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COMPACT CLOSED CATEGORIES AND Γ-CATEGORIES

(WITH AN APPENDIX BY ANDRÉ JOYAL)

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Abstract. In this paper we develop a 2-categorical approach to coherence in compact closed categories. Our approach allows us to place compact closed categories within the context of homotopical algebra. More precisely, we construct two new model categories whose fibrant objects are (two different models of) compact closed categories. We prove a strictification theorem by showing a Quillen equivalence between the two.

Contents

1. Introduction

A *compact closed* category is a symmetric monoidal category having the special property that each object has a left (and therefore a right) dual. The archetype example of compact closed categories is the category of finite dimensional vector spaces. Some other prominent examples of compact closed categories include the category of finitely generated projective modules over a commutative ring and the category of finite dimensional representations of a compact group. The category of abelian groups can be characterized as a *reflective* localization of the category of commutative monoids, namely the localization functor has a right adjoint which is the fully faithful inclusion of the full subcategory of abelian groups (local-objects). It happens that this localization is generated by a single map which is the inclusion $i_+ : \mathbb{N} \longrightarrow \mathbb{Z}$. The model category of (permutative) Picard groupoids $(\mathbf{Perm}, \mathcal{P}ic)$

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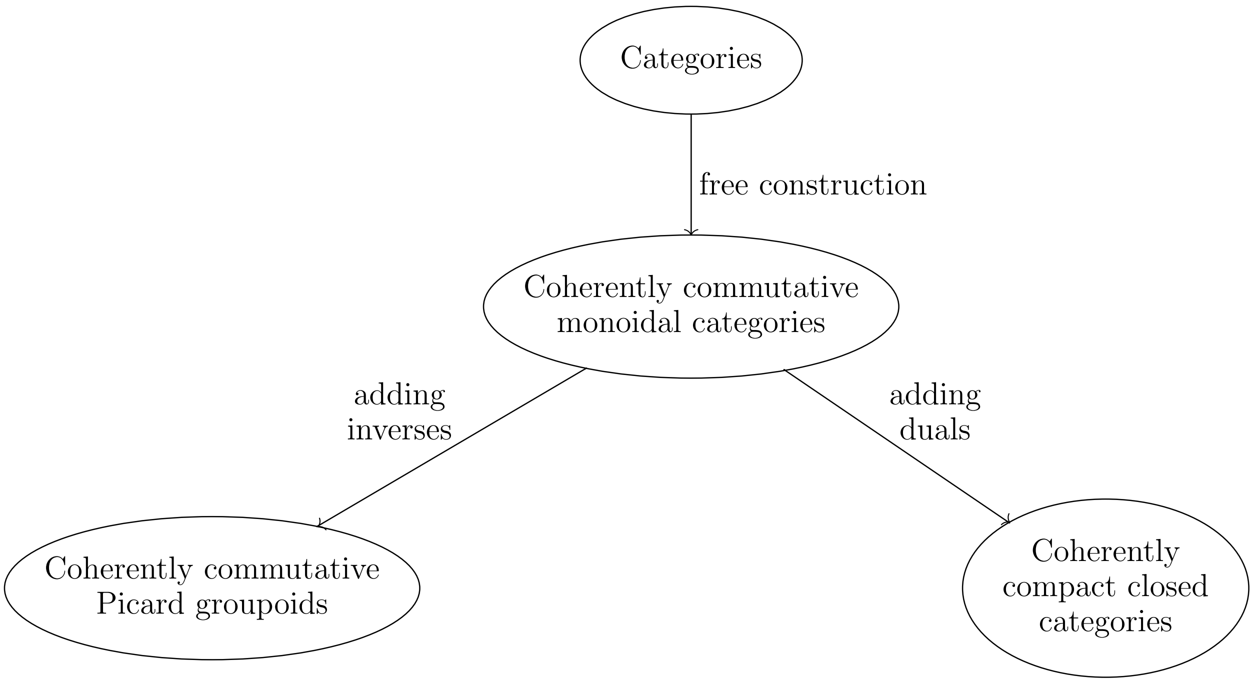
[Sha] is a left Bousfield localization of the natural model category of permutative (or strict symmetric monoidal) categories **Perm**. This localization is also generated by a single map which is the inclusion $i^{\mathcal{P}ic} : \mathcal{F}^{\otimes}(\underline{1}) \longrightarrow \mathcal{F}^{\mathcal{P}ic}(\underline{1})$ of the free permutative category on one generator into the free Picard groupoid on one generator. In section 3 of this paper we obtain a model category of (permutative) compact closed categories as a left Bousfield localization of the category of permutative categories **Perm**. The localization is again generated by a single map which is the inclusion $i : \mathcal{F}^{\otimes}(\underline{1}) \longrightarrow \mathfrak{Fr}^{cc}(\underline{1})$, where the codomain is the free compact closed category on one generator. The main objective of this paper is to compare the aforementioned model category of (permutative) compact closed categories with a model category of *coherently* compact closed categories which is a left Bousfield localization of a (model) category of coherently commutative monoidal categories [Sha20] generated by a single map which is (up to equivalence) an adjunct of the generator i. The main result of this paper is that the two aforementioned model categories are Quillen equivalent. This result maybe regarded as a *strictification theorem* for (coherently) compact closed categories.

The classical (1-dimensional) cobordism hypothesis [BD95],[Lur09] informally states that the (framed) 1-Bordism category, namely the category whose objects are framed 0-dimensional manifolds and morphisms are (diffeomorphism classes of) framed 1-dimensional manifolds with boundary, is the free compact closed category on one generator. To a purely algebraic problem, the cobordism hypothesis provides an answer which is rooted in differential topology. In this paper we seek an answer to the same underlying algebraic problem within homotopical algebra. This paper is a first in a series of papers aimed at developing a theory for compact closed (∞, n)-categories and also providing a purely algebraic description of a free compact closed (∞, n)-category on one generator. In this paper we construct a model category whose fibrant objects can be described as categories equipped with a coherently commutative multiplication structure wherein each object has a dual. This model category is intended to be a prototype for subsequently constructing model categories whose fibrant objects are models for $(n+k, n)$-categories equipped with a coherently commutative monoidal structure and which are *fully dualizable*.

Normalized coherently commutative monoidal categories were introduced in the paper [Seg74] where they were called Γ-categories. These (normalized) objects have also been referred to in the literature as *special* Γ-categories. A model category whose fibrant objects are (unnormalized) coherently commutative monoidal categories was constructed in [Sha20]. In this paper we will denote this model category by $\Gamma\mathbf{Cat}^{\otimes}$. Unlike a symmetric monoidal category, higher coherence data is specified as a part of the definition of a coherently commutative monoidal category. Moreover, in the latter, a *tensor product* of two objects is unique only up to a contractible space of choices. In section 4 of this paper we extend the notion of compact closed categories to the more generalized setting of coherently commutative monoidal categories. We name these objects *coherently compact closed categories*. We construct another model category structure on the (functor) category $\Gamma\mathbf{Cat}$ whose fibrant objects are coherently compact closed categories. This model category, denoted $\Gamma\mathbf{Cat}^{cc}$, is a (left) Bousfield localization of the model category of coherently commutative monoidal categories $\Gamma\mathbf{Cat}^{\otimes}$. We go on to show that the thickened Segal's Nerve functor [Sha20, Sec. 6.] is a right Quillen functor of a Quillen equivalence between the aforementioned model category structure of (permutative) compact closed categories on **Perm** and $\Gamma\mathbf{Cat}^{cc}$. The following picture depicts the idea of coherently compact closed categories and also depicts how various coherently commutative objects in **Cat** are related:



The addition processes depicted by the two diagonal arrows in the above picture are manifested by localizations.

The Barrat-Priddy-Quillen theorem was reformulated in the language of Γ-spaces in [Seg74]. In the same paper, Segal constructed a functor from the category of (normalized) Γ-spaces $\Gamma\mathcal{S}_\bullet$ to the category of (connective) spectra. This functor maps the unit of the symmetric monoidal structure on $\Gamma\mathcal{S}_\bullet$, namely the free Γ-space on one generator Γ^1, to the *sphere spectrum*. In section 2 of the same paper, Segal constructed a (normalized) Γ-space, which he denoted by $B\Sigma$, which can also be described as (simplicial nerve of) the (categorical) Segal's nerve [Sha20] of the (skeletal) permutative category of finite sets and bijections, denoted $\mathcal{K}(\underline{\mathcal{N}})$. The reformulated theorem states that the spectrum associated to the Γ-space $B\Sigma$ is *stably* equivalent to the sphere spectrum. In other words, the reformulation states that the Γ-space Γ^1 is equivalent to $B\Sigma$ in the *stable* model category of Γ-spaces constructed in [Sch99]. A stronger version of this theorem called the (special) Barrat-Priddy-Quillen theorem appeared in the paper [dBM17]. This theorem states that the two Γ-spaces in context are also *unstable* equivalent *i.e.* they are equivalent in a model category of special Γ-spaces. Along the same lines, our construction of the model category $\Gamma\mathbf{Cat}^{cc}$ implies that the Segal's nerve of the free compact closed category on 1-generator $\mathcal{K}(\mathfrak{Fr}^{cc}(\underline{1}))$ is a fibrant replacement of the (representable) Γ-category Γ^1 in the model category $\Gamma\mathbf{Cat}^{cc}$.

We say that a compact closed category C is freely generated by a category A if C is equipped with a functor $i_A : A \longrightarrow C$ which induces the following equivalence of groupoids, for each compact closed category D:

$$(i_A)^* : [C, D]_{\otimes} \longrightarrow J([A, D]),$$

where $[C, D]_{\otimes}$ denotes the category of symmetric monoidal functors from C to D and monoidal natural transformations between them. The main objective of section 2 is to show that the compact closed category freely generated by A in the sense of [KL80], denoted $\mathcal{F}(A)$, is also freely generated by A in the aforementioned sense.

In Appendix A we collect some folklore results regarding the notion of duality. In Appendix B we recall the basic notion and existence result of left Bousfield localization of model categories.

1.1. Acknowledgments. *The author would like to thank André Joyal for having numerous discussions with the author regarding this paper and also for writing the appendix A: Aspects of Duality which has added a lot of clarity to the paper.*

2. A 2-categorical approach to coherence in compact closed categories

A compact closed category is a symmetric monoidal category wherein each object has the special property of having a left (and hence a right) *dual*. We begin by recalling the definition of a compact closed category:

2.1. Definition. *A* compact closed *category is a symmetric monoidal category C in which each object $c \in Ob(C)$ can be assigned a triple $(c^\bullet, \eta_c, \epsilon_c)$ where $c^\bullet$ is an object of C (called* right dual *of c) and $\eta_c : 1_C \longrightarrow c \otimes c^\bullet$ and $\epsilon_c : c^\bullet \otimes c \longrightarrow 1_C$ are two maps in C such that the following two maps are identities:*

$$c^\bullet \cong c^\bullet \otimes 1_C \xrightarrow{id \otimes \eta_c} c^\bullet \otimes c \otimes c^\bullet \xrightarrow{\epsilon_c \otimes id} 1_C \otimes c^\bullet \cong c^\bullet \tag{1}$$

and

$$c \cong 1_C \otimes c \xrightarrow{\eta_c \otimes id} c \otimes c^\bullet \otimes c \xrightarrow{id \otimes \epsilon_c} c \otimes 1_C \cong c \tag{2}$$

2.2. Remark. *The symmetric monoidal structure ensures that the right dual is also a left dual and therefore we will just call $c^\bullet$ as the dual of c.*

In the paper [KL80] the authors solve the coherence problem of compact closed categories by constructing an (honest) free compact closed category generated by an ordinary category A. Their construction defines a left adjoint to the forgetful inclusion of compact closed categories into $\mathbf{Cat}$. This was possible by chosing a *duality data* for each compact closed category and defing a category of compact closed categories whose morphisms are strict symmetric monoidal functors which preserve the chosen duality data. More elaborately, the describe three categories equipped with the following inclusions:

$$\mathbf{CompPerm}^{str} \subseteq \mathbf{CompPerm} \subseteq \mathbf{Comp}$$

The objects of **Comp** are pairs $(\mathcal{D}, \mathcal{D}_D)$, where $\mathcal{D}_D$ is a choice of duality data, see definition 2.3, on D and maps are those strict symmetric monoidal functors which strictly preserve duality data. **CompPerm** is the full subcategory whose objets are pairs $(\mathcal{D}, \mathcal{D}_D)$ such that D is a permutative category. $\mathbf{CompPerm}^{str}$ is a full subcategory of **CompPerm** whose objcts are pairs $(D, \mathcal{D}_D)$ where D is a permutative category which satisfy the following three conditions:

1. For each pair of objects $d_1, d_2 \in D$, the chosen dual of $d_1 \otimes d_2$ in $\mathcal{D}_D$, *i.e.* $(d_1 \otimes d_2)^\bullet$, is $d_2^\bullet \otimes d_1^\bullet$.
2. The chosen dual of the unit object 1_D is the unit object *i.e.*
$$1_D^\bullet = 1_C.$$
3. For each object $d \in D$, the dual of $d^\bullet$ is d *i.e.*
$$d^{\bullet\bullet} = d.$$

The following three adjunctions are described in [KL80]:

$$U : \mathbf{Comp} \rightleftarrows \mathbf{Cat} : \mathcal{F} \tag{3}$$

$$U : \mathbf{CompPerm} \rightleftarrows \mathbf{Cat} : \mathcal{F}' \tag{4}$$

and

$$U : \mathbf{CompPerm}^{str} \rightleftarrows \mathbf{Cat} : \mathcal{F}'' \tag{5}$$

In this section we present a 2-categorical approach to the coherence problem of compact closed categories. More precisely, we show that for each category A, the incusion (unit) map $\iota_A : A \longrightarrow \mathcal{F}(A)$ has the following universal property: For each functor $F : A \longrightarrow D$, where D is a compact closed category, there exists a (strict) symmetric monoidal functor $F_\otimes : \mathcal{F}(A) \longrightarrow D$ such that $F = F_\otimes \circ \iota_A$ and which is unique upto a unique monoidal natural isomorphism. We establish analogous universal properties for $\mathcal{F}'(A)$ and $\mathcal{F}''(A)$. Throughout this section C will represent a compact closed (not necessarily permutative) category.

2.3. Definition. *A duality data associated with a compact closed C is the following set:*

$$\mathcal{D}_C = \bigsqcup_{c \in Ob(C)} (c^\bullet, \epsilon_c, \eta_c),$$

where $c^\bullet$ is an object in C and $\epsilon : c^\bullet \otimes c \longrightarrow 1_C$ and $\eta : 1_C \longrightarrow c \otimes c^\bullet$ are the counit *and* unit *maps such that equations (1) and (2) hold. In particular $c^\bullet$ is a dual of c in C. We will refer to $c^\bullet$ as the chosen dual of c in $\mathcal{D}_C$ and refer to the triple $(c^\bullet, \epsilon_c, \eta_c)$ as the chosen adjunction of c in $\mathcal{D}_C$.*

Let D be another compact closed category having duality data $\mathcal{D}_D = \bigsqcup_{d \in Ob(D)} (d^\bullet, \epsilon_d, \eta_d)$.

2.4. Definition. *A strict symmetric monoidal functor $F : (C, \mathcal{D}_C) \longrightarrow (D, \mathcal{D}_D)$ preserves duality data if*

$$(F(c^\bullet), F(\epsilon_c), F(\eta_c)) = (F(c)^\bullet, \epsilon_{F(c)}, \eta_{F(c)}),$$

for each object c in C, where $(c^\bullet, \epsilon_c, \eta_c)$ is the chosen adjunction of c in $\mathcal{D}_C$ and $(F(c)^\bullet, \epsilon_{F(c)}, \eta_{F(c)})$ is the chosen adjunction of c in $\mathcal{D}_D$.

Let $(F, \lambda_F, \epsilon_F) : C \longrightarrow D$ be a symmetric monoiodal functor. The following pullback square in the category of symmetric monoidal categories and symmetric monoidal dunctors, associates a category $\mathbb{P}(F)$ to this functor:

$$\begin{array}{ccc} \mathbb{P}(F) & \longrightarrow & D^J \\ {\scriptstyle p(F)}\downarrow & & \downarrow{\scriptstyle s} \\ C & \xrightarrow[F]{} & D \end{array}$$

where s is the source functor *i.e.* maps an isomorphism to its source object.

2.5. Remark. The symmetric monoidal functor s is an equivalence of categories which is surjective on objects which implies that so is $p(F)$.

The objects of $\mathbb{P}(F)$ are triples (c, d, α), where $c \in Ob(C)$, $d \in Ob(D)$ and $\alpha : F(c) \cong d$ is an isomorphism in D. A morphism from (c_1, d_1, α_1) to (c_2, d_2, α_2) is a pair $(f, g) \in C(c_1, c_2) \times D(d_1, d_2)$ such that the following diagram commutes:

$$\begin{array}{ccc} F(c_1) & \xrightarrow[\cong]{\alpha_1} & d_1 \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle g} \\ F(c_2) & \xrightarrow[\alpha_2]{\cong} & d_2 \end{array}$$

The tensor product bifunctor of $\mathbb{P}(F)$

$$- \boxdot - : \mathbb{P}(F) \times \mathbb{P}(F) \longrightarrow \mathbb{P}(F).$$

is defined on objects as follows:

$$(c_1, d_1, \alpha_1) \boxdot (c_1, d_1, \alpha_1) := (c_1 \underset{C}{\otimes} c_2, d_1 \underset{D}{\otimes} d_2, (\alpha_1 \underset{D}{\otimes} \alpha_2) \circ \lambda_F).$$

2.6. Remark. *If the domain category of a symmetric monoidal functor $F : C \longrightarrow D$ is compact closed then it follows from remark 2.5 that $\mathbb{P}(F)$ is also compact closed.*

2.7. Definition. *Let $(D, \mathcal{D}_D)$ be a compact closed category with chosen duality data. The category of isomorphisms of D, namely the functor category D^J, inherits a* canonical *duality data from D wherein the chosen duality data of an isomorphism $f : c \cong d$ is the triple*

$$(f^\bullet, \epsilon_f, \eta_f),$$

where $f^\bullet$ is the dual of f in the sense of definition A.3, namely

$$f^\bullet = (f^\dagger)^{-1}.$$

and the counit and unit maps ϵ_f and η_f are the following two commutative diagrams:

$$\begin{array}{ccc} c^\bullet \otimes c & \xrightarrow{\epsilon_c} & 1_C \\ {\scriptstyle f^\bullet \otimes f}\downarrow & & \Vert \\ d^\bullet \otimes d & \xrightarrow[\epsilon_d]{} & 1_C \end{array} \qquad \begin{array}{ccc} 1_C & \xrightarrow{\eta_c} & c \otimes c^\bullet \\ \Vert & & \downarrow{\scriptstyle f \otimes f^\bullet} \\ 1_C & \xrightarrow[\eta_d]{} & d \otimes d^\bullet \end{array}$$

where $(c^\bullet, \epsilon_c, \eta_c)$ and $(d^\bullet, \epsilon_d, \eta_d)$ are the chosen adjunctions of c and d respectively in $\mathcal{D}_D$.

2.8. Definition. *For a functor $F : (C, \mathcal{D}_C) \longrightarrow (D, \mathcal{D}_D)$, the compact closed category $\mathbb{P}(F)$ inherits a* canonical *duality data wherein the chosen dual of an object (c, d, α) is the triple $(c^\bullet, d^\bullet, \alpha^\bullet)$, where $c^\bullet$ and $d^\bullet$ are the chosen duals of c and d in $\mathcal{D}_C$ and $\mathcal{D}_D$ respectively and $\alpha^\bullet$ is the chosen dual of α in the canonical duality data associated to D^J, namely $\alpha^\bullet = (\alpha^\dagger)^{-1}$.*

2.9. Notation. *We denote by $p_t : \mathbb{P}(F) \longrightarrow D$ the following composite:*

$$\mathbb{P}(F) \xrightarrow{p_2} D^J \xrightarrow{t} D.$$

2.10. Lemma. *There exists a monoidal natural isomorphism*

$$\beta_F : F \circ p(F) \Rightarrow p_t.$$

Proof. We define β_F as follows:

$$\beta_F((c, d, \alpha)) := \alpha.$$

By the definition of maps in $\mathbb{P}(F)$, it is easy to see that this defines a natural isomorphism. Now we check that it is monoidal.

Let (c_1, d_1, α_1) and (c_2, d_2, α_2) be two objects of $\mathbb{P}(F)$. We recall that

$$(c_1, d_1, \alpha_1) \boxdot (c_2, d_2, \alpha_2) = (c_1 \otimes c_2, d_1 \otimes d_2, (\alpha_1 \otimes \alpha_2) \circ \lambda_F).$$

This implies that

$$\beta_F((c_1, d_1, \alpha_1) \boxdot (c_2, d_2, \alpha_2)) = (\alpha_1 \otimes \alpha_2) \circ \lambda_F.$$

Since p_t is strict symmetric monoidal, therefore we have the following commutative diagram:

$$\begin{array}{ccc} F(c_1 \underset{C}{\otimes} c_2) & \xrightarrow{\beta_F} & d_1 \otimes d_2 \\ {\scriptstyle \lambda_F}\big\downarrow & & \big\| \\ F(c_1) \underset{D}{\otimes} F(c_2) & \xrightarrow[\alpha_1 \underset{D}{\otimes} \alpha_2]{} & d_1 \otimes d_2 \end{array}$$

The unit object of $\mathbb{P}(F)$ is the triple $(1_C, 1_D, \epsilon_F)$. Further, $F \circ p(F)((1_C, 1_D, \epsilon_F)) = F(1_C)$ and $p_t((1_C, 1_D, \epsilon_F)) = 1_D$. Now the following commutative diagram tells us that β_F is a monoidal natural isomorphism:

$$\begin{array}{ccc} F(1_C) & \xrightarrow{\epsilon_F} & 1_D = p_t((1_C, 1_D, \epsilon_F)) \\ & {\scriptstyle \epsilon_F}\searrow & \big\| \\ & & 1_D \end{array}$$

■

We recall that any natural isomorphism $\beta : H \cong G$, where H and G are functors between categories A and B, is uniquely represented, through adjointness, by a functor $\beta : A \longrightarrow B^J$ such that the following diagram commutes:

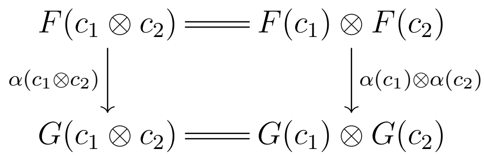
$$\begin{array}{ccc} & & B^J \\ & \overset{\beta}{\nearrow} & \big\downarrow{\scriptstyle (s,t)} \\ A & \xrightarrow[(H,K)]{} & B \times B \end{array}$$

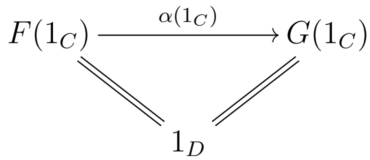
The following lemma will be useful in proving key results in this section:

2.11. Lemma. *Let $(C, \mathcal{D}_C)$ and $(D, \mathcal{D}_D)$ be a pair of compact closed categories with chosen duality data and $F : C \longrightarrow D$ and $G : C \longrightarrow D$ be a pair of strict symmetric monoidal functors which preserve duality data. A monoidal natural isomorphism $\alpha : F \cong G$ is represented by a strict symmetric monoidal functor $\alpha : C \longrightarrow D^J$ which preserves duality data, where D^J has the canonical duality data.*

Proof. We will first show that the functor $\alpha : C \longrightarrow D^J$ representing the monoidal natural transformation $\alpha : F \Rightarrow G$, is a strict symmetric monoidal functor. This will be accoplished in two steps namely, we will first show that α is a monoidal functor and then we will show that it is symmetric monoidal. Let c_1, c_2 be a pair of objects in C.

By assumption, $\alpha : F \Rightarrow G$ is a monoidal natural transformation and F and G are both strict symmetric monoidal therefore we have the following two commutative diagram:





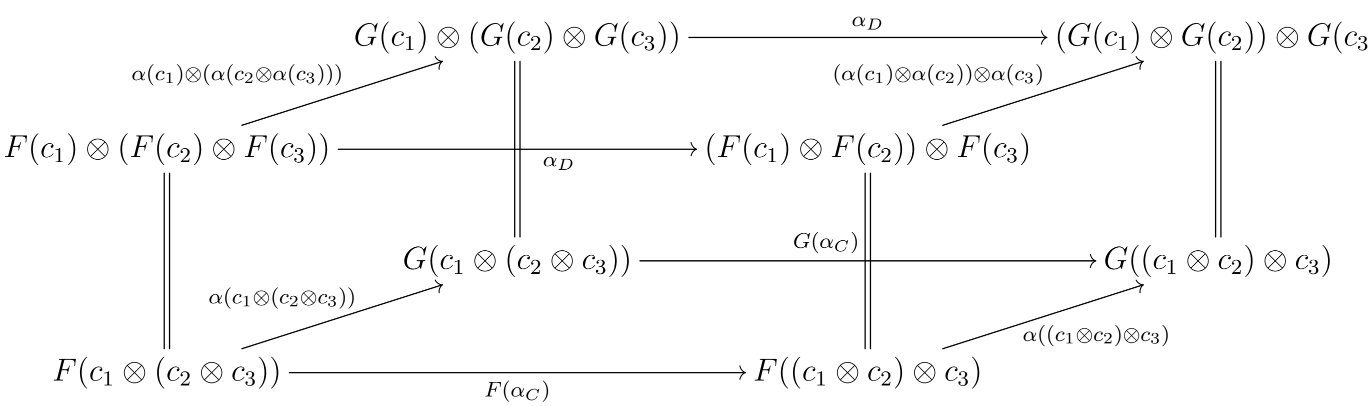
The first commutative diagram above shows that the functor α strictly preserves the tensor product, namely, for any pair of objects $c_1, c_2 \in C$ we have the followig equality:

$$\alpha(c_1 \otimes c_2) = \alpha(c_1) \otimes \alpha(c_2).$$

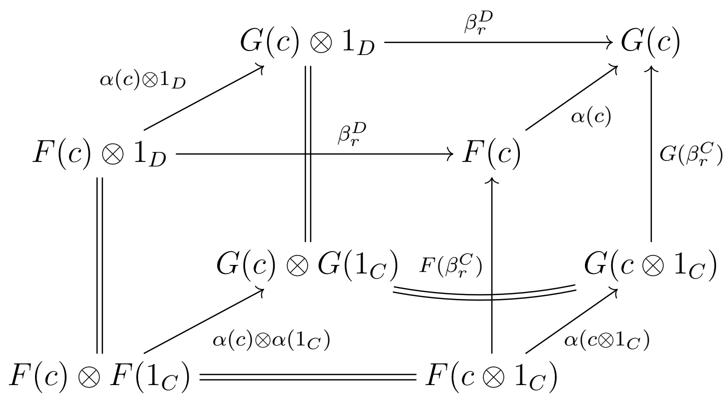
The second commutative diagram shows that the functor α is strictly unital, namely, the following equality holds:

$$\alpha(1_C) = 1_D.$$

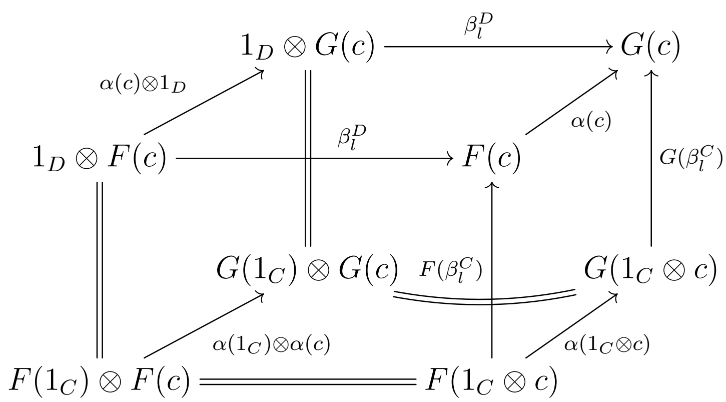
We now claim that the triple $(\alpha, id, id) : C \longrightarrow D^J$ is a monoidal functor in the sense of the definition on [ML98, Pg. 255]. In order to prove our claim we will have to verify (3) and (4) on [ML98, Pg. 255-56]. Since both F and G are strict symmetric monoidal functors and $\alpha : F \Rightarrow G$ is a monoidal natural transformation by assumption, we have the following commutative diagram for each triple of objects $c_1, c_2, c_3 \in C$:



Now condition (3) of the definition of a monoidal functor given on [ML98, Pg. 255] follows from the above commutative diagram. Condition (4) of the same definition follows from the observation that the following two diagrams commute, under the assumptions of the lemma, for each object $c \in C$:



and



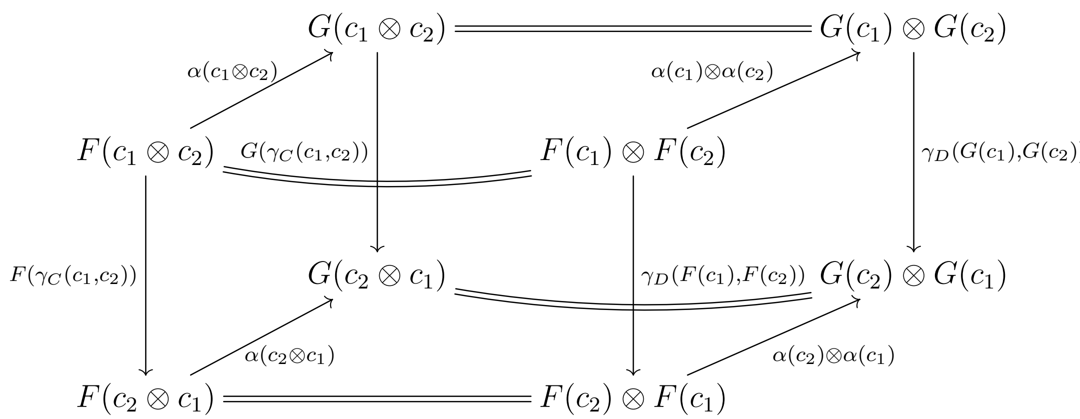
Thus we have shown that the triple

$$(\alpha, id, id) : C \longrightarrow D^J$$

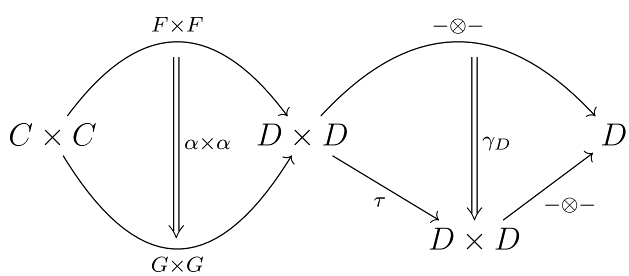
is a monoidal functor. In other words, the functor α has a strict monoidal structure. In order to show that α is a strict symmetric monoidal functor we have to establish the commutativity of the following diagram for each pair of objects $c_1, c_2 \in C$:

$$\begin{array}{ccc} \alpha(c_1 \otimes c_2) & = & \alpha(c_1) \otimes \alpha(c_2) \\ {\scriptstyle \alpha(\gamma_C(c_1,c_2))}\downarrow & & \downarrow{\scriptstyle \gamma_D(\alpha(c_1),\alpha(c_2))} \\ \alpha(c_2 \otimes c_1) & = & \alpha(c_1) \otimes \alpha(c_2) \end{array}$$

This is equivalent to showing the commutativity of the following diagram, for each pair of objects $c_1, c_2 \in C$, in the symmetric monoidal category D:



The left face of the above diagram commutes because $\alpha : F \Rightarrow G$ is a natural transformation. The right face of this diagram commutes because we have the following composite natural transformation:

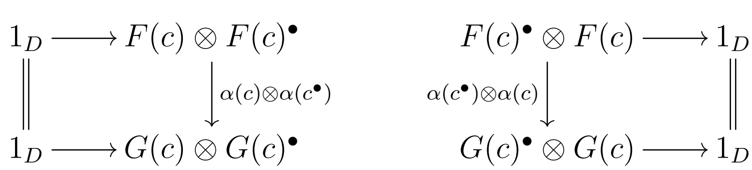


The top and the bottom faces of this diagram commute because $\alpha : F \Rightarrow G$ is a monoidal natural transformation by assumption. The front and the back face of the above diagram commute because F and G are strict symmetric monoidal functors by assumption. Thus we have shown that the functor $\alpha : C \longrightarrow D^J$, representing the monoidal natural transformation $\alpha : F \Rightarrow G$ between strict symmetric monoidal functors F and G, is a strict symmetric monoidal functor.

The next step is to show that the strict symmetric monoidal functor $\alpha : C \longrightarrow D^J$ preserves duality data. Let c be an object in C and $c^\bullet$ is its chosen dual in $\mathcal{D}_C$. It is sufficient to show that

$$\alpha(c^\bullet) = ((\alpha(c))^\bullet)^{-1},$$

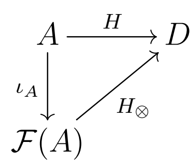
where $(\alpha(c))^\bullet$ is the dual of $\alpha(c)$ in the sense of definition A.3. Since $c^\bullet$ is the chosen dual of c therefore the duality data $\mathcal{D}_C$ specifies two maps $\epsilon_c : c^\bullet \otimes c \longrightarrow 1_C$ and $\eta_c : 1_C \longrightarrow c \otimes c^\bullet$. Now, the strict symmetric monoidal structures of F and G and the assumption that both functors preserve duality data, together imply that the following two diagrams commute:



Now it follows from lemma A.5 that $\alpha(c^\bullet) = (\alpha(c)^\dagger)^{-1}$. ■

2.12. NOTATION. *We denote by* $[C, D]_\otimes^{Du}$ *the groupoid whose objects are strict symmetric monoidal functors between compact closed categories* C *and* D *which preserve duality data and whose morphisms are monoidal natural isomorphsisms.*

For the remaining section, the domain C of the symmetric monoidal functor F will be the category $\mathcal{F}(A)$, which is constructed in [KL80], namely the free compact closed category generated by A, where A is an ordinary category. We recall from [KL80] that $\mathcal{F}(A)$ has an associated duality data and it is equipped with an inclusion functor $\iota_A : A \longrightarrow \mathcal{F}(A)$. This compact closed category has the following universal property: For any functor $H : A \longrightarrow D$, whose codomain is a compact closed category D with chosen duality data $\mathcal{D}_D$, there exists a unique strict symmetric monoidal functor $H_\otimes$ which preserves duality data, such that the following diagram commutes:

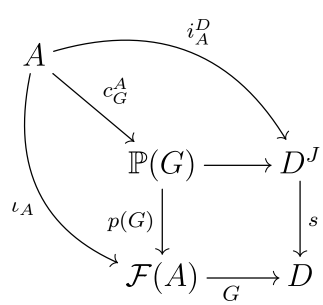


(6)

The following lemma tells us that for a fixed compact closed category $(D, \mathcal{D}_D)$, each symmetric monoidal functor in $[\mathcal{F}(A), D]_\otimes$ is isomorphic to a strict symmetric monoidal functor which preserves duality data:

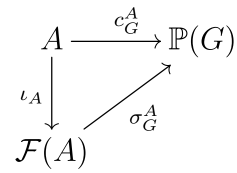
2.13. LEMMA. *A symmetric monoidal functor* $G : \mathcal{F}(A) \longrightarrow D$ *is isomorphic to the unique strict symmetric monoidal functor* $(G\iota_A)_\otimes : \mathcal{F}(A) \longrightarrow (D, \mathcal{D}_D)$ *which preserves duality data, defined in* (6).

PROOF. Let $G : \mathcal{F}(A) \longrightarrow D$ be a symmetric monoidal functor which may not preserve duality data. Since the pullback square defining $\mathbb{P}(G)$ is also a pullback square in **Cat** and there exists a functor $i_A^D : A \longrightarrow D^J$ defined by $i_A^D(a) = id_{G(a)}$, the category $\mathbb{P}(G)$ is equipped with a canonical functor $c_G^A : A \longrightarrow \mathbb{P}(G)$ which is depicted in the following commutative diagram:

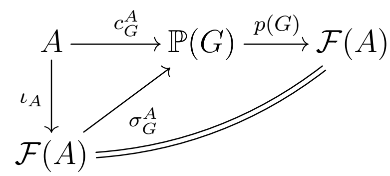


Now the choice of canonical duality data on $\mathbb{P}(G)$ and the universal property of ι_A

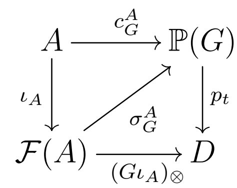
gives us a strict symmetric monoidal functor σ_G^A which preserves duality data:



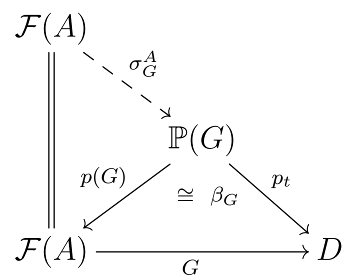
Again by the universality of ι_A, we have the following commutative diagram:



The universality of the map ι_A implies that the following diagram commutes:



Thus the composite $p_t \circ \sigma_G^A$ of strict symmetric monoidal functors which preserve duality data is the same as the strict symmetric monoidal functor $(G\iota_A)_\otimes$ which preserves duality data and which is defined in (6). Now the following diagram gives us a monoidal natural isomorphism which we denote by $\gamma_G : G \cong (G\iota_A)_\otimes$:



where β_G is the natural isomorphism from lemma 2.10.

∎

The above universal property leads us to the following theorem which states that the compact closed category $\mathcal{F}(A)$ is also free in a sense of 2-universal algebra:

2.14. Theorem. *For any compact closed category D, the following functor is an equivalence of groupoids which is surjective on objects:*

$$(\iota_A)^* : [\mathcal{F}(A); D]_\otimes \longrightarrow J([A; D]),$$

where $[\mathcal{F}(A); D]_\otimes$ is the groupoid of symmetric monoidal functors from $\mathcal{F}(A)$ to D and $J([A; D])$ is the groupoid of all functors from A and D and natural isomorphsism between them.

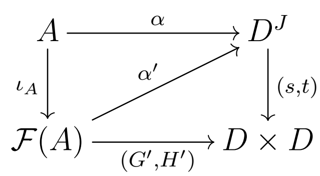
Proof. Let $G : A \longrightarrow D$ be a functor. We can choose a duality data $\mathcal{D}_D$ on D. Now the aforementioned universal property of ι_A shows that the functor $(\iota_A)^*$ is surjective on objects. Now it is sufficient to show that $(\iota_A)^*$ is fully-faithful.

Let $\alpha : G \Rightarrow H$ be a natural isomorphism in $J([A, D])$ and G' and H' be two symmetric monoidal functors from $\mathcal{F}(A)$ to D such that $(\iota_A)^*(G') = G$ and $(\iota_A)^*(H') = H$. We will show that there exists a unique monoidal natural isomorphism $\alpha' : G' \Rightarrow H'$ such that $(\iota_A)^*(\alpha') = \alpha$.

The natural isomorphism α can be viewed as a functor

$$\alpha : A \longrightarrow D^J,$$

such that $G = s \circ \alpha$ and $H = t \circ \alpha$. Since D is a permutative compact closed category by assumption, the acyclic fibration of categories $s : D^J \longrightarrow D$ implies that so is D^J. We may chose a duality data on D^J such that the source and target strict symmetric monoidal functors $s : D^J \longrightarrow D$ and $t : D^J \longrightarrow D$ are both duality data preserving. By construction of the monoidal natural isomorphism $\gamma_{G'}$ in the proof of the lemma above, we may assume that G' and H' both are strict symmetric monoidal functors which preserve duality data. Now by the universal property of ι_A, there exists a unique strict symmetric monoidal functor α' which preserves duality data such that the following diagram commutes:



The uniqueness of α' follows from our assumption that H' and G' are both strict symmetric monoidal functors which preserve duality data together with lemma 2.11. ∎

2.15. Corollary. *For each permutative compact closed category D, the inclusion functor*

$$i_\otimes : [\mathcal{F}(A), D]^{Du}_\otimes \hookrightarrow [\mathcal{F}(A), D]_\otimes$$

is an equivalence of groupoids.

Proof. The functor is fully faithful. In the proof of the above theorem we have shown that each symmetric monoidal functor can be replaced by a strict symmetric monidal functor which preserves duality data. ∎

A similar argument applied to strict symmetric monoidal functors gives us the following corollary:

2.16. Corollary. *The inclusion functor*

$$i_{str} : [\mathcal{F}(A), D]^{Du}_\otimes \hookrightarrow [\mathcal{F}(A), D]^{str}_\otimes$$

is an equivalence of groupoids.

Combining the above two corollaries we get the following proposition:

2.17. PROPOSITION. *For each permutative compact closed category D, the inclusion*

$$[\mathcal{F}(A), D]_{\otimes}^{str} \subseteq [\mathcal{F}(A), D]_{\otimes}$$

is an equivalence of categories, where $[\mathcal{F}(A), D]_{\otimes}^{str}$ is the groupoid of strict symmetric monoidal functors and monoidal natural isomorphisms and $[\mathcal{F}(A), D]_{\otimes}$ is the groupoid of symmetric monoidal functors and monoidal natural isomorphisms.

PROOF. The inclusion $i_{\otimes}$ factors as follows:

$$[\mathcal{F}(A), D]_{\otimes}^{Du} \overset{i_{str}}{\hookrightarrow} [\mathcal{F}(A), D]_{\otimes}^{str} \subseteq [\mathcal{F}(A), D]_{\otimes}.$$

Now the result follows from the two out of three property of weak equivalences in a model category, considering the facts that $i_{\otimes}$ and i_{str} are both equivalences of categories. ∎

The following lemma will be useful in proving the next theorem in this section:

2.18. LEMMA. *Let Q be a cofibrant permutative category. For any permutative category D, we have the following inclusion is an equivalence of categories:*

$$[Q, D]_{\otimes}^{str} \hookrightarrow [Q, D]_{\otimes}.$$

PROOF. The category **Perm** of srict symmetric monoidal categories and strict symmetric monoidal functors is isomorphic to the category of algebras over the (categorical) Barrat-Eccles operad in **Cat**. This implies that their is a 2-monad T in **Cat** such that the category of (strict) T-Algebras and strict morphisms $T - Alg_s$ is isomorphic to **Perm**. Further, the category $T - Alg$ of strict T-Algebras and pseudo morphisms is isomorphic to the category of strict symmetric monoidal categories and symmetric monoidal functors.

We recall that the cofibrant objects in the natural model category **Perm**, [Sha, Thm. 3.1], [Lac07, Thm. 4.5] are exactly *flexible* algebras defined in [BKP89, Rem. 4.5]. Now the result follows from [BKP89, Thm. 4.7]. ∎

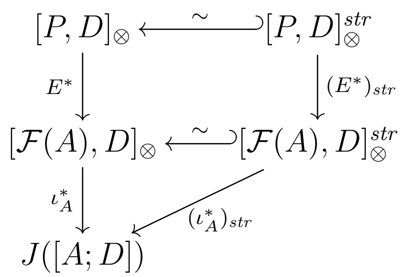
Let P be a permutative category which is cofibrant in the natural model category of permutative categories **Perm** which is equipped with a strict symmetric monoidal functor $E : \mathcal{F}(A) \longrightarrow P$ whose underlying functor is an equivalence of categories. The next lemma says that equivalence of groupoids from the previous theorem is preserved under E:

2.19. LEMMA. *The following composite functor:*

$$(\iota_A)^*_{str} E^*_{str} : [P, D]_{\otimes}^{str} \longrightarrow [\mathcal{F}(A), D]_{\otimes}^{str} \longrightarrow J([A, D]).$$

is an equivalence of groupoids, for any permutative category D.

PROOF. We have the following commutative diagram in the category of groupoids:



The top horizontal arrow in the above diagram is an equivalence of categories from Lemma 2.18. Since E is a homotopy equivalence of symmetric monoidal categories, namely there exists a symmetric monoidal functor $H : P \longrightarrow \mathcal{F}(A)$ and two monoidal natural isomorphisms $EH \cong id$ and $id \cong HE$, it follows that E^* is an equivalence of categories. Now the commutativity of the above diagram proves the result.

∎

Now we consider the free permutative compact closed category $\mathcal{F}'(A)$ generated by A and the free permutative strict compact closed category $\mathcal{F}'(A)$ generated by A which are described in [KL80, Sec. 9]. These permutative categories are equipped with the following strict symmetric monoidal functors (see [KL80, Sec. 9]) which preserves duality data:

$$\mathcal{F}(A) \xrightarrow{E_1} \mathcal{F}'(A) \xrightarrow{E_2} \mathcal{F}''(A).$$

Further, (the underlying ordinary functors of) both E_1 and E_2 are equivalences of categories.

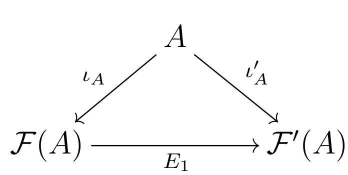
2.20. COROLLARY. *The following composite functor:*

$$(\iota_A)^*_{str}(E_1)^*_{str} : [\mathcal{F}'(A), D]^{str}_\otimes \longrightarrow [\mathcal{F}(A), D]^{str}_\otimes \longrightarrow J([A, D]).$$

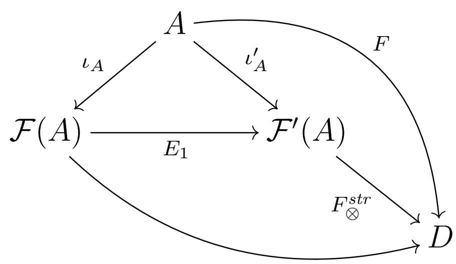
is an equivalence of groupoids which is surjective on objects, for any permutative category D.

PROOF. The permutative category $\mathcal{F}'(A)$ is cofibrant in the natural model category of permutatve categories so it follows from the above lemma that the functor $(\iota_A)^*_{str}(E_1)^*_{str}$ is an equivalence of categories.

We have the following commutative diagram in **Cat**:



where ι and ι' are the unit maps of the adjunctions $(\mathcal{F}, U)$ and $(\mathcal{F}', U)$ discussed in [KL80]. Now the universal property of ι' implies that for any functor F in $[A; D]$, there exists a strict symmetric monoidal functor $F^{str}_{\otimes}$ such that the following diagram commutes:



The above diagram implies that the functor $(\iota_A)^*_{str}(E_1)^*_{str}$ is surjective on objects. ■

2.21. Remark. *The above corollary implies that for each functor $F : A \longrightarrow D$, where D is a permutative compact closed category, there exists a strict symmetric monoidal functor $F^{str}_{\otimes} : \mathcal{F}'(A) \longrightarrow D$ such that $F = F^{str}_{\otimes} \circ \iota'_A$ which is unique upto a unique isomorphism.*

The above argument applied to the composite $E_2 \circ E_1$ gives the following corollary:

2.22. Corollary. *The following composite functor:*

$$(\iota_A)^*_{str}(E_1 \circ E_2)^*_{str} : [\mathcal{F}''(A), D]^{str}_{\otimes} \longrightarrow [\mathcal{F}(A), D]^{str}_{\otimes} \longrightarrow J([A, D]).$$

is an equivalence of groupoids which is surjective on objects, for any permutative category D.

2.23. Remark. *The above corollary implies that for each functor $F : A \longrightarrow D$, where D is a permutative compact closed category, there exists a strict symmetric monoidal functor $F^{str}_{\otimes} : \mathcal{F}''(A) \longrightarrow D$ such that $F = F^{str}_{\otimes} \circ \iota''_A$ which is unique upto a unique isomorphism.*

3. Compact closed permutative categories

The category of all (small) symmetric monoidal categories has a subcategory **Perm** which inherits a model category structure from the *natural* model category **Cat**. The objects of **Perm** are *permutative* categories (or strict symmetric monoidal categories) which are those symmetric monoidal categories whose *tensor product* is strictly associative and strictly unital. The morphisms of **Perm** are strict symmetric monoidal functors namely those symmetric monoidal functors which preserve the symmetric monoidal structure strictly. In this section we will construct another model category structure on the category of permutative categories **Perm** whose fibrant objects are (permutative) compact closed categories.

3.1. Notation. *Unless specified otherwise, in this section a compact closed category will mean a permutative category which is compact closed.*

3.2. Notation. *We denote by* $\overline{1}$ *the following set:*

$$\overline{1} := \{+, -\}$$

and refer to it as the set of orientations of a point.

We now describe a permutative category $\mathfrak{Fr}^{cc}(\underline{1})$ whose object monoid is the free monoid generated by the two object set $\overline{1} = \{+, -\}$. An object in $\mathfrak{Fr}^{cc}(\underline{1})$ is a finite sequence of elements of $\{+, -\}$, for example $(+, -, +, -, -)$. The monoidal product in $Ob(\mathfrak{Fr}^{cc}(\underline{1}))$ is given by concatenation. The product of f and g will be denoted by $f \Box g$.

3.3. Notation. *We will name objects of* $\mathfrak{Fr}^{cc}(\underline{1})$ *by lowercase letters* f, g etc. *which is suggestive of the fact that an object* f *in* $\mathfrak{Fr}^{cc}(\underline{1})$ *can be represented by a funtion* $f : \underline{n} \longrightarrow \{+, -\}$.

The set $\overline{1}$ is equipped with a bijection $in : \overline{1} \longrightarrow \overline{1}$ which changes the sign *i.e.* $in(+) = -$ and $in(-) = +$. For each $\underline{n}$ is equipped with a bijection $\Sigma_{rev}(n) : \underline{n} \longrightarrow \underline{n}$ which reverses the order: $i \mapsto n - i + 1$.

3.4. Definition. *Each object* f *in the proposed permutative category* $\mathfrak{Fr}^{cc}(\underline{1})$ *determines another object, which we denote by* $f^\bullet$, *which is obtained by replacing each* $+$ *in* f *by a* $-$ *and each* $-$ *by a* $+$ *and then reversing order. Equivalently, considering the object* f *as a function* $f : \underline{n} \longrightarrow \overline{1}$, $f^\bullet$ *is represented by by the following composite:*

$$\underline{n} \xrightarrow{\Sigma_{rev}(n)} \underline{n} \xrightarrow{f} \overline{1} \xrightarrow{in} \overline{1}.$$

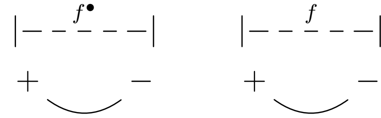
We will refer to $f^\bullet$ *as the* dual *of* f.

3.5. Remark. *For each* $f \in Ob(\mathfrak{Fr}^{cc}(\underline{1}))$ *there is a canonical bijection of the underlying sets of* f *and* $f^\bullet$ *which we denote by* $c(f)$. *This canonical bijection maps the element at index* i *in* f *to the element at index* $n - i + 1$ *in* $f^\bullet$. *We observe that the bijection* $c(f)$ *is a sign reversing involution.*

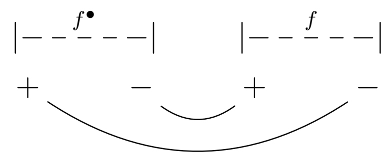
3.6. Notation. *The* length *of an object* f *in the proposed permutative category* $\mathfrak{Fr}^{cc}(\underline{1})$ *is the cardinality of the finite sequence* f e.g. *the length of* $(+, -, +, +)$ *is* 4. *We denote the length of* f *by* $|f|$. *We will denote the ordered set* $\{1, 2, \ldots, |f|$ *by* $\underline{|f|}$.

3.7. Definition. *Let* f *and* g *be two objects in the proposed category* $\mathfrak{Fr}^{cc}(\underline{1})$. *A matching* G *from* f *to* g, *denoted* $G : f \rightsquigarrow g$, *is a sign reversing involution of the set* $f^\bullet \Box g$.

3.8. Example. *Let* $f = (+, -)$ *be an object in* $\mathfrak{Fr}^{cc}(\underline{1})$. *The following pictures represent two distinct matchings from* f *to* f:



and the second matching is the following:



3.9. Remark. *A matching $F : f \rightsquigarrow g$ uniquely determines a (directed) graph $\mathcal{G}(F)$ whose vertices are the elements of the set $f^\bullet \Box g$. Each vertex in $\mathcal{G}(F)$ has degree one. A connected component in $\mathcal{G}(F)$ consists of two elements of $f^\bullet \Box g$ having opposite signs and an edge joining them. In each connected component of $\mathcal{G}(F)$, the vertex $-$ has outgoing degree one and incoming degree zero and the vertex $+$ has incoming degree one and outgoing degree zero.*

3.10. Remark. *The set of vertices of each connected component of $\mathcal{G}(F)$ is a pair of elements in $f^\bullet \Box g$ having opposite signs. The family of pairs determined by all connected components of $\mathcal{G}(F)$ partition the set $f^\bullet \Box g$.*

3.11. Definition. *A pair in a matching $F : f \rightsquigarrow g$ is a two element subset $p \subseteq f^\bullet \Box g$ whose elements are the source and target of a unique edge in $\mathcal{G}(F)$.*

The next definition classifies pairs:

3.12. Definition. *A pair p in a matching $F : f \rightsquigarrow g$ is called a domain pair if $p \cap g = \emptyset$. It is called a codomain pair if $p \cap f = \emptyset$ and it is called an external pair if it is neither a domain pair nor a codomain.pair.*

Now we describe a *composition* of two matchings: $F : f \rightsquigarrow g$ and $G : g \rightsquigarrow h$. In order to do so we consider the following coCartesian square in the category of graphs:

$$\begin{array}{ccccc} g & \xrightarrow{c(g)} & g^\bullet & \hookrightarrow & \mathcal{G}(G) \\ \downarrow & & & & \downarrow \\ \mathcal{G}(F) & & \longrightarrow & & \mathcal{G}(F)\bigsqcup_{g}\mathcal{G}(G) \end{array}$$

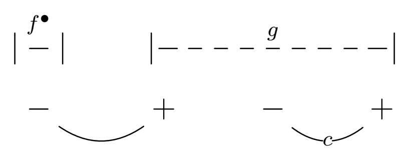
In the above diagram, we are regarding g as a graph whose set of vertices is g and whose source and target maps are identities. The connected components of $\mathcal{G}(F)\bigsqcup_{g}\mathcal{G}(G)$ can be classified into two different types:

1. Grafted edge: A connected subgraph of $\mathcal{G}(F)\bigsqcup_{g}\mathcal{G}(G)$ which has one vertex of outgoing degree one and another vertex having incoming degree one, all other vertices have degree two.

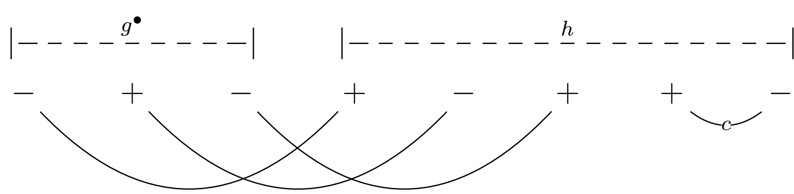
2. Circle: A connected subgraph of $\mathcal{G}(F)\bigsqcup_{g}\mathcal{G}(G)$ whose set of vertices V is in bijection with a finite sum of copies of the set $\underline{1} = \{+, -\}$ and is equipped with an inclusion $V \hookrightarrow g$, such that each vertex has incoming degree one and outgoing degree one.

All the grafted edges of $\mathcal{G}(F)\bigsqcup_{g}\mathcal{G}(G)$ uniquely determine a matching of $f^{\bullet}\Box h$ which we denote by $G \circ F$ and which is the composite of G and F.

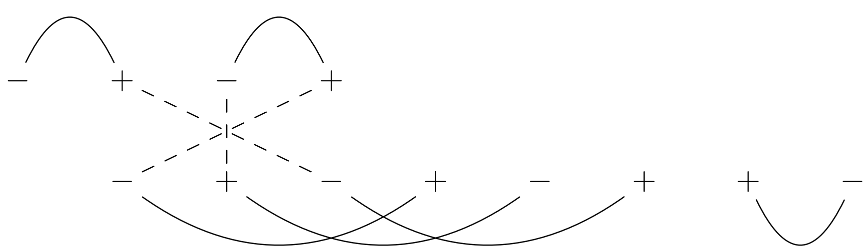
3.13. Example. *Let $f = (+)$, $g = (+, -, +)$ and $h = (+, -, +, +, -)$ be three objects in $\mathfrak{Fr}^{cc}(\underline{1})$. Let F be a matching from f to g which is represented by the following diagram:*

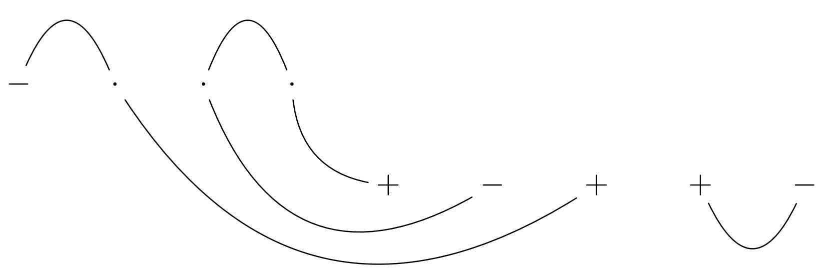


Let G be a matching from g to h which is represented by the following diagram:

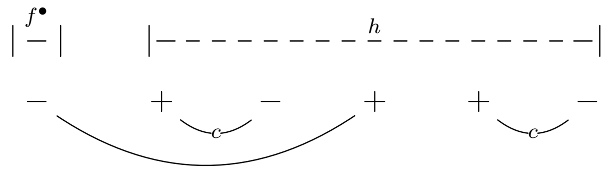


The composition of G and F can be described by the following diagrams:





Now the composite matching $G \circ F$ *is more succiently represented by the following diagram:*



We observe that in the above composition, a codomain pair of G composed *with a couple of external pairs of* H *to produce a codomain pair in* $H \circ G$.

3.14. Definition. *For two composable matchings* F *and* G *as above, we define a natural number* $Cir(H, G)$ *as follows:*

$$Cir(G, F) := \text{number of circles in } \mathcal{G}(F) \bigsqcup_{g} \mathcal{G}(G).$$

We will refer to this number as the circles created *by the composition of* H *and* G.

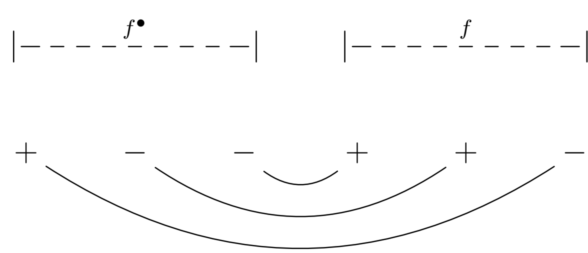
Now we have all the machinery needed to define a morphism in $\mathfrak{Fr}^{cc}(\underline{1})$:

3.15. Definition. *A morphism from* f *to* g *in* $\mathfrak{Fr}^{cc}(\underline{1})$ *is a pair* (G, k) *where* G *is a matching from* f *to* g *and* $k \in \mathbb{N}$.

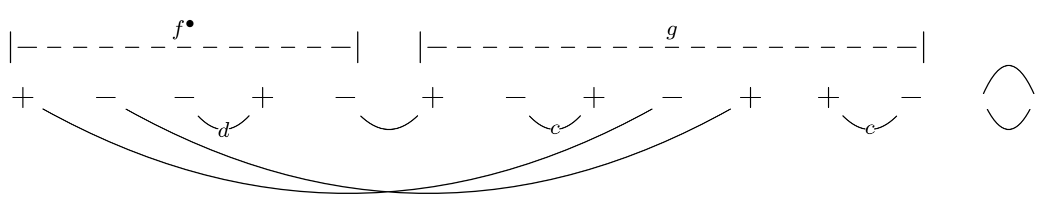
We observe that there is NO morphism between two objects f and g in $\mathfrak{Fr}^{cc}(\underline{1})$, if the sum of their lengths $|f| + |g|$ is an odd number. Let $(H, l) : g \longrightarrow h$ be another morphism in $\mathfrak{Fr}^{cc}(\underline{1})$. We define

$$(H, l) \circ (G, k) := (H \circ G, k + l + Cir(H, G))$$

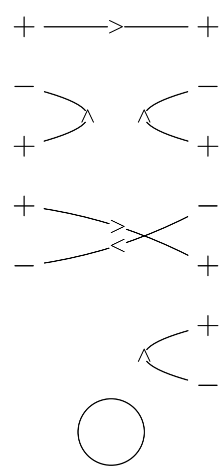
3.16. Example. *Let* $f = (+, +, -)$ *be an object of* $\mathfrak{Fr}^{cc}(\underline{1})$. *The identity map of* f *in* $\mathfrak{Fr}^{cc}(\underline{1})$, *denoted* $(id_f, 0)$ *is represented by the following diagram:*



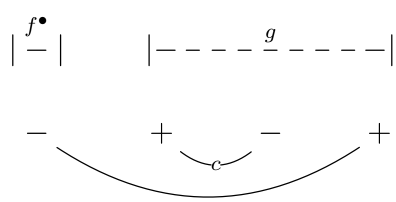
3.17. Example. *A morphism* $(H, 1)$ *from* $f = (+, -, +, +, -)$ *to* $g = (+, -, +, -, +, +, -)$ *is represented by the following diagram:*



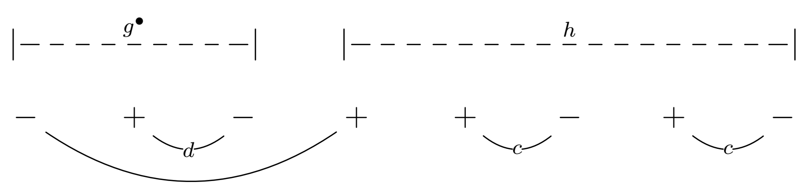
An object of $\mathfrak{Fr}^{cc}(\underline{1})$ *represents an oriented smooth* 0*-manifold by considering it as a sequence of points given* $+$ *or* $-$ *orientation. The morphism* $(H,1)$ *determines a* 1*-Bordism whose boundary is* $f^{\bullet}\Box g$*. This* 1*-Bordism is represented by the following diagram:*



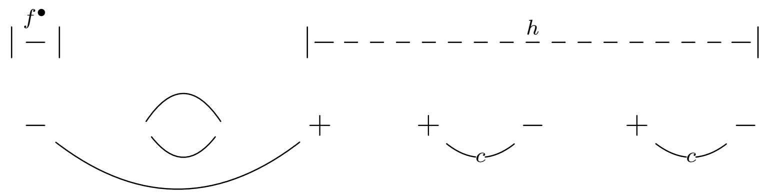
3.18. Example. *Let* $f = (+)$*,* $g = (+,-,+)$ *and* $h = (+,+,-,+,-)$ *be three objects in* $\mathfrak{Fr}^{cc}(\underline{1})$*. Let a map* $(G,0) : f \longrightarrow g$ *in* $\mathfrak{Fr}^{cc}(\underline{1})$ *be represented by the following diagram:*



We observe that G *is a matching of codomain index* 1*. Let another map* $(H,0) : g \longrightarrow h$ *in* $\mathfrak{Fr}^{cc}(\underline{1})$ *be represented by the following diagram:*



We observe that H *is a matching of codomain index* 2 *and domain index* 1*. The composite matching* $H \circ G$ *creates one circle and the map* $(H \circ G, 1) = (H,0) \circ (G,0)$ *is represented* *by the following diagram:*

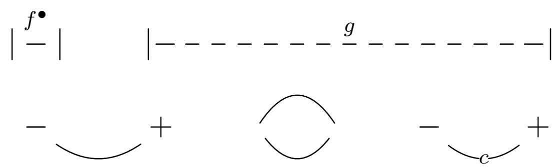


The *tensor product* of two maps $(G, k) : f \longrightarrow g$ and $(H, q) : h \longrightarrow l$ is defined as follows:

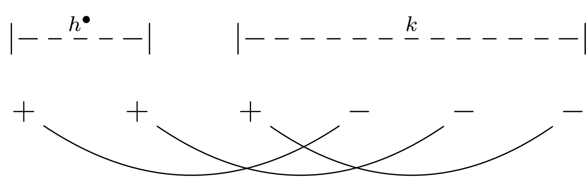
$$(G, k)\Box(H, q) := (G\Box H, k + q),$$

where $G\Box H$ is the unique matching determined by the graph $\mathcal{G}(G) \bigsqcup \mathcal{G}(H)$.

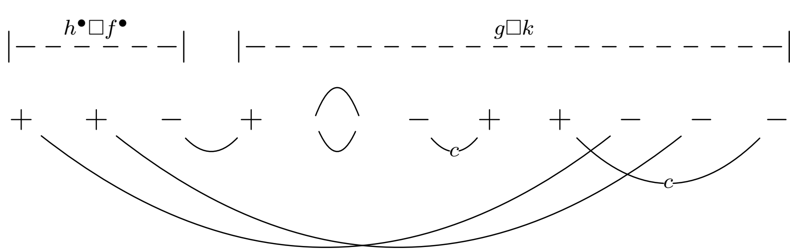
3.19. Example. *Let $f = (+)$, $g = (+, -, +)$, $h = (-, -)$ and $k = (+, -, -, -)$ be objects in $\mathfrak{Fr}^{cc}(\underline{1})$. Let $(G, 1) : f \longrightarrow g$ be a map represented by the following diagram:*



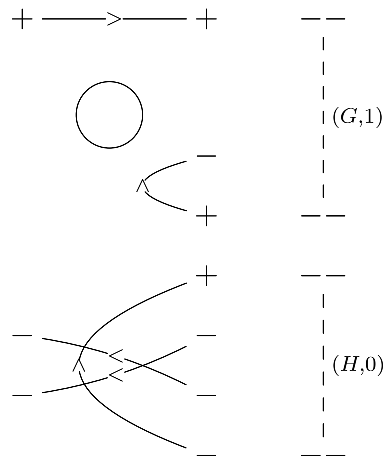
Let $(H, 0) : h \longrightarrow k$ be a map represented by the following diagram:



The map $(G, 1)\Box(H, 0) : f\Box h \longrightarrow g\Box k$ is represented by the following diagram:



The above tensor product $(G, 1)\Box(H, 0)$ represents a 1-Bordism which is a disjoint union of two 1-Bordisms represented by $(G, 1)$ and $(H, 0)$, as shown by the following diagram:



The aforementioned tensor product defines a bifunctor:

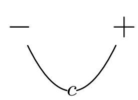
$$-\Box- : \mathfrak{Fr}^{cc}(\underline{1}) \times \mathfrak{Fr}^{cc}(\underline{1}) \longrightarrow \mathfrak{Fr}^{cc}(\underline{1}).$$

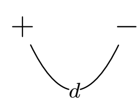
The above bifunctor endows the category $\mathfrak{Fr}^{cc}(\underline{1})$ with a strict symmetric monoidal (permutative) category structure.

3.20. Lemma. *The permutative category $\mathfrak{Fr}^{cc}(\underline{1})$ is a compact closed category wherein each object $f : \underline{n} \longrightarrow \overline{1}$ has a dual $f^\bullet$.*

Proof. An object f in $\mathfrak{Fr}^{cc}(\underline{1})$ is as a tensor product of a finite number of objects reprepresented by $(+)$ and $(-)$. Moreover, adjunctions compose which implies that if f has a dual $f^\bullet$ and g has a dual $g^\bullet$ in the permutative category $\mathfrak{Fr}^{cc}(\underline{1})$, then $g^\bullet\Box f^\bullet$ is a dual of $f\Box g$.

In light of the above observations, it is sufficient to show that $(-)$ is a dual of $(+)$ in the permutative category $\mathfrak{Fr}^{cc}(\underline{1})$. We claim that the unit $\eta : () \longrightarrow (-, +)$ and counit $\epsilon : (+, -) \longrightarrow ()$ maps for the duality in context are represented by the following two diagrams respectively:

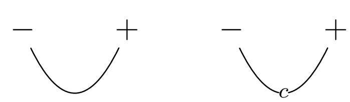




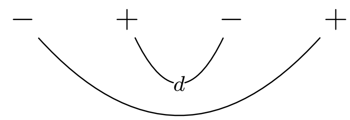
In order to prove our claim, we verify (1) and (2). In order to verify (2), we will show that the composite

$$(\epsilon\Box id_+) \circ (id_+\Box\eta) = id_+.$$

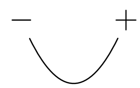
The map $(id_+\Box\eta) : (+) \longrightarrow (+, -, +)$ is represented by the following diagram:



The map $(\epsilon\Box id_+) : (+,-,+) \longrightarrow (+)$ is represented by the following diagram:



Now, one can see that the composite of the above two morphisms is the following which is also the identity map on $(+)$:



Condition (1) can be verified similarly.

■

The following corollary is an easy consequence of some earlier definitions and lemma 3.20:

3.21. Corollary. *In the compact closed category $\mathfrak{Fr}^{cc}(\underline{1})$, we have the following:*

1. *The unit object is its own dual* i.e. $()^\bullet = ()$.
2. *The dual of the dual of an object f in $\mathfrak{Fr}^{cc}(\underline{1})$, is the object f itself* i.e. $(f^\bullet)^\bullet = f$, *for each $f \in \mathfrak{Fr}^{cc}(\underline{1})$.*
3. *For a pair of objects f and g in $\mathfrak{Fr}^{cc}(\underline{1})$ (viewed as maps $f : \underline{n} \longrightarrow \overline{1}$ and $g : \underline{m} \longrightarrow \overline{1}$), the following commutative diagram implies the equality $(f\Box g)^\bullet = g^\bullet \Box f^\bullet$:*

$$\begin{array}{ccccc}
\underline{m+n} & \xrightarrow{\Sigma_{rev}(\underline{m+n})} & \underline{n+m} & \xrightarrow{f\Box g} & \overline{1} \\
& \searrow^{\Sigma_{rev}(\underline{m})+\Sigma_{rev}(\underline{n})} & & \nearrow_{g\Box f} & \downarrow{in} \\
& & \underline{m+n} & \longrightarrow & \overline{1}
\end{array}$$

3.22. Definition. *We will refer to $\mathfrak{Fr}^{cc}(\underline{1})$ either as the* free *compact closed category on one generator or as the algebraic* 1*-Bordism category.*

3.23. Lemma. *The compact closed category $\mathfrak{Fr}^{cc}(\underline{1})$ has the universal property that for each object c of a compact closed permutative category C there is a strict symmetric monoidal functor $F_c : \mathfrak{Fr}^{cc}(\underline{1}) \longrightarrow C$ such that $F_c(c) = c$, which is unique upto a unique natural isomorphism.*

Proof. The terminal category $\underline{1}$ is the category having one object and NO non-identity morphisms. Using the description of morphisms of the compact closed category $G(\underline{1})$ on [KL80, Pg. 197-8] one can readily deduce that the compact closed permutative category $G''(\underline{1})$, described in [KL80, Sec. 9], is isomorphic to $\mathfrak{Fr}^{cc}(\underline{1})$. Observations in the last paragraph of [KL80, Pg. 210] imply that $G''(\underline{1})$ is isomorphic to $\mathcal{F}''(\underline{1})$ described by the adjunction 5 above. Now the result follows from corollary (2.22).

■

3.24. Remark. *The free compact closed category over one generator* $\mathfrak{Fr}^{cc}(\underline{1})$ *is equipped with an inclusion (strict) symmetric monoidal functor* $i : \mathcal{F}^{\otimes}(\underline{1}) \longrightarrow \mathfrak{Fr}^{cc}(\underline{1})$*, where* $\mathcal{F}^{\otimes}(\underline{1})$ *is the free permutative category on one generator namely it is (isomorphic to) the category of finite sets and bijections. Both* $\mathfrak{Fr}^{cc}(\underline{1})$ *and* $\mathcal{F}^{\otimes}(\underline{1})$ *are cofibrant objects in the natural model category of permutative categories* **Perm**.

3.25. Proposition. *A permutative category is compact closed if and only if it is a* $\{i\}$*-local object.*

Proof. For any compact closed category C, it follows from lemma 3.23 and corollary A.10 that the following map, which is the evaluation map on the generator, is an equivalence of groupoids:

$$J\left([i, C]_{\otimes}^{str}\right) : J\left([\mathfrak{Fr}^{cc}(\underline{1}), C]_{\otimes}^{str}\right) \longrightarrow J\left([\mathcal{F}^{\otimes}(\underline{1}), C]_{\otimes}^{str}\right) \cong J(C)$$

where J is the right adjoint to the inclusion map $\iota : \mathbf{Gpd} \longrightarrow \mathbf{Cat}$. Thus each permutative compact closed category is an $\{i\}$-local object.

Conversely, let us assume that C is an $\{i\}$-local object. We recall that for any category D we have the following equality of object sets: $Ob(D) = Ob(J(D))$. By assumption the functor $J\left([i, C]_{\otimes}^{str}\right)$ is an equivalence of groupoids which now implies that each strict symmetric monoidal functor $F : \mathcal{F}^{\otimes}(\underline{1}) \longrightarrow C$ is isomorphic to some functor in the image of $J\left([i, C]_{\otimes}^{str}\right)$. Let F lie in the image of $J\left([i, C]_{\otimes}^{str}\right)$, then there exists a strict symmetric monoidal functor $F^{cc} : \mathfrak{Fr}^{cc}(\underline{1}) \longrightarrow C$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{Fr}^{cc}(\underline{1}) & \xrightarrow{F^{cc}} & C \\ {\scriptstyle i}\uparrow & \nearrow_{F} & \\ \mathcal{F}^{\otimes}(\underline{1}) & & \end{array}$$

In particular, $F^{cc}((+)) = F^{\otimes}((+))$. Since F^{cc} is a strict symmetric monoidal functor, it follows that $F^{cc}((-))$ is a dual of $F^{cc}((+)) = F^{\otimes}((+))$ in the permutative category C because $(-)$ is the dual of $(+)$ in $\mathfrak{Fr}^{cc}(\underline{1})$. Now the following isomorphism:

$$J\left([\mathcal{F}^{\otimes}(\underline{1}), C]_{\otimes}^{str}\right) \cong J(C),$$

and our supposed equivalence together imply that each object in C is isomorphic to an object $iF^{cc}((+))$ for some F^{cc} in $\left([\mathfrak{Fr}^{cc}(\underline{1}), C]_{\otimes}^{str}\right)$ and therefore it has a dual in C. Thus we have shown that C is compact closed. ∎

3.26. Definition. *A map of permutative categories* $F : C \longrightarrow D$ *will be called a compact closed equivalence of permutative categories if it is a* $\{i\}$*-local equivalence.*

3.27. Remark. *A strict symmetric monoidal functor* $F : C \longrightarrow D$ *between cofibrant permutative categories is a compact closed equivalence if the following functor is an equivalence of groupoids:*

$$J([F, E]_{\otimes}^{str}) : J([D, E]_{\otimes}^{str}) \longrightarrow J([C, E]_{\otimes}^{str})$$

for each permutative compact closed category E.

Now we state and prove the main result of this section which is is regarding the construction of a model category of compact closed categories. The proof uses an existence theorem of a left Bousfield localization of a (class of) model category which is reviewed in appendix B:

3.28. THEOREM. *There is a model category structure on the category of all small permutative categories and strict symmetric monoidal functors* **Perm** *in which*

1. *A cofibration is a strict symmetric monoidal functor which is a cofibration in the natural model category* **Perm**.
2. *A weak-equivalence is a compactly closed equivalence of permutative categories.*
3. *A fibration is a strict symmetric monoidal functor having the right lifting property with respect to all maps which are both cofibrations and weak equivalences.*

Further, this model category structure is combinatorial and left-proper. The fibrant objects in this model category are (permutative) compact closed categories.

PROOF. The model category structure is a left Bousfield localization of the left-proper, combinatorial natural model category structure on **Perm** with respect to a single map $i : \mathcal{F}^{\otimes}(\underline{1}) \longrightarrow \mathfrak{Fr}^{cc}(\underline{1})$. The existence of this left Bousfield localization and the characterization of cofibrations and weak-equivalences follows from [Shab, Thm. 3.4] and B.2. ■

3.29. NOTATION. *We denote the above model category by* $\mathbf{Perm}^{cc}$ *and refer to it as the model category of compact closed categories (or as the model category of permutative compact closed categories).*

4. Coherently compact closed categories

In this section we will construct another model category structure on the category of Γ-categories $\Gamma\mathbf{Cat} = [\Gamma^{op}, \mathbf{Cat}]$. The main result of this section is that the thickened Segal's nerve functor $\overline{\mathcal{K}}$ is a right Quillen functor of a Quillen equivalence between the model category of coherently compact closed categories, which will be constructed in this section, and the model category of compact closed permutative categories $\mathbf{Perm}^{cc}$ constructed in the previous section. We construct the desired model category as a left Bousfield localization of the model category of coherently commutative monoidal categories constructed in [Sha20], which we denote by $\Gamma\mathbf{Cat}^{\otimes}$. We begin by briefly recalling that the thickened Segal's nerve functor $\overline{\mathcal{K}} : \mathbf{Perm} \longrightarrow \Gamma\mathbf{Cat}$ constructed in [Sha20]:

4.1. Definition. *For each $n \in \mathbb{N}$ we will now define a permutative groupoid $\overline{\mathcal{L}}(n)$. An object of this groupoid is a finite sequence $(f_1, f_2, \dots, f_r)$ consisting of based maps in Γ^{op} having domain n^+. A morphism $(f_1, f_2, \dots, f_r) \longrightarrow (g_1, g_2, \dots, g_k)$ is an isomorphism of finite sets*

$$F : Supp(f_1) \sqcup Supp(f_2) \sqcup \cdots \sqcup Supp(f_r) \xrightarrow{\cong} Supp(g_1) \sqcup Supp(g_2) \sqcup \cdots \sqcup Supp(g_k)$$

such that the following diagram commutes

$$\begin{array}{ccc} Supp(f_1) \sqcup \cdots \sqcup Supp(f_r) & \xrightarrow{F} & Supp(g_1) \sqcup \cdots \sqcup Supp(g_k) \\ & \searrow \quad \swarrow & \\ & \underline{n} & \end{array}$$

where the diagonal maps are uniquely determined by the inclusions of components of the coproducts into $\underline{n}$ and $Supp(f)$ denotes the support of the based map f.

The (thickened) Segal's nerve functor is now defined, in degree n as follows:

$$\overline{\mathcal{K}}(C)(n^+) := [\overline{\mathcal{L}}(n), C]^{str}_{\otimes},$$

where C is a permutative category. The functor $\overline{\mathcal{K}} : \mathbf{Perm} \longrightarrow \Gamma\mathbf{Cat}$ has a left adjoint, denoted $\overline{\mathcal{L}}$, see [Sha20, Sec. 6]. The thickened Segal's nerve of the free compact closed category on one generator $\mathfrak{Fr}^{cc}(\underline{1})$, denoted by $\overline{\mathcal{K}}(\mathfrak{Fr}^{cc}(\underline{1}))$, is equipped with an inclusion map

$$j : \Gamma^1 \longrightarrow \overline{\mathcal{K}}(\mathfrak{Fr}^{cc}(\underline{1})) \tag{7}$$

which is determined by the generator of $\mathfrak{Fr}^{cc}(\underline{1})$.

4.2. Definition. *A coherently commutative monoidal category X is called a* coherently compact closed *category if the symmetric monoidal category $X(1^+)$ is a (not necessarily permutative) compact closed category.*

4.3. Proposition. *The thickened Segal's nerve $\overline{\mathcal{K}}(C)$ of a compact closed permutative category C is a coherently compact closed category.*

Proof. The above proposition follows from [Sha20, Cor. 6.13] and the fact that any symmetric monoidal category which is equivalent to a compact closed category is itself compact closed.

∎

4.4. Definition. *We will refer to the coherently compact closed category $\overline{\mathcal{K}}(\mathfrak{Fr}^{cc}(\underline{1}))$ as the thickened Segal's nerve of the algebraic 1-Bordism category.*

4.5. Definition. *A $\{j\}$-local equivalence will be called a compact closed equivalence of Γ-categories.*

4.6. THEOREM. *There is a left-proper, combinatorial model category structure on the category* $\Gamma\mathbf{Cat}$ *wherein a map is a*

1. *cofibration if it is a strict cofibration of* Γ*-categories, namely a cofibration in the strict model category of* Γ*-categories.*
2. *weak-equivalence if it is a compact closed equivalence of* Γ*-categories.*
3. *a fibration if it has the right lifting property with respect to maps which are simultaneously cofibrations and weak-equivalences.*

A Γ*-category is fibrant in this model category structure if and only if it is coherently compact closed.*

PROOF. The model category structure is obtained by carrying out a left Bousfield localization of the natural model category structure on **Perm** with respect to $\{j\}$, this follows from [Bar07, Thm. 2.11]. The same theorem implies that the model category is combinatorial and left-proper. ∎

4.7. NOTATION. *We denote the above model category by* $\Gamma\mathbf{Cat}^{cc}$ *and refer to it as the model category of coherently compact closed categories.*

4.8. LEMMA. *The adjoint pair* $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ *is a Quillen pair between the model category* $\mathbf{Perm}^{cc}$ *and the model category* $\Gamma\mathbf{Cat}^{cc}$*.*

PROOF. We recall from above that the model category $\Gamma\mathbf{Cat}^{cc}$ is a left Bousfield localization of the model category of coherently commutative monoidal categories $\Gamma\mathbf{Cat}^{\otimes}$ therefore it has the same cofibrations as $\Gamma\mathbf{Cat}^{\otimes}$, namely Q-cofibrations. Since the adjoint pair in context is a Quillen pair between **Perm** and $\Gamma\mathbf{Cat}^{\otimes}$ therefore the left adjoint $\overline{\mathcal{L}}$ preserves cofibrations between the two model categories in the context of the theorem. The fibrations between fibrant objects in $\Gamma\mathbf{Cat}^{cc}$ are strict fibrations of Γ-categories which are preserved by $\overline{\mathcal{K}}$. Now [Joy08, Prop. E.2.14] tells us that the adjoint pair $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ is a Quillen pair between $\mathbf{Perm}^{cc}$ and $\Gamma\mathbf{Cat}^{cc}$. ∎

4.9. REMARK. *An argument similar to the proof of the above lemma shows that the adjoint pair* $(\mathcal{L}, \mathcal{K})$ *defined in [Sha20, Sec. 5] is a also a Quillen pair.*

4.10. THEOREM. *A coherently commutative monoidal category* X *is a* $\{j\}$*-local object if and only if* $\overline{\mathcal{L}}(X)$ *is a compact closed permutative category.*

PROOF. Let us first assume that $\overline{\mathcal{L}}(X)$ is a compact closed category, then $\overline{\mathcal{K}\mathcal{L}}(X)$ is a $\{j\}$-local object because the adjunction $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ is a Quillen pair, by lemma (4.8), and a right Quillen functor preserves fibrant objects. Further, the unit map $\eta_X : X \longrightarrow \overline{\mathcal{K}\mathcal{L}}(X)$ is a strict equivalence of Γ-categories [Sha20, lem. 6.15]. This implies that X is a $\{j\}$-local object because $\overline{\mathcal{K}\mathcal{L}}(X)$ is one.

Conversely, let us assume that X is a $\{j\}$-local object. We consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}ap^h_{\Gamma\mathbf{Cat}^{cc}}(\overline{\mathcal{K}}(\mathfrak{Fr}^{cc}(\underline{1})), \overline{\mathcal{K}\mathcal{L}}(X)) & \xrightarrow{\mathcal{M}ap^h_{\Gamma\mathbf{Cat}^{cc}}(j, \overline{\mathcal{K}\mathcal{L}}(X))} & \mathcal{M}ap^h_{\Gamma\mathbf{Cat}^{cc}}(\Gamma^1, \overline{\mathcal{K}\mathcal{L}}(X)) \\ \cong \Big\downarrow & & \Big\downarrow \cong \\ \mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(\overline{\mathcal{L}}(\overline{\mathcal{K}}(\mathfrak{Fr}^{cc}(\underline{1}))), \overline{\mathcal{L}}(X)) & \xrightarrow[K]{} & \mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(\overline{\mathcal{L}}(\Gamma^1), \overline{\mathcal{L}}(X)) \end{array}$$

where the bottom horizontal map $K = \mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(\overline{\mathcal{L}}(j), \overline{\mathcal{L}}(X))$. Since X is a $\{j\}$-local object by assumption, the Γ-category $\overline{\mathcal{K}\mathcal{L}}(X)$ is also one. Thus the top row is a homotopy equivalence of Kan complexes. By the two out of three property of weak-equivalences in a model category, K is a homotopy equivalence of Kan-complexes. Now the following commutative diagram implies that $\overline{\mathcal{L}}(X)$ is a compact closed category:

$$\begin{array}{ccc} \mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(\overline{\mathcal{L}}(\overline{\mathcal{K}}\mathfrak{Fr}^{cc}(\underline{1})), \overline{\mathcal{L}}(X)) & \xrightarrow{\mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(\overline{\mathcal{L}}(j), \overline{\mathcal{L}}(X))} & \mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(\overline{\mathcal{L}}(\Gamma^1), \overline{\mathcal{L}}(X)) \\ \mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(\epsilon, \overline{\mathcal{L}}(X)) \Big\uparrow & & \Big\downarrow T \\ \mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(\mathfrak{Fr}^{cc}(\underline{1}), \overline{\mathcal{L}}(X)) & \longrightarrow & \mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(\mathcal{F}(\underline{1}), \overline{\mathcal{L}}(X)) \end{array}$$

where ϵ is the counit map which is a weak equivalence in the natural model category **Perm**. The downward map $T = \mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(\iota, \overline{\mathcal{L}}(X))$ is a homotopy equivalence of Kan complexes because the inclusion functor $\iota : \mathcal{F}(\underline{1}) \longrightarrow \overline{\mathcal{L}}(\Gamma^1)$ is an equivalence of categories.

■

4.11. Corollary. *An Γ-category X is a fibrant object of the model category $\Gamma\mathbf{Cat}^{cc}$ if and only if it is a coherently compact closed Γ-category.*

Proof. We begin by recalling that for each coherently commutative monoidal category Z, its degree one category $Z(1^+)$ inherits a symmetric monoidal category structure [**?**, Prop. 3.3.1.]. A Γ-category X is fibrant in the model category $\Gamma\mathbf{Cat}^{cc}$ if and only if it is a coherently commutative monoidal category and a $\{j\}$-local object. In this case, the unit map $\eta_X : X \longrightarrow \overline{\mathcal{K}}(\overline{\mathcal{L}}(X))$ is a strict equivalence of Γ-categories. Now we have we have the following commutative diagram of equivalence of categories:

$$\begin{array}{ccc} X(1^+) & \xrightarrow{\eta_X(1^+)} & \overline{\mathcal{K}}(\overline{\mathcal{L}}(X))(1^+) \\ & {}_{i_X}\searrow & \Big\downarrow ev \\ & & \overline{\mathcal{L}}(X) \end{array}$$

This diagram implies that $X(1^+)$ is compact closed because it is equivalent to a compact closed category $\overline{\mathcal{L}}(X)$ via the (symmetric monoidal) functor i_X.

■

4.12. Remark. *The model category of coherently commutative monoidal categories* $\Gamma\mathbf{Cat}^{\otimes}$ *is a* $\mathbf{Cat}$*-model category. This implies that*

$$\mathcal{M}ap^h_{\Gamma\mathbf{Cat}^{\otimes}}(j, X) = J(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(j, X)).$$

Now it is easy to see that following statements are equivalent:

1. *The inclusion map* $j : \Gamma^1 \longrightarrow \overline{\mathcal{K}}(\mathfrak{Fr}^{cc}(\underline{1}))$ *is a weak-equivalence in* $\Gamma\mathbf{Cat}^{cc}$*.*
2. *For any coherently compact closed category* X*, the following map is an equivalence of function spaces:*

$$J\mathcal{M}ap_{\Gamma\mathbf{Cat}}(j, X) : J\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\overline{\mathcal{K}}(Cob1), X) \longrightarrow J(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, X)) \xrightarrow{\cong} JN(X(1^+)).$$

4.13. Proposition. *A morphism of* Γ*-categories* $F : X \longrightarrow Y$ *is a compact closed equivalence of* Γ*-categories if and only if for each compact closed (permutative) category* Z *we have the following homotopy equivalence of function complexes:*

$$\mathcal{M}ap^h_{\Gamma\mathbf{Cat}^{\otimes}}(F, \overline{\mathcal{K}}(Z)) : \mathcal{M}ap^h_{\Gamma\mathbf{Cat}^{\otimes}}(Y, \overline{\mathcal{K}}(Z)) \longrightarrow \mathcal{M}ap^h_{\Gamma\mathbf{Cat}^{\otimes}}(X, \overline{\mathcal{K}}(Z))$$

Proof. F is a compact closed equivalence of Γ-categories if and only if for each coherently compact closed category W, the following map is a homotopy equivalence of Kan complexes:

$$\mathcal{M}ap^h_{\Gamma\mathbf{Cat}^{\otimes}}(F, W) : \mathcal{M}ap^h_{\Gamma\mathbf{Cat}^{\otimes}}(Y, W) \longrightarrow \mathcal{M}ap^h_{\Gamma\mathbf{Cat}^{\otimes}}(X, W)$$

Since $\overline{\mathcal{K}}(Z)$ is a coherently compact closed category, by the above corollary, one direction of the statement is obvious.

The other direction of the statement follows from the observation that for each coherently compact closed category W, the unit map $\eta_W : W \longrightarrow \overline{\mathcal{K}}\overline{\mathcal{L}}(W)$ is a strict equivalence of Γ-categories. ∎

It was shown in [Sha20, Thm. 6.18] that the adjoint pair $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ is a Quillen equivalence between the natural model category of permutative categories **Perm** and the model category of coherently commutative monoidal categories $\Gamma\mathbf{Cat}^{cc}$. This Quillen equivalence is a strict equivalence of the underlying homotopy theories, namely both functors preserve and reflect weak-equivalences of the model categories in context and the unit and the counit maps are natural weak-equivalences.

4.14. Lemma. *The Segal's nerve functor* $\mathcal{K}$ *preserves and reflects compact closed equivalences of permutative categories.*

Proof. Let $F : C \longrightarrow D$ be a compact closed equivalence of permutative categories. It follows from [Sha20, Thm. 6.18], [Sha20, Thm. 6.17] and the observation that each object of the natural model category **Perm** is fibrant, that for each permutative category C, the counit map $\epsilon_C : \overline{\mathcal{L}}\overline{\mathcal{K}}(C) \longrightarrow C$ is a weak-equivalence in the natural model category **Perm** *i.e.* the underlying functor of ϵ_C is an equivalence of categories. We consider the following diagram of function complexes for each compact closed permutative category Z:

$$\begin{array}{ccc}
\mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(D,Z) & \xrightarrow{\mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(F,Z)} & \mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(C,Z) \\
{\scriptstyle \mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(\epsilon_D,Z)}\downarrow & & \downarrow{\scriptstyle \mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(\epsilon_C,Z)} \\
\mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(\overline{\mathcal{L}}\overline{\mathcal{K}}(D),Z) & \longrightarrow & \mathcal{M}ap^h_{\mathbf{Perm}^{cc}}(\overline{\mathcal{L}}\overline{\mathcal{K}}(C),Z) \\
\cong\downarrow & & \downarrow\cong \\
\mathcal{M}ap^h_{\Gamma\mathbf{Cat}^{cc}}(\overline{\mathcal{K}}(D),\overline{\mathcal{K}}(Z)) & \xrightarrow[\mathcal{M}ap^h_{\Gamma\mathbf{Cat}^{cc}}(\overline{\mathcal{K}}(F),\overline{\mathcal{K}}(Z))]{} & \mathcal{M}ap^h_{\Gamma\mathbf{Cat}^{cc}}(\overline{\mathcal{K}}(C),\overline{\mathcal{K}}(Z))
\end{array}$$

The two vertical isomorphisms follow from [Hir02, Prop. 17.4.16] applied to the Quillen pair from lemma 4.8. It follows from [Hir02, Thm. 17.7.7] that the top horizontal arrow and the upper two verical arrows are homotopy equivalences of simplicial sets. Now the two out of three property of weak equivalences in a model category implies that the lower horizontal map, namely $\mathcal{M}ap^h_{\Gamma\mathbf{Cat}^{cc}}(\overline{\mathcal{K}}(F),\overline{\mathcal{K}}(Z))$ is a homotopy equivalence of simplicial sets. Now proposition 4.13 and [Hir02, Thm. 17.7.7] together imply that $\overline{\mathcal{K}}(F)$ is a weak-equivalence in $\Gamma\mathbf{Cat}^{cc}$.

Conversely, let us assume that $\overline{\mathcal{K}}(F)$ is a weak-equivalence in $\Gamma\mathbf{Cat}^{cc}$. Now the bottom horizontal arrow in the above diagram is a homotopy equivalence of simplicial sets and therefore the top horizontal arrow is one too. Now proposition 4.13 and [Hir02, Thm. 17.7.7] together imply that F is a weak-equivalence in $\mathbf{Perm}^{cc}$. ∎

The following corollary is an easy consequence of the above lemma:

4.15. Corollary. *The left Quillen functor* $\overline{\mathcal{L}}$ *preserves and reflects compact closed equivalences of* Γ*-categories.*

Now we will state and prove the main result of this paper:

4.16. Theorem. *The adjoint pair* $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ *is a Quillen equivalence between the natural model category of compact closed permutative categories* $\mathbf{Perm}^{cc}$ *and the model category of coherently commutative monoidal categories* $\Gamma\mathbf{Cat}^{cc}$*.*

Proof. Let X be a cofibrant object in $\Gamma\mathbf{Cat}^{\otimes}$ and C be a fibrant object of $\mathbf{Perm}^{cc}$. We will show that a map $F : \overline{\mathcal{L}}(X) \longrightarrow C$ in $\mathbf{Perm}^{cc}$ is a weak equivalence if and only if it's adjunct map $\phi(F) : X \longrightarrow \overline{\mathcal{K}}(C)$ is a weak-equivalence in $\Gamma\mathbf{Cat}^{cc}$.

We first recall that the Quillen pair $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ is a Quillen equivalence between the natural model category **Perm** and the model category of coherently commutative monoidal categories $\Gamma\mathbf{Cat}^{\otimes}$ [Sha20, Thm. 6.18]. We further recall that every object in the natural model category **Perm** is fibrant. Now it follows from [Hov99, Prop. 1.3.13.] that for each Q-cofibrant Γ-category X, the unit map of the adjunction $\eta_X : X \longrightarrow \overline{\mathcal{K}\mathcal{L}}(X)$ is a weak-equivalence in $\Gamma\mathbf{Cat}^{\otimes}$ and therefore a weak-equivalence in $\Gamma\mathbf{Cat}^{cc}$.

Now the result follows from proposition 4.14 and the following commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{K}\mathcal{L}}(X) & \xrightarrow{\overline{\mathcal{K}}(F)} & \overline{\mathcal{K}}(C) \\ {\scriptstyle \eta(X)}\uparrow & \nearrow_{\phi(F)} & \\ X & & \end{array}$$

■

The following corollary follows from the above theorem, remark 4.9 and the natural weak-equivalence $\mathcal{K} \Rightarrow \overline{\mathcal{K}}$ constructed in [Sha20, Cor. 6.19]:

4.17. Corollary. *The adjoint pair $(\mathcal{L}, \mathcal{K})$ is a Quillen equivalence between the natural model category of compact closed permutative categories $\mathbf{Perm}^{cc}$ and the model category of coherently commutative monoidal categories $\Gamma\mathbf{Cat}^{cc}$.*

The main result is stronger than what is stated in Theorem 4.16:

4.18. Remark. *The Quillen pair $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ induces a strict equivalence of the underlying homotopy theories on the two model categories in context. More precisely, both functors preserve weak-equivalences and the unit and counit maps are natural weak-equivalences.*

A. Aspects of Duality
by André Joyal

The results of this appendix are folklore and where possible, we will provide a reference.

A.1. On certain monoidal transformations. Let C be a symmetric monoidal category. If c is a dualisable object in C, with dual object $c^\bullet$, let us denote by $\eta_c : 1_C \longrightarrow c \otimes c^\bullet$ and $\epsilon_c : c^\bullet \otimes c \longrightarrow 1_C$ the unit and counit of the duality. If d is another dualisable object, then

A.2. Definition. *The transpose $f^\dagger : d^\bullet \longrightarrow c^\bullet$ of a morphism $f : c \longrightarrow d$ is defined to be the composite*

$$d^\bullet \xrightarrow{d^\bullet \otimes \eta_c} d^\bullet \otimes c \otimes c^\bullet \xrightarrow{c^\bullet \otimes f \otimes c^\bullet} d^\bullet \otimes d \otimes c^\bullet \xrightarrow{\epsilon_d \otimes c^\bullet} c^\bullet$$

A.3. Definition. *A morphism $g : c^\bullet \longrightarrow d^\bullet$ in a symmetric monoidal category C is a dual of the morphism $f : c \longrightarrow d$, between dualizable objects in C, if the following two diagrams commutes*

$$\begin{array}{ccc} c^\bullet \otimes c & \xrightarrow{\epsilon_c} & 1_C \\ {\scriptstyle g\otimes f}\downarrow & & \| \\ d^\bullet \otimes d & \xrightarrow[\epsilon_d]{} & 1_C \end{array} \qquad \begin{array}{ccc} 1_C & \xrightarrow{\eta_c} & c \otimes c^\bullet \\ \| & & \downarrow{\scriptstyle f\otimes g} \\ 1_C & \xrightarrow[\eta_d]{} & d \otimes d^\bullet \end{array}$$

The following proposition shows that it is sufficient to have the above two commutative diagrams in order to establish a duality between f and g in the symmetric monoidal category C^I:

A.4. PROPOSITION. *A dual of a morphism f in a symmetric monoidal category C in the sense of definition A.3 is a dual of f when regarded as an object of the symmetric monoidal category C^I.*

PROOF. Let $g : c^\bullet \longrightarrow d^\bullet$ be a dual of a morphism $f : c \longrightarrow d$ in the symmetric monoidal category C in the sense of definition A.3. We observe that the two commutative diagrams in definition A.3 above are maps in the (symmetric monoidal) morphism category C^I. We claim that these two maps are the counit and unit maps which establish a duality between f and g. The commutativity of the two diagrams in definition A.3 implies that the following two diagrams, wherein the horizontal composite maps are identities, are commutative:

$$\begin{array}{ccccc} c & \xrightarrow{\eta_c \otimes c} & c \otimes c^\bullet \otimes c & \xrightarrow{c \otimes \epsilon_c} & c \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes g \otimes f} & & \downarrow{\scriptstyle f} \\ d & \xrightarrow[\eta_d \otimes d]{} & d \otimes d^\bullet \otimes c & \xrightarrow[d \otimes \epsilon_d]{} & d \end{array} \qquad \begin{array}{ccccc} c^\bullet & \xrightarrow{c^\bullet \otimes \eta_c} & c^\bullet \otimes c \otimes c^\bullet & \xrightarrow{\epsilon_c \otimes c^\bullet} & c \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g \otimes f \otimes g} & & \downarrow{\scriptstyle g} \\ d & \xrightarrow[d^\bullet \otimes \eta_d]{} & d \otimes d^\bullet \otimes c & \xrightarrow[\epsilon_d \otimes d^\bullet]{} & d \end{array}$$

These two commutative diagrams establish the desired duality in C^I. ■

A.5. LEMMA. *If a morphism $g : c^\bullet \longrightarrow d^\bullet$ is a dual of a morphism $f : c \longrightarrow d$, then $g \circ f^\dagger = 1_{d^\bullet}$ and $f^\dagger \circ g = 1_{c^\bullet}$. Hence the morphisms f and g are invertible.*

PROOF. Let us compute $g \circ f^\dagger$. The following diagram commutes by naturality:

$$\begin{array}{ccccccc} d^\bullet & \xrightarrow{d^\bullet \otimes \eta_c} & d^\bullet \otimes c \otimes c^\bullet & \xrightarrow{d^\bullet \otimes f \otimes c^\bullet} & d^\bullet \otimes d \otimes c^\bullet & \xrightarrow{\epsilon_d \otimes c^\bullet} & c^\bullet \\ & & \downarrow{\scriptstyle d^\bullet \otimes c \otimes g} & & \downarrow{\scriptstyle d^\bullet \otimes d \otimes g} & & \downarrow{\scriptstyle g} \\ & & d^\bullet \otimes c \otimes d^\bullet & \xrightarrow[d^\bullet \otimes f \otimes d^\bullet]{} & d^\bullet \otimes d \otimes d^\bullet & \xrightarrow[\epsilon_d \otimes d^\bullet]{} & d^\bullet \end{array}$$

It follows that the morphism $g \circ f^\dagger$ is the composite

$$d^\bullet \xrightarrow{d^\bullet \otimes \eta_c} d^\bullet \otimes c \otimes c^\bullet \xrightarrow{d^\bullet \otimes f \otimes g} d^\bullet \otimes d \otimes c^\bullet \xrightarrow{\epsilon_d \otimes d^\bullet} d^\bullet$$

But we have $(f \otimes g)\eta_c = \eta_d$, since g is a dual of f. Hence the morphism $g \circ f^\dagger$ is the composite

$$d^\bullet \xrightarrow{d^\bullet \otimes \eta_d} d^\bullet \otimes d \otimes d^\bullet \xrightarrow{\epsilon_d \otimes d^\bullet} d^\bullet$$

But $(\epsilon_d \otimes d^\bullet)(d^\bullet \otimes \eta_d) = 1_{d^\bullet}$ by the duality between d and $d^\bullet$. This shows that $g \circ f^\dagger = 1_{d^\bullet}$. The proof that $f^\dagger \circ g = 1_{c^\bullet}$ is similar. ■

A.6. Lemma. *Let $\alpha : F \Rightarrow G$ be a monoidal natural transformation between symmetric monoidal functors $F, G : C \longrightarrow D$ between symmetric monoidal categories $C = (C, \otimes, 1_C)$ and $D = (D, \otimes, 1_D)$. If the symmetric monoidal category C is compact closed, then α is invertible.*

Proof. Let us show that the map $\alpha_c : F(c) \longrightarrow G(c)$ is invertible for every object $c \in C$. The object c has a dual $c^{\bullet}$, since the category C is compact closed. Let $\eta_c : 1_C \longrightarrow c \otimes c^{\bullet}$ and $\epsilon_c : c^{\bullet} \otimes c \longrightarrow 1_C$ be the unit and counit of the duality. The object $F(c)$ has then a dual $F(c)^{\bullet} := F(c^{\bullet})$. The unit $\eta_{F(c)} : 1_D \longrightarrow F(c) \otimes F(c^{\bullet})$ is defined to be the composite

$$1_D \xrightarrow{\simeq} F(1_C) \xrightarrow{F(\eta_c)} F(c \otimes c^{\bullet}) \xrightarrow{\simeq} F(c) \otimes F(c^{\bullet})$$

and the counit $\epsilon_{F(c)} : F(c^{\bullet}) \otimes F(c) \longrightarrow 1_D$ is defined to be the composite

$$F(c^{\bullet}) \otimes F(c) \xrightarrow{\simeq} F(c^{\bullet} \otimes c) \xrightarrow{F(\epsilon_c)} F(1_C) \xrightarrow{\simeq} 1_D$$

Similarly, the object $G(c)$ has a dual $G(c)^{\bullet} := G(c^{\bullet})$. The unit $\eta_{G(c)} : 1_D \longrightarrow G(c) \otimes G(c^{\bullet})$ is defined to be the composite

$$1_D \xrightarrow{\simeq} G(1_C) \xrightarrow{G(\eta_c)} G(C \otimes c^{\bullet}) \xrightarrow{\simeq} G(c) \otimes G(c^{\bullet})$$

and the counit $\epsilon_{G(c)} : G(c^{\bullet}) \otimes G(c) \longrightarrow 1_D$ is defined to be the composite

$$G(c^{\bullet}) \otimes G(c) \xrightarrow{\simeq} G(c^{\bullet} \otimes c) \xrightarrow{G(\epsilon_c)} G(1_C) \xrightarrow{\simeq} 1_D$$

Let us show that the morphism $\alpha_{c^{\bullet}} : F(c^{\bullet}) \longrightarrow G(c^{\bullet})$ is a dual of the morphism $\alpha_c : F(c) \longrightarrow G(c)$. The following diagram commutes, since the natural transformation $\alpha : F \Rightarrow G$ is monoidal

$$\begin{array}{ccccccc}
1_D & \xrightarrow{\simeq} & F(1_C) & \xrightarrow{F(\eta_c)} & F(c \otimes c^{\bullet}) & \xrightarrow{\simeq} & F(c) \otimes F(c^{\bullet}) \\
\Vert & & \downarrow{\scriptstyle \alpha_I} & & \downarrow{\scriptstyle \alpha_{c \otimes c^{\bullet}}} & & \downarrow{\scriptstyle \alpha_c \otimes \alpha_{c^{\bullet}}} \\
1_D & \xrightarrow[\simeq]{} & G(1_C) & \xrightarrow[G(\eta_c)]{} & G(c \otimes c^{\bullet}) & \xrightarrow[\simeq]{} & G(c) \otimes G(c^{\bullet})
\end{array}$$

Hence the following square commutes

$$\begin{array}{ccc}
1_D & \xrightarrow{\eta_{F(c)}} & F(c) \otimes F(c^{\bullet}) \\
\Vert & & \downarrow{\scriptstyle \alpha_C \otimes \alpha_{c^{\bullet}}} \\
1_D & \xrightarrow[\eta_{G(d)}]{} & G(C) \otimes G(d^{\bullet})
\end{array}$$

Similarly, the following square commutes

$$\begin{array}{ccc} F(c^\bullet)\otimes F(c) & \xrightarrow{\epsilon_{F(c)}} & 1_D \\ {\scriptstyle \alpha_{c^\bullet}\otimes\alpha_c}\downarrow & & \| \\ G(d^\bullet)\otimes G(d) & \xrightarrow[\epsilon_{G(d)}]{} & 1_D \end{array}$$

This shows that the morphism $\alpha_{c^\bullet} : F(c^\bullet)\longrightarrow G(c^\bullet)$ is a dual of the morphism $\alpha_c : F(c)\longrightarrow G(c)$. It then follows by Lemma A.5 that α_c is invertible. ∎

If C and D are symmetric monoidal categories, let us denote by $[C, D]_\otimes$ the category of symmetric monoidal functors from C to D and monoidal natural transformations between them. The category $[C, D]_\otimes$ is symmetric monoidal.

A different proof of the following proposition appears in [DM18, Prop. 1.13]

A.7. Proposition. *If C is a compact closed symmetric monoidal category, then the symmetric monoidal category $[C, D]_\otimes$ is a groupoid for every symmetric monoidal category D.*

Proof. Let $\alpha : F \Rightarrow G$ be a morphism in the category $[C, D]_\otimes$. The monoidal natural transformation α is invertible by Lemma A.6. Its inverse $\alpha^{-1} : G \Rightarrow F$ is monoidal (by a general result). Thus, $[C, D]_\otimes$ is a groupoid. ∎

A.8. On the compact closed symmetric monoidal category free on one generator . Let me denote by $\mathcal{B}$ the compact closed symmetric monoidal category freely generated by one object $U \in \mathcal{B}$. By definition, for every compact closed symmetric monoidal category C and every object $c \in C$ there exists a symmetric monoidal functor $F : \mathcal{B}\longrightarrow C$ such that $F(U) = c$, and the functor F is unique up to unique isomorphism: if $G : \mathcal{B}\longrightarrow C$ is another functor such that $G(U) = c$, then there exists a unique monoidal natural isomorphism $\alpha : F \longrightarrow G$ such that $\alpha_U = id_c$.

If C is a category, then the subcategory of invertible morphisms of C is a groupoid called the *core* of C. I will denote the core of C by C^{cor} or by $J(C)$. The core of a symmetric monoidal category C is a symmetric monoidal subcategory of C.

I will use the following construction in the proof of the next propsition. Let me denote by J the groupoid freely generated by one isomorphism $i : 0\longrightarrow 1$. If C is a category then an object of the category C^J is an isomorphism f in C. The source and target functors $s, t : C^J \longrightarrow C$ are connected by a natural isomorphism $h : s \longrightarrow t$ defined by putting $h(f) = f : s(f) \longrightarrow t(f)$. The category C^J is symmetric monoidal if C is symmetric monoidal. Moreover, the source and target functors $s, t : C^J \longrightarrow C$ and the natural transformation $h : s \longrightarrow t$ are symmetric monoidal. The category C^J is compact closed if C is compact closed, since the functor $s : C^J \longrightarrow C$ is an equivalence of symmetric monoidal categories.

A.9. Proposition. *Let $\mathcal{B}$ the compact closed symmetric monoidal category freely generated by one object $U \in \mathcal{B}$. If C is a compact closed symmetric monoidal category, then the evaluation functor*

$$e_U : [\mathcal{B}, C]_\otimes \longrightarrow C$$

defined by putting $ev_U(F) := F(U)$ takes its values in the core of C. Moreover, the induced functor

$$e'_U : [\mathcal{B}, C]_\otimes \longrightarrow C^{cor}$$

is an equivalence of symmetric monoidal categories.

Proof. The category $[\mathcal{B}, C]_\otimes$ is a groupoid by Proposition A.7. Hence the functor e_U takes its values in the core of C. Let us show that the induced functor e'_U is an equivalence of categories. For every object $c \in C$ there exists a symmetric monoidal functor $F : \mathcal{B} \longrightarrow C$ such that $F(U) = c$, since C is compact closed and $\mathcal{B}$ is compact closed and freely generated by the object $U \in \mathcal{B}$. We then have $e'_U(F) := e_U(F) := F(U) = c$. We have proved that the functor e'_U is surjective on objects. Let us show that the functor e'_U is fully faithful. If $F, G : \mathcal{B} \longrightarrow C$ are symmetric monoidal functors, let us show that for every isomorphism $f : F(U) \longrightarrow G(U)$ there exists a unique monoidal natural isomorphism $\alpha : F \Rightarrow G$ such that $\alpha_U = f$. We shall first prove the existence of α. The symmetric monoidal category C^J is compact closed, since the symmetric monoidal category C is compact closed by hypothesis. The isomorphism f is an object in C^J. By the freeness of $\mathcal{B}$, there exists a symmetric monoidal functor

$$H : \mathcal{B} \longrightarrow C^J$$

such that $H(U) = f$. We have $sH(U) = s(f) = F(U)$, since $f : F(U) \longrightarrow G(U)$. The functor $sH : \mathcal{B} \longrightarrow C$ is symmetric monoidal, since the functors H and s are. Thus, there exists a unique monoidal natural isomorphism $\rho : F \longrightarrow sH$ such that $\rho_U = id_{F(U)}$. Similarly, if $t : C^J \longrightarrow C$ is the target functor, then $tH(U) = t(f) = G(U)$. Thus, there exists a unique monoidal natural isomorphism $\lambda : tH \longrightarrow G$ such that $\lambda_U = id_{G(U)}$. If $h : s \longrightarrow t$ is the canonical isomorphism, then the composite $\alpha := \lambda h \rho$ is a monoidal natural isomorphism $\alpha : F \longrightarrow G$

$$F \xrightarrow{\rho} sH \xrightarrow{h \circ H} tH \xrightarrow{\lambda} G$$

We have $\alpha_U = f$, since $\rho_U = 1_{F(U)}$, $(h \circ H)_U = h(H(U)) = h(f) = f$ and $\lambda_U = 1_{G(U)}$. The existence of $\alpha : F \longrightarrow G$ is proved. Let us show that α is unique. Let $\beta : F \longrightarrow G$ a monoidal natural isomorphism such that $\beta_U = f$. Then $\gamma := \beta^{-1}\alpha : F \longrightarrow F$ is a monoidal natural isomorphism such that $\gamma_U = 1_U$. It follows that $\gamma = 1_F$, since $\mathcal{B}$ is freely generated by the object $U \in \mathcal{B}$. We have proved that the functor $e'_U : [\mathcal{B}, C]_\otimes \longrightarrow C^{cor}$ is fully faithful. It is thus an equivalence of categories, since it is surjective on objects. It is also an equivalence of symmetric monoidal categories, since it is a symmetric monoidal functor. ∎

The above proposition has the following version for strict version which can be proved by closely adapting the argument in the proof of the above proposition:

A.10. COROLLARY. *Let $\mathcal{B}_{str}$ be the compact closed permutative category freely generated by one object $U \in \mathcal{B}_{str}$. If C is a compact closed permutative category, then the evaluation functor*

$$e_U : [\mathcal{B}_{str}, C]^{str}_{\otimes} \longrightarrow C$$

defined by putting $ev_U(F) := F(U)$ takes its values in the core of C. Moreover, the induced functor

$$e'_U : [\mathcal{B}_{str}, C]^{str}_{\otimes} \longrightarrow \mathcal{C}^{cor}$$

is an equivalence of categories, where $[\mathcal{B}_{str}, C]^{str}_{\otimes}$ denotes the category of strict symmetric monoidal functors from $\mathcal{B}_{str}$ to C and monoidal natural transformations between them.

B. Localization of model categories

In this appendix we recall the notion of a *left Bousfield localization* of a model category and also recall an existence result of the same.

B.1. DEFINITION. *Let $\mathcal{M}$ be a model category and let $\mathcal{S}$ be a class of maps in $\mathcal{M}$. The left Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{S}$ is a model category structure $L_{\mathcal{S}}\mathcal{M}$ on the underlying category of $\mathcal{M}$ such that*

1. *The class of cofibrations of $L_{\mathcal{S}}\mathcal{M}$ is the same as the class of cofibrations of $\mathcal{M}$.*

2. *A map $f : A \longrightarrow B$ is a weak equivalence in $L_{\mathcal{S}}\mathcal{M}$ if it is an $\mathcal{S}$-local equivalence, namely, for every fibrant $\mathcal{S}$-local object X, the induced map on homotopy function complexes*

$$f^* : Map^h_{\mathcal{M}}(B, X) \longrightarrow Map^h_{\mathcal{M}}(A, X)$$

is a weak homotopy equivalence of simplicial sets. Recall that an object X is called fibrant $\mathcal{S}$-local if X is fibrant in $\mathcal{M}$ and for every element $g : K \longrightarrow L$ of the set $\mathcal{S}$, the induced map on homotopy function complexes

$$g^* : Map^h_{\mathcal{M}}(L, X) \longrightarrow Map^h_{\mathcal{M}}(K, X)$$

is a weak homotopy equivalence of simplicial sets.

We recall the following theorem which will be the main tool in the construction of the desired model category. This theorem first appeared in an unpublished work [Smi] but a proof was later provided by Barwick in [Bar07].

B.2. Theorem. *[Bar07, Theorem 2.11] If $\mathcal{M}$ is a left-proper, combinatorial model category and $\mathcal{S}$ is a small set of homotopy classes of morphisms of $\mathcal{M}$, the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ of $\mathcal{M}$ along any set representing $\mathcal{S}$ exists and satisfies the following conditions.*

1. *The model category $L_{\mathcal{S}}\mathcal{M}$ is left proper and combinatorial.*
2. *As a category, $L_{\mathcal{S}}\mathcal{M}$ is simply $\mathcal{M}$.*
3. *The cofibrations of $L_{\mathcal{S}}\mathcal{M}$ are exactly those of $\mathcal{M}$.*
4. *The fibrant objects of $L_{\mathcal{S}}\mathcal{M}$ are the fibrant $\mathcal{S}$-local objects Z of $\mathcal{M}$.*
5. *The weak equivalences of $L_{\mathcal{S}}\mathcal{M}$ are the $\mathcal{S}$-local equivalences.*

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