

ON SUPERCOMPACTLY AND COMPACTLY GENERATED TOPOSES

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ABSTRACT. We present and characterize the classes of Grothendieck toposes having enough supercompact objects or enough compact objects. In the process, we examine the subcategories of supercompact objects and compact objects within such toposes and classes of geometric morphism which interact well with these objects. We also present canonical classes of sites generating such toposes.

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Introduction

Grothendieck toposes can be succinctly defined as left exact reflective subcategories of presheaf toposes. If we already have an elementary topos \mathcal{F} to hand, this can be stated as the existence of a geometric inclusion $\mathcal{F} \rightarrow \mathcal{E}$, where \mathcal{E} is a presheaf topos. In practice, one often encounters (elementary) toposes constructed by other means: \mathcal{F} might be a category of *coalgebras* for a left exact comonad on another topos \mathcal{G} , for example, which amounts to the existence of a geometric surjection $\mathcal{G} \rightarrow \mathcal{F}$. At this level of generality, even if \mathcal{G} is a presheaf topos, \mathcal{F} can fail to be a Grothendieck topos. However, there is still much we can deduce from the existence of such a geometric morphism. For example, \mathcal{F} must be cocomplete, since comonadic functors create any colimits which exist in their codomains. Imposing (or identifying) further properties of the geometric morphism can allow us to deduce further properties of \mathcal{F} .

In this paper we shall present and thoroughly investigate the class of *supercompactly generated toposes*. By a supercompactly generated topos, we mean a topos with a separating set of supercompact objects (see Definition 1.1.1). This class includes all regular toposes (Example 1.2.3), as well as all presheaf toposes and some other important classes described in Proposition 1.2.4, and so is of general interest in topos theory. Our motivation for examining this broad class of toposes is that, as we shall eventually see in Theorem 1.8.5, any topos admitting a *hyperconnected* morphism from a supercompactly generated Grothendieck topos (such as a presheaf topos) will also be such a topos. This notably includes any topos of continuous actions of a topological monoid on sets, studied by the present author in [Rogers, 2021]; such a topos typically will not fall into any of the other aforementioned classes (see Example 1.2.5), whence our desire to develop a general theory unifying these examples.

Since it is convenient to do so, we also study *compactly generated toposes*, which are conceptually similar enough that we can prove analogous results about them with little extra work. For brevity, some of the section headings refer only to the supercompact naming conventions.

Some relevant results appear in Section 4.1 of the thesis of Bridge, [Bridge, 2012]¹; this is a theoretical excursion from the main topics of that thesis, but one yield of the results is that the notion of Krull-Gabriel dimension for regular toposes respects inclusions of regular subtoposes. Where there are overlaps between the basic results of that paper and the present one, the main distinction of the present paper is the emphasis on topos-theoretic machinery: the present author avoids reasoning directly with sheaves as far as possible, which allows for more concise categorical proofs.

Our developments of the relevant classes of geometric morphisms for studying these toposes draw from and generalize the definitions and results for proper and relatively proper geometric morphisms in [Moerdijk and Vermeulen, 1997, Chapters I and V]. We

¹Bridge refers to our supercompactly generated toposes as ‘*locally supercompact*’; the reason for the terminology choice of the present author is that the adverb “locally” is already overloaded in topos-theory literature.

show how those notions of compactness for geometric morphisms interact with the notion for objects inside a topos, and ultimately show that these are the ‘right’ notion of morphisms between (super)compactly generated toposes, since they are induced by morphisms between the natural sites for these toposes.

For the site-theoretic material, we rely on the recent monograph [Caramello, 2019], which contains general results about site representations of toposes; we quote some of those results without proof here. As such, the principal and finitely generated sites of Section 2 can be appreciated as illustrative applications of the abstract techniques from that monograph.

In [Kondo and Yasuda, 2017], one can find ‘*B-sites*’, which are a restricted class of the principal sites found here. These sites are subsequently constrained further so as to yield toposes of actions of ‘locally prodiscrete’ monoids², but their developments beg the question of which toposes are generated from less restricted sites. The present paper provides some answers.

We assume the reader is familiar with the basics of category theory and Grothendieck toposes; the content of the textbook [Mac Lane and Moerdijk, 1992] should be adequate.

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OVERVIEW. The structure of this paper is as follows. In Section 1.1, we recall the definitions of supercompact and compact objects in a topos. This leads us to formally define supercompactly and compactly generated toposes in Section 1.2.

In Section 1.3, we turn to the full subcategories of supercompact and compact objects in a topos, presenting the structure they inherit from their ambient toposes, with a focus on monomorphisms, epimorphisms, and the classes of *funneling* and *multifunneling* colimits, which we introduce in Definitions 1.3.3 and 1.3.6. The purpose of this investigation is to present these subcategories as canonical sites for supercompactly and compactly generated toposes in Section 1.4, which we do in Theorem 1.4.3. In Section 1.5, we investigate some extra conditions on a supercompactly generated topos which guarantee further properties of its category of supercompact objects.

In Section 1.6, we examine some classes of geometric morphism whose inverse image functors preserve supercompact or compact objects, introducing the notions of *pristine* and *polished* geometric morphisms in analogy with proper geometric morphisms (Definition 1.6.2), before focussing on relative versions of these properties (Definition 1.6.5) which are more directly useful in our analysis. Having established these definitions, we examine how some more familiar classes of geometric morphism interact with supercompactly and compactly generated toposes: surjections and inclusions in Section 1.7, then hyperconnected morphisms in Section 1.8. This exploration gives us several tools for constructing such toposes, which are summarised in Theorem 1.8.6.

²We were unable to find a satisfactory definition of the property of local prodiscreteness in the literature.

The focus of Section 2 is a broader site-theoretic investigation. In Section 2.1, we exhibit the categorical data of *principal* and *finitely generated* sites, which are natural classes of sites whose categories of sheaves are supercompactly and compactly generated toposes, respectively. In Section 2.2, we examine the morphisms between the representable sheaves on these sites, and then show in Section 2.3 how a general such site may be reduced via a canonical congruence without changing the resulting topos. We use what we have learned about these sites in Section 2.4 to characterize the categories of supercompact and compact objects which were the subject of Section 1.4 as *reductive* and *coalescent* categories, respectively (Definition 2.4.2), satisfying additional technical conditions. We make the correspondence between such categories and the toposes they generate explicit in Theorem 2.4.12, briefly examining the special cases of these categories which produce localic toposes in Section 2.5. It is natural to compare reductive and coalescent categories to the well-known classes of (locally) regular and coherent categories, which we do in Section 2.6.

Moving onto morphisms, we recall the definition of *morphisms of sites* in Section 2.7, showing that, according to the class of sites under consideration, these induce the relatively pristine, polished or proper geometric morphisms introduced in Section 1.6. More significantly, restricting to canonical sites, we are able to extend the correspondences of Theorem 2.4.12 to some 2-equivalences between 2-categories of sites and 2-categories of toposes. We also to examine comorphisms of sites in Section 2.8, which provide some results about points of various classes of topos.

Finally, to ground the discussion, we present some examples and counterexamples of reductive and coalescent sites and supercompactly and compactly generated toposes in Section 2.9.

1. Supercompact and compact objects

Throughout, when (\mathcal{C}, J) is a site, we write $\ell : \mathcal{C} \longrightarrow \text{Sh}(\mathcal{C}, J)$ for the composite of the Yoneda embedding and the sheafification functor, assuming the Grothendieck topology is clear in context, and call the images $\ell(C)$ of the objects $C \in \mathcal{C}$ the **representable sheaves**.

1.1. SUPERCOMPACT OBJECTS. The following definitions can be found in [Caramello, 2017, Definition 2.1.14]:

1.1.1. DEFINITION. *An object C of a category \mathcal{E} is **supercompact** (resp. **compact**) if any jointly epic family of morphisms $\{A_i \longrightarrow C \mid i \in I\}$ contains an epimorphism (resp. a finite jointly epic sub-family).*

Clearly every supercompact object is compact. Compact objects are more widely studied, notably in the lifting of the concept of compactness from topological spaces to toposes reviewed in [Moerdijk and Vermeulen, 1997]. Since the two classes of objects behave very similarly, we treat them in parallel.

In a topos, we may re-express the definitions of supercompact and compact objects in terms of their subobjects. As is standard, we can further convert any statement about subobjects of an object X in a topos \mathcal{E} into a statement about the subterminal object in the slice topos \mathcal{E}/X .

1.1.2. LEMMA. *An object C of a Grothendieck topos \mathcal{E} is supercompact (resp. compact) if and only if every covering of C by a family (resp. a directed family) of subobjects $A_i \hookrightarrow C$ contains an isomorphism. This occurs if and only if the global sections functor $\Gamma : \mathcal{E}/C \rightarrow \mathbf{Set}$ preserves arbitrary (resp. directed) unions of subobjects.*

PROOF. For the first part, suppose that we are given a family (resp. a directed family) of subobjects covering a supercompact (resp. compact) object C . Then one of the monomorphisms involved must be epic and hence an isomorphism (resp. this collection contains a finite covering family, but the union of these subobjects is also a member of the family and must be covering). In the opposite direction it suffices to consider images of the morphisms in an arbitrary covering family (resp. finite unions of these images).

The remainder of the proof is modelled after that in [Johnstone, 2002, C1.5.5]. Given an object $f : A \rightarrow C$ of \mathcal{E}/C which is a union (resp. a directed union) of subobjects $A_i \hookrightarrow A \rightarrow C$ and given a global section $x : C \rightarrow A$ of f , we may consider the pullbacks:

$$\begin{array}{ccc} C_i & \hookrightarrow & C \\ \downarrow & \lrcorner & \downarrow x \\ A_i & \hookrightarrow & A. \end{array}$$

By extensivity, C is the union of the C_i , and by the above one of the $C_i \hookrightarrow C$ must be an isomorphism, so that x factors through one of the A_i , which gives the result.

Conversely, given a jointly epic family (resp. directed family) of subobjects $C_i \hookrightarrow C$ considered as subterminals in \mathcal{E}/C , we may apply Γ to see that one of them must be an isomorphism, as required. ■

1.2. SUPERCOMPACTLY GENERATED TOPOSES. The collections of supercompact and compact objects in *any* Grothendieck topos are conveniently tractable:

1.2.1. LEMMA. *Let $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J)$ be a Grothendieck topos of sheaves on a small site (\mathcal{C}, J) . Then the supercompact objects are quotients of the representable sheaves $\ell(C)$ for $C \in \mathcal{C}$. In particular, they are indexed (up to isomorphism) by a set. Similarly, the compact objects are quotients of finite coproducts of the images $\ell(C)$, and so (up to isomorphism) also form a set.*

PROOF. Given a supercompact object Q , since the objects $\ell(C)$ are separating in \mathcal{E} , the collection of morphisms $\ell(C) \rightarrow Q$ (is inhabited and) jointly epimorphic. It follows that one such must be epimorphic.

Given a compact object Q , the above argument instead yields a (possibly empty) finite jointly epimorphic family of morphisms $\ell(C_i) \rightarrow Q$, which corresponds to an epimorphism $\coprod_{i \in I} \ell(C_i) \rightarrow Q$, as claimed. ■

Lemma 1.2.1 ensures that we can always consider the full subcategories on the supercompact (resp. compact) objects, equipped with the canonical topology induced by the topos (which shall be recalled in Definition 1.4.1 below) as an essentially small site $(\mathcal{C}_s, J_{can}^{\mathcal{E}}|_{\mathcal{C}_s})$ (resp. $(\mathcal{C}_c, J_{can}^{\mathcal{E}}|_{\mathcal{C}_c})$). By the Comparison Lemma, the induced canonical comparison morphism $\mathcal{E} \rightarrow \text{Sh}(\mathcal{C}_s, J_{can}^{\mathcal{E}}|_{\mathcal{C}_s})$ is an equivalence if and only if the collection of supercompact objects is separating, and similarly for the compact case. We shall continue to use \mathcal{C}_s and \mathcal{C}_c to denote these categories in the remainder; for simplicity, we shall actually assume that we have chosen a representative set of the supercompact or compact objects, such as the quotients of representables in Lemma 1.2.1, so that we are working with small sites.

1.2.2. DEFINITION. *We say a topos is **supercompactly generated** (resp. **compactly generated**) if its collection of supercompact (resp. compact) objects is separating.*

1.2.3. EXAMPLE. The syntactic category of a regular theory can be recovered from its classifying topos as the full category of regular objects. In general, we say that an object X in a topos is **regular** if X is supercompact and for any cospan

$$Y \xrightarrow{f} X \xleftarrow{g} Z$$

with Y and Z supercompact, the pullback $Y \times_X Z$ is also supercompact. In particular, classifying toposes of regular theories are special cases of supercompactly generated toposes. The same can be said when ‘supercompact’ is replaced by ‘compact’ and ‘regular’ is replaced by ‘coherent’. See [Caramello, 2012b] for this result and a more detailed discussion (note that Caramello refers to regular objects as *supercoherent* objects). We shall return to examination of categories of regular and coherent objects in Section 2.6.

Supercompactly generated Grothendieck toposes include several other important established classes of Grothendieck topos.

1.2.4. PROPOSITION.

- (i) *Every atomic topos is supercompactly generated.*
- (ii) *Every supercompactly generated topos is compactly generated and locally connected.*
- (iii) *Every presheaf topos is supercompactly generated.*

PROOF. For (i), recall that a Grothendieck topos is atomic if and only if it has a separating set of atoms, and these are easily seen to be supercompact. For (ii), we can similarly observe that any supercompact object is compact and indecomposable, then recall (by Theorem 2.7 of [Caramello, 2012b], say) that a topos is locally connected if and only if it has a separating set of indecomposable objects.

For (iii), note that the representable presheaves are irreducible projectives so they are in particular supercompact (every jointly epic family over a representable presheaf contains a *split* epimorphism). ■

1.2.5. **EXAMPLE.** Let (M, τ) be a monoid equipped with a topology. We may consider the full subcategory of the topos $[M^{\text{op}}, \mathbf{Set}]$ of right actions of M on sets on those actions which are continuous with respect to the product of τ and the discrete topology on the underlying set; call this category $\text{Cont}(M, \tau)$. In [Rogers, 2021], the results of the present paper are used to show that this category is a supercompactly generated Grothendieck topos.

Given that it is a topos, however, we can show directly that it is supercompactly generated: the supercompact objects are the *continuous cyclic M -sets*, which is to say those generated by a single element. Since we can clearly cover any continuous M -set with the cyclic sub- M -sets generated by its elements, these objects form a separating collection in the topos.

In [Rogers, 2021, Example 3.28], we give the Schanuel topos as an example of such a topos which is not a presheaf topos. We give a different example here, which is not a member of any of the classes of Proposition 1.2.4. Let \mathcal{E} be the topos of continuous actions of the profinite completion of the natural numbers $\hat{\mathbb{N}}$. The continuous cyclic $\hat{\mathbb{N}}$ -sets can be identified with the finite cyclic \mathbb{N} -sets, which are described in [Rogers, 2021, Example 3.14]: these are of the form $N_{a,b} = \{0, 1, \dots, a + b - 1\}$, with a, b non-negative integers and $b \geq 1$; the \mathbb{N} -action is by addition, such that sums greater than $a + b - 1$ are reduced modulo b into the interval $[a, a + b - 1]$. These are not closed under products, so \mathcal{E} is not a regular topos. It is not a presheaf topos, since the topos contains no indecomposable projectives: any such would necessarily be supercompact, but for any (a, b) , there is an epimorphism $N_{a,2b} \twoheadrightarrow N_{a,b}$ which is not split. Finally, it is not atomic, since the atoms are those of the form $N_{0,b}$, and $N_{a,b}$ is not covered by such an \mathbb{N} -set for $a > 0$.

1.3. **CATEGORIES OF SUPERCOMPACT OBJECTS.** We now examine properties of the categories \mathcal{C}_s and \mathcal{C}_c in a general topos \mathcal{E} .

1.3.1. **LEMMA.** *Let \mathcal{E} be a Grothendieck topos and let $\mathcal{C}_s, \mathcal{C}_c$ be the categories of supercompact and compact objects of \mathcal{E} respectively. These categories are closed in \mathcal{E} under quotients.*

PROOF. Given an epimorphism $k : D \twoheadrightarrow C$ with D supercompact and a covering family over C , pulling back this family along k we immediately conclude that one of the constituent morphisms must be an epimorphism. Thus C is a member of \mathcal{C}_s . The argument for \mathcal{C}_c is analogous, except that we end up with a finite family of morphisms. ■

Lemma 1.3.1 has as a consequence that when considering a covering family of supercompact or compact objects over an object X of \mathcal{E} , we may without loss of generality assume that the morphisms in the family are monomorphisms. That is, we may restrict attention to covering families of (super)compact *subobjects* when we so choose, because a family of morphisms with common codomain in a topos is jointly epic if and only if the union of their images is the maximal subobject.

The subcategories inherit some further structure from \mathcal{E} .

1.3.2. COROLLARY. For $\mathcal{E}, \mathcal{C}_s, \mathcal{C}_c$ as in Lemma 1.3.1, \mathcal{C}_s and \mathcal{C}_c are closed under image factorizations in \mathcal{E} , so that in particular they have image factorizations.

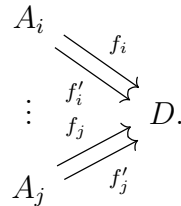
PROOF. Given a morphism $C \rightarrow C'$ between supercompact (resp. compact) objects, the image object C'' in the factorization $C \twoheadrightarrow C'' \hookrightarrow C'$ is also supercompact (resp. compact) by Lemma 1.3.1, whence the factoring morphisms lie in \mathcal{C}_s (resp. \mathcal{C}_c) since it is a full subcategory. ■

Note that the resulting orthogonal factorization systems on \mathcal{C}_s and \mathcal{C}_c are not between all monomorphisms and all epimorphisms; only between those inherited from \mathcal{E} . We spend the rest of this section deriving an intrinsic characterisation of these morphisms.

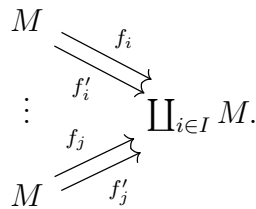
1.3.3. DEFINITION. We say a small indexing category \mathcal{D} is a **funnel** if it has a weakly terminal object. A **funneling diagram** in an arbitrary category \mathcal{C} is a functor $F : \mathcal{D} \rightarrow \mathcal{C}$ with \mathcal{D} a funnel. We call the colimit of such a diagram, if it exists, a **funneling colimit**.

Given a funneling diagram $F : \mathcal{D} \rightarrow \mathcal{C}$, its colimit is determined by an object C of \mathcal{C} equipped with an epimorphism $f : F(D_0) \rightarrow C$, where D_0 is a weakly terminal object of \mathcal{D} , since all legs of the colimit cone factor through this one.

1.3.4. EXAMPLE. An example of a funnel is the following:



Consider the topos $[M^{\text{op}}, \mathbf{Set}]$. We can present any right M -set X as the colimit of a funneling diagram of the above general shape, where the weakly terminal object is sent to a coproduct of copies of M , and all other objects are sent to copies of M :



Here, the set I indexes a set of generators of X and the morphisms (f_i, f'_i) identify pairs of elements which are to be identified.

Recall that a morphism $h : D \rightarrow C$ in a category \mathcal{C} is called a **strict epimorphism** if whenever another morphism $k : D \rightarrow E$ satisfies the condition that for each parallel pair $p, q : B \rightrightarrows D$ with $h \circ p = h \circ q$ we have $k \circ p = k \circ q$, it follows that k factors uniquely through h . The dual concept appears in [Caramello, 2016, Theorem 4.1].

1.3.5. LEMMA. *In a small category \mathcal{C} , a morphism $h : C' \rightarrow C$ is a strict epimorphism if and only if there exists a funneling diagram $F : \mathcal{D} \rightarrow \mathcal{C}$ with weakly terminal object C' whose colimit is expressed by h .*

PROOF. By definition, if h is a strict epimorphism, it is a colimit for the diagram consisting of all pairs of morphisms with domain C' which h coequalizes. Conversely, if h expresses the colimit of any funneling diagram, and k coequalizes all of the same parallel pairs that h does, then it clearly induces a cone by composition with the morphisms of the funneling diagram, whence it has a universal factorisation through h , as required. ■

Notably, strict epimorphisms include isomorphisms and regular epimorphisms. Continuing with the parallel treatment of compactly generated toposes, we arrive at the following definitions.

1.3.6. DEFINITION. *A small indexing category \mathcal{D} is a **multifunnel** if it has a (possibly empty) finite collection of objects D_1, \dots, D_n to which all other objects admit morphisms³. A colimit of a multifunneling diagram (a diagram indexed by a multifunnel) in a category \mathcal{C} shall be called a **multifunneling colimit**, and is defined by a finite jointly epic family from the images of the objects D_1, \dots, D_n . A finite jointly epic family obtained in this way will be called a **strictly epic finite family**.*

1.3.7. LEMMA. *A category has multifunneling colimits if and only if it has finite coproducts and funneling colimits.*

PROOF. Clearly finite coproducts and funneling colimits are special cases of multifunneling colimits. Conversely, given a multifunneling diagram $F : \mathcal{D} \rightarrow \mathcal{C}$ with weakly terminal objects $F(D_1), \dots, F(D_n)$, consider the coproduct of these objects. Composing the morphisms in the diagram F with the coproduct inclusions, we get a funneling diagram. The universal property of the coproduct ensures that the colimit of this diagram coincides with the colimit of F . ■

1.3.8. REMARK. Note that we can make the further simplification, implicit in the diagram of Definition 1.3.3, that all of the non-identity morphisms in a funneling diagram have the weakly terminal object as their codomains, since given $t : A_i \rightarrow A_j$, there exists some morphism $f_j : A_j \rightarrow D$, and the cocone commutativity conditions for $f_j \circ t$ and f_j ensure that $\lambda_i = \lambda_D \circ (f_j \circ t) = \lambda_j \circ t$ is automatically satisfied, so we may omit t from the diagram.

1.3.9. LEMMA. *Let \mathcal{E} be a Grothendieck topos and let $\mathcal{C}_s, \mathcal{C}_c$ be the categories of supercompact and compact objects of \mathcal{E} respectively. Then \mathcal{C}_s is closed in \mathcal{E} under funneling colimits and \mathcal{C}_c is closed in \mathcal{E} under multifunneling colimits.*

³A reader interested in the obvious generalisations of this concept to higher cardinalities might prefer to employ a name such as ‘finitely funneled’ to emphasise the finitary aspect.

PROOF. Let $F : \mathcal{D} \rightarrow \mathcal{C}_s$ be a funneling diagram with weakly terminal object $F(D)$. In \mathcal{E} , this diagram has a colimit determined by an epimorphism $f : F(D) \rightarrow C$; the colimit C is supercompact by Lemma 1.3.1, as required. The argument for multifunneling colimits in \mathcal{C}_c is analogous, except that we must pull back along each member of a finite family in the proof of Lemma 1.3.1 to obtain the finite covering subfamily of a given covering family over the colimit. ■

1.3.10. COROLLARY. *Let \mathcal{E} be a supercompactly (resp. compactly) generated topos and $\mathcal{C}_s, \mathcal{C}_c$ the usual subcategories. Then a morphism of \mathcal{C}_s is an epimorphism in \mathcal{E} if and only if it is a strict epimorphism in \mathcal{C}_s , and a finite family of morphisms into C in \mathcal{C}_c is jointly epic in \mathcal{E} if and only if it is a strictly epic finite family in \mathcal{C}_c .*

PROOF. This could be deduced by checking the conditions of the dual of [Caramello, 2016, Proposition 4.9], but rather than reproducing that result, we give a direct proof.

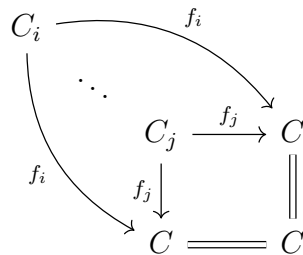
Suppose $h : C' \rightarrow C$ is epic in \mathcal{E} with C', C in \mathcal{C}_s . Any epimorphism in \mathcal{E} is regular, so is the coequalizer of some pair $p, q : D \rightrightarrows C'$. Since D is covered by supercompact objects, composing p and q with the monomorphisms $C_i \hookrightarrow D$ such that C_i is in \mathcal{C}_s we obtain a funneling diagram in \mathcal{C}_s whose colimit is still C , as required. The argument for \mathcal{C}_c is analogous.

Conversely, the inclusion of \mathcal{C}_s and \mathcal{C}_c into \mathcal{E} preserves funneling (resp. multifunneling) colimits by Lemma 1.3.9, whence the strict epimorphisms (resp. strictly epic finite families) from these categories are still epic in \mathcal{E} . ■

In fact, epimorphic families in \mathcal{C}_c are better behaved than those in \mathcal{C}_s in general:

1.3.11. LEMMA. *Let \mathcal{E} be any Grothendieck topos. Then a family of morphisms in \mathcal{C}_c with common codomain is jointly epic in \mathcal{C}_c if and only if it is so in \mathcal{E} . In particular, when \mathcal{E} is compactly generated, every jointly epimorphic family (including every epimorphism) in \mathcal{C}_c is strict.*

PROOF. Observe that a family of morphisms $f_i : C_i \rightarrow C$ in a category is jointly epimorphic if and only if the diagram:



is a colimit diagram. But the diagram (after removing the copy of C in the lower right corner) is clearly an instance of a multifunneling colimit, so by Lemma 1.3.9 its colimit is created by the inclusion of \mathcal{C}_c into \mathcal{E} , so a family is jointly epic in \mathcal{C}_c if and only if it is so in \mathcal{E} , as required. ■

Having extensively discussed the epimorphisms, we should also discuss monomorphisms in the subcategories under investigation.

1.3.12. LEMMA. *Monomorphisms in \mathcal{C}_s and \mathcal{C}_c coincide with monomorphisms in \mathcal{E} when \mathcal{E} is supercompactly or compactly generated, respectively.*

PROOF. Certainly a monomorphism of \mathcal{E} lying in \mathcal{C}_s or \mathcal{C}_c is still monic, since there are fewer morphisms which it needs to distinguish in general.

Suppose \mathcal{E} is supercompactly generated and let $s : A \hookrightarrow B$ be a monomorphism in \mathcal{C}_s , Q an object of \mathcal{E} and $f, g : Q \rightrightarrows A$ such that $sf = sg$. Covering Q with supercompact subobjects $q_i : Q_i \hookrightarrow Q$, consider $f q_i, g q_i : Q_i \rightrightarrows A$, which are morphisms in \mathcal{C}_s . These are equalized by s and hence are equal for every i . The q_i being jointly epic then forces $f = g$. Thus s is monic in \mathcal{E} , as claimed. The argument for \mathcal{C}_c is analogous, replacing supercompact subobjects with compact ones. ■

Once again, we can immediately strengthen this result for \mathcal{C}_c .

1.3.13. LEMMA. *For any Grothendieck topos \mathcal{E} , the monomorphisms of \mathcal{E} lying in \mathcal{C}_c are regular monomorphisms there. In particular, when \mathcal{E} is compactly generated, every monomorphism in \mathcal{C}_c is regular.*

PROOF. If $e : C' \hookrightarrow C$ is a morphism in \mathcal{C}_s which is monic in \mathcal{E} , consider its cokernel pair in \mathcal{E} :

$$\begin{array}{ccc} C' & \xleftarrow{e} & C \\ e \downarrow & \lrcorner & \downarrow s \\ C & \xrightarrow{t} & D. \end{array}$$

Since a topos is an adhesive category, this is also a pullback square and so e is the equalizer of s and t . Since pushouts are multifunnel colimits, D lies in \mathcal{C}_s , so the same is true there. ■

Thus we can make Corollary 1.3.2 more precise.

1.3.14. COROLLARY. *If \mathcal{E} is supercompactly generated, then \mathcal{C}_s has an orthogonal (strict epi,mono)-factorisation system. More generally, if \mathcal{E} is merely compactly generated, \mathcal{C}_c has an orthogonal (epi,mono)-factorisation system.*

For later reference, we observe that even though the categories \mathcal{C}_s and \mathcal{C}_c need not have finite products (see Example 2.9.6), we can still extend Corollaries 1.3.2 and 1.3.14 with factorizations of spans through jointly monic spans. Corollary 1.3.14 is the case $I = 1$ of the following Lemma.

1.3.15. LEMMA. *Let $\{f_j : B \rightarrow A_j \mid j \in I\}$ be a collection of morphisms with common domain in \mathcal{C}_s or \mathcal{C}_c . Then there exists a strict epimorphism $e : B \rightarrow R$ and morphisms $\{r_j : R \rightarrow A_j \mid j \in I\}$ which are jointly monic, such that $f_j = r_j \circ e$.*

PROOF. Let e be the strict epimorphism obtained from the funneling colimit of the collection of all parallel pairs of morphism which are coequalized by all of the f_j . By definition, all of the f_j factorize through it, and by construction the factors form a jointly monic family. ■

1.4. CANONICAL SITES OF SUPERCOMPACT OBJECTS. We have now done enough work to usefully apply the proof of Giraud’s theorem and obtain a canonical site of definition for a supercompactly or compactly generated Grothendieck topos.

1.4.1. DEFINITION. Recall that a sieve S on an object C of a category \mathcal{C} is **effective-epimorphic** if, when S is viewed as a full subcategory of \mathcal{C}/C , C is the colimit of the (possibly large) diagram $D_S : S \hookrightarrow \mathcal{C}/C \rightarrow \mathcal{C}$ obtained by composing with the forgetful functor. A sieve generated by a single morphism f is effective-epimorphic if and only if the morphism is a strict epimorphism. Such a sieve S is **universally** effective-epimorphic if its pullback along the functor $\mathcal{C}/D \rightarrow \mathcal{C}/C$ induced by a morphism $f : D \rightarrow C$ is effective-epimorphic for any f .

The **canonical Grothendieck topology** $J_{can}^{\mathcal{C}}$ on \mathcal{C} is the topology whose covering sieves are precisely the universally effective-epimorphic ones. If \mathcal{C} is a Grothendieck topos, this coincides with the Grothendieck topology whose covering sieves are those containing small jointly epic families.

We recall the following result, which appears as [Caramello, 2019, Lemma 4.35]:

1.4.2. LEMMA. Let \mathcal{E} be a Grothendieck topos and \mathcal{C} a small full separating subcategory of \mathcal{E} . Let S be a sieve in \mathcal{C} on an object C and let D_S be the diagram in \mathcal{C} described in Definition 1.4.1. Suppose that the colimit of D_S in \mathcal{E} lies in \mathcal{C} . Then S is universally effective-epimorphic in \mathcal{C} if and only if it is the restriction to \mathcal{C} of a sieve containing a small jointly epic family in \mathcal{E} .

1.4.3. THEOREM. Suppose \mathcal{E} is supercompactly generated. Let J_r be the Grothendieck topology on \mathcal{C}_s whose covering sieves are those containing strict epimorphisms. Then $\mathcal{E} \simeq \text{Sh}(\mathcal{C}_s, J_r)$.

Similarly, if \mathcal{E} is compactly generated, and J_c is the Grothendieck topology on \mathcal{C}_c whose covering sieves are those containing strictly epic (equivalently, jointly epic) finite families. Then $\mathcal{E} \simeq \text{Sh}(\mathcal{C}_c, J_c)$.

PROOF. By Giraud’s theorem, given a (small, full) separating subcategory \mathcal{C} of objects in a Grothendieck topos \mathcal{E} , we have an equivalence of toposes $\mathcal{E} \simeq \text{Sh}(\mathcal{C}, J_{can}^{\mathcal{E}}|_{\mathcal{C}})$, where $J_{can}^{\mathcal{E}}|_{\mathcal{C}}$ is the restriction of the canonical topology on \mathcal{E} to \mathcal{C} , whose covering sieves are the intersections of $J_{can}^{\mathcal{E}}$ -sieves with \mathcal{C} . Thus it suffices to show in each case that the restriction of the canonical topology is the topology described in the statement.

By Lemma 1.3.9, the principal (resp. finitely generated) sieves S on \mathcal{C}_s (resp. \mathcal{C}_c), whose corresponding diagrams D_S are funneling (resp. multifunneling) diagrams, have colimits contained in \mathcal{C}_s (resp. \mathcal{C}_c), so Lemma 1.4.2 applies. Thus these are effective epimorphic sieves if and only if the generating morphism is a strict epimorphism in \mathcal{C}_s (resp. the generating morphisms form a strictly epimorphic finite family in \mathcal{C}_c).

Now given any sieve S containing a jointly epic family on an object of \mathcal{C}_s (resp. \mathcal{C}_c) in \mathcal{E} , by the definition of supercompact (resp. compact) objects, S must contain an epimorphism (resp. a finite covering family). In particular, every $J_{can}^{\mathcal{E}}|_{\mathcal{C}_s}$ -covering sieve

contains a $J_{can}^{\mathcal{E}}|_{\mathcal{C}_s}$ -covering *principal* sieve. Similarly, every $J_{can}^{\mathcal{E}}|_{\mathcal{C}_c}$ -covering sieve contains a $J_{can}^{\mathcal{E}}|_{\mathcal{C}_c}$ -covering *finitely generated* sieve.

It follows that the strict epimorphisms in \mathcal{C}_s are precisely the morphisms generating universally effective-epimorphic families and similarly for strict jointly epimorphic families in \mathcal{C}_c , as required. Alternatively, see the proof of Proposition 2.1.7 below for a direct argument showing that the strict epimorphisms (resp. strictly epic finite families) are stable. ■

1.4.4. REMARK. It should be clear by now from our joint treatment of supercompactness and compactness that much of our analysis can be applied to more general notions of compactness. Indeed, Theorem 1.4.3 is an explicit special case of [Caramello, 2019, Proposition 4.36].

Suppose P is some property of pre-sieves (families of morphisms with common co-domain); then we may define P -compact objects in a topos \mathcal{E} as those for which every jointly epic covering family contains a jointly epic presieve satisfying P . If P satisfies suitable composition and stability criteria, which Caramello specifies, and the full subcategory \mathcal{C}_P of \mathcal{E} on the P -compact objects is separating and closed under certain colimits, then \mathcal{E} is equivalent to the category of sheaves on \mathcal{C}_P for the topology generated by the effective-epimorphic P -presieves in \mathcal{C}_P . For supercompactness, P is the property ‘is a singleton’, while for ordinary compactness, P is the property ‘is finite’, and our earlier results show that these do satisfy Caramello’s criteria. More generally, any property which descends along epimorphisms can yield an interesting class of toposes; this is the methodology of [Caramello, 2012a].

While we shall not attempt to extend the present paper to this most general case, we encourage the reader to explore whether any given topos of interest to them is P -compactly generated for some suitable property P , and if so to compute the corresponding site produced by Caramello’s result.

The advantage of the intrinsic expressions for the Grothendieck topologies in Theorem 1.4.3 is that it guarantees that the categories of (super)compact objects contain enough information to completely reconstruct the toposes *by themselves*. This immediately gives us results such as the following:

1.4.5. COROLLARY. *Suppose \mathcal{E} and \mathcal{E}' are supercompactly generated toposes and $\mathcal{C}_s, \mathcal{C}'_s$ are their respective categories of supercompact objects. Then $\mathcal{E} \simeq \mathcal{E}'$ if and only if $\mathcal{C}_s \simeq \mathcal{C}'_s$.*

1.5. COKERNELS, WELL-SUPPORTED OBJECTS AND TWO-VALUED TOPOSES. We saw in Lemmas 1.3.11 and 1.3.13 that when \mathcal{E} is compactly generated, every epimorphism in \mathcal{C}_c is strict and every monomorphism in \mathcal{C}_c is regular. This leads us to wonder under what extra conditions these facts hold true in \mathcal{C}_s , given that \mathcal{E} is supercompactly generated.

1.5.1. EXAMPLE. To properly motivate this section, we show that epimorphisms in \mathcal{C}_s need not coincide with those in \mathcal{E} . Let \mathcal{D} be the category

$$A \xleftarrow{l} B \xrightarrow{r} C.$$

In the topos \mathcal{E} of presheaves on \mathcal{D} it is easily calculated that the supercompact objects are precisely the representables, so \mathcal{D} coincides with \mathcal{C}_s (we shall see in Proposition 2.5.1 that this argument is valid for all posets). The morphisms l and r are trivially epic in \mathcal{D} but are not epic in \mathcal{E} .

Our main tool in this section is the following definition.

1.5.2. DEFINITION. Given a morphism $f : A \rightarrow B$, its **cokernel**⁴ $B \rightarrow B/f$ is the pushout:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \text{\textcircled{!}} \downarrow & \lrcorner & \downarrow \\ 1 & \xrightarrow{x} & B/f. \end{array}$$

Cokernels are useful for understanding epimorphisms thanks to the following result.

1.5.3. LEMMA. A morphism $f : A \rightarrow B$ of a topos \mathcal{E} is an epimorphism if and only if the lower morphism $x : 1 \rightarrow B/f$ of its cokernel is an isomorphism (or equivalently, an epimorphism).

PROOF. If f is an epimorphism we have:

$$\begin{array}{ccc} A & \xrightarrow{f} \twoheadrightarrow & B \\ \text{\textcircled{!}} \downarrow & \lrcorner & \downarrow \\ 1 & \xrightarrow{x} \twoheadrightarrow & B/f, \end{array}$$

since the pushout of an epimorphism is epic. But any quotient of 1 in a topos is an isomorphism, as required.

Conversely, if $x : 1 \rightarrow B/f$ is an isomorphism, we can consider the (epi,mono) factorization $f = m \circ e$:

$$\begin{array}{ccccc} & & f & & \\ & & \curvearrowright & & \\ A & \xrightarrow{e} \twoheadrightarrow & A' & \xrightarrow{m} \hookrightarrow & B \\ \text{\textcircled{!}} \downarrow & & \text{\textcircled{!}} \downarrow & & \downarrow \\ 1 & \xrightarrow{\sim} & 1 & \xrightarrow{\sim} & B/f. \\ & & \curvearrowleft & & \\ & & x & & \end{array}$$

By the first part, the left hand square and outside rectangle are both pushouts, which makes the right hand square a pushout. But in a topos (or any adhesive category), a pushout square in which the upper horizontal morphism is monic is also a pullback square. Thus m is an isomorphism, and f is epic. ■

⁴Not to be confused with the *cokernel pairs* mentioned in the proof of Lemma 1.3.13.

Recall that an object A of a category with a terminal object is **well-supported** if the unique morphism $!_A : A \rightarrow 1$ is an epimorphism in \mathcal{E} . Using cokernels, we obtain a partial dual to Lemma 1.3.12 even without requiring \mathcal{E} to be (super)compactly generated.

1.5.4. LEMMA. *Let \mathcal{E} be a topos and \mathcal{C}_s the usual subcategory. Let A be an object of \mathcal{C}_s which is well-supported as an object of \mathcal{E} . Then a morphism $A \rightarrow B$ of \mathcal{C}_s is an epimorphism in that category if and only if it is an epimorphism in \mathcal{E} .*

PROOF. An epimorphism of \mathcal{E} lying in \mathcal{C}_s clearly remains epic there.

Conversely, if $e : A \rightarrow B$ is epic in \mathcal{C}_s , consider the cokernel $B \rightarrow B/e$:

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ \downarrow !_A & \lrcorner & \downarrow q \\ 1 & \xrightarrow{x} & B/e. \end{array}$$

By assumption B/e is supercompact as q is epic in \mathcal{E} . The unique morphism $!_A : A \rightarrow 1$ factors through e via the unique morphism $!_B : B \rightarrow 1$. Thus we have $qe = x!_A = x!_B e$, and since these expressions are composed of morphisms lying in \mathcal{C}_s where e is epic, it follows that $q = x!_B$ whence x is epic and hence an isomorphism. Hence e is epic in \mathcal{E} , by Lemma 1.5.3. ■

It is worth noting that the existence of any well-supported supercompact object forces the terminal object of \mathcal{E} to be supercompact. Moreover, this proof concertedly fails when the object A is not well-supported:

1.5.5. LEMMA. *Let \mathcal{E} , \mathcal{C}_s be as above. Then \mathcal{C}_s is closed under cokernels if and only if every supercompact object is well-supported.*

PROOF. Given a supercompact object A , consider its **support**, which is the subterminal object $U \hookrightarrow 1$ in the factorization of the morphism $!_A$. By Lemma 1.3.1, U is supercompact, so if \mathcal{C}_s is closed under cokernels, the pushout of $U \hookrightarrow 1$ along itself must be in \mathcal{C}_s . But the colimit morphisms $1 \rightrightarrows 1 +_U 1$ are jointly epic, so one of them must be an epimorphism and hence an isomorphism, which forces $U \cong 1$. Thus A is well-supported, as required.

Conversely, if every object A of \mathcal{C}_s is well-supported then the cokernel (in \mathcal{E}) of a morphism $A \rightarrow B$ in \mathcal{C}_s is a quotient of B and so is supercompact. Thus \mathcal{C}_s is closed under cokernels, as required. ■

We shall see a relevant sufficient condition for this to occur in Proposition 1.8.1. In this setting we can also strengthen Lemma 1.3.12.

1.5.6. SCHOLIUM. *Let \mathcal{E} be a topos such that every object of \mathcal{C}_s is well-supported. Then monomorphisms in \mathcal{C}_s inherited from \mathcal{E} are regular. In particular, if \mathcal{E} is also supercompactly generated, then all monomorphisms in \mathcal{C}_s are regular.*

PROOF. By Proposition 1.8.1, the hypotheses guarantee that the cokernel of a morphism in \mathcal{C}_s also lies in that category.

As remarked in the proof of Lemma 1.5.3, the pushout square defining the cokernel of an inclusion of supercompact objects $i : A \hookrightarrow B$ is also a pullback in \mathcal{E} . It follows easily that i is the equalizer in \mathcal{E} of the morphisms $q, x!_B : B \rightrightarrows B/i$ described in the proof of Lemma 1.5.4, and is consequently also their equalizer in \mathcal{C}_s .

If \mathcal{E} is supercompactly (resp. compactly) generated then we may apply Lemma 1.3.12 to conclude that the above applies to all monomorphisms of \mathcal{C}_s (resp. \mathcal{C}_c). ■

On the other hand, we shall see in Example 1.8.3 that it is not in general possible to strengthen the properties of epimorphisms in \mathcal{C}_s beyond the consequence of the proof of Theorem 1.4.3 that they are strict. In particular, they are not regular in general.

Finally, we explicitly record how Lemma 1.5.4 simplifies the expression for the Grothendieck topology J_r induced on \mathcal{C}_s from Theorem 1.4.3.

1.5.7. COROLLARY. *Let \mathcal{E} be a supercompactly generated topos and \mathcal{C}_s its full subcategory of supercompact objects. Suppose every object of \mathcal{C}_s is well-supported. Let J_r be the topology on \mathcal{C}_s whose covering sieves are precisely those containing epimorphisms. Then $\mathcal{E} \simeq \text{Sh}(\mathcal{C}_s, J_r)$.*

1.6. PROPER, POLISHED AND PRISTINE MORPHISMS. In this section we present some classes of geometric morphism whose inverse image functors interact well with supercompact and compact objects. The material in this section contains more technical topos theoretic concepts than that in the earlier sections, so may be skipped on a first reading; the key result is Proposition 1.6.9.

The latter part of Lemma 1.1.2, regarding compact objects, states precisely that the geometric morphism $\mathcal{E}/C \rightarrow \mathbf{Set}$ is proper in the sense of [Moerdijk and Vermeulen, 1997, Definition I.1.8] (see also [Johnstone, 2002, C3.2.12 and C1.5.5]), or equivalently that \mathcal{E}/C is a compact Grothendieck topos. We can generalise Moerdijk and Vermeulen’s definition of proper morphisms to the arbitrary union case in order to capture the idea of supercompactness. In order to do this, we recall some classic topos-theoretic constructions, from [Johnstone, 1977, Chapter 2].

Recall from [Johnstone, 1977, Definition 2.11] that an **internal category** \mathbb{I} in a topos (or, more generally, a cartesian category) \mathcal{E} consists of:

- (i) An *objects of objects* I_0 and an *object of morphisms* I_1 in \mathcal{E} , and
- (ii) Morphisms $i : I_0 \rightarrow I_1$, $d, c : I_1 \rightrightarrows I_0$ and $m : I_2 \rightarrow I_1$,

where $I_2 := c \times_{I_0} d$ is the *object of composable pairs*, defined as the pullback:

$$\begin{array}{ccc}
 I_2 & \xrightarrow{\pi_2} & I_1 \\
 \pi_1 \downarrow & \lrcorner & \downarrow d \\
 I_1 & \xrightarrow{c} & I_0.
 \end{array}$$

The morphisms define the *identity morphisms*, *domains*, *codomains* and *composition*, respectively. These are required to satisfy the equations $di = ci = \text{id}_{I_0}$, $dm = d\pi_1$, $cm = c\pi_2$, $m(\text{id} \times m) = m(m \times \text{id})$ and $m(\text{id} \times i) = m(i \times \text{id}) = \text{id}_{I_1}$. These are diagrammatic translations of the axioms for ordinary categories in **Set**. An **internal functor** between internal categories is a pair of morphisms between the respective objects of objects and objects of morphisms commuting with the respective structure morphisms.

1.6.1. DEFINITION. We say an internal category \mathbb{I} in \mathcal{E} is **filtered** if the usual definition of filteredness, cf. [Mac Lane and Moerdijk, 1992, §VII.6], is satisfied internally. Since this internalization is rarely made explicit, we explain it in full here: \mathbb{I} is filtered if and only if the following three conditions are satisfied,

- \mathbb{I} is internally **inhabited**⁵, which is to say that I_0 is a well-supported object.
- The morphism $c \times_{I_0} c \hookrightarrow I_1 \times I_1 \xrightarrow{d \times d} I_0 \times I_0$, is an epimorphism; the domain is the ‘object of pairs of morphisms with common codomain’.
- Let $A \hookrightarrow I_1 \times I_1 \times I_1$ be the subobject $(d \times_{I_0} d \times \text{id}_{I_1}) \cap (c \times_{I_0} c \times_{I_0} d)$, which is the internalization of $\{f, g, h \in I_1 \mid d(f) = d(g) \wedge c(f) = c(g) = d(h)\}$. Let E be the equalizer of the morphisms $m \circ \pi_1, m \circ \pi_2 : A \rightrightarrows I_1$, the subobject consisting of the triples satisfying $h \circ f = h \circ g$. Finally, let $B \hookrightarrow I_1 \times I_1$ be $(c \times_{I_0} c) \times_{I_1 \times I_1} (d \times_{I_0} d)$, the ‘object of parallel pairs of morphisms’. We require that the projection $\pi_1 \times \pi_2 : E \rightarrow B$ be an epimorphism.

While the constructions of Definition 1.6.1 are a little technical, reasoning about filtered internal categories always amounts to a categorical re-expression of the usual external reasoning for such categories.

Given an internal category \mathbb{I} in \mathcal{E} , we recall from [Johnstone, 1977, Definition 2.14] that an (internal) **diagram** of shape \mathbb{I} consists of:

1. An object $a : F_0 \rightarrow I_0$ of \mathcal{E}/I_0 , and
2. A morphism $b : F_1 \rightarrow F_0$,

where F_1 is the defined as the pullback

$$\begin{array}{ccc} F_1 & \xrightarrow{\pi_2} & F_0 \\ \pi_1 \downarrow & \lrcorner & \downarrow a \\ I_1 & \xrightarrow{d} & I_0, \end{array}$$

such that $ab = c\pi_2$, $b(\text{id} \times i) = \text{id}_{F_0}$, and $e(e \times \text{id}) = e(\text{id} \times m)$. In **Set**, this data captures the encoding of a functor into **Set** via the Grothendieck construction.

⁵This is the constructive term for ‘non-emptiness’, meaning ‘has an element’.

There is an accompanying notion of natural transformation, and hence we obtain the diagram category $[\mathbb{I}, \mathcal{E}]$. This is a topos over \mathcal{E} , which is proved using the comonadicity theorem in [Johnstone, 1977, Corollary 2.33].

For any geometric morphism $f : \mathcal{F} \rightarrow \mathcal{E}$, since f^* preserves finite limits, we can apply f^* to the data of an internal category \mathbb{I} in \mathcal{E} to obtain an internal category $f^*(\mathbb{I})$ in \mathcal{F} . There is an induced pullback square of diagram toposes: ⁶

$$\begin{array}{ccc}
 [f^*(\mathbb{I}), \mathcal{F}] & \xrightarrow{f^{\mathbb{I}}} & [\mathbb{I}, \mathcal{E}] \\
 \pi \downarrow & \lrcorner & \downarrow \pi \\
 \mathcal{F} & \xrightarrow{f} & \mathcal{E},
 \end{array} \tag{1}$$

Each vertical morphism labelled π has inverse image functor sending an object to the ‘constant diagram of shape \mathbb{I} ’. As well as a right adjoint π_* , this functor always has an \mathcal{E} -indexed left adjoint $\pi_!$. The functors π_* and $\pi_!$ send an internal diagram to its (internal) limit and colimit, respectively.

1.6.2. DEFINITION. *Let E be an object of \mathcal{E} and \mathbb{I} an internal category (resp. inhabited internal category \mathbb{I} ; filtered internal category \mathbb{I}) in \mathcal{E}/E . Let $f : \mathcal{F} \rightarrow \mathcal{E}$ be a geometric morphism. Consider the special case of (1) where we take the lower geometric morphism to be f/E :*

$$\begin{array}{ccc}
 [f^*(\mathbb{I}), \mathcal{F}/f^*(E)] & \xrightarrow{(f/E)^{\mathbb{I}}} & [\mathbb{I}, \mathcal{E}/E] \\
 \pi \downarrow & \lrcorner & \downarrow \pi \\
 \mathcal{F}/f^*(E) & \xrightarrow{f/E} & \mathcal{E}/E.
 \end{array} \tag{2}$$

We call a geometric morphism $f : \mathcal{F} \rightarrow \mathcal{E}$ **pristine** (resp. **polished**; **proper**) if the square above satisfies the condition $\pi_! \circ (f/E)_*^{\mathbb{I}}(V) \cong (f/E)_* \circ \pi_!(V)$ for every subterminal object V of $[f^*(\mathbb{I}), \mathcal{F}/f^*(E)]$, for every choice of E and \mathbb{I} . This can be understood as stating that the direct image of f preserves \mathcal{E} -indexed (resp. \mathcal{E} -indexed inhabited; \mathcal{E} -indexed directed) unions of subobjects.

We call a Grothendieck topos **supercompact** (resp. **compact**) if its unique geometric morphism to **Set** is pristine (resp. proper), which by Lemma 1.1.2 occurs if and only if the terminal object has the corresponding property. The global sections morphism of a Grothendieck topos is polished if and only if the topos is supercompact or degenerate (that is, the terminal object is either supercompact or initial).

Note that Moerdijk and Vermeulen denote $\pi_!$ by ∞^* because, in the proper case, $\pi_!$ preserves finite limits and hence is the inverse image functor of a geometric morphism in the opposite direction. In the pristine and polished cases the left adjoint $\pi_!$ is in general not left exact, so this notation no longer makes sense.

⁶See [Johnstone, 2002, Corollary B3.2.12] for an explanation of why this square is a pullback.

1.6.3. **REMARK.** A filtered *preorder* is ordinarily called **directed**; we have of course already seen this notion in Lemma 1.1.2. It is intuitive that a filtered diagram of subobjects can be re-expressed as a directed preorder-indexed diagram by identifying parallel indexing morphisms. We do this informally in Definition 1.6.2, but it is formally justified: we shall see in Corollary 1.8.4 that hyperconnected morphisms are proper, recovering the fact ([Moerdijk and Vermeulen, 1997, Corollary I.2.5]) that a geometric morphism is proper if and only if its localic part is, and the localic part of the canonical geometric morphism $[\mathbb{I}, \mathcal{E}] \rightarrow \mathcal{E}$ is the corresponding morphism $[\mathbb{I}', \mathcal{E}] \rightarrow \mathcal{E}$, where \mathbb{I}' is the internal preorder reflection of \mathbb{I} .

1.6.4. **EXAMPLE.** The presheaf topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is supercompact if and only if \mathcal{C} is a funnel in the sense of Definition 1.3.3. Indeed, if \mathcal{C} has a weakly terminal object C_0 then there is by inspection an epimorphism $\mathbf{y}(C_0) \rightarrow 1$ in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$, and conversely if 1 is supercompact then one of the morphisms $\mathbf{y}(C) \rightarrow 1$ must be an epimorphism, since these are jointly epic, whence every object of \mathcal{C} admits at least one morphism to the corresponding object of \mathcal{C} . More generally, $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is compact if and only if \mathcal{C} is a multifunnel in the sense of Definition 1.3.6.

An immediate consequence of Definition 1.6.2 is that we can relativise the concepts of supercompactness and compactness to depend on the base topos over which we work (so far we have been implicitly working over \mathbf{Set}). Viewing a geometric morphism $\mathcal{F} \rightarrow \mathcal{E}$ as expressing \mathcal{F} as a topos over \mathcal{E} , an object X of \mathcal{F} is **\mathcal{E} -compact** if the composite geometric morphism $\mathcal{F}/X \rightarrow \mathcal{F} \rightarrow \mathcal{E}$ is proper, for example. Conversely, since in this paper we will only be concerned with objects which are supercompact relative to some fixed base topos \mathcal{S} , it makes sense to employ the broader classes of geometric morphism introduced in [Moerdijk and Vermeulen, 1997, Chapter V].

1.6.5. **DEFINITION.** Let $p : \mathcal{E} \rightarrow \mathcal{S}$ and $q : \mathcal{F} \rightarrow \mathcal{S}$ be toposes over \mathcal{S} . A geometric morphism $f : \mathcal{F} \rightarrow \mathcal{E}$ over \mathcal{S} is **\mathcal{S} -relatively pristine** (resp. **\mathcal{S} -relatively polished**; **\mathcal{S} -relatively proper**) if its direct image preserves arbitrary \mathcal{S} -indexed unions (resp. inhabited \mathcal{S} -indexed unions; directed \mathcal{S} -indexed unions) of subobjects. Explicitly, this requires that the respective conditions of Definition 1.6.2 hold for diagrams \mathbb{I} in \mathcal{E} of the form $p^*(\mathbb{I}')$, where \mathbb{I}' is a diagram category of the appropriate type in \mathcal{S} . We shall assume \mathcal{S} is \mathbf{Set} in the remainder, and so we drop the ‘ \mathcal{S} -’ prefixes.

All of these definitions appear hard to work with in general for the simple reason that internal diagram categories take a significant amount of computation to express and work with concretely (which is to say externally) in any given case. However, the \mathcal{S} -relative notions conveniently coincide with their external counterparts, in a sense made pristine in Lemma 1.6.8 below. Also, all of the notions are clearly stable under slicing and composition, from which we can extract general consequences which are sufficient for the purposes of this paper.

By inspection, we have the following relationships between Definitions 1.6.2 and 1.6.5.

1.6.6. LEMMA. Consider a commuting triangle of geometric morphisms:

$$\begin{array}{ccc}
 \mathcal{F} & \xrightarrow{f} & \mathcal{E} \\
 & \searrow q & \swarrow p \\
 & \mathbf{Set}. &
 \end{array}$$

Then:

1. If f is pristine, it is relatively pristine.
2. p and q are relatively pristine morphism if and only if they are pristine.
3. If f is relatively pristine and p is pristine, then q is pristine.

The same statements hold when ‘pristine’ is replaced by ‘polished’ or ‘proper’.

A handy consequence of this for our objects of interest is the following:

1.6.7. COROLLARY. Let $f : \mathcal{F} \rightarrow \mathcal{E}$ be a geometric morphism between Grothendieck toposes. If f is relatively pristine (resp. relatively polished, relatively proper), then f^* preserves supercompact (resp. ‘supercompact or initial’, compact) objects.

PROOF. Given an object E of \mathcal{E} and a relatively pristine (resp. relatively polished, relatively proper) morphism f , consider the triangle:

$$\begin{array}{ccc}
 \mathcal{F}/f^*(E) & \xrightarrow{f/E} & \mathcal{E}/E \\
 & \searrow & \swarrow \\
 & \mathbf{Set}. &
 \end{array}$$

If E is supercompact (resp. ‘supercompact or initial’, compact), then the global sections morphism of \mathcal{E}/E is also pristine (resp. polished, proper), so $f^*(E)$ must be supercompact (resp. supercompact or initial, compact) by Lemma 1.6.6.3. ■

In order to make more explicit arguments, we now extend the characterisation of relatively proper geometric morphisms in [Moerdijk and Vermeulen, 1997, Proposition V.3.7(i)].

1.6.8. LEMMA. A geometric morphism $f : \mathcal{F} \rightarrow \mathcal{E}$ over \mathcal{S} is relatively pristine (resp. relatively polished, relatively proper) if and only if for any \mathcal{S} -indexed⁷ (resp. \mathcal{S} -inhabited, \mathcal{S} -directed) jointly epimorphic family $\{g_i : X_i \hookrightarrow f^*(Y)\}$ of subobjects in \mathcal{F} there exists a jointly epimorphic family $\{h_j : Y_j \rightarrow Y\}$ in \mathcal{E} such that each $f^*(h_j)$ factors through some g_i .

⁷Taking \mathcal{S} to be **Set**, \mathcal{S} -indexed just means set-indexed, or small.

PROOF. Suppose f has one of the relative properties and we are given a collection of subobjects of the relevant type. By assumption, their union is preserved by $(f/Y)_*$, whence there are subobjects $h_j : Y_j \hookrightarrow Y$ (the images under $(f/Y)_*$ of the subobjects) which are also jointly epic. By construction, each of these must have image under f^* which factors through one or more of the X_i .

Conversely, given an \mathcal{S} -indexed diagram (of the relevant type) of subterminal objects $\{g_i : X_i \longrightarrow f^*(Y)\}$ in $\mathcal{F}/f^*(Y)$, let $U \hookrightarrow f^*(Y)$ be the union of the g_i , and let $Y' \hookrightarrow Y$ be its image under $(f/Y)_*$ in \mathcal{E}/Y . Applying f^* , we have a monomorphism $f^*(Y') \hookrightarrow U$, so we can pull back to get a jointly epimorphic family of the appropriate shape $\{g'_i : X'_i \longrightarrow f^*(Y')\}$ over $f^*(Y')$. This is precisely data of the required form to apply the hypotheses, and the covering family of Y' thus provided ensures that the union of the images of the g_i under $(f/Y)_*$ is precisely Y' . ■

It is an indirect consequence of Lemma 1.6.8 that the converse of Corollary 1.6.7 cannot hold in general. Indeed, if the only compact object of \mathcal{E} is the initial object, such as in Example 2.9.4 below, then the preservation of compact or supercompact objects by f^* is a vacuous condition, but the required properties in the characterisation of Lemma 1.6.8 are clearly non-trivial. However, \mathcal{E} failing to have enough supercompact (resp. compact) objects is the only obstacle.

1.6.9. PROPOSITION. *Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism, and suppose \mathcal{E} is supercompactly generated. Then f is relatively pristine if and only if f^* preserves supercompact objects, and relatively polished if and only if f^* preserves ‘supercompact or initial’ objects. If \mathcal{E} is merely compactly generated, then f is relatively proper if and only if f^* preserves compact objects.*

PROOF. Given an arbitrary (resp. inhabited) jointly epic collection of subobjects $\{g_i : X_i \hookrightarrow f^*(Y)\}$ in \mathcal{F} , consider a (possibly empty) covering of Y by supercompact subobjects $h_j : Y_j \hookrightarrow Y$ in \mathcal{E} . Since each $f^*(Y_j)$ is supercompact (resp. supercompact or initial), pulling back the inclusions g_i along $f^*(h_j)$, we conclude that one of the resulting inclusions $f^*(Y_j)$ (if there are any) must be an isomorphism by Lemma 1.1.2; in the inhabited case, this is trivially true when $f^*(Y_j)$ is initial. Hence the $f^*(h_j)$ each factor through one of the g_i , whence the criteria of Lemma 1.6.8 are fulfilled. The compactly generated case, with a directed family of subobjects, is analogous. ■

Since the initial object in any topos is strict, the distinction between relatively pristine and relatively polished morphisms is indeed as small as this proposition makes it seem.

1.6.10. LEMMA. *Given a geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$, f^* reflects the initial object if and only if f_* preserves it.*

PROOF. If f^* reflects 0, then considering the counit $f^*f_*(0) \longrightarrow 0$, we conclude that $f^*f_*(0)$ is initial (by strictness of the initial object), whence $f_*(0) \cong 0$ so f_* preserves the initial object. Conversely, if f_* preserves 0, then given C with $f^*(C) \cong 0$ the unit $C \longrightarrow f_*f^*(C) \cong 0$ is a morphism to 0, whence C is initial. ■

1.6.11. **COROLLARY.** *Let $f : \mathcal{F} \rightarrow \mathcal{E}$ be a geometric morphism, and suppose \mathcal{E} is supercompactly generated. Then f is relatively pristine if and only if it is relatively polished and f^* reflects the initial object.*

PROOF. A (relatively) pristine morphism is (relatively) polished. Considering 0 as an empty union of subobjects of any given object, it is preserved by f_* by relative pristineness, so f^* reflects 0 by Lemma 1.6.10, as required. Note that this implication holds even when \mathcal{E} is not supercompactly generated.

Conversely, given that f is relatively polished and f^* reflects 0, we have that f^* preserves ‘supercompact or initial’ objects by Proposition 1.6.9, but a supercompact object X of \mathcal{E} is sent to an initial object if and only if it is initial, which is impossible, so $f^*(X)$ is supercompact, as required. ■

We caution the reader this ostensibly small difference between relatively polished and relatively pristine morphisms greatly impacts the frequency with which these classes of morphisms occur, as witnessed in Corollary 2.8.4 and Lemma 2.8.5 below.

1.7. **INCLUSIONS AND SURJECTIONS.** Recall that a geometric morphism $f : \mathcal{F} \rightarrow \mathcal{E}$ is a **surjection** if f^* is faithful, or equivalently if f^* is a comonadic functor. Meanwhile, a geometric morphism $f : \mathcal{F} \rightarrow \mathcal{E}$ is an **inclusion** (or *embedding*) if its direct image f_* is full and faithful. See [Mac Lane and Moerdijk, 1992, §VII.4] or [Johnstone, 2002, A4.2] for general results regarding these, which we shall assume familiarity with. In this section we examine how these two types of geometric morphism interact with supercompact and compact objects, as well as some of the classes of geometric morphism introduced in the last section.

Recall that a geometric inclusion $f : \mathcal{F} \rightarrow \mathcal{E}$ is **closed** if there is some subterminal object U in \mathcal{E} such that f_*f^* sends an object X to the pushout of the product projections from $X \times U$. The following result illustrates why the pristine, polished and proper morphisms are too restrictive for analysing supercompactly generated subtoposes.

1.7.1. **LEMMA.** *An inclusion of toposes $f : \mathcal{F} \rightarrow \mathcal{E}$ is proper if and only if it is closed, if and only if it is polished. An inclusion is pristine if and only if it is an equivalence.*

PROOF. The first part is the conclusion of [Johnstone, 2002, Remark C3.2.9], where it is observed that a closed inclusion has a direct image functor preserving arbitrary inhabited (\mathcal{E} -indexed) unions of subobjects and that closed inclusions are stable under slicing, and conversely that any proper inclusion is closed.

Given that f is a closed inclusion, any nontrivial subobject of the corresponding subterminal object U in \mathcal{E} is sent by f^* to the initial object in \mathcal{F} , so the initial object is reflected if and only if U is initial, in which case f is an equivalence. ■

The relative versions of these properties are well-behaved with respect to surjections and inclusions.

1.7.2. PROPOSITION. Consider a factorisation of a geometric morphism f ,

$$\begin{array}{ccc}
 \mathcal{F} & \xrightarrow{f} & \mathcal{E} \\
 & \searrow q & \nearrow p \\
 & & \mathcal{G}.
 \end{array} \tag{3}$$

1. Suppose that p is an inclusion. Then if f is relatively pristine (resp. relatively polished, relatively proper), so is q .
2. Suppose that q is a surjection. Then if f is relatively pristine (resp. relatively polished, relatively proper), so is p .

It follows that a geometric morphism is relatively pristine (resp. relatively polished, relatively proper) if and only if both parts of its (surjection, inclusion) factorisation are.

PROOF. We use the characterisation of these properties from Lemma 1.6.8.

1. Given a jointly epic family (resp. inhabited family, directed family) $\{g_i : X_i \hookrightarrow q^*(Z)\}$ in \mathcal{E} , we may express Z up to isomorphism as $p^*(Y)$ (taking $Y = p_*(Z)$, say), so this can without loss of generality be seen as a family $\{g_i : X_i \hookrightarrow f^*(Y)\}$. Since f is relatively pristine (resp. relatively polished, relatively proper), we have a jointly epic family $\{h_j : Y_j \twoheadrightarrow Y\}$ such that each $f^*(h_j)$ factors through some g_i , and hence $\{p^*(h_j) : p^*(Y_j) \twoheadrightarrow Z\}$ is the required family to fulfill the criterion of Lemma 1.6.8.

2. Given a jointly epic family (resp. inhabited family, directed family) $\{g_i : X_i \hookrightarrow p^*(Y)\}$ in \mathcal{E} , we have $\{q^*(g_i) : q^*(X_i) \hookrightarrow f^*(Y)\}$ in \mathcal{F} being of the desired form to ensure that there is a covering family $\{h_j : Y_j \twoheadrightarrow Y\}$ such that each $f^*(h_j)$ factors through some $q^*(g_i)$. But then q^* being conservative forces the $p^*(h_j)$ to factor through g_i , as required. Indeed, the intersection of g_i with the image of $p^*(h_j)$ is preserved by q^* , and one of the sides of the resulting pullback square is sent to an isomorphism, which is reflected by q^* . ■

1.7.3. COROLLARY. Any topos \mathcal{E} with a surjective point is supercompact. Any topos with a finite jointly surjective collection of points is compact.

PROOF. The hypotheses correspond to the existence of a surjection $\mathbf{Set} \twoheadrightarrow \mathcal{E}$, or a surjection $\mathbf{Set}/K \twoheadrightarrow \mathcal{E}$ with K finite. The unique morphism $\mathbf{Set} \twoheadrightarrow \mathbf{Set}$ is (an equivalence, and hence) pristine, while the morphism $\mathbf{Set}/K \twoheadrightarrow \mathbf{Set}$ is proper. Thus by Proposition 1.7.2.2 and Lemma 1.6.6, we conclude that \mathcal{E} is supercompact (resp. compact) over \mathbf{Set} . ■

1.7.4. EXAMPLE. The first part of Corollary 1.7.3 applies to toposes of the form $[M^{\text{op}}, \mathbf{Set}]$ for M a monoid; the latter part similarly applies to $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ for \mathcal{C} any small category with a finite number of objects. Note that in both of these cases, the idempotent completions may have many more objects than the original categories,

When the codomain of the geometric morphism is supercompactly (resp. compactly) generated, the simpler characterisation of Proposition 1.6.9 comes to our aid.

1.7.5. LEMMA. *Suppose that \mathcal{E} is supercompactly (resp. compactly) generated and $f : \mathcal{F} \rightarrow \mathcal{E}$ is an inclusion. If f is relatively polished (resp. relatively proper), \mathcal{F} is also supercompactly (resp. compactly) generated.*

PROOF. By Proposition 1.6.9, the separating collection of supercompact (resp. compact) objects in \mathcal{E} is mapped by the inverse image of the relatively polished (resp. relatively proper) inclusion f to a separating collection of objects of the same kind or initial objects) in \mathcal{F} . ■

We shall see in Corollary 2.1.8 that Lemma 1.7.5 is optimal, in the sense that a topos is supercompactly (resp. compactly) generated if and only if it admits a relatively polished (resp. relatively proper) inclusion into a presheaf topos.

1.7.6. COROLLARY. *Suppose that \mathcal{E} is supercompactly (resp. compactly) generated. Then a geometric morphism $f : \mathcal{F} \rightarrow \mathcal{E}$ is relatively polished (resp. relatively proper) if and only if both parts of its (surjection, inclusion) factorisation have inverse images preserving supercompact or initial (resp. compact) objects.*

PROOF. This follows from applying Proposition 1.6.9 to the factorisation in Proposition 1.7.2, using Lemma 1.7.5 to conclude that the intermediate topos must be supercompactly (resp. compactly) generated. ■

More generally, surjections interact well with supercompact and compact objects.

1.7.7. LEMMA. *Suppose $f : \mathcal{F} \rightarrow \mathcal{E}$ is a surjective geometric morphism. Then f^* reflects supercompact, compact and initial objects.*

PROOF. Since the inverse image functor of f is comonadic, we have an equivalence between \mathcal{E} and the topos of coalgebras for the comonad on \mathcal{F} induced by f . Without loss of generality we work with coalgebras.

Given a coalgebra $(X, \alpha : X \rightarrow f^* f_*(X))$ and a jointly epic family $g_i : (U_i, \beta_i) \rightarrow (X, \alpha)$, since f^* preserves arbitrary colimits, the underlying family of morphisms $g_i : U_i \rightarrow X$ in \mathcal{F} must be jointly epic. Thus if $X = f^*(X, \alpha)$ is supercompact (resp. compact), one of the g_i must be an epimorphism (resp. there is a finite jointly epic subfamily of the g_i). Since f^* moreover creates colimits, we conclude (via the same colimit diagram employed in Lemma 1.3.11) that g_i is an epimorphism in \mathcal{E} too (resp. that the finite subfamily lifts to a jointly epic finite subfamily). Thus (X, α) is supercompact, as required.

Preservation of the initial object by f_* , which is equivalent to reflection of 0 by f^* by Lemma 1.6.10, is due to strictness of the initial object forcing $f^* f_*(0) \cong 0$. ■

It follows from Lemma 1.7.7 and Corollary 1.6.11 that a geometric surjection is relatively pristine if and only if it is relatively polished.

1.8. HYPERCONNECTED MORPHISMS. Recall that a geometric morphism $f : \mathcal{F} \rightarrow \mathcal{E}$ is said to be **hyperconnected** if f^* is full and faithful and its image is closed in \mathcal{F} under subobjects and quotients (up to isomorphism). Recall also that f is **localic** if every object in \mathcal{F} is a subquotient of one of the form $f^*(X)$. See [Johnstone, 2002, Section A4.6] for

some background on hyperconnected morphisms, as well as localic morphisms, including the (hyperconnected,localic) factorization of a general geometric morphism.

When it comes to hyperconnected morphisms into **Set**, there are various alternative characterisations; note that these rely on properties of **Set**, so are not true constructively.

1.8.1. PROPOSITION. *Let \mathcal{E} be a Grothendieck topos. Then the following are equivalent:*

1. *The unique geometric morphism $\mathcal{E} \rightarrow \mathbf{Set}$ is hyperconnected.*
2. *\mathcal{E} is **two-valued**: the only subterminal objects are the initial and terminal objects.*
3. *Every object of \mathcal{E} is either well-supported (the unique morphism $X \rightarrow 1$ is an epimorphism) or initial, but not both.*
4. *\mathcal{E} is non-degenerate and has a separating set of well-supported objects.*

PROOF. (1 \Leftrightarrow 2) The inverse image of a hyperconnected geometric morphism is full and faithful and closed under subobjects, so in particular the only subobjects of 1 in \mathcal{E} is 0. Conversely, if \mathcal{E} is two-valued, consider the (hyperconnected,localic) factorization of the unique geometric morphism $\mathcal{E} \rightarrow \mathbf{Set}$; the intermediate topos is the localic reflection of \mathcal{E} , equivalent to the topos of sheaves on the locale of subterminal objects of \mathcal{E} , so is equivalent to **Set**. Thus the morphism $\mathcal{E} \rightarrow \mathbf{Set}$ is hyperconnected.

(2 \Leftrightarrow 3) If \mathcal{E} is two-valued, the monic part of the (epi,mono) factorization of $X \rightarrow 1$ is non-trivial iff $X \cong 0$, so X is either initial or well-supported. Conversely, any proper subterminal object fails to be well-supported, so if 3 holds there can be no proper subterminals and \mathcal{E} is two-valued.

(3 \Leftrightarrow 4) Since \mathcal{E} is a Grothendieck topos, it has some generating set of objects; any such set is still generating after excluding the initial object, and since 0 is distinct from 1, any generating set contains a non-initial object, so we obtain a generating set of non-initial objects as required. Conversely, an inhabited colimit of well-supported objects is well-supported, so any non-initial object is well-supported if there is a separating set of well-supported objects. ■

In particular, since supercompact objects are not initial, we obtain a necessary and sufficient condition for the hypotheses in Section 1.5 to hold:

1.8.2. COROLLARY. *If \mathcal{E} is a two-valued (Grothendieck) topos, every supercompact object in \mathcal{E} is well-supported. In particular, a morphism $A \rightarrow B$ in the subcategory \mathcal{C}_s of supercompact objects of \mathcal{E} is an epimorphism if and only if it is epic in \mathcal{E} , so every epimorphism in \mathcal{C}_s is strict. Conversely, if \mathcal{E} is supercompactly generated, then \mathcal{C}_s is closed under cokernels if and only if \mathcal{E} is two-valued.*

Thus we can give the counterexample, promised earlier, to the hypothesis that epimorphisms in \mathcal{C}_s are regular when \mathcal{E} is two-valued.

1.8.3. **EXAMPLE.** Let M be the free monoid on two generators, viewed as a category, and let $\mathcal{E} = [M^{\text{op}}, \mathbf{Set}]$, where the objects are viewed as right M -sets. It is easily checked that this topos is two-valued, and being a presheaf topos it is supercompactly generated. The supercompact objects in this topos are precisely the cyclic right M -sets.

Given a cyclic M -set N generated by n , the relation generated by a pair of morphisms $f, g : N \rightrightarrows M$ is the one identifying $f(n)k$ with $g(n)k$ for each $k \in M$. A case-by-case analysis of the possible pairs of elements $f(n), g(n)$ demonstrates that there is no pair of which the epimorphism $M \twoheadrightarrow 1$ is a coequalizer, so this epimorphism is not regular in the category of supercompact objects of \mathcal{E} .

The main reason for our interest in hyperconnected morphisms, however, is that they create the structure of supercompactly (and compactly) generated Grothendieck toposes which we are studying in this paper.

1.8.4. **COROLLARY.** *If $f : \mathcal{F} \rightarrow \mathcal{E}$ is a hyperconnected geometric morphism between Grothendieck toposes, then it is pristine. Thus (being surjective) f^* preserves and reflects supercompact, compact and initial objects.*

PROOF. We extend the proof that hyperconnected morphisms are proper, [Moerdijk and Vermeulen, 1997, Proposition I.2.4], replacing ∞^* with $\pi_!$.

Suppose $f : \mathcal{F} \rightarrow \mathcal{E}$ is hyperconnected, and consider a diagram of the form

$$\begin{array}{ccc}
 [f^*(\mathbb{I}), \mathcal{F}] & \xrightarrow{f^{\mathbb{I}}} & [\mathbb{I}, \mathcal{E}] \\
 \pi \downarrow & \lrcorner & \downarrow \pi \\
 \mathcal{F} & \xrightarrow{f} & \mathcal{E}.
 \end{array} \tag{4}$$

Since $f^{\mathbb{I}}$ is a pullback of f , it is hyperconnected too, so that any $V \hookrightarrow 1$ in $[f^*(\mathbb{I}), \mathcal{F}]$ is of the form $(f^{\mathbb{I}})^*(U)$ for some $U \hookrightarrow 1$ in $[\mathbb{I}, \mathcal{E}]$ (the restriction of a hyperconnected morphism to the subterminal objects is an equivalence). Thus,

$$f_*\pi_!(V) = f_*\pi_!(f^{\mathbb{I}})^*(U) = f_*f^*\pi_!(U) = \pi_!(U),$$

where the last equality holds since f^* is full and faithful. But $U = (f^{\mathbb{I}})_*(f^{\mathbb{I}})^*(U) = (f^{\mathbb{I}})_*(V)$, so $f_*\pi_!(V) = \pi_!(f^{\mathbb{I}})_*(V)$, as required. The same argument applied in slices gives the result.

Preservation of supercompact and compact objects by f^* then follows from Proposition 1.6.9 and Corollary 1.6.11 (preservation of the initial object is automatic), while reflection follows from Lemma 1.7.7. ■

1.8.5. **THEOREM.** *Let $f : \mathcal{F} \rightarrow \mathcal{E}$ be a hyperconnected geometric morphism between elementary toposes. If \mathcal{F} is a Grothendieck topos, then so is \mathcal{E} . Assuming this is so, if \mathcal{F} :*

(i) *is supercompactly generated, or*

(ii) *is compactly generated, or*

(iii) has enough points, or

(iv) is two-valued,

then \mathcal{E} inherits that property.

PROOF. Let \mathcal{C} be a (full subcategory on a) small separating set of objects in \mathcal{F} . Then every object of \mathcal{F} is a colimit of a diagram in \mathcal{C} . Given an object Q of \mathcal{E} , we can express $f^*(Q)$ as such a colimit; write $\lambda_i : C_i \rightarrow f^*(Q)$ with $C_i \in \mathcal{C}$ for the legs of the colimit cone. By taking image factorizations of the λ_i we obtain an expression for $f^*(Q)$ as a colimit where the legs of the colimit cone are all monomorphisms. Since the image of f^* is closed under subobjects, we obtain an expression for $f^*(Q)$ as a colimit of objects of the form $f^*(D_i)$ with D_i in \mathcal{E} , which moreover are quotients of objects in the separating subcategory \mathcal{C} of \mathcal{F} .

Thus, since f^* creates all small colimits, the quotients of objects in \mathcal{C} lying in \mathcal{E} form a separating set. Also, \mathcal{E} is locally small since \mathcal{F} is. Thus by the version of Giraud’s Theorem that appears in [Johnstone, 2002, C2.2.8(v)], say, \mathcal{E} is a Grothendieck topos.

The inheritance of property (i) (resp. (ii)) follows from Corollary 1.8.4, taking \mathcal{C} in the above to be \mathcal{C}_s (resp. \mathcal{C}_c) and noting that the objects $f^*(D_i)$ in the argument above are supercompact (resp. compact) in \mathcal{F} by Lemma 1.3.1, whence the D_i are so in \mathcal{E} by Corollary 1.8.4.

For (iii), if \mathcal{F} has enough points, which is to say that there is a collection of geometric morphisms $\mathbf{Set} \rightarrow \mathcal{F}$ whose inverse images are jointly faithful, then composing each point with the hyperconnected morphism to \mathcal{E} gives such a collection for \mathcal{E} . Finally, for (iv), note once again that the restriction of f to subterminal objects is an equivalence. ■

In spite of our reliance on Corollary 1.8.4 here, we shall see in Example 2.9.5 that we cannot extend Theorem 1.8.5(i) or (ii) to relatively pristine or relatively proper surjections, although parts (iii) and (iv) do apply in that situation.

We can summarise the results from the last two sections as stability results for supercompactly and compactly generated toposes.

1.8.6. THEOREM. *Suppose \mathcal{E} is a supercompactly (resp. compactly) generated Grothendieck topos. If \mathcal{F} is:*

1. *The domain of a closed inclusion $f : \mathcal{F} \rightarrow \mathcal{E}$, or more generally, the domain of a relatively polished (resp. relatively proper) inclusion into \mathcal{E} ,*
2. *The domain of a local homeomorphism $g : \mathcal{F} \simeq \mathcal{E}/X \rightarrow \mathcal{E}$, or*
3. *The codomain of a hyperconnected morphism $h : \mathcal{E} \rightarrow \mathcal{F}$,*

then \mathcal{F} is also a supercompactly (resp. compactly) generated Grothendieck topos.

PROOF. Let \mathcal{C}_s and \mathcal{C}_c be the (separating) subcategories of supercompact and compact objects in \mathcal{F} , respectively.

1. For any inclusion, the images of objects in \mathcal{C}_s (resp. \mathcal{C}_c) under f^* form a separating set for \mathcal{E} . The stated properties ensure that these objects are all supercompact or initial (resp. compact), so that \mathcal{E} is supercompactly (resp. compactly) generated, by Corollary 1.6.7.

2. The objects with domain in \mathcal{C}_s (resp. \mathcal{C}_c) in any slice \mathcal{F}/X form a separating set. These lifted objects inherit the property of being supercompact (resp. compact), by Lemma 1.1.2 and the standard result $(\mathcal{F}/X)/(Y \rightarrow X) \simeq \mathcal{F}/Y$.

3. This is immediate from Theorem 1.8.5. ■

2. Principal sites

So far, we have established properties of ‘canonical’ sites for supercompactly and compactly generated toposes. In the spirit of Caramello’s work, [Caramello, 2012a], we obtain in this section a broader class of sites whose toposes of sheaves have these properties.

As in Section 1, we write $\ell : \mathcal{C} \rightarrow \text{Sh}(\mathcal{C}, J)$ for the composite of the Yoneda embedding and the sheafification functor. We will occasionally make use of the following fact regarding representable sheaves, which is easily derived from the fact that ℓ is a dense morphism of small-generated sites, in the sense described in [Caramello, 2019]:

2.0.1. FACT. *A sieve T on $\ell(C)$ in $\text{Sh}(\mathcal{C}, J)$ is jointly epimorphic if and only if the sieve $\{f : D \rightarrow C \mid \ell(f) \in T\}$ is J -covering.*

2.1. STABLE CLASSES.

2.1.1. DEFINITION. *Let \mathcal{C} be a small category. A class \mathcal{T} of morphisms in \mathcal{C} is called **stable** if it satisfies the following three conditions:*

1. \mathcal{T} contains all identities.
2. \mathcal{T} is closed under composition.
3. For any $f : C \rightarrow D$ in \mathcal{T} and any morphism g in \mathcal{C} with codomain D , there exists a commutative square

$$\begin{array}{ccc}
 A & \xrightarrow{f'} & B \\
 g' \downarrow & & \downarrow g \\
 C & \xrightarrow{f} & D,
 \end{array} \tag{5}$$

in \mathcal{C} with $f' \in \mathcal{T}$.

These correspond to the necessary and sufficient conditions for \mathcal{T} -morphisms to be singleton presieves generating a Grothendieck topology, as presented in [Mac Lane and Moerdijk, 1992, Exercise III.3]. We call the resulting Grothendieck topology the **principal topology generated by \mathcal{T}** .

In [Kondo and Yasuda, 2017], a stable class of morphisms is called *semi-localizing*, in reference to a related definition in [Gelfand and Manin, 2003]. The authors call a principal topology an *A-topology*, presumably because the atomic topology is an example of a principal topology; see Example 2.3.3. We have chosen a naming convention that we believe to be more evocative in this context.

Continuing the parallel investigation of compactness, we obtain a related concept by replacing individual morphisms by finite families of morphisms.

2.1.2. DEFINITION. *Let \mathcal{T}' be a class of finite families of morphisms with specified common codomain in \mathcal{C} . We say \mathcal{T}' is **stable** if*

- 1'. \mathcal{T}' contains the families whose only member is the identity.
- 2'. \mathcal{T}' is closed under multicomposition, in that if $\{f_i : D_i \rightarrow C \mid i = 1, \dots, n\}$ is in \mathcal{T}' and so are $\{g_{j,i} : E_j \rightarrow D_i \mid j = 1, \dots, m_i\}$ for each $i = 1, \dots, n$, then $\{f_i \circ g_{j,i}\}$ is also a member of \mathcal{T}' .
- 3'. For any $\{f_i : D_i \rightarrow C \mid i = 1, \dots, n\}$ in \mathcal{T}' and any morphism $g : B \rightarrow C$ in \mathcal{C} , there is a \mathcal{T}' -family $\{h_j : A_j \rightarrow B \mid j = 1, \dots, m\}$ such that each $g \circ h_j$ factors through one of the f_i .

*These are necessary and sufficient conditions for \mathcal{T}' -families to generate a Grothendieck topology, which we call the **topology (finitely) generated by \mathcal{T}'** .*

Note that we do not require \mathcal{C} to have pullbacks in the above definitions, so it is sensible to compare them with the usual notion of stability with respect to pullbacks.

2.1.3. LEMMA. *Let \mathcal{T} be a stable class of morphisms in \mathcal{C} with the additional ‘push-forward’ property:*

- 4. *Given any morphism f of \mathcal{C} such that $f \circ g \in \mathcal{T}$ for some morphism g of \mathcal{C} , we have $f \in \mathcal{T}$.*

This in particular is true of the class of epimorphisms, for example. Then morphisms in \mathcal{T} are stable under any pullbacks which exist in \mathcal{C} .

PROOF. Given a pullback of a \mathcal{T} morphism, the comparison between any square provided by (5) and this pullback provides a factorization of a \mathcal{T} morphism through the pullback, so by the assumed property the pullback is also in \mathcal{T} , as required. ■

As remarked in [Kondo and Yasuda, 2017], we can extend a stable class of morphisms \mathcal{T} to the stable class $\hat{\mathcal{T}}$ of morphisms whose principal sieves contain a member of \mathcal{T} (that is, morphisms f such that $f \circ g \in \mathcal{T}$ for some g) without changing the resulting principal topology. Thus we can safely assume that stable classes satisfy axiom 4 of Lemma 2.1.3 if we so choose. The advantage of enforcing this assumption is that it gives a one-to-one correspondence between stable classes of morphisms in a category and the principal Grothendieck topologies on that category, since we can recover the classes as those morphisms which generate covering sieves.

2.1.4. **REMARK.** The class of identity morphisms in any category satisfies axioms 1,2 and 3, but in order to satisfy axiom 4 it must be extended to the class of split epimorphisms, which is easily verified to satisfy all four axioms in any category. Thus the class of split epimorphisms corresponds to the trivial Grothendieck topology. It is worth noting also that every split epimorphism is regular and hence strict.

We may similarly extend a class \mathcal{T}' to a maximal class $\hat{\mathcal{T}}'$. However, for a class of finite families to be maximal, it must be closed under supersets as well as under the equivalent of the push-forward property of Lemma 2.1.3, so we have two extra axioms:

- 4'. Given a finite family $\mathfrak{f} = \{f_i : D_i \rightarrow C\}$ of morphisms in \mathcal{C} such that every morphism in some \mathcal{T}' -family over C factors through one of the f_i , we have $\mathfrak{f} \in \mathcal{T}'$.
- 5'. Any finite family $\mathfrak{f} = \{f_i : D_i \rightarrow C\}$ of morphisms in \mathcal{C} containing a \mathcal{T}' -family \mathfrak{f}' is also a member of \mathcal{T}' .

The equivalent of the pullback stability statement of Lemma 2.1.3 is as follows.

2.1.5. **SCHOLIUM.** *Suppose that \mathcal{T}' is a stable class of finite families of morphisms in a category \mathcal{C} satisfying the additional axioms 4' and 5'. Then \mathcal{T}' is stable under any pullbacks which exist in \mathcal{C} , in that given a family $\{g_i : E_i \rightarrow C\}$ in \mathcal{T}' and $h : C' \rightarrow C$ such that the pullback of g_i along h exists for each i , the family $\{h^*(g_i) : E'_i \rightarrow C'\}$ is in \mathcal{T}' .*

We leave the proof, and the verification that enforcing axioms 4' and 5' gives a one-to-one correspondence between stable classes of finite families and finitely generated Grothendieck topologies, to the reader.

2.1.6. **DEFINITION.** *Let \mathcal{C} be a small category, \mathcal{T} a stable class of its morphisms; we denote the corresponding **principal** (Grothendieck) **topology** by $J_{\mathcal{T}}$. We call a site $(\mathcal{C}, J_{\mathcal{T}})$ constructed in this way a **principal site**. Similarly, for a stable class of finite families \mathcal{T}' on \mathcal{C} , we have a corresponding **finitely generated** (Grothendieck) **topology** denoted $J_{\mathcal{T}'}$; a site of the form $(\mathcal{C}, J_{\mathcal{T}'})$ shall be called a **finitely generated site**.*

2.1.7. **PROPOSITION.** *Let \mathcal{C} be a small category and J a Grothendieck topology on it. Then the representable sheaves are all supercompact if and only if $J = J_{\mathcal{T}}$ is a principal topology for some stable class \mathcal{T} of morphisms in \mathcal{C} . In particular, the topos of sheaves on a principal site $(\mathcal{C}, J_{\mathcal{T}})$ is supercompactly generated.*

Similarly, the representable sheaves are all compact in $\text{Sh}(\mathcal{C}, J)$ if and only if $J = J_{\mathcal{T}'}$ for a stable class \mathcal{T}' of finite families of morphisms in \mathcal{C} , and the topos of sheaves on a finitely generated site $(\mathcal{C}, J_{\mathcal{T}'})$ is compactly generated.

PROOF. By Fact 2.0.1, for $J = J_{\mathcal{T}}$, given a covering family on $\ell(C)$, the sieve it generates must contain an epimorphism which is the image of a \mathcal{T} -morphism. Since this epimorphism factors through some member of the covering family, that member must also be an epimorphism. Thus $\ell(C)$ is supercompact, as required.

Conversely, given that $\ell(C)$ is supercompact for every C , let \mathcal{T} be the class of morphisms f such that $\ell(f)$ is epimorphic. We first claim that \mathcal{T} is a stable class. Indeed, axioms 1, 2 and 4 are immediate; to see that axiom 3 holds, suppose that $f : C \rightarrow D$ is in \mathcal{T} and $g : B \rightarrow D$ is any \mathcal{C} morphism. Then we may consider the pullback of $\ell(f)$ against $\ell(g)$ in $\text{Sh}(\mathcal{C}, J)$:

$$\begin{array}{ccc} A & \xrightarrow{f'} \twoheadrightarrow & \ell(B) \\ g' \downarrow & \lrcorner & \downarrow \ell(g) \\ \ell(C) & \xrightarrow{\ell(f)} \twoheadrightarrow & \ell(D). \end{array}$$

Since A is covered by objects of the form $\ell(C')$, by supercompactness of $\ell(B)$ there must be an epimorphism $\ell(C') \twoheadrightarrow \ell(B)$ factoring through the pullback, and in turn the sieve it generates must contain an epimorphism in the image of ℓ , so that we ultimately recover the square (5) required for axiom 3.

Now we show that $J = J_{\mathcal{T}}$. Given a J -covering sieve S on C (generated by a family of morphisms, for example), the sieve generated by $\ell(S) := \{\ell(g) \mid g \in S\}$ necessarily covers $\ell(C)$, and therefore $\ell(S)$ contains an epimorphism by supercompactness, so that the original sieve must have contained a member of \mathcal{T} , which gives $J \subseteq J_{\mathcal{T}}$. Conversely, $J_{\mathcal{T}}$ -covering sieves are certainly J -covering, so $J_{\mathcal{T}} \subseteq J$. Thus J is a principal topology, as claimed.

The argument for finitely generated sites is almost identical after replacing \mathcal{T} -morphisms with finite \mathcal{T}' -families in the first part, and defining \mathcal{T}' families to be those finite families which are mapped by ℓ to jointly epic families in the second part. ■

In particular, we may extend a stable class of morphisms \mathcal{T} to a stable class of families of morphisms \mathcal{T}' by viewing the morphisms in \mathcal{T} as one-element families. Then it is clear that $J_{\mathcal{T}} = J_{\mathcal{T}'}$.

Intermediate between the two classes of sites discussed so far are a class which we call **quasi-principal sites**: these are sites (\mathcal{C}, J) such that for every object $C \in \mathcal{C}$, either the empty sieve is a covering sieve on C or every covering sieve on C contains a principal sieve. Observe that if \mathcal{C}' is the full subcategory of \mathcal{C} on the latter class of objects (which we can always construct over **Set**), then $\text{Sh}(\mathcal{C}, J) \cong \text{Sh}(\mathcal{C}', J_{\mathcal{T}})$, where \mathcal{T} is the class of morphisms generating principal covering sieves.

The following result subsumes Lemma 4.11 of [Bridge, 2012]; our work up to this point allows us to avoid any direct manipulation with sheaves in the proof.

2.1.8. COROLLARY. *Let (\mathcal{C}, J) be a small site. Then the inclusion $\text{Sh}(\mathcal{C}, J) \rightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is relatively pristine (resp. relatively proper) if and only if J is a principal (resp. finitely generated) topology. The inclusion is relatively polished if and only if (\mathcal{C}, J) is a quasi-principal site.*

PROOF. Since $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is supercompactly generated, by Proposition 1.6.9, the inclusion $\text{Sh}(\mathcal{C}, J) \rightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is relatively pristine if and only if the sheafification functor preserves supercompact objects; this in particular requires all of the objects $\ell(C)$ to be

supercompact, which occurs if and only if J is principal by Proposition 2.1.7. But any other supercompact objects are quotients of representables, so $\ell(C)$ being supercompact for every C is also sufficient. As usual, the relatively proper case is analogous.

For relative polishedness, we relax the conditions above to requiring that each representable is sent to a supercompact or initial object, and note that the empty sieve is covering on C if and only if $\ell(C)$ is initial. ■

2.2. MORPHISMS BETWEEN SHEAVES ON PRINCIPAL SITES. In order to better understand the relationship between a principal site and the topos it generates, we employ some results from [Caramello, 2019], which enable us to describe morphisms in a Grothendieck topos $\text{Sh}(\mathcal{C}, J)$ in terms of those in a presenting site (\mathcal{C}, J) .

For a general site (\mathcal{C}, J) , the functor $\ell : \mathcal{C} \rightarrow \text{Sh}(\mathcal{C}, J)$ is neither full nor faithful. To describe the full collection of morphisms in the sheaf topos, several notions are introduced in [Caramello, 2019, §2], of which we introduce the relevant special cases here.

For morphisms $h, k : A \rightrightarrows B$, we say h and k are J -**locally equal** (written $h \equiv_J k$) if there is a J -covering sieve S on A such that $h \circ f = k \circ f$ for every $f \in S$. If J is principal (resp. finitely generated) then this is equivalent to saying that there is some \mathcal{T} -morphism which equalizes h and k (resp. a \mathcal{T}' -family whose members all equalize h and k). This leads naturally to the following moderately technical definitions:

2.2.1. DEFINITION. Let \mathcal{C} be a small category and \mathcal{T} a stable class of morphisms in \mathcal{C} . Then for objects A, B in \mathcal{C} , a \mathcal{T} -**span** from A to B is a span

$$\begin{array}{ccc}
 & E & \\
 f \swarrow & & \searrow g \\
 A & & B,
 \end{array} \tag{6}$$

such that f is in \mathcal{T} . A \mathcal{T} -**arch** is a \mathcal{T} -span such that for any $h, k : D \rightrightarrows E$ with $f \circ h = f \circ k$ we have $g \circ h \equiv_{J_{\mathcal{T}}} g \circ k$.

Similarly, for \mathcal{T}' a stable class of finite families of morphisms on \mathcal{C} , a \mathcal{T}' -**span** is a finite (possibly empty) family of spans:

$$\begin{array}{ccc}
 & E_i & \\
 f_i \swarrow & & \searrow g_i \\
 A & & B,
 \end{array} \tag{7}$$

such that $\{f_1, \dots, f_n\}$ is in \mathcal{T}' . A \mathcal{T}' -**multiarch** is a \mathcal{T}' -multispan such that for any $h : D \rightarrow E_i, k : D \rightarrow E_{i'}$ with $f_i \circ h = f_{i'} \circ k$ we have $g_i \circ h \equiv_{J_{\mathcal{T}'}} g_{i'} \circ k$.

The constituent morphisms in any span or multispan will be referred to as their **legs**.

2.2.2. LEMMA. Let $(\mathcal{C}, J_{\mathcal{T}})$ be a principal site. Let $\mathbf{Arch}_{\mathcal{T}}(A, B)$ be the collection of \mathcal{T} -arches from A to B in \mathcal{C} . For each \mathcal{T} -arch $(t, g) \in \mathbf{Arch}_{\mathcal{T}}(A, B)$, there is a (necessarily unique) morphism $\ell(t, g) : \ell(A) \rightarrow \ell(B)$ in $\text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ such that $\ell(t, g) \circ \ell(t) = \ell(g)$. The

mapping ℓ so defined is a surjection from $\mathbf{Arch}_{\mathcal{T}}(A, B)$ to the set of morphisms from $\ell(A)$ to $\ell(B)$ in $\mathbf{Sh}(\mathcal{C}, J_{\mathcal{T}})$.

Similarly, letting $\mathbf{mArch}_{\mathcal{T}'}(A, B)$ be the set of \mathcal{T}' -multiarches from A to B , ℓ induces a surjection from $\mathbf{mArch}_{\mathcal{T}'}(A, B)$ to $\mathbf{Hom}_{\mathbf{Sh}(\mathcal{C}, J_{\mathcal{T}'})}(A, B)$.

PROOF. This is immediate from [Caramello, 2019, Proposition 2.5]. ■

Intuitively it seems that the collections $\mathbf{Arch}_{\mathcal{T}}(A, B)$ “should” be the morphisms of a category. However, while [Caramello, 2019, Proposition 2.5(iv)] suggests a composition of arches coming from covering families that generate sieves, this composition produces a maximal family of arches presenting the composite rather than a single \mathcal{T} -arch; there is a similar problem for multiarches. We therefore examine what structure exists in general, and identify some sufficient conditions under which arches and multiarches admit a composition operation.

2.2.3. LEMMA. *Let $(\mathcal{C}, J_{\mathcal{T}})$ be a principal site. For each pair of objects A and B in \mathcal{C} , let $\mathbf{Span}_{\mathcal{T}}(A, B)$ be the collection of \mathcal{T} -spans from A to B . Then $\mathbf{Span}_{\mathcal{T}}(A, B)$ admits a canonical categorical structure, where a morphism $x : (t : E \rightarrow A, g : E \rightarrow B) \rightarrow (t' : E' \rightarrow A, g' : E' \rightarrow B)$ is a morphism $x : E \rightarrow E'$ with $t = t' \circ x$ and $g = g' \circ x$. This restricts to give a category structure on $\mathbf{Arch}_{\mathcal{T}}(A, B)$ too.*

Expanding upon this, if $(\mathcal{C}, J_{\mathcal{T}'})$ is a finitely generated site, there is a canonical categorical structure on each collection of \mathcal{T}' -multispans $\mathbf{mSpan}_{\mathcal{T}'}(A, B)$, where $\vec{x} : (t_i : E_i \rightarrow A, g_i : E_i \rightarrow B) \rightarrow (t'_j : E'_j \rightarrow A, g'_j : E'_j \rightarrow B)$ consists of an index j for each index i , and a morphism $x_i : E_i \rightarrow E'_j$ with $t_i = t'_j \circ x_i$ and $g_i = g'_j \circ x_i$. Note that any permutation of the spans forming a given \mathcal{T}' -multispan form a \mathcal{T}' -multispan which is isomorphic in this category. Once again, this structure restricts to the collections of multiarches.

2.2.4. PROPOSITION. *Suppose that \mathcal{T} is a stable class of morphisms in a small category \mathcal{C} such that axiom 3 of Definition 2.1.1 provides stability squares **weakly functorially**. That is, calling an ordered pair of morphisms $(h : A \rightarrow D, s : B \rightarrow D)$ with $s \in \mathcal{T}$ a **\mathcal{T} -cospan** from A to B , suppose the stability axiom defines a mapping from \mathcal{T} -cospans to \mathcal{T} -spans satisfying the following conditions:*

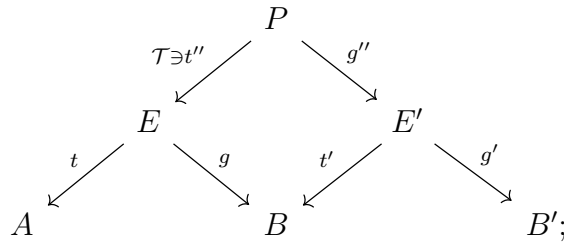
1. *For any \mathcal{T} -morphism $t : B \rightarrow A$, the \mathcal{T} -span coming from (id_A, t) is isomorphic in $\mathbf{Span}_{\mathcal{T}}(A, B)$ to (t, id_B) .*
2. *If $f : A \rightarrow D$, $g : B \rightarrow D$ and $k : A' \rightarrow A$ such that g is a \mathcal{T} -morphism, the \mathcal{T} -span obtained by applying the stability mapping along f and then k is isomorphic in $\mathbf{Span}_{\mathcal{T}}(A', B)$ to that obtained by applying it along $f \circ k$.*
3. *If $f : A \rightarrow D$, $g : B \rightarrow D$ and $e : B' \rightarrow B$ such that e and g are \mathcal{T} -morphisms, the \mathcal{T} -span obtained by applying the stability mapping to g and then e is isomorphic in $\mathbf{Span}_{\mathcal{T}}(A, B')$ to that obtained by applying it along $g \circ e$.*

Then there is a weak composition on \mathcal{T} -arches, in the sense that there are mappings

$$\circ : \mathbf{Arch}_{\mathcal{T}}(B, C) \times \mathbf{Arch}_{\mathcal{T}}(A, B) \longrightarrow \mathbf{Arch}_{\mathcal{T}}(A, C),$$

which are associative and unital up to isomorphism of \mathcal{T} -arches. Moreover, this composition is natural in the second component up to isomorphism, in the sense that for each fixed \mathcal{T} -arch (u, h) in $\mathbf{Arch}_{\mathcal{T}}(B, C)$, a morphism $x : (t, g) \longrightarrow (t', g')$ in $\mathbf{Arch}_{\mathcal{T}}(A, B)$ induces a morphism $(u, h) \circ x : (u, h) \circ (t, g) \longrightarrow (u, h) \circ (t', g')$ in $\mathbf{Arch}_{\mathcal{T}}(A, C)$, and the resulting mapping $(u, h) \circ - : \mathbf{Arch}_{\mathcal{T}}(A, B) \longrightarrow \mathbf{Arch}_{\mathcal{T}}(A, C)$ is functorial up to unit and associativity isomorphisms.

PROOF. Even without the listed conditions, stability provides a putative definition of the composition operation: given a consecutive pair of \mathcal{T} -arches, we simply apply the stability axiom to the pair of morphisms with common codomain,



that the resulting \mathcal{T} -span $(t \circ t', g' \circ g'')$ is a \mathcal{T} -arch is easily checked. The extra conditions are needed to make this operation weakly unital and associative. The naturality in the second component is a direct consequence of the second condition. ■

For brevity, we leave the analogous statement and proof of Proposition 2.2.4 for finitely generated sites to the reader, noting that the analogue of \mathcal{T} -cospans will not be duals of \mathcal{T}' -multispans, but the more restrictive shape of diagram relevant to the stability axiom 3'.

In the best case scenario where it *is* possible to construct a weak composition on arches, we *may* obtain a bicategory (see [Bénabou, 1967, Definition 1.1] for a definition of bicategory) from the principal site $(\mathcal{C}, J_{\mathcal{T}})$, whose 0-cells are the objects of \mathcal{C} , whose 1-cells are \mathcal{T} -arches and whose 2-cells are morphisms between these.

2.2.5. REMARK. The fact that \mathcal{T} -arches do not assemble into a bicategory in full generality is not merely an artefact of us having restricted ourselves to the data of the stable classes of morphisms (resp. finite families), rather than the principal (resp. finitely generated) Grothendieck topologies they generate. If we expand our collections of morphisms to multiarches indexed by arbitrary $J_{\mathcal{T}}$ -covering families, the construction in [Caramello, 2019, Proposition 2.5(iv)] does give a canonical family representing the composite, but it typically fails to be unital, since composing with an identity \mathcal{T} -span gives a strictly larger family. We can restrict to J -covering sieves to avoid this problem, but even then, without pullbacks the composition may not be weakly associative, since multi-composition of J -covering sieves is not necessarily associative in the required sense.

One situation where the hypotheses of Proposition 2.2.4 are satisfied is when \mathcal{C} has pullbacks, by Lemma 2.1.3.

2.2.6. COROLLARY. *Let $(\mathcal{C}, J_{\mathcal{T}})$ be a principal site where \mathcal{C} has pullbacks, such as the canonical sites on locally regular categories we shall see in Definition 2.6.1. Then the objects of \mathcal{C} , the \mathcal{T} -spans on \mathcal{C} and the morphisms between these assemble into a bicategory. In particular, the composition operations of Proposition 2.2.4 are natural in the first component.*

As usual, the analogous result for finitely generated sites holds, but we omit the proof.

PROOF. The verification of the conditions in Proposition 2.2.4 is straightforward; note that we actually require a specified choice of pullbacks, but the mediating isomorphisms are provided by universal properties. For the final claim, we observe that in the third condition of Proposition 2.2.4, we no longer need to restrict ourselves to the case where $e : B' \rightarrow B$ is a \mathcal{T} -morphism, since we can complete the defining rectangle with a pullback square (\mathcal{T} -morphisms are indicated with double-headed arrows):

$$\begin{array}{ccccc}
 & & (goe)' & & \\
 & \curvearrowright & & \curvearrowleft & \\
 A' & \xrightarrow{\quad} & A & \xrightarrow{\twoheadrightarrow} & B \\
 \downarrow f'' & \lrcorner & e' & \lrcorner & \downarrow f \\
 & & f' & & \\
 & \lrcorner & & \lrcorner & \\
 B' & \xrightarrow{\quad} & B & \xrightarrow{\twoheadrightarrow} & D \\
 & \curvearrowleft & & \curvearrowright & \\
 & & goe & &
 \end{array}$$

The morphism e' provides the morphism of \mathcal{T} -spans corresponding to e to make the composition natural (again, up to relevant isomorphisms) in the first component, as claimed.

The commutativity of the associativity and identity coherence diagrams which are required to formally make this a bicategory are guaranteed by the uniqueness in the universal property of the pullbacks involved. ■

As usual, the analogous result for finitely generated sites holds, but we omit the proof.

Whether it satisfies all of the requirements of a bicategory or not, however, the relationship between the \mathcal{T} -arch structure and the subcategory of $\text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ on the representables is simply that of collapsing the 2-cells, in the following sense:

2.2.7. LEMMA. *Two \mathcal{T} -arches (resp. \mathcal{T}' -multiarches) from A to B on a principal (resp. finitely generated) site are identified by ℓ if and only if they are in the same component in the category $\mathbf{Arch}_{\mathcal{T}}(A, B)$ (resp. $\mathbf{mArch}_{\mathcal{T}'}(A, B)$) described in Lemma 2.2.3.*

PROOF. In one direction, if $x : (t : E \rightarrow A, g : E \rightarrow B) \rightarrow (t' : E' \rightarrow A, g' : E' \rightarrow B)$, then by definition the unique morphism $\ell(t, g) : \ell(A) \rightarrow \ell(B)$ with $\ell(g) = \ell(t, g) \circ \ell(t)$ also satisfies $\ell(g') = \ell(t, g) \circ \ell(t')$, whence $\ell(t', g') = \ell(t, g)$. Thus ℓ identifies \mathcal{T} -spans in the same component.

Conversely, applying [Caramello, 2019, Proposition 2.5(iii)], two \mathcal{T} -arches $(t : E \rightarrow A, g : E \rightarrow B)$ and $(t' : E' \rightarrow A, g' : E' \rightarrow B)$ induce the same morphism in $\text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ if and

only if there is a \mathcal{T} -morphism $s : D \rightarrow A$ and morphisms $h : D \rightarrow E$ and $h' : D' \rightarrow E'$ with $t \circ h = s = t' \circ h'$ and $g \circ h \equiv_{J_{\mathcal{T}}} g' \circ h'$. Expanding on the latter condition, this implies the existence of some \mathcal{T} -morphism $u : E \rightarrow D$ equalizing $g \circ h$ and $g' \circ h'$. But then $(t \circ h \circ u, g \circ h \circ u) = (t' \circ h' \circ u, g' \circ h' \circ u)$ is easily shown to be a \mathcal{T} -arch, and it admits morphisms $h \circ u$ and $h' \circ u'$ to (t, g) and (t', g') respectively, whence these are in the same connected component, as required. ■

Returning to the general case, we observe that we do not need the composition operation to be well-defined at the level of \mathcal{T} -arches in order to reconstruct the full subcategory of $\text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ on the representable sheaves.

2.2.8. SCHOLIUM. *Let $(\mathcal{C}, J_{\mathcal{T}})$ be a principal site. Then the full subcategory of $\text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ on the representable sheaves is equivalent to the category whose objects are the objects of \mathcal{C} and whose morphisms $A \rightarrow B$ are indexed by the connected components of the category $\mathbf{Arch}_{\mathcal{T}}(A, B)$.*

Similarly, if $(\mathcal{C}, J_{\mathcal{T}'})$ is a finitely generated site, then the full subcategory of $\text{Sh}(\mathcal{C}, J_{\mathcal{T}'})$ on the representable sheaves is equivalent to the category whose objects are the objects of \mathcal{C} and whose morphisms $A \rightarrow B$ are indexed by the connected components of the category $\mathbf{mArch}_{\mathcal{T}'}(A, B)$.

PROOF. Observe that in the definition of composition given in the proof of Proposition 2.2.4, any choice of stability square will produce a \mathcal{T} -arch (resp. \mathcal{T}' -multiarch) lying in the same component of $\mathbf{Arch}_{\mathcal{T}}(A, C)$ (resp. $\mathbf{mArch}_{\mathcal{T}'}(A, C)$), since this \mathcal{T} -arch (resp. \mathcal{T}' -multiarch) will necessarily be mapped by ℓ to the composite of the morphisms corresponding to the pair of arches (resp. multiarches) being composed. Thus, even without weak functoriality, the composition is well-defined on connected components, as required. ■

In the subcanonical case, where all \mathcal{T} -morphisms are strict epimorphisms (resp. all \mathcal{T}' -families are jointly strictly epimorphic), the computations from this section simplify greatly. Indeed, ℓ is full and faithful in this case, which means that every component of each category $\mathbf{Arch}_{\mathcal{T}}(A, B)$ (resp. $\mathbf{mArch}_{\mathcal{T}'}(A, C)$) contains a unique (multi)arch of the form (id_A, f) . In a \mathcal{T} -arch (t, g) , g coequalizes every pair of morphisms which t coequalizes by definition, whence the morphism $\ell(t, g)$ corresponds to the unique morphism $A \rightarrow B$ factorizing g through t ; the morphisms representing multiarches are recovered analogously from the universal properties of jointly strictly epic families.

2.3. QUOTIENTS OF PRINCIPAL SITES. Rather than directly computing the category of representable sheaves in $\text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ via \mathcal{T} -arches, we might hope to simplify things by first modifying the principal site.

In Kondo and Yasuda's definition of 'B-site', they assume that the underlying category is an 'E-category', which is to say that every morphism is an epimorphism, [Kondo and Yasuda, 2017, Definitions 4.1.1, 4.2.1], which seems very restrictive. However, by taking the quotient of \mathcal{C} by a canonical congruence, we show here that we may at least assume

that \mathcal{T} is contained in the class of epimorphisms of \mathcal{C} without loss of generality, since the corresponding topos of sheaves is equivalent to that on the original site.

2.3.1. PROPOSITION. *Let $(\mathcal{C}, J_{\mathcal{T}})$ be a principal site. Then there is a canonical congruence \sim on \mathcal{C} such that $(\mathcal{C}/\sim, J_{\mathcal{T}/\sim})$ is a principal site with \mathcal{T}/\sim a subclass of the epimorphisms of \mathcal{C}/\sim , and with $\text{Sh}(\mathcal{C}, J_{\mathcal{T}}) \simeq \text{Sh}(\mathcal{C}/\sim, J_{\mathcal{T}/\sim})$.*

Similarly, if $(\mathcal{C}, J_{\mathcal{T}'})$ is a finitely generated site, there is a congruence \sim on \mathcal{C} such that $(\mathcal{C}/\sim, J_{\mathcal{T}'/\sim})$ is a finitely generated site with \mathcal{T}'/\sim a subclass of the epimorphisms of \mathcal{C}/\sim , and with $\text{Sh}(\mathcal{C}, J_{\mathcal{T}'}) \simeq \text{Sh}(\mathcal{C}/\sim, J_{\mathcal{T}'/\sim})$.

PROOF. Simply let $f \sim f' : C \rightarrow D$ whenever there is a morphism $h : C' \rightarrow C$ in \mathcal{T} with $fh = f'h$. To verify that this is a congruence, given $g \sim g' : D \rightarrow E$ equalized by $k : D' \rightarrow D$, stability of k along $fh = f'h$ gives $k' \in \mathcal{T}$:

$$\begin{array}{ccccc} C'' & \longrightarrow & D' & & \\ \downarrow k' & & \downarrow k & & \\ C' & \xrightarrow{h} & C & \xrightarrow[f']{f} & D & \xrightarrow[g']{g} & E; \end{array}$$

\mathcal{T} is closed under composition and $gfhk' = g'f'hk'$, so $gf \sim g'f'$ as required.

By the definition of the congruence it is immediate that the morphisms in \mathcal{T}/\sim are epimorphisms. The canonical functor $(\mathcal{D}, J_{\mathcal{T}}) \rightarrow (\mathcal{D}/\sim, J_{\mathcal{T}/\sim})$ is a morphism and comorphism of sites⁸, since it is cover-preserving, cover-lifting and flat by inspection. In the terminology of [Caramello, 2019], the quotient functor is $J_{\mathcal{T}}$ -full, $J_{\mathcal{T}}$ -faithful and $J_{\mathcal{T}/\sim}$ -dense; indeed, being full and essentially surjective, only the $J_{\mathcal{T}}$ -faithfulness needs to be checked, and the definition of the congruence ensures that this holds. Thus this morphism of sites induces an equivalence of sheaf toposes by [Caramello, 2019, Proposition 7.16].

The congruence for a finitely generated site has $f \sim f'$ whenever $fh_i = f'h_i$ for each h_i in a \mathcal{T}' -family. The remainder of the proof is analogous. ■

The reduced site $(\mathcal{C}/\sim, J_{\mathcal{T}/\sim})$ in Proposition 2.3.1 can be obtained in various alternative ways. By construction, the functor $\mathcal{C} \rightarrow \mathcal{C}/\sim$ is the universal functor with domain \mathcal{C} sending \mathcal{T} -morphisms to epimorphisms, so it is not surprising thanks to the proof of Proposition 2.1.7 that ℓ canonically factors through it. Less evidently, this congruence is that induced by the functor underlying the morphism of sites $(\mathcal{C}, J_{\mathcal{T}}) \rightarrow (\text{Sh}(\mathcal{C}, J_{\mathcal{T}}), J_{\text{can}})$. See Section 2.7 below for more on morphisms of sites.

After making this simplification, $J_{\mathcal{T}}$ -local equality (resp. $J_{\mathcal{T}'}$ -local equality) reduces to ordinary equality, so that for example a \mathcal{T} -arch from A to B is a \mathcal{T} -span as in (6) such that for any $h, k : D \rightrightarrows C$ with $f \circ h = f \circ k$ we have $g \circ h = g \circ k$. In particular, by considering the arches in which the \mathcal{T} -morphism is an identity, we see that the functor $\ell : (\mathcal{C}/\sim, J_{\mathcal{T}/\sim}) \rightarrow \text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ is faithful, and further that $\ell : (\mathcal{C}, J_{\mathcal{T}}) \rightarrow \text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ is faithful if and only if every \mathcal{T} -morphism is an epimorphism, which is to say that the congruence \sim is trivial.

⁸See Definitions 2.7.1 and 2.8.1 below.

2.3.2. COROLLARY. *Let $(\mathcal{C}, J_{\mathcal{T}})$ be a principal site and let \sim be the congruence on \mathcal{C} from Proposition 2.3.1. Then the functor $\ell : (\mathcal{C}, J_{\mathcal{T}}) \rightarrow \text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ is full if and only if every \mathcal{T}/\sim -morphism in \mathcal{C}/\sim is a **strict** epimorphism.*

Similarly, if $(\mathcal{C}, J_{\mathcal{T}'})$ is a finitely generated site, then $\ell : (\mathcal{C}, J_{\mathcal{T}'}) \rightarrow \text{Sh}(\mathcal{C}, J_{\mathcal{T}'})$ is full if and only if every \mathcal{T}'/\sim -family in \mathcal{C}/\sim is a strictly epimorphic family.

PROOF. If the hypothesis holds, then the site $(\mathcal{C}/\sim, J_{\mathcal{T}/\sim})$ is subcanonical, so the induced functor to the topos is full and faithful, whence ℓ with domain $(\mathcal{C}, J_{\mathcal{T}})$ is full.

Conversely, given that ℓ is full, suppose that $t : E \rightarrow A$ is in \mathcal{T} . Suppose that we have $g : E \rightarrow B$ such that whenever $h, k : E' \rightarrow E$ with $t \circ h = t \circ k$, we have $g \circ h = g \circ k$. Then (t, g) is a \mathcal{T} -arch, and there is a morphism $\ell(t, g)$ in $\text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ completing the triangle. By fullness of ℓ , this is the image of a morphism $A \rightarrow B$ in \mathcal{C}/\sim , and by the definition of \sim there is at most one such morphism, whence t/\sim is a strict epimorphism, as claimed.

The argument for finitely generated sites is analogous. ■

It follows from Corollary 2.3.2 that the restriction of the codomain of ℓ to the full subcategory of $\text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ on the objects of the form $\ell(C)$ is the universal functor sending \mathcal{T} -morphisms to strict epimorphisms.

2.3.3. EXAMPLE. Recall that a category \mathcal{C} satisfies the **right Ore condition** if any cospan can be completed to a commutative square. This is exactly the condition needed to make the class of *all* morphisms of \mathcal{C} stable, and the corresponding principal topology is more commonly called the **atomic topology**, J_{at} , while the site (\mathcal{C}, J_{at}) is called an **atomic site**. The above results show that we may reduce any atomic site to one in which every morphism is epic (hence a ‘ B -site’ in the terminology of Kondo and Yasuda).

2.4. REDUCTIVE AND COALESCENT CATEGORIES. Returning to our study of the subcategories of supercompact and compact objects from the last section, we observe that the epimorphisms they inherit from \mathcal{E} always meet most of the requirements for stability.

2.4.1. LEMMA. *For any Grothendieck topos \mathcal{E} , let $\mathcal{C}_s, \mathcal{C}_c$ the usual subcategories. Then the class \mathcal{T} of epimorphisms in \mathcal{C}_s which are inherited from \mathcal{E} satisfies axioms 1, 2 and 4 for stable classes, while the class \mathcal{T}' of finite jointly epimorphic families on \mathcal{C}_c inherited from \mathcal{E} satisfies axioms 1', 2', 4' and 5'.*

PROOF. Clearly \mathcal{C}_s and \mathcal{C}_c inherit identities from \mathcal{E} , so \mathcal{T} contains these and \mathcal{T}' contains the singleton families of the identities. Since epimorphisms are stable under composition in \mathcal{E} , \mathcal{T} is closed under composition. Multicomposition of finite jointly covering families in \mathcal{E} is similarly direct, giving the second axiom for \mathcal{T}' . Axioms 4' and 5' are straightforward. ■

By Theorem 1.4.3, if \mathcal{E} is supercompactly generated (resp. compactly generated) then axiom 3 (resp. axiom 3') must also be satisfied in each case. Note that the converse fails: stability of \mathcal{E} -epimorphisms in \mathcal{C}_s or stability of \mathcal{E} -epimorphic finite families in \mathcal{C}_c are not sufficient to guarantee that \mathcal{E} is supercompactly or compactly generated. Indeed, if \mathcal{E} a non-degenerate topos where the only compact object is the initial object, such as that

exhibited in Example 2.9.3, the stability axioms are trivially satisfied but \mathcal{E} is neither supercompactly nor compactly generated.

In the remainder of this section, we refine the concepts of principal and finitely generated sites in order to obtain a characterisation of the categories \mathcal{C}_s (resp. \mathcal{C}_c) of supercompact (resp. compact) objects in supercompactly (resp. compactly) generated toposes.

2.4.2. DEFINITION. We say a small category \mathcal{C} is **reductive** if it has funneling colimits and its class of strict epimorphisms is stable. The **reductive topology** J_r on a reductive category is the principal topology generated by its class of strict epimorphisms.

We say \mathcal{C} is **coalescent** if it has multifunneling colimits and its class of (strictly) epic finite families is stable. The **coalescent topology** J_c on a coalescent category is the finitely generated topology on its class of strictly epic finite families.

2.4.3. REMARK. Some justification for this naming and notation is warranted.

The names of the categories are intended to evoke the presence of funneling (resp. multifunneling) colimits, since any diagram in them of the respective shapes ‘is reduced’ (resp. ‘coalesces’) by composing with a suitable epimorphism (resp. jointly epimorphic family). If we consider the reductive category

$$\mathcal{C} := A \rightrightarrows B \rightarrow C,$$

featuring in Example 2.9.6 below, for example, we can identify a functor $F : \mathcal{C} \rightarrow \mathbf{Set}$ with a directed graph, equipped with a mapping on its set of vertices which identifies connected elements; if F preserves funneling colimits, then the image of the epimorphism $B \rightarrow C$ must exactly be the map reducing the graph to its set of connected components.

The names were also chosen to have their first few letters in common with *regular* and *coherent* respectively, since the regular and coherent objects in a topos are respectively subclasses of the supercompact and compact objects. Thus, while the r and c in J_r and J_c stand for *reductive* and *coalescent* respectively, we shall see that when a category is both regular and reductive, the regular topology (see Definition 2.6.3 below) coincides with the reductive topology, so the r on J_r could also mean ‘regular’; similarly for coalescent and coherent.

While not every stable class of finite families need contain a stable class of singleton morphisms, we record the fact that this does happen when the families involved are strictly epimorphic families.

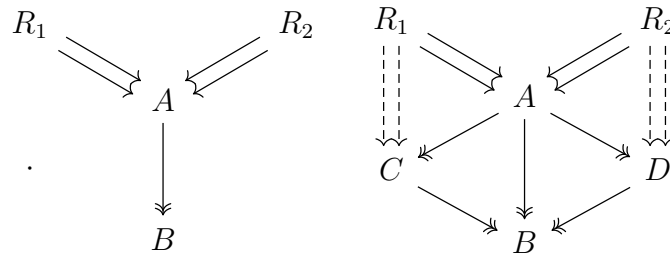
2.4.4. LEMMA. Any coalescent category is a reductive category with finite colimits and a strict initial object.

PROOF. If \mathcal{C} is a coalescent category, it certainly has the required colimits, so it suffices to show stability of strict epimorphisms. Indeed, if $t : D \rightarrow C$ is strict and $g : B \rightarrow C$ is any morphism in \mathcal{C} , then since $\{t\}$ is a strictly epic family, there is some strictly epic family $\{h_j : A_j \rightarrow B \mid i = 1, \dots, m\}$ over B such that each $g \circ h_j$ factors through t . Factoring this family through the coproduct $A_1 + \dots + A_m$ gives a strict epimorphism completing the required stability square (even in the case $m = 0$).

To see that the initial object is strict, observe that the empty family is a strict jointly epic family on the initial object, so given any morphism $A \rightarrow 0$, stability forces the empty family to be jointly epic over A , whence A is also an initial object. ■

It would be easy to mistakenly conclude based on the results presented thus far that the subcategory of supercompact objects in the category of sheaves on a reductive category \mathcal{C} should be equivalent to \mathcal{C} . The flaw in this reasoning lies in the fact that, while the functor $\ell : \mathcal{C} \rightarrow \text{Sh}(\mathcal{C}, J_r)$ is full and faithful (since (\mathcal{C}, J_r) is a subcanonical site) and this functor preserves strict epimorphisms, it *does not* preserve all funneling colimits; a similar argument applies for coalescent categories.

2.4.5. EXAMPLE. Consider the following categories, \mathcal{C} and \mathcal{C}' on the left and right respectively. It is easily checked that they are reductive, with strict epimorphisms identified with two heads.



The coequalization is that suggested by the positioning, so that in the first diagram, the coequalizer of the pair coming from R_1 is the terminal object B , but in the second diagram, C is the coequalizer of the pair coming from R_2 .

One can calculate directly that the category of supercompact objects in $\text{Sh}(\mathcal{C}, J_r) \simeq \text{Sh}(\mathcal{C}', J_r)$ is equivalent to \mathcal{C}' . Indeed, the functor $\ell : \mathcal{C} \rightarrow \text{Sh}(\mathcal{C}, J_r)$ does not preserve the coequalizer diagram $R_1 \rightrightarrows A \rightarrow B$.

We shall see a further example of a failure of ℓ to preserve coequalizers (and hence funneling colimits) in Example 2.9.13. In order to understand which colimits are preserved by ℓ , we apply criteria derived in [Caramello, 2019, Corollary 2.25], which we recall here; we refer the reader to that monograph for the proof.

2.4.6. LEMMA. Let (\mathcal{C}, J) be a site, $F : \mathcal{D} \rightarrow \mathcal{C}$ a diagram and $\vec{\lambda} = \{\lambda_D : F(D) \rightarrow C_0 \mid D \in \mathcal{D}\}$ a cocone under F with vertex C_0 . Then $\vec{\lambda}$ is sent by the canonical functor $\ell : \mathcal{C} \rightarrow \text{Sh}(\mathcal{C}, J)$ to a colimit cone if and only if:

- (i) For any object C and morphism $g : C \rightarrow C_0$ in \mathcal{C} , there is a J -covering family $\{f_i : C_i \rightarrow C \mid i \in I\}$ and for each $i \in I$, an object D_i of \mathcal{D} and an arrow $h_i : C_i \rightarrow F(D_i)$ such that $\lambda_{D_i} \circ h_i = g \circ f_i$.
- (ii) For any object C in \mathcal{C} and morphisms $g_1 : C \rightarrow F(X)$, $g_2 : C \rightarrow F(Y)$ such that $\lambda_X \circ g_1 = \lambda_Y \circ g_2$, there is a J -covering family $\{f_i : C_i \rightarrow C \mid i \in I\}$ such that for each $i \in I$, $g_1 \circ f_i$ and $g_2 \circ f_i$ lie in the same connected component of $(C_i \downarrow F)$.

Observe that the first condition can be simplified.

2.4.7. LEMMA. *Condition (i) of Lemma 2.4.6 is equivalent to the requirement that $\{\lambda_D : F(D) \rightarrow C_0 \mid D \in \mathcal{D}\}$ is a J -covering family.*

PROOF. Consider the case where g is the identity on C_0 . There must be some J -covering family $\{f_i : C_i \rightarrow C \mid i \in I\}$ and for each $i \in I$, an object D_i of \mathcal{D} and an arrow $h_i : C_i \rightarrow F(D_i)$ such that $\lambda_{D_i} \circ h_i = g \circ f_i$. Since every morphism in this covering family factors through a leg of the colimit cone, the legs of the cone must form a J -covering family. Conversely, given any morphism $g : C \rightarrow C_0$, since J -covering families are required to be stable, pulling back $\vec{\lambda}$ gives the required J -covering family over C to fulfill condition (i). ■

Applying this in the particular case of funneling or multifunneling colimits in a principal or finitely generated site (\mathcal{C}, J) , we get the following:

2.4.8. PROPOSITION. *Let $(\mathcal{C}, J_{\mathcal{T}})$ be a principal site and $F : \mathcal{D} \rightarrow \mathcal{C}$ a funneling diagram with weakly terminal object $F(D_0)$ and $\vec{\lambda} = \{\lambda_D : F(D) \rightarrow C_0 \mid D \in \mathcal{D}\}$ a cocone under F with vertex C_0 . Then $\vec{\lambda}$ is sent by the canonical functor $\ell : \mathcal{C} \rightarrow \text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ to a colimit cone if and only if:*

- (i) $\lambda_{D_0} \in \mathcal{T}$.
- (ii) *For any object C in \mathcal{C} and morphisms $g_1, g_2 : C \rightrightarrows F(D_0)$, such that $\lambda_{D_0} \circ g_1 = \lambda_{D_0} \circ g_2$, there is a \mathcal{T} -morphism $t : C' \rightarrow C$ such that $g_1 \circ t$ and $g_2 \circ t$ lie in the same connected component of $(C' \downarrow F)$.*

Similarly, if $(\mathcal{C}, J_{\mathcal{T}'})$ is a finitely generated site and $F : \mathcal{D} \rightarrow \mathcal{C}$ a multifunneling diagram with weakly terminal objects $F(D_1), \dots, F(D_n)$ and $\vec{\lambda} := \{\lambda_D : F(D) \rightarrow C_0 \mid D \in \mathcal{D}\}$ a cocone under F with vertex C_0 , then $\vec{\lambda}$ is sent by $\ell : \mathcal{C} \rightarrow \text{Sh}(\mathcal{C}, J_{\mathcal{T}'})$ to a colimit cone if and only if:

- (i) $\{\lambda_{D_1}, \dots, \lambda_{D_n}\} \in \mathcal{T}'$.
- (ii) *For any object C in \mathcal{C} and morphisms $g_1 : C \rightarrow F(D_k)$ and $g_2 : C \rightarrow F(D_l)$ with $1 \leq k, l \leq n$, such that $\lambda_{D_k} \circ g_1 = \lambda_{D_l} \circ g_2$, there is a \mathcal{T}' -family $\{t_i : C_i \rightarrow C \mid 1 \leq i \leq N\}$ such that $g_1 \circ t_i$ and $g_2 \circ t_i$ lie in the same connected component of $(C_i \downarrow F)$.*

PROOF. For (i) in each case, we apply Lemma 2.4.7 and then the fact that every morphism in the cone factors through λ_{D_0} (resp. one of the λ_{D_k}) to deduce, thanks to stability axiom 4 (resp. axioms 4' and 5') that condition (i) of Lemma 2.4.6 is equivalent to the given statement.

Condition (ii) in each case is a consequence of condition (ii) in Lemma 2.4.6, having simply taken the special case $X = Y = D_0$ (resp. $X = D_k$ and $Y = D_l$). Conversely, for arbitrary $g_1 : C \rightarrow F(X)$ and $g_2 : C \rightarrow F(Y)$ such that $\lambda_X \circ g_1 = \lambda_Y \circ g_2$, we may extend this via any of the morphisms $p_1 : X \rightarrow D_0$ and $p_2 : Y \rightarrow D_0$ in the diagram (resp. $p_1 : X \rightarrow D_k$ and $p_2 : Y \rightarrow D_l$) so that $\lambda_{D_0} \circ Fp_1 \circ g_1 = \lambda_{D_0} \circ Fp_2 \circ g_2$ (resp. $\lambda_{D_k} \circ Fp_1 \circ g_1 = \lambda_{D_k} \circ Fp_2 \circ g_2$). Given a \mathcal{T} -morphism $t : C' \rightarrow C$ such that $Fp_1 \circ g_1 \circ t$

and $Fp_2 \circ g_2 \circ t$ are in the same connected component of $(C' \downarrow F)$ (resp. a \mathcal{T}' -family $\{t_i : C_i \rightarrow C\}$ such that $Fp_1 \circ g_1 \circ t_i$ and $Fp_2 \circ g_2 \circ t_i$ are in the same connected component of $(C_i \downarrow F)$), it is clear that $g_1 \circ t$ and $g_2 \circ t$ (resp. $g_1 \circ t_i$ and $g_2 \circ t_i$ for each i) also lie in this same component, as required. ■

As an illustration of the second half of Proposition 2.4.8, we deduce the special case of binary coproducts (cf. Lemma 1.3.7).

2.4.9. COROLLARY. *For $(\mathcal{C}, J_{\mathcal{T}'})$ a finitely generated site, a cospan $\lambda_1 : X_1 \rightarrow Y \leftarrow X_2 : \lambda_2$ is mapped by $\ell : \mathcal{C} \rightarrow \text{Sh}(\mathcal{C}, J_{\mathcal{T}'})$ to a coproduct cocone if and only if:*

- (i) $\{\lambda_1, \lambda_2\} \in \mathcal{T}'$,
- (ii) Whenever $f_1 : C \rightarrow X_1$ and $f_2 : C \rightarrow X_2$ have $\lambda_1 \circ f_1 = \lambda_2 \circ f_2$, the empty family is \mathcal{T}' covering on C , and
- (iii) Whenever $f, f' : C \rightrightarrows X_1$ are coequalized by λ_1 , there is a \mathcal{T}' -covering family on C consisting of morphisms equalizing f and f' (and similarly for pairs of morphisms into X_2).

If \mathcal{T}' -covering families are jointly epic, we can replace the last condition by the condition that λ_1 and λ_2 must be monic.

PROOF. This is a direct application of Proposition 2.4.8 for multifunneling colimits in the case of a finite discrete diagram, where two morphisms are in the same connected component of $(C \downarrow F)$ if and only if they are equal (which is impossible if they have distinct codomains). ■

Imposing these conditions on funneling and multifunneling colimit cones in reductive and coalescent categories, we arrive at the following definitions.

2.4.10. DEFINITION. *We say a reductive or coalescent category \mathcal{C} is **effectual** if the respective conditions of Proposition 2.4.8 hold for every funneling (resp. multifunneling) colimit diagram.*

Explicitly, a reductive category is effectual if for every funneling diagram $F : \mathcal{D} \rightarrow \mathcal{C}$ with colimit expressed by $\lambda : F(D_0) \rightarrow C_0$, given morphisms $g_1, g_2 : C \rightrightarrows F(D_0)$ in \mathcal{C} such that $\lambda \circ g_1 = \lambda \circ g_2$, there is a strict epimorphism $t : C' \rightarrow C$ such that $g_1 \circ t$ and $g_2 \circ t$ lie in the same connected component of $(C' \downarrow F)$.

Similarly, a coalescent category is effectual if for every multifunneling diagram $F : \mathcal{D} \rightarrow \mathcal{C}$ with colimit expressed by $\{\lambda_i : F(D_i) \rightarrow C_0\}$, given morphisms $g_1 : C \rightarrow F(D_k)$ and $g_2 : C \rightarrow F(D_l)$ such that $\lambda_k \circ g_1 = \lambda_l \circ g_2$, there is a jointly strictly epimorphism family $\{t_j : C'_j \rightarrow C : 1 \leq j \leq m\}$ such that $g_1 \circ t_j$ and $g_2 \circ t_j$ lie in the same connected component of $(C'_j \downarrow F)$ for each index j .

Recall that a category \mathcal{C} with finite coproducts is said to be **positive** if and only if coproduct inclusions are monomorphisms and finite coproducts are disjoint, in the sense

that for all objects A and B of \mathcal{C} , the following square is a pullback:

$$\begin{array}{ccc} 0 & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A + B. \end{array}$$

Applying Corollary 2.4.9, we deduce:

2.4.11. COROLLARY. *An effectual coalescent category is positive.*

With these definitions to hand, we can finally express a definitive correspondence result between supercompactly or compactly generated toposes and their canonical sites from Theorem 1.4.3.

2.4.12. THEOREM. *Up to equivalence, there is a one-to-one correspondence between supercompactly generated Grothendieck toposes and essentially small effectual, reductive categories. The correspondence sends a topos to its essentially small category of supercompact objects and a reductive category to the topos of sheaves for the reductive topology on that category.*

Similarly, there is an up-to-equivalence correspondence between compactly generated Grothendieck toposes and effectual coalescent categories.

PROOF. Passing from a topos to its subcategory of supercompact (resp. compact) objects and back again gives an equivalent topos by Theorem 1.4.3; the small category must be an effectual reductive (resp. effectual coalescent) category, since the subcategory of supercompact (resp. compact) objects is closed under funneling (resp. multifunneling) colimits, so the inclusion must preserve them.

In the other direction, since the strict (resp. strict finite family) Grothendieck topology is subcanonical, a reductive (resp. coalescent) category is included faithfully as a full subcategory of the corresponding sheaf topos, and all of the representable sheaves are supercompact (resp. compact). The supercompact (resp. compact) objects are funneling (resp. multifunneling) colimits of the representable sheaves in this topos, but these colimits are preserved by ℓ by construction when the category is effectual, so the category of representable sheaves coincides with the category of supercompact (resp. compact) objects, as required. ■

We shall extend this correspondence to an equivalence of 2-categories in Section 2.7. Since geometric morphisms shall come into play at that point, we add here the following extra definition:

2.4.13. DEFINITION. *A reductive category \mathcal{C} is **augmented** if it has an initial object. The **augmented reductive topology** J_{r+} on such a category has covering sieves generated by singleton or empty strictly jointly epic families. The resulting **augmented reductive site** (\mathcal{C}, J_{r+}) is quasi-principal.*

Before moving on, a word of warning. In spite of Lemma 2.4.4, an effectual coalescent category *may fail to be an effectual reductive category*, because the condition for a funneled colimit in a coalescent category to be preserved upon passing to the corresponding topos is weaker than the condition on reductive categories; we shall see a counterexample in Example 2.9.15. Even when a coalescent category is ‘reductive-effectual’, the corresponding toposes from Theorem 2.4.12 are always distinct, since we have seen that the initial object of a topos is never supercompact but always compact.

2.5. LOCALIC SUPERCOMPACTLY GENERATED TOPOSES. Recall that a Grothendieck topos is **localic** if it is of the form $\text{Sh}(L)$, the category of (set-valued) sheaves on some locale L ; this is equivalent to the global sections geometric morphism being localic in the sense described in Section 1.8. These toposes can equivalently be characterised by the fact that their set of subterminal objects is separating, and these subterminals form a frame isomorphic to the frame of opens of the locale.

The conditions for $\text{Sh}(L)$ to be supercompactly or compactly generated reduce to conditions on the frame of subterminal objects. It is worth mentioning that the cases where $\text{Sh}(L)$ has enough coherent objects produces the notion of *coherent frame*, while having enough regular objects gives a *supercoherent frame*. These are respectively discussed by Banaschewski and a coauthor in [Banaschewski 1981] and [Banaschewski and Niefield 1991], where they employ the subobject characterizations in the form of ‘way below’ and ‘totally below’ relations, respectively.

2.5.1. PROPOSITION. *A localic topos is supercompactly generated if and only if it is equivalent to $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ for some poset \mathcal{C} . Moreover, any poset is an instance of an effectual reductive category.*

PROOF. The ‘if’ direction is a consequence of Proposition 1.2.4 and the fact such a presheaf topos is necessarily localic.

Conversely, if \mathcal{E} is supercompactly generated, each subterminal must be covered by its supercompact subobjects, so the supercompact subterminal objects generate the topos. A subterminal object is supercompact if and only if it has no proper cover by strictly smaller subterminals. This forces the canonical topology on the supercompact objects to be trivial, whence $\mathcal{E} \simeq [\mathcal{C}_s^{\text{op}}, \mathbf{Set}]$. By considering the expression of \mathcal{E} as a category of sheaves on a locale, we see that the supercompact objects must all be quotients of subterminals, and hence themselves subterminals, so \mathcal{C}_s is a poset. ■

To characterize localic compactly generated toposes, we need to present a more obscure definition.

2.5.2. DEFINITION. *A **join semilattice** is a poset having all finite joins, including the bottom element. We say a join semilattice is **distributive** if for any triple of objects (a, b, c) with $a \leq b \vee c$, there are elements $b' \leq b$ and $c' \leq c$ such that $a = b' \vee c'$; see [Grätzer, 2002, §II.5]. This in particular holds in any distributive lattice.*

Distributivity, which inductively extends to arbitrary finite joins, is precisely the condition ensuring that the collection of finite join covers is a stable class of finite families.

Note that finite join covers are coproduct injections, so that *any distributive join semilattice is a coalescent category*. Considering the join of any element with itself, we see that any non-degenerate distributive lattice is an example of a coalescent category which fails to be positive (and hence fails to be effectual).

2.5.3. PROPOSITION. *A localic topos is compactly generated if and only if it is the category of sheaves on a distributive join semilattice with respect to the topology that makes finite joins covering.*

PROOF. Unlike in the supercompact case, the compact subterminal objects no longer populate all of \mathcal{C}_c , but a similar argument applies: if \mathcal{C} is the subcategory of \mathcal{E} on the compact *subterminal* objects, then \mathcal{C} is a distributive join semilattice since a finite union of compact subterminal objects is compact, and so $\mathcal{E} = \text{Sh}(\mathcal{C}, J)$ where J is the topology on \mathcal{C} whose covering families are finite joins. ■

By applying Theorem 1.8.6, we can immediately conclude that the above characterisations apply to the localic reflections of supercompactly and compactly generated toposes.

2.5.4. COROLLARY. *The localic reflection of any supercompactly generated topos is a (localic) presheaf topos. The localic reflection of a compactly generated topos is a topos of sheaves on a distributive join semilattice.*

2.6. LOCALLY REGULAR AND COHERENT CATEGORIES. As we observed earlier, supercompactness and compactness have been studied in the context of regular and coherent toposes, which are toposes of sheaves on regular and coherent categories respectively, equipped with suitable Grothendieck topologies. Here we recall the definitions of these classes of categories, as well as some more general classes, for comparison with reductive and coalescent categories.

2.6.1. DEFINITION. *Recall that an epimorphism e in a category \mathcal{C} is **extremal** if whenever $e = m \circ g$ with m a monomorphism, then m is an isomorphism.*

*A category is **locally regular** if it is closed under connected finite limits, it has an orthogonal (extremal epi, mono) factorisation system, and every span factors through a jointly monic pair via an extremal epimorphism. Such a category is **regular** if it also has finite products (equivalently, a terminal object). Clearly, a slice (also called an ‘over-category’) of a locally regular category is regular.*

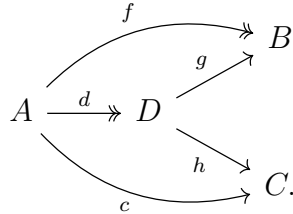
*We say a category is **locally coherent** if it is locally regular and finite unions of subobjects (including the minimal subobject) exist and are stable under pullback. Such a category is **coherent** if it also has finite products.*

2.6.2. LEMMA. *Every extremal epimorphism in a locally regular category is regular.*

PROOF. We adapt the proof, [Johnstone, 2002, Proposition A1.3.4], that in a regular category, covers are regular epimorphisms; we omit the composition symbol for conciseness in this proof.

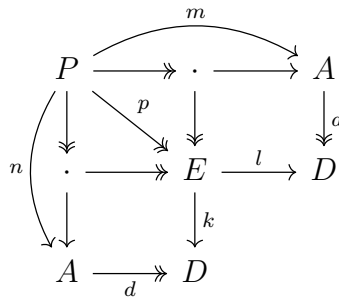
Let $f : A \rightarrow B$ be an extremal epimorphism in a locally regular category and let $a, b : R \rightrightarrows A$ be its kernel pair. We show that f coequalizes a and b .

Suppose $c : A \rightarrow C$ has $ca = cb$, and factorize the span (f, c) as an extremal epimorphism followed by a jointly monic pair:



If we can show that g is monic, then extremality of f will force it to be an isomorphism, so that $c = hg^{-1}f$ factors through f .

Given $k, l : E \rightrightarrows D$ with $gk = gl$, consider the following diagram composed of pullback squares:



The morphism labelled p is a composite of extremal epimorphisms (by stability) and hence is itself an extremal epimorphism. From this diagram and the preceding assumptions, we have $fm = gdm = gkp = glp = gdn = fn$, whence (m, n) factors through (a, b) via some morphism $q : P \rightarrow R$, and we have:

$$hkp = hdm = cm = caq = cbq = cn = hdn = hlp,$$

whence $hk = hl$ by epicness of p , but since (g, h) was jointly monic, we have $k = l$, which completes the proof. ■

2.6.3. DEFINITION. *The **regular topology** on a regular or locally regular category is simply the principal topology generated by the extremal epimorphisms, and similarly the **coherent topology** on a coherent or locally coherent category is the finitely generated topology generated by the finite jointly extremal epic families.*

By Lemma 2.6.2, we may replace ‘extremal’ with ‘regular’ in the descriptions of the stable classes in this definition, whence we see that these sites are subcanonical. Accordingly we obtain a regular (resp. locally regular, coherent, locally coherent) topos of sheaves on such a site, where here the adjective merely indicates that the topos can be generated by such a site; any topos is automatically a coherent (and hence regular, locally regular and locally coherent) category⁹.

⁹The reason for the somewhat unfortunate naming convention which we are extending here is explained in [Johnstone, 2002, D3.3].

By the previous results in this section, any locally regular topos is supercompactly generated, and any locally coherent topos is compactly generated. We are therefore led to wonder when the classes of categories coincide.

2.6.4. **THEOREM.** *A small category is locally regular with funneling colimits if and only if it is a reductive category with pullbacks.*

A small category is locally coherent with multifunneling colimits if and only if it is a coalescent category with pullbacks.

In each case, we can remove the “locally” adjective in exchange for adding a terminal object.

PROOF. In one direction, by Lemma 2.6.2 we have that in a locally regular category, the classes of extremal, strict and regular epimorphisms all coincide, since any strict epimorphism is extremal, and they form a stable class by assumption, whence a locally regular category with funneling colimits is a reductive category with pullbacks.

Conversely, given a reductive category \mathcal{C} with pullbacks, we must show that \mathcal{C} has equalizers, since a category has connected finite limits if and only if it has both pullbacks and equalizers; the remaining conditions follow from Corollary 1.3.14 and Lemma 1.3.15, thanks to Lemma 2.1.3. Given a pair of morphisms $h, k : A \rightrightarrows B$ in \mathcal{C} , consider their coequalizer $c : B \rightarrow C$. Then \mathcal{C}/C is regular, since it has pullbacks and a terminal object (so all finite limits), and it inherits the required factorization system, including that for spans, from \mathcal{C} . Therefore there exists an equalizer of h and k as morphisms over C , and it is clear that this will also be their equalizer in \mathcal{C} .

For the locally coherent case, by considering the strictly epic finite families of *subobjects*, we see that the fact that unions are stable under pullback ensures that strictly epic finite families form a stable class, as required.

Conversely, given a coalescent category \mathcal{C} with pullbacks, the argument above, Corollary 1.3.14 and Lemma 1.3.15 give that \mathcal{C} is locally regular. For finite unions of subobjects, observe that it suffices to consider nullary and binary unions. The former are guaranteed by the strict initial object of a coalescent category, seen in Lemma 2.4.4. For the latter, observe that the union can be expressed as the pushout (a multifunneling colimit) along the intersection of the two subobjects (the pullback of the monomorphisms defining the subobjects), since any subobject containing the given pair of subobjects forms a cone under this diagram. The fact that strictly epic families are stable under pullback ensures that these unions are too, again thanks to Lemma 2.1.3. ■

2.6.5. **EXAMPLE.** A distributive join semilattice has pullbacks and a top element if and only if it is a distributive lattice, so the (bounded) distributive lattices are precisely the coherent distributive join semilattices. ■

The reader may have noticed that we did not include the properties of effectiveness for regular and coherent categories in Definition 2.6.1:

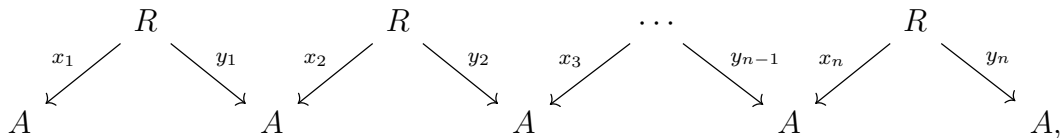
2.6.6. DEFINITION. A (locally) regular category is **effective**¹⁰ if all equivalence relations are kernel pairs.

We chose a similar name, ‘effectual’, for the concepts appearing in Definition 2.4.10 because both effectuality and effectiveness are conditions equivalent to the relevant categories being recoverable from the associated topos. Indeed, a locally regular, effective category \mathcal{C} can be recovered from the topos of sheaves on \mathcal{C} for the regular topology, $\text{Sh}(\mathcal{C}, J_r)$, as the category of *regular objects*, which were defined in Example 1.2.3. Similarly, if \mathcal{C} is locally coherent, positive and effective, it can be recovered from $\text{Sh}(\mathcal{C}, J_c)$ as the category of *coherent objects*, which are defined analogously. As a special case, we recover the familiar correspondences between effective regular categories and regular toposes, or between effective, positive coherent categories (also known as *pretoposes*) and coherent toposes. These results are comparable to Theorem 2.4.12. The concepts of effectuality and effectiveness are directly related:

2.6.7. PROPOSITION. Let \mathcal{C} be a reductive category with pullbacks. Then if \mathcal{C} is effectual as a reductive category, it is also effective as a regular category.

PROOF. First suppose that \mathcal{C} is effectual, let $a, b : R \rightrightarrows A$ be an equivalence relation on A , and let $\lambda : A \rightrightarrows B$ be its coequalizer. We must show that (a, b) is the kernel pair of c . Given $g_1, g_2 : C \rightrightarrows A$ with $\lambda \circ g_1 = \lambda \circ g_2$, by effectuality of \mathcal{C} there is a strict epimorphism $t : C' \twoheadrightarrow C$ such that $g_1 \circ t$ and $g_2 \circ t$ lie in the same connected component of $(C' \downarrow F)$, where $F : \mathcal{D} \longrightarrow \mathcal{C}$ is the diagram picking out the parallel pair (a, b) .

The only form a connecting zigzag can have in $(C' \downarrow F)$ is (omitting the morphisms from C' and any identity morphisms):



with each x_i and y_i equal to a or b . By reflexivity of R we may construct a zigzag consisting of $n > 0$ spans. We show by downward induction that there must always be a zigzag of exactly 1 span, which corresponds to a factorization of the span $(g_1 \circ t, g_2 \circ t)$ through the relation (a, b) .

Clearly if $x_1 = y_1$ we may omit the first zigzag, and similarly for all of the others, so we may assume that $x_i \neq y_i$. By symmetry of R , if $(x_i, y_i) = (b, a)$, this factorizes through (a, b) , which gives an alternative zigzag of the same length in $(C' \downarrow F)$, so we may assume $x_i = a$ and $y_i = b$. If $n \geq 2$, we may factor through the pullback of x_2 along y_1 , and transitivity of R means that we get a strictly shorter zigzag; iterating this, we reach a zigzag with $n = 1$, as required.

Finally, taking the (regular epimorphism, relation) factorization of $(g_1 \circ t, g_2 \circ t)$, we conclude that the resulting relation (and hence (g_1, g_2)) must factor uniquely through R , as required. ■

¹⁰Referred to as **Barr-exactness** in older texts; we follow Johnstone in our terminology.

As we shall see in Example 2.9.14, the converse of Proposition 2.6.7 fails, which is why we did not employ the same name for these concepts.

2.6.8. **REMARK.** While we provided a direct proof of Proposition 2.6.7 for completeness, we could more succinctly have reasoned as follows. When a category is both regular and reductive, the strict and regular topologies on the category coincide. If such a category is effectual, therefore, all of the supercompact objects in its topos of sheaves are regular, and hence it must also be effective, since it is equivalent to the category of regular objects in its associated topos.

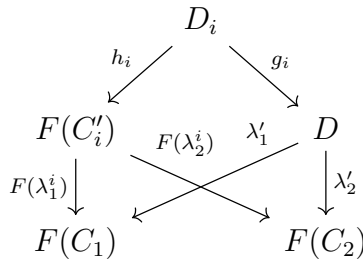
More generally, one might take an interest in the regular objects in a supercompactly generated topos, or the coherent objects in a compactly generated topos. However, this class of objects need not be stable under pullback in general, and hence may not assemble into a locally regular (resp. locally coherent) category. Nonetheless, by considering the induced Grothendieck topology on this subcategory, we obtain a supercompactly generated subtopos of the original topos. Iterating this process recursively, in the countable limit we obtain a maximal pullback-stable class of regular objects, although the resulting subcategory still may not be a locally regular category in the sense of Definition 2.6.1, since that definition also required the presence of equalizers. Since it is unclear to us whether this class of objects or the corresponding subtopos have an interesting universal property, and since we lack interesting specific examples of this construction, we terminate our analysis here.

2.7. **MORPHISMS OF SITES.** Morphisms of sites are most easily defined on sites whose underlying category has finite limits. However, there is no reason for this property to hold in a general principal or finitely generated site. We must therefore use the more general definition of morphism of sites, which we quote from [Caramello, 2019, Definition 3.2].

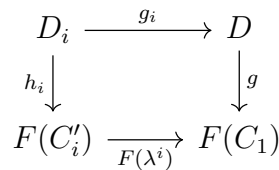
2.7.1. **DEFINITION.** *Let (\mathcal{C}, J) and (\mathcal{D}, K) be sites. Then a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a **morphism of sites** if it satisfies the following conditions:*

1. *F sends every J -covering family in \mathcal{C} to a K -covering family in \mathcal{D} .*
2. *Every object D of \mathcal{D} admits a K -covering family $\{g_i : D_i \rightarrow D \mid i \in I\}$ by objects D_i admitting morphisms $h_i : D_i \rightarrow F(C'_i)$ to objects in the image of F .*
3. *For any objects C_1, C_2 of \mathcal{C} and any span $(\lambda'_1 : D \rightarrow F(C_1), \lambda'_2 : D \rightarrow F(C_2))$ in \mathcal{D} , there exists a K -covering family $\{g_i : D_i \rightarrow D \mid i \in I\}$, a family of spans in \mathcal{C} , $\{(\lambda_1^i : C'_i \rightarrow C_1, \lambda_2^i : C'_i \rightarrow C_2) \mid i \in I\}$, and a family of morphisms in \mathcal{D} ,*

$\{h_i : D_i \rightarrow F(C_i)\}$, such that the following diagram commutes:

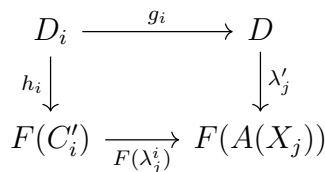


4. For any pair of arrows $f_1, f_2 : C_1 \rightrightarrows C_2$ in \mathcal{C} and any arrow $\lambda' : D \rightarrow F(C_1)$ of \mathcal{D} satisfying $F(f_1) \circ \lambda' = F(f_2) \circ \lambda'$, there exist a K -covering family in \mathcal{D} $\{g_i : D_i \rightarrow D \mid i \in I\}$, and a family of morphisms of \mathcal{C} $\{\lambda^i : C'_i \rightarrow C_1 \mid i \in I\}$, satisfying $f_1 \circ \lambda^i = f_2 \circ \lambda^i$ for all $i \in I$ and of morphisms of \mathcal{D} $\{h_i : D_i \rightarrow F(C'_i) \mid i \in I\}$, making the following squares commutative:



2.7.2. REMARK. It is not difficult to show by induction on finite diagrams that the last three conditions are equivalent to the following more condensed condition:

Given a finite diagram $A : \mathbb{I} \rightarrow \mathcal{C}$ and a cone L' over $F \circ A$ in \mathcal{D} with apex D , there is a K -covering family of morphisms $\{g_i : D_i \rightarrow D \mid i \in I\}$ and cones L_i over A in \mathcal{C} with apex C'_i such that $L \circ g_i$ factors through $F(L_i)$ for each $i \in I$, in the sense that there exist morphisms $h_i : D_i \rightarrow F(C'_i)$ with



for each i , where λ_j^i and λ_j^i are the j th legs of cones L' and L_i respectively.

Moreover, when the domain site *does* have finite limits, these three conditions reduce to the requirement that F preserves finite limits.

A functor is a morphism of sites precisely if $\ell_{\mathcal{D}} \circ F : \mathcal{C} \rightarrow \text{Sh}(\mathcal{D}, K)$ is a J -continuous flat functor, so that this composite extends along $\ell_{\mathcal{C}}$ to provide the inverse image functor of a geometric morphism $\text{Sh}(\mathcal{D}, K) \rightarrow \text{Sh}(\mathcal{C}, J)$.

2.7.3. COROLLARY. Suppose (\mathcal{C}, J) and (\mathcal{D}, K) are principal (resp. quasi-principal, finitely generated) sites. Then any morphism of sites $F : (\mathcal{C}, J) \rightarrow (\mathcal{D}, K)$ induces a relatively pristine (resp. relatively polished, relatively proper) geometric morphism $f : \text{Sh}(\mathcal{D}, K) \rightarrow \text{Sh}(\mathcal{C}, J)$.

PROOF. Since the conditions on the sites ensure that the representables are supercompact (resp. supercompact or initial, compact), and the restriction of the inverse image functor to these is precisely $\ell_{\mathcal{D}} \circ F$, we conclude that f^* preserves these objects, whence f is relatively pristine (resp. relatively polished, relatively proper) by Proposition 1.6.9. ■

2.7.4. REMARK. More generally, suppose (\mathcal{C}, J) is any small-generated site such that $\text{Sh}(\mathcal{C}, J)$ is supercompactly (resp. compactly) generated and (\mathcal{D}, K) is a principal (resp. finitely generated) site. The geometric morphism induced by a morphism of sites $F : (\mathcal{C}, J) \longrightarrow (\mathcal{D}, K)$ has inverse image functor sending any funneling (resp. multifunneling) colimit of representables to a colimit of supercompact (resp. compact) objects of the same shape, whence by Lemma 1.3.9 it must in particular preserve supercompact (resp. compact) objects. As such, we can replace the hypotheses of Corollary 2.7.3 with these weaker conditions if we so choose.

Beyond morphisms, it is natural to make the extra step of forming a 2-category of sites. Indeed, any natural transformation between functors underlying morphisms of sites induces a natural transformation between the inverse images of the corresponding geometric morphisms; for subcanonical sites, this mapping is full and faithful. Thus, for example, any equivalence of sites (an equivalence of categories which respects the Grothendieck topologies) lifts to an equivalence between the corresponding toposes.

Having formed these 2-categories of sites, we may examine 2-functors between them. We say that a principal site $(\mathcal{C}, J_{\mathcal{T}})$ is *epimorphic* (resp. *strictly epimorphic*) if \mathcal{T} is contained in the class of epimorphisms (resp. strict epimorphisms). An (effectual) reductive site is an (effectual) reductive category equipped with its reductive topology. We employ the *ad hoc* notation of **EffRedSite**, **RedSite**, **StrEpPSite**, **EpPSite** and **PSite** for the 2-categories of effectual reductive sites, reductive sites, strictly epimorphic principal sites, epimorphic principal sites and all principal sites respectively, each endowed with morphisms of sites as 1-cells and natural transformations as 2-cells. Clearly, we have forgetful 2-functors:

$$\mathbf{EffRedSite} \longrightarrow \mathbf{RedSite} \longrightarrow \mathbf{StrEpPSite} \longrightarrow \mathbf{EpPSite} \longrightarrow \mathbf{PSite}. \tag{8}$$

We apply analogous terminology and notation for the comparable kinds of finitely generated sites and coalescent sites. For example, we write **EffCoalSite**, **PosCoalSite** and **FGSite** for the 2-categories of effectual coalescent sites, positive coalescent sites and finitely generated sites, respectively. There results an analogous chain of 2-functors:

$$\mathbf{EffCoalSite} \rightarrow \mathbf{PosCoalSite} \rightarrow \mathbf{CoalSite} \rightarrow \mathbf{StrEpFGSite} \rightarrow \mathbf{EpFGSite} \rightarrow \mathbf{FGSite}. \tag{9}$$

Consolidating the results of Section 2.2, we find that several of these forgetful functors have adjoints.

2.7.5. COROLLARY. *Let $(\mathcal{C}, J_{\mathcal{T}})$ be a principal site, \sim the canonical congruence of Proposition 2.3.1, $\ell(\mathcal{C})$ the full subcategory of $\text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ on the representable sheaves and \mathcal{C}_s*

the (essentially small) category of supercompact objects in that topos. Then the canonical functors underly morphisms of sites:

$$(\mathcal{C}_s, J_r) \longleftarrow (\ell(\mathcal{C}), J_{can}|_{\ell(\mathcal{C})}) \longleftarrow (\mathcal{C}/\sim, J_{\mathcal{T}/\sim}) \longleftarrow (\mathcal{C}, J_{\mathcal{T}})$$

which are the units of reflections to the forgetful functors

$$\mathbf{EffRedSite} \longrightarrow \mathbf{StrEpPSite} \longrightarrow \mathbf{EpPSite} \longrightarrow \mathbf{PSite}$$

found in Diagram (8).

Similarly, if $(\mathcal{C}, J_{\mathcal{T}'})$ is a finitely generated site, then with analogous notation, we have morphisms of sites:

$$(\mathcal{C}_c, J_c) \longleftarrow (\ell(\mathcal{C}), J_{can}|_{\ell(\mathcal{C})}) \longleftarrow (\mathcal{C}/\sim, J_{\mathcal{T}'/\sim}) \longleftarrow (\mathcal{C}, J_{\mathcal{T}'})$$

which are units for the reflections of the forgetful functors

$$\mathbf{EffCoalSite} \longrightarrow \mathbf{StrEpFGSite} \longrightarrow \mathbf{EpFGSite} \longrightarrow \mathbf{FGSite}$$

appearing in Diagram (9).

All of these units induce equivalences at the level of the associated toposes.

PROOF. We omit the straightforward checks that these are indeed morphisms of sites. The universality of the middle and right hand units has been discussed in and beneath Proposition 2.3.1 and Corollary 2.3.2; it remains only to show that the final morphism $(\ell(\mathcal{C}), J_{can}|_{\ell(\mathcal{C})}) \longrightarrow (\mathcal{C}_s, J_r)$ is universal.

Let \mathcal{C}' be an effectual, reductive category. A morphism of sites $F : (\mathcal{C}, J_{\mathcal{T}}) \longrightarrow (\mathcal{C}', J_r)$ corresponds to a geometric morphism $\mathrm{Sh}(\mathcal{C}', J_r) \longrightarrow \mathrm{Sh}(\mathcal{C}, J_{\mathcal{T}})$ whose inverse image functor restricts to F on the representable sheaves, and so sends these to supercompact objects in $\mathrm{Sh}(\mathcal{C}', J_r)$. Since a quotient of a supercompact object is supercompact and inverse image functors preserve quotients, F extends uniquely (up to isomorphism) to a morphism of sites $(\mathcal{C}_s, J_r) \longrightarrow (\mathcal{C}', J_r)$ inducing the same geometric morphism, as required.

As usual, the proof for finitely generated sites is analogous. ■

The morphisms appearing in Corollary 2.7.5 allow us to give another characterisation of reductive and coalescent categories.

2.7.6. LEMMA. *Let $(\mathcal{C}, J_{\mathcal{T}})$ be a strictly epimorphic principal site in which \mathcal{T} is the class of all strict epimorphisms of \mathcal{C} . Then, assuming the axiom of choice, \mathcal{C} is reductive if and only if the (underlying functor of the) composed unit morphism $(\mathcal{C}, J_{\mathcal{T}}) \longrightarrow (\mathcal{C}_s, J_r)$ has a left adjoint.*

Similarly, a strictly epimorphic finitely generated site $(\mathcal{C}, J_{\mathcal{T}'})$ where \mathcal{T}' consists of the strict jointly epic families has \mathcal{C} coalescent if and only if the morphism of sites $(\mathcal{C}, J_{\mathcal{T}'}) \longrightarrow (\mathcal{C}_c, J_c)$ has a left adjoint.

PROOF. If \mathcal{C} is reductive and \mathcal{T} is its class of strict epimorphisms, consider a supercompact object C in $\text{Sh}(\mathcal{C}, J_{\mathcal{T}}) = \text{Sh}(\mathcal{C}, J_r)$. C is a quotient of some representable $\ell(C_0)$, and so is the colimit of some funneling diagram in $\ell(\mathcal{C})$ with weakly terminal object $\ell(C_0)$. Lifting this to a funneling diagram in \mathcal{C} , call its colimit $L(C)$. There is a universal morphism $\eta : C \rightarrow \ell(L(C))$, since the image of the strict epimorphism $\ell(C_0 \rightarrow L(C))$ forms a cone under the original funnel in $\text{Sh}(\mathcal{C}, J_{\mathcal{T}})$. This η is the universal morphism from C to a representable object, since given $C \rightarrow \ell(D)$, we have that the composite $\ell(C_0) \rightarrow C \rightarrow \ell(D)$ is a morphism in the image of ℓ (since ℓ is full and faithful on a strict principal site) forming a cone under the same funnel, so there is a factoring morphism $\ell(L(C)) \rightarrow \ell(D)$, as required. This universality means that $\ell(L(C))$ is well-defined up to isomorphism, and we can use choice to select a representative for each C ; the universality then ensures that L is functorial, and is a left adjoint to the inclusion $\mathcal{C} \rightarrow \mathcal{C}_s$, as required.

Conversely, suppose we have a left adjoint functor $L : \mathcal{C}_s \rightarrow \mathcal{C}$. Given a funnel in \mathcal{C} , consider its colimit in \mathcal{C}_s ; this is preserved by L , so the colimit exists in \mathcal{C} , which is enough to make \mathcal{C} a reductive category.

The argument for coalescent categories is analogous, passing via a finite coproduct to define $L(C)$ in the first part. ■

In other words, any reductive category is a coreflective subcategory of an effectual reductive category, and similarly any coalescent category is a coreflective subcategory of an effectual coalescent category.

2.7.7. THEOREM. *Let $\mathbf{SG}\mathcal{I}\mathcal{D}\mathfrak{P}_{\text{relprec}}$ be the 2-category of supercompactly generated Grothendieck toposes, relatively pristine geometric morphisms and all geometric transformations. Then the object mapping:*

$$\begin{aligned} \mathbf{PSite} &\longrightarrow \mathbf{SC}\mathcal{I}\mathcal{D}\mathfrak{P}_{\text{relprec}} \\ (\mathcal{C}, J_{\mathcal{T}}) &\mapsto \text{Sh}(\mathcal{C}, J_{\mathcal{T}}). \end{aligned}$$

extends to a 2-functor between these 2-categories. This functor is faithful on 2-cells if we restrict the domain to $\mathbf{EpPSite}$. It is full and faithful on 2-cells and faithful (up to isomorphism) on 1-cells if we restrict the domain to $\mathbf{StrEpPSite}$. Finally, it is a 2-categorical equivalence if we restrict the domain to $\mathbf{EffRedSite}$.

Analogously, letting $\mathbf{CG}\mathcal{I}\mathcal{D}\mathfrak{P}_{\text{relprop}}$ be the 2-category of compactly generated Grothendieck toposes and relatively proper geometric morphisms, there is a 2-functor whose effect on objects is:

$$\begin{aligned} \mathbf{FGSite} &\longrightarrow \mathbf{CG}\mathcal{I}\mathcal{D}\mathfrak{P}_{\text{relprop}} \\ (\mathcal{C}, J_{\mathcal{T}'}) &\mapsto \text{Sh}(\mathcal{C}, J_{\mathcal{T}'}). \end{aligned}$$

This restricts to an equivalence between the 2-category $\mathbf{EffCoalSite}$ and $\mathbf{CG}\mathcal{I}\mathcal{D}\mathfrak{P}_{\text{relprop}}$.

Finally, there is an equivalence between the 2-category $\mathbf{SG}\mathcal{I}\mathcal{D}\mathfrak{P}_{\text{relpol}}$ of supercompactly generated Grothendieck toposes, relatively polished geometric morphisms and all geometric transformations, and the 2-category $\mathbf{EffRed}^+\mathbf{Site}$ of effective augmented reductive sites of Definition 2.4.13.

PROOF. The claim of 2-functoriality is fulfilled thanks to Corollary 2.7.3.

Faithfulness when we restrict to **EpPSite** is by virtue of the observations after Proposition 2.3.1 that ℓ is faithful (on \mathcal{T} -arches and hence) on morphisms coming from the site.

Fullness on 2-cells and faithfulness on 1-cells when we restrict further to **StrEpPSite** is more directly derived from full faithfulness of ℓ for such subcanonical sites, and the fact that natural transformations between inverse image functors are determined by their components at the representables. Any such natural transformation (including a natural isomorphism) restricts along ℓ to a natural transformation between the underlying morphisms of sites.

The equivalence when we restrict to **EffRedSite** is thanks to Theorem 2.4.12. Indeed, since an effectual reductive category is equivalent to the category of supercompact objects in the corresponding topos, any relatively pristine morphism between such toposes restricts to a morphism of sites between the underlying effective reductive sites.

As ever, the argument for finitely generated sites and quasi-principal sites is analogous. ■

2.8. COMORPHISMS OF SITES AND POINTS. We would be remiss not to also discuss comorphisms of sites.

2.8.1. DEFINITION. *Given sites (\mathcal{C}, J) and (\mathcal{D}, K) , a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a **comorphism of sites** $F : (\mathcal{C}, J) \rightarrow (\mathcal{D}, K)$ if it has the **cover-lifting property**, so that for any object C of \mathcal{C} and K -covering sieve S on $F(C)$, there exists a J -covering sieve R on C with $F(R) \subseteq S$.*

Such a comorphism of sites induces a geometric morphism $f : \mathrm{Sh}(\mathcal{C}, J) \rightarrow \mathrm{Sh}(\mathcal{D}, K)$, whose inverse image functor maps a sheaf X to $\mathbf{a}_J(\mathrm{Hom}_{\mathcal{D}}(F(-), X))$.

While comorphisms of sites could in principle provide a way to extend the correspondence of Theorem 2.4.12 to a covariant duality, we introduce them for another purpose. First, we make the following observation regarding toposes as canonical sites:

2.8.2. LEMMA. *Let $f : \mathcal{F} \rightarrow \mathcal{E}$ be a geometric morphism. The inverse image functor f^* is a comorphism of sites $(\mathcal{E}, J_{\mathrm{can}}) \rightarrow (\mathcal{F}, J_{\mathrm{can}})$ if and only if the direct image f_* has a right adjoint.*

PROOF. Suppose f^* is a comorphism of sites. The induced geometric morphism has inverse image functor defined by $X \mapsto \mathbf{a}_{J_{\mathrm{can}}}(\mathrm{Hom}_{\mathcal{F}}(f^*(-), X))$, and by inspection, this coincides with f_* . As such, f_* is the inverse image functor of a geometric morphism, and so has a right adjoint. Conversely, if f_* has a right adjoint, then it is a morphism of sites since it is left exact and preserves jointly epimorphic families, whence its left adjoint f^* is a comorphism of sites by [Mac Lane and Moerdijk, 1992, Lemma VII.10.3]. ■

Indeed, comparing this definition with Lemma 1.6.8, we obtain the following:

2.8.3. PROPOSITION. *Suppose \mathcal{F} is a Grothendieck topos in which every small jointly epimorphic family $\{x_i : X_i \rightarrow X\}$ can be refined to a jointly epimorphic family of monomorphisms $\{x'_j : X'_j \hookrightarrow X\}$, in the sense that each x'_j factors through some x_i . Then $f : \mathcal{F} \rightarrow \mathcal{E}$ is relatively pristine if and only if its inverse image functor is a comorphism of sites $(\mathcal{E}, J_{can}) \rightarrow (\mathcal{F}, J_{can})$, if and only if f_* has a right adjoint.*

PROOF. Recall that the covering sieves for the canonical topology on a topos are those containing small jointly epimorphic families. Given a jointly epimorphic family on an object of the form $f^*(Y)$, we may by assumption refine it to a jointly epimorphic family of monomorphisms, whence from the characterization of relative pristine morphisms in Lemma 1.6.8, f is relatively pristine if and only if f^* is a comorphism of sites. The final line follows from Lemma 2.8.2. ■

Any localic topos satisfies the conditions of Proposition 2.8.3, but so does the topos of actions of the two-element monoid, for example. We apply it in the case that \mathcal{F} is **Set**.

Recall that a geometric morphism is **totally connected** if it is locally connected and the extra left adjoint preserves finite limits; see [Johnstone, 2002, §C3.6] for basic results about these morphisms. A topos is said to be totally connected if its global sections geometric morphism is.

2.8.4. COROLLARY. *A Grothendieck topos has a relatively pristine point if and only if it is totally connected, and such a point is unique up to isomorphism.*

PROOF. By Proposition 2.8.3, a geometric morphism from **Set** is relatively pristine if and only if its direct image functor has a right adjoint, which must then coincide with the direct image of the unique global sections morphism. As such, this happens if and only if the inverse image functor of the global sections morphism has a left adjoint preserving finite limits, making the topos totally connected. ■

In light of Corollary 2.8.4, considering the points of a localic topos demonstrates how large the (ostensibly subtle) difference between relatively polished and relatively pristine morphisms can be.

2.8.5. LEMMA. *Every point of a localic topos is relatively polished.*

PROOF. We use Lemma 1.6.8. Since **Set** is atomic, given a point $p : \mathbf{Set} \rightarrow \mathcal{E}$ and an object X of \mathcal{E} , it suffices to consider the minimal inhabited covering of $p^*(X)$ by singletons $\{x_i : 1 \hookrightarrow p^*(X)\}$ or, if $p^*(X)$ is empty, by the identity $\{0 \hookrightarrow p^*(X)\}$. The characterisation of localic toposes as being generated by subterminal objects means we have a covering of X by subterminals $\{y_j : X_j \rightarrow X\}$, and since $f^*(X_j)$ is necessarily subterminal for every j , these factorize through this minimal covering, as required. ■

Returning to relatively pristine points in the case of supercompactly generated toposes, we find a convenient characterization of when such a point exists.

2.8.6. PROPOSITION. *Given a principal site $(\mathcal{C}, J_{\mathcal{T}})$, the topos $\text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ has a relatively pristine point (and so is totally connected) if and only if \mathcal{C}^{op} is filtered.*

PROOF. By Theorem 2.7.7, any relatively pristine point of $\text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ comes from a morphism of sites $(\mathcal{C}_s, J_s) \rightarrow (1, J_{\text{can}})$, where \mathcal{C}_s is the usual subcategory in $\text{Sh}(\mathcal{C}, J_{\mathcal{T}})$. Composing with the canonical morphism of sites $(\mathcal{C}, J_{\mathcal{T}}) \rightarrow (\mathcal{C}_s, J_s)$ from Corollary 2.7.5, we conclude that such a point exists if and only if there exists a morphism of sites $(\mathcal{C}, J_{\mathcal{T}}) \rightarrow (1, J_{\text{can}})$. Since there is exactly one functor $\mathcal{C} \rightarrow 1$, which automatically preserves covers (since all $J_{\mathcal{T}}$ are inhabited), the result follows after observing that the remaining conditions of Definition 2.7.1 correspond to filteredness of \mathcal{C}^{op} . ■

Note that since the condition on \mathcal{C} is independent of the choice of principal topology it is equipped with, we may deduce that any relatively pristine subtopos of $\text{Sh}(\mathcal{C}, J_{\mathcal{T}})$ is totally connected when the hypotheses hold.

We immediately recover the following result, which although deducible from [Johnstone, 2002, Example C3.6.17(c)], does not seem to have been recorded explicitly anywhere that we know of.

2.8.7. COROLLARY. *Any regular topos is totally connected.*

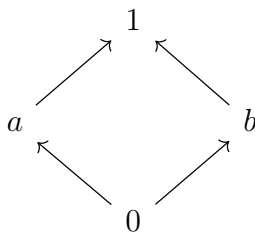
PROOF. It suffices to observe that if \mathcal{C} has finite limits, \mathcal{C}^{op} is filtered, so Proposition 2.8.6 applies. ■

In particular, by [Johnstone, 2002, Theorem C3.6.16(iv)], the category of models of a regular theory in any Grothendieck topos has a terminal object (easily described as the model in which every sort is interpreted as the terminal object). The logic of supercompactly generated toposes more generally is a subject for a future paper.

2.9. EXAMPLES OF REDUCTIVE CATEGORIES. In this final subsection, we present examples of reductive and coalescent categories, as well as principal and finitely generated sites and their toposes of sheaves, in order to address some hypotheses about relationships between the concepts presented in this paper. We begin with localic examples, then examples relating to categories of finite sets, then abelian categories.

2.9.1. EXAMPLE. In the proof of Corollary 1.8.4, we observe that hyperconnected morphisms are pristine, so that any supercompactly generated two-valued topos is supercompact. Taking \mathcal{C} to be any non-trivial poset with a maximal element and taking \mathcal{E} to be the category of presheaves on this poset provides an example of a supercompactly generated, supercompact topos which is not two-valued.

2.9.2. EXAMPLE. The objects of a reductive category need not be supercompact within this category, in spite of Corollary 1.3.10. For example, consider the four-element lattice:



By Proposition 2.5.1, it forms a reductive category. However, the colimit of the span $a \leftarrow 0 \rightarrow b$ is 1, which is to say that the arrows from a and b to 1 form a strictly epic family containing no strict epimorphism. Even if we relax to mere epimorphisms here, the empty family is strictly epic over 0 yet has no inhabited subfamilies.

By considering the lattice of subsets of \mathbb{N} as a distributive join semilattice, we similarly find that objects of coalescent categories need not be compact in those categories.

2.9.3. EXAMPLE. A familiar example of a localic, locally connected topos which is not supercompactly (or even compactly) generated is the topos of sheaves on the real numbers: no non-trivial open sets in the reals are compact.

2.9.4. EXAMPLE. As a more original non-example, here is a localic, totally connected topos which is not compactly generated. Consider the poset P whose objects are the natural numbers (excluding 0), and with order given by $n \leq m$ iff n is divisible by m , so that 1 is terminal. Endow this poset with the Grothendieck topology J whose covering sieves on a natural number n are those containing cofinitely many prime multiples of n .

All of these sieves are connected and effective epimorphic; that is, (P, J) is a localic, locally connected, subcanonical site. For every n , $\ell(n)$ is therefore an indecomposable subterminal object of $\text{Sh}(P, J)$. Since P has finite limits, the topos $\text{Sh}(P, J)$ is moreover totally connected. To show that $\text{Sh}(P, J)$ fails to be compactly generated it suffices to show that none of the $\ell(n)$ are. But by construction each $\ell(n)$ has a nontrivial infinite covering family by other representables which contains no finite subcovers. Thus this topos has no supercompact objects, and the only compact object is the initial object.

2.9.5. EXAMPLE. As promised earlier, we demonstrate that it is not possible to extend Theorem 1.8.5(i) or (ii) to relatively pristine or relatively proper surjections.

Consider the poset P constructed as a fractal tree with countably many roots and branches. Explicitly, it has elements *non-empty* finite sequences of natural numbers, $\coprod_{n=1}^{\infty} \mathbb{N}^n$, with $\vec{x} \leq \vec{y}$ if \vec{y} is an initial segment of \vec{x} . The Alexandroff locale L corresponding to P has opens which are downward-closed subsets in this ordering, so that for any sequence \vec{x} in an open set, all extensions of \vec{x} also lie in that open.

Consider the collection of opens U such that whenever $(x_1, \dots, x_{k-1}, x_k) \in U$, we have $(x_1, \dots, x_{k-1}, y) \in U$ for cofinitely many values of y . This collection is clearly closed under finite intersections and arbitrary unions (we needed the sequences in P to be non-empty to ensure that the empty intersection of opens was included here), which makes it a subframe of $\mathcal{O}(L)$. This corresponds to some locale L' such that there is a surjective locale map $L \rightarrow L'$, and hence a geometric surjection $s : \text{Sh}(L) \rightarrow \text{Sh}(L')$. Moreover, this surjection is relatively pristine. Indeed, if X is a sheaf on L' and we are given a covering of $s^*(X)$ in $\text{Sh}(L)$, we may without loss of generality assume that $s^*(X)$ is covered by supercompact opens of L , and each supercompact open contains the open of L' consisting of the strict extensions of sequences it contains; X is necessarily the union of these in $\text{Sh}(L')$.

However, $\text{Sh}(L')$ is not supercompactly or even compactly generated, since the opens of the form $s^*(U)$ are not compact, with the exception of the initial open, despite s^* preserving supercompact objects.

2.9.6. **EXAMPLE.** There exist reductive categories without equalizers, products or pullbacks (or even pullbacks of monomorphisms), so which in particular are not locally regular. Indeed, simplifying Example 2.4.5, consider the category of presheaves on $f, g : A \rightrightarrows B$. The subcategory of supercompact objects in this topos is simply the coequalizer diagram $A \rightrightarrows B \rightarrow C$, so that in particular the pair of monomorphisms f, g has neither an equalizer nor a pullback, and the product $B \times B$ does not exist. We obtain a similar example from any finite category containing a parallel pair of morphisms lacking an equalizer.

2.9.7. **EXAMPLE.** Any discrete category (a category with no non-identity morphisms) with more than one object is a reductive and locally regular category which is not regular.

The free finite cocompletion of a discrete category \mathcal{C} is a coalescent category, since it is equivalent to $[\mathcal{C}^{\text{op}}, \mathbf{FinSet}]$, where \mathbf{FinSet} is the category of finite sets¹¹, and this is a coalescent category, whence $[\mathcal{C}^{\text{op}}, \mathbf{FinSet}]$ has funneling colimits computed pointwise. \mathbf{FinSet} is in some sense the archetypal coalescent category, since by inspection, $\text{Sh}(\mathbf{FinSet}, J_c) \simeq \mathbf{Set}$. Since \mathbf{FinSet} has pullbacks, these free finite cocompletions are coherent categories.

2.9.8. **EXAMPLE.** We may also consider the category \mathbf{FinSet}_+ of *inhabited* finite sets; since a funneling colimit of inhabited finite sets is inhabited and finite, and taking the pullback in \mathbf{FinSet} of an epimorphism in \mathbf{FinSet}_+ along a morphism with inhabited domain gives another epimorphism with inhabited domain, we have that \mathbf{FinSet}_+ is another example of a reductive category without pullbacks (since the two inclusions $1 \rightrightarrows 2$ ‘should’ have empty intersection).

By reintroducing the empty set and declaring that the empty sieve is covering over it, we obtain an augmented reductive site with underlying category \mathbf{FinSet} having an equivalent topos of sheaves; since the coalescent topology on \mathbf{FinSet} is a refinement of the augmented reductive one, we see that we have a (relatively proper) inclusion of toposes:

$$\mathbf{Set} \simeq \text{Sh}(\mathbf{FinSet}, J_c) \hookrightarrow \text{Sh}(\mathbf{FinSet}_+, J_r),$$

which is not an equivalence since the sheaves represented by the finite sets of cardinality at least 2 are supercompact in the latter topos but merely compact in the former.

2.9.9. **EXAMPLE.** As a dual construction, we have that $\text{Sh}(\mathbf{FinSet}^{\text{op}}, J_c)$ embeds into the *classifying topos for the theory of objects*, $[\mathbf{FinSet}, \mathbf{Set}]$. We mention this as a counterexample to an extension of Lemma 2.8.5 to the idea that a point of any Grothendieck topos is relatively polished: the points of $[\mathbf{FinSet}, \mathbf{Set}]$ correspond to sets objects of \mathbf{Set} , and the correspondence sends a geometric morphism to the set which is the image of the representable functor $\mathbf{y}(1)$. But $\mathbf{y}(1)$ is supercompact, so a point corresponding to any set with more than one element fails to be polished.

¹¹This expression for the free finite cocompletion applies if and only if \mathcal{C} has finite hom-sets, since this is necessary and sufficient for the representable presheaves to lie in $[\mathcal{C}^{\text{op}}, \mathbf{FinSet}]$. This is trivially the case when \mathcal{C} is discrete.

2.9.10. EXAMPLE. For yet another related example, consider the simplex category Δ , whose objects are *inhabited* finite ordinals,

$$[n] = \{0, \dots, n - 1\}, \quad n \geq 1$$

and whose morphisms are the order-preserving maps between these. Clearly this does not have pullbacks (since the intersection of the two inclusions $[1] \rightrightarrows [2]$ would be the empty ordinal which is not an object of Δ).

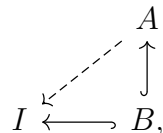
Δ has funneling colimits: given any collection of morphisms into the object $[n]$ of Δ , their colimit is the quotient of $[n]$ identifying $f(x), f(x) + 1, \dots, g(x)$ (or $g(x), g(x) + 1, \dots, f(x)$) for each parallel pair $f, g : [n'] \rightrightarrows [n]$ in the diagram and each $x \in [n']$. Moreover, each epimorphism $g : [n] \twoheadrightarrow [m]$ is split (so in particular is strict) by the monomorphism $\min(g^{-1}) : [m] \rightrightarrows [n]$, say.

In particular, by Remark 2.1.4 the collection of strict epimorphisms is stable, which makes Δ a reductive category with $\text{Sh}(\Delta, J_r) = [\Delta^{\text{op}}, \mathbf{Set}]$, the topos of *simplicial sets*. While this tells us little about the theory of strict linear intervals classified by $[\Delta^{\text{op}}, \mathbf{Set}]$ (referred to as ‘orders’ in [Mac Lane and Moerdijk, 1992][§8.VIII]), it does recover a non-trivial fact about the topos of simplicial sets: every quotient of a representable simplicial set is also representable, so that every simplicial set is a union of its representable subsets.

2.9.11. EXAMPLE. To contrast Examples 2.9.7 and 2.9.10, we recall an example of a supercompactly generated topos which is not equivalent to a presheaf topos; this should be contrasted with Proposition 2.5.1, below.

Consider the Schanuel topos, $\text{Sh}(\mathbf{FinSet}_{\text{mono}}^{\text{op}}, J_{at})$. We see that $(\mathbf{FinSet}_{\text{mono}}^{\text{op}}, J_{at})$ is an atomic site with pullbacks, but moreover it is a reductive and regular site, since all of the morphisms in the category are regular epimorphisms which are stable under pullback.

We know of two ways to show that this topos is not a presheaf topos. The first is to show that the site is effectual, which explicitly involves identifying an algorithm which, given a cofunnel F in $\mathbf{FinSet}_{\text{mono}}$ with weakly initial object $F(D_0)$ and $x, y : F(D_0) \rightrightarrows C$ equalized by its limit, constructs an inclusion $i : C \hookrightarrow C'$ and a connecting zigzag between $i \circ x$ and $i \circ y$ in $(F \downarrow C')$. It then follows that the supercompact objects (equivalently, atoms) in the Schanuel topos are precisely the representable sheaves coming from $\mathbf{FinSet}_{\text{mono}}^{\text{op}}$. However, if I is a finite set, A is any set of cardinality larger than that of I , and we have inclusions from a further finite set B into both A and I , there can be no monomorphism completing the triangle,



whence I is not injective in $\mathbf{FinSet}_{\text{mono}}$ and hence is not projective in $\mathbf{FinSet}_{\text{mono}}^{\text{op}}$. It follows that no object of $\mathbf{FinSet}_{\text{mono}}^{\text{op}}$ is projective, whence the Schanuel topos has no indecomposable projective objects, but is non-degenerate, and so is not a presheaf topos

(the category of representable presheaves in a presheaf topos being recoverable up to idempotent completion as the projective indecomposable objects, which in particular must be supercompact).

An alternative proof, which we thank Olivia Caramello for describing to us, is to observe that the category of representables in a presheaf topos can be identified, up to idempotent-completion, with the full subcategory of finitely presentable objects in its category of points. We also have that a presheaf topos is atomic if and only if the representing category is a groupoid. The Schanuel topos classifies infinite decidable objects, so its category of points corresponds to the category of infinite sets (and monomorphisms); since any inhabited full subcategory of this category is not a groupoid (every infinite set has an injective endomorphism which is not invertible) and the topos is non-degenerate, it again follows that this cannot be a presheaf topos.

2.9.12. **EXAMPLE.** Since any abelian category is effective regular, any small abelian category with funneling colimits is reductive. This is the case for the finitely presented (right) modules of a (right) Noetherian ring, say, since these coincide with finitely generated modules and so is closed under quotients in the large category of modules. For example, the category of finitely generated abelian groups is a reductive category with finite limits and colimits.

In order to construct a small abelian category which does *not* have funneling colimits, we look for a coherent ring R whose collection of finitely generated ideals is not closed under infinite intersections (note that if the ring is an integral domain, coherence ensures that it will be closed under finite intersections). Let I be an infinitely generated ideal which is obtained as such an intersection. In the large category of modules, we may identify R/I as the colimit of the funneling diagram consisting of the inclusions of finitely generated sub-ideals of I , along with the parallel zero maps, into the ring, viewed as a (right) module over itself. If the colimit of this diagram existed in the category of finitely presented R -modules, it would have to be a quotient of R by some finitely generated ideal, but by construction there is no initial finitely presented quotient under this diagram, whence the colimit does not exist.

Consider the ring R of eventually constant sequences valued in the field on two elements (with point-wise operations). Observe that all finitely generated ideals in this ring are principal. An ideal generated by a sequence g is the cokernel of the module homomorphism $R \rightarrow R$ sending x to $x \cdot (1 - g)$, so this is indeed a coherent ring. For each index i we have a ‘basis element’ e_i which is 1 at i and 0 elsewhere. The ideal I_i generated by $(1 - e_i)$ consists of those sequences which are 0 at i . Consider $\bigcap_{j=1}^{\infty} I_{2j}$: this consists of sequences which are non-zero only at odd indices, but by the eventually constant criterion, no single sequence can generate this ideal, whence the ideal fails to be finitely generated.

We would like to thank Jens Hemelaer for helping us to identify the sufficient structure needed to find this counterexample and Ryan C. Schwiebert for identifying a ring realising that structure (via math.stackexchange.com).

Note that a non-trivial abelian category cannot be coherent or coalescent since the initial object is not strict in such a category (cf Lemma 2.4.4).

2.9.13. **EXAMPLE.** As an example of a regular and reductive category that fails to be effective, let alone effectual, we adapt [Johnstone, 2002, before Example A1.3.7].

Consider the category \mathbf{TF}_{fg} of finitely-generated torsion-free abelian groups. By considering it as a reflective subcategory of the category of finitely generated abelian groups, we find that it is regular and has all coequalizers. We can moreover check that it has funneling colimits, since the full category of abelian groups is cocomplete and any quotient of a finitely generated group is finitely generated (to obtain the quotient in \mathbf{TF}_{fg} , the torsion parts of such a quotient are annihilated). Thus it is a reductive category with finite limits. However, as Johnstone points out, the equivalence relation

$$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \equiv b \pmod{2}\} \cong \mathbb{Z} \times \mathbb{Z}$$

is not a kernel pair of any morphism in \mathbf{TF}_{fg} , so this category is not effective regular.

2.9.14. **EXAMPLE.** For an example of a regular and reductive category that is effective but not effectual, we modify [Johnstone, 2002, Example D3.3.9]. Let \mathbf{Set}_ω be the full subcategory of \mathbf{Set} on the finite and countable sets. This has pullbacks and funneling colimits which are stable under pullback, inherited from \mathbf{Set} . Moreover, all epimorphisms are regular, so this is a regular and reductive category (indeed, it is a coherent and coalescent category too!). It also inherits the property of being effective from \mathbf{Set} .

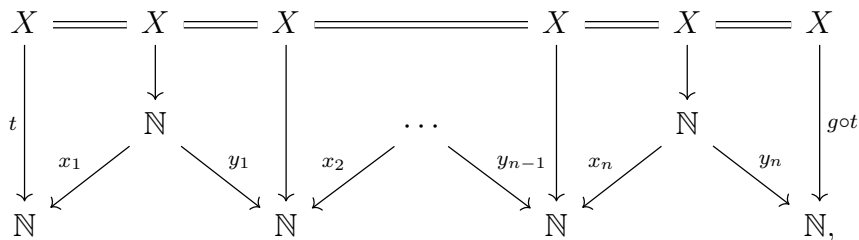
However, it is not effectual. Johnstone exhibits the following coequalizer diagram:

$$\mathbb{N} \begin{array}{c} \xrightarrow{\text{id}} \\ \xrightarrow{s} \end{array} \mathbb{N} \twoheadrightarrow 1, \tag{10}$$

where s is the successor function. He concludes by considering the natural number object in $\text{Sh}(\mathbf{Set}_\omega, J_c)$ that this coequalizer is not preserved by the canonical functor $\ell : \mathbf{Set}_\omega \rightarrow \text{Sh}(\mathbf{Set}_\omega, J_c)$; we could deduce the failure of effectuality from that. Instead, we prove it directly by considering the morphisms

$$\mathbb{N} \begin{array}{c} \xrightarrow{\text{id}} \\ \xrightarrow{g:=2 \times -} \end{array} \mathbb{N},$$

which are clearly coequalized by the epimorphism in (10). If \mathbf{Set}_ω were effectual, there would exist some epimorphism $t : X \twoheadrightarrow \mathbb{N}$ such that t and $g \circ t$ lie in the same connected component of $(X \downarrow F)$, where F is the parallel arrow diagram (id, s) whose coequalizer is shown in (10). However, given any finite zigzag,



where x_i and y_i are id or s , composing with any morphism $m : 1 \rightarrow X$ such that $t \circ m > n$ as elements of \mathbb{N} , we conclude that since the difference between the image of m in the first copy of \mathbb{N} and the last copy of \mathbb{N} is at most n , we have a contradiction. That is, no zigzag of finite length is sufficient to connect t and $g \circ t$ in $(X \downarrow F)$.

2.9.15. **EXAMPLE.** Finally, we provide an example of an effectual coalescent category which is not effectual as a reductive category. Consider the following category, \mathcal{C} :

$$\begin{array}{ccc}
 R_1 & \xrightarrow{c_1} & Z & \xleftarrow{c_2} & R_2 \\
 & \searrow & \downarrow & \downarrow & \swarrow \\
 & & A & &
 \end{array}$$

where the composition is such as to make Z the coproduct of R_1 and R_2 . We make this into a strictly epimorphic finitely generated site with the stable class \mathcal{T}' consisting of the singleton families on the identity morphisms and the pair $\{c_1, c_2\}$. This embeds into the effectual coalescent category \mathcal{C}_c of compact objects in $\text{Sh}(\mathcal{C}, J_{\mathcal{T}'})$. Consider the funneling diagram consisting of the pairs $R_1 \rightrightarrows A$ and $R_2 \rightrightarrows A$. By construction, the colimit of this diagram will also coequalize all of the morphisms $Z \rightarrow A$, since c_1 and c_2 are jointly epimorphic in \mathcal{C}_c . However, there is no strict epimorphism $C' \rightarrow Z$ verifying the definition of effectuality for a reductive category. Indeed, in the completion of \mathcal{C}_c to an effectual reductive category, c_1 and c_2 are no longer jointly epimorphic, so that the funneling colimit described fails to coequalize the morphisms $Z \rightarrow A$.

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