{rst})$ is a triangle as above.

The family $(\mathfrak{L}_n)_{n\geq 0}$ forms a cosimplicial cdgl which allows the definition of the spatial realization of $L \in \mathbf{cdgl}$ by,

$$\langle L \rangle := \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L).$$

The cofaces δ^i and the codegeneracies σ^i of the cosimplicial cdgl $\mathfrak{L}_\bullet$ are defined by

$$\delta^i a_{i_0\dots i_p} = a_{j_0\dots j_p} \quad \text{with} \quad j_k = \begin{cases} i_k & \text{if } i_k < i, \\ i_k+1 & \text{if } i_k \geq i, \end{cases} \tag{1}$$

$$\sigma^i a_{i_0\dots i_p} = \begin{cases} 0 \quad \text{if} \quad \{i, i+1\} \subset \{i_0, \dots, i_p\}, \\ a_{j_0\dots j_p} \quad \text{otherwise, with} \quad j_k = \begin{cases} i_k & \text{if } i_k \leq i, \\ i_k - 1 & \text{if } i_k > i. \end{cases} \end{cases} \tag{2}$$

2. Proof of the main theorem

The simplicial bar construction on a group G, is the simplicial set $B_\bullet G$ with set of n-simplices $B_n G = G^n$. Its elements are denoted $[g_1|\dots|g_n]$, with $g_i \in G$. The faces d_i and degeneracies s_i of $B_\bullet G$ are defined as follows:

$$\begin{aligned} d_0[g_1|\dots|g_n] &= [g_2|\dots|g_n], \\ d_i[g_1|\dots|g_n] &= [g_1|\dots|g_i g_{i+1}|\dots|g_n], \quad \text{for} \quad 0 < i < n, \\ d_n[g_1|\dots|g_n] &= [g_1|\dots|g_{n-1}]. \end{aligned} \tag{3}$$

The degeneracy s_i inserts the identity e of G in position i.

Let $L \in \mathbf{cdgl}$ be generated in degree 0 and $f\colon \mathfrak{L}_n \to L$ a morphism in $\mathbf{cdgl}$. For degree reasons, we have $f(a_{i_0\dots i_k}) = 0$ if $k \neq 1$. Moreover, since f commutes the differential, from the definition of the differential in $\mathfrak{L}_2$, we get

$$0 = df(a_{0rs}) = f(a_{0r}) * f(a_{rs}) * f(a_{0s})^{-1}.$$

Therefore, for any $r, s > 0$, we have

$$f(a_{rs}) = f(a_{0r})^{-1} * f(a_{0s}).$$

The map f being entirely defined by its values on the a_{0i}, we have a bijection

$$\Phi\colon \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_n, L) \to L^n, \quad \text{defined by} \quad f \mapsto (f(a_{01}), f(a_{02}), \dots, f(a_{0n})).$$

We now determine the image of the faces and degeneracies on $\mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L)$, induced from (1) and (2). For the face operators, as only the elements (a_{0r}) play a role, it suffices to consider,

$$\delta^i(a_{0r}) = \begin{cases} a_{0r} & \text{if } r < i \\ a_{0(r+1)} & \text{if } r \geq i \end{cases} \quad \text{for} \quad i > 0, \quad \text{and} \quad \delta^0(a_{0r}) = a_{1(r+1)}.$$

Let $f\colon \mathfrak{L}_n \to L$ be specified by $(f(a_{01}),\dots,f(a_{0n}))$. Then $d_0 f = f\circ\delta^0\colon \mathfrak{L}_{n-1}\to L$ is described by

$$\begin{aligned} d_0(f(a_{01}),\dots,f(a_{0(n-1)})) &= (f\circ\delta^0(a_{01}),\dots,f\circ\delta^0(a_{0(n-1)})) \\ &= (f(a_{12}),\dots,f(a_{1n})) \\ &= (f(a_{01})^{-1}*f(a_{02}),\dots,f(a_{01})^{-1}*f(a_{0n})). \end{aligned}$$

Therefore, the face operator d_0 on $L^\bullet$, induced from Φ, is

$$d_0(x_1,\dots,x_n) = (x_1^{-1}*x_2,\dots,x_1^{-1}*x_n).$$

Similar arguments give, for $i>0$,

$$d_i(x_1,\dots,x_n) = (x_1,\dots,\hat{x}_i,\dots,x_n).$$

As for the degeneracies, starting from

$$\sigma^i(a_{0r}) = \begin{cases} a_{0r} & \text{if } r\leq i \\ a_{0(r-1)} & \text{if } r>i \end{cases} \qquad \text{for} \quad i>0,$$

$$\sigma^0(a_{0r}) = a_{0(r-1)} \quad \text{if} \quad r>1, \quad \text{and} \quad \sigma^0(a_{01}) = 0,$$

we get

$$s_0(x_1,\dots,x_n) = (0,x_1,\dots,x_n)$$

and for $i>0$,

$$s_i(x_1,\dots,x_n) = (x_1,\dots,x_i,x_i,\dots,x_n).$$

Now a straightforward and easy computation shows that the morphism

$$\Psi\colon \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L) \to B_\bullet(\exp L)$$

defined by

$$\Psi(f) = [f(a_{01})|f(a_{01})^{-1}f(a_{02})|f(a_{02})^{-1}f(a_{03})|\dots|f(a_{0(n-1)})^{-1}f(a_{0n})]$$

is an isomorphism of simplicial sets.

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