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LADDERS AND COMPLETION OF TRIANGULATED CATEGORIES

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ABSTRACT. We provide a method to construct recollements and ladders of triangulated categories. For a ladder of height $n(n \geq 2)$ of triangulated categories with good metrics, if all the functors are compression functors, then there is a ladder of height $n-1$ of the corresponding completion categories. In particular, for a recollement(a ladder of height 1) of triangulated categories with good metrics, if all the functors are compression functors, then there is a half recollement of the corresponding completion categories.

1. Introduction

Recollements of triangulated categories were introduced by Beilinson, Bernstein and Deligne in their fundamental work on perverse sheaves [1]. Ladder, introduced by Beilinson, Ginzburg and Schechtman (see [2]), is an extended concept of recollement. Roughly, a ladder of triangulated categories is a collection of recollements of triangulated categories. Nowadays, they both play important roles in studying the representation theory of algebras [5, 7, 8]. However, a question remains: How to construct recollements or ladders? We observe that triangulated categories are building blocks of recollements and ladders. So, it seems reasonable to apply the method of constructing triangulated categories to construct recollements or ladders. In the literature, there are many ways to construct triangulated categories. The ways of taking the stable category of a Frobenius category [6, ChapterI] and a Verdier quotient of a triangulated category over a subcategory [15] are well known. Also, there are construction methods provided by B. Keller [9] and P. Balmer [3]. In the recent papers of H. Krause [11] and A. Neeman [12, 13], they construct new triangulated categories by taking Cauchy completion and completion with respect to a good metric, respectively. In this paper, inspired by Neeman's method of constructing triangulated categories, we want to know the answer to the question: Can Neeman's method be applied to construct recollements or ladders of triangulated categories?

We are going to give a positive answer to the above question. Let's first note some observations in additive categories. Suppose that $F : \mathcal{A} \to \mathcal{B}$ is an additive functor of additive categories, then it can induce an exact additive functor $\widehat{F} : \mathsf{Mod}\text{–}\mathcal{B} \to \mathsf{Mod}\text{–}\mathcal{A}$ easily by taking B to $B \circ F$. The operation shares good properties: suppose that $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$ is an adjoint pair. Then (1) $\widehat{F} : \mathsf{Mod}\text{–}\mathcal{B} \rightleftarrows \mathsf{Mod}\text{–}\mathcal{A} : \widehat{G}$ is an adjoint pair, (2) if G is fully-faithful, then so is $\widehat{F}$ (see Corollary 3.2).

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Now, we focus on triangulated categories with good metrics. Let $\mathcal{T}$ be a triangulated category with a *good metric* ([13, Definition 10]), Neeman considers three full subcategories $\mathfrak{L}(\mathcal{T}), \mathfrak{C}(\mathcal{T})$ and $\mathfrak{S}(\mathcal{T})$ of the category Mod- $\mathcal{T}$ (see Section 2 for detailed definitions). Of these the category $\mathfrak{S}(\mathcal{T})$ is triangulated and called the *completion category* of $\mathcal{T}$, in such a way that the inclusion $\mathfrak{S}(\mathcal{T}) \longrightarrow \mathsf{Mod}\text{–}\mathcal{T}$ commutes with the suspension. Let $F : \mathcal{T} \to \mathcal{S}$ be a triangular functor. Then there is the following diagram

$$\begin{tikzcd}
\mathfrak{S}(\mathcal{S}) \arrow[d, hook] & \mathfrak{S}(\mathcal{T}) \arrow[d, hook] \\
\mathsf{Mod}\text{–}\mathcal{S} \arrow[r, "\widehat{F}"] & \mathsf{Mod}\text{–}\mathcal{T}
\end{tikzcd}$$

It is natural to ask: (1) whether $\widehat{F}$ takes $\mathfrak{S}(\mathcal{S})$ to $\mathfrak{S}(\mathcal{T})$; (2) if (1) holds, whether $\widehat{F}$ is a triangular functor. Our first main theorem answer these questions.

1.1. Theorem. *Let $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$ be an adjoint pair of triangulated functors between triangulated categories. Suppose moreover that $\mathcal{S}$ and $\mathcal{T}$ both have good metrics, and that F and G are both compressions. Then the functor $\widehat{F} : \mathsf{Mod}\text{–}\mathcal{S} \longrightarrow \mathsf{Mod}\text{–}\mathcal{T}$ has the properties*

(1) *$\widehat{F}$ takes $\mathfrak{L}(\mathcal{S}) \subset \mathsf{Mod}\text{–}\mathcal{S}$ to $\mathfrak{L}(\mathcal{T}) \subset \mathsf{Mod}\text{–}\mathcal{T}$.*

(2) *$\widehat{F}$ takes $\mathfrak{C}(\mathcal{S}) \subset \mathsf{Mod}\text{–}\mathcal{S}$ to $\mathfrak{C}(\mathcal{T}) \subset \mathsf{Mod}\text{–}\mathcal{T}$.*

(3) *The restriction of $\widehat{F}$ to $\mathfrak{S}(\mathcal{S}) = \mathfrak{L}(\mathcal{S}) \cap \mathfrak{C}(\mathcal{S})$ induces a triangulated functor, which we will (by abuse of notation) also call $\widehat{F} : \mathfrak{S}(\mathcal{S}) \longrightarrow \mathfrak{S}(\mathcal{T})$.*

In the above theorem.if moreover G has a compression right adjoint, then we obtain an adjoint pair $\widehat{F} : \mathfrak{S}(\mathcal{S}) \rightleftarrows \mathfrak{S}(\mathcal{T}) : \widehat{G}$.

So much for preparations. Now, let us answer the question of constructing recollements and ladders.

1.2. Theorem. *Let $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{R}$ be triangulated categories with good metrics. Assume that the following diagram is a recollement of triangulated categories*

$$\begin{tikzcd}
\mathcal{R} \arrow[r, "i_* = i_!" description] & \mathcal{T} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^! = j^*" description] & \mathcal{S} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

If all the functors in the diagram are compressions, then there is a right recollement of the corresponding completion categories

$$\begin{tikzcd}
\mathfrak{S}(\mathcal{R}) \arrow[r, "\widehat{i^*}" description] & \mathfrak{S}(\mathcal{T}) \arrow[l, bend left, "\widehat{i_*}"] \arrow[r, "\widehat{j_!}" description] & \mathfrak{S}(\mathcal{S}) \arrow[l, bend left, "\widehat{j^!}"]
\end{tikzcd}.$$

Next, we generalize the above case to the version of ladders.

1.3. Corollary. *Let $\mathcal{L}$ be a ladder of height $n(n \geq 2)$ of triangulated categories with good metrics. If all the functors in $\mathcal{L}$ are compressions, then there is a ladder of height $n-1$ of the corresponding completion categories.*

The contents of this paper are organized as follows. In Section 2, we fix notation and recall some definitions and facts used in the paper.

In Section 3, we discuss the induced functors of additive functors of additive categories. In Section 4, we discuss the induced functors of compression functors of triangulated categories with good metrics and then prove our main results.

2. Preliminaries

In this section, we briefly recall some notations, definitions, and basic facts used in the paper.

Let $\mathcal{A}$ be an additive category. By Mod–$\mathcal{A}$ we denote the category of right $\mathcal{A}$-modules, i.e. the objects of Mod–$\mathcal{A}$ consist of all additive functors from $\mathcal{A}^{op}$ to the category of abelian groups. As we all know, Mod–$\mathcal{A}$ is a cocomplete abelian category (i.e. all small colimits exist), and $\mathcal{A}$ can be regarded as a full subcategory of Mod–$\mathcal{A}$ through the Yoneda functor $Y : \mathcal{A} \to \mathsf{Mod}\text{–}\mathcal{A}$, $a \mapsto \mathsf{Hom}(-, a)$. From this viewpoint, for a triangulated category $\mathcal{T}$ with shift functor Σ, Σ can be lifted to Mod–$\mathcal{T}$, that is

$$\begin{aligned} \Sigma : \mathsf{Mod}\text{–}\mathcal{T} &\to \mathsf{Mod}\text{–}\mathcal{T} \\ A &\mapsto (t \mapsto A(\Sigma^{-1}t)). \end{aligned}$$

Let $\mathcal{X}, \mathcal{Y}$ be full subcategories of $\mathcal{T}$. We define

$$\mathcal{X} * \mathcal{Y} := \{t \in \mathcal{T} \mid \exists \text{ a triangle } x \to t \to y \to \Sigma x \text{ with } x \in \mathcal{X}, y \in \mathcal{Y}\}.$$

Now, we recall some definitions and results on completion of a triangulated category.

2.1. Definition. ([13, Definition 10]) A *good metric* on $\mathcal{T}$ is a sequence of full subcategories $\{\mathcal{M}_i \subset \mathcal{T} \mid i \in \mathbb{N}\}$ such that

(1) Each $\mathcal{M}_i$ contains 0 and $\mathcal{M}_1 = \mathcal{T}$;
(2) $\Sigma^{-1}\mathcal{M}_{i+1} \cup \mathcal{M}_{i+1} \cup \Sigma\mathcal{M}_{i+1} \subset \mathcal{M}_i$ for every i;
(3) $\mathcal{M}_i * \mathcal{M}_i = \mathcal{M}_i$ for every i.

A good metric is a special metric, which is defined by Neeman in [12, Definition 1.2]. One can easily check that each $\mathcal{M}_i$ is closed under isomorphisms.

Let $\mathcal{T}$ be a triangulated category with a good metric $\{\mathcal{M}_i\}_{i\in\mathbb{N}}$. A sequence $a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} a_3 \to \cdots$ with objects in $\mathcal{T}$ is a *Cauchy sequence* (see [12, Definition 1.6]) if for every pair of integers $i > 0$ and $j \in \mathbb{Z}$, there exists an integer $N > 0$ such that, in any triangle $a_n \xrightarrow{f_{m-1}\circ\cdots\circ f_n} a_m \longrightarrow a_{n,m} \longrightarrow \Sigma a_n$ with $N \leqslant n < m$, the oject $\Sigma^j a_{n,n} \in \mathcal{M}_i$. Simply, we denote the sequence by a_*. With this definition, we define three full subcategories $\mathfrak{L}(\mathcal{T}), \mathfrak{C}(\mathcal{T}), \mathfrak{S}(\mathcal{T})$ of the category Mod–$\mathcal{T}$ as follows.

- The objects of $\mathfrak{L}(\mathcal{T})$ are the functors in $\mathsf{Mod}\text{–}\mathcal{T}$ which can be expressed as $\varinjlim Y(a_i)$, where a_* is a Cauchy sequence in $\mathcal{T}$ with respect to $\{\mathcal{M}_i\}_{i\in\mathbb{N}}$
- $\mathfrak{C}(\mathcal{T}) := \{A \in \mathsf{Mod}\text{–}\mathcal{T} \ \mid \forall\ j \in \mathbb{Z}, \exists\ i \in \mathbb{Z} \text{ s.t. } \mathsf{Hom}(Y(\Sigma^j\mathcal{M}_i), A) = 0\}$
- $\mathfrak{S}(\mathcal{T}) := \mathfrak{L}(\mathcal{T}) \cap \mathfrak{C}(\mathcal{T})$

where, we call $\mathfrak{S}(\mathcal{T})$ the *completion* of $\mathcal{T}$ with respect to $\{\mathcal{M}_i\}_{i\in\mathbb{N}}$.

A functor from a triangulated category to an abelian category is called a *cohomological functor* (see [16, Definition 10.2.7]) if it takes each triangle to a long exact sequence. As we all know that representable functors are cohomological (see [16, Exercise 10.2.3]). In fact, functors in $\mathfrak{S}(\mathcal{T})$ are also cohomological (see [12, Remark 1.11]).

As proved in [12, Theorem 2.11], the completion category $\mathfrak{S}(\mathcal{T})$ is also a triangulated category. A sequence $A \overset{\alpha}{\to} B \overset{\beta}{\to} C \overset{\gamma}{\to} \Sigma A$ is a distinguished triangle if it is isomorphic to the colimit of the image under Y of some Cauchy sequence of triangles $a_* \overset{f_*}{\to} b_* \overset{g_*}{\to} c_* \overset{h_*}{\to} \Sigma a_*$ in the category $\mathcal{T}$.

We begin to introduce recollement and ladder in this subsection.

2.2. Definition. ([1]) Let $\mathcal{S}$ and $\mathcal{R}$ be triangulated subcategories of $\mathcal{T}$. $\mathcal{T}$ is a *recollement* of $\mathcal{S}$ and $\mathcal{R}$ if there are six triangular functors as in the following diagram

$$\begin{tikzcd} \mathcal{R} \arrow[rr, "i_*=i_!" description] && \mathcal{T} \arrow[ll, bend right, "i^*"'] \arrow[ll, bend left, "i^!"] \arrow[rr, "j^!=j^*" description] && \mathcal{S} \arrow[ll, bend right, "j_!"'] \arrow[ll, bend left, "j_*"] \end{tikzcd}$$

such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs.
(2) i_*, j_* and $j_!$ are fully faithful functors,
(3) $i^! j_* = 0$, and
(4) for each object $t \in \mathcal{T}$, there are two triangles in $\mathcal{T}$:

$$i_! i^!(t) \to t \to j_* j^*(t) \to i_! i^!(\Sigma t)$$

and

$$j_! j^!(t) \to t \to i_* i^*(t) \to j_! j^!(\Sigma t).$$

A *right recollement* is a diagram of form

$$\begin{tikzcd} \mathcal{R} \arrow[rr, "i^*" description] && \mathcal{T} \arrow[ll, bend left, "i_*"] \arrow[rr, "j_!" description] && \mathcal{S} \arrow[ll, bend left, "j^!"] \end{tikzcd}.$$

in which all functors satisfy the conditions in the definition of recollement.

We take the definition of a ladder from [7, Section 3] which has a minor modification of the definition in [2, Section 1.5].

2.3. Definition. A *ladder* $\mathcal{L}$ is a finite or infinite diagram of triangulated categories and triangular functors

$$\begin{tikzcd}[column sep=huge]
\mathcal{C}' \arrow[r, bend left=30, "i_{n-1}" description] \arrow[r, bend right=30, "i_{n+1}" description] & \mathcal{C} \arrow[l, bend right=60, "j_{n-2}" description] \arrow[l, "j_n" description] \arrow[l, bend left=60, "j_{n+2}" description] \arrow[r, bend left=30, "j_{n-1}" description] \arrow[r, bend right=30, "j_{n+1}" description] & \mathcal{C}'' \arrow[l, bend right=60, "i_{n-2}" description] \arrow[l, "i_n" description] \arrow[l, bend left=60, "i_{n+2}" description]
\end{tikzcd}$$

such that any three consecutive rows form a recollement. The *height* of a ladder is the number of recollements contained in it (counted with multiplicities).

A recollement is a ladder of height 1.

3. Induced functors of additive functors

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of additive categories. Then we can define an additive functor

$$\begin{aligned} \widehat{F} : \mathsf{Mod}\text{–}\mathcal{B} &\to \mathsf{Mod}\text{–}\mathcal{A} \\ B &\mapsto B \circ F \\ (B \overset{\alpha}{\to} B') &\mapsto (B \circ F \overset{\alpha_{F(-)}}{\to} B' \circ F) \end{aligned}$$

Obviously, $\widehat{F}$ is an exact functor. The results below will tell us that the operation '$\widehat{\ }$' admits good properties.

3.1. Lemma. *The following are true:*

(1) *If* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ *are additive functors of additive categories, then* $\widehat{G \circ F} = \widehat{F} \circ \widehat{G}$.

(2) *Given two additive functors* $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ *and a natural transformation* $\Phi : F \longrightarrow G$, *then composition with* Φ *induces a natural transformation* $\widehat{\Phi} : \widehat{G} \longrightarrow \widehat{F}$.

(3) *Given three additive functors* $F, G, H : \mathcal{A} \longrightarrow \mathcal{B}$ *and two natural transformations* $F \xrightarrow{\Phi} G \xrightarrow{\Psi} H$, *then* $\widehat{\Psi \circ \Phi} = \widehat{\Phi} \circ \widehat{\Psi}$.

Proof. (1)Let C be an object in $\mathsf{Mod}\text{–}\mathcal{C}$. By definition, $\widehat{G \circ F}(C) = C \circ G \circ F = \widehat{G}(C) \circ F = \widehat{F} \circ \widehat{G}(C)$. Hence, we have $\widehat{G \circ F} = \widehat{F} \circ \widehat{G}$.

(2) For a natural transformation $\Phi : F \to G$, we define

$$\widehat{\Phi} := (\widehat{\Phi}_B : \ \widehat{G}(B) \overset{\mathsf{id}_B \circ \Phi}{\longrightarrow} \widehat{F}(B))_{B \in \mathsf{Mod}\text{–}\mathcal{B}}.$$

Next, we show $\widehat{\Phi}$ is really a natural transformation. For a morphism $\beta : Y \to Z$ in $\mathsf{Mod}\text{–}\mathcal{B}$, it follows from the naturality of β that there is the following commutative diagram for any $b \in \mathcal{B}$,

$$\begin{tikzcd} Y(G(b)) \arrow[r, "Y(\Phi_b)"] \arrow[d, "\beta_{G(b)}"'] & Y(F(b)) \arrow[d, "\beta_{F(b)}"] \\ Z(G(b)) \arrow[r, "Z(\Phi_b)"'] & Z(F(b)) \end{tikzcd}$$

This implies the following diagram is commutative,

$$\begin{tikzcd} \widehat{G}(Y) \arrow[r, "\widehat{\Phi}_Y"] \arrow[d, "\widehat{G}(\beta)"'] & \widehat{F}(Y) \arrow[d, "\widehat{F}(\beta)"] \\ \widehat{G}(Z) \arrow[r, "\widehat{\Phi}_Z"'] & \widehat{F}(Z) \end{tikzcd}$$

Thus, $\widehat{\Phi}$ is a natural transformation.

(3) Let B be an object in $\mathsf{Mod}\text{–}\mathcal{B}$. Then $\widehat{\Psi \circ \Phi}(B) = B \circ \Psi \circ \Phi = \widehat{\Psi}(B) \circ \widehat{\Phi} = \widehat{\Phi} \circ \widehat{\Psi}(B)$. Hence, we have $\widehat{\Psi \circ \Phi} = \widehat{\Phi} \circ \widehat{\Psi}$. We finish the proof. ∎

3.2. COROLLARY. *Given a pair of adjoint additive functors of additive categories*

$$F : \mathcal{A} \rightleftarrows \mathcal{B} : G$$

with unit and counit of adjunction $\eta : \mathsf{id} \longrightarrow GF$ *and* $\varepsilon : FG \longrightarrow \mathsf{id}$.

(1) *The pair of functors*

$$\widehat{F} : \ \mathsf{Mod}\text{–}\mathcal{B} \rightleftarrows \mathsf{Mod}\text{–}\mathcal{A} : \widehat{G}$$

is also an adoint pair, where the unit and counit of adjunction are $\widehat{\varepsilon} : \mathsf{id} \longrightarrow \widehat{G}\widehat{F}$ *and* $\widehat{\eta} : \widehat{F}\widehat{G} \longrightarrow \mathsf{id}$.

(2) *If G is fully-faithful, then so is $\widehat{F}$.*

PROOF. (1) By [4, Theorem 3.1.5], it is equivalent to prove $(\widehat{G} * \widehat{\eta}) \circ (\widehat{\varepsilon} * \widehat{G}) = \mathsf{id}_{\widehat{G}}$ and $(\widehat{\eta} * \widehat{F}) \circ (\widehat{F} * \widehat{\varepsilon}) = \mathsf{id}_{\widehat{F}}$, where $(\widehat{G} * \widehat{\eta})_A := \widehat{G}(\widehat{\eta}_A)$, $(\widehat{\varepsilon} * \widehat{G})_A := \widehat{\varepsilon}_{\widehat{G}(A)}$, $(\widehat{\eta} * \widehat{F})_B := \widehat{\eta}_{\widehat{F}(B)}$and $(\widehat{F} * \widehat{\varepsilon})_B := \widehat{F}(\widehat{\varepsilon}_B)$ for $A \in \mathsf{Mod}\text{–}\mathcal{A}$, $B \in \mathsf{Mod}\text{–}\mathcal{B}$.

By Lemma 3.1 , we get $\widehat{G} * \widehat{\eta} = \widehat{\eta * G}$, $\widehat{\varepsilon} * \widehat{G} = \widehat{G * \varepsilon}$, $\widehat{\eta} * \widehat{F} = \widehat{F * \eta}$ and $\widehat{F} * \widehat{\varepsilon} = \widehat{\varepsilon * F}$. Hence, there are following two equations

$$(\widehat{G} * \widehat{\eta}) \circ (\widehat{\varepsilon} * \widehat{G}) = \widehat{\eta * G} \circ \widehat{G * \varepsilon} = \widehat{(G * \varepsilon) \circ (\eta * G)} = \widehat{\mathsf{id}_G} = \mathsf{id}_{\widehat{G}}$$

$$(\widehat{\eta} * \widehat{F}) \circ (\widehat{F} * \widehat{\varepsilon}) = \widehat{F * \eta} \circ \widehat{\varepsilon * F} = \widehat{(\varepsilon * F) \circ (F * \eta)} = \widehat{\mathsf{id}_F} = \mathsf{id}_{\widehat{F}}$$

which imply that $(\widehat{G}, \widehat{F})$ is an adjoint pair.

(2) G being fully faithful is equivalent to $\eta : FG \longrightarrow \mathrm{id}$ being an isomorphism, which implies that $\widehat{\eta} : \mathrm{id} \longrightarrow \widehat{G}\widehat{F}$ is an isomorphism, and this is equivalent to $\widehat{F}$ being fully faithful. We finish the proof. ∎

4. Construction of recollements and ladders

In this section, we will prove our main results of constructing recollements and ladders. To reach the goal, we firstly consider the induced functors between triangulated categories with good metrics. From now on, we assume that $\mathcal{T}$ and $\mathcal{S}$ are triangulated categories with good metrics $\{\mathcal{M}_i\}_{i\in\mathbb{N}}$ and $\{\mathcal{N}_j\}_{j\in\mathbb{N}}$, respectively.

4.1. Definition. Let F be a triangular functor from $\mathcal{T}$ to $\mathcal{S}$. F is a *compression functor* if for any $k>0$, there exists $n>0$ such that $F(\mathcal{M}_n)\subset\mathcal{N}_k$.

4.2. Lemma. *Let F be a triangular functor from $\mathcal{T}$ to $\mathcal{S}$. If F is a compression functor, then F preserve Cauchy sequences.*

Proof. Let a_* be a Cauchy sequence in $\mathcal{T}$. For any pair of integers $i>0$ and $j\in\mathbb{Z}$, there is an integer $n_i>0$ satisfies $F(\mathcal{M}_{n_i})\subset\mathcal{N}_i$ by the assumption of F. Moreover, for the pair (n_i,j), there is an integer $N>0$ such that, for any triangle $a_n\to a_m\to a_{n,m}\to\Sigma a_n$ with $m>n\geqslant N$, the object $\Sigma^j a_{n,m}\in\mathcal{M}_{n_i}$. So $\Sigma^j F(a_{n,m})\simeq F(\Sigma^j(a_{n,m}))\in F(\mathcal{M}_{n_i})\subset\mathcal{N}_i$. Hence $F(a_*)$ is a Cauchy sequence. We finish the proof. ∎

4.3. Theorem. *Let $F:\mathcal{T}\rightleftarrows\mathcal{S}:G$ be an adjoint pair of triangulated functors. Suppose that F and G are both compressions. Then the functor $\widehat{F}:\mathsf{Mod}\text{–}\mathcal{S}\longrightarrow\mathsf{Mod}\text{–}\mathcal{T}$ has the properties*

(1) *$\widehat{F}$ takes $\mathfrak{L}(\mathcal{S})\subset\mathsf{Mod}\text{–}\mathcal{S}$ to $\mathfrak{L}(\mathcal{T})\subset\mathsf{Mod}\text{–}\mathcal{T}$.*

(2) *$\widehat{F}$ takes $\mathfrak{C}(\mathcal{S})\subset\mathsf{Mod}\text{–}\mathcal{S}$ to $\mathfrak{C}(\mathcal{T})\subset\mathsf{Mod}\text{–}\mathcal{T}$.*

(3) *The restriction of $\widehat{F}$ to $\mathfrak{S}(\mathcal{S})=\mathfrak{L}(\mathcal{S})\cap\mathfrak{C}(\mathcal{S})$ induces a triangulated functor, which we will (by abuse of notation) also call $\widehat{F}:\mathfrak{S}(\mathcal{S})\longrightarrow\mathfrak{S}(\mathcal{T})$.*

Proof. Assume that G is the right adjoint of F.

(1) Let $A\in\mathfrak{L}(\mathcal{S})$. Write $A\simeq\varinjlim Y(a_i)$. Then

$$\widehat{F}(A)\simeq\widehat{F}(\varinjlim Y(a_i))=\varinjlim \mathsf{Hom}_{\mathcal{S}}(F(-),a_i)\simeq\varinjlim Y(G(a_i))$$

Since G is a compression functor, then $G(a_*)$ is a Cauchy sequence in $\mathcal{T}$ by Lemma 4.2, hence we have $\widehat{F}(A)\in\mathfrak{L}(\mathcal{T})$.

(2) Let $A\in\mathfrak{C}(\mathcal{S})$. That is, for any $\lambda\in\mathbb{Z}$, there is an index k such that

$$\mathsf{Hom}_{\mathsf{Mod}\text{–}\mathcal{S}}(Y(\Sigma^\lambda\mathcal{N}_k),A)=0.$$

By Yoneda Lemma, $A(\Sigma^\lambda\mathcal{N}_k)=0$. Since F is a compression functor, then there is an integer n_k such that $F(\mathcal{M}_{n_k})\subset\mathcal{N}_k$. Hence, for any $t\in\mathcal{M}_{n_k}$,

$$\mathsf{Hom}(Y(\Sigma^\lambda t),\widehat{F}A)\simeq\widehat{F}A(\Sigma^\lambda t)\simeq A(\Sigma^\lambda F(t))=0.$$

This implies that $\widehat{F}(A)\in\mathfrak{C}(\mathcal{T})$.

(3) It follows from (1) and (2) that $\widehat{F}:\mathfrak{G}(\mathcal{S})\longrightarrow\mathfrak{G}(\mathcal{T})$ is well-defined. For any $A\in\mathfrak{G}(\mathcal{S})$ and $t\in\mathcal{T}$, we have

$$\Sigma\widehat{F}(A)(t)=\Sigma A(F(t))=A(\Sigma^{-1}F(t))\simeq A(F(\Sigma^{-1}t))=\widehat{F}\Sigma(A)(t).$$

So $\widehat{F}$ commutes with Σ.

Let $A\xrightarrow{\alpha}B\xrightarrow{\beta}C\xrightarrow{\gamma}\Sigma A$ be a triangle in $\mathfrak{G}(\mathcal{S})$. By definition, it is isomorphic to the colimit of the image under Y of a triangle of Cauchy sequences $a_*\xrightarrow{f_*}b_*\xrightarrow{g_*}c_*\xrightarrow{h_*}\Sigma a_*$ in the category $\mathcal{S}$. Since G is a compression functor, then

$$G(a_*)\xrightarrow{G(f_*)}G(b_*)\xrightarrow{G(g_*)}G(c_*)\xrightarrow{G(h_*)}G(\Sigma a_*)\qquad(\#)$$

is a triangle of Cauchy sequences by Lemma 4.2. Take colimit of the image under Y of $(\#)$, we obtain the following sequence

$$\varinjlim Y(G(a_i))\longrightarrow\varinjlim Y(G(b_i))\longrightarrow\varinjlim Y(G(c_i))\longrightarrow\varinjlim Y(G(\Sigma a_i))$$

Notice that $\varinjlim Y(G(a_i))\simeq\varinjlim\widehat{F}(Y(a_i))\simeq\widehat{F}(\varinjlim Y(a_i))\simeq\widehat{F}(A)$. So there is the following commutative diagram

$$\begin{tikzcd}
\varinjlim Y(G(a_i)) \arrow[r] \arrow[d, "\simeq"] & \varinjlim Y(G(b_i)) \arrow[r] \arrow[d, "\simeq"] & \varinjlim Y(G(c_i)) \arrow[r] \arrow[d, "\simeq"] & \varinjlim Y(G(\Sigma a_i)) \arrow[d, "\simeq"] \\
\widehat{F}(A) \arrow[r, "\widehat{F}(\alpha)"] & \widehat{F}(B) \arrow[r, "\widehat{F}(\beta)"] & \widehat{F}(C) \arrow[r, "\widehat{F}(\gamma)"] & \Sigma\widehat{F}(A)
\end{tikzcd}$$

which implies that $\widehat{F}(A)\xrightarrow{\widehat{F}(\alpha)}\widehat{F}(B)\xrightarrow{\widehat{F}(\beta)}\widehat{F}(C)\xrightarrow{\widehat{F}(\gamma)}\Sigma\widehat{F}(A)$ is a triangle in $\mathfrak{G}(\mathcal{T})$. Hence, $\widehat{F}$ is a triangular functor. We finish the proof. ∎

Suppose we are given a triple of triangulated functors

$$\begin{tikzcd}
\mathcal{T} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "H"'] & G \arrow[l] & \mathcal{S} \arrow[l, no head]
\end{tikzcd}$$

with (F,G) and (G,H) adjoint pairs. According to Corollary 3.2, the following are adjoints

$$\begin{tikzcd}
\mathsf{Mod}\text{–}\mathcal{S} \arrow[rr, bend left, "\widehat{F}"] \arrow[rr, bend right, "\widehat{H}"'] & \widehat{G} \arrow[l] & \mathsf{Mod}\text{–}\mathcal{T} \arrow[l, no head]
\end{tikzcd}$$

If F,G,H are all compressions, then Theorem 4.3 tells us that the top two restrict to adjoints

$$\begin{tikzcd}
\mathfrak{G}(\mathcal{S}) \arrow[rr, bend left, "\widehat{F}"] & \widehat{G} \arrow[l] & \mathfrak{G}(\mathcal{T}) \arrow[l, no head]
\end{tikzcd}$$

where the unit and counit of adjunction are the restrictions of $\widehat{\varepsilon}$ and $\widehat{\eta}$. Moreover, by Corollary 3.2, G being fully faithful implies that $\widehat{F}$ is fully faithful.

4.4. THEOREM. *Let $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{R}$ be triangulated categories with good metrics. Assume that the following diagram is a recollement of triangulated categories*

$$\begin{tikzcd} \mathcal{R} \arrow[r, "i_* = i_!"] & \mathcal{T} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^! = j^*"] & \mathcal{S} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

If all the functors in the diagram are compression functors, then there is a right recollement of the corresponding completion categories

$$\begin{tikzcd} \mathfrak{S}(\mathcal{R}) \arrow[r, "\widehat{i^*}"] & \mathfrak{S}(\mathcal{T}) \arrow[l, bend left, "\widehat{i_*}"] \arrow[r, "\widehat{j_!}"] & \mathfrak{S}(\mathcal{S}) \arrow[l, bend left, "\widehat{j^!}"] \end{tikzcd} .$$

PROOF. B Corollary 3.2 and Theorem 3.1, $(\widehat{i^*}, \widehat{i_*})$ and $(\widehat{j_!}, \widehat{j^!})$ are adjoint pairs of triangular functors. $\widehat{i^*}$ and $\widehat{j^!}$ are fully-faithful.

To complete the proof, we need show $\mathsf{ker}(\widehat{i_*}) = \mathsf{im}(\widehat{j^!})$. Note that $\widehat{i_*}\widehat{j^!} = \widehat{j^! i_*} = \widehat{0} = 0$. Hence $\mathsf{im}(\widehat{j^!}) \subset \mathsf{ker}(\widehat{i_*})$. Let B be an object in $\mathsf{ker}(\widehat{i_*})$ and t an object in $\mathcal{T}$. Consider the following decomposition of t

$$j_! j^!(t) \longrightarrow t \longrightarrow i_* i^*(t) \longrightarrow j_! j^!(\Sigma t).$$

Since B is a cohomological functor, then there is a long exact sequence

$$\cdots \longrightarrow B(i_* i^*(t)) \longrightarrow B(t) \longrightarrow B(j_! j^!(t)) \longrightarrow B(i_* i^*(\Sigma^{-1} t)) \longrightarrow \cdots .$$

By the choice of B, $B(i_* i^*(t)) = \widehat{i^*}\widehat{i_*}(B)(t) = 0$ and $B(i_* i^*(\Sigma^{-1}t)) = \widehat{i^*}\widehat{i_*}(B)(\Sigma^{-1}t) = 0$. Then $B(t) \simeq B(j_! j^!(t)) \simeq \widehat{j^!}(\widehat{j_!}(B))(t)$. Hence $B \in \mathsf{im}(\widehat{j^!})$. We finish the proof. ■

4.5. EXAMPLE. Let A, B and C be finite dimensional algebras over a field k. Assume that there is a recollement among their homotopy categories of finitely generated projective modules

$$\begin{tikzcd} K^b(B-\mathsf{proj}) \arrow[r, "i_* = i_!"] & K^b(A-\mathsf{proj}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^! = j^*"] & K^b(C-\mathsf{proj}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Consider good metrics (see [13, Example 12(i)]), where $* = B,\ A,\ C$,

$$\mathcal{M}_i^{K^b(*-\mathsf{proj})} := \begin{cases} K^b(*-\mathsf{proj}) & i = 1 \\ \{X^\bullet \in K^b(*-\mathsf{proj}) \mid H^j(X^\bullet) = 0, -i < j\} & i \geq 2. \end{cases}$$

With respect to these good metrics, $\mathfrak{S}(K^b(*-\mathsf{proj})) = \mathcal{D}^b(*-\mathsf{smod})$ (see [13, Example 23]). In fact, all functors in the recollement are compression functors. As a representation, we prove for i^*. Without loss of generality, we may assume the degrees of non-zero homological groups of $i^*(A)$ are in $[-N,\ N]$ for some $N \in \mathbb{N}$.

Then for any $k \in \mathbb{N}$,

$$i^*(\Sigma^{N+k}(A)) = \Sigma^{N+k}(i^*(A)) \in \mathcal{M}_k^{K^b(A-\mathsf{proj})}.$$

This implies that $i^*(\mathcal{M}_{N+k}^{K^b(A-\mathsf{proj})}) \subset \mathcal{M}_k^{K^b(B-\mathsf{proj})}$. So i^* is a compression functor, and then there is a right recollement by Theorem 4.4

$$\begin{tikzcd}
\mathcal{D}^b(B-\mathsf{mod}) \arrow[r, "\widehat{i^*}"] & \mathcal{D}^b(A-\mathsf{mod}) \arrow[l, bend left, "\widehat{i_*}"] \arrow[r, "\widehat{j_!}"] & \mathcal{D}^b(C-\mathsf{mod}) \arrow[l, bend left, "\widehat{j^!}"]
\end{tikzcd}.$$

4.6. Corollary. *Let $\mathcal{L}$ be a ladder of height $n (n \geq 2)$ of triangulated categories with good metrics. If all the functors in $\mathcal{L}$ are compression functors, then there is a ladder of height $n-1$ of the corresponding completion categories.*

Proof. At first, we assume n is finite and the ladder $\mathcal{L}$ has the following form

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, bend left, "i_2"] \arrow[r, bend right, "i_{n+1}"'] & \mathcal{C} \arrow[l, bend right=50, "i_1"'] \arrow[l, bend left=50, "i_{n+2}"] \arrow[r, bend left, "j_2"] \arrow[r, bend right, "j_{n+1}"'] & \mathcal{C}'' \arrow[l, bend right=50, "j_1"'] \arrow[l, bend left=50, "j_{n+2}"]
\end{tikzcd}$$

It follows from Theorem 4.4 that there is a diagram

$$\begin{tikzcd}
\mathfrak{G}(\mathcal{C}'') \arrow[r, bend left, "\widehat{j_2}"] \arrow[r, bend right, "\widehat{j_n}"'] & \mathfrak{G}(\mathcal{C}) \arrow[l, bend right=50, "\widehat{j_1}"'] \arrow[l, bend left=50, "\widehat{j_{n+1}}"] \arrow[r, bend left, "\widehat{i_2}"] \arrow[r, bend right, "\widehat{i_n}"'] & \mathfrak{G}(\mathcal{C}') \arrow[l, bend right=50, "\widehat{i_1}"'] \arrow[l, bend left=50, "\widehat{i_{n+1}}"]
\end{tikzcd}$$

It follows from the proof of Theorem 4.4 that the above diagram is a ladder of triangulated categories of height $n-1$.

The case of n being infinite is proved similarly. ■

4.7. Example. ([7, Example 3.4]) Let k be a field, B and C be finite-dimensional k-algebras and M be a finitely generated C-B-bimodule with finite projective dimension over B and C. Consider the matrix algebra $A = \begin{pmatrix} B & 0 \\ {}_CM_B & C \end{pmatrix}$. Put $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. One can check that both Ae_1A and Ae_2A are stratifying ideals of A. Hence they produce a ladder of height 2

$$\begin{tikzcd}
K^b(B-\mathsf{proj}) \arrow[r, "?\otimes_B^{\mathbf{L}} e_1A"] \arrow[r, bend right=50] & K^b(A-\mathsf{proj}) \arrow[l, bend right=50] \arrow[l, "?\otimes_A Ae_1"'] \arrow[r] \arrow[r, bend right=50] & K^b(C-\mathsf{proj}) \arrow[l, bend right=50, "?\otimes_C^{\mathbf{L}} e_2A"'] \arrow[l, bend left=10]
\end{tikzcd}.$$

Consider good metrics on $K^b(B-\mathsf{proj})$ as in the Example 4.5. Then, it follows from Corollary 4.6 that there is a ladder of height 1, i.e. a recollement among their derived categories

$$\begin{tikzcd}
\mathcal{D}^b(C-\mathsf{mod}) \arrow[r] & \mathcal{D}^b(A-\mathsf{mod}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathcal{D}^b(B-\mathsf{mod}) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}.$$

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