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A HOMOTOPY THEORY OF COHERENTLY COMMUTATIVE MONOIDAL QUASI-CATEGORIES

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Abstract. The main objective of this paper is to construct a symmetric monoidal closed model category of coherently commutative monoidal quasi- categories. We construct another model category structure whose fibrant objects are (essentially) those coCartesian fibrations which represent objects that are known as symmetric monoidal quasi-categories in the literature. We go on to establish a zig zag of Quillen equivalences between the two model categories.

Contents

1. Introduction

A *symmetric monoidal category* is a category equipped with a multiplicative structure which is associative, unital and commutative only up to natural (coherence) isomorphisms. A *quasi-category* is a simplicial set which satisfies the weak Kan condition, namely every inner horn has a filler. In this paper we study quasi-categories which are equipped with a *coherently commutative* multiplicative structure and thereby generalize the notion of

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symmetric monoidal categories to higher categories. Such quasi-categories most commonly arise as (simplicial) nerves of simplicial model categories which are equipped a compatible symmetric monoidal structure see [NS17]. These quasi-categories played a prominent role in Jacob Lurie's work on the *cobordism hypothesis*. The *coherence theorem* for symmetric monoidal categories states that the category of (small) symmetric monoidal categories is equivalent to the category of strict symmetric monoidal categories, which are also known as *permutative* categories, and strict symmetric monoidal functors **Perm**. The category **Perm** is isomorphic to the category of algebras over the *categorical Barrat-Eccles operad* in **Cat**. We recall that the categorical Barrat-Eccles operad is an E_∞-operad in **Cat**. In a subsequent paper we intend to prove a similar theorem for the quasi-categories equipped with a coherently commutative multiplicative structure.

There are several different models present in the literature which were developed to encode a coherently commutative multiplicative structure on simplicial sets. The most commonly used model is based on operads. An E_∞-simplicial set is a simplicial set equipped with a *coherently commutative multiplicative* structure which is encoded by an action of an E_∞-operad. In other words an E_∞-simplicial set is an *algebra* over an E_∞-operad in the category of simplicial sets $\mathcal{S}$. There are two model category structures on the category $\mathcal{S}$ namely the *standard* or the *Kan* model category structure $(\mathcal{S}, \mathbf{Kan})$ and the *Joyal* model category structure $(\mathcal{S}, \mathbf{Q})$, which is also referred to as the model category structure of *quasi-categories*. In this paper we will only be working with the later model category structure. The category of E_∞-simplicial sets inherits a model category structure from the Joyal model category structure, see [BM03]. A fibrant object in this model category can be described as a quasi-category equipped with a coherently commutative multiplicative structure which is encoded by an action of an E_∞-operad. However this model category is NOT symmetric monoidal closed. The main objective of this paper is to overcome this shortcoming by presenting a new model for *coherently commutative monoidal quasi-categories* based on Γ-spaces. Another model to encode a coherently commutative multiplicative structure on simplicial sets was presented by Jacob Lurie in his book [Lur] which he called *symmetric monoidal quasi-categories*. He modelled these objects as *coCartesian fibrations* over a quasi-category which is the nerve of a skeletal category of based finite sets Γ^{op} whose objects are $n^+ = \{0, 1, 2, \ldots, n\}$. In this paper we take a dual perspective, namely we model these as functors from Γ^{op} into $(\mathcal{S}, \mathbf{Q})$. However Lurie does not construct a model category structure on his symmetric monoidal quasi-categories. Yet another model to encode a coherently commutative multiplicative structure on simplicial sets was presented by Kodjabachev and Sagave in the paper [KS15]. The authors present a rigidification of an E_∞-quasi-category by replacing it by a commutative monoid in a symmetric monoidal functor category. They go on to construct a zig-zag of Quillen equivalences between a suitably defined model category structure on the category of commutative monoids mentioned above and a model category of E_∞-simplicial sets. However they were unable to show the existence of a symmetric monoidal closed model category structure.

A Γ-space is a functor from the category Γ^{op} into the category of simplicial sets $\mathcal{S}$. The category $\Gamma\mathcal{S}$ of Γ-spaces is the category of functors and natural transformations $[\Gamma^{op}, \mathcal{S}]$. A *normalized* Γ-space is a functor $X : \Gamma^{op} \longrightarrow \mathcal{S}_\bullet$ such that $X(0^+) = *$. The category of normalized Γ-spaces $\Gamma\mathcal{S}_\bullet$ is the full subcategory of the functor category $[\Gamma^{op}; \mathcal{S}_\bullet]$ whose objects are normalized Γ-spaces. In the paper [Seg74] Segal introduced a notion of normalized Γ-spaces and showed that they give rise to a homotopy category which is equivalent to the homotopy category of connective spectra. Segal's Γ-spaces were renamed *special* Γ-spaces by Bousfield and Friedlander in [BF78] who constructed a model category structure on the category of all normalized Γ-spaces $\Gamma\mathcal{S}_\bullet$. The two authors go on to prove that the homotopy category obtained by inverting *stable weak equivalences* in $\Gamma\mathcal{S}_\bullet$ is equivalent to the homotopy category of connective spectra. In the paper [Sch99] Schwede constructed a symmetric monoidal closed model category structure on the category of *normalized* Γ-spaces which he called the *stable* Q-model category. The fibrant objects in this model category can be described as *coherently commutative group objects* in the category of (pointed) simplicial sets $\mathcal{S}_\bullet$, where the latter category is endowed with the *Kan* model category structure. The objective of Schwede's construction was to establish normalized Γ-spaces as a model for *connective spectra*. In this paper we extend the ideas in [Sch99] to study coherently commutative monoidal objects in the model category of quasi-categories and thereby generalizing the theory of *symmetric monoidal categories*. We construct a new symmetric monoidal closed model category structure on the category of Γ-spaces $\Gamma\mathcal{S}$. Our model category is constructed along the lines of Schwede's construction and we call it the *JQ* model category structure and denote it by $(\Gamma\mathcal{S}_\otimes, \mathbf{Q})$. The fibrant objects in $(\Gamma\mathcal{S}_\otimes, \mathbf{Q})$ can be described as *coherently commutative monoidal quasi-categories*. We will show that the $(\Gamma\mathcal{S}_\otimes, \mathbf{Q})$ is symmetric monoidal closed under the *Day convolution product*.

The category of pointed simplicial sets $\mathcal{S}_\bullet$ inherits a model category structure from the Joyal model category $(\mathcal{S}_\bullet, \mathbf{Q})$. This model category is symmetric monoidal closed under the *smash product of pointed simplicial sets*, see [JT08]. We construct a new model category structure on the category of nomalized Γ-spaces $\Gamma\mathcal{S}_\bullet$. The fibrant object of this model category can be described as *normalized coherently commutative monoidal quasi-categories*. We will refer to this model category as the *JQ* model category of *normalized* Γ-spaces. In the paper [Lyd99] Lydakis constructed a *smash product of* Γ*-spaces* and showed that it endows $\Gamma\mathcal{S}_\bullet$ with a closed symmetric monoidal structure. We will show that the *JQ* model category structure of normalized Γ-spaces is compatible with the smash product of Γ-spaces *i.e.* it is symmetric monoidal closed under the smash product. Another significant result of this paper is that the obvious forgetful functor $U : \Gamma\mathcal{S}_\bullet \longrightarrow \Gamma\mathcal{S}$ is the right Quillen functor of a Quillen equivalence between the *JQ* model category and the normalized *JQ* model category. This result is indicative of the presence of a *weak semi-additive* structure in the *JQ* model category. Even though a prominent objective of this paper is to show that normalized coherently commutative monoidal quasi-categories can be replaced by unnormalized ones, we still develop a full theory of normalized coherently commutative monoidal quasi-categories in Appendix C. We do so because traditionally, Γ-spaces have been studied as normalized objects and we want to establish a continuity between our theory and the existing literature on this subject.

A *marked* simplicial set is a pair $(S, \mathcal{E})$, where S is a simplicial set and $\mathcal{E}$ is a set of edges of S which are called *marked edges*. We review the theory of marked simplicial sets in Section 2.2. In Section 6 we construct a model category of *coherently commutative monoidal marked quasi-categories* which we denote by $(\Gamma\mathcal{S}^+_\otimes, \mathbf{Q})$. This model category serves as an intermediary in achieving the main result of this paper. We establish the following Quillen equivalence:

$$\Gamma(-)^\flat : (\Gamma\mathcal{S}_\otimes, \mathbf{Q}) \rightleftarrows (\Gamma\mathcal{S}^+_\otimes, \mathbf{Q}) : U \tag{1}$$

The notion of a symmetric monoidal quasi-category based on coCartesian fibrations of simplicial sets was introduced in [Lur]. A coCartesian fibration $p : X \longrightarrow N(\Gamma^{op})$ determines a fibrant object $(X^\natural, p)$ in a *coCartesian model category structure* on the over-category $\mathcal{S}^+/N(\Gamma^{op})$, by *marking* its *coCartesian edges*. We construct another model category structure on $\mathcal{S}^+/N(\Gamma^{op})$ by localizing the coCartesian model category structure. We denote this model category by $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$. An object (X, p) in $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$ is fibrant if and only if it is isomorphic to an object $(Y^\natural, q)$ determined by a coCartesian fibration $q : Y \longrightarrow N(\Gamma^{op})$ which is a symmetric monoidal quasi-category. We go on further to establish a Quillen equivalence

$$\mathfrak{F}^+_\bullet(\Gamma^{op}) : (\mathcal{S}^+/N(\Gamma^{op}), \otimes) \rightleftarrows (\Gamma\mathcal{S}^+_\otimes, \mathbf{Q}) : N^+_\bullet(\Gamma^{op}), \tag{2}$$

where the adjoint functor are from [Lur09, Prop. 3.2.5.18]. The right Quillen functor $N^+_\bullet(\Gamma^{op})$, known as the *marked relative nerve funtor*, can be viewed as a higher categorical version of the Grothendieck construction functor. The main result of this paper is to establish a zig-zag of Quillen equivalences between the model categories $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$ and $(\Gamma\mathcal{S}_\otimes, \mathbf{Q})$. Now the main result is obtained by composing (2) and (1).

This paper has seven sections and three appendices. In Section 2 we collect all the machinery needed to write this paper. More precicely, we review the theory of Γ-spaces, the theory of *marked* simplicial sets and the *coCartesian model category structure*. In Section 3 we construct a model category structure on the category of Γ-spaces $\Gamma\mathcal{S}$ called the strict JQ model category structure. This model category structure is an unnormalized version of the strict Q model category structure constructed on the category of normalized Γ-spaces [Sch99, Lem. 1.4]. The fibrant objects of the former can be described as Γ-quasi-categories whereas those of the latter model category can be described as normalized Γ-Kan complexes. We go on to show that this model category is symmetric monoidal closed under the Day convolution product. In Section 4 we construct a symmetric monoidal closed model category of coherently commutative monoidal quasi-categories which is a left Bousfield localization of the strict JQ model category. In Appendix C we do the same for normalized Γ-spaces, namely we construct a symmetric monoidal closed model category of normalized Γ-spaces. The main goal of Section 5 is to establish a Quillen equivalence between the model category of coherently commutative monoidal quasi-categories and the model category of normalized coherently commutative monoidal quasi-categories. In Section 6 we construct a model category of coherently commutative monoidal *marked* quasi-categories. We show that the Quillen equivalence

$$(-)^{\flat} : (\mathcal{S}, \mathbf{Q}) \rightleftarrows (\mathcal{S}^+, \mathbf{Q}) : U,$$

from Theorem 2.9, extends to the Quillen equivalence (1). The main result of this paper appears in Section 7. In this section we construct a model category whose fibrant objects are (essentially) symmetric monoidal quasi-categories defined in [Lur]. This model category is a left Bousfield localization of the coCartesian model category structure on $\mathcal{S}^+/N(\Gamma^{op})$. The main result of this paper establishes a zig zag of Quillen equivalences between the model category of symmetric monoidal quasi-categories and the model category of coherently commutative monoidal quasi-categories.

2. The Setup

In this section we will collect the machinery needed for various constructions in this paper.

2.1. Notation. *We will denote the terminal object in a model category by* 1*. In the case of* $\mathcal{S}$ *we may also denote a terminal object by* $*$.

2.2. A review of marked simplicial sets. In this subsection we will review the theory of marked simplicial sets. The material presented in this subsection is essentially a reproduction of that in [Lur09, Sec. 3.1]. Later in this paper we will develop a theory of coherently commutative monoidal objects in the category of marked simplicial sets.

2.3. Definition. *A* marked *simplicial set is a pair* $(X, \mathcal{E})$*, where* X *is a simplicial set and* $\mathcal{E}$ *is a set of edges of* X *which contains every degenerate edge of* X*. We will say that an edge of* X *is* marked *if it belongs to* $\mathcal{E}$*. A morphism* $f : (X, \mathcal{E}) \longrightarrow (X', \mathcal{E}')$ *of marked simplicial sets is a simplicial map* $f : X \longrightarrow X'$ *having the property that* $f(\mathcal{E}) \subseteq \mathcal{E}'$*. We denote the category of marked simplicial sets by* $\mathcal{S}^+$*.*

Every simplicial set S may be regarded as a marked simplicial set in many ways. We mention two extreme cases: We let $S^{\sharp} = (S, S_1)$ denote the marked simplicial set in which every edge is marked. We denote by $S^{\flat} = (S, s_0(S_0))$ denote the marked simplicial set in which only the degenerate edges of S have been marked.

The category $\mathcal{S}^+$ is *cartesian-closed*, *i.e.* for each pair of objects $X, Y \in Ob(\mathcal{S}^+)$, there is an internal mapping object $[X, Y]^+$ equipped with an *evaluation map* $[X, Y]^+ \times X \longrightarrow Y$ which induces a bijection:

$$\mathcal{S}^+(Z, [X, Y]^+) \xrightarrow{\cong} \mathcal{S}^+(Z \times X, Y),$$

for every $Z \in \mathcal{S}^+$.

2.4. Notation. *We denote by* $[X, Y]^{\flat}$ *the underlying simplicial set of* $[X, Y]^+$*.*

The mapping space $[X, Y]^{\flat}$ is characterized by the following bijection:

$$\mathcal{S}(K, [X, Y]^{\flat}) \xrightarrow{\cong} \mathcal{S}^+(K^{\flat} \times X, Y),$$

for each simplicial set K.

2.5. Notation. *We denote by* $[X,Y]^{\sharp}$ *the simplicial subset of* $[X,Y]^{\flat}$ *consisting of all simplices* $\sigma \in [X,Y]^{\flat}$ *such that every edge of* σ *is a marked edge of* $[X,Y]^{+}$.

The mapping space $[X,Y]^{\sharp}$ is characterized by the following bijection:

$$\mathcal{S}(K, [X,Y]^{\sharp}) \xrightarrow{\cong} \mathcal{S}^{+}(K^{\sharp} \times X, Y),$$

for each simplicial set K.

The Joyal model category structure on $\mathcal{S}$ has the following analog for marked simplicial sets:

2.6. Theorem. *There is a left-proper, combinatorial model category structure on the category of marked simplicial sets* $\mathcal{S}^{+}$ *in which a morphism* $p : X \longrightarrow Y$ *is a*

1. *cofibration if the simplicial map between the underlying simplicial sets is a cofibration in* $(\mathcal{S}, \mathbf{Q})$*, namely a monomorphism.*
2. *a weak equivalence if the induced simplicial map on the mapping spaces*

$$[p, K^{\natural}]^{\flat} : [X, K^{\natural}]^{\flat} \longrightarrow [Y, K^{\natural}]^{\flat}$$

is a categorical equivalence, for each quasi-category K.

3. *fibration if it has the right lifting property with respect to all maps in* $\mathcal{S}^{+}$ *which are simultaneously cofibrations and weak equivalences.*

Further, the above model category structure is enriched over the Joyal model category, i.e. it is a $(\mathcal{S}, \mathbf{Q})$*-model category.*

The above theorem follows from [Lur09, Prop. 3.1.3.7].

2.7. Notation. *We will denote the model category structure in Theorem 2.6 by* $(\mathcal{S}^{+}, \mathbf{Q})$ *and refer to it either as the* Joyal *model category of* marked *simplicial sets or as the model category of marked quasi-categories.*

2.8. Theorem. *The model category* $(\mathcal{S}^{+}, \mathbf{Q})$ *is a cartesian closed model category.*

Proof. The theorem follows from [Lur09, Corollary 3.1.4.3] by taking $S = T = \Delta[0]$. ■

There is an obvious forgetful functor $U : \mathcal{S}^{+} \longrightarrow \mathcal{S}$. This forgetful functor has a left adjoint $(-)^{\flat} : \mathcal{S} \longrightarrow \mathcal{S}^{+}$.

2.9. Theorem. *The adjoint pair of functors* $((-)^{\flat}, U)$ *determine a Quillen equivalence between the Joyal model category of marked simplicial sets and the Joyal model category of simplicial sets.*

The proof of the above theorem follows from [Lur09, Prop. 3.1.5.3].

2.10. Review of Γ-spaces. In this subsection we will briefly review the theory of Γ-spaces. We begin by introducing some notations which will be used throughout the paper.

2.11. Notation. *We will denote by* $\underline{n}$ *the finite set* $\{1, 2, \ldots, n\}$ *and by* n^+ *the based set* $\{0, 1, 2, \ldots, n\}$ *whose basepoint is the element* 0.

2.12. Notation. *We will denote by* $\mathcal{N}$ *the skeletal category of finite unbased sets whose objects are* $\underline{n}$ *for all* $n \geq 0$ *and maps are functions of unbased sets. The category* $\mathcal{N}$ *is a (strict) symmetric monoidal category whose symmetric monoidal structure will be denoted by* $+$. *For two objects* $\underline{k}, \underline{l} \in \mathcal{N}$ *their* tensor product *is defined as follows:*

$$\underline{k} + \underline{l} := \underline{k + l}.$$

2.13. Notation. *We will denote by* Γ^{op} *the skeletal category of finite based sets whose objects are* n^+ *for all* $n \geq 0$ *and maps are functions of based sets.*

2.14. Notation. *Given a morphism* $f : n^+ \longrightarrow m^+$ *in* Γ^{op}, *we denote by* $Supp(f)$ *the largest subset of* $\underline{n}$ *whose image under* f *does not contain the basepoint of* m^+. *The set* $Supp(f)$ *inherits an order from* $\underline{n}$ *and therefore could be regarded as an object of* $\mathcal{N}$. *We denote by* $Supp(f)^+$ *the based set* $Supp(f) \sqcup \{0\}$ *regarded as an object of* Γ^{op} *with order inherited from* $\underline{n}$.

2.15. Definition. *A map* $f : n^+ \longrightarrow m^+$ *in* Γ^{op} *is called* inert *if its restriction to the set* $Supp(f)^+$ *is a bijection.*

2.16. Definition. *A morphism* f *in* Γ^{op} *is called* active *if* $f^{-1}(\{0\}) = \{0\}$ i.e. *the pre-image of* $\{0\}$ *is the singleton set* $\{0\}$.

2.17. Notation. *A map* $f : \underline{n} \longrightarrow \underline{m}$ *in the category* $\mathcal{N}$ *uniquely determines an active map in* Γ^{op} *which we will denote by* $f^+ : n^+ \longrightarrow m^+$. *This map agrees with* f *on non-zero elements of* n^+.

2.18. Remark. *Each morphism in* Γ^{op} *can be factored into a composite of an inert map followed by an active map in* Γ^{op}. *The factorization is unique up to a unique isomorphism.*

2.19. Definition. *Each* $n^+ \in \Gamma^{op}$ *determines* n projection *maps* $\delta^n_i : n^+ \longrightarrow 1^+$ *for* $1 \leq i \leq n$ *which are defined by* $\delta^n_i(i) = 1$ *and* $\delta^n_i(j) = 0$ *for* $j \neq i$ *and* $j \in n^+$.

2.20. Definition. *Each* $n^+ \in \Gamma^{op}$ *determines a* multiplication *map* $m_n : n^+ \longrightarrow 1^+$ *which is the unique active map from* n^+ *to* 1^+.

2.21. Definition. *A* Γ*-space is a functor from* Γ^{op} *into the category of simplicial sets* $\mathcal{S}$.

2.22. Definition. *A* normalized Γ*-space is* X *a* Γ*-space which satisfies the normalization condition namely* $X(0^+) \cong *$.

In this paper we will also study functors from Γ^{op} into the category of marked simplicial sets:

2.23. Definition. *A* marked Γ*-space is a functor from* Γ^{op} *to the category of marked simplicial sets* $\mathcal{S}^+$*. A morphism of marked* Γ*-spaces is a natural transformation between marked* Γ*-spaces.*

2.24. Notation. *We denote the category of all marked* Γ*-spaces and morphisms of* Γ*-spaces by* $\Gamma\mathcal{S}^+$*.*

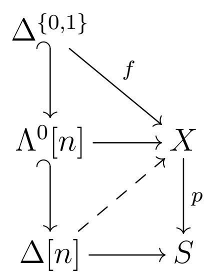
The adjoint pair $((-)^\flat, U)$ defined above induces an adjunction

$$\Gamma(-)^\flat : \Gamma\mathcal{S} \rightleftharpoons \Gamma\mathcal{S}^+ : U \tag{3}$$

where the left adjoint $\Gamma(-)^\flat$ is the induced map $[\Gamma^{op}, (-)^\flat] : \Gamma\mathcal{S} = [\Gamma^{op}, \mathcal{S}] \longrightarrow [\Gamma^{op}, \mathcal{S}^+] = \Gamma\mathcal{S}^+$.

2.25. Review of the coCartesian model structure. In this subsection we will review the theory of coCartesian fibrations over the simplicial set $N(\Gamma^{op})$. We also review a model category structure on the category $\mathcal{S}^+/N(\Gamma^{op})$ in which the fibrant objects are (essentially) coCartesian fibrations. We begin by recalling the notion of a p-coCartesian edge:

2.26. Definition. *Let* $p : X \longrightarrow S$ *be an inner fibration of simplicial sets. Let* $f : x \longrightarrow y \in (X)_1$ *be an edge in* X*. We say that* f *is* p*-coCartesian if, for all* $n \geq 2$ *and every (outer) commutative diagram, there exists a (dotted) lifting arrow which makes the entire diagram commutative:*



(4)

2.27. Remark. *Let* M *be a (ordinary) category equipped with a functor* $p : M \longrightarrow I$*, then an arrow* f *in* M*, which maps isomorphically to* I*, is coCartesian in the usual sense if and only if* f *is* $N(p)$*-coCartesian in the sense of the above definition, where* $N(p) : N(M) \longrightarrow \Delta[1]$ *represents the nerve of* p*.*

This definition leads us to the notion of a coCartesian fibration of simplicial sets:

2.28. Definition. *A map of simplicial sets* $p : X \longrightarrow S$ *is called a* coCartesian *fibration if it satisfies the following conditions:*

1. *p is an inner fibration of simplicial sets.*
2. *for each edge* $p : x \longrightarrow y$ *of* S *and each vertex* $\underline{x}$ *of* X *with* $p(\underline{x}) = x$*, there exists a* p*-coCartesian edge* $\underline{f} : \underline{x} \longrightarrow \underline{y}$ *with* $p(\underline{f}) = f$*.*

A coCartesian fibration roughly means that it is up to weak equivalence determined by a *functor* from S to a suitably defined ∞-category of ∞-categories. This idea is explored in detail in [Lur09, Ch. 3]. Next we will review the *relative nerve*:

2.29. Definition. *([Lur09, 3.2.5.2]). Let D be a category, and $f : D \longrightarrow \mathcal{S}$ a functor. The nerve of D relative to f is the simplicial set $N_f(D)$ whose n-simplices are sets consisting of:*

(i) a functor $d : [n] \longrightarrow D$; we write $d(i, j)$ for the image of $i \leq j$ in $[n]$.

(ii) for every nonempty subposet $J \subseteq [n]$ with maximal element j, a map

$$\tau^J : \Delta^J \longrightarrow f(d(j)),$$

(iii) such that for nonempty subsets $I \subseteq J \subseteq [n]$ with respective maximal elements $i \leq j$, the following diagram commutes:

$$\begin{array}{ccc} \Delta^I & \xrightarrow{\tau^I} & f(d(i)) \\ \big\downarrow & & \big\downarrow{\scriptstyle f(d(i,j))} \\ \Delta^J & \xrightarrow[\tau^J]{} & f(d(j)) \end{array}$$

For any f, there is a canonical map $p_f : N_f(D) \longrightarrow N(D)$ down to the ordinary nerve of D, induced by the unique map to the terminal object $\Delta^0 \in \mathcal{S}$ [Lur09, 3.2.5.4]. When f takes values in quasi-categories, this canonical map is a coCartesian fibration.

2.30. Remark. *A vertex of the simplicial set $N_f(D)$ is a pair (c, g), where $c \in Ob(D)$ and $g \in f(c)_0$. An edge $\underline{e} : (c, g) \longrightarrow (d, k)$ of the simplicial set $N_f(D)$ consists of a pair (e, h), where $e : c \longrightarrow d$ is an arrow in D and $h : f(e)_0(g) \longrightarrow k$ is an edge of $f(d)$.*

An immediate consequence of the above definition is the following proposition:

2.31. Proposition. *Let $f : D \longrightarrow \mathcal{S}$ be a functor, then the fiber of $p_f : N_f(D) \longrightarrow N(D)$ over any $d \in Ob(D)$ is isomorphic to the simplicial set $f(d)$.*

We now recall a result which will be used in the last section of this paper:

2.32. Theorem. *[Lur09, Thm. 3.2.5.18.] The relative nerve functor $N^+_\bullet(\Gamma^{op})$ has a left adjoint $\mathfrak{F}^+_\bullet(\Gamma^{op})$. The adjoint pair $(\mathfrak{F}^+_\bullet(\Gamma^{op}), N^+_\bullet(\Gamma^{op}))$ is a Quillen equivalence between the coCartesian model category $(\mathcal{S}^+/N(\Gamma^{op}), \mathbf{cC})$ and the strict JQ model category of marked Γ-spaces.*

The latter model category in the statement of the above theorem is constructed in Section 6.

2.33. Notation. *To each coCartesian fibration $p : X \longrightarrow N(\Gamma^{op})$ we can associate a marked simplicial set denoted $X^\natural$ which is composed of the pair $(X, \mathcal{E})$, where $\mathcal{E}$ is the set of p-coCartesian edges of X*

2.34. Notation. *Let X, Y be two objects in $(\mathcal{S}^+, \mathbf{Q})$. We will denote by $[X, Y]^\flat_{N(\Gamma^{op})} \subseteq [X, Y]^\flat_{\mathcal{S}^+}$ and $[X, Y]^\sharp_{N(\Gamma^{op})} \subseteq [X, Y]^\sharp_{\mathcal{S}^+}$ the simplicial subsets whose vertices are those maps which are compatible with the projections to $N(\Gamma^{op})$.*

2.35. Definition. A morphism $F : X \longrightarrow Y$ in the category $\mathcal{S}^+/N(\Gamma^{op})$ is called a *coCartesian*-equivalence if for each coCartesian fibration $Z \longrightarrow N(\Gamma^{op})$, the induced simplicial map

$$\left[F, Z^\natural\right]^\flat_{\mathcal{S}^+} : \left[Y, Z^\natural\right]^\flat_{\mathcal{S}^+} \longrightarrow \left[X, Z^\natural\right]^\flat_{\mathcal{S}^+}$$

is a categorical equivalence of simplicial-sets(quasi-categories).

Next we will recall a model category structure on the overcategory $\mathcal{S}^+/N(\Gamma^{op})$ from [Lur09, Prop. 3.1.3.7.] in which fibrant objects are (essentially) coCartesian fibrations.

2.36. Theorem. *There is a left-proper, combinatorial model category structure on the category $\mathcal{S}^+/N(\Gamma^{op})$ in which a morphism is*

1. *a cofibration if it is a monomorphism when regarded as a map of simplicial sets.*
2. *a weak equivalences if it is a coCartesian equivalence.*
3. *a fibration if it has the right lifting property with respect to all maps that are simultaneously cofibrations and weak equivalences.*

We have defined a function object for the category $\mathcal{S}^+/N(\Gamma^{op})$ above. The simplicial set $[X, Y]^\flat_{\Gamma^{op}}$ has vertices, all maps from X to Y in $\mathcal{S}^+/N(\Gamma^{op})$. An n-simplex in $[X, Y]^\flat_{\Gamma^{op}}$ is a map $\Delta[n]^\flat \times X \longrightarrow Y$ in $\mathcal{S}^+/N(\Gamma^{op})$, where $\Delta[n]^\flat \times (X, p) = (\Delta[n]^\flat \times X, pp_2)$, where p_2 is the projection $\Delta[n]^\flat \times X \longrightarrow X$. The enriched category $\mathcal{S}^+/N(\Gamma^{op})$ admits tensor and cotensor products. The *tensor product* of an object $X = (X, p)$ in $\mathcal{S}^+/N(\Gamma^{op})$ with a simplicial set A, is the following object:

$$A^\flat \times X = (A^\flat \times X, pp_2).$$

The *cotensor product* of X by A is an object of $\mathcal{S}^+/N(\Gamma^{op})$ denoted $X^{[A]}$. If

$$q : X^{[A]} \longrightarrow N(\Gamma^{op})^\sharp$$

is the structure map, then a simplex $x : \Delta[n]^\flat \longrightarrow X^{[A]}$ over a simplex

$$y = qx : \Delta[n] \longrightarrow N(\Gamma^{op})^\sharp$$

is a map $A^\flat \times (\Delta[n]^\flat, y) \longrightarrow (X, p)$. The object $(X^{[A]}, q)$ can be constructed by the following pullback square in $\mathcal{S}^+$:

$$\begin{array}{ccc} X^{[A]} & \longrightarrow & [A^\flat, X]^+ \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle [A^\flat, p]^+} \\ N(\Gamma^{op})^\sharp & \longrightarrow & [A^\flat, N(\Gamma^{op})^\sharp]^+ \end{array}$$

where the bottom map is the diagonal. There are canonical isomorphisms:

$$\left[A^\flat \times X, Y\right]^\flat_D \cong \left[A, [X, Y]^\flat_D\right] \cong \left[X, Y^{[A]}\right]^\flat_D \tag{5}$$

2.37. Remark. *The coCartesian model category structure on $\mathcal{S}^+/N(\Gamma^{op})$ is a simplicial model category structure with the simplicial Hom functor:*

$$[-,-]^{\sharp}_{\Gamma^{op}} : \mathcal{S}^+/N(\Gamma^{op})^{op} \times \mathcal{S}^+/N(\Gamma^{op}) \longrightarrow \mathcal{S}.$$

This is proved in [Lur09, Corollary 3.1.4.4.]. The coCartesian model category structure is a $(\mathcal{S}, \mathbf{Q})$-model category structure with the function object given by:

$$[-,-]^{\flat}_{\Gamma^{op}} : \mathcal{S}^+/N(\Gamma^{op})^{op} \times \mathcal{S}^+/N(\Gamma^{op}) \longrightarrow \mathcal{S}.$$

This is Remark [Lur09, 3.1.4.5.].

2.38. Remark. *The coCartesian model category is a $(\mathcal{S}^+, \mathbf{Q})$-model category with the Hom functor:*

$$[-,-]^{+}_{D} : \mathcal{S}^+/N(D)^{op} \times \mathcal{S}^+/N(D) \longrightarrow \mathcal{S}^+.$$

This follows from [Lur09, Corollary 3.1.4.3] by taking $S = N(D)$ and $T = \Delta[0]$, where S and T are specified in the statement of the corollary.

3. The strict JQ model category

Schwede introduced two model category structures on the category of (normalized) Γ-spaces which he called the *strict Q-model category* structure and the *stable Q-model category* structure in [Sch99]. The strict Q-model category structure is obtained by restricting the projective model category structure on the functor category $[\Gamma^{op}, \mathcal{S}_\bullet]$, where the codomain category $\mathcal{S}_\bullet$ is endowed with the Kan model category structure. In this section we study the *projective* model category structure on the category of Γ-spaces namely the functor category $[\Gamma^{op}, \mathcal{S}]$. Following Schwede we will refer to this projective model category as the *strict JQ model category*. We will show that the strict JQ model category is a $\mathcal{S}$-model category, where $\mathcal{S}$ is endowed with the Joyal model category structure. We go on further to show that the strict JQ model category is a symmetric monoidal closed model category. We begin by recalling the notion of a *categorical equivalence* of simplicial sets which is essential for defining weak equivalences of the desired model category structure.

3.1. Definition. *A morphism of simplicial sets $f : A \longrightarrow B$ is called a categorical equivalence if for any quasi-category X, the induced morphism on the homotopy categories of mapping spaces*

$$ho(\mathcal{M}ap_{\mathcal{S}}(f, X)) : ho(\mathcal{M}ap_{\mathcal{S}}(B, X)) \longrightarrow ho(\mathcal{M}ap_{\mathcal{S}}(A, X)),$$

is an equivalence of (ordinary) categories.

3.2. Remark. *Categorical equivalences are weak equivalences in a cofibrantly generated model category structure on simplicial sets called the* Joyal model category structure *which we will denote by $(\mathcal{S}, \mathbf{Q})$, see [Joy08, Theorem 6.12] for the definition of the Joyal model category structure.*

3.3. DEFINITION. *We call a map of* Γ*-spaces*

1. *A* strict JQ fibration *if it is degreewise a* pseudo-fibration *i.e. a fibration of simplicial sets in the Joyal model category structure on simplicial sets.*

2. *A* strict JQ equivalence *if it is degreewise a* categorical equivalence *i.e. a weak equivalence of simplicial sets in the Joyal model category structure on simplicial sets, see [Lur09].*

3. *A* strict JQ cofibration *if it has the left lifting property with respect to all maps of* Γ*-spaces which are simultaneously JQ equivalences and JQ fibrations.*

3.4. THEOREM. *Strict JQ equivalences, strict JQ fibrations and strict JQ cofibrations provide the category of* Γ*-spaces with a combinatorial, left-proper model category structure on the category of* Γ*-spaces* $\Gamma\mathcal{S}$*.*

The model structure in the above theorem follows from [Lur09, Proposition A 3.3.2] and the left properness is a consequence of the left properness of the Joyal model category.

3.5. ENRICHMENT OF THE STRICT JQ MODEL CATEGORY. The goal of this section is to show that the strict JQ model category is a *(symmetric) monoidal closed model category i.e.* it is enriched over itself in the sense of Definition 3.9. We will prove this in two steps; we first establish the existence of a Quillen bifunctor

$$- \underset{\mathcal{S}}{\boxtimes} - : \Gamma\mathcal{S} \times \mathcal{S} \longrightarrow \Gamma\mathcal{S},$$

where the category $\Gamma\mathcal{S}$ is endowed with the strict JQ model category structure and $\mathcal{S}$ is endowed with the Joyal model category structure $(\mathcal{S}, \mathbf{Q})$. Then we will use this Quillen bifunctor to prove the desired enrichment. We begin by reviewing the notion of a *monoidal model category.* Our review is largely taken from [Hov99, Ch. 4].

3.6. DEFINITION. *A* monoidal model category *is a closed monoidal category* $\mathcal{C}$ *with a model category structure, such that* $\mathcal{C}$ *satisfies the following conditions:*

1. *The monoidal structure* $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ *is a Quillen bifunctor.*

2. *Let* $QS \overset{q}{\longrightarrow} S$ *be the cofibrant replacement for the unit object* S*, obtained by using the functorial factorization system to factorize* $0 \longrightarrow S$ *into a cofibration followed by a trivial fibration. Then the natural map*

$$QS \otimes X \overset{q \otimes 1}{\longrightarrow} S \otimes X$$

is a weak equivalence for all cofibrant X*. Similarly, the natural map* $X \otimes QS \overset{1 \otimes q}{\longrightarrow} X \otimes S$ *is a weak equivalence for all cofibrant* X*.*

3.7. EXAMPLE. *The model category of simplicial sets with the Joyal model category structure,* $(\mathcal{S}, \mathbf{Q})$ *is a monoidal model category.*

3.8. Example. *The stable Q-model category is a monoidal model category with respect to the smash product defined in [Lyd99].*

3.9. Definition. *Let* $\mathbf{S}$ *be a monoidal model category. An* $\mathbf{S}$-enriched model category *is an* $\mathbf{S}$ *enriched category* $\mathbf{A}$ *equipped with a model category structure (on its underlying category) such that*

1. *The category* $\mathbf{A}$ *is tensored and cotensored over* $\mathbf{S}$.
2. *There is a Quillen adjunction of two variables, (see Definition A.2),*

$$(\otimes, \mathbf{hom_A}, \mathcal{M}ap_{\mathbf{A}}, \phi, \psi) : \mathbf{A} \times \mathbf{S} \longrightarrow \mathbf{A}.$$

When $\mathbf{A}$ *is itself a monoidal model category which is also an* $\mathbf{A}$*-enriched model category, we will say that* $\mathbf{A}$ *is* enriched over itself as a model category.

3.10. Example. *Both strict and stable Q-model category structures, constructed in [Sch99], on the category* $\Gamma\mathcal{S}$ *are simplicial, i.e. both strict and stable Q- model categories are* $(\mathcal{S}, \mathbf{Kan})$*-enriched model categories.*

3.11. Remark. *The strict JQ model category structure is NOT simplicial.*

For each pair (F, K), where $F \in Ob(\Gamma\mathcal{S})$ and $K \in Ob(\mathcal{S})$, one can construct a Γ-space which we denote by $F \underset{\mathcal{S}}{\boxtimes} K$ and which is defined as follows:

$$(F \underset{\mathcal{S}}{\boxtimes} K)(n^+) := F(n^+) \times K,$$

where the product on the right is taken in the category of simplicial sets. This construction is functorial in both variables. Thus we have a functor

$$- \underset{\mathcal{S}}{\boxtimes} - : \Gamma\mathcal{S} \times \mathcal{S} \longrightarrow \Gamma\mathcal{S}.$$

Now we will define a couple of function objects for the category $\Gamma\mathcal{S}$. The first function object enriches the category $\Gamma\mathcal{S}$ over $\mathcal{S}$ *i.e.* there is a bifunctor

$$\mathcal{M}ap_{\Gamma\mathcal{S}}(-,-) : \Gamma\mathcal{S}^{op} \times \Gamma\mathcal{S} \longrightarrow \mathcal{S}$$

which assigns to each pair of objects $(X, Y) \in Ob(\Gamma\mathcal{S}) \times Ob(\Gamma\mathcal{S})$ a simplicial set $\mathcal{M}ap_{\Gamma\mathcal{S}}(X, Y)$ which is defined in degree zero as follows:

$$\mathcal{M}ap_{\Gamma\mathcal{S}}(X, Y)_0 := \Gamma\mathcal{S}(X, Y)$$

and the simplicial set is defined in degree n as follows:

$$\mathcal{M}ap_{\Gamma\mathcal{S}}(X, Y)_n := \Gamma\mathcal{S}(X \underset{\mathcal{S}}{\boxtimes} \Delta[n], Y)$$

For any Γ-space X, the functor $X \underset{\mathcal{S}}{\boxtimes} - : \mathcal{S} \longrightarrow \Gamma\mathcal{S}$ is left adjoint to the functor $\mathcal{M}ap_{\Gamma\mathcal{S}}(X, -) : \Gamma\mathcal{S} \longrightarrow \mathcal{S}$. The *counit* of this adjunction is the evaluation map $ev : X \underset{\mathcal{S}}{\boxtimes} \mathcal{M}ap_{\Gamma\mathcal{S}}(X, Y) \longrightarrow Y$ and the *unit* is the obvious simplicial map $K \longrightarrow \mathcal{M}ap_{\Gamma\mathcal{S}}(X, X \underset{\mathcal{S}}{\boxtimes} K)$.

To each pair of objects $(K, X) \in Ob(\mathcal{S}) \times Ob(\Gamma\mathcal{S})$ we can define a Γ-space X^K, in degree n, as follows:

$$(X^K)(n^+) := [K, X(n^+)] \ .$$

This assignment is functorial in both variables and therefore we have a bifunctor

$$-^- : \mathcal{S}^{op} \times \Gamma\mathcal{S} \longrightarrow \Gamma\mathcal{S}.$$

For any Γ-space X, the functor $X^- : \mathcal{S} \longrightarrow \Gamma\mathcal{S}^{op}$ is left adjoint to the functor $\mathcal{M}ap_{\Gamma\mathcal{S}}(-, X) : \Gamma\mathcal{S}^{op} \longrightarrow \mathcal{S}$. The following proposition summarizes the above discussion.

3.12. Proposition. *There is an adjunction of two variables*

$$(- \underset{\mathcal{S}}{\boxtimes} -, -^-, \mathcal{M}ap_{\Gamma\mathcal{S}}(-, -)) : \Gamma\mathcal{S} \times \mathcal{S} \longrightarrow \Gamma\mathcal{S}. \tag{6}$$

3.13. Theorem. *The strict model category of Γ-spaces, $\Gamma\mathcal{S}$, is a $(\mathcal{S}, \mathbf{Q})$-model category.*

Proof. We will show that the adjunction of two variables (6) is a Quillen adjunction for the strict JQ model category structure on $\Gamma\mathcal{S}$ and the model category $(\mathcal{S}, \mathbf{Q})$. In order to do so, we will verify condition (2) of Lemma A.3. Let $g : K \longrightarrow L$ be a cofibration in $\mathcal{S}$ and let $p : Y \longrightarrow Z$ be a strict fibration of Γ-spaces. We have to show that the induced map

$$\mathbf{hom}^{\square}_{\Gamma\mathcal{S}}(g, p) : Y^L \longrightarrow Z^L \underset{Z^K}{\times} Y^K$$

is a fibration in $\Gamma\mathcal{S}$ which is acyclic if either of g or p is acyclic. It would be sufficient to check that the above morphism is degreewise a fibration in $(\mathcal{S}, \mathbf{Q})$, i.e. for all $n^+ \in \Gamma^{op}$, the morphism

$$\mathbf{hom}^{\square}_{\Gamma\mathcal{S}}(g, p)(n^+) = \mathbf{hom}^{\square}_{\mathcal{S}}(g, p(n^+)) : Y(n^+)^L \longrightarrow Z(n^+)^L \underset{Z(n^+)^K}{\times} Y(n^+)^K,$$

is a fibration in $(\mathcal{S}, \mathbf{Q})$. This follows from the observations that the simplicial morphism $p(n^+) : Y(n^+) \longrightarrow Z(n^+)$ is a fibration in $(\mathcal{S}, \mathbf{Q})$ and the model category $(\mathcal{S}, \mathbf{Q})$ is a cartesian closed model category whose internal hom is provided by the bifunctor $-^- : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S}$. ∎

Let X and Y be two Γ-spaces, the *Day convolution product* of X and Y denoted by $X * Y$ is defined as follows:

$$X * Y(n^+) := \int^{(k^+, l^+) \in \Gamma^{op}} \Gamma^{op}(k^+ \wedge l^+, n^+) \times X(k^+) \times Y(l^+). \tag{7}$$

Equivalently, one may define the Day convolution product of X and Y as the left Kan extension of their *external tensor product* $X\overline{\times}Y$ along the smash product functor

$$- \wedge - : \Gamma^{op} \times \Gamma^{op} \longrightarrow \Gamma^{op}.$$

we recall that the external tensor product $X\overline{\times}Y$ is a bifunctor

$$X\overline{\times}Y : \Gamma^{op} \times \Gamma^{op} \longrightarrow \mathcal{S}$$

which is defined on objects by

$$X\overline{\times}Y(m^+, n^+) = X(m^+) \times Y(n^+).$$

It follows from [Lyd99, Thm. 2.2] that the functor $- * \Gamma^n$ has a right adjoint which we denote by $-(n^+ \wedge -) : \Gamma\mathcal{S} \longrightarrow \Gamma\mathcal{S}$. We will denote the Γ-space $-(n^+ \wedge -)(X)$ by $X(n^+ \wedge -)$ and define it by the following composite:

$$\Gamma^{op} \xrightarrow{n^+ \wedge -} \Gamma^{op} \xrightarrow{X} \mathcal{S}. \tag{8}$$

The following proposition sums up this observation:

3.14. Proposition. *There is a natural isomorphism*

$$\phi : -(n^+ \wedge -) \cong \underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(\Gamma^n, -).$$

In particular, for each Γ*-space* X *there is an isomorphism of* Γ*-spaces*

$$\phi(X) : X(n^+ \wedge -) \cong \underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(\Gamma^n, X).$$

Proof. Consider the functor $n^+ \wedge - : \Gamma^{op} \longrightarrow \Gamma^{op}$. We observe that a *left Kan extension* of $\Gamma^1 : \Gamma^{op} \longrightarrow \mathcal{S}$ along $n^+ \wedge -$ is the Γ-space $\Gamma^n : \Gamma^{op} \longrightarrow \mathcal{S}$. This implies that we have the following bijection

$$\Gamma\mathcal{S}(\Gamma^n, X) \cong \Gamma\mathcal{S}(\Gamma^1, X(n^+ \wedge -)).$$

We observe that this natural bijection extends to a natural isomorphism of Γ-spaces:

$$\underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(\Gamma^n, X) \cong \underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(\Gamma^1, X(n^+ \wedge -)).$$

■

3.15. Proposition. *The category of all* Γ*-spaces* $\Gamma\mathcal{S}$ *is a symmetric monoidal category under the Day convolution product* (7)*. The unit of the symmetric monoidal structure is the representable* Γ*-space* Γ^1*.*

Next we define an internal function object of the category $\Gamma\mathcal{S}$ which we will denote by

$$\underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(-, -) : \Gamma\mathcal{S}^{op} \times \Gamma\mathcal{S} \longrightarrow \Gamma\mathcal{S}. \tag{9}$$

Let X and Y be two Γ-spaces, we define the Γ-space $\underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(X, Y)$ as follows:

$$\underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(X, Y)(n^+) := \mathcal{M}ap_{\Gamma\mathcal{S}}(X * \Gamma^n, Y).$$

3.16. Proposition. *The category* $\Gamma\mathcal{S}$ *is a closed symmetric monoidal category under the Day convolution product. The internal hom is given by the bifunctor* (9) *defined above.*

The above proposition implies that for each $n \in \mathbb{N}$ the functor $- * \Gamma^n : \Gamma\mathcal{S} \longrightarrow \Gamma\mathcal{S}$ has a right adjoint $\underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(\Gamma^n, -) : \Gamma\mathcal{S} \longrightarrow \Gamma\mathcal{S}$.

The next theorem shows that the strict model category $\Gamma\mathcal{S}$ is compatible with the Day convolution product.

3.17. Theorem. *The strict JQ model category* $\Gamma\mathcal{S}$ *is a symmetric monoidal closed model category under the Day convolution product.*

Proof. Using the adjointness which follows from Proposition 3.16 one can show that if a map $f : U \longrightarrow V$ is a (acyclic) cofibration in the strict JQ model category $\Gamma\mathcal{S}$ then the induced map $f * \Gamma^n : U * \Gamma^n \longrightarrow V * \Gamma^n$ is also a (acyclic) cofibration in the strict JQ model category for all $n \in \mathbb{N}$. By (3) of Lemma A.3 it is sufficient to show that whenever f is a cofibration and $p : Y \longrightarrow Z$ is a fibration then the map

$$\underline{\mathcal{M}ap}^{\square}_{\Gamma\mathcal{S}}(f, p) : \underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(V, Y) \longrightarrow \underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(V, Z) \underset{\underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(U,Z)}{\times} \underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(U, Y).$$

is a fibration in $\Gamma\mathcal{S}$ which is acyclic if either f or p is a weak equivalence. The above map is a (acyclic) fibration if and only if the simplicial map

$$\underline{\mathcal{M}ap}^{\square}_{\Gamma\mathcal{S}}(f * \Gamma^n, p)(n^+) : \mathcal{M}ap_{\Gamma\mathcal{S}}(V * \Gamma^n, Y) \longrightarrow \mathcal{M}ap_{\Gamma\mathcal{S}}(V * \Gamma^n, Z) \underset{\mathcal{M}ap_{\Gamma\mathcal{S}}(U * \Gamma^n, Z)}{\times} \mathcal{M}ap_{\Gamma\mathcal{S}}(U * \Gamma^n, Y)$$

is a (acyclic) fibration in $(\mathcal{S}, \mathbf{Q})$ for all $n \in \mathbb{N}$. Since $f * \Gamma^n$ is a cofibration (which is acyclic whenever f is acyclic as observed above) therefore it follows from Theorem 3.13 that the simplicial map $\underline{\mathcal{M}ap}^{\square}_{\Gamma\mathcal{S}}(f * \Gamma^n, p)(n^+)$ is an (acyclic) fibration of simplicial sets for all $n \in \mathbb{N}$. ∎

The following corollary is an easy consequence of the above theorem and we leave the proof as an exercise for the interested reader.

3.18. Corollary. *Let* F' *be a* Q*-cofibrant* Γ*-space and* $p : F \longrightarrow G$ *is be a strict JQ fibration. Then the morphism induced by* p *on the function objects*

$$\underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(F', p) : \underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(F', F) \longrightarrow \underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(F', G)$$

is a strict JQ fibration.

3.19. Definition. *A morphism in* $\Gamma\mathcal{S}$ *is called a* trivial fibration *of* Γ*-spaces if it has the right lifting property with respect to all maps in the following class of maps*

$$\{\Gamma^n \underset{\mathcal{S}}{\boxtimes} f;\ f \textit{ is a simplicial monomorphism and } n \geq 0\}$$

3.20. Proposition. *A trivial fibration is a strict JQ equivalence.*

Proof. Let $p : X \longrightarrow Y$ be a trivial fibration of Γ-spaces and $f : A \longrightarrow B$ be a simplicial monomorphism. Then whenever the outer diagram commutes in the following diagram:

$$\begin{array}{ccc} \Gamma^n \underset{\mathcal{S}}{\boxtimes} A & \longrightarrow & X \\ {\scriptstyle \Gamma^n \underset{\mathcal{S}}{\boxtimes} f}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Gamma^n \underset{\mathcal{S}}{\boxtimes} B & \longrightarrow & Y \end{array}$$

there exists a dotted arrow which makes the whole diagram commutative, for each $n \geq 0$. By adjointness, we get the following commutative diagram in the category of simplicial sets:

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^n, X) \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^n, p)} \\ B & \longrightarrow & \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^n, Y) \end{array}$$

We observe that the map

$$\mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^n, X) : \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^n, X) \longrightarrow \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^n, Y)$$

is the same as the simplicial map $p(n^+) : X(n^+) \longrightarrow Y(n^+)$ up to isomorphism namely we have the following commutative diagram:

$$\begin{array}{ccccc} A & \longrightarrow & \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^n, X) & \overset{\cong}{\longrightarrow} & X(n^+) \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^n, p)} & & \downarrow{\scriptstyle p(n^+)} \\ B & \longrightarrow & \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^n, Y) & \underset{\cong}{\longrightarrow} & Y(n^+) \end{array}$$

This observation and the above simplicial commutative diagram together imply that for each $n \geq 0$, the simplicial map $p(n^+)$ has the right lifting property with respect to all simplicial monomorphisms, in other words $p(n^+)$ is a trivial fibration of simplicial sets. By [Joy08, Prop. 1.22], this implies that the simplicial map $p(n^+)$ is a weak equivalence in the Joyal model category of simplicial sets. Thus, we have shown that p is a strict JQ equivalence of Γ-spaces. ∎

3.21. Proposition. *A strict JQ fibration is a trivial fibration of* Γ*-spaces if and only if it is a strict JQ equivalence.*

4. The model category of coherently commutative monoidal quasi-categories

The objective of this section is to construct a new model category structure on the category $\Gamma\mathcal{S}$. This new model category is obtained by localizing the strict JQ model category defined above. We will refer to this new model category structure as the *model category structure of coherently commutative monoidal quasi-categories*. The guiding principle of this new model structure is to endow its homotopy category with a semi-additive structure. In other words we want this new model category structure to have finite *homotopy biproducts*. We go on further to show that this new model category is symmetric monoidal closed with respect to the *Day convolution product*, see [Day70]. We begin by recalling the notion of a *left Bousfield localization*. A detailed treatment of this notion can be found in [Hir02, Sec. 3.3]:

4.1. DEFINITION. *([Hir02, Defn. 3.3.1.(1))]) Let $\mathcal{M}$ be a model category and let $\mathcal{S}$ be a class of maps in $\mathcal{M}$. The left Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{S}$ is a model category structure $L_{\mathcal{S}}\mathcal{M}$ on the underlying category of $\mathcal{M}$ such that*

1. *The class of cofibrations of $L_{\mathcal{S}}\mathcal{M}$ is the same as the class of cofibrations of $\mathcal{M}$.*

2. *A map $f : A \longrightarrow B$ is a weak equivalence in $L_{\mathcal{S}}\mathcal{M}$ if it is an $\mathcal{S}$-local equivalence, namely, for every fibrant $\mathcal{S}$-local object X, the induced map on homotopy function complexes*
$$f^* : Map^h_{\mathcal{M}}(B, X) \longrightarrow Map^h_{\mathcal{M}}(A, X)$$
is a homotopy equivalence of simplicial sets. Recall that an object X is called fibrant $\mathcal{S}$-local if X is fibrant in $\mathcal{M}$ and for every element $g : K \longrightarrow L$ of the set $\mathcal{S}$, the induced map on homotopy function complexes
$$g^* : Map^h_{\mathcal{M}}(L, X) \longrightarrow Map^h_{\mathcal{M}}(K, X)$$
is a weak homotopy equivalence of simplicial sets.

where $\mathcal{M}ap^h_{\mathcal{M}}(-, -)$ is the simplicial function object associated with the strict model category $\mathcal{M}$, see [DK80a], [DK80b] and [DK80c].

We want to construct a left Bousfield localization of the strict model category of Γ-spaces. For each pair $k^+, l^+ \in \Gamma^{op}$, we have the obvious *projection maps* in $\Gamma\mathcal{S}$
$$\delta^{k+l}_k : (k+l)^+ \longrightarrow k^+ \quad \textit{and} \quad \delta^{k+l}_l : (k+l)^+ \longrightarrow l^+.$$
The maps
$$\Gamma^{op}(\delta^{k+l}_k, -) : \Gamma^k \longrightarrow \Gamma^{k+l} \quad \textit{and} \quad \Gamma^{op}(\delta^{k+l}_l, -) : \Gamma^l \longrightarrow \Gamma^{k+l}$$
induce a map of Γ-spaces on the coproduct which we denote as follows:
$$h^l_k : \Gamma^l \sqcup \Gamma^l \longrightarrow \Gamma^{l+k}.$$

We now define a det of maps $\mathcal{E}_\infty\mathcal{S}$ in $\Gamma\mathcal{S}$:
$$\mathcal{E}_\infty\mathcal{S} := \{h^l_k : \Gamma^l \sqcup \Gamma^l \longrightarrow \Gamma^{l+k} : l, k \in \mathbb{Z}^+\}$$

4.2. Definition. *We call a* Γ*-space* X *a* $(\Delta \times \mathcal{E}_\infty\mathcal{S})$-local object *if it is a fibrant object in the strict JQ model category and for each map* $h^l_k \in \mathcal{E}_\infty\mathcal{S}$*, the induced simplicial map*

$$\mathcal{M}ap^h_{\Gamma\mathcal{S}}(\Delta[n] \underset{\mathcal{S}}{\boxtimes} h^l_k, X) : \mathcal{M}ap^h_{\Gamma\mathcal{S}}(\Delta[n] \underset{\mathcal{S}}{\boxtimes} \Gamma^{k+l}, X) \longrightarrow \mathcal{M}ap^h_{\Gamma\mathcal{S}}(\Delta[n] \underset{\mathcal{S}}{\boxtimes} (\Gamma^k \sqcup \Gamma^l), X),$$

is a homotopy equivalence of simplicial sets for all $n \geq 0$ *where* $\mathcal{M}ap^h_{\Gamma\mathcal{S}}(-,-)$ *is the simplicial function complexes associated with the strict model category* $\Gamma\mathcal{S}$*, see [DK80a], [DK80b] and [DK80c].*

Appendix B tell us that the Kan complex $J(\mathcal{M}ap_{\Gamma\mathcal{S}}(X,Y))$, which is the maximal Kan complex contained in the quasicategory $\mathcal{M}ap_{\Gamma\mathcal{S}}(X,Y)$, is a model for the homotopy function complex $\mathcal{M}ap^h_{\Gamma\mathcal{S}}(X,Y)$, whenever X is cofibrant and Y is fibrant in the strict JQ model category.

The following proposition gives a characterization of $\mathcal{E}_\infty\mathcal{S}$-local objects

4.3. Proposition. *A strict JQ fibrant* Γ*-space* X *is a* $(\Delta \times \mathcal{E}_\infty\mathcal{S})$*-local object in* $\Gamma\mathcal{S}$ *if and only if it satisfies the Segal condition i.e. the simplicial map*

$$(X(\delta^{(k+l)}_k), X(\delta^{(k+l)}_l)) : X((k+l)^+) \longrightarrow X(k^+) \times X(l^+)$$

is a categorical equivalence of quasi-categories for all $k^+, l^+ \in Ob(\Gamma^{op})$*.*

Proof. We begin the proof by observing that each element of the set $\mathcal{E}_\infty\mathcal{S}$ is a map of Γ-spaces between cofibrant Γ-spaces. Theorem B.10 implies that X is a $(\Delta \times \mathcal{E}_\infty\mathcal{S})$-local object if and only if the following simplicial map

$$\mathcal{M}ap_{\Gamma\mathcal{S}}(h^k_l, X) : \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^{(k+l)}, X) \longrightarrow \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^k \sqcup \Gamma^l, X)$$

is a categorical equivalence of quasi-categories.

We observe that we have the following commutative square in $(\mathcal{S}, \mathbf{Q})$

$$\begin{array}{ccc} \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^{(k+l)}, X) & \xrightarrow{\mathcal{M}ap_{\Gamma\mathcal{S}}(h^k_l, X)} & \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^k \sqcup \Gamma^l, X) \\ \cong\big\downarrow & & \big\downarrow\cong \\ X((k+l)^+) & \xrightarrow[(X(\delta^{(k+l)}_k), X(\delta^{(k+l)}_l))]{} & X(k^+) \times X(l^+) \end{array}$$

This implies that the simplicial map $(X(\delta^{(k+l)}_k), X(\delta^{(k+l)}_l))$ is a categorical equivalence of quasi-categories if and only if the functor $\mathcal{M}ap_{\Gamma\mathcal{S}}(h^k_l, X)$ is one. ∎

4.4. Definition. *We will refer to a* $(\Delta \times \mathcal{E}_\infty \mathcal{S})$*-local object as a* coherently commutative monoidal quasi-category*.*

4.5. Definition. *Let* X *be a coherently commutative monoidal quasi-category. We will refer to the homotopy category of the quasi-category* $X(1^+)$*,* $ho(X(1^+))$*, as the* homotopy category *of* X *and denote it by* $ho(X)$*.*

4.6. Definition. *A morphism of* Γ*-spaces* $F : X \longrightarrow Y$ *is a* $(\Delta \times \mathcal{E}_\infty \mathcal{S})$*-local equivalence if for each coherently commutative monoidal category* Z *the following simplicial map*

$$\mathcal{M}ap^h_{\Gamma\mathcal{S}}(F, Z) : \mathcal{M}ap^h_{\Gamma\mathcal{S}}(Y, Z) \longrightarrow \mathcal{M}ap^h_{\Gamma\mathcal{S}}(X, Z)$$

is a homotopy equivalence of simplicial sets.

4.7. Proposition. *A morphism between two cofibrant* Γ*-spaces* $F : X \longrightarrow Y$ *is an* $(\Delta \times \mathcal{E}_\infty \mathcal{S})$*-local equivalence if and only if the simplicial map*

$$\mathcal{M}ap_{\Gamma\mathcal{S}}(F, Z) : \mathcal{M}ap_{\Gamma\mathcal{S}}(Y, Z) \longrightarrow \mathcal{M}ap_{\Gamma\mathcal{S}}(X, Z)$$

is an equivalence of quasi-categories for each coherently commutative monoidal quasi-category Z*.*

Proof. Let us first assume that $F : X \longrightarrow Y$ is an $(\Delta \times \mathcal{E}_\infty \mathcal{S})$-local equivalence. Then for any coherently commutative monoidal quasi-category Z the following simplicial map

$$\mathcal{M}ap^h_{\Gamma\mathcal{S}}(F, Z) : \mathcal{M}ap^h_{\Gamma\mathcal{S}}(Y, Z) \longrightarrow \mathcal{M}ap^h_{\Gamma\mathcal{S}}(X, Z)$$

is a homotopy equivalence of Kan complexes. We observe that for each $n > 0$, the Γ-space $Z^{\Delta[n]}$ is a coherently commutative monoidal quasi-category because it satisfies the Segal condition, see 4.3, namely we have the following diagram in which the first map is an equivalence of quasi-categories

$$Z((k+l)^+)^{\Delta[n]} \longrightarrow (Z(k^+) \times Z(l^+))^{\Delta[n]} \cong Z(k^+)^{\Delta[n]} \times Z(l^+)^{\Delta[n]}.$$

This implies that for each $n > 0$ the following simplicial map is an equivalence of quasi-categories:

$$\mathcal{M}ap^h_{\Gamma\mathcal{S}}(F, Z^{\Delta[n]}) : \mathcal{M}ap^h_{\Gamma\mathcal{S}}(Y, Z^{\Delta[n]}) \longrightarrow \mathcal{M}ap^h_{\Gamma\mathcal{S}}(X, Z^{\Delta[n]}).$$

By Proposition B.8, we have

$$\mathcal{M}ap^h_{\Gamma\mathcal{S}}(F, Z^{\Delta[n]}) = J(\mathcal{M}ap_{\Gamma\mathcal{S}}(F, Z^{\Delta[n]})).$$

By adjointness we have the following isomorphisms in the category of arrows of simplicial sets:

$$\mathcal{M}ap_{\Gamma\mathcal{S}}(F, Z^{\Delta[n]}) \cong \mathcal{M}ap_{\Gamma\mathcal{S}}(F \times \Delta[n], Z) \cong \mathcal{M}ap_{\Gamma\mathcal{S}}(F, Z)^{\Delta[n]}$$

Since the map $J(\mathcal{M}ap_{\Gamma\mathcal{S}}(F, Z^{\Delta[n]}))$ is a homotopy equivalence of Kan complexes, the above isomorphisms imply that so is the simplicial map $J(\mathcal{M}ap_{\Gamma\mathcal{S}}(F, Z)^{\Delta[n]})$. Now Lemma B.10 says that the simplicial map $\mathcal{M}ap_{\Gamma\mathcal{S}}(F, Z)$ is an equivalence of quasi-categories.

Conversely, let us assume that the simplicial map $\mathcal{M}ap_{\Gamma\mathcal{S}}(F, Z)$ is an equivalence of quasi-categories. Since the functor J takes equivalences of quasi-categories to homotopy equivalences of Kan complexes, therefore $J(\mathcal{M}ap_{\Gamma\mathcal{S}}(F, Z)) = \mathcal{M}ap^h_{\Gamma\mathcal{S}}(F, Z)$ is a homotopy equivalence of Kan complexes. Thus, we have shown that $F : X \longrightarrow Y$ is a $(\Delta \times \mathcal{E}_\infty\mathcal{S})$-local equivalence.

■

4.8. Definition. *We will refer to a* $(\Delta \times \mathcal{E}_\infty\mathcal{S})$*-local equivalence either as an* equivalence of coheretly commutative monoidal categories *or as a JQ equivalence.*

The main result of this section is about constructing a new model category structure on the category $\Gamma\mathcal{S}$, by localizing the strict model category of Γ-spaces with respect to morphisms in the set $\mathcal{E}_\infty\mathcal{S}$. We recall the following theorem which will be the main tool in the construction of the desired model category. This theorem first appeared in an unpublished work [Smi] but a proof was later provided by Barwick in [Bar10].

4.9. Theorem. *[Bar10, Theorem 4.7] If $\mathcal{M}$ is a left-proper, combinatorial model category and $\mathcal{S}$ is a small set of homotopy classes of morphisms of $\mathcal{M}$, the left Bousfield localization $L_\mathcal{S}\mathcal{M}$ of $\mathcal{M}$ along any set representing $\mathcal{S}$ exists and satisfies the following conditions.*

1. *The model category $L_\mathcal{S}\mathcal{M}$ is left proper and combinatorial.*
2. *As a category, $L_\mathcal{S}\mathcal{M}$ is simply $\mathcal{M}$.*
3. *The cofibrations of $L_\mathcal{S}\mathcal{M}$ are exactly those of $\mathcal{M}$.*
4. *The fibrant objects of $L_\mathcal{S}\mathcal{M}$ are the fibrant $\mathcal{S}$-local objects Z of $\mathcal{M}$.*
5. *The weak equivalences of $L_\mathcal{S}\mathcal{M}$ are the $\mathcal{S}$-local equivalences.*

4.10. Theorem. *There is a closed, left proper, combinatorial model category structure on the category of Γ-spaces, $\Gamma\mathcal{S}$, in which*

1. *The class of cofibrations is the same as the class of JQ cofibrations of Γ-spaces.*
2. *The weak equivalences are equivalences of coherently commutative monoidal quasi-categories.*

An object is fibrant in this model category if and only if it is a coherently commutative monoidal category. A fibration between two coherently commutative monoidal quasi-categories is a strict JQ equivalence.

Proof. The strict model category of Γ-spaces is a combinatorial model category therefore the existence of the model structure follows from Theorem 4.9 stated above. The last statement follows from (1). ■

4.11. Notation. *The model category constructed in Theorem 4.10 will be referred to either as the* model category of coherently commutative monoidal quasi-categories *or as the* JQ model category of Γ-spaces. *We will denote this model category by* $(\Gamma\mathcal{S}_{\otimes}, \mathbf{Q})$.

The rest of this section is devoted to proving that the model category of coherently commutative monoidal quasi-categories is a symmetric monoidal closed model category. In order to do so we will need some general results which we state and prove now.

4.12. Proposition. *A cofibration, $f : A \longrightarrow B$, between cofibrant objects in a model category $\mathcal{C}$ is a weak equivalence in $\mathcal{C}$ if and only if it has the right lifting property with respect to all fibrations between fibrant objects in $\mathcal{C}$.*

Proof. The unique terminal map $B \longrightarrow *$ can be factored into an acyclic cofibration $\eta_B : B \longrightarrow R(B)$ followed by a fibration $R(B) \longrightarrow *$. The composite map $\eta_B \circ f$ can again be factored as an acyclic cofibration followed by a fibration $R(f)$ as shown in the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & R(A) \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle R(f)} \\ B & \xrightarrow[\eta_B]{} & R(B) \end{array}$$

Since $R(B)$ is fibrant and $R(f)$ is a fibration, therefore $R(A)$ is a fibrant object in $\mathcal{C}$. Thus, $R(f)$ is a fibration between fibrant objects in $\mathcal{C}$ and now by assumption, the dotted arrow exists which makes the whole diagram commutative. Since both η_A and η_B are acyclic cofibrations, therefore the two out of six property of model categories implies that the map f is a weak equivalence in the model category $\mathcal{C}$.

■

4.13. Proposition. *If X is a coherently commutative monoidal quasi-category, then so is the Γ-space $X(n^+ \wedge -)$, for each $n \in \mathbb{N}$.*

Proof. We begin by observing that $X(n^+ \wedge -)(1^+) = X(n^+)$ and since X is fibrant, the pointed category $X(n^+)$ is equivalent to $\prod_1^n X(1^+)$. Notice the isomorphisms $(n^+ \wedge (k+l)^+) \cong \bigvee_1^n (k+l)^+ \cong (\bigvee_1^n k^+) \vee (\bigvee_1^n l^+) \cong ((\bigvee_1^n k^+) + (\bigvee_1^n l^+))$. The two projection maps $\delta_k^{k+l} : (k+l)^+ \longrightarrow k^+$ and $\delta_l^{k+l} : (k+l)^+ \longrightarrow l^+$ induce an equivalence of categories $X((\bigvee_1^n k^+) + (\bigvee_1^n l^+)) \longrightarrow X(\bigvee_1^n k^+) \times X(\bigvee_1^n l^+)$. Composing with the isomorphisms above, we get the following equivalence of pointed simplicial sets $X(n^+ \wedge -)((k+l)^+) \longrightarrow X(n^+ \wedge -)(k^+) \times X(n^+ \wedge -)(l^+)$. ■

4.14. Corollary. *For each coherently commutative monoidal category X, the mapping object $\underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(\Gamma^n, X)$ is also a coherently commutative monoidal category for each $n \in \mathbb{N}$.*

Proof. The corollary follows from the above Proposition and Proposition 3.14. ■

The category Γ^{op} is a symmetric monoidal category with respect to the smash product of pointed sets. In other words the smash product of pointed sets defines a bi-functor $- \wedge - : \Gamma^{op} \times \Gamma^{op} \longrightarrow \Gamma^{op}$. For each pair $k^+, l^+ \in Ob(\Gamma^{op})$, there are two natural transformations

$$\delta_k^{k+l} \wedge - : (k+l)^+ \wedge - \Rightarrow k^+ \wedge - \quad \text{and} \quad \delta_l^{k+l} \wedge - : (k+l)^+ \wedge - \Rightarrow l^+ \wedge -.$$

Horizontal composition of either of these two natural transformations with a Γ-space X determines a morphism of Γ-spaces

$$id_X \circ (\delta_k^{k+l} \wedge -) =: X(\delta_k^{k+l} \wedge -) : X((k+l)^+ \wedge -) \longrightarrow X(k^+ \wedge -).$$

4.15. Proposition. *Let X be an coherently commutative monoidal quasi-category, then for each pair $(k,l) \in \mathbb{N} \times \mathbb{N}$, the following morphism*

$$(X(\delta_k^{k+l} \wedge -), X(\delta_l^{k+l} \wedge -)) : X((k+l)^+ \wedge -) \longrightarrow X(k^+ \wedge -) \times X(l^+ \wedge -)$$

is a strict equivalence of Γ-spaces.

Using the previous two propositions, we now show that the mapping space functor $\underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(-,-)$ provides the homotopically correct function object when the domain is cofibrant and codomain is fibrant.

4.16. Lemma. *Let W be a Q-cofibrant Γ-space and let X be a coherently commutative monoidal quasi-category. Then the mapping object $\underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(W,X)$ is also a coherently commutative monoidal quasi-category.*

Proof. We begin by recalling that

$$\underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(W,X)((k+l)^+) = \mathcal{M}ap_{\Gamma\mathcal{S}}(W * \Gamma^{k+l}, X).$$

We recall that the Γ^{k+l} is a cofibrant Γ-space and by assumption W is also a cofibrant Γ-space therefore it follows from Theorem 3.17 that $W * \Gamma^{k+l}$ is also a cofibrant Γ-space. Since X is a coherently commutative monoidal quasi-category *i.e.* a fibrant object in the model category of coherently commutative monoidal quasi-categories, therefore it follows from Theorem 3.17 that the mapping object $\mathcal{M}ap_{\Gamma\mathcal{S}}(W * \Gamma^{k+l}, X)$ is a quasi-category, for all $k, l \geq 0$.

We recall that the map $h_l^k : \Gamma^k \sqcup \Gamma^l \longrightarrow \Gamma^{k+l}$ is a weak equivalence in the model category of coherently commutative monoidal quasi-categories, now by using adjointness and Corollary 4.14, it is easy to show that the top arrow in the following commutative diagram is a categorical equivalence of quasi-categories:

$$\begin{array}{ccc} \mathcal{M}ap_{\Gamma\mathcal{S}}(W * \Gamma^{k+l}, X) & \xrightarrow{\mathcal{M}ap_{\Gamma\mathcal{S}}(W * h_l^k, X)} & \mathcal{M}ap_{\Gamma\mathcal{S}}((W * \Gamma^k) \sqcup (W * \Gamma^l), X) \\ & & \downarrow \cong \\ & & \mathcal{M}ap_{\Gamma\mathcal{S}}(W * \Gamma^k, X) \times \mathcal{M}ap_{\Gamma\mathcal{S}}(W * \Gamma^l, X) \end{array}$$

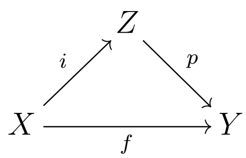
Now the result follows from the above diagram.

∎

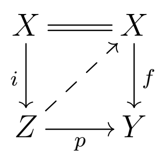
The following lemma will be used in the proof of the main result of this section:

4.17. Lemma. *Let $\mathcal{M}$ be a model category and $\mathcal{L}(\mathcal{M})$ be a left Bousfield localization of $\mathcal{M}$. Then a map $f : X \longrightarrow Y$ between two fibrant objects in $\mathcal{L}(\mathcal{M})$, is a fibration in $\mathcal{L}(\mathcal{M})$ if and only if it is a fibration in $\mathcal{M}$.*

Proof. The implication ($\Rightarrow$) is clear because each fibration in $\mathcal{L}(\mathcal{M})$ is a fibration in $\mathcal{M}$. Conversely, let us assume that f is an $\mathcal{M}$-fibration between two $\mathcal{L}(\mathcal{M})$-fibrant objects. We will show that f is also an $\mathcal{L}(\mathcal{M})$-fibration. Let us choose a factorization



where p is an $\mathcal{L}(\mathcal{M})$-fibration and i is an acyclic cofibration in $\mathcal{L}(\mathcal{M})$. We observe that i is a map between two $\mathcal{L}(\mathcal{M})$-fibrant objects. The identity functor $id : \mathcal{L}(\mathcal{M}) \longrightarrow \mathcal{M}$ is a right Quillen functor and therefore preserves weak equivalences between fibrant objects. Thus i is also an acyclic cofibration in $\mathcal{M}$ because cofibrations in $\mathcal{M}$ and $\mathcal{L}(\mathcal{M})$ are the same. This implies that the following (outer) commutative diagram has a (dotted) diagonal filler:



This implies that f is a retract of p and therefore f is an $\mathcal{L}(\mathcal{M})$-fibration. ■

A map of Γ-spaces $F : X \longrightarrow Y$ is an acyclic strict JQ fibration of Γ-spaces if and only if it has the right lifting property with respect to all maps in the set

$$\mathcal{I} = \{\Gamma^n \underset{\mathcal{S}}{\boxtimes} \partial_i \mid \forall n, i \in Ob(\mathcal{N})\}, \tag{10}$$

where $\partial_i : \partial\Delta[i] \longrightarrow \Delta[i]$ is the boundary inclusion map. This set thus forms a set of generating cofibrations.

Finally, we get to the main result of this section. All the lemmas proved above will be useful in proving the following theorem:

4.18. Theorem. *The model category of coherently commutative monoidal quasi-categories is a symmetric monoidal closed model category under the Day convolution product.*

Proof. In light of [Hov99, Cor. 4.2.5], it is sufficient to show that for each pair of *generating* JQ cofibrations $i : U \longrightarrow V$ and $j : Y \longrightarrow Z$, the following *pushout product* morphism

$$i \Box j : U * Z \coprod_{U * Y} V * Y \longrightarrow V * Z$$

is a JQ cofibration which is also a JQ equivalence whenever either i or j is one. We first deal with the case of both i and j being a generating JQ cofibration. The closed symmetric monoidal model structure on the strict JQ model category, see Theorem 3.17, implies that $i\Box j$ is a JQ cofibration.

Now, let us assume that j is an acyclic JQ cofibration *i.e.* the JQ cofibration j is also a JQ equivalence of coherently commutative monoidal quasi-categories. Since all generating JQ cofibrations, and therefore the map $i\Box j$, have cofibrant domains and codomains, it follows from Proposition 4.12 the JQ cofibration $i\Box j$ is a JQ equivalence if and only if it has the left lifting property with respect to all strict JQ fibrations of Γ-spaces between coherently commutative monoidal quasi-categories. Let $p : W \longrightarrow X$ be a strict JQ fibration between two coherently commutative monoidal quasi-categories. By adjointness, a (dotted) lifting arrow would exists in the following diagram

$$\begin{array}{ccc} U * Z \coprod_{U*Y} V * Y & \longrightarrow & W \\ \big\downarrow & \nearrow & \big\downarrow p \\ V * Z & \longrightarrow & Y \end{array}$$

if and only if a (dotted) lifting arrow exists in the following adjoint commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & \underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(V, W) \\ j \big\downarrow & \nearrow & \big\downarrow (j^*, p^*) \\ Z & \longrightarrow & \underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(U, X) \underset{\underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(U,Y)}{\times} \underline{\mathcal{M}ap}_{\Gamma\mathcal{S}}(V, Y) \end{array}$$

The map (j^*, p^*) is a strict JQ fibration of Γ-spaces by Lemma A.3 and Theorem 3.17. Further the observation that both V and U are JQ cofibrant and the above Lemma 4.16 together imply that (j^*, p^*) is a strict JQ fibration between coherently commutative monoidal quasi-categories and therefore, by the above lemma, a fibration in the JQ model category. Since j is an acyclic cofibration in the JQ model category by assumption therefore the (dotted) lifting arrow exists in the above diagram. Thus, we have shown that if i is a JQ cofibration and j is a JQ cofibration which is also a weak equivalence in the JQ model category then $i\Box j$ is an acyclic cofibration in the JQ model category. A similar argument shows that whenever i is an acyclic JQ cofibration and j is merely a JQ cofibration, the map $i\Box j$ is an acyclic JQ cofibration.

■

Finally, we will show that the construction of the model category of coherently commutative monoidal quasi-categories achieves the goal of inducing a semi-additive structure on its homotopy category:

4.19. Theorem. *The homotopy category of the model category of coherently commutative monoidal quasi-categories is semi-additive.*

Proof. In light of [GS16, Prop. 6(ii)], it is sufficient to show that for each cofibrant coherently commutative monoidal quasi-category X, its coproduct with itself is homotopy equivalent to its product with itself. This follows from observing that the following map is a homotopy equivalence:

$$X \sqcup X \cong X * (\Gamma^1 \sqcup \Gamma^1) \overset{X * h_1^1}{\longrightarrow} X * (\Gamma^2) \cong X \times X.$$

■

5. Equivalence with normalized Γ-spaces

A normalized Γ-space is a functor $X : \Gamma^{op} \longrightarrow \mathcal{S}_\bullet$ such that $X(0^+) = 1$. The category of all (small) normalized Γ-spaces $\Gamma\mathcal{S}_\bullet$ is the category whose objects are normalized Γ-spaces. This category is defined by the following equalizer diagram in **Cat**:

$$\begin{array}{ccccc} \Gamma\mathcal{S}_\bullet & \longrightarrow & [\Gamma^{op}; \mathcal{S}_\bullet] & \xrightarrow{[0;\mathcal{S}_\bullet]} & [1; \mathcal{S}_\bullet] \\ & & & \searrow \quad 1 \quad \nearrow_{0} & \end{array}$$

where $[0; \mathcal{S}_\bullet]$ is the functor which precomposes a functor in $[\Gamma^{op}; \mathcal{S}_\bullet]$ with the unique (pointed) functor $1 \longrightarrow \Gamma^{op}$ whose image is $0^+ \in \Gamma^{op}$ and the upward diagonal functor 0 maps the terminal category 1 to the terminal object of the codomain functor category. Traditionally, Γ-spaces have been studied as normalized objects [Seg74], [BF78], [Lyd99], [Sch99]. This is because Γ-spaces are a model for connective spectra which are normalized object. In order to carry forward this tradition, we develop a theory of normalized coherently commutative monoidal quasi-categories in Appendix C. All results from the previous two sections have analogs, for normalized Γ-spaces, in Appendix C.

Even though we carry the aforementioned tradition forward, the purpose of this section is to show that the theory of normalized coherently commutative monoidal quasi-categories is equivalent to the theory of unnormalized coherently commutative monoidal quasi-categories. More precisely, in this section we will establish a Quillen equivalence between the model category of coherently commutative monoidal quasi-categories constructed above and the model category of normalized coherently commutative monoidal quasi-categories which is constructed in Appendix C.10. A similar result in the context of coherently commutative monoidal (special Γ-) spaces was proved in [dBM17]. In the cited paper, the authors prove the aforementioned result for the (Quillen-equivalent) Reedy model category structures on the two categories of Γ-spaces in context. Here we will prove our result working directly with the two aforementioned model categories.

The category of normalized Γ-spaces is equipped with a forgetful functor

$$U : \Gamma\mathcal{S}_\bullet \longrightarrow \Gamma\mathcal{S}. \tag{11}$$

This functor maps a normalized Γ-space X to the following composite

$$\Gamma^{op} \xrightarrow{X} \mathcal{S}_\bullet \xrightarrow{U_\mathcal{S}} \mathcal{S},$$

where the second functor is the obvious forgetful functor which forgets the basepoint of a simplicial set. The forgetful functor U has some very desirable homotopical properties: we will show in this section that U preserves weak equivalences namely it maps JQ equivalences of normalized Γ-spaces to JQ equivalences. This functor also preserves cofibrations even though it is a right Quillen functor.

5.1. Proposition. *The forgetful functor $U : \Gamma\mathcal{S}_\bullet \longrightarrow \Gamma\mathcal{S}$ preserves acyclic fibrations.*

Proof. A morphism of normalized Γ-spaces $p : X \longrightarrow Y$ is an acyclic fibration in the JQ model category of normalized Γ-spaces if and only if there is a lifting arrow in the following commutative diagram for each $n \in \mathbb{N}$

$$\begin{array}{ccc} \Gamma^n \underset{\mathcal{S}_\bullet}{\wedge} \partial\Delta[i]^+ & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Gamma^n \underset{\mathcal{S}_\bullet}{\wedge} \Delta[i]^+ & \longrightarrow & Y \end{array}$$

because the collection $I_\bullet = \{\Gamma^n \underset{\mathcal{S}_\bullet}{\wedge} \partial\Delta[i]^+ \longrightarrow \Gamma^n \underset{\mathcal{S}_\bullet}{\wedge} \Delta[i]^+ : n, i \in \mathbb{N}\}$ is a set of generating cofibrations for the combinatorial JQ model category of normalized Γ-spaces $\Gamma\mathcal{S}_\bullet$. By adjointness the lifting arrow exists in the above diagram if and only if a lifting arrow exists in the following (adjoint) commutative diagram in $\mathcal{S}_\bullet$

$$\begin{array}{ccc} \partial\Delta[i]^+ & \longrightarrow & \mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(\Gamma^n, X) \cong X(n^+) \\ \downarrow & \nearrow & \downarrow p \\ \Delta[i]^+ & \longrightarrow & \mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(\Gamma^n, Y) \cong Y(n^+) \end{array}$$

We recall the adjunction $(-)^+ : \mathcal{S} \rightleftarrows \mathcal{S}_\bullet : U_\mathcal{S}$ and observe that $U(X)(n^+) = U_\mathcal{S}(X(n^+))$. This implies that the lifting arrow in the above commutative diagram of pointed simplicial sets will exists if and only if a lifting arrow exists in the following (adjoint) commutative diagram in $\mathcal{S}$

$$\begin{array}{ccc} \partial\Delta[i] & \longrightarrow & U_\mathcal{S}(X(n^+)) \\ \downarrow & \nearrow & \downarrow U_\mathcal{S}(p_n) \\ \Delta[i] & \longrightarrow & U_\mathcal{S}(Y(n^+)) \end{array}$$

We observe that for any normalized Γ-space Z, $U_\mathcal{S}(Z(n^+)) \cong U(Z)(n^+)$. Therefore a lifting arrow exists in the above diagram if and only if a lifting arrow exists in the following commutative diagram:

$$\begin{array}{ccc} \partial\Delta[i] & \longrightarrow & \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^n, U(X)) \\ \downarrow & \nearrow & \downarrow {\scriptstyle \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^n, U(p_n))} \\ \Delta[i] & \longrightarrow & \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^n, U(Y)) \end{array}$$

By adjointness, this lifting arrow would exist if and only if there exists a lifting arrow in the following (adjoint) commutative diagram:

$$\begin{array}{ccc} \Gamma^n \underset{\mathcal{S}}{\boxtimes} \partial\Delta[i] & \longrightarrow & U(X) \\ \downarrow & \nearrow & \downarrow \\ \Gamma^n \underset{\mathcal{S}_\bullet}{\boxtimes} \Delta[i] & \longrightarrow & U(Y) \end{array}$$

The collection $I = \{\Gamma^n \underset{\mathcal{S}}{\boxtimes} \partial\Delta[i] \longrightarrow \Gamma^n \underset{\mathcal{S}_\bullet}{\boxtimes} \Delta[i] : n, i \in \mathbb{N}\}$ is a set of generating cofibrations for the combinatorial model category $\Gamma\mathcal{S}$. Thus, we have shown that the map of Γ-spaces $U(p)$ has the right lifting property with respect to the set of generating cofibrations of the JQ model category and hence $U(p)$ is an acyclic fibration.

■

A similar argument as in the proof of the above proposition when applied to the collection of generating acyclic cofibrations of the strict JQ model category of normalized Γ-spaces $\Gamma\mathcal{S}_\bullet$ gives a proof of the following proposition:

5.2. Proposition. *The forgetful functor $U : \Gamma\mathcal{S}_\bullet \longrightarrow \Gamma\mathcal{S}$ preserves strict JQ fibrations.*

We would like to construct a left adjoint of the functor U. For a given Γ-space X we will construct another Γ-space $X[0]$ which is equipped with a map $\iota : X[0] \longrightarrow X$.

5.3. Definition. *Let X be a Γ-space, the* unital part *of X is the constant Γ-space $X[0]$ which is defined by*

$$X[0](n^+) := X(0^+)$$

for all $n^+ \in Ob(\Gamma^{op})$. The map ι is defined, in degree n by the following simplicial map:

$$\iota(n^+) := X(0_n) : X(0^+) \longrightarrow X(n^+),$$

where $0_n : 0^+ \longrightarrow n^+$ is the unique map in Γ^{op} between 0^+ and n^+.

We notice that if X is a normalized Γ-space then the unital part of $U(X)[0]$ is the terminal Γ-space. We want to use the above construction to associate with a Γ-space a normalized Γ-space which is equipped with a map from the original Γ-space.

5.4. Definition. *Let X be a Γ-space, we define another Γ-space $U(X^{nor})$ by the following pushout square:*

$$\begin{array}{ccc} X[0] & \xrightarrow{\iota} & X \\ \downarrow & & \downarrow \eta_X \\ 1 & \longrightarrow & U(X^{nor}) \end{array} \tag{12}$$

where 1 is the terminal Γ-space. The bottom horizontal arrow in the above pushout square is an object of the category $1/\Gamma\mathcal{S}$. Since $U(X^{nor})(0^+) = *$ therefore the image is an object of the category $(1/\Gamma\mathcal{S})_\bullet$, see (C.11). Its image under the isomorphism of categories from remark (C.12) determines a normalized Γ-space which we denote by X^{nor} and call it the normalization of X.

The above construction is functorial in X and hence we have defined a functor $(-)^{nor}$: $\Gamma\mathcal{S} \longrightarrow \Gamma\mathcal{S}_\bullet$.

5.5. Proposition. *For any Γ-space X the map $\iota : X[0] \longrightarrow X$ defined above is a degree-wise mononorphism of simplicial sets.*

Proof. We want to show that for each $n^+ \in Ob(\Gamma^{op})$ the simplicial map

$$\iota(n^+) : X(0^+) \longrightarrow X(n^+)$$

is a monomorphism. We observe that the object 0^+ is the zero object in Γ^{op} therefore the unique composite arrow

$$0^+ \longrightarrow n^+ \longrightarrow 0^+$$

is the identity map of 0^+, for all $n^+ \in Ob(\Gamma^{op})$. This implies that the simplicial map $X(0^+) \longrightarrow X(n^+) \longrightarrow X(0^+)$ is the identity map of $X(0^+)$, in other words the simplicial map $\iota(n^+)$ has a left inverse which implies that the map $\iota(n^+) : X(0^+) \longrightarrow X(n^+)$ is a monomorphism. ∎

5.6. Proposition. *For each coherently commutative monoidal quasi-category X the map of Γ-spaces $\eta_X : X \longrightarrow U(X^{nor})$ defined in (12) is a (strict) JQ equivalence.*

Proof. Since X is a coherently commutative monoidal quasi-category by assumption therefore the unique terminal map $X[0] \longrightarrow 1$ is a strict JQ equivalence because $X(0^+)$ is homotopy equivalent to the terminal simplicial set in the Joyal model category. The Γ-space $U(X^{nor})$ is defined as a pushout, see (12), and pushouts in the category Γ-space are degreewise therefore we have the following pushout diagram in the category $\mathcal{S}$:

$$\begin{array}{ccc} X[0](n^+) & \xrightarrow{\iota(n^+)} & X(n^+) \\ \downarrow & & \downarrow \eta_X(n^+) \\ 1 & \longrightarrow & U(X^{nor})(n^+) \end{array} \tag{13}$$

for each $n^+ \in Ob(\Gamma^{op})$. By Proposition 5.5 the simplicial map $\iota(n^+)$ is a monomorphism. Since monomorphisms are cofibrations in the Joyal model category which is a left proper model category therefore a pushout of a weak equivalence along a monomorphism is a weak equivalence in Joyal model category. Thus, we have shown that the map $\eta_X(n^+) : X(n^+) \longrightarrow U(X^{nor})(n^+)$ is a weak equivalence in the Joyal model category which proves that the unit map $\eta_X : X \longrightarrow U(X^{nor})$ is a strict JQ equivalence whenever X is a coherently commutative monoidal quasi-category.

■

5.7. Corollary. *The functor $(-)^{nor}$ maps coherently commutative monoidal quasi-categories to normalized coherently commutative monoidal quasi-categories.*

5.8. Corollary. *The functor $U : \Gamma\mathcal{S}_\bullet \longrightarrow \Gamma\mathcal{S}$ preserves weak equivalences.*

Proof. A map $f : X \longrightarrow Y$ is a JQ equivalence of normalized Γ-spaces if and only if each cofibrant replacement of f is also the same, so we may assume that f is a map between cofibrant objects. It would be sufficient to show that for each coherently commutative monoidal quasi-category Z the following simplicial map is an equivalence of quasi-categories:

$$\mathcal{M}ap_{\Gamma\mathcal{S}}(U(f), Z) : \mathcal{M}ap_{\Gamma\mathcal{S}}(U(Y), Z) \longrightarrow \mathcal{M}ap_{\Gamma\mathcal{S}}(U(X), Z)$$

We have the following commutative diagram of simplicial mapping objects:

$$\begin{array}{ccc}
\mathcal{M}ap_{\Gamma\mathcal{S}}(U(Y), Z) & \xrightarrow{\mathcal{M}ap_{\Gamma\mathcal{S}}(U(f), Z)} & \mathcal{M}ap_{\Gamma\mathcal{S}}(U(X), Z) \\
{\scriptstyle \mathcal{M}ap_{\Gamma\mathcal{S}}(U(Y), \eta_Z)} \downarrow & & \downarrow {\scriptstyle \mathcal{M}ap_{\Gamma\mathcal{S}}(U(X), \eta_Z)} \\
\mathcal{M}ap_{\Gamma\mathcal{S}}(U(Y), U(Z^{nor})) & \longrightarrow & \mathcal{M}ap_{\Gamma\mathcal{S}}(U(X), U(Z^{nor})) \\
\cong \downarrow & & \downarrow \cong \\
U(\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(Y, Z^{nor})) & \xrightarrow[U(\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(f, Z^{nor}))]{} & U(\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X, Z^{nor}))
\end{array}$$

Since f is a JQ equivalence of normalized Γ-spaces by assumption and Z^{nor} is a normalized coherently commutative monoidal quasi-category therefore the simplicial map $U(\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(f, Z^{nor}))$ is an equivalence of quasi-categories. The vertical arrows in the bottom rectangle are isomorphisms by Corollary C.3. The vertical arrows in the top rectangle are equivalences of quasi-categories because η_Z is a weak equivalence by Proposition 5.6, $U(X)$ and $U(Y)$ are cofibrant and Z and $U(Z^{nor})$ are both fibrant. Now the two out of three property of weak equivalences in a model category tells us that the top horizontal map in the above diagram namely $\mathcal{M}ap_{\Gamma\mathcal{S}}(U(f), \eta_Z)$ is a weak equivalence in the Joyal model category. By Lemma 4.7 we have shown that the map $U(f)$ is a JQ equivalence. ■

We claim that $(-)^{nor}$ is a left adjoint of the forgetful functor $U : \Gamma\mathcal{S}_\bullet \longrightarrow \Gamma\mathcal{S}$. The unit of this adjunction is given by the quotient map $\eta_X : X \longrightarrow U(X^{nor})$. For a normalized Γ-space Y we have a canonical isomorphism (depicted by the dotted arrow) in the following diagram

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, equal] & U(Y) \arrow[d] \arrow[ddr, bend left, "id"] & \\ 1 \arrow[r] \arrow[drr, bend right] & U(U(Y)^{nor}) \arrow[dr, dashed, "\epsilon_Y"] & \\ & & U(Y) \end{tikzcd} \tag{14}$$

The diagram (14) is a composite arrow in the category $(1/\Gamma\mathcal{S})_\bullet$. The image of the map ϵ_Y under the isomorphism of categories from remark (C.12) gives us the counit map which we also denote by ϵ_Y.

The next proposition verifies our claim made above:

5.9. Proposition. *The functor* $(-)^{nor} : \Gamma\mathcal{S} \longrightarrow \Gamma\mathcal{S}_\bullet$ *is a left adjoint to the forgetful functor* $U : \Gamma\mathcal{S}_\bullet \longrightarrow \Gamma\mathcal{S}$.

Proof. We will prove this proposition by showing that the unit map η_X constructed above is universal. Let X be a Γ-space and Y be a normalized Γ-space and $f : X \longrightarrow U(Y)$ be a map in $\Gamma\mathcal{S}$. We will show the existence of a unique map $g : X^{nor} \longrightarrow Y$ in $\Gamma\mathcal{S}_\bullet$ such that the following diagram commutes in the category $\Gamma\mathcal{S}$:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & U(X^{nor}) \arrow[d, "U(g)"] \\ & U(Y) \end{tikzcd} \tag{15}$$

The map $1 \longrightarrow U(Y)$ in the diagram below is the image of the normalized Γ-space Y under the isomorphism of categories in remark following (C.11):

$$\begin{tikzcd} X[0] \arrow[r] \arrow[d] & X \arrow[d, "\eta_X"'] \arrow[ddr, bend left, "f"] & \\ 1 \arrow[r] \arrow[drr, bend right] & U(X^{nor}) \arrow[dr, dashed, "U(g)"] & \\ & & U(Y) \end{tikzcd}$$

Since $U(Y)(0^+) = *$ therefore f maps $X(0^+)$ to a point. This implies that the outer solid diagram in the figure above commutes. Since the square in the above diagram is a pushout square therefore there exists a unique (dotted) arrow which makes the whole diagram commutative. The lower commutative triangle in the diagram above is a map in the category $(1/\Gamma\mathcal{S})_\bullet$. The image of this map under the isomorphism of categories from remark following (C.11) is a map $g : X^{nor} \longrightarrow Y$ in $\Gamma\mathcal{S}_\bullet$ whose image under the forgetful functor $U(g)$ makes the diagram (15) commute. ■

This proposition has the following consequence:

5.10. Corollary. *The forgetful functor $U : \Gamma\mathcal{S}_\bullet \longrightarrow \Gamma\mathcal{S}$ maps JQ cofibrations of normalized Γ-spaces to JQ cofibrations.*

Proof. Let $i : V \longrightarrow W$ be a JQ cofibration of normalized Γ-spaces. We will show that $U(i)$ is a JQ cofibration. By adjointness $U(i)$ is a cofibration if and only if $U(i)^{nor}$ is a JQ cofibration of normalized Γ-spaces. The following commutative square in $\Gamma\mathcal{S}_\bullet$ shows that $U(i)$ is a cofibration because i is one by assumption:

$$\begin{array}{ccc} U(V)^{nor} & \xrightarrow[\cong]{\epsilon_V} & V \\ {\scriptstyle U(i)^{nor}}\downarrow & & \downarrow{\scriptstyle i} \\ U(W)^{nor} & \xrightarrow[\epsilon_W]{\cong} & W \end{array}$$

■

Next we show that the adjunction $(-)^{nor} \dashv U$ is compatible with the model category structures *i.e.* it is a Quillen adjunction.

5.11. Lemma. *The pair of adjoint functors $((-)^{nor}), U)$ is a Quillen pair.*

Proof. A pair of adjoint functors between two model categories is a Quillen pair if and only if the left adjoint preserves cofibrations and the right adjoint preserves fibrations between fibrant objects, see [JT06, Prop. 7.15]. Let $i : A \longrightarrow B$ be a cofibration in $\Gamma\mathcal{S}$ and let $p : X \longrightarrow Y$ be an acyclic fibration in $\Gamma\mathcal{S}_\bullet$ then by Proposition 5.1, there is a lifting arrow in the following (outer) commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & U(X) \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle U(p)} \\ B & \longrightarrow & U(Y) \end{array}$$

By adjointness this lifting arrow exists if and only if there exists a lifting arrow in the following (adjoint) commutative diagram:

$$\begin{array}{ccc} A^{nor} & \longrightarrow & X \\ {\scriptstyle i^{nor}}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B^{nor} & \longrightarrow & Y \end{array}$$

Thus, we have shown that for each cofibration i in $\Gamma\mathcal{S}$, its image i^{nor} in $\Gamma\mathcal{S}_\bullet$ has the left lifting property with respect to acyclic fibrations in the JQ model category of normalized Γ-spaces $\Gamma\mathcal{S}_\bullet$. Hence we have shown that the left adjoint preserves cofibrations.

By Lemma 4.17 JQ fibrations between JQ fibrant normalized Γ-spaces are just strict JQ fibrations of normalized Γ-spaces. Now Proposition 5.2 tells us that U preserves fibrations between fibrant normalized Γ-spaces. Hence by [JT06, Prop. 7.15] the adjunction in context is a Quillen pair.

■

By definition, the counit map $\epsilon_Y : U(Y)^{nor} \longrightarrow Y$ of the adjunction $((-)^{nor}), U)$ is an isomorphism for each normalized Γ-space Y. Now we want to show that the unit map of the same adjunction is a JQ equivalence.

5.12. Lemma. *The unit map $\eta_X : X \longrightarrow U(X^{nor})$ is a JQ equivalence for each Γ-space X.*

Proof. We have already seen in Proposition 5.6 that the result holds when the Γ-space X is a coherently commutative monoidal quasi-category. Now we tackle the general case wherein X is an arbitrary Γ-space. Since the unit map η is a natural transformation therefore we have the following commutative diagram in the category $\Gamma\mathcal{S}$:

$$\begin{array}{ccc} X & \longrightarrow & R(X) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_{R(X)}} \\ U(X^{nor}) & \longrightarrow & U(R(X)^{nor}) \end{array}$$

where $X \longrightarrow R(X)$ is a fibrant replacement of X and therefore it is an acyclic JQ cofibration and $R(X)$ is a coherently commutative monoidal quasi-category. Thus we have shown that the top and right vertical arrow in the commutative diagram above are JQ equivalences. Now we want to show that the bottom horizontal arrow is also a JQ equivalence. The functor $(-)^{nor}$ is a left Quillen functor, see (5.11), therefore it preserves acyclic JQ cofibrations. Now Proposition 5.10 says that U preserves weak equivalences which implies that the bottom horizontal map is a JQ equivalence.

■

An easy consequence of the above lemma and the fact that the counit of the Quillen pair $((-)^{nor}, U)$ is a natural isomorphism is the following theorem:

5.13. Theorem. *The Quillen pair*

$$(-)^{nor} : \Gamma\mathcal{S} \rightleftarrows \Gamma\mathcal{S}_\bullet : U$$

is a Quillen equivalence.

6. The model category of coherently commutative monoidal marked quasi-categories

The objective of this section is to construct a new model category structure on the category $\Gamma\mathcal{S}^+ = [\Gamma^{op}; \mathcal{S}^+]$, where $\mathcal{S}^+$ is the model category of marked simplicial sets. We will refer to an object of $\Gamma\mathcal{S}^+$, namely a functor from Γ^{op} to $\mathcal{S}^+$, as a *marked* Γ-space. This new model category can be described as the model category of coherently commutative objects in $\mathcal{S}^+$. We begin by describing a *projective* model category structure on $\Gamma\mathcal{S}^+$:

6.1. Definition. *We call a map of marked* Γ*-spaces*

1. *A* strict *JQ* fibration *of marked* Γ*-spaces if it is degreewise a* pseudo-fibration *of marked simplicial sets* i.e. *a fibration in the Joyal model category structure on marked simplicial sets.*
2. *A* strict *JQ* equivalence *of marked* Γ*-spaces if it is degreewise a* categorical equivalence *of marked simplicial sets* i.e. *a weak equivalence in the Joyal model category structure on marked simplicial sets.*
3. *A* strict *JQ* cofibration *of marked* Γ*-spaces if it has the left lifting property with respect to maps which are simultaneously strict JQ fibrations and strict JQ equivalences.*

6.2. Theorem. *Strict JQ equivalences, strict JQ fibrations and strict JQ cofibrations of marked* Γ*-spaces provide the category* $\Gamma\mathcal{S}^+$ *with a combinatorial, left-proper model category structure.*

The model structure in the above theorem follows from [Lur09, Proposition A 3.3.2] and the left properness is a consequence of the left properness of the Joyal model category.

6.3. Notation. *We will refer to the above model category as the* strict *JQ* model category of marked simplicial sets.

Next we will construct function objects for the above model category. For each pair (F, K), where $F \in Ob(\Gamma\mathcal{S}^+)$ and $K \in Ob(\mathcal{S})$, one can construct a Γ-space which we denote by $F \underset{\mathcal{S}}{\boxtimes} K$ and which is defined as follows:

$$(F \underset{\mathcal{S}}{\boxtimes} K)(n^+) := F(n^+) \times K^\flat,$$

where the product on the right is taken in the category of simplicial sets. This construction is functorial in both variables. Thus we have a functor

$$- \underset{\mathcal{S}}{\boxtimes} - : \Gamma\mathcal{S}^+ \times \mathcal{S} \longrightarrow \Gamma\mathcal{S}^+.$$

Now we will define a couple of function objects for the category $\Gamma\mathcal{S}^+$. The first function object enriches the category $\Gamma\mathcal{S}$ over $\mathcal{S}$ *i.e.* there is a bifunctor

$$\mathcal{M}ap_{\Gamma\mathcal{S}^+}(-,-) : (\Gamma\mathcal{S}^+)^{op} \times \Gamma\mathcal{S}^+ \longrightarrow \mathcal{S}$$

which assigns to each pair of objects $(X, Y) \in Ob(\Gamma\mathcal{S}^+) \times Ob(\Gamma\mathcal{S}^+)$, a simplicial set $\mathcal{M}ap_{\Gamma\mathcal{S}^+}(X, Y)$ which is defined in degree zero as follows:

$$\mathcal{M}ap_{\Gamma\mathcal{S}^+}(X, Y)_0 := \Gamma\mathcal{S}^+(X, Y)$$

and the simplicial set is defined in degree n as follows:

$$\mathcal{M}ap_{\Gamma\mathcal{S}^+}(X,Y)_n := \Gamma\mathcal{S}^+(X \underset{\mathcal{S}}{\boxtimes} \Delta[n], Y) \tag{16}$$

For any marked Γ-space X, the functor $X \underset{\mathcal{S}}{\boxtimes} - : \mathcal{S} \longrightarrow \Gamma\mathcal{S}^+$ is left adjoint to the functor $\mathcal{M}ap_{\Gamma\mathcal{S}^+}(X,-) : \Gamma\mathcal{S}^+ \longrightarrow \mathcal{S}$. The *counit* of this adjunction is the evaluation map $ev : X \boxtimes \mathcal{M}ap_{\Gamma\mathcal{S}^+}(X,Y) \longrightarrow Y$ and the *unit* is the obvious simplicial map $K \longrightarrow \mathcal{M}ap_{\Gamma\mathcal{S}^+}(X, X \underset{\mathcal{S}}{\boxtimes} K)$.

To each pair of objects $(K,X) \in Ob(\mathcal{S}) \times Ob(\Gamma\mathcal{S}^+)$ we can define a Γ-space X^K, in degree n, as follows:

$$(X^K)(n^+) := [K^\flat, X(n^+)] \ .$$

This assignment is functorial in both variables and therefore we have a bifunctor

$$-^- : \mathcal{S}^{op} \times \Gamma\mathcal{S}^+ \longrightarrow \Gamma\mathcal{S}^+.$$

For any Γ-space X, the functor $X^- : \mathcal{S} \longrightarrow (\Gamma\mathcal{S}^+)^{op}$ is left adjoint to the functor

$$\mathcal{M}ap_{\Gamma\mathcal{S}^+}(-,X) : (\Gamma\mathcal{S}^+)^{op} \longrightarrow \mathcal{S}.$$

The following proposition summarizes the above discussion.

6.4. Proposition. *There is an adjunction of two variables*

$$(- \underset{\mathcal{S}}{\boxtimes} -, -^-, \mathcal{M}ap_{\Gamma\mathcal{S}^+}(-,-)) : \Gamma\mathcal{S}^+ \times \mathcal{S} \longrightarrow \Gamma\mathcal{S}^+. \tag{17}$$

The following theorem follows from [Lur09, Remark A.3.3.4]. A direct proof can also be easily given by a straightforward verification of Lemma A.3(2).

6.5. Theorem. *The strict model category of marked Γ-spaces, $\Gamma\mathcal{S}^+$, is a $(\mathcal{S}, \mathbf{Q})$- model category with respect to the adjunction of two variables (17).*

The adjoint functors $((-)^b, U)$, see 2.9, induce an adjunction

$$\Gamma(-)^b : \Gamma\mathcal{S} \rightleftharpoons \Gamma\mathcal{S}^+ : U. \tag{18}$$

This adjunction is a Quillen equivalence in light of [Lur09, Remark A 3.3.2] and 2.9:

6.6. Theorem. *The adjoint pair $(\Gamma(-)^\flat, U)$ determines a Quillen equivalence between the strict JQ model structure on $\Gamma\mathcal{S}$ and the model category structure in Theorem 6.2.*

The following three lemmas will be useful in proving various results in this section:

6.7. Lemma. *For each pair (X,Y) consisting of a Γ-space X and a marked Γ-space Y, the above adjunction gives the following simplicial isomorphism:*

$$\mathcal{M}ap_{\Gamma\mathcal{S}^+}(\Gamma(X)^\flat, Y) \cong \mathcal{M}ap_{\Gamma\mathcal{S}}(X, U(Y)).$$

Proof. By definition of the function space $\mathcal{M}ap_{\Gamma\mathcal{S}^+}(\Gamma(X)^\flat, Y)$, see (16), and [GJ99, Lemma 2.9] it is sufficient to observe that for each $n \in \mathbb{N}$

$$\Gamma(X)^\flat \times \Delta[n]^\flat = \Gamma(X \underset{\mathcal{S}}{\boxtimes} \Delta[n])^\flat.$$

∎

The next lemma is a consequence of the definition of the left adjoint functor $\Gamma(-)^\flat$:

6.8. LEMMA. *The left adjoint functor $\Gamma(-)^\flat$ maps strict JQ equivalences of Γ-spaces to strict JQ equivalences of marked Γ-spaces.*

PROOF. Let $F : X \longrightarrow Y$ be a strict JQ equivalence of Γ-spaces. For each $n \in \mathbb{N}$ we have a categorical equivalence of simplicial sets $F(n^+) : X(n^+) \longrightarrow Y(n^+)$. We recall that each simplicial set is cofibrant in the Joyal model category therefore $F(n^+)$ is a weak equivalence between cofibrant objects.

In degree n, the map of marked Γ-spaces $\Gamma(F)^\flat$ is the following map of marked simplicial sets:

$$\Gamma(F)^\flat(n^+) : X(n^+)^\flat \longrightarrow Y(n^+)^\flat.$$

In other words $\Gamma(F)^\flat(n^+) = F(n^+)^\flat$. Since $(-)^\flat$ is a left adjoint of a Quillen equivalence therefore it preserves weak equivalences between cofibrant objects. Thus $F(n^+)^\flat = \Gamma(F)^\flat(n^+)$ is a weak equivalence in the Joyal model category of marked simplicial sets for each $n^+ \in \Gamma^{op}$ and hence $\Gamma(F)^\flat$ is a strict JQ equivalence of marked Γ-spaces. ∎

6.9. LEMMA. *For any strict JQ fibrant marked Γ-spaces X, the counit map $\epsilon_X : \Gamma(X)^\flat \longrightarrow X$ is a strict JQ equivalence of marked Γ-spaces.*

PROOF. Since $(\Gamma(-)^\flat, U)$ is a Quillen pair therefore it induces a *derived* adjunction $(\Gamma^L(-)^\flat, U^R)$ on the homotopy categories of the two model categories in context. For a strict JQ fibrant Γ-space X, the counit of this derived adjunction ϵ_X^D is defined as follows:

$$\begin{array}{ccc} \Gamma(U(X))^\flat & \xrightarrow{\epsilon_X} & X \\ {\scriptstyle \Gamma(q)^\flat}\uparrow & \nearrow{\scriptstyle \epsilon_X^D} & \\ \Gamma(Q(U(X)))^\flat & & \end{array}$$

where $q : QU((X)) \longrightarrow U(X)$ is a cofibrant replacement of $U(X)$ in the strict JQ model category of Γ-spaces. By the previous Lemma $\Gamma(q)^\flat$ is a strict JQ equivalence of marked Γ-spaces. Since the Quillen pair $(\Gamma(-)^\flat, U)$ is also a Quillen equivalence between the strict model categories therefore ϵ_X^D is a strict JQ equivalence of marked Γ-spaces. By the 2 out of 3 property of weak equivalences in a model category, we conclude that the counit map ϵ_X is a strict JQ equivalence of marked simplicial sets. ∎

Now we will construct another model category structure on $\Gamma\mathcal{S}^+$. The guiding principle of this new model structure is to endow its homotopy category with a semi-additive structure. In other words we want this new model category structure to have finite *homotopy biproducts*. We want to construct a left Bousfield localization of the strict JQ model category of marked Γ-spaces. For each pair $k^+, l^+ \in \Gamma^{op}$, we have the obvious *projection maps* in $\Gamma\mathcal{S}$

$$\delta_k^{k+l} : (k+l)^+ \longrightarrow k^+ \quad \textit{and} \quad \delta_l^{k+l} : (k+l)^+ \longrightarrow l^+.$$

The maps

$$\Gamma^{op}(\delta^{k+l}_k, -) : \Gamma^k \longrightarrow \Gamma^{k+l} \quad \textit{and} \quad \Gamma^{op}(\delta^{k+l}_l, -) : \Gamma^l \longrightarrow \Gamma^{k+l}$$

induce a map of Γ-spaces on the coproduct which we denote as follows:

$$\Gamma(h^l_k)^\flat : \Gamma(\Gamma^l)^\flat \sqcup \Gamma(\Gamma^l)^\flat \longrightarrow \Gamma(\Gamma^{l+k})^\flat.$$

We now define a set of maps $\Gamma\mathcal{E}_\infty\mathcal{S}$ in $\Gamma\mathcal{S}^+$:

$$\Gamma\mathcal{E}_\infty\mathcal{S} := \{\Gamma(h^l_k)^\flat : \Gamma(\Gamma^k)^\flat \sqcup \Gamma(\Gamma^l)^\flat \longrightarrow \Gamma(\Gamma^{k+l})^\flat : l, k \in \mathbb{Z}^+\}$$

6.10. Definition. *We call a* Γ*-space* X *a* $(\Delta\times\Gamma\mathcal{E}_\infty\mathcal{S})$-local object *if it is a fibrant object in the strict JQ model category and for each map* $h^l_k \in \Gamma\mathcal{E}_\infty\mathcal{S}$*, the induced simplicial map*

$$\mathcal{M}ap^h_{\Gamma\mathcal{S}^+}(\Delta[n] \underset{\mathcal{S}}{\boxtimes} \Gamma(h^l_k)^\flat, X) : \mathcal{M}ap^h_{\Gamma\mathcal{S}^+}(\Delta[n] \underset{\mathcal{S}}{\boxtimes} \Gamma(\Gamma^{k+l})^\flat, X) \longrightarrow \mathcal{M}ap^h_{\Gamma\mathcal{S}^+}(\Delta[n] \underset{\mathcal{S}}{\boxtimes} (\Gamma(\Gamma^k)^\flat \sqcup \Gamma(\Gamma^l)^\flat), X),$$

is a homotopy equivalence of simplicial sets for all $n \geq 0$ *where* $\mathcal{M}ap^h_{\Gamma\mathcal{S}^+}(-,-)$ *is the simplicial function complexes associated with the strict JQ model category* $\Gamma\mathcal{S}^+$*, see [DK80a], [DK80b] and [DK80c].*

Appendix B tell us that the Kan complex $J(\mathcal{M}ap_{\Gamma\mathcal{S}^+}(X,Y))$, namely the maximal Kan complex contained in the quasicategory $\mathcal{M}ap_{\Gamma\mathcal{S}^+}(X,Y)$, is a model for $\mathcal{M}ap^h_{\Gamma\mathcal{S}^+}(X,Y)$, whenever X is cofibrant and Y is fibrant.

The following proposition gives a characterization of $\mathcal{E}_\infty\mathcal{S}$-local objects

6.11. Proposition. *A* Γ*-space* X *is a* $(\Delta\times\Gamma\mathcal{E}_\infty\mathcal{S})$*-local object in* $\Gamma\mathcal{S}^+$ *if and only if its underlying* Γ*-space* $U(X)$ *satisfies the Segal condition, namely the functor*

$$(U(X)(\delta^{(k+l)}_k), U(X)(\delta^{(k+l)}_l)) : U(X)((k+l)^+) \longrightarrow U(X)(k^+) \times U(X)(l^+)$$

is a categorical equivalence of quasi-categories for all $k^+, l^+ \in Ob(\Gamma^{op})$.

Proof. We begin the proof by observing that each element of the set $\Gamma\mathcal{E}_\infty\mathcal{S}$ is a map of marked Γ-spaces between cofibrant marked Γ-spaces. Theorem B.10 and 6.7 together imply that X is a $(\Delta\times\Gamma\mathcal{E}_\infty\mathcal{S})$-local object if and only if the following simplicial map

$$\mathcal{M}ap_{\Gamma\mathcal{S}}(h^k_l, U(X)) : \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^{k+l}, U(X)) \longrightarrow \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^k \sqcup \Gamma^l, U(X))$$

is a categorical equivalence of quasi-categories.

We observe that we have the following commutative square in $(\mathcal{S}, \mathbf{Q})$

$$\begin{array}{ccc} \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^{(k+l)}, U(X)) & \xrightarrow{\mathcal{M}ap_{\Gamma\mathcal{S}}(h^k_l, U(X))} & \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^k \sqcup \Gamma^l, U(X)) \\ \cong\downarrow & & \downarrow\cong \\ U(X)((k+l)^+) & \xrightarrow[(U(X)(\delta^{k+l}_k), U(X)(\delta^{k+l}_l))]{} & U(X)(k^+) \times U(X)(l^+) \end{array}$$

This implies that the simplicial map $(U(X)(\delta^{k+l}_k), U(X)(\delta^{k+l}_l))$ is a categorical equivalence of quasi-categories if and only if the functor $\mathcal{M}ap_{\Gamma\mathcal{S}}(h^k_l, U(X))$ is one. ■

6.12. Definition. *We will refer to a* $(\Delta \times \Gamma\mathcal{E}_\infty\mathcal{S})$*-local object as a* coherently commutative monoidal marked quasi-category.

6.13. Definition. *A morphism of marked* Γ*-spaces* $F : X \longrightarrow Y$ *is a* $(\Delta \times \Gamma\mathcal{E}_\infty\mathcal{S})$*-local equivalence if for each coherently commutative monoidal marked quasi-category* Z *the following simplicial map*

$$\mathcal{M}ap^h_{\Gamma\mathcal{S}^+}(F, Z) : \mathcal{M}ap^h_{\Gamma\mathcal{S}^+}(Y, Z) \longrightarrow \mathcal{M}ap^h_{\Gamma\mathcal{S}^+}(X, Z)$$

is a homotopy equivalence of simplicial sets.

The following proposition follows from an argument similar to the one in the proof of Proposition 4.7:

6.14. Proposition. *A morphism between two cofibrant marked* Γ*-spaces* $F : X \longrightarrow Y$ *is an* $(\Delta \times \Gamma\mathcal{E}_\infty\mathcal{S})$*-local equivalence if and only if the simplicial map*

$$\mathcal{M}ap_{\Gamma\mathcal{S}^+}(F, Z) : \mathcal{M}ap_{\Gamma\mathcal{S}^+}(Y, Z) \longrightarrow \mathcal{M}ap_{\Gamma\mathcal{S}^+}(X, Z)$$

is an equivalence of quasi-categories for each coherently commutative monoidal marked quasi-category Z*.*

6.15. Definition. *We will refer to a* $(\Delta \times \Gamma\mathcal{E}_\infty\mathcal{S})$*-local equivalence either as an* equivalence of coherently commutative monoidal marked quasi-categories *or as a JQ equivalence of marked* Γ*-spaces.*

We construct a new model category structure on the category $\Gamma\mathcal{S}^+$, by localizing the strict JQ model category of marked Γ-spaces with respect to morphisms in the set $\Gamma\mathcal{E}_\infty\mathcal{S}$.

6.16. Theorem. *There is a left proper, combinatorial model category structure on the category of marked* Γ*-spaces,* $\Gamma\mathcal{S}^+$*, in which*

1. *The class of cofibrations is the same as the class of JQ cofibrations of marked* Γ*-spaces.*
2. *The weak equivalences are equivalences of coherently commutative monoidal marked quasi-categories.*

An object is fibrant in this model category if and only if it is a coherently commutative monoidal marked quasi-category. A fibration between two coherently commutative monoidal marked quasi-categories is a strict JQ equivalence of marked simplicial sets.

Proof. The strict model category of Γ-spaces is a combinatorial model category therefore the existence of the model structure follows from Theorem 4.9. The last statement follows from (1). ∎

6.17. Notation. *The model category constructed in Theorem 6.16 will be referred to either as the model category of* coherently commutative monoidal marked quasi-categories *or as the JQ model category of marked* Γ*-spaces. We will denote this model category by* $(\Gamma\mathcal{S}^+_\otimes, \mathbf{Q})$.

6.18. Theorem. *The adjoint pair*

$$\Gamma(-)^\flat : (\Gamma\mathcal{S}_\otimes, \mathbf{Q}) \rightleftarrows (\Gamma\mathcal{S}^+_\otimes, \mathbf{Q}) : U$$

is a Quillen pair.

Proof. As observed above, the adjunction (3) determines a Quillen equivalence between the strict JQ model category of marked Γ-spaces and the strict JQ model category of Γ-spaces. Since the two model categories in context are constructed as a left Bousfield localization of the corresponding strict model categories, therefore the cofibrations are the same as the corresponding strict model categories. This implies that the left adjoint $\Gamma(-)^\flat$ preserves cofibrations. We observe that the fibrations between fibrant objects, in the JQ model category of marked Γ-spaces, are the same as strict JQ fibrations of marked Γ-spaces. Now it follows from the aforementioned Quillen equivalence that the right adjoint U preserves fibrations between fibrant objects. In light of Proposition [Joy08, Prop. E.2.14.], we conclude that $(\Gamma(-)^\flat, U)$ is a Quillen pair between the two JQ model categories in the context of the theorem. ∎

6.19. Proposition. *The right adjoint functor* U *maps JQ equivalences of marked* Γ*-spaces to JQ equivalences of* Γ*-spaces.*

Proof. Let us first assume that $F : X \longrightarrow Y$ is a JQ equivalence of marked Γ-spaces between strict JQ fibrant and cofibrant marked Γ-spaces, *i.e.*, X and Y are both cofibrant and fibrant objects in the strict JQ model category of marked Γ-spaces. By Lemma 6.9 we have the following commutative square:

$$\begin{array}{ccc} \Gamma(U(X))^\flat & \xrightarrow{\Gamma(F)^\flat} & \Gamma(U(Y))^\flat \\ {\scriptstyle \epsilon_X}\downarrow & & \downarrow{\scriptstyle \epsilon_Y} \\ X & \xrightarrow[F]{} & Y \end{array}$$

wherein the vertical maps are strict JQ equivalences of marked Γ-spaces. This implies that $\Gamma(F)^\flat$ is a JQ equivalence whenever F is one. Now for each coherently commutative monoidal quasi-category Z, we have the following commutative diagram of maps between mapping spaces in $\mathcal{S}$:

$$\begin{CD}
\mathcal{M}ap_{\Gamma\mathcal{S}^+}(\Gamma(U(Y))^\flat, \Gamma(Z)^\flat) @>{\mathcal{M}ap_{\Gamma\mathcal{S}^+}(\Gamma(U(F))^\flat, \Gamma(Z)^\flat)}>> \mathcal{M}ap_{\Gamma\mathcal{S}^+}(\Gamma(U(X))^\flat, \Gamma(Z)^\flat) \\
@V{\cong}VV @VV{\cong}V \\
\mathcal{M}ap_{\Gamma\mathcal{S}}(U(Y), U(\Gamma(Z)^\flat)) @>>{\mathcal{M}ap_{\Gamma\mathcal{S}}(U(F), U(\Gamma(Z)^\flat))}> \mathcal{M}ap_{\Gamma\mathcal{S}}(U(X), U(\Gamma(Z)^\flat)) \\
@| @| \\
\mathcal{M}ap_{\Gamma\mathcal{S}}(U(Y), Z) @>>{\mathcal{M}ap_{\Gamma\mathcal{S}}(U(F), Z)}> \mathcal{M}ap_{\Gamma\mathcal{S}}(U(X), Z)
\end{CD}$$

where the vertical isomorphisms follow from Lemma 6.7 and the vertical equalities follow from the observation that the unit map of the adjunction in the context is the identity. The above commutative diagram of mapping spaces implies that $U(F)$ is a JQ equivalence of Γ-spaces whenever F is a JQ equivalence of marked Γ-spaces.

Let $F : X \longrightarrow Y$ be a JQ equivalence of marked Γ-spaces. By [Hir02, Prop. 8.1.17] we can choose a cofibrant- fibrant replacement functor (R, r) in the strict JQ model category structure on marked Γ-spaces. This functor gives us the following commutative square:

$$\begin{CD}
R(X) @>{R(F)}>> R(Y) \\
@A{r_X}AA @AA{r_Y}A \\
X @>>{F}> Y
\end{CD}$$

The vertical maps are strict JQ equivalences (acyclic cofibrations) of marked Γ-spaces and the top horizontal arrow is a JQ equivalence of marked Γ-spaces between objects which are both cofibrant and fibrant in the strict JQ model category of marked Γ-spaces. It is easy to see that U maps strict JQ equivalences of marked Γ-spaces to strict JQ equivalences of Γ-spaces. Further, $U(R(F))$ is a (strict) JQ equivalences of Γ-spaces from the arguments made earlier in the proof. Now the 2 out of 3 property of weak equivalences in a model category tells us that $U(F)$ is a JQ equivalences of Γ-spaces. ∎

Now we state and prove the main result of this section:

6.20. Theorem. *The Quillen pair of Theorem* (6.18) *is a Quillen equivalence.*

Proof. Let X be a Q-cofibrant Γ-space and Y be a JQ fibrant marked Γ-space. We will show that a map $F : \Gamma(X)^\flat \longrightarrow Y$ is a JQ equivalence of marked Γ-spaces if and only if its adjunct map $\phi(F) : X \longrightarrow U(Y)$ is a JQ equivalence of Γ-spaces. We consider the following commutative diagram provided by the adjunction $(\Gamma(-)^\flat, U)$

$$\begin{array}{ccc}
U(\Gamma(X)^\flat) & \xrightarrow{F} & U(Y) \\
\uparrow{\scriptstyle \eta_X} & \nearrow_{\phi(F)} & \\
X & &
\end{array}$$

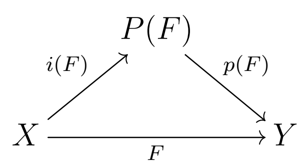
where η_X is the unit map. Now the result follows from this diagram, Proposition 6.19 and the observation that the unit η_X is the identity map.

■

The next lemma is a consequence of the fact that the left adjoint functor $\Gamma(-)^\flat$ preserves fibrant objects, which follows easily from the above characterization of coherently commutative monoidal marked Γ-spaces. The following lemma tells us a strong property of the left adjoint $\Gamma(-)^\flat$ which is not possessed by every left Quillen functor of a Quillen adjunction:

6.21. Lemma. *The left adjoint functor $\Gamma(-)^\flat$ preserves weak equivalences namely the JQ equivalences. Further, the left adjoint $\Gamma(-)^\flat$ reflects weak equivalences whose codomain is fibrant.*

Proof. Let $F : X \longrightarrow Y$ be a morphism of Γ-spaces then it can be factored as follows:



where $i(F)$ is an acyclic JQ cofibration and $p(F)$ is a JQ fibration of Γ-spaces. Let us first assume that F is a JQ equivalence of Γ-spaces. Now the 2 out of 3 property of weak equivalences in a model category tells us that $p(F)$ is a acyclic JQ fibration which is the same as a strict acyclic JQ fibration. Theorem 6.18 tells us that $\Gamma(-)^\flat$ is a left Quillen functor and therefore it preserves acyclic JQ cofibrations. By Lemma 6.8 it also preserves strict JQ equivalences. In other words both $\Gamma(i(F))^\flat$ and $\Gamma(p(F))^\flat$ are JQ equivalences of marked Γ-spaces. Therefore their composite $\Gamma(F)^\flat = \Gamma(i(F))^\flat \circ \Gamma(p(F))^\flat$ is also a JQ equivalence. Hence we have shown that the left adjoint preserves all weak equivalences.

Now let us assume that Y is a JQ fibrant marked Γ-space. Let us further assume that $\Gamma(F)^\flat : \Gamma(X)^\flat \longrightarrow \Gamma(Y)^\flat$ is a JQ equivalence of marked Γ-spaces. Applying the left adjoint functor $\Gamma(-)^\flat$ to the above factorization of F gives us the equality $\Gamma(F)^\flat = \Gamma(i(F))^\flat \circ \Gamma(p(F))^\flat$. Since $\Gamma(-)^\flat$ is a left Quillen functor therefore $\Gamma(i(F))^\flat$ is an acyclic JQ cofibration of marked Γ-spaces. By assumption $\Gamma(F)^\flat$ is a weak equivalence, therefore $\Gamma(p(F))^\flat$ is a JQ equivalence of marked Γ-spaces. By Proposition 6.11, both $\Gamma(P(F))^\flat$ and $\Gamma(Y)^\flat$ are JQ fibrant marked Γ-spaces. The right adjoint U is a right Quillen functor therefore it preserves weak equivalences between fibrant objects, hence $U(\Gamma(p(F))^\flat) = p(F)$ is a JQ equivalence of marked Γ-spaces. Since $i(F)$ is an acyclic JQ cofibration of Γ-spaces therefore $F = p(F) \circ i(F)$ is a JQ equivalence of Γ-spaces.

■

7. Comparison with Symmetric monoidal quasi-categories

In this section we compare our functor-based model for coherently commutative monoidal quasi-categories with a recent coCartesian fibration based model for similar objects which have been named *symmetric monoidal* quasi-categories. The main result of this paper is presented in this section as Corollary 7.14. The result states that a homotopy theory of symmetric monoidal quasi-categories is equivalent to a homotopy theory of coherently commutative monoidal quasi-categories. We begin by recalling the following definition from [Lur]:

7.1. Definition. *A* symmetric monoidal *quasi-category is a coCartesian fibration* $p : X \longrightarrow N(\Gamma^{op})$ *such that for each pair of objects* $k^+, l^+ \in \Gamma^{op}$ *the projection maps* $\delta^{k+l}_k : (k+l)^+ \longrightarrow k^+$ *and* $\delta^{k+l}_l : (k+l)^+ \longrightarrow l^+$ *induce morphisms of quasi-categories on the fibers*

$$X((k+l)^+) \longrightarrow X(k^+) \quad \textit{and} \quad X((k+l)^+) \longrightarrow X(l^+)$$

which determine a categorical equivalence $X((k+l)^+) \longrightarrow X(k^+) \times X(l^+)$.

The main result of this section implies that the underlying coCartesian fibrations of symmetric monoidal quasi-categories can be *rectified* (up to equivalence) into an honest functor. We recall that a coCartesian fibration $p : X \longrightarrow N(\Gamma^{op})$ can be viewed as an object $(X^\natural, p)$ in $\mathcal{S}^+/N(\Gamma^{op})$ wherein the marked edges of $X^\natural$ are exactly the p-coCartesian edges. An object is fibrant in the coCartesian model category $(\mathcal{S}^+/N(\Gamma^{op}), \mathbf{cC})$ if and only if it is isomorphic to some $(X^\natural, p)$. In this section we will construct another model category structure on $\mathcal{S}^+/N(\Gamma^{op})$, denoted $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$, in which an object is fibrant if and only if it is isomorphic to some $(X^\natural, p)$ whose underlying coCartesian fibration represents a symmetric monoidal quasi-category. A prominent theorem of this section shows that the relative nerve functor defined in [Lur09, Sec. 3.2.5] is a right Quillen functor of a Quillen equivalence between the JQ model category of marked Γ-spaces $\Gamma\mathcal{S}^+$ and the model category $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$. This theorem is instrumental in proving the main result of the paper.

For each $k^+ \in \Gamma^{op}$ we define a simplicial set $N(k^+/\Gamma^{op})$ which is the nerve of the overcategory k^+/Γ^{op}. This simplicial set is a quasi-category and is equipped with an obvious projection map

$$p : N(k^+/\Gamma^{op}) \longrightarrow N(\Gamma^{op}).$$

This map is a pseudo-fibration *i.e.* a fibration in the Joyal model category. We regard this projection map as a morphism of marked simplicial sets as follows:

$$p : N(k^+/\Gamma^{op})^\sharp \longrightarrow N(\Gamma^{op})^\sharp$$

7.2. Remark. *We observe that* $N^+_{\Gamma^k}(\Gamma^{op}) \cong N(k^+/\Gamma^{op})$. *We will denote the quasi-category* $N^+_{\Gamma^k}(\Gamma^{op})$ *by* $N(k^+/\Gamma^{op})$.

7.3. Notation. *We will denote the value of the functor $N^+_\bullet(\Gamma^{op})$ on a marked Γ-space X either as $N^+_X(\Gamma^{op})$ or as $N^+_\bullet(\Gamma^{op})(X)$.*

For each pair of objects $k^+, l^+ \in \Gamma^{op}$ we can define a map in the overcategory $\mathcal{S}^+/N(\Gamma^{op})$ by the following commutative triangle:

$$\begin{array}{ccc} N(k^+/\Gamma^{op})^\sharp \sqcup N(k^+/\Gamma^{op})^\sharp & \longrightarrow & N((k+l)^+/\Gamma^{op})^\sharp \\ & \searrow \quad \swarrow & \\ & N(\Gamma^{op})^\sharp & \end{array}$$

where the diagonal maps are the obvious projections. We denote the above morphism by $\Upsilon(k,l)$ We define a set of maps

$$\Upsilon\mathcal{S} = \{\Upsilon(k,l) : k, l \in \mathbb{N}\}$$

The category $\mathcal{S}^+/N(\Gamma^{op})$ is tensored over $\mathcal{S}$ therefore we can define the following set of maps:

$$\Delta \times \Upsilon\mathcal{S} = \{\Upsilon(k,l) \underset{\mathcal{S}}{\boxtimes} \Delta[n] : n, k, l \in \mathbb{N}\} \tag{19}$$

The following proposition is an easy consequence of the enrichment of the coCartesian model category $\mathcal{S}^+/N(\Gamma^{op})$ over $(\mathcal{S}, \mathbf{Q})$ and the main result of Appendix B.

7.4. Proposition. *A coCartesian fibration $p : X \longrightarrow N(\Gamma^{op})$ viewed as an object of $\mathcal{S}^+/N(\Gamma^{op})$, namely $(X^\natural, p)$, is a $(\Delta\times\Upsilon\mathcal{S})$-local object if and only if the following simplicial morphism of mapping spaces is a categorical equivalence of quasi-categories:*

$$\begin{aligned} \left[\Upsilon(k,l), X^\natural\right]^\flat_{\Gamma^{op}} : \left[N((k+l)^+/\Gamma^{op}), X^\natural\right]^\flat_{\Gamma^{op}} \longrightarrow \left[N(k^+/\Gamma^{op}) \sqcup N(l^+/\Gamma^{op}), X^\natural\right]^\flat_{\Gamma^{op}} \cong \\ \left[N(k^+/\Gamma^{op}), X^\natural\right]^\flat_{\Gamma^{op}} \times \left[N(l^+/\Gamma^{op}), X^\natural\right]^\flat_{\Gamma^{op}} \end{aligned}$$

for each pair $k^+, l^+ \in \Gamma^{op}$.

7.5. Notation. *The fiber of a coCartesian fibration $X \longrightarrow S$ over $s \in S_0$ will be denoted by $X(s)$.*

7.6. Remark. *Each fibrant object in $(\mathcal{S}^+/N(\Gamma^{op}), \mathbf{cC})$ is isomorphic to an object which represents a coCartesian fibration [Lur09, Prop. 3.1.4.1]. The above proposition encodes the idea that for each pair of objects $k^+, l^+ \in \Gamma^{op}$, the fiber over $(k+l)^+$, $X((k+l)^+)$ is equivalent (as quasi-categories) to the product of fibers over k^+ and l^+, $X(k^+) \times X(l^+)$. We recall from [Sha, Lemma 3.12] that there is a categorical equivalence*

$$\left[N(k^+/\Gamma^{op}), X^\natural\right]^\flat_{\Gamma^{op}} \xrightarrow{\left[id_{k^+}, X^\natural\right]^\flat_{\Gamma^{op}}} \left[\Delta[0], X^\natural\right]^\flat_{\Gamma^{op}} \cong U(X(k^+))$$

for each $k^+ \in \Gamma^{op}$.

Next we define another model category structure on the overcategory $\mathcal{S}^+/N(\Gamma^{op})$:

7.7. THEOREM. *There is a left proper, combinatorial model category structure on the category $\mathcal{S}^+/N(\Gamma^{op})$, in which a map is a*

1. *cofibration if it is a cofibration in* $(\mathcal{S}^+/N(\Gamma^{op}), \mathbf{cC})$ *namely its underlying simplicial map is a monomorphism.*

2. *weak equivalence if it is a* $(\Delta \times \Upsilon\mathcal{S})$*-local equivalence.*

3. *fibration if it has the right lifting property with respect to all maps which are simultaneously cofibrations and weak equivalences.*

PROOF. The desired model category is obtained by a left Bousfield localization of the coCartesian model category $(\mathcal{S}^+/N(\Gamma^{op}), \mathbf{cC})$. The model category structure follows from Theorem 4.9 with the set of generators for the localization being the set $\Delta \times \Upsilon\mathcal{S}$.

■

It follows from Remark 7.6 and [Lur09, Prop. 3.1.4.1] that each fibrant object in $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$ is isomorphic to a (fibrant) object which represents a symmetric monoidal quasi-category.

7.8. NOTATION. *We will refer to the above model category as the model category of* symmetric monoidal quasi-categories *and denote this model category by* $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$.

7.9. THEOREM. *The adjoint pair*

$$N_\bullet^+(\Gamma^{op}) : (\mathcal{S}^+/N(\Gamma^{op}), \otimes) \rightleftarrows (\Gamma\mathcal{S}_\otimes, \mathbf{Q}) : \mathfrak{F}_\bullet^+(\Gamma^{op})$$

is a Quillen pair.

PROOF. In light of [Joy08, Prop. E.2.14], it is sufficient to show that the left adjoint $\mathfrak{F}_\bullet^+(\Gamma^{op})$ maps cofibrations to cofibrations and the right adjoint $N_\bullet^+(\Gamma^{op})$ maps fibrations between fibrant objects to fibrations. We recall from [Lur09, Prop. 3.2.5.18(2)] that the adjoint pair $(\mathfrak{F}_\bullet^+(\Gamma^{op}), N_\bullet^+(\Gamma^{op}))$ is a Quillen pair with respect to the coCartesian model category structure $(\mathcal{S}^+/N(\Gamma^{op}), \mathbf{cC})$ and the strict JQ model category structure on $\Gamma\mathcal{S}^+$. Since the two model category structures in context are left Bousfield localizations, which preserves cofibrations, therefore the left adjoint $\mathfrak{F}_\bullet^+(\Gamma^{op})$ will (still) preserve cofibrations. The fibrations between fibrant objects in the JQ model category are strict JQ fibrations, by Lemma 4.17, therefore the right adjoint $N_\bullet^+(\Gamma^{op})$ will map such a fibration to a fibration in $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$. Further the right adjoint functor $N_\bullet^+(\Gamma^{op})$ maps fibrant objects in the JQ model category to fibrant objects in $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$. This implies that the right adjoint preserves fibrations between fibrant objects.

■

7.10. Definition. *An object Z in $\mathcal{S}^+/N(\Gamma^{op})$ is called a* local *object if the following composite map:*

$$Z \xrightarrow{\eta_Z} N^+_\bullet(\Gamma^{op})\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(Z)\right) \xrightarrow{r} N^+_\bullet(\Gamma^{op})\left(R\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(Z)\right)\right)$$

is a weak equivalence in $(\mathcal{S}^+/N(\Gamma^{op}), \mathbf{cC})$, where (R, r) is a fibrant replacement replacement functor in the JQ model category and η_Z is the unit map.

7.11. Remark. *The notion of a local object is invariant under coCartesian equivalences.*

The following lemma will be useful in writing the proof of the main theorem of this section:

7.12. Lemma. *Each fibrant object Z in $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$ is a local object.*

Proof. In light of [Lur09, Prop. 3.1.4.1] we may assume that the underlying simplicial map $U(p) : U(Z) \longrightarrow U(N(\Gamma^{op})^\sharp)$ is a coCartesian fibrations and the marked edges of Z are the p-coCartesian edges *i.e.*, $Z = U(Z)^\natural$. We begin by making the observation that for each $k^+ \in \Gamma^{op}$, the marked simplicial set $\mathfrak{F}^+_Z(\Gamma^{op})(k^+)$ is equivalent to the fiber over k^+ of Y. Since Z is fibrant therefore $\mathfrak{F}^+_Z(\Gamma^{op})$ satisfies the Segal condition. Now a fibrant replacement in the strict JQ model category will produce a coherently commutative monoidal marked quasi-category. It follows that any fibrant replacement of $\mathfrak{F}^+_Z(\Gamma^{op})$ in the JQ model category is a fibrant replacement in the strict JQ model category. This gives us the following map in $\mathcal{S}^+/N(\Gamma^{op})$:

$$Z \xrightarrow{\eta_Z} N^+_\bullet(\Gamma^{op})\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(Z)\right) \xrightarrow{r} N^+_\bullet(\Gamma^{op})\left(R\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(Z)\right)\right).$$

We recall that for each marked Γ-space X, the fiber over each $k^+ \in \Gamma^{op}$ of

$$p : N^+_X(\Gamma^{op}) \longrightarrow N(\Gamma^{op})$$

is isomorphic to $X(k^+)$. This implies that the above map is a *pointwise* equivalence *i.e.* for each $k^+ \in \Gamma^{op}$ the above map induces a categorical equivalence of (marked) simplicial sets on the fiber over k^+. Since both $Z = U(Z)^\natural$ and $N^+_\bullet(\Gamma^{op})\left(R\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(Z)\right)\right) = U(N^+_\bullet(\Gamma^{op})\left(R\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(Z)\right)\right))^\natural$, it follows from [Lur09, Prop. 3.3.1.5] that the above map is a coCartesian equivalence. Now Remark 7.11 implies that Z is a local object. ∎

Now we state and prove the main result of this section:

7.13. Theorem. *The Quillen pair of Theorem 7.9 is a Quillen equivalence.*

Proof. We will prove this theorem by verifying [Hov99, Prop. 1.3.13(b)]. We choose a fibrant replacement functor (R, r) in the JQ model category of marked Γ-spaces. We will first show that the following composite map

$$X \xrightarrow{\eta_X} N^+_\bullet(\Gamma^{op})\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(X)\right) \longrightarrow N^+_\bullet(\Gamma^{op})(R\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(X)\right))$$

is a weak equivalence in the model category $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$, for each cofibrant object X. We choose another fibrant replacement functor $(R^{\otimes}, r^{\otimes})$ in $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$. Now we have the following commutative diagram in $\Gamma\mathcal{S}^+$:

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & N^+_\bullet(\Gamma^{op})\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(X)\right) & \xrightarrow{A} & N^+_\bullet(\Gamma^{op})(R\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(X)\right)) \\ {\scriptstyle r^{\otimes}_X}\downarrow & & \downarrow & & \downarrow{\scriptstyle C} \\ R^{\otimes}(X) & \xrightarrow[\eta_{R^{\otimes}(X)}]{} & N^+_\bullet(\Gamma^{op})\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(R^{\otimes}(X))\right) & \xrightarrow[B]{} & N^+_\bullet(\Gamma^{op})(R\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(R^{\otimes}(X))\right)) \end{array}$$

where A is the map $N^+_\bullet(\Gamma^{op})\left(r_{\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(X)\right)}\right)$, B is the map

$$N^+_\bullet(\Gamma^{op})\left(r_{\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(R^{\otimes}(X))\right)}\right)$$

and the downward map C is the map

$$N^+_\bullet(\Gamma^{op})\left(R\left(\mathfrak{F}^+_\bullet(\Gamma^{op})(r^{\otimes}(X))\right)\right).$$

Since the object $R^{\otimes}(X)$ is a fibrant object in $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$, it follows from Lemma 7.12 that the bottom row of the above diagram is a coCartesian equivalence. Since $r^{\otimes}_X$ is an acyclic cofibration therefore the left Quillen functor $\mathfrak{F}^+_\bullet(\Gamma^{op})$ preserves it. Thus, $R(\mathfrak{F}^+_\bullet(\Gamma^{op})(r^{\otimes}(X)))$ is a weak equivalence between fibrant objects which the right Quillen functor $N^+_\bullet(\Gamma^{op})$ will preserve. Thus, the rightmost vertical arrow is also a weak equivalence in $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$. Now the 2 out of 3 property of weak equivalences implies that the top row of the above diagram is a weak equivalence in $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$.

Next we choose a cofibrant replacement functor (Q, q) in $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$. We will show that the following map is a weak equivalence for each coherently commutative monoidal marked quasi-category Y:

$$\mathfrak{F}^+_\bullet(\Gamma^{op})(Q\left(N^+_\bullet(\Gamma^{op})(Y)\right)) \xrightarrow{G} \mathfrak{F}^+_{\Gamma^{op}}(\Gamma^{op})(N^+_\bullet(\Gamma^{op})(Y)) \xrightarrow{\epsilon_Y} Y$$

where G is the map $\mathfrak{F}^+_\bullet(\Gamma^{op})(q_{N^+_\bullet(\Gamma^{op})(Y)})$. The Quillen equivalence [Lur09, Prop. 3.2.5.18(2)] implies that for each fibrant object Y, the counit map ϵ_Y is a coCartesian equivalence and hence a weak equivalence in $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$. Since $q_{N^+_\bullet(\Gamma^{op})(Y)}$ is a weak equivalence between fibrant objects in $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$ therefore it is a coCartesian equivalence which will be preserved by the left Quillen functor $\mathfrak{F}^+_\bullet(\Gamma^{op})$. Thus, we have shown that the above composite map is a weak equivalence in $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$.

■

Now we state the main result of this paper:

7.14. Corollary. *The model category $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$ is Quillen equivalent to the model category $(\Gamma\mathcal{S}_{\otimes}, \mathbf{Q})$ by the following zig-zag of Quillen equivalences:*

$$(\mathcal{S}^+/N(\Gamma^{op}), \otimes) \underset{\mathfrak{F}^+_\bullet(\Gamma^{op})}{\overset{N^+_\bullet(\Gamma^{op})}{\leftrightarrows}} (\Gamma\mathcal{S}^+_{\otimes}, \mathbf{Q}) \underset{\Gamma(-)^\flat}{\overset{U}{\rightleftarrows}} (\Gamma\mathcal{S}_{\otimes}, \mathbf{Q})$$

Finally, we give another characterization for symmetric monoidal quasi-categories. We recall a right Quillen functor $\mathfrak{R}^+_{\Gamma^{op}} : \mathcal{S}^+/N(\Gamma^{op}) \longrightarrow \Gamma\mathcal{S}^+$ defined in the paper [Sha]:

$$\mathfrak{R}^+_{\Gamma^{op}}(X)(k^+) = [N(k^+/\Gamma^{op}), X]^+_{\Gamma^{op}}.$$

The following corollary provides another characterization of fibrant objects. It is an easy consequence of the above theorem:

7.15. Corollary. *A coCartesian fibration $p : X \longrightarrow N(\Gamma^{op})$ viewed as an object of $(\mathcal{S}^+/N(\Gamma^{op}), \otimes)$ is a fibrant object if and only if the marked Γ-space $\mathfrak{R}^+_{\Gamma^{op}}(X)$ is a coherently commutative monoidal marked quasi-category.*

A. Quillen Bifunctors

The objective of this section is to recall the notion of *Quillen Bifunctors*. In order to do so, we begin with the definition of a *two-variable adjunction*:

A.1. Definition. *Suppose $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are categories. An* adjunction of two variables *from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ is a quintuple $(\otimes, \mathbf{hom}_{\mathcal{C}}, \mathcal{M}ap_{\mathcal{C}}, \phi, \psi)$, where*

$$\otimes : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}, \quad \mathbf{hom}_{\mathcal{C}} : \mathcal{D}^{op} \times \mathcal{E} \longrightarrow \mathcal{C}, \text{ and } \quad \mathcal{M}ap_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{E} \longrightarrow \mathcal{D}$$

are functors and ϕ, ψ are the following natural transformations

$$\mathcal{C}(C, \mathbf{hom}_{\mathcal{C}}(D, E)) \underset{\cong}{\xrightarrow{\phi^{-1}}} \mathcal{E}(\mathcal{C} \otimes D, E) \underset{\cong}{\xrightarrow{\psi}} \mathcal{D}(D, \mathcal{M}ap_{\mathcal{C}}(C, E)).$$

The following definition is based on Quillen's $SM7$ axiom, see [Qui67].

A.2. Definition. *Given model categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, an adjunction of two variables, $(\otimes, \mathbf{hom}_{\mathcal{C}}, \mathcal{M}ap_{\mathcal{C}}, \phi, \psi) : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$, is called a* Quillen adjunction of two variables*, if, given a cofibration $f : U \longrightarrow V$ in $\mathcal{C}$ and a cofibration $g : W \longrightarrow X$ in $\mathcal{D}$, the induced map*

$$f \Box g : (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \longrightarrow V \otimes X$$

is a cofibration in $\mathcal{E}$ that is trivial if either f or g is. We will refer to the left adjoint of a Quillen adjunction of two variables as a Quillen bifunctor.

The following lemma provides three equivalent characterizations of the notion of a Quillen bifunctor. These will be useful in this paper in establishing enriched model category structures.

A.3. LEMMA. *[Hov99, Lemma 4.2.2] Given model categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, an adjunction of two variables, $(\otimes, \mathbf{hom}_{\mathcal{C}}, \mathcal{M}ap_{\mathcal{C}}, \phi, \psi) : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$. Then the following conditions are equivalent:*

(1) $\otimes : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$ is a Quillen bifunctor.

(2) Given a cofibration $g : W \longrightarrow X$ in $\mathcal{D}$ and a fibration $p : Y \longrightarrow Z$ in $\mathcal{E}$, the induced map

$$\mathbf{hom}^{\square}_{\mathcal{C}}(g,p) : \mathbf{hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathbf{hom}_{\mathcal{C}}(X,Z) \underset{\mathbf{hom}_{\mathcal{C}}(W,Z)}{\times} \mathbf{hom}_{\mathcal{C}}(W,Y)$$

is a fibration in $\mathcal{C}$ that is trivial if either g or p is a weak equivalence in their respective model categories.

(3) Given a cofibration $f : U \longrightarrow V$ in $\mathcal{C}$ and a fibration $p : Y \longrightarrow Z$ in $\mathcal{E}$, the induced map

$$\mathcal{M}ap^{\square}_{\mathcal{C}}(f,p) : \mathcal{M}ap_{\mathcal{C}}(V,Y) \longrightarrow \mathcal{M}ap_{\mathcal{C}}(V,Z) \underset{\mathcal{M}ap_{\mathcal{C}}(W,Z)}{\times} \mathcal{M}ap_{\mathcal{C}}(W,Y)$$

is a fibration in $\mathcal{C}$ that is trivial if either f or p is a weak equivalence in their respective model categories.

B. On local objects in a model category enriched over quasi-categories

A very detailed sketch of this appendix was provided to the author by André Joyal. This appendix contains some key results which have made this research possible.

B.1. INTRODUCTION. A model category C is enriched over quasi-categories if the category C is simplicial, tensored and cotensored, and the hom functor

$$\mathcal{M}ap_C(-,-) : C^{op} \times C \longrightarrow \mathcal{S}.$$

is a Quillen functor of two variables, where $\mathcal{S} = (\mathcal{S}, \mathbf{Q})$ is the model structure for quasi-categories. The purpose of this appendix is to introduce the notion of local object with respect to a map in a model category enriched over quasi-categories.

B.2. PRELIMINARIES. Recall that a Quillen model structure on a category C is determined by its class of cofibrations together with its class of fibrant objects. For example, the category of simplicial sets $\mathcal{S} = [\Delta^{op}, Set]$ admits two model structures in which the cofibrations are the monomorphisms: the fibrant objects are the Kan complexes in one, and they are the quasi-categories in the other. We call the former the model structure for Kan complexes and the latter the model structure for quasi-categories. We shall denote them respectively by $(\mathcal{S}, \mathbf{Kan})$ and $(\mathcal{S}, \mathbf{Q})$.

Recall that a simplicial category is a category enriched over simplicial sets. There is a notion of simplicial functor between simplicial categories, and a notion of strong natural transformation between simplicial functors. If $C = (C, \mathcal{M}ap_C(-,-))$ is a simplicial category, then so is the category $\mathbf{SFunc}(C;\mathcal{S})$ of simplicial functors $C \longrightarrow \mathcal{S}$ [GJ99, Thm. 4.4]. A simplicial functor $F : C \longrightarrow \mathcal{S}$ isomorphic to a simplicial functor $\mathcal{M}ap_C(A,-) : C \longrightarrow \mathcal{S}$ is said to be *representable*. Recall the Yoneda lemma for simplicial functors [GJ99, IX, Lem. 1.2]: if $F : C \longrightarrow \mathcal{S}$ is a simplicial functor and $A \in C$, then the map $y : Nat(\mathcal{M}ap_C(A,-), F) \longrightarrow F(A)_0$ defined by putting $y(\alpha) = \alpha(A)(1_A)$ for a natural transformation of simplicial functors [GJ99, Pg. 432.] $\alpha : \mathcal{M}ap_C(A,-) \longrightarrow F$, is bijective. The simplicial functor F is said to be *represented by a pair* (A, a), with $a \in F(A)_0$, if the unique natural transformation of simplicial functors $\alpha : \mathcal{M}ap_C(A,-) \longrightarrow F$ such that $\alpha(A)(1_A) = a$ is invertible. We say that a simplicial category $C = (C, \mathcal{M}ap_C(-,-))$ is *tensored by* Δ if the simplicial functor

$$\mathcal{M}ap_C(A,-)^{\Delta[n]} : C \longrightarrow \mathcal{S}$$

is representable (by an object denoted $\Delta[n] \underset{\mathcal{S}}{\boxtimes} A$) for every object $A \in C$ and every $n \geq 0$. If C has finite colimits and is tensored by Δ, then it is tensored by finite simplicial sets: the simplicial functor is representable (by an object $K \underset{\mathcal{S}}{\boxtimes} A$) for every object $A \in C$ and every finite simplicial set K. Dually, we say that a simplicial category C is *cotensored by* Δ if the simplicial functor

$$\mathcal{M}ap_{\mathcal{S}}(-,A)^{\Delta[n]} : C^{op} \longrightarrow \mathcal{S}$$

is representable (by an object denoted $X^{\Delta[n]}$) for every object $X \in C$ and every $n \geq 0$. If C has finite limits and is cotensored by Δ, then it is cotensored by finite simplicial sets: the simplicial functor

$$\mathcal{M}ap_{\mathcal{S}}(-,A)^K : E^{op} \longrightarrow \mathcal{S}$$

is representable by an object X^K for every object $X \in C$ and every finite simplicial set K. Recall that a model category C is said to be simplicial if the category C is simplicial, tensored and cotensored by Δ and the functor $\mathcal{M}ap_C(-,-) : C^{op} \times C \longrightarrow \mathcal{S}$ is a Quillen functor of two variables, where $\mathcal{S} = (\mathcal{S}, \mathbf{Kan})$. The last condition implies that if $A \in C$ is cofibrant and $X \in C$ is fibrant, then the simplicial set $\mathcal{M}ap_C(A,X)$ is a Kan complex. For this reason, we shall say that a simplicial model category is enriched over Kan complexes.

B.3. Definition. *We shall say that a model category C is enriched over quasi-categories if the category C is simplicial, tensored and cotensored over Δ and the functor*

$$\mathcal{M}ap_C(-,-) : C^{op} \times C \longrightarrow \mathcal{S}$$

is a Quillen functor of two variables, where $\mathcal{S} = (\mathcal{S}, \mathbf{Q})$.

The last condition of Definition B.3 implies that if $A \in C$ is cofibrant and $X \in C$ is fibrant, then the simplicial set $\mathcal{M}ap_C(A,X)$ is a quasi-category. If C is a category with finite limits than so is the category $[\Delta^{op}, C]$ of simplicial objects in C. The evaluation functor $ev_0 : [\Delta^{op}, C] \longrightarrow C$ defined by putting $ev_0(X) = X_0$ has a left adjoint sk^0 and a right adjoint $cosk^0$. If $A \in C$, then $sk^0(A)_n = A$ and $cosk^0(A)_n = A^{[n]} = A^{n+1}$ for every $n \geq 0$ (the simplicial object $sk^0(A)$ is the constant functor $cA : \Delta^{op} \longrightarrow C$ with values A). The category $[\Delta^{op}, C]$ is simplicial. If $X, Y \in [\Delta^{op}, C]$ then we have

$$(\mathcal{M}ap_{[\Delta^{op},C]}(X, Y))_n = Nat(X \circ p_n, Y \circ p_n)$$

for every $n \geq 0$, where p_n is the forgetful functor $\Delta/[n] \longrightarrow \Delta$. If $A \in C$ and $cA := sk^0(A)$, then

$$(\mathcal{M}ap_{[\Delta^{op},C]}(cA, X))_n = C(A, X_n)$$

for every $n \geq 0$. The simplicial category $[\Delta^{op}, C]$ is tensored and cotensored by Δ. By construction, if $X \in [\Delta^{op}, C]$ and K is a finite simplicial set, then

$$(K \underset{\mathcal{S}}{\boxtimes} X)_n = K_n \underset{\mathcal{S}}{\boxtimes} X_n \quad and \quad (X^K)_n = \int_{[k] \longrightarrow [n]} X_k^{K_k}$$

The object $M_n(X) := (X^{\partial\Delta[n]})_n$ is called the nth matching object of X. If $S(n)$ denotes the poset of non-empty proper subsets of $[n]$ then we have

$$M_n(X) = \varinjlim_{S(n)} X \circ s(n)$$

where $s(n) : S(n) \longrightarrow \Delta$ is the canonical functor. From the inclusion $\partial\Delta[n] \subset \Delta[n]$ we obtain a map $X^{\Delta[n]} \longrightarrow X^{\partial\Delta[n]}$ hence also a map $X_n \longrightarrow M_n(X)$. We observe that the nth matching object of X is the value of the right adjoint of the *restriction functor*

$$i_*^n : [\Delta^{op}, C] \longrightarrow [\Delta^{op}_{\leq n}, C].$$

Thus another way to describe it is the following:

$$M_n(X) = \varinjlim_{[k] \longrightarrow [n]} X([k]),$$

where the limit is taken over all monomorphisms in Δ having codomain $[n]$. Now we obtain the first description given above by identifying a monomorphism $[k] \longrightarrow [n]$ with a proper subset of $[n]$.

If C is a model category, then a map $f : X \longrightarrow Y$ in $[\Delta^{op}, C]$ is called a *Reedy fibration* if the map $X_n \longrightarrow Y_n \underset{M_n(Y)}{\times} M_n(X)$ obtained from the square

$$\begin{array}{ccc} X_n & \longrightarrow & M_n(X) \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle M_n(f)} \\ Y_n & \longrightarrow & Mn(Y) \end{array}$$

is a fibration for every $n \geq 0$. There is then a model structure on the category $[\Delta^{op}, C]$ called the *Reedy model structure* whose fibrations are the Reedy fibrations and whose weak equivalences are the level-wise weak equivalences. A simplicial object $X : \Delta^{op} \longrightarrow C$ is Reedy fibrant if and only if the canonical map $X_n \longrightarrow M_n(X)$ is a fibration for every $n \geq 0$. The Reedy model structure is simplicial. If X is Reedy fibrant and $A \in C$ then the simplicial set $C(A, X) := \mathcal{M}ap_{[\Delta^{op},C]}(cA, X)$ is a Kan complex.

B.4. Definition. *Let C be a model category. Then a simplicial object $Z : \Delta^{op} \longrightarrow C$ is called a frame (see [Hov99]) if the following two conditions are satisfied:*

1. *Z is Reedy fibrant;*
2. *$Z(f)$ is a weak equivalence for every map $f \in \Delta$.*

The frame Z is cofibrant if the canonical map $sk^0 Z_0 \longrightarrow Z$ is a cofibration in the Reedy model structure. A *coresolution* of an object $X \in C$ is a frame $Fr(X) : \Delta^{op} \longrightarrow C$ equipped with a weak equivalence $X \longrightarrow Fr(X)_0$. Every fibrant object $X \in C$ has a (cofibrant) coresolution $Fr(X) : \Delta^{op} \longrightarrow C$ with $Fr(X)_0 = X$. Let C be a model category. If $A, X \in C$, then the homotopy mapping space $\mathcal{M}ap^h_C(A, X)$ is defined to be the simplicial set

$$\mathcal{M}ap^h_C(A, X) = C(A^c, Fr(X)) \tag{20}$$

where $A^c \longrightarrow A$ is a cofibrant replacement of A and $Fr(X)$ is a coresolution of X. The simplicial set $C(A^c, Fr(X))$ is a Kan complex and it is homotopy unique. If C is enriched over Kan complexes, A is cofibrant and X is fibrant, then the simplicial set $\mathcal{M}ap^h_C(A, X)$ is homotopy equivalent to the simplicial set $\mathcal{M}ap_C(A, X)$ (see [Hir02]).

B.5. Function spaces for quasi-categories. If C is a category, we shall denote by $J(C)$ the sub-category of invertible arrows in C. The sub-category $J(C)$ is the largest sub-groupoid of C. More generally, if X is a quasi-category, we shall denote by $J(X)$ the largest sub-Kan complex of X. By construction, we have a pullback square

$$\begin{array}{ccc} J(X) & \longrightarrow & X \\ \downarrow & & \downarrow h \\ J(\tau_1(X)) & \longrightarrow & \tau_1(X) \end{array}$$

where $\tau_1(X)$ is the fundamental category of X and h is the canonical map. The function space X^A is a quasi-category for any simplicial set X. We shall denote by $X^{(A)}$ the full sub-simplicial set of X^A whose vertices are the maps $A \longrightarrow X$ that factor through the inclusion $J(X) \subseteq X$. The simplicial set $X^{(\Delta[1])}$ is a path-space for X.

B.6. Lemma. *If X is a quasi-category, then the simplicial object $P(X) \in [\Delta^{op}, sSet]$ defined by putting $P(X)_n = X^{(\Delta[n])}$ for every $n \geq 0$ is a cofibrant coresolution of X.*

B.7. Proposition. *If X is a quasi-category and A is a simplicial set, then*

$$\mathcal{M}ap^h_{\mathcal{S}}(A, X) \simeq J(X^A).$$

Proof. By Lemma B.6 and the fact that each simplicial set is cofibrant, we have

$$\mathcal{M}ap^h_{\mathcal{S}}(A, X)_n = \mathcal{S}(A, P(X)_n) = \mathcal{S}(A, X^{(\Delta[n])})$$

But a map $f : A \longrightarrow X^{\Delta[n]}$ factors through the inclusion $X^{(\Delta[n])} \subseteq X^{\Delta[n]}$ if and only if the transposed map $f^t : \Delta[n] \longrightarrow X^A$ factors through the inclusion $J(X^A) \subseteq X^A$, by [Joy08, Prop. 5.2]. Thus, $\mathcal{S}(A, X^{(\Delta[n])}) = \mathcal{S}(\Delta[n], J(X^A)) = J(X^A)_n$ and this shows that $\mathcal{M}ap^h_{\mathcal{S}}(A, X) \simeq J(X^A)$. ∎

B.8. PROPOSITION. *Let C be a model category enriched over quasi-categories. If $A \in C$ is cofibrant and $X \in C$ is fibrant, then the function space $\mathcal{M}ap^h_C(A, X)$ is equivalent to the Kan complex $J(\mathcal{M}ap_C(A, X))$.*

PROOF. The functor $\mathcal{M}ap_C(A, -) : C \longrightarrow \mathcal{S}$ is a right Quillen functor with values in the model category $(\mathcal{S}, \mathbf{Q})$, since A is cofibrant. It thus takes a coresolution $Fr(X)$ of $X \in C$ to a coresolution $\mathcal{M}ap_C(A, Fr(X))$ of the quasi-category $\mathcal{M}ap_C(A, X)$. We have $\mathcal{M}ap^h_{\mathcal{S}}(1, \mathcal{M}ap_C(A, X)) \simeq \mathcal{S}(1, P(\mathcal{M}ap_C(A, X)))$, since the (terminal) simplicial set 1 is cofibrant. By Lemma B.6, the quasi-category $\mathcal{M}ap_C(A, X)$ has a cofibrant coresolution $P(\mathcal{M}ap_C(A, X))$. We have $\mathcal{M}ap^h_{\mathcal{S}}(1, \mathcal{M}ap_C(A, X)) \simeq \mathcal{S}(1, \mathcal{M}ap_C(A, Fr(X)))$, since the simplicial set 1 is cofibrant. There exists a level-wise weak categorical equivalence $\phi : P(\mathcal{M}ap_C(A, X)) \longrightarrow \mathcal{M}ap_C(A, Fr(X))$ such that the map $\phi(0)$ is the identity, since the coresolution $P(\mathcal{M}ap_C(A, X))$ is cofibrant. Moreover, the map

$$\mathcal{S}(1, \phi) : \mathcal{S}(1, P(\mathcal{M}ap_C(A, X))) \longrightarrow \mathcal{S}(1, \mathcal{M}ap_C(A, Fr(X)))$$

is a weak homotopy equivalence. But we have

$$\mathcal{S}(1, P(\mathcal{M}ap_C(A, X))) = J(\mathcal{M}ap_C(A, X))$$

by Lemma B.6, Proposition B.7 and (20). Moreover,

$$\mathcal{S}(1, \mathcal{M}ap_C(A, Fr(X))) = C(A, Fr(X)),$$

since

$$\mathcal{S}(1, \mathcal{M}ap_C(A, Fr(X)))_n = \mathcal{S}(1, \mathcal{M}ap_C(A, Fr(X))_n) = \mathcal{S}(1, \mathcal{M}ap_C(A, Fr(X)_n)) = C(A, Fr(X)_n)$$

for every $n \geq 0$. ∎

B.9. LOCAL OBJECTS. Let Σ be a set of maps in a model category C. An object $X \in C$ is said to be Σ-local if the map

$$\mathcal{M}ap^h_C(u, X) : \mathcal{M}ap^h_C(A', X) \longrightarrow \mathcal{M}ap^h_C(A, X)$$

is a homotopy equivalence for every map $u : A \longrightarrow A'$ in Σ. Notice that if an object X is weakly equivalent to a Σ-local object, then X is Σ-local. If the model category C is simplicial (= enriched over Kan complexes) and Σ is a set of maps between cofibrant objects, then a fibrant object $X \in C$ is Σ-local iff the map $\mathcal{M}ap_C(u, X) : \mathcal{M}ap_C(A', X) \longrightarrow \mathcal{M}ap_C(A, X)$ is a homotopy equivalence for every map $u : A \longrightarrow A'$ in Σ. We refer the interested reader to [Hir02, Ch. 17] for a proof of this fact and more details on homotopy function complexes.

B.10. Lemma. *Let C be a model category enriched over quasi-categories. If $u : A \longrightarrow B$ is a map between cofibrant objects in C, then the following conditions on a fibrant object $X \in C$ are equivalent*

1. *the map $\mathcal{M}ap_C(u, X) : \mathcal{M}ap_C(B, X) \longrightarrow \mathcal{M}ap_C(A, X)$ is a categorical equivalence;*
2. *the object X is local with respect to the map $\Delta[n] \underset{\mathcal{S}}{\boxtimes} u : \Delta[n] \underset{\mathcal{S}}{\boxtimes} A \longrightarrow \Delta[n] \underset{\mathcal{S}}{\boxtimes} B$ for every $n \geq 0$.*

Proof. ($1 \Rightarrow 2$) The map $\mathcal{M}ap_C(u, X)^{\Delta[n]} : \mathcal{M}ap_C(B, X)^{\Delta[n]} \longrightarrow \mathcal{M}ap_C(A, X)^{\Delta[n]}$ is a categorical equivalence for every $n \geq 0$, since the map $\mathcal{M}ap_C(u, X)$ is a categorical equivalence by the hypothesis. Hence the map $\mathcal{M}ap_C(\Delta[n] \underset{\mathcal{S}}{\boxtimes} u, X)$ is a categorical equivalence, since

$$\mathcal{M}ap_C(\Delta[n] \underset{\mathcal{S}}{\boxtimes} u, X) \cong \mathcal{M}ap_C(u, X)^{\Delta[n]}.$$

It follows that the map $J(\mathcal{M}ap_C(\Delta[n] \underset{\mathcal{S}}{\boxtimes} u, X))$ is a homotopy equivalence, since the functor $J : QCat \longrightarrow Kan$ takes a categorical equivalences to homotopy equivalences by [Joy08, Prop. 6.27] and [Joy08, Prop. 6.26]. But we have

$$\mathcal{M}ap^h_E(\Delta[n] \underset{\mathcal{S}}{\boxtimes} u, X) = J(\mathcal{M}ap_C(\Delta[n] \underset{\mathcal{S}}{\boxtimes} u, X))$$

by Proposition B.8, since $\Delta[n] \underset{\mathcal{S}}{\boxtimes} u$ is a map between cofibrant objects. Hence the map $\mathcal{M}ap^h_E(\Delta[n] \underset{\mathcal{S}}{\boxtimes} u, X)$ is a homotopy equivalence for every $n \geq 0$. This shows that the object X is local with respect to the map $\Delta[n] \underset{\mathcal{S}}{\boxtimes} u$ for every $n \geq 0$.

($1 \Leftarrow 2$) By Proposition B.8, we have

$$\mathcal{M}ap^h_C(\Delta[n] \underset{\mathcal{S}}{\boxtimes} u, X) = J(\mathcal{M}ap_C(\Delta[n] \underset{\mathcal{S}}{\boxtimes} u, X))$$

for every $n \geq 0$, since $\Delta[n] \underset{\mathcal{S}}{\boxtimes} u$ is a map between cofibrant objects. Hence the map $J(\mathcal{M}ap_C(\Delta[n] \underset{\mathcal{S}}{\boxtimes} u, X))$ is a homotopy equivalence for every $n \geq 0$. But we have

$$\mathcal{M}ap_C(\Delta[n] \underset{\mathcal{S}}{\boxtimes} u, X) = \mathcal{M}ap_C(u, X)^{\Delta[n]}.$$

Hence the map $J(\mathcal{M}ap_C(u, X)^{\Delta[n]})$ is a homotopy equivalence for every $n \geq 0$. By Theorem 4.11 and Proposition 4.10 of [JT08] a map between quasi-categories $f : U \longrightarrow V$ is a categorical equivalence if and only if the map $J(f^{\Delta[n]}) : J(U^{\Delta[n]}) \longrightarrow J(V^{\Delta[n]})$ is a homotopy equivalence for every $n \geq 0$. This shows that the map $\mathcal{M}ap_C(u, X)$ is a categorical equivalence. ∎

Finally, we want to discuss left Bousfield localization of a monoidal model category which is also enriched over quasi-categories.

B.11. Definition. *A closed monoidal category $\mathcal{M}$ which is also enriched over simplicial-sets is called* compatible with simplicial enrichment *if we have the following natural isomorphism for each pair of objects $A, B \in \mathcal{M}$ and each simplicial-set K:*

$$(A \underset{\mathcal{M}}{\otimes} B) \underset{\mathcal{S}}{\boxtimes} K \cong (A \underset{\mathcal{S}}{\boxtimes} K) \underset{\mathcal{M}}{\otimes} B$$

B.12. Remark. In a closed monoidal category $\mathcal{M}$ which is compatible with simplicial enrichment, the following natural isomorphism follows from [GJ99, Lem. 2.9]:

$$\mathcal{M}ap_{\mathcal{M}}(A \underset{\mathcal{M}}{\otimes} B, D) \cong \mathcal{M}ap_{\mathcal{M}}(B, \underline{\mathcal{M}ap}_{\mathcal{M}}(A, D)),$$

where $\underline{\mathcal{M}ap}_{\mathcal{M}}(-,-)$ is the internal hom for $\mathcal{M}$.

B.13. Lemma. *Let $\mathcal{M}$ be a monoidal model category enriched over quasi-categories whose underlying closed monoidal category is compatible with the simplicial enrichment. Let S be a set of maps in $\mathcal{M}$ such that each morphism in S is between cofibrant objects of $\mathcal{M}$ and let $\mathcal{L}_S(\mathcal{M})$ be a left Bousfield localization of $\mathcal{M}$ with respect to maps in S. Then the following map is a weak equivalence in $\mathcal{L}_S(\mathcal{M})$:*

$$A^c \underset{\mathcal{M}}{\otimes} f : A^c \underset{\mathcal{M}}{\otimes} K \longrightarrow A^c \underset{\mathcal{M}}{\otimes} L$$

for each morphism $f : K \longrightarrow L$ in S and $A \in Ob(\mathcal{M})$, where A^c is a cofibrant replacement of A.

Proof. In light of Proposition B.8 and the observation that $A^c \underset{\mathcal{M}}{\otimes} f$ is a map between cofibrant objects, it is sufficient to show that for each fibrant object X in $\mathcal{L}_S(\mathcal{M})$, the following simplicial map is a homotopy equivalence of Kan complexes:

$$J(\mathcal{M}ap_{\mathcal{M}}(A^c \underset{\mathcal{M}}{\otimes} f, X)) : J(\mathcal{M}ap_{\mathcal{M}}(A^c \underset{\mathcal{M}}{\otimes} L, X)) \longrightarrow J(\mathcal{M}ap_{\mathcal{M}}(A^c \underset{\mathcal{M}}{\otimes} K, X)).$$

By the remark above, this map is a homotopy equivalence if and only if the following simplicial map is a homotopy equivalence of Kan complexes:

$$J(\mathcal{M}ap_{\mathcal{M}}(f, \underline{\mathcal{M}ap}_{\mathcal{M}}(A^c, X))) : J(\mathcal{M}ap_{\mathcal{M}}(L, \underline{\mathcal{M}ap}_{\mathcal{M}}(A^c, X))) \longrightarrow \\ J(\mathcal{M}ap_{\mathcal{M}}(K, \underline{\mathcal{M}ap}_{\mathcal{M}}(A^c, X))).$$

Since f is an S-local equivalence and the internal function object $\underline{\mathcal{M}ap}_{\mathcal{M}}(A^c, X)$ is fibrant in $\mathcal{L}_S(\mathcal{M})$, it follows from Proposition B.8 that $J(\mathcal{M}ap_{\mathcal{M}}(f, \underline{\mathcal{M}ap}_{\mathcal{M}}(A^c, X)))$ is a homotopy equivalence of Kan complexes. ∎

The above lemma simplifies the charaterization of monoidal left Bousfield localization, presented in [Whi14], for a monoidal model category whose underlying closed monoidal category is compatible with simplicial enrichment:

B.14. COROLLARY. *Let $\mathcal{M}$ be a cofibrantly generated monoidal model category which is enriched over quasi-categories, whose underlying closed monoidal category is compatible with simplicial enrichment and cofibrant objects in $\mathcal{M}$ be flat. Let S be a set of maps in $\mathcal{M}$. Then the left Bousfield localization $\mathcal{L}_S(\mathcal{M})$ is monoidal.*

PROOF. The set of generating maps S can be replaced by another set of generating maps S' whose maps are between cofibrant objects. This can be done by using a cofibrant replacement functor. Now the corollary follows from the above lemma and [Whi14, Thm. 4.6] ∎

C. The model category of normalized coherently commutative monoidal quasi-categories

A normalized Γ-space is a functor $X : \Gamma^{op} \longrightarrow \mathcal{S}_\bullet$ such that $X(0^+) = 1$, where 1 is the terminal simplicial set (which we also denote by $*$). The category of all normalized Γ-spaces $\Gamma\mathcal{S}_\bullet$ is the category whose objects are normalized Γ-spaces. This category is defined by the following equalizer diagram in **Cat**:

$$\begin{array}{ccccc} \Gamma\mathcal{S}_\bullet \longrightarrow & [\Gamma^{op};\mathcal{S}_\bullet] & \xrightarrow{[0;\mathcal{S}_\bullet]} & & [1;\mathcal{S}_\bullet] \\ & & \searrow & & \nearrow_{0} \\ & & & 1 & \end{array}$$

where $[0;\mathcal{S}_\bullet]$ is the functor which precomposes a functor in $[\Gamma^{op};\mathcal{S}_\bullet]$ with the unique (pointed) functor $1 \longrightarrow \Gamma^{op}$ whose image is $0^+ \in \Gamma^{op}$ and the upward diagonal functor 0 maps the terminal category 1 to the identity functor on the terminal simplicial sets. It follows from [Hov99, Prop. 1.1.8] that the category of (pointed) simplicial sets $\mathcal{S}_\bullet$ inherits a model category structure from the Joyal model category $(\mathcal{S}, \mathbf{Q})$. We denote this model category by $(\mathcal{S}_\bullet, \mathbf{Q})$. In this appendix we reproduce the theory developed in sections 3 and 4 of this paper, in the setting of normalized Γ-spaces. As mentioned earler in this paper, Γ-spaces have been tradionally studied as normalized objects and we want to carry forward this tradition. Here we will construct two model category structures on $\Gamma\mathcal{S}_\bullet$, namely, the *strict JQ model category structure of normalized* Γ*-spaces* and the *JQ model category structure of normalized* Γ*-spaces*. These two model categories are normalized versions of the strict JQ and JQ model categories of (unnormalized) Γ-spaces constructed earlier in this paper. We recall that the Kan model category of (pointed) simplicial sets $(\mathcal{S}_\bullet, \mathbf{Kan})$ can be obtained as left Bousfield localization of $(\mathcal{S}_\bullet, \mathbf{Q})$. We conjecture that, along the same lines, the strict Q model category structure and the stable Q model category structure constructed in [Sch99] can be obtained as a left Bousfield localization of the strict JQ and the JQ model category structures constructed in this appendix respectively. Thus, one prominent reason for developing the theory in this appendix is to establish a strong connection with the traditional theory of Γ-spaces.

Moreover, we want to show that the theory of coherently commutative monoidal quasi-categories is a generalization of that of simplicial abelian monoids. In a future paper, we would like to show that a (model) category of simplicial abelian monoids can be embedded as a full sub-category in the (JQ model category of) normalized Γ-spaces.

Our first goal in this section is to construct a combinatorial model category structure on the category $\Gamma\mathcal{S}_\bullet$ which is a version of the strict JQ model category structure, defined in Section 3, for normalized Γ-spaces.

C.1. Definition. *A morphism $F : X \longrightarrow Y$ of Γ-spaces is called*

1. *a* strict JQ equivalence *of normalized Γ-spaces if it is degreewise weak equivalence in the Joyal model category structure on $\mathcal{S}_\bullet$* i.e. *$F(n^+) : X(n^+) \longrightarrow Y(n^+)$ is a weak categorical equivalence of (pointed) simplicial sets.*

2. *a* strict JQ fibration *of normalized Γ-spaces if it is degreewise a fibration in the Joyal model category structure on $\mathcal{S}_\bullet$* i.e. *$F(n^+) : X(n^+) \longrightarrow Y(n^+)$ is an pseudo-fibration of (pointed) simplicial sets.*

3. *a JQ* cofibration *of normalized Γ-spaces if it has the left lifting property with respect to all morphisms which are both strict JQ equivalence and strict JQ fibrations of normalized Γ-spaces.*

In order to describe the generating cofibrations and generating acyclic cofibrations in the proposed combinatorial model category structure on $\Gamma\mathcal{S}_\bullet$, we want to describe an enrichment of $\Gamma\mathcal{S}_\bullet$ over $\mathcal{S}_\bullet$:

We recall that the *smash product* of two (pointed) simplicial sets (X, x) and (Y, y), where the simplicial maps $x : 1 \longrightarrow X$ and $y : 1 \longrightarrow Y$ specify the respective basepoints, is defined by the following pushout square:

$$\begin{array}{ccc} X \vee Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X \wedge Y \end{array} \tag{21}$$

where the top horizontal arrow is the canonical map between the coproduct and product of the two (pointed) simplicial sets. To any pair of objects $(X, K) \in Ob(\Gamma\mathcal{S}_\bullet) \times Ob(\mathcal{S}_\bullet)$ we can assign a Γ-space $X \underset{\mathcal{S}_\bullet}{\wedge} K$ which is defined in degree n as follows:

$$(X \underset{\mathcal{S}_\bullet}{\wedge} K)(n^+) := X(n^+) \wedge K, \tag{22}$$

where the pointed category on the right is the smash product of (pointed) simplicial sets, see (21). This assignment is functorial in both variables and therefore we have a bifunctor

$$- \underset{\mathcal{S}_\bullet}{\wedge} - : \Gamma\mathcal{S}_\bullet \times \mathcal{S}_\bullet \longrightarrow \Gamma\mathcal{S}_\bullet.$$

Next,we define a couple of function objects for the category Γ-space. The first function object enriches the category $\Gamma\mathcal{S}_\bullet$ over $\mathcal{S}_\bullet$ *i.e.* there is a bifunctor

$$\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(-,-) : \Gamma\mathcal{S}_\bullet^{op} \times \Gamma\mathcal{S}_\bullet \longrightarrow \mathcal{S}_\bullet$$

which assigns to any pair of objects $(X, Y) \in Ob(\Gamma\mathcal{S}_\bullet) \times Ob(\Gamma\mathcal{S}_\bullet)$, a pointed simplicial set $\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X, Y)$ which is defined in degree zero as follows:

$$\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X, Y)_0 := \Gamma\mathcal{S}_\bullet(X, Y).$$

The mapping simplicial set is defined in degree n as follows:

$$\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X, Y)_n := \Gamma\mathcal{S}_\bullet(X \wedge \Delta[n]^+, Y)$$

For any Γ-space X, the functor $X \underset{\mathcal{S}_\bullet}{\wedge} - : \mathcal{S}_\bullet \longrightarrow \Gamma\mathcal{S}_\bullet$ is left adjoint to the functor

$$\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X, -) : \Gamma\mathcal{S}_\bullet \longrightarrow \mathcal{S}_\bullet.$$

The counit of this adjunction is the evaluation map

$$ev : X \underset{\mathcal{S}_\bullet}{\wedge} \mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X, Y) \longrightarrow Y$$

and the unit is the obvious functor

$$K \longrightarrow \mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X, X \underset{\mathcal{S}_\bullet}{\wedge} K),$$

where Y is a normalized Γ-space and K is a (pointed) simplicial set.

The mapping object $\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X, Y)$ is a (pointed) simplicial set whose basepoint is the composite map $X \longrightarrow \Gamma^0 \longrightarrow Y$, where Γ^0 is the zero object in $\Gamma\mathcal{S}_\bullet$. Let $U(\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X, Y))$ denote the simplicial set obtained by forgetting the basepoint of $\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X, Y)$. We also recall the forgetful functor U which forgets the normalization of a Γ-space, see (11).

C.2. Lemma. *Let X and Y be two normalized Γ-spaces. The mapping simplicial set $U(\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X, Y))$ is an equalizer of the following diagram:*

$$\begin{array}{ccc} \mathcal{M}ap_{\Gamma\mathcal{S}}(U(X), U(Y)) & \longrightarrow & \mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^0, U(Y)) \\ & \searrow \quad\quad \nearrow_{0} & \\ & 1 & \end{array}$$

Proof. Each normalized Γ-space X uniquely determines a morphism $0_X : \Gamma^0 \longrightarrow U(X)$. It is sufficient to observe that a morphism $f : U(X) \longrightarrow U(Y)$ lies in the image of the forgetful functor U if and only if the following diagram commutes:

$$\begin{array}{ccc} & \Gamma^0 & \\ {}^{0_X}\swarrow & & \searrow^{0_Y} \\ U(X) & \xrightarrow[f]{} & U(Y) \end{array}$$

∎

C.3. Corollary. *For each pair of normalized* Γ*-spaces* X *and* Y *we have the following canonical isomorphism of mapping simplicial sets*

$$U(\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X,Y)) \cong \mathcal{M}ap_{\Gamma\mathcal{S}}(U(X),U(Y)).$$

Proof. It is sufficient to observe that for any normalized Γ-space Y, Yoneda lemma tells us that the mapping simplicial set $\mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^0, U(Y)) \cong 1$. ■

To each pair of objects $(K,X) \in Ob(\mathcal{S}_\bullet) \times Ob(\Gamma\mathcal{S}_\bullet)$ we can assign a Γ-space X^K which is defined in degree n as follows:

$$(X_\bullet^K)(n^+) := X(n^+)_\bullet^K$$

where the (pointed) simplicial set on the right is is defined by the following equalizer diagram:

$$X(n^+)_\bullet^K \longrightarrow \mathcal{M}ap_{\mathcal{S}}(K, X(n^+)) \xrightarrow{\mathcal{M}ap_{\mathcal{S}}(c, X(n^+))} \mathcal{M}ap_{\mathcal{S}}(1, X(n^+)), \qquad \mathcal{M}ap_{\mathcal{S}}(K, X(n^+)) \longrightarrow 1 \xrightarrow{0} \mathcal{M}ap_{\mathcal{S}}(1, X(n^+))$$

where $c : 1 \longrightarrow K$ is the basepoint map. This assignment is functorial in both variables and therefore we have a bifunctor

$$-^- : \mathcal{S}_\bullet^{op} \times \Gamma\mathcal{S}_\bullet \longrightarrow \Gamma\mathcal{S}_\bullet.$$

For any Γ-space X, the functor $X_\bullet{}^- : \mathcal{S}_\bullet \longrightarrow \Gamma\mathcal{S}_\bullet^{op}$ is left adjoint to the functor $\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(-,X) : \Gamma\mathcal{S}_\bullet^{op} \longrightarrow \mathcal{S}_\bullet$.

The following proposition summarizes the above discussion.

C.4. Proposition. *There is an adjunction of two variables*

$$(- \underset{\mathcal{S}_\bullet}{\wedge} -, -_\bullet{}^-, \mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(-,-)) : \Gamma\mathcal{S}_\bullet \times \mathcal{S}_\bullet \longrightarrow \Gamma\mathcal{S}_\bullet. \tag{23}$$

Now we are ready to describe the generating cofibrations and generating acyclic cofibrations of the desired model category: A map of Γ-spaces $F : X \longrightarrow Y$ is a strict acyclic fibration of normalized Γ-spaces if and only if it has the right lifting property with respect to all maps in the set

$$I_\bullet = \{\Gamma^n \underset{\mathcal{S}_\bullet}{\wedge} \partial_i^+ : \Gamma^n \underset{\mathcal{S}_\bullet}{\wedge} \partial\Delta[i]^+ \longrightarrow \Gamma^n \underset{\mathcal{S}_\bullet}{\wedge} \Delta[i]^+ : n, i \in \mathbb{N}\}, \tag{24}$$

where the product with pointed simplicial sets is defined in (22). From the results of [Joy08, Appendix D], one can deduce that a set of generating acyclic cofibrations in the Joyal model category $\mathcal{J}_S$ consists of representatives of isomorphism classes of monomorphisms between finite simplicial sets which are also weak equivalences in $(\mathcal{S}, \mathbf{Q})$. A morphism of normalized Γ-spaces F is a strict JQ fibration of normalized Γ-spaces if and only it has the right lifting property with respect to all maps in the set

$$\mathcal{J} = \{\Gamma^n \underset{\mathcal{S}_\bullet}{\wedge} j^+ \mid \forall n \in Ob(\mathcal{N}), j \in \mathcal{J}_S\}. \tag{25}$$

C.5. Remark. *The category $\Gamma\mathcal{S}_\bullet$ is a locally presentable category. The small object argument (for presentable categories), [Lur09, Proposition A.1.2.5], implies that the sets $\mathcal{I}$ and $\mathcal{J}$ provide two functorial factorization systems on the category Γ-space. The first one factors each morphism in Γ-space into a composite of a strict cofibration of Γ-spaces followed by a strict acyclic fibration of Γ-spaces and the second functorial factorization system factors each morphism in Γ-space into a composite of a strict acyclic cofibration of Γ-spaces followed by a strict fibration of Γ-spaces.*

The main aim of this subsection is to construct a model category structure on the category of all Γ-spaces whose three classes of morphisms are the ones defined above.

C.6. Theorem. *Strict JQ equivalences, strict JQ fibrations and strict JQ cofibrations of normalized Γ-spaces provide the category $\Gamma\mathcal{S}_\bullet$ with a combinatorial model category structure.*

Proof. The category of all functors from Γ^{op} to $\mathcal{S}_\bullet$, namely $[\Gamma^{op}, \mathcal{S}_\bullet]$ has a model category structure, called the *projective model category structure*, in which a map is a weak equivalence (resp. fibration) if and only if it is a weak equivalence (resp. fibration) degreewise, see [Lur09, Prop. A.3.3.2] for a proof. The category $\Gamma\mathcal{S}_\bullet$ is a subcategory of $[\Gamma^{op}, \mathcal{S}_\bullet]$. This implies that the axioms $CM(2), CM(3)$ and $CM(4)$, see [Qui67], [GJ99, Chap. 2] are satisfied by Γ-space because they are satisfied by the projective model category $[\Gamma^{op}, \mathcal{S}_\bullet]$. Finally, $CM(5)$ follows from Remark C.5 above. The category Γ-space is locally presentable. The sets $\mathcal{I}$ and $\mathcal{J}$ defined above form the sets of generating cofibrations and generating acyclic cofibrations respectively of the strict model category structure. ∎

C.7. Notation. *We will refer to the above model category as the* strict JQ model category of normalized Γ-spaces *and we denote it by $\Gamma\mathcal{S}_\bullet^{str}$.*

C.8. Theorem. *The strict model category of normalized Γ-spaces, $\Gamma\mathcal{S}_\bullet$, is a $\mathcal{S}_\bullet$-enriched model category.*

Proof. We will show that the adjunction of two variables (6) is a Quillen adjunction for the strict JQ model category structure on $\Gamma\mathcal{S}_\bullet$ and the Joyal model category structure on $\mathcal{S}_\bullet$. In order to do so, we will verify condition (2) of Lemma A.3. Let $g : C \longrightarrow D$ be a cofibration in $\mathcal{S}_\bullet$ and let $p : Y \longrightarrow Z$ be a strict fibration of Γ-spaces. We have to show that the induced map

$$\mathbf{hom}^{\square}_{\Gamma\mathcal{S}_\bullet}(g, p) : {Y_\bullet}^X \longrightarrow {Z_\bullet}^D \underset{{Z_\bullet}^C}{\times} {Y_\bullet}^C$$

is a fibration in $\mathcal{S}_\bullet$ which is acyclic if either of g or p is acyclic. It would be sufficient to check that the above morphism is degreewise a fibration in $\mathcal{S}_\bullet$, i.e. for all $n^+ \in \Gamma^{op}$, the morphism

$$\mathbf{hom}^{\square}_{\Gamma\mathcal{S}_\bullet}(g, p)(n^+) : \mathcal{M}ap_{\mathcal{S}_\bullet}(D, Y(n^+)) \longrightarrow \mathcal{M}ap_{\mathcal{S}_\bullet}(D, Z(n^+)) \underset{\mathcal{M}ap_{\mathcal{S}_\bullet}(C, Z(n^+))}{\times} \mathcal{M}ap_{\mathcal{S}_\bullet}(C, Y(n^+)),$$

is a fibration in $\mathcal{S}_\bullet$. This follows from the observations that the functor

$$p(n^+) : Y(n^+) \longrightarrow Z(n^+)$$

is a fibration in $\mathcal{S}_\bullet$ and the natural model category $\mathcal{S}_\bullet$ is a $\mathcal{S}_\bullet$-enriched model category whose enrichment is provided by the bifunctor $\mathcal{M}ap_{\mathcal{S}_\bullet}(-,-)$. ■

The adjunction $-^+ : \mathcal{S} \rightleftharpoons \mathcal{S}_\bullet : U$ provides us with an enrichment of the strict JQ model category of normalized Γ-space, over the Joyal model category of simplicial sets $(\mathcal{S}, \mathbf{Q})$.

C.9. Corollary. *The strict model category of normalized* Γ*-spaces,* $\Gamma\mathcal{S}_\bullet$*, is a* $(\mathcal{S}, \mathbf{Q})$*-enriched model category.*

A proof of this corollary follows from the above theorem and [Bar10, Lem. 1.31].

C.10. The JQ model category of normalized Γ-spaces. The objective of this subsection is to construct a new model category structure on the category $\Gamma\mathcal{S}_\bullet$ which is an analog of the JQ model category structure on $\Gamma\mathcal{S}$. This new model category is constructed along the same lines as the JQ model category structre, namely, it is obtained by localizing the strict JQ model category of normalized Γ-spaces (see Section C) and we refer to it as the JQ *model category of normalized* Γ*-spaces*. We go on further to show that this new model category is symmetric monoidal closed with respect to the smash product which is a categorical version of the smash product constructed in [Lyd99].

C.11. Notation. *We denote by* $1/\Gamma\mathcal{S}$ *the overcategory whose objects are maps in* $\Gamma\mathcal{S}$ *having domain the terminal* Γ*-space* 1*. We denote by* $(1/\Gamma\mathcal{S})_\bullet$ *the subcategory of* $1/\Gamma\mathcal{S}$ *whose objects are those maps* $1 \longrightarrow X$ *in* $\Gamma\mathcal{S}$ *whose codomain* Γ*-space satisfies the following normalization condition:*

$$X(0^+) = *.$$

C.12. Remark. *We observe that the category of normalized pointed objects* $(1/\Gamma\mathcal{S})_\bullet$ *is isomorphic to the category of normalized* Γ*-spaces* $\Gamma\mathcal{S}_\bullet$*.*

We want to construct a left Bousfield localization of the strict model category of Γ-spaces. For each pair $k^+, l^+ \in \Gamma^{op}$, we have the obvious *projection maps* in Γ^{op}

$$\delta^{k+l}_k : (k+l)^+ \longrightarrow k^+ \quad \textit{and} \quad \delta^{k+l}_l : (k+l)^+ \longrightarrow l^+.$$

The following two inclusion maps between representable Γ-spaces

$$\Gamma^{op}(\delta^{k+l}_k, -) : \Gamma^k \longrightarrow \Gamma^{k+l} \quad \textit{and} \quad \Gamma^{op}(\delta^{k+l}_l, -) : \Gamma^l \longrightarrow \Gamma^{k+l}$$

together induce a map of Γ-spaces on the coproduct which we denote as follows:

$$h^l_k : \Gamma^l \vee \Gamma^k \longrightarrow \Gamma^{l+k}.$$

We now define a set of maps $\mathcal{E}_\infty\mathcal{S}_\bullet$ in $\Gamma\mathcal{S}_\bullet$:

$$\mathcal{E}_\infty\mathcal{S}_\bullet := \{h^l_k : \Gamma^l \vee \Gamma^k \longrightarrow \Gamma^{l+k} : l, k \in \mathbb{Z}^+\}$$

Next we define the set of arrows in $\Gamma\mathcal{S}_\bullet$ with respect to which we will localize the strict JQ model category of normalized Γ-spaces:

$$\Delta \times \mathcal{E}_\infty\mathcal{S}_\bullet := \{\Delta[n]^+ \underset{\mathcal{S}_\bullet}{\wedge} h^l_k : h^l_k \in \Delta \times \mathcal{E}_\infty\mathcal{S}_\bullet\}$$

C.13. Definition. *We call a* Γ*-space* X *a* $(\Delta\times\mathcal{E}_\infty\mathcal{S}_\bullet)$-local object *if it is a fibrant object in the strict JQ model category of normalized* Γ*-spaces and for each map* $\Delta[n]^+ \underset{\mathcal{S}_\bullet}{\wedge} h^l_k \in \Delta \times \mathcal{E}_\infty\mathcal{S}_\bullet$*, the induced simplicial map*

$$\mathcal{M}ap^h_{\Gamma\mathcal{S}_\bullet}(\Delta[n]^+ \underset{\mathcal{S}_\bullet}{\wedge} h^l_k, X) : \mathcal{M}ap^h_{\Gamma\mathcal{S}_\bullet}(\Delta[n]^+ \underset{\mathcal{S}_\bullet}{\wedge} \Gamma^{k+l}, X) \longrightarrow \mathcal{M}ap^h_{\Gamma\mathcal{S}_\bullet}(\Delta[n]^+ \underset{\mathcal{S}_\bullet}{\wedge} (\Gamma^k \vee \Gamma^l), X),$$

is a homotopy equivalence of simplicial sets for all $n \geq 0$ *where* $\mathcal{M}ap^h_{\Gamma\mathcal{S}_\bullet}(-,-)$ *is the simplicial function complexe associated with the strict model category* $\Gamma\mathcal{S}_\bullet$*, see [DK80a], [DK80b] and [DK80c].*

Corollary C.9 above and Appendix B tell us that a model for $\mathcal{M}ap^h_{\Gamma\mathcal{S}_\bullet}(X,Y)$ is the Kan complex $J(\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X,Y))$ which is the maximal Kan complex contained in the quasicategory $\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X,Y)$.

The following proposition gives a characterization of $\Delta \times \mathcal{E}_\infty\mathcal{S}_\bullet$-local objects

C.14. Proposition. *A normalized* Γ*-space* X *is a* $(\Delta \times \mathcal{E}_\infty\mathcal{S}_\bullet)$*-local object if and only if it satisfies the Segal condition, namely the functor*

$$(X(\delta^{k+l}_k), X(\delta^{k+l}_l)) : X((k+l)^+) \longrightarrow X(k^+) \times X(l^+)$$

is an equivalence of (pointed) quasi-categories for all $k^+, l^+ \in Ob(\Gamma^{op})$.

Proof. Throughout this proof we slightly abuse notation by denoting the $\mathcal{S}$ valued functor $U(\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(-,-))$ by $\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(-,-)$. We begin the proof by observing that each element of the set $\Delta \times \mathcal{E}_\infty\mathcal{S}_\bullet$ is a map of Γ-spaces between cofibrant Γ-spaces. Theorem B.10 implies that X is a $(\Delta \times \mathcal{E}_\infty\mathcal{S}_\bullet)$-local object if and only if the following map of simplicial sets

$$\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(h^k_l, X) : \mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(\Gamma^{k+l}, X) \longrightarrow \mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(\Gamma^k \vee \Gamma^l, X)$$

is an equivalence of quasi-categories.

We observe that we have the following commutative square in $(\mathcal{S}, \mathbf{Q})$

$$\begin{array}{ccc}
\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(\Gamma^{k+l}, X) & \xrightarrow{\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(h^k_l, X)} & \mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(\Gamma^k \vee \Gamma^l, X) \\
\cong \downarrow & & \downarrow \cong \\
X((k+l)^+) & \xrightarrow[(X(\delta^{k+l}_k), X(\delta^{k+l}_l))]{} & X(k^+) \times X(l^+)
\end{array}$$

By the two out of three property of weak equivalences in a model category the simplicial map $(X(\delta^{k+l}_k), X(\delta^{k+l}_l))$ is an equivalence of quasi-categories if and only if the map $\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(h^k_l, X)$ is an equivalence of quasi-categories. ∎

C.15. Definition. *We will refer to a* $(\Delta \times \mathcal{E}_\infty \mathcal{S}_\bullet)$*-local object as a* normalized coherently commutative monoidal quasi-category.

C.16. Definition. *A morphism of normalized* Γ*-spaces* $F : X \longrightarrow Y$ *is a* $(\Delta \times \mathcal{E}_\infty \mathcal{S}_\bullet)$*-local equivalence if for each normalized coherently commutative monoidal quasi-category* Z *the following simplicial map*

$$\mathcal{M}ap^h_{\Gamma\mathcal{S}_\bullet}(F, Z) : \mathcal{M}ap^h_{\Gamma\mathcal{S}_\bullet}(Y, Z) \longrightarrow \mathcal{M}ap^h_{\Gamma\mathcal{S}_\bullet}(X, Z)$$

is a homotopy equivalence of simplicial sets. We may sometimes refer to a $(\Delta \times \mathcal{E}_\infty \mathcal{S}_\bullet)$*-local equivalence as an equivalence of normalized coherently commutative monoidal quasi-categories.*

An argument similar to the proof of Proposition 4.7 proves the following proposition:

C.17. Proposition. *A morphism between two JQ cofibrant normalized* Γ*-spaces* $F : X \longrightarrow Y$ *is an* $(\Delta \times \mathcal{E}_\infty \mathcal{S}_\bullet)$*-local equivalence if and only if the simplicial map*

$$\mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(F, Z) : \mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(Y, Z) \longrightarrow \mathcal{M}ap_{\Gamma\mathcal{S}_\bullet}(X, Z)$$

is an equivalence of quasi-categories for each normalized coherently commutative monoidal quasi-category Z.

The main objective of the current subsection is to construct a new model category structure on the category of normalized Γ-spaces $\Gamma\mathcal{S}_\bullet$ by localizing the strict JQ model category of normalized Γ-spaces with respect to morphisms in the set $\Delta \times \mathcal{E}_\infty \mathcal{S}_\bullet$. The desired model structure follows from Theorem 4.9

C.18. Theorem. *There is a closed, left proper, combinatorial model category structure on the category of normalized* Γ*-spaces,* $\Gamma\mathcal{S}_\bullet$*, in which*

1. *The class of cofibrations is the same as the class of JQ cofibrations of normalized* Γ*-spaces.*

2. *The weak equivalences are equivalences of normalized coherently commutative monoidal quasi-categories.*

An object is fibrant in this model category if and only if it is a normalized coherently commutative monoidal quasi-category. Further, this model category structure makes $\Gamma\mathcal{S}_\bullet$ *a closed symmetric monoidal model category under the smash product.*

Proof. The strict JQ model category of normalized Γ-spaces is a combinatorial model category therefore the existence of the model structure follows from Theorem 4.9. The statement characterizing fibrant objects also follows from Theorem 4.9. An argument similar to the proof of Theorem 4.18 using the enrichment of the strict JQ model category of normalized Γ-spaces over the $(\mathcal{S}_\bullet, \mathbf{Q})$ established in Proposition 3.12 shows that the localized model category has a symmetric monoidal closed model category structure under the smash product. ∎

C.19. Notation. *The model category constructed in Theorem C.18 will be referred to either as* the *JQ* model category of normalized Γ-spaces *or as the model category of normalized coherently commutative monoidal quasi-categories.*

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