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REMARKS ON COMBINATORIAL AND ACCESSIBLE MODEL CATEGORIES

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Abstract. Using full images of accessible functors, we prove some results about combinatorial and accessible model categories. In particular, we give an example of a weak factorization system on a locally presentable category which is not accessible.

1. Introduction

Twenty years ago, M. Hovey asked for examples of model categories which are not cofibrantly generated. This is the same as asking for examples of weak factorization systems which are not cofibrantly generated. One of the first examples was given in [1]: it is a weak factorization system $(\mathcal{L}, \mathcal{R})$ on the locally presentable category of posets where $\mathcal{L}$ consists of embeddings. The reason is that posets injective to embeddings are precisely complete lattices which do not form an accessible category. Hence $\mathcal{L}$ is not cofibrantly generated. Since then, the importance of accessible model categories and accessible weak factorization systems has emerged, and, the same question appears again, i.e., to give an example of a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a locally presentable category which is not accessible. Now, $\mathcal{L}$-injective objects do not necessarily form an accessible category but only a full image of an accessible functor. Such full images are accessible only under quite restrictive assumptions (see [5]). But, for an accessible weak factorization system, $\mathcal{L}$-injective objects form the full image of a forgetful functor from algebraically $\mathcal{L}$-injective objects (see Corollary 3.3). Moreover, such full images are closed under reduced products modulo κ-complete filters for some regular cardinal κ (see Theorem 2.5). We use this property to present a non-accessible factorization system on the category of Boolean algebras having, again, $\mathcal{L}$ consisting of embeddings. Full images of accessible functors are also used for showing that accessible weak factorization systems on a locally presentable category are closed under small intersections (see Theorem 3.9). Another proof of this fact is given in [10].

Given a cofibrantly generated weak factorization system $(\mathcal{L}, \mathcal{R})$ on a locally presentable category $\mathcal{K}$, [19] constructs a class $\mathcal{W}_{\mathcal{L}}$ and shows that, assuming Vopěnka's principle, $\mathcal{W}_{\mathcal{L}}$ is the smallest class of weak equivalences making $\mathcal{K}$ a model category with $\mathcal{L}$ as the class

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of cofibrations. It is still open whether Vopěnka's principle is needed for this. Recently, S. Henry [9] has radically generalized results of D.-C. Cisinski [6] and M. Olschok [12] and has given mild assumptions under which Vopěnka's principle is not needed. Using full images of accessible functors we show that $(\mathcal{K}, \mathcal{L}, \mathcal{W}_{\mathcal{L}})$ is a model category iff its transfinite construction from [19] converges, i.e., it stops at some ordinal (see Theorem 4.7).

Finally, we show that weak equivalences in an accessible model category form a full image of an accessible functor, which corrects an error in [17] (see Theorem 5.1).

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2. Full images

Let $F : \mathcal{M} \to \mathcal{K}$ be an accessible functor. Recall that this means that both $\mathcal{M}$ and $\mathcal{K}$ are accessible and F preserves λ-directed colimits for some regular cardinal λ. The full subcategory of $\mathcal{K}$ consisting of objects FM, $M \in \mathcal{M}$ is called a *full image* of F. While accessible categories are, up to equivalence, precisely categories of models of basic theories, full images of accessible functors are, up to equivalence, precisely categories of structures which can be axiomatized using additional operation and relation symbols (see [14]); they are also called pseudoaxiomatizable. In both cases, we use infinitary first-order theories.

Let $\mathcal{M}$ be a full subcategory of a category $\mathcal{K}$ and K an object in $\mathcal{K}$. We say that $\mathcal{M}$ satisfies the *solution-set condition* at K if there exists a set of morphisms $(K \to M_i)_{i \in I}$ with M_i in $\mathcal{M}$ for each $i \in I$ such that every morphism $f : K \to M$ with M in $\mathcal{M}$ factorizes through some f_i, i.e., $f = gf_i$. $\mathcal{M}$ is called *cone-reflective* in $\mathcal{K}$ if it satisfies the solution-set condition at each object K in $\mathcal{K}$ (see [2]). Given a set $\mathcal{X}$ of objects of $\mathcal{K}$, we say that $\mathcal{M}$ satisfies the solution set condition at $\mathcal{X}$ if it satisfies this condition at each $X \in \mathcal{X}$.

2.1. Proposition. *([16, Proposition 2.4]) The full image of an accessible functor is cone-reflective in $\mathcal{K}$.*

2.2. Proposition. *Let $\mathcal{K}$ be a locally presentable category, I a set and $\mathcal{X}_i \subseteq \mathcal{K}$, $i \in I$, full images of accessible functors. Then $\cup_{i \in I} \mathcal{X}_i$ is a full image of an accessible functor.*

Proof. Let $\mathcal{X}_i$ be full images of accessible functors $F_i : \mathcal{M}_i \to \mathcal{K}$, $i \in I$. Then $\cup_{i \in I} \mathcal{X}_i$ is a full image of an accessible functor $F : \coprod_{i \in I} \mathcal{X}_i \to \mathcal{K}$ induced by F_i. ∎

2.3. Notation. Let $\mathcal{X}$ be a class of morphisms in $\mathcal{K}$. Then $\overline{\mathcal{X}}$ will denote its 2-out-of-3 closure, i.e., the smallest class of morphisms such that

1. $f, g \in \overline{\mathcal{X}}$ implies $gf \in \overline{\mathcal{X}}$,
2. $gf, f \in \overline{\mathcal{X}}$ implies $g \in \overline{\mathcal{X}}$ and
3. $gf, g \in \overline{\mathcal{X}}$ implies $f \in \overline{\mathcal{X}}$.

We will consider these classes as full subcategories in $\mathcal{K}^{\to}$.

2.4. Proposition. *Let $\mathcal{K}$ be a locally presentable category and $\mathcal{X} \subseteq \mathcal{K}^{\to}$ a full image of an accessible functor. Then $\overline{\mathcal{X}}$ is a full image of an accessible functor.*

Proof. $\overline{\mathcal{X}}$ can be obtained from $\mathcal{X}$ by a sequence of pseudopullbacks. Let $\mathcal{X}_0 = \mathcal{X}$. We take composable pairs of $\mathcal{X}_0$ and their compositions form $\mathcal{X}_1$. Then we take those pairs (g, f) for which f and the composition gf belong to $\mathcal{X}_1$. Their g's form $\mathcal{X}_2$. Further we take those pairs (g, f) for which g and gf belong to $\mathcal{X}_2$. Their f's form $\mathcal{X}_3$. By iterating this construction, we get $\overline{\mathcal{X}} = \cup_{i<\omega}\mathcal{X}_i$, Thus the result will follow from [16, Lemma 2.6] and Proposition 2.2 as soon as we show that $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ are always full images of accessible functors.

Assume that $\mathcal{X}$ is a full image of an accessible functor $F : \mathcal{M} \to \mathcal{K}^{\to}$. We have the pseudopullback

$$\begin{tikzcd}
\mathcal{K}^{\to\to} \arrow[r, "P_2"] \arrow[d, "P_1"'] & \mathcal{K}^{\to} \arrow[d, "\mathrm{dom}"] \\
\mathcal{K}^{\to} \arrow[r, "\mathrm{cod}"'] & \mathcal{K}
\end{tikzcd}$$

where dom takes the codomain and dom the domain of a morphism. Consider the pseudopullback

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\bar{P}_2"] \arrow[d, "\bar{P}_1"'] & \mathcal{M} \arrow[d, "\mathrm{dom}\cdot F"] \\
\mathcal{M} \arrow[r, "\mathrm{cod}\cdot F"'] & \mathcal{K}
\end{tikzcd}$$

Let $H : \mathcal{P} \to \mathcal{K}^{\to\to}$ be the induced functor and comp : $\mathcal{K}^{\to\to} \to \mathcal{K}^{\to}$ the composition functor. Then $\mathcal{X}_1$ is the full image of the accessible functor comp $\cdot\, H$ (we use [11, Theorem 5.1.6]).

Now, consider the pseudopullbacks

$$\begin{tikzcd}
\mathcal{Q}_1 \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "F"] \\
\mathcal{K}^{\to\to} \arrow[r, "\mathrm{comp}"'] & \mathcal{K}^{\to}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{P}_1 \arrow[r] \arrow[d, "\bar{G}"'] & \mathcal{X} \arrow[d, "G"] \\
\mathcal{K}^{\to\to} \arrow[r, "\mathrm{comp}"'] & \mathcal{K}^{\to}
\end{tikzcd}$$

where G is the inclusion. Let $H_1 : \mathcal{Q}_1 \to \mathcal{P}_1$ be the induced functor. We get the accessible functor $\bar{G} \cdot H_1$ whose full image consists of composable pairs having composition in $\mathcal{X}$.

Next, consider the pseudopullbacks

$$\begin{tikzcd}
\mathcal{Q}_2 \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "F"] \\
\mathcal{K}^{\to\to} \arrow[r, "P_1"'] & \mathcal{K}^{\to}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{P}_2 \arrow[r] \arrow[d, "\bar{\bar{G}}"'] & \mathcal{X} \arrow[d, "G"] \\
\mathcal{K}^{\to\to} \arrow[r, "P_1"'] & \mathcal{K}^{\to}
\end{tikzcd}$$

Let $H_2 : \mathcal{Q}_2 \to \mathcal{P}_2$ be the induced functor. We get the accessible functor $\bar{\bar{G}} \cdot H_1$ whose full image consists of composable pairs having the first member in $\mathcal{X}$. Since the intersection of two full images of accessible functor is a full image of an accessible functor (see [16, Lemma 2.6]), we get an accessible functor $H : \mathcal{N} \to \mathcal{K}^{\to\to}$ whose full image consists of composable pairs having the composite and the first member in $\mathcal{X}$. Then $\mathcal{X}_2$ is the full image of $P_2 \cdot H : \mathcal{N} \to \mathcal{K}^{\to}$.

Analogously, we show that $\mathcal{X}_3$ is a full image of an accessible functor. ■

2.5. THEOREM. *Let $F : \mathcal{M} \to \mathcal{K}$ be a limit preserving κ-accessible functor where $\mathcal{M}$ is locally κ-presentable. Then the full image of F is closed in $\mathcal{K}$ under reduced products modulo κ-complete filters.*

PROOF. Let I be a set, $K_i = F(M_i)$, $i \in I$ and let $\mathcal{F}$ be a κ-complete filter on I. Then the reduced product $\prod_{\mathcal{F}} K_i$ is a κ-directed colimit of projections $K_i^A \to K_i^B$ where $A \supseteq B \in \mathcal{F}$. Then $K = F(M)$ where $M = \prod_{\mathcal{F}} K_i$. ■

3. Accessible weak factorization systems

A functorial weak factorization system in a locally presentable category is called *accessible* if the factorization functor $F : \mathcal{K}^{\to} \to \mathcal{K}^{\to\to}$ is accessible (see [17]). Here, $\mathcal{K}^{\to\to}$ denotes the category of composable pairs of morphisms. Any cofibrantly generated weak factorization system in a locally presentable category is accessible.

3.1. THEOREM. *Let $(\mathcal{L}, \mathcal{R})$ be an accessible weak factorization system in a locally presentable category $\mathcal{K}$. Then $\mathcal{R}$ is a full image of a limit-preserving accessible functor $\mathcal{M} \to \mathcal{K}^{\to}$ where $\mathcal{M}$ is locally presentable.*

PROOF. $\mathcal{R}$ is the full image of an accessible right adjoint $\mathrm{Alg}(R) \to \mathcal{K}^{\to}$ (see [17, Remarks 2.3(2) and 4.2(1)]). ■

3.2. Remark. But $\mathcal{R}$ does not need to be accessible, see [17, Example 2.6]. Thus [17, Remark 5.2 (2)] is not correct (I am indebted to M. Shulman for pointing this out). Neither it is accessibly embedded in $\mathcal{K}^2$. Assuming the existence of a proper class of almost strongly compact cardinals, $\mathcal{R}$ is preaccessible and *preaccessibly embedded* to $\mathcal{K}^{\to}$; see the proof of [16, Proposition 2.2]. The latter means that the embedding $\mathcal{R} \to \mathcal{K}^2$ preserves λ-directed colimits for some λ.

3.3. Corollary. *Let $(\mathcal{L}, \mathcal{R})$ be an accessible weak factorization system in a locally presentable category $\mathcal{K}$. Then $\mathcal{L}$-Inj is a full image of a limit-preserving accessible functor $\mathcal{M} \to \mathcal{K}$ where $\mathcal{M}$ is locally presentable.*

Proof. An object K is $\mathcal{L}$-injective if and only if $K \to 1$ is in $\mathcal{R}$. The functor R restricts to a functor $\overline{R}$ on $\mathcal{K} \downarrow 1$ and yields an accessible right adjoint $\operatorname{Alg}(\overline{R}) \to \mathcal{K}$. Now, $\mathcal{L}$-Inj is the full image of this restriction. ∎

The next result improves Proposition 3.4 in [16].

3.4. Theorem. *Let $(\mathcal{L}, \mathcal{R})$ be an accessible weak factorization system in a locally presentable category $\mathcal{K}$. Then $\mathcal{L}$ is a full image of a colimit-preserving (accessible) functor $\mathcal{M} \to \mathcal{K}^{\to}$ where $\mathcal{M}$ is locally presentable.*

Proof. $\mathcal{L}$ is the full image of an accessible functor $\operatorname{Coalg}(L) \to \mathcal{K}^{\to}$ (see [17, Remarks 2.3(2) and 4.2(1)]). ∎

3.5. Remark. If $(\mathcal{L}, \mathcal{R})$ is a cofibrantly generated weak factorization system in a locally presentable category then $\mathcal{L}$ does not need to be accessible. An example is given in [16, Example 3.5(2)] under the axiom of constructibility. In this example, $\mathcal{L}$ is accessible assuming the existence of an almost strongly compact cardinal. We do not know any example of non-accessible $\mathcal{L}$ in ZFC. Example [16] 3.3(1) is not correct because split monomorphisms are not cofibrantly generated in posets (this was pointed out by T. Campion).

3.6. Remark. A weak factorization system $(\mathcal{L}, \mathcal{R})$ is accessible iff $\operatorname{Coalg}(L)$ is locally presentable, which is a kind of smallness property of $\operatorname{Coalg}(L)$. Following [17, Theorem 4.3], $(\mathcal{L}, \mathcal{R})$ is accessible iff $\mathcal{L}$ is small generated.

3.7. Examples. (1) Let $\mathcal{L}$ be the class of regular monomorphisms (= embeddings) in the category **Bool** of Boolean algebras. Then $\mathcal{L}$-injective Boolean algebras are precisely complete Boolean algebras and **Bool** has enough $\mathcal{L}$-injectives (see [8]). Thus $(\mathcal{L}, \mathcal{L}^{\square})$ is a weak factorization system (see [1, Proposition 1.6]). We will show that this weak factorization system is not accessible. Following Theorem 2.5 and Corollary 3.3, it suffices to show that complete Boolean algebras are not closed under reduced products modulo κ-complete filters for any regular cardinal κ. I have learnt the following example from M. Goldstern.

Let I be a set of cardinality κ and $\mathcal{F}$ be the filter of subsets $X \subseteq I$ such that the cardinality of $I \setminus X$ is $< \kappa$. Then the reduced product $\prod_{\mathcal{F}} 2$ is isomorphic to the Boolean algebra $U(\kappa) = \mathcal{P}(I)/[\kappa]^{<\kappa}$ where $[\kappa]^{<\kappa}$ is the ideal J consisting of subsets of cardinality $< \kappa$. Let A_i, $i < \kappa$ be pairwise disjoint subsets of I of cardinality κ. Let X be an upper bound of A_i, $i < \kappa$ in $U(\kappa)$. Choose $a_i \in A_i$, $i < \kappa$. Then $X \setminus \{a_i | i < \kappa\}$ is an upper bound of A_i in $U(\kappa)$ smaller than X. Hence A_i, $i < \kappa$ do not have a supremum in $U(\kappa)$.

(2) Let $\mathcal{L}$ be the class of regular monomorphisms (= embeddings) in the category **Pos** of posets. Then $\mathcal{L}$-injective posets are precisely complete lattices and **Pos** has enough $\mathcal{L}$-injectives (see [3]). Since the forgetful functor **Bool** $\to$ **Pos** preserves products and directed colimits, complete lattices are not closed under reduced products modulo κ-complete filters for any regular cardinal κ. It suffices to take the same reduced products as in (1).

3.8. Remark. We can order weak factorization systems: $(\mathcal{L}_1, \mathcal{R}_1) \leq (\mathcal{L}_2, \mathcal{R}_2)$ if $\mathcal{L}_1 \subseteq \mathcal{L}_2$. Following [17, Theorem 4.3], accessible weak factorization systems have small joins: if $\mathcal{L}_i$ is generated by $\mathcal{C}_i$, $i \in I$ then $\cup_{i \in I} \mathcal{L}_i$ is generated by $\cup_{i \in I} \mathcal{C}_i$. S. Henry [10, Theorem B.8] showed that they have small meets. We will give another proof.

3.9. Theorem. *Let $(\mathcal{L}_i, \mathcal{R}_i)$, $i \in I$ be a set of accessible weak factorization systems in a locally presentable category. Then $(\cap_{i \in I} \mathcal{L}_i, \mathcal{R})$ is an accessible weak factorization system.*

Proof. Let $\mathcal{P}$ be the pseudopullback of all forgetful functors $\mathrm{Coalg}(L_i) \to \mathcal{K}^{\to}$ (see [17]). Then $\mathcal{P}$ is locally presentable and the full image of $U : \mathcal{P} \to \mathcal{K}^{\to}$ is $\mathcal{L} = \cap_{i \in I} \mathcal{L}_i$ (see [16, Lemma 2.6]). There is a regular cardinal λ such that $\mathcal{P}$ is locally λ-presentable and U preserves λ-filtered colimits. Let $\mathcal{C}$ be the (representative) full subcategory of λ-presentable objects in $\mathcal{P}$. Following [17, Theorem 4.3], $\mathcal{C}^{\boxplus} = \mathcal{P}^{\boxplus}$ and thus, for $\mathcal{R} = |\mathcal{P}^{\boxplus}|$, $(^{\square}\mathcal{R}, \mathcal{R})$ is an accessible weak factorization system (see [17, Remark 3.6 and Theorem 4.3]. It remains to show that $^{\square}\mathcal{R} = \mathcal{L}$.

Since

$$\mathcal{R}_i = |\mathrm{Coalg}(L_i)^{\boxplus}| \subseteq |\mathcal{P}^{\boxplus}| = \mathcal{R},$$

we have $\cup_{i \in I} \mathcal{R}_i \subseteq \mathcal{R}$. Hence

$$^{\square}\mathcal{R} \subseteq {}^{\square}(\cup_{i \in I} \mathcal{R}_i) \subseteq \cap_{i \in I} {}^{\square}\mathcal{R}_i \subseteq \mathcal{L}.$$

On the other hand,

$$\mathcal{L} = |\mathcal{P}| \subseteq |^{\boxplus}(\mathcal{P}^{\boxplus})| \subseteq {}^{\square}|\mathcal{P}^{\boxplus}| \subseteq {}^{\square}\mathcal{R}.$$

∎

3.10. Remark. Another proof was suggested by the referee. Consider the weak factorization system $(\prod_i \mathcal{L}_i, \prod_i \mathcal{R}_i)$ on $\mathcal{K}^I$. This is clearly accessible and the diagonal functor $\Delta : \mathcal{K} \to \mathcal{K}^I$ is a left adjoint. Then $(\cap_i \mathcal{L}_i, \mathcal{R})$ is the left-lifting of $(\prod_i \mathcal{L}_i, \prod_i \mathcal{R}_i)$ and, following [7, Theorem 2.6], it is an accessible weak factorization system.

4. Combinatorial model categories

4.1. Convention. In what follows, $(\mathcal{L}, \mathcal{R})$ will be a weak factorization system in a locally presentable category $\mathcal{K}$ cofibrantly generated by $\mathcal{I}$.

Denote by $\mathrm{Comb}(\mathcal{L})$ the class of all combinatorial model structures with $\mathcal{L}$ as cofibrations. We can order it by $(\mathcal{L}, \mathcal{W}_1) \leq (\mathcal{L}, \mathcal{W}_2)$ iff $\mathcal{W}_1 \subseteq \mathcal{W}_2$.

4.2. Proposition. *([16, Corollary 4.7])* $\mathrm{Comb}(\mathcal{L})$ *has small meets given as*

$$\wedge_{i \in I}(\mathcal{L}, \mathcal{W}_i) = (\mathcal{L}, \cap_{i \in I} \mathcal{W}_i).$$

4.3. Remark. (1) Consider $(\mathcal{L}, \mathcal{W}_i) \in \mathrm{Comb}(\mathcal{L})$ where $I \neq \emptyset$ such that $(\mathcal{L}, \mathcal{W}_{i_0})$ is left proper for some $i_0 \in I$. Each $\mathcal{W}_i \cap \mathcal{L}$ is cofibrantly generated by a set $\mathcal{S}_i$. Put $\mathcal{S} = \cup_{i \in I}(\mathcal{S}_i) \setminus \mathcal{S}_{i_0}$. Then the left Bousfield localization of $(\mathcal{L}, \mathcal{W}_{i_0})$ at $\mathcal{S}$, yields the join $\vee_{i \in I}(\mathcal{L}, \mathcal{W}_i)$ in $\mathrm{Comb}(\mathcal{L})$.

(2) Assuming Vopěnka's principle, $\mathrm{Comb}(\mathcal{L})$ is a large complete lattice, i.e., it has all joins and meets. In particular, $\mathrm{Comb}(\mathcal{L})$ has a smallest element. There are given as $\vee_{i \in I}(\mathcal{L}, \mathcal{W}_i) = (\mathcal{L}, \cup_{i \in I} \mathcal{W}_i)$ and $(\wedge_{i \in I}(\mathcal{L}, \mathcal{W}_i) = (\mathcal{L}, \cap_{i \in I} \mathcal{W}_i)$. This follows from Smith's theorem because, assuming Vopěnka's principle, every full subcategory of a locally presentable category has a small dense subcategory. Thus it is cone-reflective. Note that Vopěnka's principle is equivalent to the statement that any full subcategory of a locally presentable category is cone-reflective (see [2, Corollary 6.7], or [18, Theorem 4.2]).

4.4. Definition. *([19, Definition 2.1]) Let* $\mathcal{W}_{\mathcal{L}}$ *be the smallest class* $\mathcal{W}$ *of morphisms such that*

1. $\mathcal{R} \subseteq \mathcal{W}$,
2. $\mathcal{W}$ *satisfies the 2-out-of-3 condition, and*
3. $\mathcal{L} \cap \mathcal{W}$ *is closed under pushout, transfinite composition and retracts.*

If $(\mathcal{L}, \mathcal{W}_{\mathcal{L}})$ *is a model structure it is called left-determined.*

4.5. Remark. (1) Retracts are meant in the category of morphisms $\mathcal{K}^{\to}$. [19] assumes in (2) that $\mathcal{W}$ is closed under retracts. But this can be omitted following [13] (or Lemma 1 in Model category, nLab). On the other hand, we assume it in (3).

In what follows $\mathrm{cof}(\mathcal{X})$ will denote the closure of $\mathcal{X}$ under pushout, transfinite composition and retracts while $\mathrm{cell}(\mathcal{X})$ the closure under pushout and transfinite composition.

(2) If $(\mathcal{L}, \mathcal{W}_{\mathcal{L}})$ is a combinatorial model category, it is the smallest element in $\mathrm{Comb}(\mathcal{L})$. It always happens assuming Vopěnka's principle. But, without it, we do not know whether the smallest element in $\mathrm{Comb}(\mathcal{L})$ might exist without being equal to $\mathcal{W}_{\mathcal{L}}$.

Recently, S. Henry [9] proved the existence of a left-determined model structure in ZFC under mild assumption.

4.6. Notation. We put $\mathcal{W}_0 = \mathcal{R}$, $\mathcal{W}_{i+1} = \overline{\mathcal{W}_i}$ if i is an even ordinal, $\mathcal{W}_{i+1} = \operatorname{cof}(\mathcal{L} \cap \mathcal{W}_i) \cup \mathcal{W}_i$ if i is an odd ordinal and $\mathcal{W}_i = \cup_{j<i}\mathcal{W}_j$ if $0 < i$ is a limit ordinal. Recall that any limit ordinal is even and $i+1$ is odd iff i is even. Then $\mathcal{W}_\mathcal{L} = \cup_i \mathcal{W}_i$ where i runs over all ordinals.

We say that $\mathcal{W}_i$ *stops* if $\mathcal{W}_\mathcal{L} = \mathcal{W}_i$ for an ordinal i.

4.7. Theorem. *$(\mathcal{L}, \mathcal{W}_\mathcal{L})$ is a combinatorial model category iff $\mathcal{W}_i$ stops.*

Proof. I. Assume that $(\mathcal{L}, \mathcal{W}_\mathcal{L})$ is a combinatorial model structure. Then $\mathcal{L} \cap \mathcal{W}_\mathcal{L}$ is cofibrantly generated by a set $\mathcal{S}$. There is an odd ordinal i such that $\mathcal{S} \subseteq \mathcal{W}_i$. Thus $\mathcal{L} \cap \mathcal{W}_\mathcal{L} \subseteq \mathcal{W}_{i+1}$. Hence $\mathcal{W}_\mathcal{L} \subseteq \mathcal{W}_{i+2}$ and the construction stops.

II. Assume that $\mathcal{W}_i$ stops. At first, we replace $\mathcal{W}_i$ by $\mathcal{W}_i^*$ which are full images of accessible functors. They are defined in the same way as $\mathcal{W}_i$ for i even. $\mathcal{R}$ is a full image of an accessible functor (following [16, Proposition 3.3]) and 2-out-of-3 closure and union keep full images of accessible functors (see Proposition 2.4 and Proposition 2.2). Let i be odd. We will follow the proof of Smith's theorem given in [4]. Since $\mathcal{W}_i$ is cone-reflective (see Proposition 2.1), satisfies the 2-out-of-3 property and contains $\mathcal{R}$, [4, Lemma 1.9] produces a set $\mathcal{J}_i$ from [4, Lemma 1.8] for $\mathcal{L}$ and $\mathcal{W}_i$. This means that $\mathcal{J}_i \subseteq \mathcal{L} \cap \mathcal{W}_i$ and every $f \in \mathcal{W}_i$ can be factored as hg with $g \in \operatorname{cell}(\mathcal{J}_i)$ and $h \in \mathcal{R}$. We put $\mathcal{W}_{i+1}^* = \operatorname{cof}(\mathcal{J}_i) \cup \mathcal{W}_i^*$. Then $\mathcal{W}_{i+1}^* \subseteq \mathcal{W}_{i+1}$. Following Theorem 3.4 and Proposition 2.2, $\mathcal{W}_{i+1}^*$ is a full image of an accessible functor. Take $f \in \mathcal{W}_i$ and its expression as $f = hg$ with $g \in \operatorname{cell}(\mathcal{J}_i)$ and $h \in \mathcal{R}$. T Since $g \in \mathcal{W}_{i+1}^*$ and $h \in \mathcal{R} \subseteq \mathcal{W}_i^*$, we have $f \in \mathcal{W}_{i+2}^*$. Hence $\mathcal{W}_i \subseteq \mathcal{W}_{i+2}^*$. Consequently, $\mathcal{W} = \cup_i \mathcal{W}_i^*$.

Since $\mathcal{W}_i$ stops, $\mathcal{W}_i^*$ stops as well. Hence $\mathcal{W}$ is a full image of an accessible functor and thus it is cone-reflective (see Proposition 2.1). Smith's theorem implies that $(\mathcal{L}, \mathcal{W})$ is a combinatorial model category. ∎

5. Accessible model categories

A model structure $(\mathcal{C}, \mathcal{W})$ on a locally presentable category $\mathcal{K}$ is *accessible* if both $(\mathcal{C}, \mathcal{C}^\square)$ and $(\mathcal{C} \cap \mathcal{W}, (\mathcal{C} \cap \mathcal{W})^\square)$ are accessible weak factorization systems.

5.1. Theorem. *Let $(\mathcal{C}, \mathcal{W})$ be an accessible model structure on a locally presentable category $\mathcal{K}$. Then $\mathcal{W}$ is a full image of an accessible functor.*

Assuming the existence of a proper class of almost strongly compact cardinals, $\mathcal{W}$ is preaccessible and preaccessibly embedded to $\mathcal{K}^\to$.

Proof. The first claim is what [17, Remark 5.2(2)] proves, using [16, Lemma 2.6] and Theorem 3.1. The second claim follows from [16, Proposition 2.2]. ∎

5.2. Remark. To correct [17, Proposition 5.3], one has to replace (4) by

(4') $\mathcal{W}$ is preaccessible and preaccessibly embedded to $\mathcal{K}^\to$.

Indeed, in the proof, $\mathcal{P}$ is preaccessible and preaccessibly embedded to $\mathcal{K}^\to$ and thus it has a small dense subcategory $\mathcal{J}$ of λ-presentable objects. This is what the proof needs.

Note that we can apply [17, Lemma 3.3] because the forgetful functor $\mathcal{P} \to \mathcal{K}^{\to}$ preserves λ-directed colimits.

References

[1] J. Adámek, H. Herrlich, J. Rosický and W. Tholen, Weak factorization systems and topological functors, Appl. Categ. Structures 120 (2002), 237–249.

[2] J. Adámek and J. Rosický, Locally Presentable and Accessible Categories, Cambridge University Press 1994.

[3] B. Banaschewski and G. Bruns, Categorical characterization of the Mac Neille completion, Arch. Math. (Basel) 18 (1967), 369–377.

[4] T. Beke, Sheafifiable homotopy model categories, Math. Proc. Cambridge Philos. Soc. 129 (2000), 447–475.

[5] A. Brooke-Taylor and J. Rosický, Accessible images revisited, Proc. AMS 145 (2017), 1317–1327.

[6] D.-C. Cisinski, Théorie homotopiques dans les topos, J. Pure Appl. Alg. 174 (2002), 43–82.

[7] R. Garner, M. Kędziorek and E. Riehl, Lifting accessible model structures, J. Topology 13 (2020), 59–76.

[8] P. Halmos, Lectures in Boolean algebras, Van Nostrand 1963.

[9] S. Henry, Minimal model structures, arXiv:2011.13408.

[10] S. Henry, Combinatorial and accessible weak model categories, arXiv:2005.02360.

[11] M. Makkai and R. Paré, Accessible categories: The foundation of Categorical Model Theory, Contemp. Math. 104, AMS 1989.

[12] M. Olschok, Left determined model structures for locally presentable categories, Appl. Categ. Structures 19 (2011), 901–938.

[13] E. Riehl, A concise definition of a model category, http://www.math.jhu.edu/ eriehl/modelcat.pdf.

[14] J. Rosický, More on directed colimits of models, Appl. Cat. Struct. 2 (1994), 71-76.

[15] J. Rosický, Generalized Brown representability in homotopy categories, Theory Appl. Categ. 14 (2005), 451–479.

[16] J. Rosický, On combinatorial model categories, Appl. Categ. Structures 17 (2009), 303-316.

[17] J. Rosický, Accessible model categories, Appl. Categ. Structures 25 (2017), 187–196.

[18] J. Rosický and W. Tholen, Accessibility and the solution set-condition, J. Pure Appl. Algebra 98 (1995), 189–208.

[19] J. Rosický and W. Tholen, Left-determined model categories and universal homotopy theories, Trans. Amer. Math. Soc. 355(9) (2003), 3611–3623.

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