

## DISCRETE DOUBLE FIBRATIONS

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ABSTRACT. Presheaves on a small category are well-known to correspond via a category of elements construction to ordinary discrete fibrations over that same small category. Work of R. Paré proposes that presheaves on a small double category are certain lax functors valued in the double category of sets with spans. This paper isolates the discrete fibration concept corresponding to this presheaf notion and shows that the category of elements construction introduced by Paré leads to an equivalence of virtual double categories.

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### 1. Introduction

A discrete fibration is functor  $F: \mathcal{F} \rightarrow \mathcal{C}$  such that for each arrow  $f: C \rightarrow FY$  in  $\mathcal{C}$ , there is a unique arrow  $f^*Y \rightarrow Y$  in  $\mathcal{F}$  whose image under  $F$  is  $f$ . These correspond to ordinary presheaves

$$\mathbf{DFib}(\mathcal{C}) \simeq [\mathcal{C}^{op}, \mathbf{Set}]$$

via a well-known “category of elements” construction [MacLane, 1998]. This is a “representation theorem” in the sense that discrete fibrations over  $\mathcal{C}$  correspond to set-valued representations  $\mathcal{C}$ .

Investigation of higher-dimensional structures leads to the question of analogues of such well-established developments in lower-dimensional settings. In the case of fibrations, for example, ordinary fibrations have their analogues in “2-fibrations” over a fixed base 2-category [Buckley, 2014]. Discrete fibrations over a fixed base 2-category have their 2-dimensional version in “discrete 2-fibrations” [Lambert, 2020]. The justification that the notion of (discrete) fibration is the correct one in each case consists in the existence

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of a representation theorem taking the form of an equivalence with representations of the base structure via a category of elements construction.

In the double-categorical world, R. Paré proposes [Paré, 2011] that certain span-valued, lax functors on a double category  $\mathbb{B}$  are presheaves on  $\mathbb{B}$ . This paper aims to isolate the notion of “discrete double fibration” corresponding to this notion of presheaf. As justification of the proposed definition, Paré’s double category of elements will be used to exhibit a representation theorem on the pattern of those reviewed above. The technically interesting part of the results is that ultimately an equivalence of virtual double categories is achieved. For this the language of monoids and modules in virtual double categories as in [Leinster, 2004] will be used.

1.1. OVERVIEW AND MOTIVATION. To give away the answer completely, a discrete double fibration should be a category object in the category of discrete fibrations. This follows the approach to double category theorizing that a “double  $---$ ” is a category object in the category of  $---$ ’s. The point of the paper is thus to show that this expected definition comports with Paré’s notion of presheaf by exhibiting the representation theorem as discussed above. This will be proved directly for all the 1-categorical structure involved.

However, lax functors and their transformations are fragments of a higher double-dimensional structure. That is, modules and their multimodulations make lax functors into a virtual double category. The question, then, arises as to the corresponding virtual double category structure on the discrete fibration side of the desired equivalence. It turns out that there is already well-established language for these structures from [Leinster, 2004].

For a category with finite limits  $\mathcal{C}$ , the category of monoids in the bicategory of spans in  $\mathcal{C}$  is equivalent to the category of category objects in  $\mathcal{C}$ . In fact the correspondence is much stronger than this, since the data on either side of the equivalence is the same modulo rearrangement of tuples. This general result leads to the special case for  $\mathcal{C} = \mathbf{DFib}$ , yielding the equivalence

$$\mathbf{Mon}(\mathbf{Span}(\mathbf{DFib})) \simeq \mathbf{Cat}(\mathbf{DFib}).$$

This is the main clue leading to a natural candidate for virtual double category structures on discrete double fibrations. For monoids in the bicategory part of any double category  $\mathbb{D}$  form the 1-categorical part of the virtual double category of monoids and modules in  $\mathbb{D}$ . That is, there is a virtual double category  $\mathbf{Mod}(\mathbb{D})$  of modules whose underlying 1-category is precisely monoids in the bicategory part of  $\mathbb{D}$ . This leads to the natural candidate for virtual double category structure on discrete double fibrations as  $\mathbf{Mod}(\mathbf{Span}(\mathbf{DFib}))$ . The ultimate objective of the paper is thus to show that an appropriate slice of such modules is equivalent to the virtual double category of lax presheaves via the double category of elements construction.

1.2. ORGANIZATION AND RESULTS. Section 2 reviews lax functors, their transformations and the elements construction from [Paré, 2011]. The main definition of “discrete double fibration” is given as a double functor  $P: \mathbb{E} \rightarrow \mathbb{B}$  for which both  $P_0$  and  $P_1$  are discrete

fibrations. This is equivalently a category object in the category of discrete fibrations. The section culminates in the first main contribution of the paper, proved as Theorem 2.27 below. This result says that for any strict double category  $\mathbb{B}$ , there is an equivalence of 1-categories

$$\mathbf{DFib}(\mathbb{B}) \simeq \mathbf{Lax}(\mathbb{B}^{op}, \mathbf{Span})$$

between the category of discrete double fibrations over  $\mathbb{B}$  and span-valued lax functors on  $\mathbb{B}$ , induced by the double category of elements construction. This is achieved by constructing a pseudo-inverse for the double category of elements.

Section 3 introduces the virtual double category structure on lax span-valued functors as in [Paré, 2011] and [Paré, 2013] with the goal of extending the theorem above to an equivalence of virtual double categories. It turns out that a convenient language for setting up the virtual double category structure on discrete double fibrations is that of monoids and modules in virtual double categories as in [Leinster, 2004] and later [Cruttwell & Shulman, 2010].

Section 4 extends the elements construction and its pseudo-inverse to functors of virtual double categories and culminates in the second result, proved as Theorem 4.14. That is, for any strict double category  $\mathbb{B}$ , there is an equivalence of virtual double categories

$$\mathbb{DFib}(\mathbb{B}) \simeq \mathbf{Lax}(\mathbb{B}^{op}, \mathbf{Span})$$

induced by the elements functor.

**1.3. CONVENTIONS AND NOTATION.** Double categories go back to Ehresmann [Ehresmann, 1963]. Other references include [Grandis & Paré, 1999], [Grandis & Paré, 2004], and [Shulman, 2008]. One quirk of our presentation is that, owing to different conventions concerning which arrows are “vertical” and which are “horizontal,” we prefer to use the more descriptive language of “arrows,” “proarrows,” and “cells.” Arrows are of course the ordinary arrows between objects (the horizontal arrows in [Grandis & Paré, 2004] and [Grandis & Paré, 2004] vs. the vertical ones in [Shulman, 2008] and [Cruttwell & Shulman, 2010]); whereas the proarrows are the objects of the bicategory part (that is, the vertical ones of [Grandis & Paré, 1999] and [Grandis & Paré, 2004] vs. the horizontal ones of [Shulman, 2008] and [Cruttwell & Shulman, 2010]). In this language, the arrows of the canonical example of categories and profunctors would be the ordinary functors while the proarrows are the profunctors. This choice is inspired by the language of a “proarrow equipment” developed in [Wood, 1982] and [Wood, 1985], which sought to axiomatize this very situation as a 2-category “equipped with certain proarrows.”

Throughout use blackboard letters  $\mathbb{A}$ ,  $\mathbb{B}$ ,  $\mathbb{C}$ ,  $\mathbb{D}$  for double categories. The 0-part of such  $\mathbb{D}$  is the category of objects and arrows, denoted by ‘ $\mathbb{D}_0$ ’. The 1-part is the category of proarrows and cells, denoted by ‘ $\mathbb{D}_1$ ’. The category structures of  $\mathbb{D}_0$  and  $\mathbb{D}_1$  are referred to as the “internal” structure of  $\mathbb{D}$ . So-called “external structure” is given by the functors relating  $\mathbb{D}_0$  and  $\mathbb{D}_1$ . External composition is written with a tensor ‘ $\otimes$ ’. Internal and external compositions satisfy an “interchange law,” essentially stating that  $\otimes$  is a bifunctor. The external unit is denoted with  $u: \mathbb{D}_0 \rightarrow \mathbb{D}_1$ . External source and target

functors are  $\text{src}, \text{tgt}: \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$ . Internal composition is denoted by juxtaposition in the usual order. The terms “domain” and “codomain” refer to internal structure, whereas “source” and “target” refer to external structure. Generally, external composition is associative only up to coherent natural isomorphism. For emphasis these are sometimes called “pseudo-” double categories. Double categories for which external composition is strictly associative are called “strict” double categories. These are equivalently category objects in categories.

The notation ‘ $\mathbb{D}^{op}$ ’ indicates the opposite of  $\mathbb{D}$ , obtained by formally reversing only the arrows of  $\mathbb{D}$ . One result requires the notion of the “transpose” of a double category  $\mathbb{D}$ , denoted here by ‘ $\mathbb{D}^\dagger$ ’. This is obtained by reversing both the arrows and proarrows of  $\mathbb{D}$ . When  $\mathbb{D}$  is strict,  $\mathbb{D}^\dagger$  is a strict double category and transposing extends to a functor  $(-)^\dagger: \mathbf{Dbl} \rightarrow \mathbf{Dbl}$  on the 1-category of strict double categories and double functors.

Script letters  $\mathcal{C}$  and  $\mathcal{D}$  denote ordinary categories with objects sets  $\mathcal{C}_0$  and arrow sets  $\mathcal{C}_1$ . Well-known 1-, 2-, and bi-categories referred to throughout are given in boldface such as **Set**, **Cat**, **Span**, and **Rel**. We always refer to the highest level of structure commonly associated with the objects of the category. Thus, **Cat** is the 2-category of categories, unless specified otherwise. The main exception is that **DFib** is the ordinary category of discrete fibrations. Throughout we take for granted the notions of monoid and category internal to a given category with finite limits. Some 2-categorical concepts are referenced but not in a crucial way. Common double-categories are presented in mixed typeface and named by their proarrows. In particular **Prof** is the double category of profunctors; **Rel** is the double category of sets and relations; **Span** (rather than **Set**) is the double category of sets and spans. Note that these are genuinely pseudo-double categories.

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## 2. Lax Functors, Elements, and Discrete Double Fibrations

In this section, the double category of elements is recalled, and it is seen how its construction leads naturally to the definition of a discrete double fibration. As groundwork, first review the definitions of lax functor and their natural transformations.

2.1. DEFINITION. [See e.g. §7.2 of [Grandis & Paré, 1999] or §1.1.2 of [Paré, 2013]] A **lax functor** between double categories  $F: \mathbb{A} \rightarrow \mathbb{B}$  consists of assignments

$$A \mapsto FA \quad f \mapsto Ff \quad m \mapsto Fm \quad \alpha \mapsto F\alpha$$

on objects, arrows, proarrows and cells, respecting domains, codomains, sources and targets; and laxity cells

$$\begin{array}{ccc}
 FA \xrightarrow{u_{FA}} FA & & FA \xrightarrow{Fm} FB \xrightarrow{Fn} FC \\
 \parallel \quad \phi_A \quad \parallel & & \parallel \quad \phi_{n,m} \quad \parallel \\
 FA \xrightarrow{Fu_A} FA & & FA \xrightarrow{F(n \otimes m)} FC
 \end{array}$$

for each object  $A$  and composable proarrows  $m: A \rightrightarrows B$  and  $n: B \rightrightarrows C$ , all subject to the following axioms.

1. [Internal Functoriality] The object and arrows assignments  $A \mapsto FA$ ,  $f \mapsto Ff$  define an ordinary functor  $F_0: \mathbb{A}_0 \rightarrow \mathbb{B}_0$ ; and the proarrow and cell assignments define an ordinary functor  $F_1: \mathbb{A}_1 \rightarrow \mathbb{B}_1$  with respect to internal identities and composition of cells.
2. [Naturality] For any  $f: A \rightarrow B$ , there is an equality

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \cdot & \xrightarrow{u_{FA}} & \cdot \\
 Ff \downarrow & u_{Ff} & \downarrow Ff \\
 \cdot & \xrightarrow{\quad} & \cdot \\
 \parallel & \phi_B & \parallel \\
 \cdot & \xrightarrow{Fu_B} & \cdot
 \end{array} & = & \begin{array}{ccc}
 \cdot & \xrightarrow{u_{FA}} & \cdot \\
 \parallel & \phi_A & \parallel \\
 \cdot & \xrightarrow{\quad} & \cdot \\
 Ff \downarrow & Fu_f & \downarrow Ff \\
 \cdot & \xrightarrow{Fu_B} & \cdot
 \end{array}
 \end{array}$$

and for any externally composable cells  $\alpha: m \rightrightarrows p$  and  $\beta: n \rightrightarrows q$ , there is an equality

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \cdot & \xrightarrow{Fm} & \cdot & \xrightarrow{Fn} & \cdot \\
 Ff \downarrow & F\alpha & \downarrow F\beta & Fh & \downarrow \\
 \cdot & \xrightarrow{Fp} & \cdot & \xrightarrow{Fq} & \cdot \\
 \parallel & \phi_{q,p} & \parallel & & \parallel \\
 \cdot & \xrightarrow{F(q \otimes p)} & \cdot & & \cdot
 \end{array} & = & \begin{array}{ccc}
 \cdot & \xrightarrow{Fm} & \cdot & \xrightarrow{Fn} & \cdot \\
 \parallel & \phi_{n,m} & \parallel & & \parallel \\
 \cdot & \xrightarrow{F(n \otimes m)} & \cdot & & \cdot \\
 Ff \downarrow & F(\beta \otimes \alpha) & \downarrow Ff & & \downarrow Ff \\
 \cdot & \xrightarrow{F(q \otimes p)} & \cdot & & \cdot
 \end{array}
 \end{array}$$

of composite cells.

3. [Unit and Associativity] Given a proarrow  $m: A \rightrightarrows B$ , the unit laxity cells satisfy

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \cdot & \xrightarrow{u_{FA}} & \cdot & \xrightarrow{Fm} & \cdot \\
 \parallel & \phi_A & \downarrow Fu_m & \parallel & \parallel \\
 \cdot & \xrightarrow{Fu_A} & \cdot & \xrightarrow{Fm} & \cdot \\
 \parallel & \phi_{m,u_A} & \parallel & & \parallel \\
 \cdot & \xrightarrow{F(m \otimes u_A)} & \cdot & & \cdot
 \end{array} & = & \begin{array}{ccc}
 \cdot & \xrightarrow{u_{FA}} & \cdot & \xrightarrow{Fm} & \cdot \\
 \parallel & \cong & \parallel & & \parallel \\
 \cdot & \xrightarrow{Fm} & \cdot & & \cdot \\
 \parallel & \cong & \parallel & & \parallel \\
 \cdot & \xrightarrow{F(m \otimes u_A)} & \cdot & & \cdot
 \end{array}
 \end{array}$$

and

$$\begin{array}{ccc}
 \begin{array}{c} \cdot \xrightarrow{Fm} \cdot \xrightarrow{u_{FB}} \cdot \\ \parallel \quad \downarrow \quad \parallel \\ Fu_m \quad \phi_B \\ \parallel \quad \downarrow \quad \parallel \\ \cdot \xrightarrow{Fm} \cdot \xrightarrow{Fu_B} \cdot \\ \phi_{u_B, m} \\ \parallel \quad \downarrow \quad \parallel \\ \cdot \xrightarrow{F(u_B \otimes m)} \cdot \end{array} & = & \begin{array}{c} \cdot \xrightarrow{Fm} \cdot \xrightarrow{u_{FB}} \cdot \\ \parallel \quad \cong \quad \parallel \\ \cdot \xrightarrow{Fm} \cdot \\ \parallel \quad \cong \quad \parallel \\ \cdot \xrightarrow{F(u_B \otimes m)} \cdot \end{array}
 \end{array}$$

and for any three composable proarrows  $m: A \rightarrow B$ ,  $n: B \rightarrow C$ , and  $p: C \rightarrow D$ , the laxity cells are associative in the sense that

$$\begin{array}{ccc}
 \begin{array}{c} \cdot \xrightarrow{Fm} \cdot \xrightarrow{Fn} \cdot \xrightarrow{Fp} \cdot \\ \parallel \quad \parallel \quad \parallel \\ Fu_m \quad \phi_{p,n} \\ \parallel \quad \parallel \quad \parallel \\ \cdot \xrightarrow{F(p \otimes n \otimes m)} \cdot \\ \phi_{p \otimes n, m} \end{array} & = & \begin{array}{c} \cdot \xrightarrow{Fm} \cdot \xrightarrow{Fn} \cdot \xrightarrow{Fp} \cdot \\ \parallel \quad \parallel \quad \parallel \\ \phi_{n,m} \quad Fu_p \\ \parallel \quad \parallel \quad \parallel \\ \cdot \xrightarrow{F(p \otimes (n \otimes m))} \cdot \\ \phi_{p, n \otimes m} \end{array}
 \end{array}$$

are equal, modulo composition with the image under  $F$  of the associativity iso cell

$$(p \otimes n) \otimes m \cong p \otimes (n \otimes m)$$

given with the structure of  $\mathbb{A}$ .

The structural isomorphisms reduce to strict identities when  $\mathbb{A}$  and  $\mathbb{B}$  are strict double categories. A **pseudo double functor** is a lax double functor where the laxity cells are isomorphisms; a **strict double functor** is one whose laxity cells are identities.

2.2. **REMARK.** Generally speaking, use lower-case Greek letters for the laxity cells corresponding to the Latin capital letter for the functor. Thus,  $F$  is a lax functor with laxity cells denoted by ‘ $\phi$ ’ with subscripts; ‘ $\gamma$ ’ is used for a lax functor  $G$ .

2.3. **EXAMPLE.** [Cf. §1.2 of [Paré, 2011]] The usual object functor  $(-)_0: \mathbf{Cat} \rightarrow \mathbf{Set}$  extends to a lax double functor  $(-)_0: \mathbf{Prof} \rightarrow \mathbf{Span}$  in the following way. On a category  $\mathcal{C}$  take the object set  $\mathcal{C}_0$  to be the image as usual; likewise take the object part  $F_0: \mathcal{C}_0 \rightarrow \mathcal{D}_0$  to be the image of a given functor  $F: \mathcal{C} \rightarrow \mathcal{D}$ . On a profunctor  $P: \mathcal{C} \rightarrow \mathcal{D}$  – that is, a functor  $P: \mathcal{C}^{op} \times \mathcal{D} \rightarrow \mathbf{Set}$  – take the image to be the disjoint union

$$(P)_0 := \coprod_{C,D} P(C, D).$$

The assignment on a given transformation of profunctors is induced by the universal property of the coproduct. This is a bona fide lax functor

2.4. DEFINITION. [Cf. §1.1.5 [Paré, 2013]] Let  $F, G: \mathbb{A} \rightrightarrows \mathbb{B}$  denote lax functors with laxity cells  $\phi$  and  $\gamma$ . A **natural transformation**  $\tau: F \rightarrow G$  assigns to each object  $A$  an arrow  $\tau_A: FA \rightarrow GA$  and to each proarrow  $m: A \rightrightarrows B$  a cell

$$\begin{array}{ccc} FA & \xrightarrow{Fm} & FB \\ \tau_A \downarrow & \tau_m & \downarrow \tau_B \\ GA & \xrightarrow{Gm} & GB \end{array}$$

in such a way that the following axioms are satisfied.

1. [Naturality] Given any arrow  $f: A \rightarrow B$ , the usual naturality square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \tau_A \downarrow & & \downarrow \tau_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes; and given any cell  $\alpha: m \rightrightarrows n$ , the corresponding naturality square commutes in the sense that the compositions

$$\begin{array}{ccc} \cdot & \xrightarrow{Fm} & \cdot \\ \downarrow & F\alpha & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \\ \parallel & \tau_n & \parallel \\ \cdot & \xrightarrow{Gn} & \cdot \end{array} = \begin{array}{ccc} \cdot & \xrightarrow{Fm} & \cdot \\ \parallel & \tau_m & \parallel \\ \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & G\alpha & \downarrow \\ \cdot & \xrightarrow{Gn} & \cdot \end{array}$$

are equal.

2. [Functoriality] For any object  $A$ , the compositions

$$\begin{array}{ccc} \cdot & \xrightarrow{u_{FA}} & \cdot \\ \downarrow & u_{\tau_A} & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \\ \parallel & \gamma_A & \parallel \\ \cdot & \xrightarrow{Gu_A} & \cdot \end{array} = \begin{array}{ccc} \cdot & \xrightarrow{u_{FA}} & \cdot \\ \parallel & \phi_A & \parallel \\ \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \tau_{u_A} & \downarrow \\ \cdot & \xrightarrow{Gu_A} & \cdot \end{array}$$

are equal; and for any composable proarrows  $m: A \rightrightarrows B$  and  $n: B \rightrightarrows C$ , the composites

$$\begin{array}{ccc} \cdot & \xrightarrow{Fm} & \cdot & \xrightarrow{Fn} & \cdot \\ \tau_A \downarrow & \tau_m & \downarrow \tau_n & \downarrow \tau_C & \\ \cdot & \xrightarrow{Gm} & \cdot & \xrightarrow{Gn} & \cdot \\ \parallel & \gamma_{n,m} & \parallel & & \\ \cdot & \xrightarrow{G(n \otimes m)} & \cdot & & \end{array} = \begin{array}{ccc} \cdot & \xrightarrow{Fm} & \cdot & \xrightarrow{Fn} & \cdot \\ \parallel & \phi_{n,m} & \parallel & & \\ \cdot & \xrightarrow{\quad} & \cdot & & \\ \tau_A \downarrow & \tau_{n \otimes m} & \downarrow \tau_C & & \\ \cdot & \xrightarrow{G(n \otimes m)} & \cdot & & \end{array}$$

are equal.

Let  $\mathbf{Lax}(\mathbb{A}, \mathbb{B})$  denote the category of lax functors  $\mathbb{A} \rightarrow \mathbb{B}$  and their transformations. The composition is “component-wise” and identity transformations are those with identity morphisms in each of their components.

In several cases, this category can be described in terms of well-known structures. These equivalences are given by the functor induced by the double category of elements in Lemma 2.9.

2.5. **EXAMPLE.** A lax functor  $\mathbf{1} \rightarrow \mathbf{Span}$  is essentially a small category. In fact, taking elements induces an equivalence

$$\mathbf{Lax}(\mathbf{1}, \mathbf{Span}) \simeq \mathbf{Cat}.$$

In terms of monoids and their morphisms (i.e. a 1-category of monoids and their morphisms in a bicategory), this means that

$$\mathbf{Lax}(\mathbf{1}, \mathbf{Span}) \simeq \mathbf{Mon}(\mathbf{Span}(\mathbf{Set})).$$

as 1-categories. This is a primordial example or special case of our main results, Theorem 2.27 and Theorem 4.14.

2.6. **EXAMPLE.** [See §3.13 of [Paré, 2011]] If  $\mathcal{C}$  is an ordinary category, let  $\mathcal{V}\mathcal{C}$  denote the “vertical double category” formed from the objects of  $\mathcal{C}$  and using the morphisms of  $\mathcal{C}$  as the proarrows with only identity arrows and cells. There is then an equivalence

$$\mathbf{Lax}(\mathcal{V}\mathcal{C}, \mathbf{Span}) \simeq \mathbf{Cat}/\mathcal{C}$$

of ordinary categories given by taking elements.

The fibration properties of the double category of elements construction lead to the axiomatization of our notion of “discrete double fibration.” Let us recall the details of this double category and its associated projection functor.

2.7. **CONSTRUCTION.** [§3.7 [Paré, 2011]] Let  $F: \mathbb{B}^{op} \rightarrow \mathbf{Span}$  denote a lax functor on a double category  $\mathbb{B}$  with laxity cells  $\phi$ . The **double category of elements**  $\mathbb{E}\mathbf{l}(F)$  has

1. as objects those pairs  $(B, x)$  with  $x \in FB$ ;
2. as arrows  $f: (B, x) \rightarrow (C, y)$  those arrows  $f: B \rightarrow C$  of  $\mathbb{B}$  such that  $f^*y = x$  holds under the action of the transition function  $f^*: FC \rightarrow FB$ ;
3. as proarrows  $(B, x) \rightrightarrows (C, y)$  pairs  $(m, s)$  consisting of a proarrow  $m: B \rightrightarrows C$  and an element  $s \in Fm$  of the vertex of  $FB \xleftarrow{(-)_0} Fm \xrightarrow{(-)_1} FC$  such that  $s_0 = x$  and  $s_1 = y$  both hold; and



4. as cells  $(m, s) \Rightarrow (n, t)$  those cells  $\theta$  of  $\mathbb{B}$  as at left below for which the associated morphism of spans on the right

$$\begin{array}{ccc}
 A & \xrightarrow{m} & B \\
 f \downarrow & \theta & \downarrow g \\
 C & \xrightarrow{n} & D
 \end{array}
 \qquad
 \begin{array}{ccccc}
 FC & \xleftarrow{(-)_0} & Fn & \xrightarrow{(-)_1} & FD \\
 f^* \downarrow & & \downarrow \theta^* & & \downarrow g^* \\
 FA & \xleftarrow{(-)_0} & Fm & \xrightarrow{(-)_1} & FB
 \end{array}$$

satisfies  $\theta^*(t) = s$ .

The 1-category structure on objects and morphisms is the same as the ordinary category of elements. Internal composition of cells uses the internal composition in  $\mathbb{B}$  and the strict equalities of the form  $\theta^*(t) = s$  in the definition. External composition uses the given laxity morphisms. For example, given composable proarrows  $(m, s): (B, x) \mapsto (C, y)$  and  $(n, t): (C, y) \mapsto (D, z)$ , the composite is defined as

$$(n, t) \otimes (m, s) := (n \otimes m, \phi_{n,p}(s, t)).$$

This is a well-defined proarrow. Identities and external composition of cells is similar. This makes  $\mathbb{E}\mathbb{I}(F)$  a double category. It is only as strict as  $\mathbb{B}$ . There is a **projection double functor**  $\Pi: \mathbb{E}\mathbb{I}(F) \rightarrow \mathbb{B}$  taking the indexing objects, morphisms, proarrows and cells to  $\mathbb{B}$ . This is strict even if  $\mathbb{B}$  is pseudo.

It is remarked in [Paré, 2011] that taking elements extends to a functor. Here are the details.

2.8. CONSTRUCTION. [Morphism from a Transformation of Lax Functors] Let  $\tau: F \rightarrow G$  denote a transformation of lax functors  $F, G: \mathbb{B}^{op} \rightrightarrows \mathbf{Span}$  as in Definition 2.4. Define what will be a strict functor of double categories  $\mathbb{E}\mathbb{I}(\tau): \mathbb{E}\mathbb{I}(F) \rightarrow \mathbb{E}\mathbb{I}(G)$  over  $\mathbb{B}$ . On objects and arrows, take

$$(D, x) \mapsto (D, \tau_D x) \qquad f \mapsto f$$

The arrow assignment is well-defined because  $f^* \tau_D = \tau_C f^*$  holds by the strict naturality condition in the definition. This assignment is functorial by construction. Now, for proarrows, send

$$(m, s) \mapsto (m, \tau_m s) \qquad \alpha \mapsto \alpha$$

similarly to the object and arrow assignments. Well-definition again follows from naturality. Functoriality is by construction. The rest of the important content is in the next result.

Suppose for convenience throughout the rest of the section that  $\mathbb{B}$  is strict.

2.9. LEMMA. *Given a transformation  $\tau: F \rightarrow G$  of span-valued, lax functors, the assignments above yield a morphism from  $\Pi: \mathbb{E}\mathbb{I}(F) \rightarrow \mathbb{B}$  to  $\Pi: \mathbb{E}\mathbb{I}(G) \rightarrow \mathbb{B}$  over  $\mathbb{B}$ . This association is functorial, meaning that the elements functor*

$$\mathbb{E}\mathbb{I}(-): \mathbf{Lax}(\mathbb{B}^{op}, \mathbf{Span}) \longrightarrow \mathbf{Dbl}/\mathbb{B}$$

*valued in the slice of double categories over  $\mathbb{B}$  is well-defined.*

PROOF. It is left to see that  $\mathbb{E}\mathbf{l}(\tau): \mathbb{E}\mathbf{l}(F) \rightarrow \mathbb{E}\mathbf{l}(G)$  preserves external composites and units. The definition of external composition in the elements construction incorporates the laxity morphisms coming with  $F$  and  $G$ . So, the preservation of external composition reduces to the statement that these laxity morphisms interact in the proper way with the components of  $\tau$  used in the definition. But this is precisely what the “Functoriality” condition of Definition 2.4 axiomatizes. Unit preservation follows similarly. ■

Those double functors in the image of this functor possess certain fibration properties leading to the definition of a “discrete double fibration.” First the properties:

2.10. LEMMA. *Let  $F: \mathbb{B}^{op} \rightarrow \mathbf{Span}$  denote a lax double functor. The projection functors  $\Pi_0$  and  $\Pi_1$  underlying the canonical projection  $\Pi: \mathbb{E}\mathbf{l}(F) \rightarrow \mathbb{B}$  are discrete fibrations.*

PROOF. Here are the required lifts. Given  $(D, x)$  and  $f: C \rightarrow D$ , the lift is

$$f: (C, f^*x) \rightarrow (D, x)$$

where  $f^*: FD \rightarrow FC$  is the transition function. Similarly, given  $(n, s)$  for a proarrow  $n$  and  $s \in Fn$  and a cell  $\alpha: m \Rightarrow n$ , the cartesian cell above  $\alpha$  with codomain  $(n, s)$  is

$$\begin{array}{ccc} (A, f^*x) & \xrightarrow{(m, \alpha^*s)} & (B, g^*y) \\ f \downarrow & \alpha & \downarrow g \\ (C, x) & \xrightarrow{(n, s)} & (D, y) \end{array}$$

where  $\alpha^*: Fn \rightarrow Fm$  is the transition function between the vertices of the spans. Notice that by the fact that  $\alpha^*$  induces a morphism of spans, the source and target of  $(m, \alpha^*s)$  are well-defined. ■

Such functors are equivalently characterized with a distinctly double categorical flavor.

2.11. PROPOSITION. *A double functor  $P: \mathbb{E} \rightarrow \mathbb{B}$  between strict double categories for which  $P_0$  and  $P_1$  are discrete fibrations is equivalently a category object in  $\mathbf{DFib}$ , and thus equivalently a monoid in the bicategory of spans in  $\mathbf{DFib}$ .*

PROOF. In the first place  $\mathbf{DFib}$  has strict pullbacks, so the statement itself makes sense. On the one hand, a category object in any subcategory of an arrow category closed under finite limits is an internal functor. Thus, a category in  $\mathbf{DFib}$  is a double functor. Its object and arrow parts must be objects of  $\mathbf{DFib}$ , that is, discrete fibrations. On the other hand, a double functor is a category object in the arrow category of  $\mathbf{Cat}$ . But if  $P_0$  and  $P_1$  are discrete fibrations, then such  $P$  lives in  $\mathbf{Cat}(\mathbf{DFib})$ . A monoid in spans in categories is equivalently a double category; similarly a monoid in spans in the arrow category of  $\mathbf{Cat}$  is equivalently a double functor. Thus, the components of such a double functor are discrete fibrations if and only if the double functor is in fact a monoid in spans in  $\mathbf{DFib}$ . ■

2.12. DEFINITION. A **discrete double fibration** is a category object in **DFib**. A morphism of discrete double fibrations is a pair of double functors  $(H, K)$  making a commutative square. Let **DDFib** denote the category of discrete double fibrations. Let **DFib**( $\mathbb{B}$ ) denote the full subcategory of discrete double fibrations with fixed target  $\mathbb{B}$  and morphisms with  $K = 1_{\mathbb{B}}$ . This is equivalently the fiber of the codomain projection  $\text{cod}: \mathbf{DDFib} \rightarrow \mathbf{Dbl}$  over  $\mathbb{B}$ .

2.13. REMARK. Although the official definition has a succinct and philosophically appropriate phrasing, as it is convenient, any of the equivalent characterizations of discrete double fibrations in Proposition 2.11 may be used throughout.

2.14. REMARK. The name used is “discrete double fibration” because this is a discretization of a more general concept of “double fibration.” This will be a double functor  $P: \mathbb{E} \rightarrow \mathbb{B}$  whose underlying functors  $P_0$  and  $P_1$  are fibrations but that satisfy some further compatibility conditions.

2.15. REMARK. The results so far means that by fiat the elements functor is valued in discrete double fibrations

$$\mathbf{El}(-): \mathbf{Lax}(\mathbb{B}^{op}, \mathbf{Span}) \longrightarrow \mathbf{DFib}(\mathbb{B})$$

over  $\mathbb{B}$ . The purpose of this section is now to exhibit the pseudo-inverse yielding an equivalence of categories. First some general results.

2.16. EXAMPLE. [Domain Projection] For any double category, the domain projection functor

$$\text{dom}: \mathbb{B}/X \longrightarrow \mathbb{B}$$

from the double slice  $\mathbb{B}/X$  is a discrete double fibration. This is the image under the elements functor  $\mathbf{El}(-): \mathbf{Lax}(\mathbb{B}^{op}, \mathbf{Span}) \rightarrow \mathbf{DFib}(\mathbb{B})$  of the canonical representable functor on  $X$ .

2.17. PROPOSITION. A double functor  $P: \mathbb{E} \rightarrow \mathbb{B}$  is a discrete double fibration over a strict double category  $\mathbb{B}$  if, and only if, the transpose square

$$\begin{array}{ccc} \mathbb{E}_1^\dagger & \xrightarrow{\text{cod}} & \mathbb{E}_0^\dagger \\ P_1^\dagger \downarrow & & \downarrow P_0^\dagger \\ \mathbb{B}_1^\dagger & \xrightarrow{\text{cod}} & \mathbb{B}_0^\dagger \end{array}$$

is a pullback in **Cat**.

PROOF. Straightforward verification. ■

2.18. **REMARK.** Lemma 2.17 is the analogue of the characterization of ordinary discrete fibrations, saying that a functor  $F: \mathcal{F} \rightarrow \mathcal{C}$  is a discrete fibration if, and only if, the square

$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{d_1} & \mathcal{F}_0 \\ F_1 \downarrow & & \downarrow F_0 \\ \mathcal{C}_1 & \xrightarrow{d_1} & \mathcal{C}_0 \end{array}$$

is a pullback in **Set**. This characterization will be important in the monadicity developments in forthcoming work.

The next goal is to construct a pseudo-inverse for the elements functor. This will be a functor from discrete double fibrations back to lax span-valued double functors. For a discrete double fibration  $P: \mathbb{E} \rightarrow \mathbb{B}$ , begin correspondences leading to a lax double functor  $F_P: \mathbb{B}^{op} \rightarrow \mathbf{Span}$  in the following way.

2.19. **CONSTRUCTION.** [Transition Morphisms] On objects, take  $B \mapsto \mathbb{E}_B$ , the set of objects of  $\mathbb{E}_0$  over  $B \in \mathbb{B}_0$  via  $P_0$ . Call this the “fiber” over  $B$ . Since  $P_0$  is a discrete fibration, for every horizontal morphism  $f: B \rightarrow C$ , there is a corresponding transition function  $f^*: \mathbb{E}_C \rightarrow \mathbb{E}_B$  given on  $x \in \mathbb{E}_C$  by taking the domain  $f^*x$  of the unique horizontal morphism over  $f$  with codomain  $x$ . Given a proarrow  $m: A \twoheadrightarrow B$ , required is a span from  $\mathbb{E}_A$  to  $\mathbb{E}_B$ . For this, let  $\mathbb{E}_m$  denote the fiber of  $P_1$  over  $m$ , and send  $m$  to the span

$$\mathbb{E}_A \xleftarrow{\text{src}} \mathbb{E}_m \xrightarrow{\text{tgt}} \mathbb{E}_B.$$

Finally, associated to a cell  $\alpha$  of  $\mathbb{B}$  of the form

$$\begin{array}{ccc} A & \xrightarrow{m} & B \\ f \downarrow & \alpha & \downarrow g \\ C & \xrightarrow{n} & D \end{array}$$

will be a cell of **Span**, that is, a morphism of the spans associated to  $m$  and  $n$ . In light of the definitions so far, required is a function  $\mathbb{E}_n \rightarrow \mathbb{E}_m$  making the diagram

$$\begin{array}{ccccc} \mathbb{E}_C & \longleftarrow & \mathbb{E}_n & \longrightarrow & \mathbb{E}_D \\ \downarrow & & \downarrow \alpha^* & & \downarrow \\ \mathbb{E}_A & \longleftarrow & \mathbb{E}_m & \longrightarrow & \mathbb{E}_B \end{array}$$

commute. But such a function  $\mathbb{E}_n \rightarrow \mathbb{E}_m$  is given by the fact that  $P_1$  is a discrete fibration. That is, given  $u \in \mathbb{E}_n$ , there is a unique cell of  $\mathbb{E}$  with codomain  $u$  over  $\alpha$ , which, by the fact that  $P_0$  is a discrete fibration, must be of the form

$$\begin{array}{ccc} f^*x & \twoheadrightarrow & g^*y \\ \downarrow & \Downarrow & \downarrow \\ x & \xrightarrow{u} & y \end{array}$$

with source and target the unique lifts over  $f$  and  $g$  respectively. Thus, send  $u$  to the vertical source of this lifted cell, which, by construction, is over  $m$ , hence well-defined. The cell diagram in **Span** displayed above then commutes by construction of  $\mathbb{E}_n \rightarrow \mathbb{E}_m$ .

2.20. CONSTRUCTION. [Laxity Cells I] Let  $D$  denote an object of  $\mathbb{B}$ . The laxity cell for the corresponding unit proarrow is of the form

$$\begin{array}{ccccc} \mathbb{E}_D & \xleftarrow{1} & \mathbb{E}_D & \xrightarrow{1} & \mathbb{E}_D \\ \parallel & & \downarrow & & \parallel \\ \mathbb{E}_D & \xleftarrow{\text{src}} & \mathbb{E}_{u_D} & \xrightarrow{\text{tgt}} & \mathbb{E}_D \end{array}$$

The functor between vertices  $\phi_D: \mathbb{E}_D \rightarrow \mathbb{E}_{u_D}$  is given by  $x \mapsto u_x$ . Notice that the unit condition of Definition 2.1 is satisfied because the external composition for  $\mathbb{E}$  is strict.

2.21. CONSTRUCTION. [Laxity Cells II] Let  $m: A \rightarrow B$  and  $n: B \rightarrow C$  denote proarrows of  $\mathbb{B}$ . Required for  $F_P$  are laxity cells for such  $m$  and  $n$  of the form

$$\begin{array}{ccccc} \mathbb{E}_A & \xleftarrow{\text{src}} & \mathbb{E}_m & \xrightarrow{\text{tgt}} & \mathbb{E}_B & \xleftarrow{\text{src}} & \mathbb{E}_n & \xrightarrow{\text{tgt}} & \mathbb{E}_C \\ \parallel & & & & \phi_{n,m} & & & & \parallel \\ \mathbb{E}_A & \xleftarrow{\text{src}} & \mathbb{E}_{n \otimes m} & \xrightarrow{\text{tgt}} & \mathbb{E}_C \end{array}$$

between spans. The composed span in the domain of the cell has vertex given by the pullback. The cell amounts to a functor between vertices respecting the source and target functors. This is given by external composition of proarrows and cells over  $m$  and  $n$  respectively:

$$\begin{array}{ccc} \mathbb{E}_m \times_{\mathbb{E}_B} \mathbb{E}_n & & u: x \rightarrow y, v: y \rightarrow z \\ \phi_{n,m} \downarrow & & \downarrow \\ \mathbb{E}_{n \otimes m} & & v \otimes u \end{array}$$

This is strictly functorial and a well-defined span morphism by the assumptions on  $P: \mathbb{E} \rightarrow \mathbb{B}$ . The laxity coherence law follows from the fact that external composition for  $\mathbb{E}$  is associative.

2.22. LEMMA. Let  $P: \mathbb{E} \rightarrow \mathbb{B}$  denote a discrete double fibration. The assignments

$$D \mapsto \mathbb{E}_D \quad f \mapsto f^* \quad m \mapsto \mathbb{E}_m \quad \alpha \mapsto \alpha^*$$

with unit and laxity cells as above define a lax functor  $F_P: \mathbb{B}^{op} \rightarrow \mathbf{Span}$ .

PROOF. What remains to check is the naturality conditions in Definition 2.1. Take cells  $\alpha: m \Rightarrow p$  and  $\beta: n \Rightarrow q$  of  $\mathbb{B}$ . The first condition amounts to the commutativity of the square made by the functors between vertices:

$$\begin{array}{ccc} \mathbb{E}_p \times_{\mathbb{E}_N} \mathbb{E}_q & \xrightarrow{\phi} & \mathbb{E}_{q \otimes p} \\ \alpha^* \times \beta^* \downarrow & & \downarrow (\beta \otimes \alpha)^* \\ \mathbb{E}_m \times_{\mathbb{E}_B} \mathbb{E}_n & \xrightarrow{\phi} & \mathbb{E}_{n \otimes m} \end{array}$$

But this follows by uniqueness assumptions. Naturality for the unit cells follows similarly. ■

2.23. CONSTRUCTION. [Pseudo-Inverse on Morphisms of Discrete Double Fibrations] Start with a morphism  $H: P \rightarrow B$  of discrete double fibrations  $P: \mathbb{E} \rightarrow \mathbb{B}$  and  $Q: \mathbb{G} \rightarrow \mathbb{B}$ . Define correspondences for what will be a transformation  $F_H: F_P \rightarrow F_Q$  between lax double functors  $F_P$  and  $F_Q$  arising as in Lemma 2.22. On objects  $B$ , define  $(F_H)_B$  to be the restriction of  $H$  to the fibers  $H_0: \mathbb{E}_B \rightarrow \mathbb{G}_B$ . This is well defined since  $QH = P$  holds. Similarly, for a proarrow  $m: B \rightarrow C$ , take the component proarrow  $(F_H)_m$  in **Span** to be the span

$$\begin{array}{ccccc} \mathbb{E}_B & \longleftarrow & \mathbb{E}_m & \longrightarrow & \mathbb{E}_C \\ H_0 \downarrow & & \downarrow H_1 & & \downarrow H_0 \\ \mathbb{G}_B & \longleftarrow & \mathbb{G}_m & \longrightarrow & \mathbb{G}_C \end{array}$$

Again this is well-defined since  $QH = P$  holds.

2.24. PROPOSITION. *The assignments in Construction 2.23 make a transformation  $F_H: F_P \rightarrow F_Q$  of lax functors as in Definition 2.4. These assignments result in a functor*

$$F_{(-)}: \mathbf{Lax}(\mathbb{B}^{op}, \mathbf{Span}) \longrightarrow \mathbf{DFib}(\mathbb{B}).$$

PROOF. The first naturality condition holds because  $H$  commutes with  $P$  and  $Q$  and because  $Q_0$  is a discrete fibration; the second naturality condition holds again because  $QH = P$  is true and because  $Q_1$  is also a discrete fibration. Proarrow functoriality follows from the fact that  $H$  is a strict double functor. ■

The pseudo-inverse construction of the last subsection in fact induces an equivalence of categories, leading to the first representation theorem.

2.25. CONSTRUCTION. To define a natural isomorphism  $\eta: 1 \cong F_{\mathbb{E}l(-)}$  between functors, needed are component transformations of lax functors indexed by lax  $H: \mathbb{B}^{op} \rightarrow \mathbf{Span}$ . For such a lax functor  $H$ , the required transformation of lax functors  $\eta_H: H \rightarrow F_{\mathbb{E}l(H)}$  is given by

$$\eta_{H,A}: HA \longrightarrow \mathbb{E}l(H)_A \quad X \mapsto (A, x)$$

on objects  $A \in |\mathbb{B}_0|$  and by

$$\eta_{H,m}: Hm \longrightarrow \mathbb{E}l(F)_m \quad s \mapsto (m, s)$$

for a proarrow  $m: B \rightarrow C$ , giving a morphism of spans

$$\begin{array}{ccccc} HB & \longleftarrow & Hm & \longrightarrow & HC \\ \eta \downarrow & & \downarrow \eta & & \downarrow \eta \\ \mathbb{E}\mathbf{I}(H)_B & \longleftarrow & \mathbb{E}\mathbf{I}(H)_m & \longrightarrow & \mathbb{E}\mathbf{I}(H)_C. \end{array}$$

Since the maps  $\eta_{H,A}$  and  $\eta_{H,v}$  just add in indices, they are both bijections of sets. Moreover, these components result in a transformation of lax functors as in Definition 2.4 by construction. Thus, such  $\eta_H$  is the vertical component of a supposed transformation of ordinary functors. Naturality in  $H$  is proved in the theorem below.

2.26. CONSTRUCTION. A natural isomorphism  $\epsilon: \mathbb{E}\mathbf{I}(F_{(-)}) \cong 1$  has components  $\epsilon_P: \mathbb{E}\mathbf{I}(F_P) \rightarrow \mathbb{E}$  indexed by discrete double fibrations  $P: \mathbb{E} \rightarrow \mathbb{B}$ . On objects and arrows take

$$(D, x) \mapsto x \quad (C, x) \xrightarrow{f} (D, y) \mapsto f^*y \xrightarrow{!} y$$

where  $f^*y \rightarrow y$  is the unique arrow above  $f$  with codomain  $y$ . It is well-defined because  $f^*y = x$  holds. These are bijections by uniqueness of these lifts; they are functorial and respect the fibering over  $\mathbb{B}$ . On proarrows and cells, take

$$(m, s) \mapsto s \quad \alpha \mapsto \alpha^*s \Rightarrow s$$

where  $\alpha^*s \Rightarrow s$  is the unique lift in  $\mathbb{E}$  of  $\alpha$  with codomain  $s$ . Again these are bijections, functorial and fiber-respecting. Naturality in  $P$  is proved in the following result.

2.27. THEOREM. *For any strict double category  $\mathbb{B}$ , there is an equivalence of categories*

$$\mathbf{DFib}(\mathbb{B}) \simeq \mathbf{Lax}(\mathbb{B}^{op}, \mathbf{Span})$$

*induced by the double category of elements construction.*

PROOF. It remains to check the naturality of the isomorphisms in Constructions 2.25 and 2.26. If  $H$  and  $G$  are lax functors with a transformation  $\tau: H \rightarrow G$ , the diagram

$$\begin{array}{ccc} H & \xrightarrow{\eta} & F_{\mathbb{E}\mathbf{I}(H)} \\ \tau \downarrow & & \downarrow \\ G & \xrightarrow{\eta} & F_{\mathbb{E}\mathbf{I}(G)} \end{array}$$

commutes because indexing commutes with applying the components of  $\tau$ . Naturality in  $P$  also follows. Let  $H: P \rightarrow Q$  be a morphism of double fibrations  $P: \mathbb{E} \rightarrow \mathbb{B}$  and  $Q: \mathbb{G} \rightarrow \mathbb{B}$ . For naturality, it is required that the square

$$\begin{array}{ccc} \mathbb{E}\mathbf{I}(F_P) & \xrightarrow{\epsilon} & \mathbb{E} \\ \mathbb{E}\mathbf{I}(F_H) \downarrow & & \downarrow H \\ \mathbb{E}\mathbf{I}(F_Q) & \xrightarrow{\epsilon} & \mathbb{G} \end{array}$$

commutes. Chasing an object  $(D, x)$  or a proarrow  $(m, s)$  around each side of the square, the result either way is  $Hx$  in the former case and is  $hs$  in the latter case. Thus, to check are the arrows and cells. Given an arrow  $f: (C, x) \rightarrow (D, y)$ , the counter-clockwise direction gives the unique lift  $!: f^*Hy \rightarrow Hy$  whereas the clockwise direction gives the image of one  $H(!): Hf^*y \rightarrow Hy$ . These are strictly equal, however, by uniqueness of lifts. A formally similar argument works to show that the square commutes also at the level of cells. ■

2.28. **REMARK.** The primordial examples of §§2.5-2.6 are thus special cases of the theorem.

### 3. Monoids, Modules and Multimodulations

The main representation theorem of the paper asserts an equivalence of certain virtual double categories. To define these first recall the definition, presented here in the form of [Cruttwell & Shulman, 2010]. Virtual double categories have been known under the name “f.c.-multicategories,” that is, as an example of a “generalized multicategory,” in this case relative to the free-category monad in [Leinster, 2004].

3.1. **DEFINITION.** [See §2.1 of [Cruttwell & Shulman, 2010]] A **virtual double category**  $\mathbb{D}$  consists of an underlying category  $\mathbb{D}_0$ , giving the objects and morphisms of  $\mathbb{D}$ , together with, for any two objects  $C$  and  $D$ , a class of proarrows  $v: C \dashrightarrow D$  and for any “arity”  $k$  multicells

$$\begin{array}{ccccc} A_0 & \xrightarrow{m_1} & A_1 & \xrightarrow{m_2} & \cdots & \xrightarrow{m_k} & A_k \\ f \downarrow & & & \mu & & & \downarrow g \\ B_0 & \xrightarrow{\quad} & & \dashrightarrow & & & B_1 \end{array}$$

all subject to the unit, composition and associativity axioms, as detailed in the reference. The list of proarrows  $(m_1, m_2, \dots, m_k)$  is a “ $k$ -ary multisource.” The definition allows nullary multisources with  $k = 0$ . A **functor** of virtual double categories  $F: \mathbb{C} \rightarrow \mathbb{D}$  sends objects to objects, arrows to arrows, proarrows to proarrows and multicells to multicells in such a way as to preserve domains, codomains, multisources (so arities in particular), targets, identities and compositions.

3.2. **EXAMPLE.** Every double category is a virtual double category by forgetting external composition of proarrows and cells. In particular, for  $\mathcal{C}$  with pullbacks  $\mathbf{Span}(\mathcal{C})$  is a virtual double category and given a span

$$\mathbb{X} = (X_0 \xleftarrow{d_0} X_1 \xrightarrow{d_1} X_0)$$

the projection

$$\mathbf{Span}(\mathcal{C}) \longrightarrow \mathbf{Span}(\mathcal{C})/\mathbb{X}$$

given by taking the product is a functor of virtual double categories.



3.3. DEFINITION. [Cf. §5.1 [Cruttwell & Shulman, 2010]] A multicell

$$\begin{array}{c} \cdot \xrightarrow{m_1} \cdot \xrightarrow{m_2} \cdot \dots \xrightarrow{m_k} \cdot \\ \parallel \qquad \qquad \qquad \parallel \\ \mu \\ \parallel \qquad \qquad \qquad \parallel \\ \cdot \xrightarrow{n} \cdot \end{array}$$

in a virtual double category  $\mathbb{D}$  is **opcartesian** if any cell of the form

$$\begin{array}{c} \cdot \xrightarrow{p_1} \cdot \xrightarrow{p_2} \cdot \dots \xrightarrow{p_s} \cdot \xrightarrow{m_1} \cdot \xrightarrow{m_2} \cdot \dots \xrightarrow{m_k} \cdot \xrightarrow{q_1} \cdot \dots \xrightarrow{q_t} \cdot \\ f \downarrow \qquad \qquad \qquad \rho \qquad \qquad \qquad \downarrow g \\ \cdot \xrightarrow{n} \cdot \end{array}$$

factors through  $\mu$  via a unique multicell  $\gamma$  in the sense that  $\rho$  is equal to

$$\begin{array}{c} \cdot \xrightarrow{p_1} \cdot \xrightarrow{p_2} \cdot \dots \xrightarrow{p_s} \cdot \xrightarrow{m_1} \cdot \xrightarrow{m_2} \cdot \dots \xrightarrow{m_k} \cdot \xrightarrow{q_1} \cdot \dots \xrightarrow{q_t} \cdot \\ \parallel \qquad \parallel \qquad \dots \parallel \qquad \qquad \qquad \mu \qquad \qquad \qquad \parallel \qquad \parallel \qquad \parallel \\ \cdot \xrightarrow{p_1} \cdot \xrightarrow{p_2} \cdot \dots \xrightarrow{p_s} \cdot \xrightarrow{n} \cdot \xrightarrow{q_1} \cdot \dots \xrightarrow{q_t} \cdot \\ f \downarrow \qquad \qquad \qquad \gamma \qquad \qquad \qquad \downarrow g \\ \cdot \xrightarrow{n} \cdot \end{array}$$

A **unit proarrow** for an object  $D \in \mathbb{D}$  in a virtual double category is a proarrow  $u_D: D \rightarrow D$  and a nullary opcartesian cell

$$\begin{array}{c} D \xrightarrow{=} D \\ \parallel \qquad \qquad \qquad \parallel \\ \downarrow \\ D \xrightarrow{u_D} D. \end{array}$$

A virtual double category **has units** if it is equipped with a choice of unit for each object.

3.4. REMARK. The unit proarrow of any double category is a unit in this sense. Existence of units is one of the requirements for a *virtual equipment* as described in [Cruttwell & Shulman, 2010]. The full structure will not be needed here. All the interesting examples possess units, so without further ado all virtual double categories will be assumed to have units. Functors of virtual double categories will be assumed to be **normal** in the sense that they preserve nullary opcartesian cells.

3.5. REMARK. The universal property of any unit  $u_D$  on a given object  $D$  implies that  $\mathbb{D}$  possesses generic multicells

$$\begin{array}{c} \cdot \xrightarrow{u_D} \cdot \xrightarrow{u_D} \cdot \dots \xrightarrow{u_D} \cdot \\ \parallel \qquad \qquad \qquad \parallel \\ \downarrow \\ \parallel \qquad \qquad \qquad \parallel \\ \cdot \xrightarrow{u_D} \cdot \end{array}$$

of any arity  $k$ . These are given from the unique factorization. Moreover, by uniqueness of these factorizations, a correct multicomposite of any such cells gives the generic multicell of the proper arity determined in this way.

A notion of transformation gives the 2-categorical structure on virtual double categories.

**3.6. DEFINITION.** Let  $F, G: \mathbb{C} \rightrightarrows \mathbb{D}$  denote functors of virtual double categories. A **transformation**  $\tau: F \rightarrow G$  assigns to each object  $C$  of  $\mathbb{C}$  an arrow  $\tau_C: FC \rightarrow GC$  and to each proarrow  $m: C \rightrightarrows D$  a cell

$$\begin{array}{ccc} FC & \xrightarrow{Fm} & FD \\ \tau_C \downarrow & \tau_m & \downarrow \tau_D \\ GC & \xrightarrow{Gm} & GD \end{array}$$

in such a way that

1. [Arrow Naturality] for each arrow  $f: C \rightarrow D$ , the square

$$\begin{array}{ccc} FC & \xrightarrow{Ff} & FD \\ \tau_C \downarrow & & \downarrow \tau_D \\ GC & \xrightarrow{Gf} & GD \end{array}$$

commutes; and

2. [Cell Naturality] for each multicell

$$\begin{array}{ccc} \cdot & \xrightarrow{m_1} \cdot & \xrightarrow{m_2} \cdot & \dots & \xrightarrow{m_k} \cdot & \cdot \\ f \downarrow & & \mu & & & \downarrow g \\ \cdot & \xrightarrow{\quad} \cdot & \xrightarrow{\quad} \cdot & \dots & \xrightarrow{\quad} \cdot & \cdot \\ & & n & & & \end{array}$$

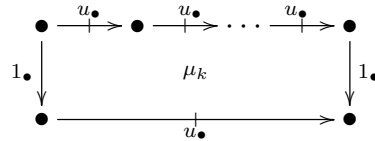
the composed multicells on either side of

$$\begin{array}{ccc} \cdot & \xrightarrow{Fm_1} \cdot & \xrightarrow{Fm_2} \cdot & \dots & \xrightarrow{Fm_k} \cdot & \cdot \\ \downarrow \tau_{m_1} & \downarrow \tau_{m_2} & \downarrow \tau_{m_k} & & \downarrow & \downarrow \\ \cdot & \xrightarrow{Gm_1} \cdot & \xrightarrow{Gm_2} \cdot & \dots & \xrightarrow{Gm_k} \cdot & \cdot \\ & & G\mu & & & \\ \downarrow & & & & & \downarrow \\ \cdot & \xrightarrow{\quad} \cdot & \xrightarrow{\quad} \cdot & \dots & \xrightarrow{\quad} \cdot & \cdot \\ & & Gn & & & \end{array} = \begin{array}{ccc} \cdot & \xrightarrow{Fm_1} \cdot & \xrightarrow{Fm_2} \cdot & \dots & \xrightarrow{Fm_k} \cdot & \cdot \\ \downarrow & & F\mu & & \downarrow & \downarrow \\ \cdot & \xrightarrow{\quad} \cdot & \xrightarrow{Fn} \cdot & \dots & \xrightarrow{\quad} \cdot & \cdot \\ \downarrow & & \tau_n & & \downarrow & \downarrow \\ \cdot & \xrightarrow{\quad} \cdot & \xrightarrow{\quad} \cdot & \dots & \xrightarrow{\quad} \cdot & \cdot \\ & & Gn & & & \end{array}$$

are equal.

Denote the 2-category of virtual double categories, their functors and transformations by **vDbI**.

3.7. EXAMPLE. [Terminal Object] The terminal object  $\mathbf{1}$  in  $\mathbf{vDbl}$  is peculiar and is needed later. It has a single object  $\bullet$ , an identity arrow  $1_\bullet$  and an “identity proarrow”  $u_\bullet$ . Whereas one might expect there to be only a single multicell  $u_\bullet \Rightarrow u_\bullet$ , in fact required are generic multicells with multisources of all arities  $k$



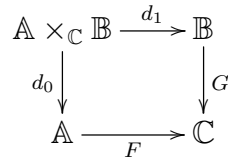
as otherwise the natural definitions on objects, arrows and proarrows will not extend to a unique functor of virtual double categories  $\mathbb{D} \rightarrow \mathbf{1}$ . Multi-composition is defined to give the generic multicell with the appropriate arity. Notice, then, that in particular  $\mathbf{1}$  has units by fiat. A **point** of a virtual double category is a (normalized) functor  $D: \mathbf{1} \rightarrow \mathbb{D}$ . A point thus consists of an object  $D$ , its identity arrow  $1_D$ , its unit proarrow  $u_D$  and the corresponding generic multicells.

3.8. PROPOSITION.  $\mathbf{vDbl}$  has (strict 2-)pullbacks.

PROOF. Given two functors  $F: \mathbb{A} \rightarrow \mathbb{C}$  and  $G: \mathbb{B} \rightarrow \mathbb{C}$ , the pullback has as its underlying category

$$(\mathbb{A} \times_{\mathbb{C}} \mathbb{B})_0 = \mathbb{A}_0 \times_{\mathbb{B}_0} \mathbb{C}_0.$$

Proarrows are pairs  $(m, n)$  for a proarrow  $m$  of  $\mathbb{A}$  and one  $n$  of  $\mathbb{B}$  satisfying  $Fm = Gn$ . Similarly, multicells are pairs  $(\mu, \nu)$  with  $\mu$  in  $\mathbb{A}$  and  $\nu$  in  $\mathbb{B}$  such that  $F\mu = G\nu$ . Composition uses composition in  $\mathbb{A}$  and  $\mathbb{B}$ . This is a virtual double category fitting into a commutative square

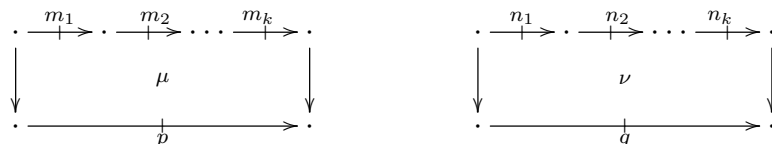


with the expected 2-categorical universal property [Street, 1976]. ■

Further limits of a 2-categorical variety abound in  $\mathbf{vDbl}$ . Comma categories are well-known in  $\mathbf{Cat}$  (e.g. §I.6 [MacLane, 1998]). The formal abstraction and elementary phrasing of its universal property in an arbitrary 2-category appear in §1 of [Street, 1974].

3.9. PROPOSITION. The 2-category  $\mathbf{vDbl}$  has comma objects.

PROOF. Given functors  $F: \mathbb{A} \rightarrow \mathbb{C}$  and  $G: \mathbb{B} \rightarrow \mathbb{C}$ , the (purported) comma  $F/G$  has its underlying 1-category as  $(F/G)_0 = F_0/G_0$ . Proarrows are triples  $(m, \alpha, n)$  with  $m$  and  $n$  proarrows of  $\mathbb{A}$  and  $\mathbb{B}$ , respectively, and  $\alpha$  a cell  $\alpha: Fm \Rightarrow Gn$  of  $\mathbb{C}$ . A multicell is thus a pair of multicells



from  $\mathbb{A}$  and  $\mathbb{B}$ , respectively, of the same arity and satisfying the equation

$$\begin{array}{ccc}
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} & \begin{array}{c} \xrightarrow{Fm_1} \\ \xrightarrow{Fm_2} \\ \dots \\ \xrightarrow{Fm_k} \end{array} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \\
 \downarrow \alpha_1 & \downarrow \alpha_2 & \downarrow \alpha_k \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} & \begin{array}{c} \xrightarrow{Gn_1} \\ \xrightarrow{Gn_2} \\ \dots \\ \xrightarrow{Gn_k} \end{array} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \\
 \downarrow & \downarrow G\nu & \downarrow \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{Gq} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array}
 \end{array} = \begin{array}{ccc}
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} & \begin{array}{c} \xrightarrow{Fm_1} \\ \xrightarrow{Fm_2} \\ \dots \\ \xrightarrow{Fm_k} \end{array} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \\
 \downarrow & \downarrow F\mu & \downarrow \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{Fp} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \\
 \downarrow & \downarrow \beta & \downarrow \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{Gq} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array}
 \end{array}$$

Composition is given by that in  $\mathbb{A}$  and  $\mathbb{B}$ . So defined,  $F/G$  comes with evident projection functors to  $\mathbb{A}$  and  $\mathbb{B}$ . The expected transformation

$$\begin{array}{ccc}
 F/G & \xrightarrow{d_1} & \mathbb{B} \\
 d_0 \downarrow & \Rightarrow & \downarrow G \\
 \mathbb{A} & \xrightarrow{F} & \mathbb{C}
 \end{array}$$

has components  $f$  for objects  $(A, f, B)$  and  $\alpha$  for proarrows  $(m, \alpha, n)$ . It satisfies the required 2-categorical universal property in the reference by construction. ■

3.10. REMARK. Thus  $\mathbf{vDbl}$  is a “representable 2-category” [Gray, 1974], [Street, 1974].

3.11. EXAMPLE. [Slice Virtual Double Category] Let  $D$  denote an object in a virtual double category  $\mathbb{D}$ . The **slice** virtual double category over  $D$  is defined as the comma  $1/D$

$$\begin{array}{ccc}
 1/D & \xrightarrow{d_1} & \mathbf{1} \\
 \downarrow & \Rightarrow & \downarrow D \\
 \mathbb{D} & \xrightarrow{1} & \mathbb{D}
 \end{array}$$

Denote this as usual by  $\mathbb{D}/D$ . The coslice is defined analogously.

The virtual double category structure on  $\mathbf{Lax}(\mathbb{B}^{op}, \mathbf{Span})$  will be given by taking so-called “modules” as proarrows and “multimodulations” as the multicells. Here we revisit the the definitions for lax functors between arbitrary double categories. A path of proarrows is a sequence  $\mathbf{m} = (m_1, \dots, m_k)$  such that, reading left to right, the target of one proarrow is the source of the next. The external composite is denoted by  $[\mathbf{m}]$ .

3.12. DEFINITION. [Cf. §3.2 [Paré, 2011]] A **module** between lax functors  $M: F \rightarrow G: \mathbb{A} \rightrightarrows \mathbb{B}$  of double categories assigns

1. to each proarrow  $m: A \rightarrow B$  of  $\mathbb{A}$ , a proarrow  $Mm: FA \rightarrow GB$  of  $\mathbb{B}$ ;

2. to each cell of  $\mathbb{A}$  as at left, one of  $\mathbb{B}$  as at right

$$\begin{array}{ccc}
 A & \xrightarrow{m} & B \\
 f \downarrow & \theta & \downarrow g \\
 C & \xrightarrow{n} & D
 \end{array}
 \qquad
 \begin{array}{ccc}
 FA & \xrightarrow{Mm} & GB \\
 Ff \downarrow & M\theta & \downarrow Gg \\
 FC & \xrightarrow{Mn} & GD
 \end{array}$$

3. for each pair of proarrows  $m: A \rightrightarrows B$  and  $n: B \rightrightarrows C$ , action multicells of  $\mathbb{B}$

$$\begin{array}{ccc}
 \cdot & \xrightarrow{Fm} & \cdot & \xrightarrow{Mn} & \cdot \\
 \parallel & & \lambda_{m,n} & & \parallel \\
 \cdot & \xrightarrow{M(n \otimes m)} & \cdot & & \cdot
 \end{array}
 \qquad
 \begin{array}{ccc}
 \cdot & \xrightarrow{Mm} & \cdot & \xrightarrow{Gn} & \cdot \\
 \parallel & & \rho_{m,n} & & \parallel \\
 \cdot & \xrightarrow{M(n \otimes m)} & \cdot & & \cdot
 \end{array}$$

in such a way that the following axioms hold.

1. [Functoriality] For any (internal) composite of monocells  $\beta\alpha$  and any proarrow  $m$ , the equations  $M\beta M\alpha = M(\beta\alpha)$  and  $M1_m = 1_{Mm}$  hold;
2. [Naturality] For any external composite  $\beta \otimes \alpha$ , the equalities

$$\begin{array}{ccc}
 \cdot & \xrightarrow{Fm} & \cdot & \xrightarrow{Mn} & \cdot \\
 \parallel & & \lambda & & \parallel \\
 \cdot & \xrightarrow{M(\beta \otimes \alpha)} & \cdot & & \cdot \\
 \downarrow & & \downarrow & & \downarrow \\
 \cdot & \xrightarrow{M(q \otimes p)} & \cdot & & \cdot \\
 \parallel & & \lambda & & \parallel \\
 \cdot & \xrightarrow{M(q \otimes p)} & \cdot & & \cdot
 \end{array}
 =
 \begin{array}{ccc}
 \cdot & \xrightarrow{Fm} & \cdot & \xrightarrow{Mn} & \cdot \\
 \downarrow & F\alpha & \downarrow & M\beta & \downarrow \\
 \cdot & \xrightarrow{M(\beta \otimes \alpha)} & \cdot & & \cdot \\
 \parallel & & \lambda & & \parallel \\
 \cdot & \xrightarrow{M(q \otimes p)} & \cdot & & \cdot
 \end{array}$$

and

$$\begin{array}{ccc}
 \cdot & \xrightarrow{Mm} & \cdot & \xrightarrow{Gn} & \cdot \\
 \parallel & & \rho & & \parallel \\
 \cdot & \xrightarrow{M(\beta \otimes \alpha)} & \cdot & & \cdot \\
 \downarrow & & \downarrow & & \downarrow \\
 \cdot & \xrightarrow{M(q \otimes p)} & \cdot & & \cdot \\
 \parallel & & \rho & & \parallel \\
 \cdot & \xrightarrow{M(q \otimes p)} & \cdot & & \cdot
 \end{array}
 =
 \begin{array}{ccc}
 \cdot & \xrightarrow{Mm} & \cdot & \xrightarrow{Gn} & \cdot \\
 \downarrow & M\alpha & \downarrow & G\beta & \downarrow \\
 \cdot & \xrightarrow{M(\beta \otimes \alpha)} & \cdot & & \cdot \\
 \parallel & & \rho & & \parallel \\
 \cdot & \xrightarrow{M(q \otimes p)} & \cdot & & \cdot
 \end{array}$$

both hold.

3. [Associativity] For any composable sequence of proarrows  $(m, n, p)$  of  $\mathbb{A}$ , the equalities (modulo suppressed associativity isomorphisms)

$$\begin{array}{ccc}
 \cdot & \xrightarrow{Fm} & \cdot & \xrightarrow{Fn} & \cdot & \xrightarrow{Mw} & \cdot \\
 \parallel & & 1 & & \parallel & & \lambda & & \parallel \\
 \cdot & \xrightarrow{M(p \otimes (n \otimes m))} & \cdot & & \cdot & & \cdot & & \cdot \\
 \parallel & & \lambda & & \parallel & & & & \parallel \\
 \cdot & \xrightarrow{M(p \otimes (n \otimes m))} & \cdot & & \cdot & & \cdot & & \cdot
 \end{array}
 =
 \begin{array}{ccc}
 \cdot & \xrightarrow{Fm} & \cdot & \xrightarrow{Fn} & \cdot & \xrightarrow{Mw} & \cdot \\
 \parallel & & F\gamma & & \parallel & & 1 & & \parallel \\
 \cdot & \xrightarrow{M((p \otimes n) \otimes m)} & \cdot & & \cdot & & \cdot & & \cdot \\
 \parallel & & \lambda & & \parallel & & & & \parallel \\
 \cdot & \xrightarrow{M((p \otimes n) \otimes m)} & \cdot & & \cdot & & \cdot & & \cdot
 \end{array}$$

and

$$\begin{array}{ccc}
 \begin{array}{c} \cdot \xrightarrow{Mm} \cdot \xrightarrow{Gn} \cdot \xrightarrow{Gp} \cdot \\ \parallel \quad \quad \quad \parallel \quad \quad \quad \parallel \\ \rho \quad \quad \quad 1 \\ \parallel \quad \quad \quad \parallel \\ \cdot \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \\ \parallel \quad \quad \quad \parallel \\ \rho \\ \parallel \quad \quad \quad \parallel \\ \cdot \xrightarrow{\quad} \cdot \\ M(p \otimes (n \otimes m)) \end{array} & = & \begin{array}{c} \cdot \xrightarrow{Mm} \cdot \xrightarrow{Gn} \cdot \xrightarrow{Gp} \cdot \\ \parallel \quad \quad \quad \parallel \quad \quad \quad \parallel \\ 1 \quad \quad \quad G\gamma \\ \parallel \quad \quad \quad \parallel \\ \cdot \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \\ \parallel \quad \quad \quad \parallel \\ \rho \\ \parallel \quad \quad \quad \parallel \\ \cdot \xrightarrow{\quad} \cdot \\ M((p \otimes n) \otimes m) \end{array}
 \end{array}$$

are valid.

4. [Compatibility] The composites

$$\begin{array}{ccc}
 \begin{array}{c} \cdot \xrightarrow{Fm} \cdot \xrightarrow{Mn} \cdot \xrightarrow{Gp} \cdot \\ \parallel \quad \quad \quad \parallel \quad \quad \quad \parallel \\ \lambda \quad \quad \quad 1 \\ \parallel \quad \quad \quad \parallel \\ \cdot \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \\ \parallel \quad \quad \quad \parallel \\ \rho \\ \parallel \quad \quad \quad \parallel \\ \cdot \xrightarrow{\quad} \cdot \\ M(p \otimes (n \otimes m)) \end{array} & = & \begin{array}{c} \cdot \xrightarrow{Fm} \cdot \xrightarrow{Mn} \cdot \xrightarrow{Gp} \cdot \\ \parallel \quad \quad \quad \parallel \quad \quad \quad \parallel \\ 1 \quad \quad \quad \rho \\ \parallel \quad \quad \quad \parallel \\ \cdot \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \\ \parallel \quad \quad \quad \parallel \\ \lambda \\ \parallel \quad \quad \quad \parallel \\ \cdot \xrightarrow{\quad} \cdot \\ M((p \otimes n) \otimes m) \end{array}
 \end{array}$$

are equal.

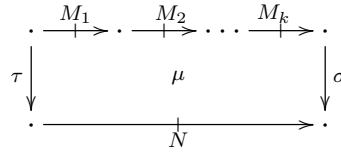
5. [Unit] For any proarrow  $m: A \rightarrow B$  of  $\mathbb{A}$ , the composites

$$\begin{array}{ccc}
 \begin{array}{c} \cdot \xrightarrow{u_{FA}} \cdot \xrightarrow{Mm} \cdot \\ \parallel \quad \quad \quad \parallel \\ \gamma \quad \quad \quad 1 \\ \parallel \quad \quad \quad \parallel \\ \cdot \xrightarrow{Fu_A} \cdot \xrightarrow{\quad} \cdot \\ \parallel \quad \quad \quad \parallel \\ \lambda \\ \parallel \quad \quad \quad \parallel \\ \cdot \xrightarrow{\quad} \cdot \\ M(m \otimes u_A) \end{array} & \text{and} & \begin{array}{c} \cdot \xrightarrow{Mm} \cdot \xrightarrow{u_{GB}} \cdot \\ \parallel \quad \quad \quad \parallel \\ 1 \quad \quad \quad \gamma \\ \parallel \quad \quad \quad \parallel \\ \cdot \xrightarrow{Gu_B} \cdot \xrightarrow{\quad} \cdot \\ \parallel \quad \quad \quad \parallel \\ \rho \\ \parallel \quad \quad \quad \parallel \\ \cdot \xrightarrow{\quad} \cdot \\ M(u_B \otimes m) \end{array}
 \end{array}$$

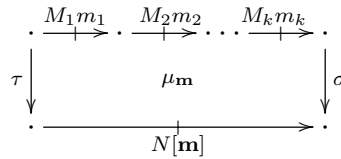
are equal to the respective canonical composition multicells for  $(m_{FA}, Mm)$  and  $(Mu, m_{GA})$ .

Multicells are given by the notion of a “multimodulation,” recalled next. A path of cells  $\theta: (\theta_1, \dots, \theta_k)$  is a sequence of cells  $\theta_1: m_1 \Rightarrow n_1, \dots, \theta_k: m_k \Rightarrow n_k$  such that the target of a given cell is equal to the source of the next. Externally composable sequences of modules  $M_1, \dots, M_k$  between lax functors will be thought of as proarrows  $F^i \rightarrow F^{i+1}$  from a lax functor  $F^i$  to one  $F^{i+1}$ , notated with superscripts so as not to confuse these with the components of the functors.

3.13. DEFINITION. [See §4.1 of [Paré, 2011] and §1.2.3 of [Paré, 2013]] A **multimodulation**

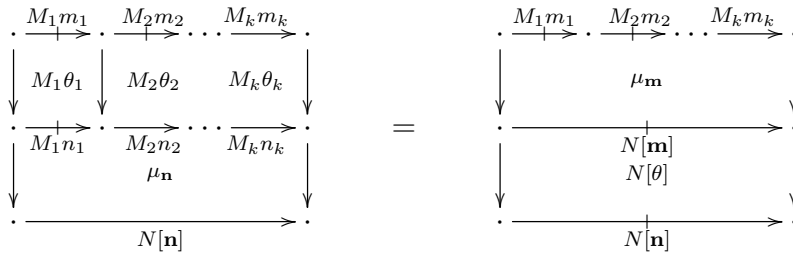


from modules  $M_i$  to  $N$ , with source and target transformations  $\tau$  and  $\sigma$ , assigns to each path  $\mathbf{m} = (m_1, \dots, m_k)$  of proarrows of  $\mathbb{A}$ , a multicell



in such a way that the following axioms are satisfied.

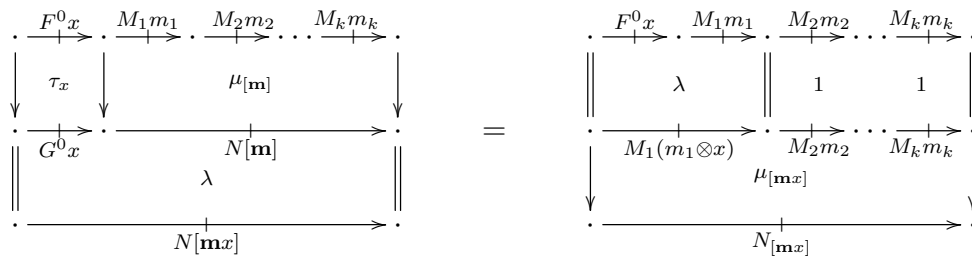
- [Naturality] for any path of cells  $\theta_1: m_1 \Rightarrow n_1, \dots, \theta_k: m_k \Rightarrow n_k$ , the two composites



are equal.

- The following equivariance axioms are satisfied:

- [Left and Right Equivariance] For any paths of proarrows  $y\mathbf{m} = (m_1, \dots, m_k, y)$  and  $\mathbf{m}x = (x, m_1, \dots, m_k)$ , the composites on either side of



and

$$\begin{array}{ccc}
 \begin{array}{c}
 \cdot \xrightarrow{M_1 m_1} \cdots \xrightarrow{M_k m_k} \xrightarrow{F^n y} \cdot \\
 \downarrow \quad \quad \quad \downarrow \tau_y \\
 \cdot \xrightarrow{N[\mathbf{m}]} \xrightarrow{G^n y} \cdot \\
 \downarrow \quad \quad \quad \downarrow \\
 \cdot \xrightarrow{N[y\mathbf{m}]} \cdot
 \end{array}
 & = &
 \begin{array}{c}
 \cdot \xrightarrow{M_1 m_1} \cdots \xrightarrow{M_k m_k} \xrightarrow{F^n y} \cdot \\
 \downarrow \quad \quad \quad \downarrow \rho \\
 \cdot \xrightarrow{M_1 m_1} \cdots \xrightarrow{M_k (y \otimes m_k)} \cdot \\
 \downarrow \quad \quad \quad \downarrow \\
 \cdot \xrightarrow{N[y\mathbf{m}]} \cdot
 \end{array}
 \end{array}$$

are equal.

- (b) [Inner Equivariance] For any path of proarrows  $(m_1, \dots, m_i, x, m_{i+1}, \dots, m_k)$ , the composite

$$\begin{array}{c}
 \cdot \xrightarrow{M_1 m_1} \cdots \xrightarrow{M_i m_i} \xrightarrow{F^i x} \cdots \xrightarrow{M_k m_k} \cdot \\
 \downarrow \quad \quad \quad \downarrow \rho \quad \quad \quad \downarrow \\
 \cdot \xrightarrow{N[\mathbf{m}]} \cdot
 \end{array}$$

is equal to

$$\begin{array}{c}
 \cdot \xrightarrow{M_1 m_1} \cdots \xrightarrow{F^i x} \xrightarrow{M_{i+1} m_{i+1}} \cdots \xrightarrow{M_k m_k} \cdot \\
 \downarrow \quad \quad \quad \downarrow \lambda \quad \quad \quad \downarrow \\
 \cdot \xrightarrow{N[\mathbf{m}]} \cdot
 \end{array}$$

for  $i = 1, \dots, k - 1$ .

3.14. THEOREM. [Paré] *Double functors  $F: \mathbb{A} \rightarrow \mathbb{B}$  and horizontal transformations, together with modules and their multi-modulations giving the proarrows and multicells, give the data of a virtual double category denoted by  $\mathbb{Lax}(\mathbb{A}, \mathbb{B})$ .*

PROOF. See the lead-up to Theorem 1.2.5 of [Paré, 2013]. ■

3.15. REMARK. Our main interest is of course in the virtual double category  $\mathbb{Lax}(\mathbb{B}^{op}, \mathbb{Span})$ . By the Theorem, in general, this is not a genuine double category. This is because composition of proarrows need not exist. The paper [Paré, 2013] is a dedicated study of this issue.

The additional structure on  $\mathbf{DFib}(\mathbb{B})$  making it a virtual double category goes by well-known terminology from another context. We already know that  $\mathbf{DFib}(\mathbb{B})$  could have been defined as  $\mathbf{Cat}(\mathbf{DFib})/\mathbb{B}$ . Another way of look at  $\mathbf{Cat}(\mathbf{DFib})$  is that it is



the category  $\mathbf{Mon}(\mathbf{Span}(\mathbf{DFib}))$  of monoids in spans in discrete fibrations. In general  $\mathbf{Mon}(\mathbf{Span}(\mathcal{C}))$  for a category  $\mathcal{C}$  with finite limits is the underlying category of the virtual double category  $\mathbf{Prof}(\mathcal{C}) = \mathbf{Mod}(\mathbf{Span}(\mathcal{C}))$  of modules in spans in  $\mathcal{C}$  as in [Leinster, 2004] or [Cruttwell & Shulman, 2010]. So, we take the modules and their multicells from this context as the virtual double category structure on  $\mathbf{DFib}(\mathbb{B})$ . Here are the definitions.

3.16. DEFINITION. [Cf. §5.3.1 [Leinster, 2004] or §2.8 of [Cruttwell & Shulman, 2010]] Let  $\mathbb{D}$  denote a virtual double category. Define its **virtual double category of monoids and modules**, denoted by  $\mathbf{Mod}(\mathbb{D})$ , by taking

1. objects: **monoids**, namely, triples  $(r, \mu, \eta)$  consisting of a proarrow  $r: A \twoheadrightarrow A$  and cells

$$\begin{array}{ccc} A & \xrightarrow{r} & A & \xrightarrow{r} & A & & A & \xlongequal{\quad} & A \\ \parallel & & \mu & & \parallel & & \parallel & & \eta & & \parallel \\ A & \xrightarrow{r} & A & & A & & A & \xrightarrow{r} & A \end{array}$$

satisfying the usual axioms for a monoid, namely, the multiplication law  $\mu(1, \mu) = \mu(\mu, 1)$  and the unit laws  $\mu(1, \eta) = 1$  and  $\mu(\eta, 1) = 1$ ;

2. arrows: monoid homomorphisms  $(r, \mu, \eta) \rightarrow (s, \nu, \epsilon)$ , namely, those pairs  $(f, \phi)$  consisting of an arrow  $f: A \rightarrow B$  and a cell

$$\begin{array}{ccc} A & \xrightarrow{r} & A \\ f \downarrow & \phi & \downarrow f \\ B & \xrightarrow{s} & B \end{array}$$

satisfying the unit axiom  $\phi\eta = \epsilon f$  and multiplication axiom  $\nu(\phi, \phi) = \phi\mu$ .

3. proarrows: so-called **modules**  $(r, \mu, \eta) \twoheadrightarrow (s, \nu, \epsilon)$ , namely, triples  $(m, \lambda, \rho)$  with  $m: A \twoheadrightarrow B$  a proarrow and  $\lambda, \rho$  left and right action cells

$$\begin{array}{ccc} A & \xrightarrow{r} & A & \xrightarrow{m} & B & & A & \xrightarrow{m} & B & \xrightarrow{s} & B \\ \parallel & & \lambda & & \parallel & & \parallel & & \rho & & \parallel \\ A & \xrightarrow{r} & A & & B & & A & \xrightarrow{m} & B & & B \end{array}$$

satisfying the module axioms  $\lambda(\mu, 1) = \lambda(1, \lambda)$  and  $\rho(1, \mu) = \rho(\rho, 1)$  for the multiplication and  $\lambda(\eta, 1) = 1$  and  $\rho(1, \eta) = 1$  for the units; a sequence of modules consists of finitely many modules  $(m_i, \lambda_i, \rho_i)$  for which  $\text{src } m_{i+1} = \text{tgt } m_i$  and  $s_{i+1} = r_i$  both hold;

4. multicells from a sequence of modules  $(m_i, \lambda_i, \rho_i)$  to one  $(n, \lambda, \rho)$  consist of those multicells in  $\mathbb{A}$

$$\begin{array}{ccc} \cdot & \xrightarrow{m_1} & \dots & \xrightarrow{m_p} & \cdot \\ f \downarrow & & \gamma & & \downarrow g \\ \cdot & \xrightarrow{\quad} & \cdot & & \cdot \end{array}$$

satisfying the equivariance axioms expressed by the equalities of composite cells:

(a) [Left]

$$\begin{array}{ccc}
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{r_1} \xrightarrow{m_1} \xrightarrow{m_2} \dots \xrightarrow{m_p} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \downarrow f & \begin{array}{c} \phi \\ \downarrow f \\ \gamma \end{array} & \downarrow g \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{s_1} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & \lambda & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & n & \parallel
 \end{array} = \begin{array}{ccc}
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{r_1} \xrightarrow{m_1} \xrightarrow{m_2} \dots \xrightarrow{m_p} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & \lambda & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{m_1} \xrightarrow{m_2} \dots \xrightarrow{m_p} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & \gamma & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & n & \parallel
 \end{array} \downarrow f \quad \downarrow g$$

(b) [Right]

$$\begin{array}{ccc}
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{m_1} \dots \xrightarrow{m_{p-1}} \xrightarrow{m_p} \xrightarrow{r_p} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \downarrow f & \begin{array}{c} \gamma \\ \downarrow g \\ \psi \end{array} & \downarrow g \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{s_q} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & \rho & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & n & \parallel
 \end{array} = \begin{array}{ccc}
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{m_1} \dots \xrightarrow{m_{p-1}} \xrightarrow{m_p} \xrightarrow{r_p} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & 1 & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{m_1} \dots \xrightarrow{m_{p-1}} \xrightarrow{m_p} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & \rho & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & \gamma & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & n & \parallel
 \end{array} \downarrow f \quad \downarrow g$$

(c) [Inner]

$$\begin{array}{ccc}
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{m_1} \dots \xrightarrow{m_{i-1}} \xrightarrow{m_i} \xrightarrow{r_i} \xrightarrow{m_{i+1}} \dots \xrightarrow{m_p} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & 1 & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & \rho & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & 1 & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & \gamma & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & n & \parallel
 \end{array} \downarrow f \quad \downarrow g$$

is equal to

$$\begin{array}{ccc}
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{m_1} \dots \xrightarrow{m_i} \xrightarrow{r_i} \xrightarrow{m_{i+1}} \xrightarrow{m_{i+2}} \dots \xrightarrow{m_p} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & 1 & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & \lambda & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & 1 & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & \gamma & \parallel \\
 \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} & \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
 \parallel & n & \parallel
 \end{array} \downarrow f \quad \downarrow g$$

for  $i = 1, \dots, p - 1$ .

Compositions and identities are given by those in  $\mathbb{A}$ .

3.17. PROPOSITION. For any virtual double category  $\mathbb{D}$ ,  $\mathbf{Mod}(\mathbb{D})$  has units.

PROOF. Any object in  $\mathbf{Mod}(\mathbb{D})$  is a monoid. Its equipped multiplication gives the unit proarrow. For more see §5.5 of [Cruttwell & Shulman, 2010]. ■

3.18. **REMARK.** The definition above omits most of the diagrams and states just the equations out of space considerations. However, upon writing down all the diagrams, one might notice a formal similarity between these axioms and those for modules and multimodulations in  $\mathbf{Lax}(\mathbb{B}^{op}, \mathbf{Span})$ . It is the point of the next section to show that this is in fact an equivalence of virtual double categories. First, however, let us consider some examples.

Let  $\mathcal{C}$  denote a category with finite limits. Then  $\mathbf{Span}(\mathcal{C})$  is a double category. As in [Cruttwell & Shulman, 2010], denote  $\mathbf{Mod}(\mathbf{Span}(\mathcal{C}))$  by  $\mathbf{Prof}(\mathcal{C})$ . Several choices of  $\mathcal{C}$  are of interest. First note that modules in  $\mathbf{Span}(\mathcal{C})$  are already known by well-established terminology.

3.19. **DEFINITION.** [Cf. §2.41 of [Johnstone, 2014]] Let  $\mathcal{C}$  denote a category with finite limits; and let  $\mathbb{C}$  and  $\mathbb{D}$  denote internal categories. An **internal profunctor**  $M: \mathbb{C} \leftrightarrow \mathbb{D}$  is a module, i.e. a proarrow in  $\mathbf{Prof}(\mathcal{C})$ . A **multicell of internal profunctors** is thus a multicell as above.

3.20. **EXAMPLE.** The virtual double category  $\mathbf{Prof}(\mathbf{Set})$  is  $\mathbf{Prof}$ . That is, a monoid in  $\mathbf{Span}(\mathbf{Set})$  is a category. A unit proarrow for such  $\mathcal{C}$  is thus the span  $\mathcal{C}_0 \leftarrow \mathcal{C}_1 \rightarrow \mathcal{C}_0$  formed from the domain and codomain maps with actions given by composition.

3.21. **EXAMPLE.** Letting  $\mathcal{C} = \mathbf{Cat}$  as a 1-category,  $\mathbf{Prof}(\mathbf{Cat})$  consists of usual double categories and double functors as the objects and arrows. Internal profunctors  $M: \mathbb{A} \leftrightarrow \mathbb{B}$  between double categories consist of a span  $\mathbb{A}_0 \xleftarrow{\partial_0} \mathcal{M} \xrightarrow{\partial_1} \mathbb{B}_0$  and left and right action functors

$$L: \mathbb{A}_1 \times_{\mathbb{A}_0} \mathcal{M} \longrightarrow \mathcal{M} \qquad R: \mathcal{M} \times_{\mathbb{B}_0} \mathbb{B}_1 \longrightarrow \mathcal{M}$$

satisfying the axioms above. A multicell of internal profunctors  $(M_1, \dots, M_k) \Rightarrow N$  thus consists of a functor

$$m: \mathcal{M}^1 \times_{\mathbb{A}_0^1} \cdots \times_{\mathbb{A}_0^k} \mathcal{M}^k \longrightarrow \mathcal{N}$$

from the vertex of the composite of  $(M_1, \dots, M_k)$ , making a morphism of spans, and satisfying the various equivariance requirements as in the definition. Notice that owing to the peculiarities of the cell structure of  $\mathbf{1}$  as in Example 3.7, a point  $\mathbb{D}: \mathbf{1} \rightarrow \mathbf{Prof}(\mathbf{Cat})$  is a double category  $\mathbb{D}$ , with the identity double functor  $1: \mathbb{D} \rightarrow \mathbb{D}$ , the unit proarrow  $u: \mathbb{D} \leftrightarrow \mathbb{D}$ , namely, the span formed by the external source and target functors with actions given by external composition, and finally multicells of all arities given by iterated external composition. This can all be generalized to  $\mathbf{Cat}(\mathcal{C})$  for arbitrary  $\mathcal{C}$  with finite limits.

3.22. **EXAMPLE.** Let  $\mathcal{C} = \mathbf{Cat}^2$ , the “arrow category” of  $\mathbf{Cat}$  and consider  $\mathbf{Prof}(\mathbf{Cat}^2)$ . A monoid is then a double functor, a morphism is a commutative square of double functors. A module consists of two modules in the former sense – one between domains of the two double categories and one between the codomains; the vertices of these modules are related by a functor making a morphism of spans. Multicells have a similar “two-tiered” structure.

3.23. **EXAMPLE.** Letting  $\mathcal{C} = \mathbf{DFib}$ , the virtual double category  $\mathbb{P}\mathbf{rof}(\mathbf{DFib})$  is the sub-virtual double category of the previous example where all the objects are not just double functors but are instead *discrete double fibrations*. There is a codomain functor

$$\text{cod}: \mathbb{P}\mathbf{rof}(\mathbf{DFib}) \longrightarrow \mathbb{P}\mathbf{rof}(\mathbf{Cat})$$

taking an object  $P: \mathbb{E} \rightarrow \mathbb{B}$  its codomain  $\mathbb{B}$  and every proarrow  $M: P \rightarrow Q$  to the module between double categories giving the codomains of  $M$ . Take  $\mathbb{P}\mathbf{rof}(\mathbf{DFib})/\mathbb{B}$  to be the pullback of  $\text{cod}$  in  $\mathbf{vDbl}$  along the point  $\mathbb{B}: \mathbf{1} \rightarrow \mathbb{P}\mathbf{rof}(\mathbf{Cat})$ .

3.24. **DEFINITION.** The virtual double category of discrete double fibrations over a strict double category  $\mathbb{B}$  is  $\mathbb{P}\mathbf{rof}(\mathbf{DFib})/\mathbb{B}$ . Denote this by  $\mathbb{D}\mathbf{Fib}(\mathbb{B})$ .

3.25. **REMARK.** A module between discrete double fibrations  $M: P \rightarrow Q$  thus consists of a discrete fibration  $M: \mathcal{M} \rightarrow \mathbb{B}_1$  and a morphism of spans

$$\begin{array}{ccccc} \mathbb{E}_0 & \xleftarrow{\partial_0} & \mathcal{M} & \xrightarrow{\partial_1} & \mathbb{G}_0 \\ P_0 \downarrow & & \downarrow M & & \downarrow Q_0 \\ \mathbb{B}_0 & \xleftarrow{\text{src}} & \mathbb{B}_1 & \xrightarrow{\text{tgt}} & \mathbb{B}_0 \end{array}$$

and left and right actions functors making commutative squares

$$\begin{array}{ccc} \mathbb{E}_1 \times_{\mathbb{E}_0} \mathcal{M} & \xrightarrow{L} & \mathcal{M} \\ P_1 \times M \downarrow & & \downarrow M \\ \mathbb{B}_1 \times_{\mathbb{B}_0} \mathbb{B}_1 & \xrightarrow{-\otimes-} & \mathbb{B}_1 \end{array} \qquad \begin{array}{ccc} \mathcal{M} \times_{\mathbb{G}_0} \mathbb{G}_1 & \xrightarrow{R} & \mathcal{M} \\ M \times Q_1 \downarrow & & \downarrow M \\ \mathbb{B}_1 \times_{\mathbb{B}_0} \mathbb{B}_1 & \xrightarrow{-\otimes-} & \mathbb{B}_1 \end{array}$$

that satisfy the action requirements as in the definition. A multicell  $\mu: (M^1, \dots, M^k) \Rightarrow N$  between such modules consists of a functor  $\mu$  making a commutative square

$$\begin{array}{ccc} \mathcal{M}^1 \times_{\mathbb{E}_0^1} \cdots \times_{\mathbb{E}_0^k} \mathcal{M}^k & \xrightarrow{\mu} & \mathcal{N} \\ \downarrow & & \downarrow N \\ \mathbb{B}_1 \times_{\mathbb{B}_0} \cdots \times_{\mathbb{B}_0} \mathbb{B}_1 & \xrightarrow{-\otimes \cdots \otimes-} & \mathbb{B}_1 \end{array}$$

satisfying the equivariance requirements above.

3.26. **REMARK.** As in §3.9 in [Cruttwell & Shulman, 2010], the mod-construction  $\mathbf{Mod}(-)$  defines an endo-2-functor  $\mathbf{Mod}(-): \mathbf{vDbl} \rightarrow \mathbf{vDbl}$ . Another way to look at the codomain functor from the previous example is that it is induced from the codomain functor  $\text{cod}: \mathbf{DFib} \rightarrow \mathbf{Cat}$ , passing first through  $\mathbf{Span}(-)$  and then  $\mathbf{Mod}(-)$ .

### 4. The Full Representation Theorem

This section extends the result of Theorem 2.27, culminating in a proof that elements construction extends to an equivalence of virtual double categories

$$\mathbb{D}\mathbf{Fib}(\mathbb{B}) \simeq \mathbf{Lax}(\mathbb{B}^{op}, \mathbf{Span})$$

for any strict double category  $\mathbb{B}$ . This appears below as Theorem 4.14.

The elements functor of Lemma 2.9 extends to one between virtual double categories. Needed are assignments on modules and multimodulations. Suppose throughout that  $\mathbb{B}$  is strict.

4.1. CONSTRUCTION. [Elements from a Module] Let  $M: F \rightrightarrows G$  denote a module between lax double functors as in Definition 3.12. Construct a category  $\mathbb{E}\mathbf{I}(M)$  in the following way. Objects are pairs  $(m, s)$  with  $m: B \rightrightarrows C$  a proarrow of  $\mathbb{B}$  and  $s \in Mm$ . A morphism  $(m, s) \rightarrow (n, t)$  is a cell  $\alpha$  with source  $m$  and target  $n$  for which the equation  $M\alpha(t) = s$  holds. So defined,  $\mathbb{E}\mathbf{I}(M)$  is a category since  $M$  is strictly functorial on cells. Notice that there are thus projection functors

$$\mathbb{E}\mathbf{I}(F)_0 \xleftarrow{\partial_0} \mathbb{E}\mathbf{I}(M) \xrightarrow{\partial_1} \mathbb{E}\mathbf{I}(G)_0$$

taking an object  $(m, s)$  to  $\partial_0(v, s) = (B, \partial_0 s)$  and  $\partial_1(v, s) = (C, \partial_1 s)$  and extended to morphisms as follows. Given a cell

$$\begin{array}{ccc} A & \xrightarrow{m} & B \\ f \downarrow & \alpha & \downarrow g \\ C & \xrightarrow{n} & D \end{array}$$

take  $\partial_0 \alpha$  to be the morphism  $f: (A, \partial_0 s) \rightarrow (B, \partial_0 t)$  and analogously for  $\partial_1 \alpha$ . These are well-defined by the commutativity conditions coming with the morphism of spans  $M\alpha$ . The assignments are then functorial by that assumed for  $M$ .

4.2. CONSTRUCTION. [Actions] Form the pullback of  $\partial_0: \mathbb{E}\mathbf{I}(M) \rightarrow \mathbb{E}\mathbf{I}(F)_0$  along the target projection  $\mathbb{E}\mathbf{I}(F)_1 \rightarrow \mathbb{E}\mathbf{I}(F)_0$  and give assignments

$$L: \mathbb{E}\mathbf{I}(F)_1 \times_{\mathbb{E}\mathbf{I}(F)_0} \mathbb{E}\mathbf{I}(M) \rightarrow \mathbb{E}\mathbf{I}(M)$$

that will amount to an action. Summarize these assignments on objects and arrows at once by the picture:

$$\begin{array}{ccc} (A, x) \xrightarrow{(m,u)} (B, y) & (p, r) & (p \otimes m, \lambda(u, r)) \\ f \downarrow & \alpha & \downarrow g \\ (C, z) \xrightarrow{(n,v)} (D, w) & (q, s) & (q \otimes n, \lambda(v, s)) \end{array} \quad \mapsto \quad \begin{array}{c} \downarrow \beta \\ \downarrow \beta \otimes \alpha \end{array}$$

where  $\lambda$  is the action cell coming with  $M$ . Of course  $\alpha$  and  $\beta$  are composable by the construction of the pullback, but it needs to be seen that the composite  $\beta \otimes \alpha$  does give a morphism of  $\mathbb{E}\mathbf{I}(M)$ . But this is equivalent to the validity of the equation

$$M(\beta \otimes \alpha)(\lambda(v, s)) = \lambda(u, r)$$

But this holds by the naturality condition for  $\lambda$  in Definition 3.12, since  $u = F\alpha(v)$  and  $M\beta(s) = r$  both hold by the construction of morphisms in  $\mathbb{E}\mathbf{I}(M)$ . So defined,  $L$  is a functor by the strict interchange law in  $\mathbb{B}$ ; by the fact that  $M$  is strictly horizontally functorial; and by the normalization hypothesis for units. A functor  $R$  for a right action of  $\mathbb{E}\mathbf{I}(G)_1$  on  $\mathbb{E}\mathbf{I}(M)$  is constructed analogously. It remains to see that the action axioms are satisfied and that they are suitably compatible, yielding an internal profunctor.

4.3. PROPOSITION. *The assignments of Construction 4.2 are well-defined functors yielding an internal profunctor between discrete double fibrations  $\mathbb{E}\mathbf{I}(M): \mathbb{E}\mathbf{I}(F) \rightarrow \mathbb{E}\mathbf{I}(G)$ .*

PROOF. The action functors  $L$  and  $R$  are unital by the normalization assumption for vertical composition with units in  $\mathbb{B}$ . Required are action iso cells such as

$$\begin{array}{ccc} \mathbb{E}\mathbf{I}(F)_1 \times_{\mathbb{E}\mathbf{I}(F)_0} \mathbb{E}\mathbf{I}(F)_1 \times_{\mathbb{E}\mathbf{I}(F)_0} \mathbb{E}\mathbf{I}(M) & \xrightarrow{1 \times L} & \mathbb{E}\mathbf{I}(F)_1 \times_{\mathbb{E}\mathbf{I}(F)_0} \mathbb{E}\mathbf{I}(M) \\ \otimes \times 1 \downarrow & & \downarrow L \\ \mathbb{E}\mathbf{I}(F)_1 \times_{\mathbb{E}\mathbf{I}(F)_0} \mathbb{E}\mathbf{I}(M) & \xrightarrow{L} & \mathbb{E}\mathbf{I}(M) \end{array}$$

and similarly for  $R$ . But chasing an object of the domain around either of the square as above and comparing, commutativity is given by associativity of proarrow composition in  $\mathbb{B}$ . Lastly, the actions  $L$  and  $R$  should be compatible in the sense that

$$\begin{array}{ccc} \mathbb{E}\mathbf{I}(F)_1 \times_{\mathbb{E}\mathbf{I}(F)_0} \mathbb{E}\mathbf{I}(M) \times_{\mathbb{E}\mathbf{I}(G)_0} \mathbb{E}\mathbf{I}(G)_1 & \xrightarrow{1 \times R} & \mathbb{E}\mathbf{I}(F)_1 \times_{\mathbb{E}\mathbf{I}(F)_0} \mathbb{E}\mathbf{I}(M) \\ L \times 1 \downarrow & & \downarrow L \\ \mathbb{E}\mathbf{I}(M) \times_{\mathbb{E}\mathbf{I}(G)_0} \mathbb{E}\mathbf{I}(G)_1 & \xrightarrow{R} & \mathbb{E}\mathbf{I}(M) \end{array}$$

commutes. But again chasing objects and arrows around each side of the square shows that commutativity follows from the compatibility assumption in Definition 3.12. ■

4.4. CONSTRUCTION. [Elements from a Multimodulation] Start with a multimodulation of contravariant lax **Span**-valued functors

$$\begin{array}{ccccc} F^0 & \xrightarrow{M_1} & F^1 & \xrightarrow{M_2} & \dots & \xrightarrow{M_k} & F^k \\ \tau \downarrow & & & \mu & & & \downarrow \sigma \\ G^0 & \xrightarrow{\quad} & & N & \xrightarrow{\quad} & & G^1 \end{array}$$

as in Definition 3.13. This means that there are projection spans  $\mathbb{E}\mathbf{I}(F^{i-1})_0 \leftarrow \mathbb{E}\mathbf{I}(M_i) \rightarrow \mathbb{E}\mathbf{I}(F^i)_0$  and one for  $\mathcal{N}$ , each with appropriate left and right actions as in Construction 4.1. Define what will be a functor

$$\mathbb{E}\mathbf{I}(\mu): \mathbb{E}\mathbf{I}(M_1) \times_{\mathbb{E}\mathbf{I}(F^1)_0} \cdots \times_{\mathbb{E}\mathbf{I}(F^{k-1})_0} \mathbb{E}\mathbf{I}(M_k) \longrightarrow \mathbb{E}\mathbf{I}(N)$$

in the following way. On objects take

$$((m_1, s_1), \dots, (m_k, s_k)) \mapsto ([\mathbf{m}], \mu_{\mathbf{m}}(s))$$

where  $\mu_{\mathbf{m}}$  is the given function coming with  $\mu$  and  $s = (s_1, \dots, s_k)$ . An arrow of the supposed source is a sequence of externally composable cells  $\theta_i: (m_i, s_i) \rightarrow (n_i, t_i)$ . Assign to such a sequence the morphism of  $\mathcal{N}$  represented by their composite

$$(\theta_1, \dots, \theta_k) \mapsto \theta_k \otimes \theta_{k-1} \otimes \cdots \otimes \theta_1.$$

This does define a morphism  $([\mathbf{m}], \mu_{\mathbf{m}}(s)) \rightarrow ([\mathbf{n}], \mu_{\mathbf{n}}(t))$  of  $\mathbb{E}\mathbf{I}(N)$  by the strict composition for the module  $N$  as in Definition 3.12. This functor has several naturality and equivariance properties, coming from the assumed properties of the original multimodulation  $\mu$ . For example, notice that  $\mathbb{E}\mathbf{I}(\mu)$  commutes with the projections and the 0-level of the induced double functors  $\mathbb{E}\mathbf{I}(\tau)_0$  and  $\mathbb{E}\mathbf{I}(\sigma)_0$  by construction. Further properties are summarized in the next result.

4.5. PROPOSITION. *The functor  $\mathbb{E}\mathbf{I}(\mu)$  of Construction 4.1 defines a multicell between internal profunctors of the form*

$$\begin{array}{ccc} \mathbb{E}\mathbf{I}(F^0) \xrightarrow{\mathbb{E}\mathbf{I}(M_1)} \mathbb{E}\mathbf{I}(F^1) \xrightarrow{\mathbb{E}\mathbf{I}(M_2)} \cdots \xrightarrow{\mathbb{E}\mathbf{I}(M_k)} \mathbb{E}\mathbf{I}(F^k) & & \\ \downarrow & \mathbb{E}\mathbf{I}(\mu) & \downarrow \\ \mathbb{E}\mathbf{I}(G^0) \xrightarrow{\mathbb{E}\mathbf{I}(N)} \mathbb{E}\mathbf{I}(G^1) & & \end{array}$$

*This completes assignments for the elements functor  $\mathbb{E}\mathbf{I}(-): \mathbf{Lax}(\mathbb{B}^{op}, \mathbf{Span}) \rightarrow \mathbf{DFib}(\mathbb{B})$  between virtual double categories.*

PROOF. The appropriate commutativity at the 0-level was observed above. That the action is left equivariant is the statement that the square

$$\begin{array}{ccc} \mathbb{E}\mathbf{I}(F^0)_1 \times_{\mathbb{E}\mathbf{I}(F^0)_0} \mathbb{E}\mathbf{I}(M_1) \times_{\mathbb{E}\mathbf{I}(F^1)_0} \cdots \times_{\mathbb{E}\mathbf{I}(F^{k-1})_0} \mathbb{E}\mathbf{I}(M_k) & \xrightarrow{\mathbb{E}\mathbf{I}(\tau) \times \mathbb{E}\mathbf{I}(\mu)} & \mathbb{E}\mathbf{I}(G^0) \times_{\mathbb{E}\mathbf{I}(G^0)_0} \mathbb{E}\mathbf{I}(N) \\ \downarrow L \times 1 & & \downarrow L \\ \mathbb{E}\mathbf{I}(M_1) \times_{\mathbb{E}\mathbf{I}(F^1)_0} \cdots \times_{\mathbb{E}\mathbf{I}(F^{k-1})_0} \mathbb{E}\mathbf{I}(M_k) & \xrightarrow{\mathbb{E}\mathbf{I}(\mu)} & \mathbb{E}\mathbf{I}(N) \end{array}$$

commutes. But chasing an object of the upper left corner around both sides of the square reveals that commutativity at the object level is precisely the left equivariance condition in Definition 3.13. Right and inner equivariance follow by the same type of argument. The assignments are already known to be well-defined on objects and morphisms. It follows easily that these assignments are suitably functorial in the sense of virtual double categories. ■

Likewise, the pseudo-inverse of Lemma 2.24 extends to a functor of virtual double categories.

4.6. CONSTRUCTION. [Pseudo-Inverse for Modules] Let  $M: P \rightrightarrows Q$  denote an internal profunctor between discrete double fibrations  $P: \mathbb{E} \rightarrow \mathbb{B}$  and  $Q: \mathbb{G} \rightarrow \mathbb{B}$  as in Remark 3.25. Construct what will be a module  $F_M: F_P \rightrightarrows F_Q$  between the associated lax functors  $F_P$  and  $F_Q$  from Lemma 2.22 in the following way. Let  $\mathcal{M}_m$  denote the inverse image of the proarrow  $m: B \rightrightarrows C$  of  $\mathbb{B}$  under the discrete fibration  $\Pi: \mathcal{M} \rightarrow \mathbb{B}_1$  coming with  $M$ . To each such proarrow  $m$  assign the span of sets

$$\mathbb{E}_B \xleftarrow{\partial_0} \mathcal{M}_m \xrightarrow{\partial_1} \mathbb{G}_C$$

Note that this is well-defined by the first assumed commutativity condition for  $M$ . To each cell of  $\mathbb{B}$ , assign the morphism of spans

$$\begin{array}{ccc} A \xrightarrow{m} B & & \mathbb{E}_B \longleftarrow \mathcal{M}_n \longrightarrow \mathbb{G}_C \\ f \downarrow \quad \theta & \downarrow & f^* \downarrow \quad \theta^* \quad \downarrow g^* \\ C \xrightarrow{n} D & & \mathbb{E}_A \longleftarrow \mathcal{M}_m \longrightarrow \mathbb{G}_D \end{array} \quad \mapsto$$

with  $\theta^*: \mathcal{M}_n \rightarrow \mathcal{M}_m$  given by taking an object of  $\mathcal{M}$  over  $n$  to the domain of the unique morphism of  $\mathcal{M}$  above  $\theta$  via  $\Pi: \mathcal{M} \rightarrow \mathbb{B}_1$ . This is well-defined and makes a span morphism. To complete the data, start with composable vertical arrows  $m: A \rightrightarrows B$  and  $n: B \rightrightarrows C$  and give assignments  $\lambda$  and  $\rho$  by using the given actions  $L$  and  $R$ , taking

$$\lambda_{m,n}: \mathbb{E}_m \times_{\mathbb{E}_D} \mathcal{M}_n \rightarrow \mathcal{M}_{n \otimes m} \quad (\tilde{m}, \tilde{n}) \mapsto L(\tilde{m}, \tilde{n})$$

and similarly for  $\rho$ . These are well-defined by the second row of commutativity conditions in Definition 3.12. Additionally, these maps commute with the projections, in the sense that the diagrams

$$\begin{array}{ccc} \mathbb{E}_A \longleftarrow \mathcal{M}_m \times_{\mathbb{G}_B} \mathbb{G}_n \longrightarrow \mathbb{G}_C & & \mathbb{E}_A \longleftarrow \mathbb{E}_m \times_{\mathbb{E}_B} \mathcal{M}_n \longrightarrow \mathbb{G}_C \\ \parallel & \downarrow \rho & \parallel \\ \mathbb{E}_A \longleftarrow \mathcal{M}_{n \otimes m} \longrightarrow \mathbb{G}_C & & \mathbb{E}_A \longleftarrow \mathcal{M}_{n \otimes m} \longrightarrow \mathbb{G}_C \\ & & \downarrow \lambda \\ & & \mathbb{E}_A \longleftarrow \mathcal{M}_{n \otimes m} \longrightarrow \mathbb{G}_C \end{array}$$

both commute.

4.7. PROPOSITION. *The assignments of Construction 4.6 make  $F_M: F_P \rightrightarrows F_Q$  a module in the sense of Definition 3.12.*

PROOF. All of the requirements for  $F_M$  to be a module between lax functors are met by the corresponding properties of the original module  $M$ , together with the fact that  $\Pi: \mathcal{M} \rightarrow \mathbb{B}_1$  is a discrete fibration. ■



4.8. CONSTRUCTION. [Pseudo-Inverse Assignment on Modulations] Start with a modulation  $U$  in  $\mathbb{DFib}(\mathbb{B})$  as in Remark 3.25. Thus, in particular, we have a functor

$$U: \mathcal{M}^1 \times_{\mathbb{E}_0^1} \cdots \times_{\mathbb{E}_0^{k-1}} \mathcal{M}^k \longrightarrow \mathcal{N}$$

commuting with the projections to the end factors and commuting with the  $(k - 1)$ -fold proarrow composition on  $\mathbb{B}$ . Required is a multi-modulation

$$\begin{array}{ccccccc} F_{P^0} & \xrightarrow{F_{M_1}} & F_{P^1} & \xrightarrow{F_{M_1}} & \cdots & \xrightarrow{F_{M_k}} & F_{P^k} \\ F_H \downarrow & & & F_U & & & \downarrow F_K \\ F_{Q^0} & \xrightarrow{\quad\quad\quad} & & & \xrightarrow{\quad\quad\quad} & & F_{Q^1} \\ & & & F_N & & & \end{array}$$

Unpacking the constructions at a path of proarrows  $\mathbf{m} = (m_1, \dots, m_k)$ , this is just to ask for a corresponding set function

$$\mathcal{M}_{m_1}^1 \times_{\mathbb{E}_{A_1}^1} \cdots \times_{\mathbb{E}_{A_{k-1}}^{k-1}} \mathcal{M}_{m_k}^k \longrightarrow \mathcal{N}_{[\mathbf{m}]}$$

which is given by the arrow part of  $U$ , namely,  $U_1$ . Well-definition follows from the fact that  $U$  commutes with the projections to  $\mathbb{B}$  and the  $(k - 1)$ -fold iterated proarrow composition of  $\mathbb{B}$ .

4.9. PROPOSITION. *The choice of  $(F_U)_{\mathbf{m}} = U_1$  in Construction 4.8 results in a multimodulation of modules between lax functors as in Definition 3.13. This extends the functor  $F_{(-)}$  to a functor of virtual double categories*

$$F_{(-)}: \mathbb{DFib}(\mathbb{B}) \longrightarrow \mathbb{Lax}(\mathbb{B}^{op}, \mathbb{Span}).$$

PROOF. The horizontal naturality condition holds by construction of the transition functions corresponding to cells  $\theta$  and because  $\Pi_1: \mathcal{N} \rightarrow \mathbb{B}_1$  is a discrete fibration. Right, left and inner equivariance then follow from the corresponding properties of the original functor  $U$ . ■

The extended elements construction and the purported pseudo-inverse induce an equivalence of virtual double categories, leading to the full representation theorem, namely, Theorem 4.14 below.

4.10. CONSTRUCTION. Extend the isomorphism of Construction 2.25 to an isomorphism of functors of virtual double categories. The required multimodulation for the cell-components is straightforward to produce. Given  $M: H \rightarrow G$  and a proarrow  $m: B \rightarrow C$  of  $\mathbb{B}$ , define a function

$$\eta_{M,m}: Mm \longrightarrow \mathbb{EI}(M)_m \quad s \mapsto (m, s)$$

again just adding in an index. This is a bijection fitting into a morphism of spans

$$\begin{array}{ccccc}
 HB & \longleftarrow & Mm & \longrightarrow & GC \\
 \eta_{H,B} \downarrow & & \downarrow \eta_{M,m} & & \downarrow \eta_{G,C} \\
 \mathbb{E}\mathbf{l}(H)_B & \longleftarrow & \mathbb{E}\mathbf{l}(M)_m & \longrightarrow & \mathbb{E}\mathbf{l}(G)_C
 \end{array}$$

that defines the required invertible modulation

$$\begin{array}{ccc}
 H & \xrightarrow{M} & G \\
 \eta_H \downarrow & \eta_M & \downarrow \eta_G \\
 \mathbb{E}\mathbf{l}(F_H) & \xrightarrow[\mathbb{E}\mathbf{l}(M)]{} & \mathbb{E}\mathbf{l}(F_G)
 \end{array}$$

as in Definition 3.13 by construction of the elements functor and its purported pseudo-inverse. This is easy to check from the definitions.

4.11. PROPOSITION. *The assignments in Construction 4.10 yield a natural isomorphism of functors of virtual double categories  $\eta: 1 \cong \mathbb{E}\mathbf{l}(F_{(-)})$ .*

PROOF. As discussed above, the components of the purported transformation are all well-defined, so it remains only to check the “cell naturality” condition of Definition 3.6. Start with a generic multimodulation

$$\begin{array}{ccccccc}
 H^0 & \xrightarrow{M_1} & H^1 & \xrightarrow{M_2} & \dots & \xrightarrow{M_k} & H^k \\
 \tau \downarrow & & & \mu & & & \downarrow \sigma \\
 G^0 & \xrightarrow{\quad} & & N & \xrightarrow{\quad} & & G^1
 \end{array}$$

in  $\mathbf{Lax}(\mathbb{B}^{op}, \mathbf{Span})$ . By construction, the composite on right side of the condition sends an  $k$ -tuple  $s = (s_1, \dots, s_k)$  to  $((m_1, s_1), \dots, (m_k, s_k))$  and then to  $([\mathbf{m}], \mu_{[\mathbf{m}]}(s))$ ; where as that on the left side sends the same element to  $\mu_{[\mathbf{m}]}(s)$  first and then to  $([\mathbf{m}], \mu_{[\mathbf{m}]}(s))$ . The point is that by construction evaluating and indexing commute. In any case, the two sides are equal, and  $\eta$  so defined is a natural isomorphism as claimed. ■

4.12. CONSTRUCTION. Extend the natural isomorphism of Construction 2.26 to one of functors of virtual double categories  $\epsilon: \mathbb{E}\mathbf{l}(F_{(-)}) \cong 1$ . Take an internal profunctor  $M: P \rightarrow Q$  of discrete double fibrations  $P: \mathbb{E} \rightarrow \mathbb{B}$  and  $Q: \mathbb{G} \rightarrow \mathbb{B}$ . The required span morphism

$$\begin{array}{ccc}
 \mathbb{E}_0 \longleftarrow \mathbb{E}\mathbf{l}(F_M) \longrightarrow \mathbb{G}_0 & & (m, s) \xrightarrow{\alpha} (n, t) \\
 \parallel & \downarrow \epsilon_M & \downarrow \\
 \mathbb{E}_0 \longleftarrow \mathcal{M} \longrightarrow \mathbb{G}_0 & & s \xrightarrow{\beta} t
 \end{array}$$

is given by the fact that  $M: \mathcal{M} \rightarrow \mathbb{B}_1$  is a discrete fibration. It is a functor and an isomorphism by uniqueness and equivariant by construction, making a morphism of internal profunctors.

4.13. PROPOSITION. *The assignments in Construction 4.12 are a natural isomorphism of functors of virtual double categories.*

PROOF. The one condition to check is the “Cell Naturality” of Definition 3.6. Taking a multicell between internal profunctors

$$\begin{array}{ccccc}
 P^0 & \xrightarrow{M_1} & P^1 & \xrightarrow{M_2} & \dots & \xrightarrow{M_k} & P^k \\
 H \downarrow & & & \mu & & & \downarrow K \\
 Q^0 & \xrightarrow{\quad} & & N & \xrightarrow{\quad} & & Q^1
 \end{array}$$

the statement of the condition reduces to checking that the equation

$$\mu \circ (\epsilon_{M_1} \times \dots \times \epsilon_{M_k}) = \epsilon_N \circ \mathbb{E}\mathbf{l}(F_\mu)$$

holds. But this is true by definition of  $\mathbb{E}\mathbf{l}(F_\mu)$  and the components of  $\epsilon$ . For on the one hand, a  $k$ -tuple  $((m_1, s_1), \dots, (m_k, s_k))$  is sent to  $s = (s_1, \dots, s_k)$  and then to  $\mu(s)$ . On the other hand, the same  $k$ -tuple is sent to  $([\mathbf{m}], s)$  by  $\mathbb{E}\mathbf{l}(F_\mu)$  and then to  $\mu(s)$  by  $\epsilon_N$ . The same kind of check works at the level of arrows. Thus,  $\epsilon$  is a natural isomorphism. ■

4.14. THEOREM. *There is an equivalence of virtual double categories*

$$\mathbf{Lax}(\mathbb{B}^{op}, \mathbf{Span}) \simeq \mathbf{DFib}(\mathbf{B})$$

for any strict double category  $\mathbb{B}$  induced by the elements functor  $\mathbb{E}\mathbf{l}(-)$ .

PROOF. This is proved by Propositions 4.10 and 4.12. ■

## 5. Prospectus and Applications

Let us close with a preview of forthcoming work relating to the present results.

5.1. MONADICITY. It is well-known that ordinary discrete fibrations over a fixed base are monadic over a slice of the category of sets. This fact is of central importance in the elementary axiomatization of results relating to presheaves and sheaves in the language of an elementary topos [Diaconescu, 1973], [Diaconescu, 1975]. In this development, “base-valued functors” (i.e. presheaves) are axiomatized as certain algebras for a monad on a slice of the ambient topos.

Any parallel development in a double categorical setting of these presheaf results will require an analogous monadicity result. Forthcoming work will establish that discrete double fibrations over a fixed base double category are monadic over a certain slice of the double category of categories. Pursing some notion of “double topos” as a forum for formal category theory, this will give a setting for elementary axiomatization of elements of presheaf and Yoneda theory for double categories.

5.2. **DOUBLE FIBRATIONS.** The main definition of the paper anticipates the natural question about whether there is a more general notion of a “double fibration” of which a discrete double fibration is a special case. For recall that each ordinary discrete fibration  $F: \mathcal{F} \rightarrow \mathcal{C}$  between 1-categories is a (split) fibration in a more general sense. Split fibrations of course have lifting properties with respect to certain compatibly chosen “cartesian arrows” and correspond via a category of elements construction to contravariant category-valued 2-functors on the base category.

Since discrete double fibrations are essentially category objects in discrete fibrations, the most double-categorical conjecture is that (split) double fibrations are just category objects in (split) fibrations. That this turns out to be the “correct” definition for such a fibration concept over an arbitrary base double category is the subject of forthcoming work with G. Cruttwell, D. Pronk and M. Szlyd. The evidence of the correctness of this definition is the exhibition of a representation theorem like Theorem 2.27 in the present paper, but suitably upping the dimension of the representing structure.

5.3. **A COMPREHENSIVE FACTORIZATION.** For any ordinary functor  $F: \mathcal{A} \rightarrow \mathcal{B}$ , there is the so-called “comprehensive factorization,” that is, a factorization  $F = PE$  realizing  $F$  as an initial functor  $E$  followed by a discrete opfibration  $P$  [Street & Walters, 1973]. The discrete opfibration arises as the category of elements of a certain left extension of  $F$  valued in sets. The initial functor arises from the universal property of the comma category. Ultimately, these are the two classes of an orthogonal factorization system on **Cat**. The notion of “discrete double fibration” presented in this paper clearly raises the question of the existence of a corresponding comprehensive factorization for a given double functor, perhaps leading to an orthogonal factorization system on **Dbl**. Based on the sketch of the 1-categorical case, this would involve a definition and useful characterization of some notion of an initial double functor and more detailed work on the construction of left extensions as colimits in sets with spans. Ultimately, this could lead to a study of higher torsors and non-abelian cohomology following the developments of [Street & Verity, 2010].

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