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THE BICATEGORY OF TOPOLOGICAL CORRESPONDENCES

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ABSTRACT. It is known that a topological correspondence (X, λ) from a locally compact groupoid with a Haar system (G, α) to another one, (H, β), produces a C^*-correspondence $\mathcal{H}(X, \lambda)$ from $C^*(G, \alpha)$ to $C^*(H, \beta)$. We described the composition of two topological correspondences in one of our earlier articles. In the present article, we prove that second countable locally compact Hausdorff groupoids with Haar systems form a bicategory $\mathfrak{T}$ when equipped with topological correspondences as 1-arrows and isomorphisms of topological correspondences as 2-arrows.

On the other hand, it well-known that C^*-algebras form a bicategory $\mathfrak{C}$ with C^*-correspondences as 1-arrows, and the unitary isomorphisms of Hilbert C^*-modules that intertwine the representations serve as the 2-arrows. In this article, we show that a topological correspondence going to a C^*-one is a bifunctor $\mathfrak{T} \longrightarrow \mathfrak{C}$. Finally, we show that in the sub-bicategory of $\mathfrak{T}$ consisting of the Macho-Stadler–O'uchi correspondences, invertible 1-arrows are exactly the groupoid equivalences.

1. Introduction

1.1. C^*-CORRESPONDENCES: C^*-correspondence is a well-established notion in operator algebras: by a C^*-correspondence $A \longrightarrow B$, where A and B are C^*-algebras, we mean a pair $(\mathcal{H}, \phi)$ where $\mathcal{H}$ is a Hilbert B-module, and $\phi\colon A \longrightarrow \mathbb{B}(\mathcal{H})$ is a nondegenerate representation of A on the C^*-algebra of adjointable operators on the Hilbert C^*-module $\mathcal{H}$. To cite a few remarkable usage of C^*-correspondences, one can notice that they appear in KK-theory of Kasparov (e.g. [Kasparov, 1988]), or in the study of Cuntz–Pimsner algebras e.g. [Katsura, 2004, Pimsner, 1997]. The memoir [Echterhoff-Kaliszewski-Quigg, 2006] by Echterhoff, Kaliszewski, Quigg and Raeburn is an excellent example that justifies the importance of C^*-correspondences in the study of C^*-algebras. In this memoir, a C^*-correspondence from A to B is called a "right-Hilbert A-B-bimodule".

An imprimitivity bimodule of Rieffel [Rieffel, 1974] is a particular type of—specifically, an *invertible*—C^*-correspondence; here the definite meaning of invertibility could be taken as either an invertible arrow in a certain category of C^*-algebras as in [Landsman, 2000,

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Proposition 4.7] or the well-known natural equivalences described in Theorems 4.1–4.3 in [Echterhoff-Kaliszewski-Quigg, 2006].

1.2. Bicategories: Since there is literature (e.g. [Buss-Meyer-Zhu, 2013, Landsman, 2000, MacLane-Saunders, 1998]) which explains bicategories comprehensively and still avoids complicated equations and symbols, we do not venture to describe bicategories (that too not as effectively as the others have done) here.

Bicategories and 2-categories have been known for a very long time. Bénabou's notes [Bénabou, 1967] (1978) are our main reference for bicategories. Apart from these notes, Mac Lane's book [MacLane-Saunders, 1998, XII §3, §6 and §7] and Leinster's notes [Leinster, 1998] are the standard references.

Briefly speaking, bicategories are categories in which morphisms are enriched. To loosely explain, call the morphisms of the category as 1-arrows. Then one has morphisms—called the 2-arrows—between the 1-arrows. And the 2-arrows, in turn, form a category. In a bicategory, the identities and associativity of composition of 1-arrows are replaced by the identity and associativity isomorphisms, respectively. All this data satisfies certain consistency conditions.

In bicategories, associativity of 1-arrows holds *up to* isomorphism. If equality holds in associativity, the bicategory is called a 2-category.

Bicategories are not new to algebraists and geometers. Rings, modules and module homomorphisms constitute a bicategory. The Morita equivalence of rings can be formulated elegantly in the language of bicategories, e.g. [MacLane-Saunders, 1998, Bénabou, 1967, Landsman, 2000]. In a topological space, the points, paths and path homotopies make up a 2-category ([MacLane-Saunders, 1998]).

1.3. Bicategories and C*-correspondences: As it is well-known, a nondegenerate *-homomorphism $\phi\colon A \longrightarrow \mathcal{M}(B)$ produces the C*-correspondence $(B,\phi)\colon A \longrightarrow B$ wherein B is considered as a Hilbert B-module in the obvious standard way, and ϕ is the nondegenerate representation of A on the Hilbert module. Therefore, the C*-correspondences generalise the *-homomorphisms of C*-algebras, as well as, the nondegenerate *-homomorphisms into the multiplier C*-algebras. Which suggests to view C*-correspondences as generalised morphisms of C*-algebras. In fact, the usual categories of C*-algebras—the one with *-homomorphisms and the one with *-homomorphisms into multiplier algebras—*sit* inside all the (bi)categories described in the following discussion. This very observation has lead many mathematicians to compose categories of C*-algebras in which morphisms are or are obtained from C*-correspondences. In all these categorise, the interior tensor product of Hilbert C*-modules ([Lance, 1995, Chapter 4]) is used to define the composition of morphisms. Some instances of these categories are discussed below.

In [Schweizer, 2001, §1] and [Schweizer, 2000, §1], Schweizer defines the category of C*-algebras in which the morphisms are unitary equivalence classes of C*-correspondences.

In the famous monograph [Echterhoff-Kaliszewski-Quigg, 2006], Echterhoff, Kaliszewski, Quigg and Raeburn develop a categorical framework for C*-dynamical systems. In this framework, the unitary equivalence classes of appropriate C*-correspondences play the role of the morphisms in certain categories of C*-dynamical systems. The crossed products of C*-algebras, induction, restriction, etc. are seen as functors of these categories. The main theorems ([Echterhoff-Kaliszewski-Quigg, 2006, Chapter 4]) prove that various imprimitivity bimodules implement an equivalence of some of these functors.

In fact, the setup of Echterhoff, Kaliszewski, Quigg and Raeburn has a straightforward bicategorical interpretation. This is because the hom-sets in **CAT**—the category of all small categories—can be enriched with natural transformations that yields a bicategory obtained from **CAT** (in fact, **CAT** is a 2-category). Reader may refer to [MacLane-Saunders, 1998, XII.3] for details about the bicategory obtained by enriching the hom-sets of **CAT**.

In [Buss-Meyer-Zhu, 2013], Buss, Meyer and Zhu take a different categorical approach towards C*-correspondences; they do not consider isomorphism classes of C*-correspondences as morphisms of C*-algebras but the C*-correspondences themselves. They define the unitary isomorphisms of C*-correspondences as the *morphisms* between the C*-correspondences. This constitutes a bicategory of C*-correspondences. In this framework, Buss, Meyer and Zhu characterise (i) the saturated Fell bundles over a discrete group G and (ii) the Busby–Smith twisted actions of G in terms of the weak actions of G on appropriate bicategories of C*-correspondences.

They also prove that two weak actions of G are equivalent *iff* (i) the corresponding Fell bundles are equivalent or (ii) the Busby–Smith twisted actions are equivalent. They also extend these results to locally compact groups by enriching the bicategories with topological assumptions. This is an instance that highlights the importance of considering the C*-correspondences—rather than their isomorphism classes—as the morphisms of C*-algebras.

We note that Buss, Meyer and Zhu give a concrete categorical meaning for referring to the Fell bundles as generalised C*-dynamical systems. We are interested in this bicategory of C*-correspondences. Landsman [Landsman, 2000] defines a similar bicategory of C*-correspondences. However, his 2-arrows are more general.

Brouwer constructs a bicategory of von Neumann algebras in [Brouwer, 2003] which is similar to that of C*-algebras Buss, Meyer and Zhu construct.

1.4. Bicategories of C*-correspondences: It is clear from earlier discussion that C*-correspondences and their categorical nature, either as a bicategory or natural transformations, is not new. In fact, Rieffel's motivation for defining imprimitivity bimodules ([Rieffel, 1974]) was to define isomorphism of representation categories of C*-algebras. Or, in other words, his aim was to define the Morita equivalence of C*-algebras. Landsman phrases and proves this fact explicitly in categorical language in [Landsman, 2000, Theorem 4.6 and Proposition 4.7].

In general, one can talk about representations of C*-algebras on Hilbert modules. Blecher takes this general case and proves Morita equivalence for C*-algebras considering the Hilbert module representations in [Blecher 1997, §5].

One remark is that Landsman [Landsman, 2000] describes the most general bicategory of C*- and von Neumann- correspondences; unlike Buss, Meyer and Zhu ([Buss-

Meyer-Zhu, 2013]), Landsman defines the 2-arrows as the bimodules maps of the Hilbert C*-modules which are not necessarily unitaries.

1.5. C*-correspondence obtained from topological or dynamics data: For group(oid)s G and H, we call a space X a G-H-bispace if X is a respectively a left G- and right H- space, and the actions commute in the usual sense. All groupoids in the following discussion are locally compact, Hausdorff, second countable and equipped with Haar systems.

Various topological structures or dynamical systems are known which have natural C*-correspondences or imprimitivity bimodules associated with them: Rieffel's classical list [Rieffel, 1982] of bispaces and transformation groupoids, and the imprimitivity bimodules associated with them is well-known. Graphs and topological graphs have associated C*-correspondences, e.g. [Katsura, 2003, Katsura-I, 2004, Katsura, 2009]. In a very general setting, an appropriate groupoid bispace—popularly called a groupoid equivalence, and defined by Renault, Williams and Muhly in [Muhly-Renault-Williams, 1987]—produces an imprimitivity bimodule between the associated groupoid C*-algebras.

In a similar fashion as Muhly, Renault and Williams, a groupoid bispace—with some conditions on the groupoid actions—produces a C*-correspondence between groupoid C*-algebras. We call such bispaces as *topological correspondences* of groupoids with Haar system. Some examples of topological correspondences are as follows: the topological correspondences defined by Marta-Stadler and O'uchi [Stadler-Ouchi, 1999]; the generalised morphisms defined by Buneci and Stachura [Buneci-Stachura, 2005]; and the generalised morphisms of locally Hausdorff groupoids defined by Tu [Tu, 2004].

The correspondences Marta-Stadler and O'uchi, and Tu define are topological variant of the well-known Hilsum-Skandalis maps [Hilsum-Skandalis, 1987].

Topological correspondences as morphisms of groupoids: Moerdijk seems to be the first one to explicitly consider topological correspondences as morphisms of groupoids in his article [Moerdijk, 1988] published in 1988. In this article, Moerdijk represents a topos by a topological groupoid. Then the "geometric morphisms" naturally show up as counterparts of topos morphisms. In [Moerdijk, 1988], Moerdijk presents a universal property involving a category of fractions for the aforementioned construction; he shows that that the category of topos can be obtained as a category of fractions from the category of topological groupoids equipped with geometric correspondences as morphisms. Much later, Pronk generalises this universal property in a bicategorical setup [Pronk, 1996, Thm 27, 28, 34, Cor 35].

As hinted above, in the world of geometry, "geometrical correspondences" naturally occur as morphisms of groupoids, such as for differentiable stacks [Pronk, 1996], [Deligne-Mumford, 1969]; for foliations [Hilsum-Skandalis, 1987],[Haefliger, 1984].

In particular, the roles of equivalence of groupoids [Muhly-Renault-Williams, 1987] in various categories—basically as some sort of *invertible arrows*—have been observed by multiple authors in various contexts since long time, for example [Muhly-Renault-Williams, 1987],[Mrcun, 1999, Proposition 1.7 and Corollary 1.8], [Moerdijk-Mrcun2005], and [Hilsum-Skandalis, 1987]

Topological correspondences not only appear in geometry, but parallels can be observed between the “correspondences” in algebraic geometry [Fulton, 1998] and topological correspondences of spaces (called as quivers in [Muhly-Tomforde, 2005]).

We end this discussion by recommending Section 2 of [Moerdijk-Mrcun2005] to the reader; we find this writeup an excellent and concise review of topological correspondences as morphisms in the context of Lie groupoids. It cites all important instances wherein these groupoids morphisms show up.

1.6. Present article: In our first article [Holkar-1, 2017] of this series, we investigated the most general definition of a topological correspondence. This led us to Definition 2.1 in [Holkar-1, 2017]. There we describe some data and conditions on a groupoid bispace that are sufficient to produce a C*-correspondences and broad enough to generalise the topological correspondences mentioned in the last paragraph. Briefly speaking, our topological correspondence $(G,\alpha)\longrightarrow(H,\beta)$ of locally compact groupoids with Haar systems is a pair (X,λ) where X is a G-H-bispace with the H-action proper, and λ is an H-invariant family of measures along the momentum map $X\longrightarrow H^{(0)}$ such that each $\lambda_u, u\in H^{(0)}$, is (G,α)-quasi-invariant. In that article, we provide a bank of examples of topological correspondences. These examples are analogues of the standard examples of C*-correspondences. Examples 3.1 and 3.4 in [Holkar-1, 2017], respectively, show that maps of spaces and group homomorphisms can be seen as topological correspondences. Now on, by a topological correspondence, we mean the topological correspondence defined in [Holkar-1, 2017].

In the next article [Holkar-2, 2017] of the series, we describe composition of topological correspondences which is the topological counterpart of the interior tensor product of C*-correspondences. Examples 4.1 and 4.3 in [Holkar-2, 2017] show that the composition of maps of spaces and composition of group homomorphisms agree with the compositions of the topological correspondences associated with them.

The present article is the last installment in the series wherein we prove that topological correspondences form a bicategory $\mathfrak{T}$, see Theorem 3.14. In the bicategory $\mathfrak{T}$, the objects are the locally compact Hausdorff second countable groupoids with Haar systems, 1-arrows are the topological correspondences, and the 2-arrows are the isomorphisms of topological correspondences defined in Definition 3.8.

Let $\mathfrak{C}$ the bicategory of C*-correspondences which Buss, Meyer an Zhu define in [Buss-Meyer-Zhu, 2013]. In $\mathfrak{C}$, the objects are C*-algebras, C*-correspondences are the 1-arrows, and the unitary isomorphisms of the correspondences are the 2-arrows. We prove that $\mathfrak{T}$ is the topological analogue of the bicategory of topological correspondences $\mathfrak{C}$ in the sense that the C*-functor is a bifunctor $\mathfrak{T}\longrightarrow\mathfrak{C}$, see Theorem 3.28.

The Macho-Stadler–O'uchi correspondences are *purely* topological since they do not demand existence of a family of measures on the bispace; this family of measures is induced by the Haar system on one of the groupoids. In the context of the Hausdorff groupoids, Macho-Stadler–O'uchi correspondences stand out as a very general notion of correspondences. The Macho-Stadler–O'uchi correspondences with open surjective momentum maps constitute a sub-bicategory $\mathfrak{GO}$ of $\mathfrak{T}$. We prove in Theorem 3.33 that the equivalence of groupoids is equivalent to invertibility of 1-arrows in $\mathfrak{GO}$. Theorem 3.33 is similar to Landsman's result [Landsman, 2000, Proposition 4.7].

In fact, one may expect that, a version of, Theorem 3.33 holds for the bicategory of topological correspondences. That is, groupoid equivalences play the role of the invertible 1-arrows in the bicategory of topological correspondences. However, we could not succeed in showing a certain isomorphism of families of measures required for such a result. We discuss the details regarding this in the last remark of the article.

Since the process of composing topological correspondences ([Holkar-2, 2017]) is intricate, those technicalities show up in this article also. In fact, topological correspondences form a bicategory is quite obvious result that not only an expert but also others may expect. However, the complications involved in the composition obscure the proofs of this expected *obvious* result. Therefore, we take over the task of writing the proofs elaborately. And, as it turns out, the proofs are intricate involving technicalities. An attempt is made to make the reader refer to earlier articles, [Holkar-1, 2017] and [Holkar-2, 2017], as little as possible. This also adds to the length of the article.

Structure of the article: In the first section, Section 2, we recall the main results and ideas from the earlier articles [Holkar-1, 2017] and [Holkar-2, 2017]. The most technical idea in [Holkar-2, 2017] that will be used very frequently is assigning appropriate family of measures in the composite of topological correspondences. This idea is discussed between Example 2.11 on page 853 and Definition 2.17 on page 857. As a matter of fact, using this idea at appropriate places makes the article lengthy and technical. We also recall some useful examples from our earlier articles.

Above review of earlier articles is followed by the definition of bicategory and bifunctor from Bénabou's notes [Bénabou, 1967]. Experts may find this definition an unnecessary repetition, however, we do repeat it to establish our notation, and use the definition as a checklist for proving Theorem 3.14.

The second section, Section 3.1, starts by defining an isomorphism of topological correspondences. Then we discuss some examples of isomorphisms of topological correspondences, and prove a few useful technical lemmas.

In the last section, Section 3.12, we define the bicategory of topological correspondences and prove Theorem 3.14. In Section 3.23, Theorem 3.28—which shows that the C^*-assignment is a bifunctor—is proved. Finally, in Section 3.32, we show that in the sub-bicategory of $\mathfrak{T}$ consisting of the Macho-Stadler–O'uchi correspondences, a 1-arrow is invertible if and only if it is an equivalence of the groupoids.

2. Recap

All the groupoids and spaces in this article are locally compact, Hausdorff and second countable. The symbols $\approx$ and $\simeq$ stand for "homeomorphic" and "isomorphic", respectively.

Let G be a topological groupoid. By $G^{(0)}$ we denote the space of units of G equipped with the subspace topology; $\mathrm{inv}_G\colon G \longrightarrow G$ denotes the inversion map on G, $\mathrm{inv}_G(\gamma) = \gamma^{-1}$ for $\gamma \in G$. Let X be a left (or right) G-space; we tacitly assume that r_X (respectively, s_X) is the momentum map for the action. The fibre product $G \times_{s_G, G^{(0)}, r_X} X$ of G and X over $G^{(0)}$ along s_G and r_X is denoted by $G \times_{G^{(0)}} X$. If X is a right G-space, then $X \times_{G^{(0)}} G$ has a similar meaning. The transformation groupoid for the action of G on X is denoted by $G \ltimes X$ (respectively, $X \rtimes G$). By r_G and s_G we denote the range and the source maps of G, respectively, which are also the momentum maps for the left and right multiplication action of G on itself. If X and Y are, respectively, left and right G spaces, then we denote the fibre product $X \times_{s_X, G^{(0)}, r_Y} Y$ by $X \times_{G^{(0)}} Y$, that is, $X \times_{G^{(0)}} Y = \{(x, y) \in X \times Y : s_X(x) = r_Y(y)\}$. The equivalence class of $x \in X$ in the quotient X/G is denoted by $[x]$; similarly, if $f\colon X \longrightarrow Y$ is a G-equivariant map, the map it induces $X/G \longrightarrow Y/G$ is denoted by $[f]$.

Assume that H is another groupoid acting on the left G-space X from right. We call X an H-G-bispace if the actions commute, that is, for all composable pairs $(\eta, x) \in H \times_{H^{(0)}} X$ and $(x, \gamma) \in X \times_{G^{(0)}} G$ we have that $(\eta x, \gamma) \in X \times_{G^{(0)}} G$ and $(\eta, x\gamma) \in H \times_{H^{(0)}} X$, and $\eta(x\gamma) = (\eta x)\gamma$.

For $A, B \subseteq G^{(0)}$, $G^A := \{\gamma \in G\colon r_G(\gamma) \in A\}, G_B := \{\gamma \in G\colon s_G(\gamma) \in B\}$ and $G^A_B := G^A \cap G_B$. If $A = \{a\}$ and $B = \{b\}$ are singletones, then we write G^a, G_b, G^a_b instead of G^A, G_B, G^A_B, respectively.

If α is a Haar system on G, then $\mathrm{C}^*(G, \alpha)$ denotes the full C^*-algebra of the groupoid (G, α) equipped with a Haar system.

We assume that the reader is familiar with the standard material about proper actions of groupoids, and invariant and quasi-invariant families of measures on a G-space where G is a groupoid which is well-known to experts and appears in most of the works. We refer the reader to Section 1.2 of [Holkar-1, 2017] for the exact information we require.

2.1. Lemma. [Proposition 1.3.21 in [Holkar, 2014]] *Let (G, α) be a locally compact groupoid with a Haar system and X a proper G-space. The Haar system induces a continuous G-invariant family of measures $\alpha_X = \{\alpha_{X[x]}\}_{[x] \in X}$ along the quotient mapping $q\colon X \longrightarrow X/G$; α_X is defined as*

$$\int_X f \,\mathrm{d}\alpha_{X[y]} = \int_G f(y\gamma)\,\mathrm{d}\alpha^{s_X(y)}(\gamma)$$

$y \in Y$ and for $f \in \mathrm{C_c}(X/G)$.

2.2. Proposition and Definition. [Cutoff function] *Let X be a proper G-space for a groupoid with Haar systems (G, α) such that X/G is paracompact. Then there is a positive function e, called a cutoff function, on X with the following properties:*

(i) e is not identically zero on any equivalence class for the action of G on X;

(ii) for every compact subset $K \subseteq X/G$, the intersection $q^{-1}(K) \cap \mathrm{supp}(e)$ is a compact subset of X; $q\colon X \longrightarrow X/G$ the quotient map.

(iii) $\int_G c(x\gamma)\,\mathrm{d}\alpha^u(\gamma) = 1$ *for all* $x \in X$.

Proof. Let e' be a function satisfying first two conditions; its existence is assured by [Bourbaki, 2004, Lemma 1, Appendix I]. Define $a\colon X/G \longrightarrow \mathbb{R}$ by $a = \alpha_X(e')$ where α_X is the family of measures along q in Lemma 2.1 above. Then $0 < a([x]) < \infty$ for all $[x] \in X/G$, see proof of Lemma 2.5 in [Holkar-2, 2017] for details. Finally, $e := e'/a \circ q$ is the function that satisfies all required properties. ■

Let A, B be C* algebras, $\mathcal{H}$ a Hilbert B-module and $\phi\colon A \longrightarrow \mathbb{B}(\mathcal{H})$ a nondegenerate representation; we call the pair $(\mathcal{H}, \phi)$ a C*-correspondence from A to B. Most frequently, we simply call $\mathcal{H}$ a C*-correspondence from A to B; and in such cases, we write $a \cdot \xi$ or simply $a\xi$ instead of $\phi(a)\xi$ for $a \in A$ and $\xi \in \mathcal{H}$. Let C be another C*-algebra and $\mathcal{K}\colon B \longrightarrow C$ a C*-correspondence. Then $\mathcal{H} \hat{\otimes}_B \mathcal{K}$ is the interior tensor product of the Hilbert modules (see [Lance, 1995, Chapter 4]); we may also write $\mathcal{H} \hat{\otimes} \mathcal{K}$ when the middle C*-algebra is clear.

We denote the multiplicative group of positive real numbers by $\mathbb{R}_+^*$.

We shall use the following important remark regarding equivariant groupoid cochain complexes in many arguments regarding 2-arrows in the bicategory topological correspondences.

2.3. Remark. [Remark 1.14 in [Holkar-1, 2017]] Let G, H be Borel groupoids and A an abelian group. If $b, b' \in \mathrm{C}^0_H(G, A)$ are A-valued H-invariant 0-cochains with coboundaries $d^0(b) = d^0(b')$, then $c := (b - b') \in \mathrm{C}^0_H(G, A)$ is a 0-cochain with

$$d^0(c) = d^0(b) - d^0(b') = 0.$$

If required, the reader may refer the discussion of groupoid equivariant cohomology of a Borel groupoid in Section 1 of [Holkar-1, 2017] for definitions in previous remark.

2.4. Topological correspondences.

2.5. Definition. [Topological correspondence ([Holkar-1, 2017] Definition 2.1)] *A topological correspondence from* (G, α)*, a locally compact groupoid with a Haar system, to* (H, β)*, a locally compact groupoid equipped with a Haar system, is a pair* (X, λ) *where:*

i) X *is a locally compact* G-H*-bispace,*

ii) the action of H *is proper,*

iii) $\lambda = \{\lambda_u\}_{u \in H^{(0)}}$ *is an* H*-invariant continuous family of measures along the momentum map* $s_X\colon X \longrightarrow H^{(0)}$*,*

iv) there is a continuous function $\Delta\colon G \ltimes X \to \mathbb{R}^+$ *such that for each* $u \in H^{(0)}$ *and* $F \in \mathrm{C_c}(G \times_{G^{(0)}} X)$*,*

$$\int_{X_u} \int_{G^{r_X(x)}} F(\gamma^{-1}, x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\,\mathrm{d}\lambda_u(x) = \int_{X_u} \int_{G^{r_X(x)}} F(\gamma, \gamma^{-1}x)\,\Delta(\gamma, \gamma^{-1}x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\,\mathrm{d}\lambda_u(x).$$

The function Δ is unique and is called *the adjoining function* of the correspondence. The last condition above implies that for each $u \in H^{(0)}$ the measure λ_u on X is (G,α)-quasi-invariant, and the function Δ implements the quasi-invariance. The adjoining function is H-invariant, see [Holkar-1, 2017, Remark 2.5].

A topological correspondence $(G,\alpha) \longrightarrow (H,\beta)$ produces a C*-correspondence

$$\mathrm{C}^*(G,\alpha) \longrightarrow \mathrm{C}^*(H,\beta);$$

following is the recipe: for $\phi \in \mathrm{C_c}(G)$, $f \in \mathrm{C_c}(X)$ and $\psi \in \mathrm{C_c}(H)$ define the functions $\phi \cdot f$ and $f \cdot \psi$ on X as follows:

$$\begin{cases} (\phi \cdot f)(x) := \int_{G^{r_X(x)}} \phi(\gamma) f(\gamma^{-1}x)\, \Delta^{1/2}(\gamma,\gamma^{-1}x)\, \mathrm{d}\alpha^{r_X(x)}(\gamma), \\ (f \cdot \psi)(x) := \int_{H^{s_X(x)}} f(x\eta)\psi(\eta^{-1})\, \mathrm{d}\beta^{s_X(x)}(\eta). \end{cases} \tag{1}$$

For $f, g \in \mathrm{C_c}(X)$ define the complex valued function $\langle f, g\rangle$ on H by

$$\langle f, g\rangle(\eta) := \int_{X_{r_H(\eta)}} \overline{f(x)} g(x\eta)\, \mathrm{d}\lambda_{r_H(\eta)}(x). \tag{2}$$

We often write ϕf and $f\psi$ instead of $\phi \cdot f$ and $f \cdot \psi$, respectively. Lemma 2.6 in [Holkar-1, 2017] shows that $\phi f, f\psi \in \mathrm{C_c}(X)$ and $\langle f\,, g\rangle \in \mathrm{C_c}(H)$. One can check that Equation (1) makes $\mathrm{C_c}(X)$ into a bimodule over the pre-C*-algebras $\mathrm{C_c}(G)$ and $\mathrm{C_c}(H)$. And Equation (2) defines a $\mathrm{C_c}(H)$-valued sesquilinear form on $\mathrm{C_c}(X)$.

2.6. Theorem. [Holkar-1, 2017, Theorem 2.10] *Let (G,α) and (H,β) be locally compact groupoids with Haar systems and (X,λ) a topological correspondence $(G,\alpha) \longrightarrow (H,\beta)$. Then the bimodule $\mathrm{C_c}(X)$ over the pre-C*-algebras $\mathrm{C_c}(G)$ and $\mathrm{C_c}(H)$ equipped with the $\mathrm{C_c}(H)$-valued sesquilinear form defined by Equation (2) completes to a C*-correspondence $\mathcal{H}(X,\lambda)\colon C^*(G,\alpha) \longrightarrow C^*(H,\beta)$.*

The main tasks in the proof of this theorem are to show that the sesquilinear form is positive and that the left action of $\mathrm{C_c}(G,\alpha)$ extends to a nondegenerate representation of $\mathrm{C}^*(G,\alpha)$. The first claim requires Renaults disintegration theorem. Whereas the latter one follows by a computation involving the GNS construction. This theorem holds for locally Hausdorff, locally compact groupoids equipped with Haar systems described in [Renault, 1985].

We reproduce some examples from [Holkar-1, 2017] below.

2.7. Example. [Example 3.1 in [Holkar-1, 2017]] Let $f\colon X \longrightarrow Y$ be a mapping of spaces. Consider X and Y as the trivial étale groupoid. Then X is a Y-X-bispace for the trivial left and right actions; the momentum maps being $Y \xleftarrow{f} X \xrightarrow{\mathrm{Id}_X} X$. These actions are proper. The pointmasses $\{\delta_x\}_{x\in X}$ constitute an X-invariant continuous family of measures along Id_X. This family of measures is also Y-invariant. Therefore, $(X, \{\delta_x\}_{x\in X})$ is a topological correspondence from $X \longrightarrow Y$. The associated C*-correspondence is $(\mathrm{C}_0(X), \phi^*)\colon \mathrm{C}_0(Y) \longrightarrow \mathrm{C}_0(X)$ where $f^*\colon \mathrm{C}_0(Y) \longrightarrow \mathrm{C}_b(X)$ is the *-homomorphism induced by f.

2.8. Example. [Example 3.4 in [Holkar-1, 2017]] Let $f\colon G \longrightarrow H$ be a homomorphism of locally compact groups. Assume that α and β are the Haar measures on G and H, respectively. Let δ_G and δ_H denote the modular functions of G and H, respectively.

Then H is a G-H-bispace where G acts on H via f and the right action is by right multiplication. Then the right action is proper. Equipe H with the right invariant measures β^{-1}. Then a direct computation involving modular functions show that β^{-1} is (G,α)-quasi-invariant with the adjoining function being $\delta_H \circ f/\delta_G$. The associated C^*-correspondence is the *-homomorphism $f_*\colon \mathrm{C}^*(G) \longrightarrow \mathrm{C}^*(H)$ induced by f.

2.9. Example. [Macho-Stadler and O'uchi correspondences, Example 3.8 in [Holkar-1, 2017]] Macho-Stadler and O'uchi define ([Stadler-Ouchi, 1999]) as a topological correspondence as follows: let (G,α) and (H,β) be groupoids with Haar systems, and X a G-H-bispace. Assume that

i) the left and right actions are proper,

ii) s_X, the momentum map for the right action, is open

iii) the momentum map for the right action induces a bijection $[s_X]\colon G\backslash X \longrightarrow H^{(0)}$.

In this case, there is a natural choice of family of measures λ along s_X so that (X,λ) is a topological correspondence in the sense of Definition 2.5. To find this family of measures, firstly, one needs to notice that (ii) and (iii) are equivalent to saying that $[s_X]$ is a homeomorphism onto $H^{(0)}$; and due to this, we can identify that quotient map $X \longrightarrow G\backslash X$ with $X \longrightarrow H^{(0)}$. Now by Lemma 2.1, we get that α_X is a G-invariant family of measures along the quotient map $X \longrightarrow H^{(0)}$. This family of measures is also H-invariant. In this case, we simply write $\mathcal{H}(X)$ instead of $\mathcal{H}(X,\lambda)$ for the Hilbert module in the associated C^*-correspondence.

2.10. Example. [Groupoid equivalence, Example 3.9 in [Holkar-1, 2017]] Consider the famous equivalence of groupoids defined by Muhly, Renault and Williams in [Muhly-Renault-Williams, 1987]. Then one can immediately see that an equivalence is a *symmetrised* version of Macho-Stadler and O'uchi correspondence with an extra demand that the both the actions are free. To be precise, in an equivalence both—the left and right—actions are free and proper; the momentum map for the right action satisfies Conditions (ii) and (iii) in Example 2.9; and the momentum map of the left action *also* satisfies these two conditions in an appropriate sense. Thus an equivalence is a *particular* example of a topological correspondence of Macho-Stadler and O'uchi. Therefore, an equivalence of groupoids is a topological correspondence in the sense of Definition 2.5. We shall prove, in Theorem 3.33, that equivalences are exactly the *invertible* Macho-Stadler–O'uchi correspondences.

Here is an additional remark about equivalences of groupoids: if X is a (G,α)-(H,β)-correspondence, then, the same argument as in Example 2.9 shows that X is equipped with two G-H-invariant family of measures along r_X also. Therefore, an appropriate modification in Equation (2) gives a $\mathrm{C}^*(G,\alpha)$-value inner product on $\mathcal{H}(X)$.

This inner product, due to associativity of actions, satisfies the condition $a\langle b\,,c\rangle_* = {}_*\langle a\,,b\rangle c$ for $a,b,c \in \mathcal{H}(X)$—this is a crucial condition for $\mathcal{H}(X)$ to be a $\mathrm{C}^*(G,\alpha)$-$\mathrm{C}^*(H,\beta)$-imprimitivity bimodule (cf.[Rieffel, 1974, Definition 6.10(1)]).

Examples 3.3, and 3.11 in [Holkar-1, 2017] show that, respectively, the topological quiver ([Muhly-Tomforde, 2005]) and generalised morphisms of Buneci and Stachura ([Buneci-Stachura, 2005]) are topological correspondences. Following are new examples which we shall be required in this article.

2.11. Example. [The identity topological correspondence] Let (G,α) be a locally compact topological groupoid with a Haar system. Define the space $X = G$. Then X is a (G,α)-(G,α)-equivalence when equipped with the left and right multiplication actions; in particular X is a Macho-Stadler–O'uchi correspondence on (G,α). What is the family of measures λ on X along the right momentum map s_G that makes it a topological correspondence in the sense of Definition 2.5? Following Example 2.9, the family of measures is given by

$$\int_X k\,\lambda_u := \int_G k(\gamma^{-1}x)\,\mathrm{d}\alpha^{r_G(x)}(\gamma)$$

for $k \in \mathrm{C_c}(X)$ and $u \in G^{(0)}$ and, where $x \in X$ is any element with $s_G(x) = u$. Since λ_u does not depend on $x \in s_G^{-1}(u)$, we choose $x = u \in G^{(0)}$ which shows that

$$\int_X k\,\lambda_u := \int_G k(\gamma^{-1})\,\mathrm{d}\alpha^{r_G(x)}(\gamma),$$

that is, $\lambda = \alpha^{-1}$.

Moreover, $\mathcal{H}(X,\alpha^{-1})$ and $\mathrm{C}^*(G,\alpha)$ are same *as* Hilbert $\mathrm{C}^*(G,\alpha)$-module, and the isomorphism is implemented by the identity map $\mathrm{Id}_G\colon X \longrightarrow G$. To see this, we firstly notice that $\mathrm{C_c}(X)$ is a dense complex vector subspace of $\mathrm{C}^*(G,\alpha)$, as well as, $\mathcal{H}(X,\alpha^{-1})$. For $f \in \mathrm{C_c}(X)$ and $\psi \in \mathrm{C_c}(G)$, Equation 1 gives us

$$f\cdot\psi(x) = \int_G f(x\eta)\psi(\eta^{-1})\,\mathrm{d}\alpha^{s_G(x)}(\eta) = f*\psi(x) \tag{3}$$

where $x \in X$, and $f*\psi$ is the convolution of $f,\psi \in \mathrm{C_c}(G) \subseteq \mathrm{C}^*(G,\alpha)$. If $g \in \mathrm{C_c}(X)$ is another function and $\eta \in G$, then Equation 2 says

$$\langle f\,,g\rangle(\gamma) = \int_G \overline{f(x)}g(x\gamma)\,\mathrm{d}\alpha^{-1}_{r_G(\gamma)}(x)$$

which equals

$$\int_G \overline{f(x^{-1})}g(x^{-1}\gamma)\,\mathrm{d}\alpha^{r_G(\gamma)}(x) = \int_G f^*(x)g(x^{-1}\gamma)\,\mathrm{d}\alpha^{r_G(\gamma)}(x) = f^**g(\eta)$$

where f^* is the involution of $f \in \mathrm{C_c}(G) \subseteq \mathrm{C}^*(G,\alpha)$ and f^**g denotes the convolution as earlier. From the construction of $\mathcal{H}(X,\alpha^{-1})$ (proof of Theorem 2.6), it is clear that $\mathcal{H}(X,\alpha^{-1}) = \mathrm{C}^*(G,\alpha)$ as Hilbert $\mathrm{C}^*(G,\alpha)$-modules. Finally, for ϕ and f as above, one can show that $\psi\cdot f = \psi*f$ in the same was as in Equation (3); this shows that $\mathcal{H}(X,\alpha^{-1})$ and $\mathrm{C}^*(G,\alpha)$ are isomorphic as C^*-correspondences on $\mathrm{C}^*(G,\alpha)$.

2.12. Example. [Example 3.3 in [Holkar-1, 2017]] Let $X \xleftarrow{b} Z \xrightarrow{f} Y$ be maps of spaces. Assume that λ is a family of measures along f; we call b and f as the backwards and forwards map, respectively. The quintuple (Z, b, f, λ) is called a topological quiver from X to Y. Considering X and Y as the trivial groupoids, one can easily check the Z is an X-Y-bispace, and $(Z, \lambda)\colon X \longrightarrow Y$ is a topological correspondence. The adjoining function in this case is trivial. This topological correspondence produces the C*-correspondence $(\mathcal{H}(Z), \phi)\colon \mathrm{C}_0(X) \longrightarrow \mathrm{C}_0(Y)$ where $\mathcal{H}(X)$ is the field of Hilbert spaces over Y associated with the family of measures λ, and the representation is the *-homomorphism $\phi\colon \mathrm{C_c}(X) \longrightarrow \mathrm{C_c}(Z) \subseteq \mathbb{B}(\mathcal{H}(Z))$ induced by b; here we view $\mathrm{C_c}(Z) \subseteq \mathbb{B}(\mathcal{H}(Z))$ because $\mathrm{C_c}(Z)$ acts on $\mathcal{H}(Z)$ by pointwise multiplication. See [Muhly-Tomforde, 2005] for details.

2.13. Composition of topological correspondences. Let (G_i, χ_i) be a locally compact Hausdorff second countable groupoid with a Haar system where $i \in \{1, 2, 3\}$. Let $(X, \alpha)\colon (G_1, \chi_1) \longrightarrow (G_2, \chi_2)$ and $(Y, \beta)\colon (G_2, \chi_2) \longrightarrow (G_3, \chi_3)$ be topological correspondences, and let Δ_1 and Δ_2 be their adjoining functions, respectively; we are assuming that X and Y are Hausdorff and second countable. In general, the composition method described here works when the groupoids are locally Hausdorff locally compact, and the spaces are locally compact Hausdorff and the quotient $(X \times_{{G_2}^{(0)}} Y)/G_2$ is paracompact. In this section, we quickly recall the process of forming the composite $(Y, \beta) \circ (X, \alpha)$ from [Holkar-2, 2017].

We, basically, need to find a G_1-G_2-bispace Ω—obtained using X and Y—and a family of measures $\mu = \{\mu_u\}_{u \in G_3^{(0)}}$—obtained from α and β—such that (i) (Ω, μ) is a correspondence from (G_1, χ_1) to (G_3, χ_2), and (ii) the C*-correspondences $\mathcal{H}(\Omega, \mu) \simeq \mathcal{H}(X, \alpha) \hat{\otimes}_{\mathrm{C}^*(G_2, \chi_2)} \mathcal{H}(Y, \beta)$.

To find Ω, let Z denote the fibre product $X \times_{{G_2}^{(0)}} Y$; then Z is a G_1-G_3-bispace with the obvious left and right actions, namely, $\gamma(x, y)\eta = (\gamma x, y\eta)$ for appropriate $\gamma \in G_1, \eta \in G_3$ and $(x, y) \in X \times_{G_2^{(0)}} Y$. Moreover, Z carries the diagonal action of G_2, that is, $(x, y)\gamma = (x\gamma, \gamma^{-1}y)$ for $(x, y, \gamma) \in Z \times_{{G_2}^{(0)}} G_2$. Since the action of G_2 on X is proper (by hypothesis), so is that of G_2 on Z. We define $\Omega = Z/G_2$. The quotient space Ω is a G_1-G_3-bispace with the actions induced from those on Z. Moreover, the right action of G_3 in Ω is proper ([Holkar-2, 2017, Lemma 3.4]). Now we define a desired family of measures $\mu = \{\mu_u\}_{u \in G_3^{(0)}}$ on Ω in steps. The proper transformation groupoid $Z \rtimes G_2$ is the central object in this discussion; Ω is the quotient of action of the action of this groupoid on its space of units. We list all necessary families of measures first:
(1) It is well-known that the Haar system χ_2 of G_2 induces a Haar system χ on the transformation groupoid $Z \rtimes G_2$: for $f \in \mathrm{C_c}(Z \rtimes G_2)$ and $v \in Z$,

$$\int_{Z \rtimes G_2} f \,\mathrm{d}\chi^v := \int_{G_2} f(\gamma^{-1}, v) \,\mathrm{d}\chi_2^{s_Z(v)}(\gamma).$$

Let χ^{-1} be the corresponding right invariant Haar system on $Z \rtimes G_2$, that is, $\int_{Z \rtimes G_2} f \,\mathrm{d}\chi_v^{-1} = \int_{Z \rtimes G_2} f \circ \mathrm{inv}_{Z \rtimes G_2} \,\mathrm{d}\chi^v$ for $f \in \mathrm{C_c}(Z \rtimes G_2)$ and $v \in (Z \rtimes G_2)^{(0)}$. (2) Let $\pi\colon Z \longrightarrow \Omega$ be the quotient map. Since Z is a proper G_2-space, we define a family of measures along π, $\lambda = \{\lambda_\omega\}_{\omega\in\Omega}$ as ${\xi_2}_Z$ discussed Lemma 2.1. Thus, for $f \in \mathrm{C_c}(Z)$ and $\omega = [x, y] \in \Omega$,

$$\int_Z f \, \mathrm{d}\lambda^\omega := \int_{G_2^{r_Y(y)}} f(x\gamma, \gamma^{-1}y) \, \mathrm{d}\chi_2^{r_Y(y)}(\gamma).$$

(3) Fix $u \in {G_3}^{(0)}$. Consider the measure[1] $m_u = \alpha \times \beta_u$ on the space Z, that is,

$$\int_Z f \, \mathrm{d}m_u = \int_Y \int_X f(x, y) \, \mathrm{d}\alpha_{r_Y(y)}(x) \, \mathrm{d}\beta_u(y)$$

for $f \in \mathrm{C_c}(Z)$.

Figure 1 shows the maps in (1)–(3) above and the families of measures along with them.

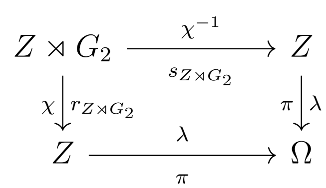


Figure 1

The following lemma is the key observation for finding μ.

2.14. Lemma. *In above setting, in Figure 1, following holds:*

(i) Let μ' be measure on Ω. Then $m' := \mu' \circ \lambda \colon \mathrm{C_c}(Z) \longrightarrow \mathbb{C}$ is a G_2-invariant measure on Z.

(ii) Conversely, if m' is an G_2-invariant measure on Z, then there is a unique *measure m' on Ω such that $m' = \mu' \circ \lambda$.*

This lemma is nothing but Proposition 3.1 in [Holkar-2, 2017] applied to the transformation groupoid $Z \rtimes G_2$ of the proper diagonal of G_2 on X. The second claim of the lemma uses *cutoff* function to construct μ' from m'—this is where the paracompactness of Ω is used. And this is the part which forces second countability on groupoid for forming a category. For the same of details, in!(ii) above, the measure μ' is given by

$$\mu'(f) = m'(\lambda(f \circ \pi \cdot e)) \tag{4}$$

where $f \in \mathrm{C_c}(\Omega)$ and $e \colon Z \longrightarrow \mathbb{R}^+$ is a cuttoff function; the measure μ does not depend on the choice of the cutoff function.

[1] Instead of $\alpha \times \beta_u$, $\alpha \times_{G_2^{(0)}} \beta_u$ is a better notation. However, we use the prior for the sake of simplicity.

Now we can discuss construction of μ following; the counter for listing measures is continued:
(4) Fix $u \in {G_3}^{(0)}$, and the measure m_u on Z. However, this measure does not turn out to be G_2-invariant. Therefore, we cannot use Lemma 2.14 directly for inducing a measure on Ω . A direct computation shows that m_u is $(Z \rtimes G_2, \chi)$-quasi-invariant (for details see the first part of Lemma 3.6 [Holkar-2, 2017]). Thus there is an $\mathbb{R}_+^*$-valued continuous 1-cocycle D (denoted by Δ in [Holkar-2, 2017]) on $Z \rtimes G_2$ with the property that $m_u \circ \chi = D\,(m_u \circ \chi^{-1})$. The cocycle D is given by

$$D\colon ((x,y),\gamma) \mapsto \Delta_2(\gamma^{-1}, y). \tag{5}$$

(5) Thus the cocycle D is an obstruction for using Lemma 2.14. This obstruction is overcome by observing the fact that every $\mathbb{R}$-valued 1-cocycle on a proper groupoid with a Haar system is a coboundary (Proposition 2.7 in [Holkar-2, 2017]). To see how this observation help us, first of all, we notice that one can change the real valued cochains to positive real valued ones via the isomorphism $\mathbb{R} \longrightarrow \mathbb{R}_*^+, x \mapsto \exp(x)$. Now Proposition 2.7 in [Holkar-2, 2017] mentioned above says that

$$D = \frac{b_u \circ s_{Z\rtimes G_2}}{b_u \circ r_{Z\rtimes G_2}} \tag{6}$$

for a 0-cochain b_u on $Z \rtimes G_2$. Note that these 0-cochains are nothing but positive functions on Z in this case.
(6) At this point, we notice that the measure $(b_u m_u)$ on Z is G_2-invariant (second part of Lemma 3.6 [Holkar-2, 2017]). Now Lemma 2.14 produces a measure μ_u on $\Omega = Z/G_2$ which gives the disintegration of measures $b_u m_u \circ \chi = \mu_u$. We write[2] b instead of b_u. Following Equation (4), the measure μ_u is given by

$$\int_\Omega f[x,y]\,\mathrm{d}\mu_u'([x,y]) = \int_Y \int_X f \circ \pi(x,y) e(x,y)\, b(x,y)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\beta_u(y) \tag{7}$$

for $f \in \mathrm{C_c}(\Omega)$. In the above equation, $\pi\colon Z \longrightarrow \Omega$ is the quotient map, e is a cutoff function on Z, cf. Lemma 2.14(ii).

To give the final touches, Proposition 3.10 in [Holkar-2, 2017] shows that $\mu = \{\mu_u\}_{u\in G_3}$ is a G_3-invariant continuous family of measures on Ω; the invariance of m induces this invariance. And [Holkar-2, 2017, Proposition 3.12] shows that each μ_u is (G_1, χ_1)-quasi-invariant; the adjoining function is

$$\Delta_{1,2}(\eta, [x,y]) := b(\eta x, y)^{-1} \Delta_1(\eta, x) b(x,y)$$

where $\Delta_1\colon G_1 \ltimes X \longrightarrow \mathbb{R}_+^*$ is the adjoining function of the topological correspondence (X, α).

[2]Each b_u is defined on fibres of the map $Z \longrightarrow G_3^{(0)}, (x,y) \mapsto s_Y(y)$. Therefore, b can be seen as 'b_u's *patched up* over Z.

2.15. Remark. At this point, we notice that the composite of measures μ on the composite is not *uniquely* determined; it depends on the choice of the 0-cochain b in that gives the decomposition (6). In fact, the choice involved for b is a crucial point that implies to the bicategorical structure. Proposition 3.10 proves that any two compositions of topological correspondences are isomorphic.

2.16. Remark. The ideas that a 1-cocycle obstructs a composition, and that, for a proper groupoid, the obstruction can be removed by decomposing the cocycle using 0-cochains shall be frequently used in many computations.

2.17. Definition. [Composite] *Let*

$$(X,\alpha)\colon (G_1,\chi_1)\to(G_2,\chi_2) \quad \textit{and} \quad (Y,\beta)\colon (G_2,\chi_2)\to(G_3,\chi_3)$$

be topological correspondences with Δ_1 *and* Δ_2 *as the adjoining function, respectively. A composite of these correspondences* $(\Omega,\mu):(G_1,\chi_1)\to(G_3,\chi_3)$ *is defined by:*

i) the space $\Omega := (X\times_{{G_2}^{(0)}} Y)/G_2$,

ii) a family of measures $\mu=\{\mu_u\}_{u\in G^{(0)}{}_3}$ *that lifts to* $\{b(\alpha\times\beta_u)\}_{u\in {G_3}^{(0)}}$ *on* Z *for a cochain* $b\in \mathrm{C}^0_{G_3}((X\times_{{G_2}^{(0)}}Y)\rtimes G_2,\mathbb{R}^*_+)$ *satisfying* $d^0(b)=D$ *where* $D\colon (X\times_{{G_2}^{(0)}}Y)\rtimes G_2\longrightarrow\mathbb{R}^*_+$ *is* $D((x,y),\gamma)=\Delta_2(\gamma^{-1},y)$.

Notice that a composite of topological correspondences is not defined uniquely; it dependes on the choice of a 0-cocycle on the transformation groupoid $(X\times_{{G_2}^{(0)}} Y)\rtimes G_2$. However, as the next result shows, the C*-correspondence associated with any composite of the topological correspondences (X,α) and (X,β) above, is isomorphic to the interior tensor product $\mathcal{H}(X,\alpha)\,\hat{\otimes}_{\mathrm{C}^*(G_2,\chi_2)}\mathcal{H}(Y,\beta)$.

2.18. Theorem. [Theorem 3.14, [Holkar-2, 2017]] *Let*

$$(X,\alpha)\colon (G_1,\chi_1)\to(G_2,\chi_2) \quad \textit{and} \quad (Y,\beta)\colon (G_2,\chi_2)\to(G_3,\chi_3)$$

be topological correspondences of locally compact groupoids with Haar systems. In addition, assume that X *and* Y *are Hausdorff and second countable. Let* $(\Omega,\mu)\colon (G_1,\chi_1)\to(G_3,\chi_3)$ *be a composite of the correspondences. Then* $\mathcal{H}(\Omega,\mu)$ *and* $\mathcal{H}(X,\alpha)\,\hat{\otimes}_{C^*(G_2,\chi_2)}\mathcal{H}(Y,\beta)$ *are isomorphic C*-correspondences from* $C^*(G_1,\chi_1)$ *to* $C^*(G_3,\chi_3)$.

For the sake of clarity, here is the definition of an isomorphism of C*-correspondences:

2.19. Definition. *Let* A *and* B *be C*-algebras and* $\mathcal{H},\mathcal{K}\colon A\longrightarrow B$ *two C*-correspondences. An isomorphism (of C*-correspondences) from* $\mathcal{H}$ *to* $\mathcal{K}$ *is a complex linear mapping* $T\colon \mathcal{H}\longrightarrow\mathcal{K}$ *that is a* unitary *isomorphism of Hilbert B-modules with the property that for given* $a\in A$ *and* $\xi\in\mathcal{H}$, $T(a\cdot\xi)=a\cdot T(\xi)$.

Following is an important remark regarding equivariant groupoid cochain complexes which we shall use in many arguments regarding 2-arrows in the bicategory topological correspondences.

2.20. Remark. [Remark 1.14 in [Holkar-1, 2017]] Let G, H be Borel groupoids and A an abelian group. If $b, b' \in \mathrm{C}^0_H(G, A)$ are A-valued H-invariant 0-cochains with coboundaries $d^0(b) = d^0(b')$, then $c' := (b - b') \in \mathrm{C}^0_H(G, A)$ is a 0-cochain with

$$d^0(c') = d^0(b) - d^0(b') = 0, \text{ equivalently, } d^0(c') + d^0(b') = d^0(b).$$

Moreover, this function c' is constant on the G-orbits of $G^{(0)}$. Therefore, it induces a function $[c']$ on $G^{(0)}/G$. Thus $c' = [c'] \circ q$ for the quotient map $q \colon G \longrightarrow G^{(0)}/G$.

2.21. Example. [Example 4.1 in [Holkar-2, 2017]] Let $f \colon X \longrightarrow Y$ and $g \colon Y \longrightarrow Z$ be maps of spaces. Let $(X, \{\delta_x\}_{x\in X}) \colon Y \longrightarrow X$ and $(Y, \{\delta_y\}_{y\in Y}) \colon Z \longrightarrow Y$ be the topological correspondences associated with the maps of spaces, as in Example 2.7. Then, the space in the composite $Z \longrightarrow Y$ of these correspondences is the fibre product $Y \times_{\mathrm{Id}_Y, Y, f} X \approx X$ where the homeomorphism is given by $(f(x), x) \mapsto x$ for $x \in X$. The right and left momentum maps on $Z \xleftarrow{Y} \times_{\mathrm{Id}_X, Y, f} X \xrightarrow{X}$ are, respectively, $(f(x), x) \mapsto x$ and $(f(x), x) \mapsto g(f(x))$. Which after identifying the fibre product with X become Id_X and $g \circ f$. The family of measures in this case is unique and it consists of the point masses $\{\delta_x\}_{x\in X}$ along the identity map $X \longrightarrow X$. Thus the composite $(X, \{\delta_x\}_{x\in X})$ corresponds to the topological correspondence associated with the composite mapping $g \circ f \colon X \longrightarrow Z$.

Composites of are described similarly using fibre products, cf. Example 4.2 in [Holkar-2, 2017]. One can observe it quickly that a composite of topological quivers is also a quiver. The family of measure on the composite is uniquely determined in this case.

2.22. Example. [Example 4.3 in [Holkar-2, 2017]] Let $A \xrightarrow{f} B \xrightarrow{g} C$ be homomorphisms of groups. Assume that α, β and κ be the Haar systems on A, B and C, respectively. Let $(B, \beta^{-1}) \colon A \longrightarrow B$ and $(C, \kappa^{-1}) \colon B \longrightarrow C$ be the topological correspondences associated with f and g, respectively, as in Example 2.8. The space in the composite of the topological correspondences $A \longrightarrow C$ is the quotient $(B \times C)/B \approx C$ where the quotient is taken for the diagonal action of B on $B \times C$; the A-C-equivariant homeomorphism is the standard one, see Example 2.11 for the homeomorphism. The right invariant measures on the composite C is κ^{-1}, and the adjoining function of this correspondences is $(D_C \circ g \circ f)/\Delta_A$, see Example 4.3 [Holkar-2, 2017] for details. This shows that the composite of the topological correspondences associated with the homomorphisms f, g is the topological correspondence associated with $g \circ f$.

2.23. Bicategory. We follow Bénabou's notation and terminology, from [Bénabou, 1967], for bicategories. Bénabou's convention for composition is the other way round than the standard one. For a bicategory, 1-arrows are denoted by the usual arrows, whereas, thick arrows stand for the 2-arrows.

2.24. Definition. [Bicategory, [Bénabou, 1967, Definition 1.1]] *A bicategory* $\mathfrak{S}$ *is determined by the following data:*

i) a set $\mathfrak{S}_0$ *called set of objects or vertices;*

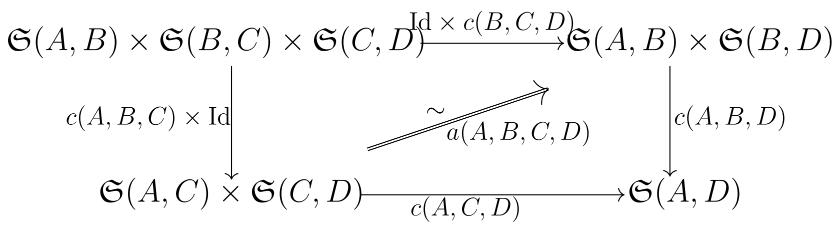
ii) for each pair (A, B) of objects, a category $\mathfrak{S}(A, B)$;

iii) for each triple (A, B, C) of objects of $\mathfrak{S}$ a composition functor

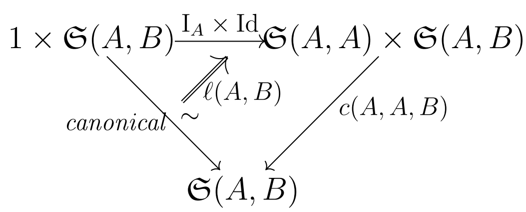
$$c(A, B, C)\colon \mathfrak{S}(A, B) \times \mathfrak{S}(B, C) \to \mathfrak{S}(A, C);$$

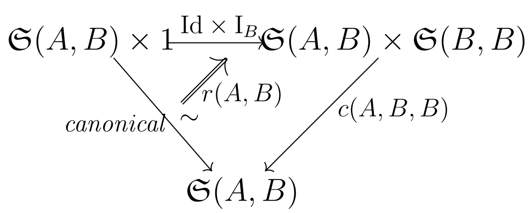
iv) for each object A of $\mathfrak{S}$ an object I_A of $\mathfrak{S}(A, A)$ called identity arrow *of A (the identity map of I_A in $\mathfrak{S}(A, A)$ is denoted $i_A\colon I_A \implies I_A$ and is called* identity 2-cell *of A);*

v) for each quadruple (A, B, C, D) of objects of $\mathfrak{S}$, a natural isomorphism $a(A, B, C, D)$ called associativity isomorphism *between the two composite functors making the following diagram commute:*



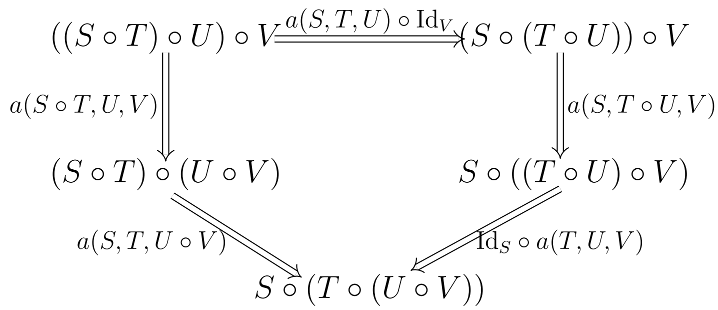
vi) for each pair (A, B) of objects of $\mathfrak{S}$, two natural isomorphisms $\ell(A, B)$ and $r(A, B)$, called left and right identities such that the following diagrams commute:



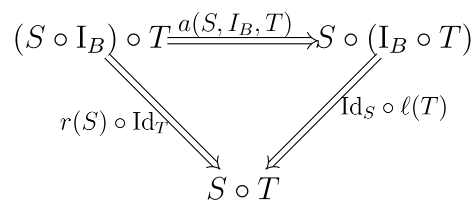


This data satisfies the following conditions:

vii) associativity coherence: if (S, T, U, V) is an object of $\mathfrak{S}(A, B) \times \mathfrak{S}(B, C) \times \mathfrak{S}(C, D) \times \mathfrak{S}(D, E)$, then the following diagram commutes:



viii) identity coherence*: if* (S,T) *is an object of* $\mathfrak{S}(A,B) \times \mathfrak{S}(B,C)$*, then the following diagram commutes:*



In modern literature, a vertex, an arrow (or a 1-cell) and a 2-cell are called an object, a 1-arrow and a 2-arrow, respectively. Let A and B be two objects and let t, u be two arrows in the category $\mathfrak{S}(A,B)$. Then we call the rule of composition of t and u in $\mathfrak{S}(A,B)$ the vertical composition of 1-arrows. The composite functor c in (iii) above gives the horizontal composition of 2-arrows. Let (S,T) and (S',T') be two objects in $\mathfrak{S}(A,B) \times \mathfrak{S}(B,C)$, respectively, and let $s \colon S \longrightarrow S'$ and $t \colon T \longrightarrow T'$ be 2-arrows. Then s and t induce a 2-arrow $s \cdot_h t \colon S \circ T \longrightarrow S' \circ T'$. The 2-arrow $s \cdot_h t$ is called the vertical composite of the 2-arrows s and t.

2.25. Example. In Section 2.2 of [Buss-Meyer-Zhu, 2013] Buss, Meyer and Zhu form a bicategory of C*-algebraic correspondences. In this bicategory the objects are the C*-algebras, 1-arrows are the C*-algebraic correspondences and 2-arrows are the equivariant unitary intertwiners of C*-correspondences.

2.26. Example. The C*-correspondences of commutative (or commutative and unital) C*-algebras is a sub-bicategory of the bicategory in Example 2.25.

2.27. Definition. [Morphisms of bicategories, [Bénabou, 1967, Definition 4.1]] *Let* $\mathfrak{S}$ *and* $\mathfrak{S}'$ *be bicategories. A morphism* $\mathfrak{V} = (V, v)$ *from* $\mathfrak{S}$ *to* $\mathfrak{S}'$ *consists of:*

i) a map $V \colon \mathfrak{S}_0 \to \mathfrak{S}'_0$ *sending an object* A *to* $V(A)$*;*

ii) a family of functors $V(A,B) \colon \mathfrak{S}(A,B) \to \mathfrak{S}'(V(A), V(B))$ *sending a 1-cell* S *to* $V(A)$ *and a 2-cell* s *to* $V(s)$*;*

iii) for each object A of $\mathfrak{S}$, a 2-cell $v_A \in \mathfrak{S}(V(A), V(B))$

$$v_A\colon I_{V(A)} \Rightarrow V(I_A);$$

iv) a family of natural transformations

$$v(A,B,C)\colon c(V(A),V(B),V(C)) \circ (V(A,B) \times V(B,C)) \to V(A,C) \circ c(A,B,C).$$

If (S,T) is an object of $\mathfrak{S}(A,B) \times \mathfrak{S}'(B,C)$, the (S,T)-components of $v(A,B,C)$

$$v(A,B,C)(S,T)\colon V(S) \circ V(T) \Rightarrow V(S \circ T)$$

shall be abbreviated v or $v(S,T)$.

This data satisfies the following coherence conditions:

v) If (S,T,U) is an object of $\mathfrak{S}(A,B) \times \mathfrak{S}(B,C) \times \mathfrak{S}(C,D)$ the diagram in Figure 2 is commutative.

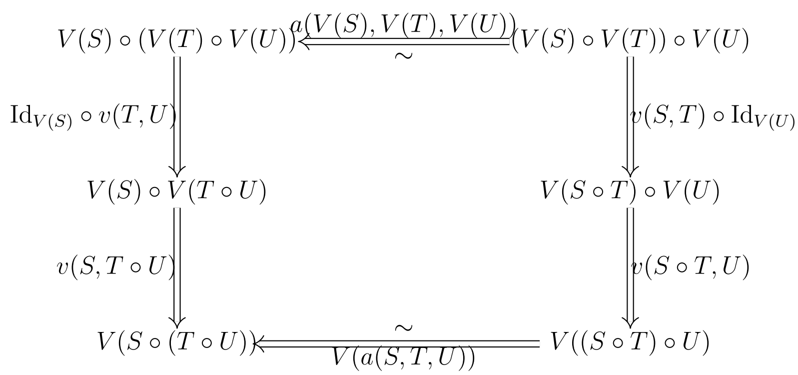


Figure 2: Associativity coherence for a transformation between bicategories

vi) If S is an object of $\mathfrak{S}(A,B)$ then the diagram in Figure 3, for the right identity commutes.

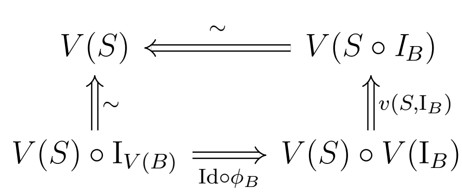


Figure 3: Coherence of the right identity (and a similar diagram is drawn for the left identity)

A similar diagram for the left identity commutes.

3. The bicategory of topological correspondences

3.1. Isomorphism of topological correspondences. As it will be proved later, isomorphisms of topological correspondences are the 2-arrows in the bicategory of topological correspondences. In this section, we define these isomorphisms. As examples of it, we discuss the identity isomorphisms of the left and right identity correspondences (Example 3.9), and show that any two composites of two topological correspondences are isomorphic (Proposition 3.10). The section ends with two remarks: first is that isomorphism is an equivalence relation on topological correspondences, and the second one describes the horizontal composite of these 2-arrows. Now we begin by discussing some elementary propreties of families of measures.

Let X be a locally compact, Hausdorff space, and let λ and λ' be equivalent Radon measures on it. Thus the Radon–Nikodym derivatives $\mathrm{d}\lambda/\mathrm{d}\lambda'$ and $\mathrm{d}\lambda'/\mathrm{d}\lambda$ are, respectively, λ- and λ'-almost everywhere positive. Moreover, the equality $\mathrm{d}\lambda/\mathrm{d}\lambda' \cdot \mathrm{d}\lambda'/\mathrm{d}\lambda = 1$ holds λ or λ'-almost everywhere. Assume that Y is another space and $\pi\colon X \longrightarrow Y$ is a homeomorphism. Then the measure $\lambda\colon \mathrm{C_c}(X) \longrightarrow \mathbb{C}$ induces the measure $\pi_*(\lambda)\colon \mathrm{C_c}(Y) \longrightarrow \mathbb{C}$ on Y as follows: for $f \in \mathrm{C_c}(Y)$, $\pi_*(\lambda)(f) = \lambda(f \circ \pi)$. We call $\pi_*(\lambda)$ the push-forward (measure) of λ.

3.2. Definition. *Let $\pi\colon X \longrightarrow Y$ be an open surjection, and λ and λ' families of measures along π. We call λ and λ' equivalent if*

(i) for each $y \in Y$, λ_y and λ'_y are equivalent,

(ii) there is a continuous positive function $\phi\colon X \longrightarrow \mathbb{R}$ such that $\phi(x) = \frac{\mathrm{d}\lambda_{\pi(x)}}{\mathrm{d}\lambda'_{\pi(x)}}(x)$ for all $x \in X$.

In (ii) above, $\mathrm{d}\lambda_{\pi(x)}/\mathrm{d}\lambda'_{\pi(x)}$ is the Radon–Nikodym derivative of $\lambda_{\pi(x)}$ with respect to $\lambda'_{\pi(x)}$. In this case, we write $\lambda \sim \lambda'$; we call the function $\mathrm{d}\lambda/\mathrm{d}\lambda'$ the Radon–Nikodym derivative of λ with respect to λ'.

In (ii) of above definition, we demand not only that the Radon–Nikodym derivative of measures on individual fibres $\pi^{-1}(x), x \in X$ are continuous, but also that the family of the Radon–Nikodym derivatives is continuous *in the direction* of X; this transverse continuity is used in Proposition 3.25. In above definition, *continuously equivalent* families of measures is a better terminology than *equivalent.* However, we shall not encounter any instance of families of measures which are not continuously equivalent. Therefore, we choose the present terminology.

Let X_1, X_2 and Z be spaces, $X_1 \xrightarrow{\pi_1} Z \xleftarrow{\pi_2} X_2$ maps, and let λ be a family of measures along π_1. Let $f\colon X_1 \longrightarrow X_2$ be a homeomorphism such that $\pi_1 = \pi_2 \circ f$. Then $\{f_*(\lambda_z)\}_{z\in Z}$ is continuous family of measures along π_2 which we denote by $f_*(\lambda)$; we write $f_*(\lambda)_z$ for $f_*(\lambda_z)$.

3.3. Lemma. *Let* X_1, X_2 *and* Z *be spaces, let* $\pi_i\colon X_i \longrightarrow Z$ *be a map for* $i = 1, 2$. *Let* $a\colon X_1 \longrightarrow X_2$ *be a homeomorphism such that* $\pi_1 = \pi_2 \circ a$. *Assume that* λ *and* μ *are equivalent families of measures along* π_1 *with Radon–Nikodym derivative* $\mathrm{d}\lambda/\mathrm{d}\mu$. *Then* $a_*(\lambda) \sim a_*(\mu)$ *and the Radon–Nikodym derivative* $\mathrm{d}a_*(\lambda)/\mathrm{d}a_*(\mu) = (\mathrm{d}\lambda/\mathrm{d}\mu) \circ a^{-1}$.

Proof. Follows from a direct computation. ∎

3.4. Lemma. *Let* X_1, X_2, X_3 *and* Z *be space. For* $i = 1, 2, 3$, *let* $\pi_i : X_i \to Z$ *be a map and* λ_i *a family of measures along* π_i. *For* $i = 1, 2$, *let* $a_i : X_i \to X_{i+1}$ *be homeomorphisms such that* $\pi_i = \pi_{i+1} \circ a_i$. *If* λ_1 *is a family of measures along* π_1, *then* $(a_2 \circ a_1)_*(\lambda_1) = a_{2*}(a_{1*}(\lambda_1))$.

Proof. This follows directly from the definition of push-forward of a measure. ∎

3.5. Lemma. [Chain rule] *Let* X_1, X_2, X_3 *and* Z *be space. For* $i = 1, 2, 3$, *let* $\pi_i : X_i \to Z$ *be a map and* λ_i *a family of measures along* π_i. *For* $i = 1, 2$, *let* $a_i : X_i \to X_{i+1}$ *be homeomorphisms such that* $\pi_i = \pi_{i+1} \circ a_i$. *If* $a_{i*}(\lambda_i)$ *is equivalent to* λ_{i+1} *for* $i = 1, 2$, *then* $(a_2 \circ a_1)_*(\lambda_1)$ *is equivalent to* λ_3. *Moreover, the Radon–Nikodym derivative*

$$\frac{\mathrm{d}(a_2 \circ a_1)_*(\lambda_1)}{\mathrm{d}\lambda_3} = \frac{\mathrm{d}a_{1*}(\lambda_1)}{\mathrm{d}\lambda_2} \circ a_2^{-1} \cdot \frac{\mathrm{d}a_{2*}(\lambda_2)}{\mathrm{d}\lambda_3}.$$

Proof. This is a straightforward computation: for $z \in Z$ and $f \in \mathrm{C_c}(X_3)$,

$$\begin{aligned}
&\int_{X_3} f(x)\frac{\mathrm{d}a_{1*}(\lambda_{1z})}{\mathrm{d}\lambda_{2z}} \circ a_2^{-1}(x)\frac{\mathrm{d}a_{2*}(\lambda_{2z})}{\mathrm{d}\lambda_{3z}}(x)\,\mathrm{d}\lambda_{3z}(x)\\
&= \int_{X_2} f \circ a_2(y)\,\frac{\mathrm{d}a_{1*}(\lambda_{1z})}{\mathrm{d}\lambda_{2z}}(y)\,\mathrm{d}\lambda_{2z}(y) = \int_{X_1} f \circ a_2 \circ a_1(w)\,\mathrm{d}\lambda_{1z}(w)\\
&= \int_{X_3} f(x)\,\mathrm{d}(a_2 \circ a_1)_*(\lambda_1)(x).
\end{aligned}$$

∎

3.6. Corollary. [Of the chain rule] *For* $i = 1, 2$, *let* X_i, π_i, λ_i, Z *and* a_1 *be as in Lemma 3.5 above. If* $a_{1*}(\lambda_1) \sim \lambda_2$, *then* $\lambda_1 \sim a_{1*}^{-1}(\lambda_2)$.

Proof. Apply the chain rule (Lemma 3.5) to $X_1 \xrightarrow{a_1} X_2 \xrightarrow{a_1^{-1}} X_1$ to get

$$1 = \frac{\mathrm{d}a_{1*}(\lambda_1)}{\mathrm{d}\lambda_2} \circ a_1 \cdot \frac{\mathrm{d}a_{1*}^{-1}(\lambda_2)}{\mathrm{d}\lambda_1}.$$

Since $\frac{\mathrm{d}a_{1*}(\lambda_1)}{\mathrm{d}\lambda_2}$ is positive continuous and a_1 is a homeomorphism, $\frac{\mathrm{d}a_{1*}(\lambda_1)}{\mathrm{d}\lambda_2} \circ a_1$ is also positive continuous function. Thus $\mathrm{d}a_{1*}^{-1}(\lambda_2)/\mathrm{d}\lambda_1 > 0$ and continuous, that is, $a_{1*}^{-1}(\lambda_2) \sim \lambda_1$. ∎

Following lemma will prove useful in many computations later.

3.7. LEMMA. *Let* (G, α) *be a locally compact groupoid (not necessarily Hausdorff or second countable) equipped with a Haar system. Let* m *and* m' *be* G*-invariant equivalent measures on* $G^{(0)}$ *with continuous Radon–Nikodym derivative* $\mathrm{d}m/\mathrm{d}m'$*. Then the following hold.*

(i) On G*,* $m \circ \alpha \sim m' \circ \alpha$ *and* $m \circ \alpha^{-1} \sim m' \circ \alpha^{-1}$*. Moreover, the Radon–Nikodym derivatives*

$$\frac{\mathrm{d}m \circ \alpha^{-1}}{\mathrm{d}m' \circ \alpha^{-1}} = \frac{\mathrm{d}m}{\mathrm{d}m'} \circ s_G \quad \textit{and} \quad \frac{\mathrm{d}m \circ \alpha}{\mathrm{d}m' \circ \alpha} = \frac{\mathrm{d}m}{\mathrm{d}m'} \circ r_G.$$

(ii) the Radon–Nikodym derivative $\mathrm{d}m/\mathrm{d}m'$ *is* G*-invariant.*

Additionally, assume that G *is proper,* $G^{(0)}/G$ *paracompact. Let* $q\colon G^{(0)} \longrightarrow G^{(0)}/G$ *be the quotient map. Let* μ *and* μ' *be the measures on* $G^{(0)}/G$ *which give the disintegration* $\mu \circ \alpha_G = m$ *and* $\mu' \circ \alpha_G = m'$*. Let* $[\mathrm{d}m/\mathrm{d}m']$ *denote the function which* $\mathrm{d}m/\mathrm{d}m'$ *induce on* $G^{(0)}/G$ *(cf. (ii) above). Then*

(iii) $\mu \sim \mu'$ *and the Radon–Nikodym derivative* $\mathrm{d}\mu/\mathrm{d}\mu' = [\mathrm{d}m/\mathrm{d}m]$*.*

Recall the meaning of α_G from Lemma 2.1. A groupoid G acts on it space of orbits (from right) by $u \cdot \gamma = s_G(\gamma)$ for $u \in G^{(0)}$ and $\gamma \in G^u$. For a proper groupoid, this action is proper. A function $\phi\colon G^{(0)} \longrightarrow \mathbb{C}$ is called invariant (under this action) if $\phi(u \cdot \gamma) = \phi(u)$, that is, $\phi(s_G(\gamma)) = \phi(r_G(\gamma))$ for all $\gamma \in G$, or equivalently $\phi \circ r_G = \phi \circ s_G$ on G.

PROOF PROOF OF LEMMA 3.7. (i): To check that $m \circ \alpha^{-1} \sim m' \circ \alpha^{-1}$, let $f \in \mathrm{C_c}(G)$. Then

$$\begin{aligned}\int_G f(\gamma)\,\mathrm{d}m \circ \alpha^{-1}(\gamma) &:= \int_{H^{(0)}} \int_G f(\gamma^{-1})\,\mathrm{d}\alpha^x(\gamma)\,\mathrm{d}m(x)\\ &= \int_{H^{(0)}} \int_G f(\gamma^{-1})\,\mathrm{d}\alpha^x(\gamma) \frac{\mathrm{d}m}{\mathrm{d}m'}(x)\,\mathrm{d}m'(x).\end{aligned}$$

Since $x = r_G(\gamma)$, we may write the last term above as

$$\int_{H^{(0)}} \int_G f(\gamma^{-1}) \frac{\mathrm{d}m}{\mathrm{d}m'}(r_G(\gamma))\,\mathrm{d}\alpha^x(\gamma)\,\mathrm{d}m'(x)$$

which, in turn, equals

$$\int_{H^{(0)}} \int_G f(\gamma^{-1}) \frac{\mathrm{d}m}{\mathrm{d}m'}(s_G(\gamma^{-1}))\,\mathrm{d}\alpha^x(\gamma)\,\mathrm{d}m'(x) := \int_G f(\gamma) \frac{\mathrm{d}m}{\mathrm{d}m'} \circ s_G(\gamma)\,\mathrm{d}m' \circ \alpha^{-1}(\gamma).$$

Thus $m \circ \alpha^{-1} \sim m' \circ \alpha^{-1}$ and the Radon–Nikodym derivative $\mathrm{d}m \circ \alpha^{-1}/\mathrm{d}m' \circ \alpha^{-1} = \mathrm{d}m/\mathrm{d}m' \circ s_G$. The other claim can be proved along similar lines.

(ii): Since m (or m') is an invariant measure on $G^{(0)}$, we have $m \circ \alpha = m \circ \alpha^{-1}$ (and similar for m'). Which along with (i) above says that

$$\frac{\mathrm{d}m}{\mathrm{d}m'} \circ s_G = \frac{\mathrm{d}m \circ \alpha^{-1}}{\mathrm{d}m' \circ \alpha^{-1}} = \frac{\mathrm{d}m \circ \alpha}{\mathrm{d}m' \circ \alpha} = \frac{\mathrm{d}m}{\mathrm{d}m'} \circ r_G.$$

In other words, $\mathrm{d}m/\mathrm{d}m'$ is an invariant function on $G^{(0)}$.
(iii): First of all, we note that the functions $\frac{\mathrm{d}m}{\mathrm{d}m'}$ and $[\frac{\mathrm{d}m}{\mathrm{d}m'}]$ have same images in $\mathbb{R}$. Now, given m (or m'), recall the definition of μ (or μ', respectively) from Equation (4) in Lemma 2.14(ii). Let $f \in \mathrm{C_c}(G^{(0)}/G)$ and $e\colon G^{(0)} \longrightarrow \mathbb{R}^* \cup \{0\}$ be a cutoff function for the quotient map q. Then

$$\mu(f) := m((f \circ q)\cdot e) = m'\left(f \circ q \cdot \frac{\mathrm{d}m}{\mathrm{d}m'} \cdot e\right) := \mu'\left(f \cdot \left[\frac{\mathrm{d}m}{\mathrm{d}m'}\right]\right).$$

Thus $\mu \sim \mu'$ and $\frac{\mathrm{d}\mu}{\mathrm{d}\mu'} = [\frac{\mathrm{d}m}{\mathrm{d}m'}]$. ∎

3.8. Definition. [Isomorphism of topological correspondences] *Let* (X, λ) *and* (X', λ') *be topological correspondences from* (G, α) *to* (H, β)*. An isomorphism* $(X, \lambda) \longrightarrow (X', \lambda')$ *is a* G*-*H*-equivariant homeomorphism* $\phi\colon X \to X'$ *with* $\phi_*(\lambda) \sim \lambda'$*.*

Following is an example of isomorphism of correspondences; Proposition 3.10 gives a class of isomorphism correspondences.

3.9. Example. [The identity isomorphism of identity correspondence] This example describes the left and right identity isomorphisms in the bicategory of topological correspondences. Let (G, α) and (H, β) be topological groupoids with Haar systems. Recall from Example 2.11 that (G, α^{-1}) is a topological correspondence on (G, α), and (H, β^{-1}) is a topological correspondence on (H, β). Let $(X, \lambda)\colon (G, \alpha) \longrightarrow (H, \beta)$ be a topological correspondence. What are composites of (G, α^{-1}) and (X, λ), and (X, λ) and (H, β^{-1})?
The left identity isomorphism: Let $(G \circ X, \mu)$ be a composite of (G, α^{-1}) and (X, λ). Firstly, note that the bispace $G \circ X$ is homeomorphic to X. The quotient $(G \times_{G^{(0)}} X)/G$ is isomorphic to X; the map $i\colon (G \times_{G^{(0)}} X)/G \longrightarrow X$ given by $i([\gamma^{-1}, x]) = \gamma^{-1}x$, where $[\gamma^{-1}, x] \in (G \times_{G^{(0)}} X)/G$ is the equivalence class of $(\gamma^{-1}, x) \in G \times_{G^{(0)}} X$, induces this homeomorphism. The inverse of i is given by $i^{-1}(x) = [r_X(x), x]$. Moreover, the map i is G-H-equivariant; we identify X as the quotient space $(G \times_{G^{(0)}} X)/G$ and call the map

$$q\colon G \times_{G^{(0)}} X \longrightarrow X, \quad q\colon (\gamma^{-1}, x) \mapsto \gamma^{-1}x$$

the quotient map. Let's recall how the family of measures μ is constructed: analogous to Figure 1, we draw Figure 4 depicting data in this case: the family of measures along the quotient map on the right side of the square is $\alpha_{G \times_{G^{(0)}} X}$ (cf. Lemma 2.1) which we denote here by $[\alpha]$, that is,

$$\int_{G \times_{G^{(0)}} X} f(t)\, \mathrm{d}[\alpha]_x(t) = \int_G f(\gamma, \gamma^{-1}x)\, \mathrm{d}\alpha^{r_X(x)}(\gamma)$$

$$
\begin{array}{ccc}
(G \times_{G^{(0)}} X) \rtimes G & \xrightarrow[s_{(G \times_{G^{(0)}} X)\rtimes G}]{\tilde{\alpha}^{-1}} & G \times_{G^{(0)}} X \\
\tilde{\alpha} \Big\downarrow r_{(G\times_{G^{(0)}} X)\rtimes G} & & q \Big\downarrow [\alpha]:=\alpha_{G\times_{G^{(0)}} X} \\
G \times_{G^{(0)}} X & \xrightarrow[q]{[\alpha]:=\alpha_{G\times_{G^{(0)}} X}} & X \approx G \circ X
\end{array}
$$

Figure 4

where $f \in \mathrm{C_c}(G \times_{G^{(0)}} X)$ and $x \in X$.

Let T denote the transformation groupoid $(G \times_{G^{(0)}} X) \rtimes G$ for the diagonal action of G on $G \times_{G^{(0)}} X$. Then we know that T is a proper groupoid with $T^{(0)} = G \times_{G^{(0)}} X$. Moreover, the Haar system α on G induces a Haar system $\tilde{\alpha}$ on T. Fix $u \in H^{(0)}$. Define the measure $m_u = \alpha^{-1} \times \lambda_u$ on $T^{(0)}$ as in (1) on page 855. Then m_u is $(T, \tilde{\alpha})$-quasi-invariant measure on $T^{(0)}$ (cf. (2) on page 856). The modular function Δ_1 of T associated with m_u is

$$\Delta_1(\gamma^{-1}, x, \eta) = \Delta(\eta^{-1}, x) \qquad \text{(see Equation (5))} \tag{8}$$

where Δ is the adjoining function of (X, λ). Now get a 0-cochain $b\colon T^{(0)} \longrightarrow \mathbb{R}^+$ on T such that $d^0(b) = \Delta_1$ and bm_u is a $(T, \tilde{\alpha})$-invariant measure on $T^{(0)}$ (cf. (5) on page 856 and (6) on page 856). This measure gives rise to a measures μ_u on $X \approx T^{(0)}/T$ such that $(X, \{\mu_u\}_{u\in H^{(0)}})$ composite of (G, α) and (X, λ). Write $\{\mu_u\}_{u\in H^{(0)}} = \mu$. In what follows, we show that the identity map on X produces an isomorphism of topological correspondences (X, μ) and (X, λ).

Fix $u \in H^{(0)}$. The measure $\lambda_u \circ [\alpha]$ on $T^{(0)}$ is also $(T, \tilde{\alpha})$-invariant due to Lemma 2.14(i). Moreover, the second part of the same Lemma shows that λ_u is the unique measure on X that disintegrates $\lambda_u \circ [\alpha]$ along $[\alpha]$. What is the relation between $\lambda_u \circ [\alpha]$ and bm_u? We claim that they are equivalent invariant measures on $T^{(0)}$ with a continuous Radon–Nikodym derivative. If the claim holds, then Lemma 3.7(iii) applied to $(T, \tilde{\alpha}), bm_u$ and λ_u shows that $\mu_u \sim \lambda_u$ and the Radon–Nikodym derivative $\mathrm{d}\mu_u/\mathrm{d}\lambda_u$ is continuous. This proves that the identity map of X induces the isomorphism of topological correspondences (X, μ) and (X, λ).

Now we prove the claim that bm_u and $\lambda_u \circ [\alpha]$ are equivalent invariant measures on $T^{(0)}$ having continuous Radon–Nikodym derivative. The first observation is that $\lambda_u \circ [\alpha]$ and m_u are equivalent. This follows because for any $f \in \mathrm{C_c}(T^{(0)})$,

$$
\begin{aligned}
m_u(f) &= \int_X \int_G f(\gamma^{-1}, x)\, \mathrm{d}\alpha^u(\gamma)\mathrm{d}\lambda_u(x) \\
&= \int_X \int_G f(\gamma, \gamma^{-1}x)\Delta(\gamma, \gamma^{-1}x)\, \mathrm{d}\alpha^u(\gamma)\mathrm{d}\lambda_u(x) = \lambda_u \circ [\alpha](f \cdot \Delta).
\end{aligned}
$$

The second equality above is the definition of adjoining function Δ (see Definition 2.5(iv)) of (X, λ). Thus $m_u \sim \lambda_u \circ [\alpha]$ with the Radon–Nikodym derivative $\mathrm{d}m_u/\mathrm{d}\lambda_u \circ [\alpha] = \Delta > 0$.

Since b is a 0-cochain in the $\mathbb{R}^+$-valued cohomology of T (with $d^0(b) = \Delta_1 > 0$), we get that $bm_u \sim m_u$. Therefore, $bm_u \sim \lambda_u \circ [\alpha]$ and the Radon–Nikodym derivative $\mathrm{d}bm_u/\mathrm{d}\lambda_u \circ [\alpha] = b\Delta$. Moreover, Lemma 3.7(iii) says $\mathrm{d}\mu_u/\mathrm{d}\lambda_u = [\mathrm{d}bm_u/\mathrm{d}\lambda_u \circ [\alpha]] = [b\Delta]$; note that both b and Δ are H-invariant, therefore, $[b\Delta]$ makes sense.

Before finishing the discussion, we simplify the function $b\Delta\colon T^{(0)} \longrightarrow \mathbb{R}^+$ which will prove useful in later computations. For $(\eta^{-1}, x) \in T^{(0)}$,

$$(b\Delta)(\eta^{-1}, x) = b(\eta^{-1}, x)\Delta(\eta^{-1}, x).$$

Using Equation (8) and the fact that $\Delta = b \circ s_G / b \circ r_G = d^0(b)$, we can see that last term above equals

$$\begin{aligned} b(\eta^{-1}, x)\Delta_1(\eta^{-1}, x, \eta) &= b(\eta^{-1}, x)\frac{b \circ s_T(\eta^{-1}, x, \eta)}{b \circ r_T(\eta^{-1}, x, \eta)} \\ &= b(\eta^{-1}, x)\frac{b(s_H(\eta), \eta^{-1}x)}{b(\eta^{-1}, x)} = b(s_H(\eta), \eta^{-1}x). \end{aligned}$$

Now one may identify the function $[b\Delta]$ on $X \approx G \circ X$ also: notice that by using the homeomorphism $X \longrightarrow (G \times_{G^{(0)}} X)/G, x \mapsto [(r_X(x), x)]$, the Radon–Nikodym derivative $\frac{\mathrm{d}\mu_u}{\mathrm{d}\lambda_u} = [b\Delta]\colon X \longrightarrow \mathbb{R}$ is given by

$$\frac{\mathrm{d}\mu_u}{\mathrm{d}\lambda_u}(\eta^{-1}, x) = [b\Delta(\eta^{-1}, x)] = b(s_H(\eta), \eta^{-1}x) \tag{9}$$

for all $(\eta^{-1}, x) \in H \ltimes X$.

The right identity isomorphism: In a similar fashion as for the left identity isomorphism, one may prove that any composite $(X, \lambda) \circ (H, \beta^{-1})\colon (G, \alpha) \longrightarrow (H, \beta)$ is isomorphic to (X, λ). The equivariant homeomorphism of spaces $(X \times_{H^{(0)}} H)/H \approx X$ is clear. Let T denote the transformation groupoid $(X \times_{G^{(0)}} G) \rtimes G$. While constructing the family of measures on the composite, note that the adjoining function of (H, β^{-1}) is the constant function 1 (Example 2.11). Therefore, the modular function Δ_1 in Equation (5) is also the constant function 1. Choose any 0-cocycle b on the transformation groupoid $T := (X \times_{H^{(0)}} H) \rtimes H$, that is, $d^0(b) = \Delta_1 = 1$; then bm_u is an invariant measure (where $m_u := \lambda \times \beta_u^{-1}$) on the space of units of T. Then

$$1 = d^0(b) := \frac{b \circ s_T}{b \circ r_T}, \quad \text{that is,} \quad b \circ r_T = b \circ s_T.$$

Thus b induces a function $[b]$ on $X \approx (X \times_{H^{(0)}} H)/H$. With this observation, the discussion for left identity correspondence holds here word-to-word. And we can conclude that any composite $(X \circ H, \mu)$ of (X, λ) and (H, β^{-1}) is isomorphic to (X, λ) with the homeomorphism given above. Moreover, the Radon–Nikodym derivative $\mathrm{d}\mu_u/\mathrm{d}\lambda_u = [b]$.

Recall from Definition 2.17 that a composite of two topological correspondences is not defined uniquely; it depends on a 0-cochain on a certain transformation groupoid. Now we show that any two composites of topological correspondences are isomorphic in the sense of Definition 3.8.

3.10. Proposition. *Let*

$$(X,\alpha)\colon (G_1,\lambda_1)\to(G_2,\lambda_2)$$
$$(Y,\beta)\colon (G_2,\lambda_2)\to(G_3,\lambda_3)$$

be topological correspondences, and let

$$(\Omega,\mu),(\Omega,\mu')\colon (G_1,\lambda_1)\to(G_3,\lambda_3)$$

be two composites of the correspondences. Assume that (Ω,μ) is obtained by using a 0-cochain b and (Ω',μ') is obtained by using a 0-cochain b' on the transformation groupoid $(X\times_{G_2^{(0)}} Y)\rtimes G_2$. Then (Ω,μ) and (Ω,μ') are isomorphic topological correspondences.

Proof. Let $Z := X\times_{G_2^{(0)}} Y$ and $\pi\colon Z\to\Omega := Z/G_2$ be the quotient map. Since $b,b'\in \mathrm{C}^0_{G_3}(Z\rtimes G_2,\mathbb{R}^*_+)$ are 0-cochains with coboundaries $d^0(b)=d^0(b')=\Delta$, Remark 2.20 gives a function $c\colon\Omega\longrightarrow\mathbb{R}^*_+$ with the property that $b'=(c\circ\pi)b$. Since the quotient map π is open, the continuity of b, b' implies that any function c with above property is continuous. Let $f\in\mathrm{C_c}(\Omega)$ and $u\in G_3^{(0)}$. Then using Equation (7) we write

$$\begin{aligned}\int_\Omega f[x,y]\,\mathrm{d}\mu'_u([x,y]) &= \int_Y\int_X f\circ\pi(x,y)e(x,y)\,b'(x,y)\,\mathrm{d}\alpha_{r_Y(y)}(x)\,\mathrm{d}\beta_u(y)\\ &= \int_Y\int_X f\circ\pi(x,y)e(x,y)\,c\circ\pi(x,y)b(x,y)\,\mathrm{d}\alpha_{r_Y(y)}(x)\,\mathrm{d}\beta_u(y)\\ &= \int_\Omega f[x,y]c[x,y]\,\mathrm{d}\mu_u([x,y])\end{aligned}$$

where $e\colon Z\longrightarrow\mathbb{R}^+$ is a cutoff function. This calculation shows that for every $u\in G_3^{(0)}$, $\mu'_u\sim\mu_u$ with the Radon–Nikodym derivative $\frac{\mathrm{d}\mu'_u}{\mathrm{d}\mu_u}=c$, where $c\colon\Omega\longrightarrow\mathbb{R}^*_+$ is a function satisfying the property in Remark 2.20. ∎

Let (G,α) and (H,β) be locally compact groupoids equipped with Haar systems. For $i=1,2,3$, let (X_i,λ_i) be a topological correspondence from (G,α) to (H,β). Assume that, for $i=1,2$, $\phi_i\colon X_i\to X_{i+1}$ is an isomorphism of correspondences. Then the composite $\phi_2\circ\phi_1\colon X_1\to X_3$ gives an isomorphism of correspondences—this follows because of the fact that the composite of G-H-equivariant maps is also an equivariant map and the chain rule for equivalent families of measures.

Isomorphism is an equivalence relation on the set of topological correspondences from (G,α) to (H,β): let (X,λ_1), (Y,λ_2) and (Z,λ_3) be correspondences from (G,α) to (H,β).

Reflexivity is given by the identity function on X.

Symmetry if ϕ is an isomorphism from (X,λ_1) to (Y,λ_2), then ϕ^{-1} is G-H-equivariant homeomorphism. Now use Corollary 3.6 to see that $\phi^{-1}_*(\lambda_2)\sim\lambda_1$.

Transitivity Follows from the discussion just before this paragraph.

3.11. Remark. [Horizontal composite of topological correspondences] Let $(G,\alpha),(H,\beta)$ and (K,κ) be groupoids with Haar systems. Let (X,λ) and (Y,κ) be correspondences from (G,α) to (H,β), and (X',λ') and (Y',κ') be correspondences from (H,β) to (K,μ). Let $(X\circ X',\lambda\circ\lambda')$ and $(Y\circ Y',\kappa\circ\kappa')$ be some composites of (X,λ) and (X',λ'), and (Y,κ) and (Y',κ'), respectively, see Figure 5. Let b_1 and b_2 be the cochains in appropriate groupoid cohomologies used to produce $\lambda\circ\lambda'$ and $\kappa\circ\kappa'$, respectively.

Additionally, assume that $\phi\colon X\longrightarrow Y$ and $\phi'\colon X'\longrightarrow Y'$ are isomorphisms of correspondences. Then the map $[\phi\times\phi']\colon (X\times_{H^{(0)}}X')/H\longrightarrow(Y\times_{H^{(0)}}Y')/H$ is an isomorphism of correspondences where $[\phi\times\phi']([x,y])=[\phi(x),\phi'(y')]$, see Figure 5.

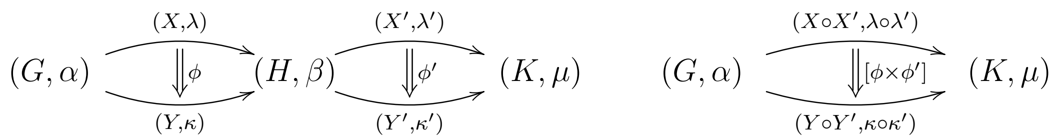


Figure 5

To check this, first of all, note that $\phi\times\phi'\colon X\times_{H^{(0)}}X'\longrightarrow Y\times_{H^{(0)}}Y'$ and $[\phi\times\phi']\colon X\circ X'\longrightarrow Y\circ Y'$ are well-defined G-K-equivariant homeomorphisms. Then, since $\phi_*(\lambda)\sim\kappa$ and ${\phi'}_*(\lambda')\sim\kappa'$, we get

$$(\phi\times\phi')_*(\lambda\times\lambda')\sim\kappa\times\kappa';$$

the Radon–Nikodym derivative

$$\frac{\mathrm{d}(\phi\times\phi')_*(\lambda\times\lambda')}{\mathrm{d}\,(\kappa\times\kappa')}=\frac{\mathrm{d}\phi_*(\lambda)}{\mathrm{d}\kappa}\;\frac{\mathrm{d}\phi'_*(\lambda')}{\mathrm{d}\kappa'}.$$

This discussion along with the fact that b_1 and b_2 are continuous positive functions allows us to say that

$$b_1\cdot(\phi\times\phi')_*(\lambda\times\lambda')\sim b_2\cdot(\kappa\times\kappa'); \tag{10}$$

the Radon–Nikodym derivative

$$\frac{\mathrm{d}(b_1\,(\phi\times\phi')_*(\lambda\times\lambda'))}{\mathrm{d}\,(b_2\,\kappa\times\kappa')_u}=\frac{\mathrm{d}\phi_*(\lambda)}{\mathrm{d}\kappa}\;\frac{\mathrm{d}\phi'_*(\lambda'_u)}{\mathrm{d}\kappa'}\;\frac{b_1}{b_2}.$$

Now use Lemma 3.7(iii) on the transformation groupoid $(Y\times_{H^{(0)}}Y')\rtimes H$ to conclude that $\lambda\circ\lambda'\sim\kappa\circ\kappa'$. This remark shows in the bicategory of topological correspondences, the two 2-arrows between a two composable 1-arrows induce a 2-arrow between a composite of 2-arrows.

3.12. The bicategory of topological correspondences. Now, with the help of the discussion in Section 3.1, we can define the data to form the bicategory of topological correspondences:

Objects or vertices second countable, locally compact, Hausdorff groupoids with Haar systems.

1-*arrows or edges* topological correspondences in which the space is locally compact, Hausdorff and second countable.

2-*arrows or* 2-*cells* isomorphisms of topological correspondences (Definition 3.8).

Vertical composition of 2-arrows 2-arrows are merely functions between spaces; their composition is the usual composition of functions.

1-*identity arrow* the identity 1-arrow on (G, α) is (G, α^{-1}), see Example 2.11.

2-*identity arrow* the identity 2-arrow on a topological correspondence (X, λ) is the identity map $\mathrm{Id}_X : X \to X$.

Composition of 1-arrows composition of correspondences as in Definition 2.17.

Horizontal composition of 2-arrows following Remark 3.11, we call $[\phi \times \phi']$ the horizontal product of ϕ and ϕ'.

The associativity isomorphism described in Theorem 3.14 below.

The identity isomorphism described in Example 3.9 earlier.

3.13. Remark. Ideally, one would expect that the identity 1-arrow over (G, α) is (G, α) itself. However, the odd choice of (G, α^{-1}) as the identity arrow above is result of the conflict between the definition of correspondence 2.5 and the traditional choice of *left* Haar systems or *left invariant* measures in general. Contrary to the conventional left invariant measures, we chose right invariant ones to define a topological correspondence which has introduced the current identity arrow. Had we switched the *sides of the conditions* in Definition 2.5, the family of measures λ on the bispace would have been left invariant. As a consequence (G, α) would be the 1-identity arrow. Initially, we wanted the C*-functor taking a topological correspondence to a C*-one to be covariant for certain reasons. Since, traditionally, a Hilbert module is considered as a right module, the covariance was achieved by the current defintion.

Thus now we have the data required in *i–iv* in Definition 2.24. The following theorem describes how this data fulfills the necessary conditions to give us the bicategory of topological correspondences. The proof of associativity isomorphisms is very long. Therefore, we break it into pieces and describe after the next theorem.

3.14. Theorem. *The above data along with (obvious) associativity and identity isomorphisms form the bicategory* $\mathfrak{T}$ *of topological correspondences.*

Proof. In the following discussion, the number of each topic indicates what topic in Definition 2.24 it is.
v)*Associativity isomorphism*: firstly, we list our data and notation for the associativity isomorphism.

i) For $i \in \{1, 2, 3\}$, (X_i, λ_i) is a correspondence from (G_i, α_i) to (G_{i+1}, α_{i+1}) with Δ_i as the adjoining function;

ii) $(X_{i(i+1)}, \mu_{i(i+1)})$ denotes a composite of (X_i, λ_i) and (X_{i+1}, λ_{i+1}) for $i = 1, 2$. Thus X_{ii+1} denotes the quotient space $(X_i \times_{G^{(0)}_{i+1}} X_{i+1})/G_{i+1}$. We write the cochain in $\mathrm{C}^0_{G_{i+2}}((X_i \times_{G^{(0)}_{i+1}} X_{i+1}) \rtimes G_{i+1}, \mathbb{R}^*_+)$ that produces the family of measures $\mu_{i(i+1)}$ as $b_{i(i+1)}$. We write π_{ii+1} for the quotient map $X_i \times_{G^{(0)}_{i+1}} X_{i+1} \longrightarrow X_{ii+1}$.

Each space X_i above is locally compact, Hausdorff, second countable and the action of the groupoid G_{i+1} on it is proper for $i \in \{1, 2, 3\}$. Therefore, the diagonal action of G_{i+1} on the fibre product $X_i \times_{H^{(0)}_{i+1}} X_{i+1}$ is proper.

iii) Moreover, the similar diagonal action of $G_2 \times G_3$ on $X_1 \times_{G^{(0)}_2} X_2 \times_{G^{(0)}_3} X_3$ is proper; let T denote the proper transformation groupoid $(X_1 \times_{G^{(0)}_2} X_2 \times_{G^{(0)}_3} X_3) \rtimes (G_2 \times G_3)$ for this diagonal action. Let π_{123} denote the quotient map $T^{(0)} \longrightarrow X_{123}$. The Haar system $\alpha_2 \times \alpha_3$ on $G_2 \times G_3$ induces a Haar system on T which we denote by $\alpha_2 \times \alpha_3$ itself. The quotient $T^{(0)}/T$ is denoted by X_{123}, and π_{123} is the quotient map $T^{(0)} \longrightarrow X_{123}$; cf. Figure 7 on page 875.

Let $(X_{(12)3}, \mu_{(12)3})$ be the given composite of (X_{12}, μ_{12}) and (X_3, λ_3), and let similar be the meaning of $(X_{1(23)}, \mu_{1(23)})$. Let $\pi'_{(12)3}\colon X_{12} \times_{G^{(0)}_3} X_3 \longrightarrow X_{(12)3}$ be the quotient map, and similar be the meaning of $\pi'_{1(23)}$. The proof starts now by defining two functions a' and a'' below. All of these spaces and maps are described in Figure 7. The map a in this figure is the associativity isomorphism that we explain now.

Define

$$a'\colon X_{123} \longrightarrow X_{(12)3}, \text{ by} \qquad a'\colon [x_1, x_2, x_2] \mapsto [[x_1, x_2], x_3],$$
$$a''\colon X_{123} \to X_{1(23)}, \text{ by} \qquad a''\colon [x_1, x_2, x_2] \mapsto [x_1, [x_2, x_3]]$$

where $[x_1, x_2, x_3] \in X_{123}$. We show that a' is well-defined and the well-definedness of a'' can be proven along similar lines. Let

$$\pi_{12} \times \mathrm{Id}_{X_3}\colon T^{(0)} \longrightarrow X_{12} \times_{G^{(0)}_3} X_3, \quad \pi'_{(12)3}\colon X_{12} \times_{G^{(0)}_3} X_3 \longrightarrow X_{(12)3}$$

be the quotient maps for the diagonal actions of H_2 and H_3 on $T^{(0)}$ and $X_{12} \times_{G^{(0)}_3} X_3$, respectively. Then $\pi_{(12)3} := \pi'_{(12)3} \circ (\pi_{12} \times \mathrm{Id}_{X_3})$ is a well-defined continuous surjection; define $\pi_{1(23)}$ similarly; cf. Figure 7. Note that for $(x_1, x_2, x_3) \in T^{(0)}$ and appropriate $(\gamma_1, \gamma_2) \in G_1 \times G_2$,

$$\begin{aligned}\pi_{(12)3}(x_1\gamma_1, \gamma_1^{-1}x_2\gamma_2, \gamma_2^{-1}x_3) &= [[x_1\gamma_1, \gamma_1^{-1}x_2\gamma_2], \gamma_2^{-1}x_3] \\ &= [[x_1\gamma_1, \gamma_1^{-1}x_2]\gamma_2, \gamma_2^{-1}x_3] = [[x_1, x_2], x_3] = \pi_{(12)3}(x_1, x_2, x_3).\end{aligned}$$

Therefore, due to the the universal property of the quotient space, $\pi_{(12)3}$ induces a continuous surjection $X_{123} \longrightarrow (X_1 \circ X_2) \circ X_3$ which we call a'. At this step, we note that Figure 7 on page 875 commutes.

We claim that both a' and a'' are homeomorphisms. We prove that a' is a homeomorphism, and the claim for a'' can be proved similarly. Surjectivity of a' is already justified. To show that a' is one-to-one, assume that for some $[x_1, x_2, x_3], [y_1, y_2, y_3] \in X_{123}$ $a'[x_1, x_2, x_3] = a'[y_1, y_2, y_3]$, that is, $[[x_1, x_2], x_3] = [[y_1, y_2], y_3]$. Then there is $\gamma_2 \in G_2$ with the property that

$$([x_1, x_2\gamma_2], \gamma_2^{-1}x_3) = ([x_1, x_2]\gamma_2, \gamma_2^{-1}x_3) = ([y_1, y_2], y_3).$$

Now there is $\gamma_1 \in G_1$ such that

$$(x_1\gamma_1, \gamma_1^{-1}x_2\gamma_2, \gamma_2^{-1}x_3) = (y_1, y_2, y_3).$$

Thus

$$[x_1, x_2, x_3] = [y_1, y_2, y_3] \in X_{123}.$$

Next we show that a' is open. Let $U \subseteq X_{123}$ be open. Then $\pi_{123}^{-1}(U)$ is open. Since all the groupoids we are working with have open range maps, the quotient maps $\pi_{12} \times \mathrm{Id}_{X_3}$ and $\pi'_{(12)3}$ are open, [Muhly-Williams, 1995, Lemma 2.1]. Finally, using the commutativity of Figure 7, we infer that $a'(U) = \pi_{(12)3}(\pi^{-1}(U)) \subseteq X_{(12)3}$ is open.

Since the quotient maps in Figure 7 are G_1-G_4-equivariant, so are a' and a''. Eventually, we define the associativity isomorphism $a(X_1, X_2, X_3)$ as

$$a(X_1, X_2, X_3) = a'' \circ a'^{-1}, \text{ that is,}$$
$$a(X_1, X_2, X_3)\colon [[x_1, x_2], x_3] \mapsto [x_1, [x_2, x_3]]$$

where $[[x_1, x_2], x_3] \in (X_1 \circ X_2) \circ X_3$. Whenever the spaces X_i, in the discussion, are clear, we write a instead of $a(X_1, X_2, X_3)$. We shall write a in rest of the discussion in this part of the proof.

We still need to show that a induces an equivalence of measures to conclude that it is an isomorphism of correspondences; this is proved independently in Proposition 3.21 and Remark 3.22 on page 874. This proof requires a long discussion that starts immediately after the present proof.

vi) *Identity isomorphisms*: let $i = 1, 2$ and (G_i, α_i) a groupoid with a Haar system, and let (X, λ) be a correspondence from (G_1, α_1) to (G_2, α_2) with Δ as the adjoining function. As we chose, (G_i, α_i^{-1}) is the identity arrow on (G_i, α_i). The claim is that the G_1-G_2-equivariant homeomorphisms of spaces

$$\ell(G_1)\colon (G_1 \times_{G_1^{(0)}} X)/G_1 \to X, \qquad [\gamma^{-1}, x] \mapsto \gamma^{-1}x$$
$$r(G_1)\colon (X \times_{G_2^{(0)}} G_2)/G_2 \to X, \qquad [x, \gamma] \mapsto x\gamma$$

are, respectively, the left and right identity coherences. This claim is proved in Example 3.9.

vii) *Horizontal composition of 2-arrows*: Let $(X_i, \lambda_i), (X_i', \lambda_i')$ be correspondences from (G_i, α_i) to (G_{i+1}, α_{i+1}) for $i = 1, 2$ and let $\phi_i \colon X_i \longrightarrow X_i'$ be isomorphisms of correspondences. Let (X_{12}, μ) be a composite of (X_1, λ_1) and (X_2, λ_2), and (X_{12}', μ') a composite of (X_1', λ_1') and (X_2', λ_2').

Now the map

$$\phi_1 \times \phi_2 \colon X_1 \times_{G_2^{(0)}} X_2 \longrightarrow X_1' \times_{G_2^{(0)}} X_2', \quad \phi_1 \times \phi_2(x, y) = (\phi_1(x), \phi_2(y)),$$

where $(x, y) \in X_1 \times_{G_2^{(0)}} X_2$, is a G_1-G_3-equivariant homeomorphism for the left and right obvious actions of G_1 and G_3. Moreover, this map is also G_2-equivariant for the diagonal actions of G_2 on the fibre products. Therefore, the map induces a well-defined G_1-G_3-equivariant homeomorphism $[\phi_1 \times \phi_2] \colon X_{12} \longrightarrow X_{12}'$. We define $[\phi_1 \times \phi_2]$ the horizontal composition of the 2-arrows ϕ_1 and ϕ_2 which is often written as $\phi_2 \cdot_h \phi_2$. To check that this definition makes sense, one needs to check that $[\phi_1 \times \phi_2]$ induces an isomorphism of topological correspondences (X_{12}, μ) and (X_{12}', μ') which is verified in Remark 3.11.

vii) *Associativity coherence*: let (G_i, α_i) be groupoids equipped with Haar systems for $i = 1, \dots, 5$. Let (X_i, λ_i) be a correspondence from G_i to G_{i+1} for $i = 1, \dots, 4$. Let $(X_{(12)3}, \mu_{(12)3})$ and $(X_{1(23)}, \mu_{1(23)})$ have meanings as in case of the associativity isomorphism, see page 870. And let $(X_{((12)3)4}, \mu_{((12)3)4})$ and other subscripts of X and μ with parentheses have similar meanings. Let $a(\_, \_, \_)$ denote the associativity isomorphism when the blanks filled appropriately, as discussed for associativity isomorphism. Then the associativity coherence demands that the pentagon in Figure 6 should commute.

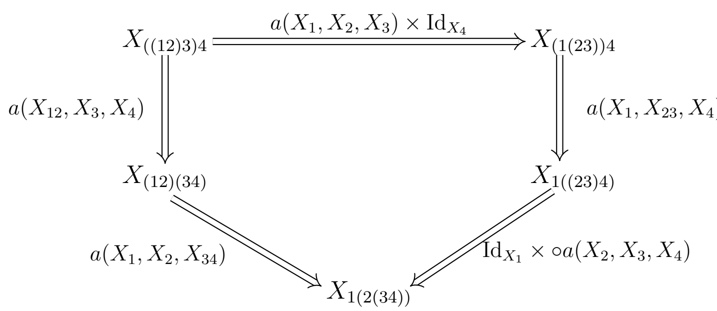


Figure 6: Associativity coherence

Let $[[[x_1, x_2], x_3], x_4]$ be a point in $X_{((12)3)4}$. Following the left top vertex of the pentagon along the right top side till the vertex at the bottom, we get that

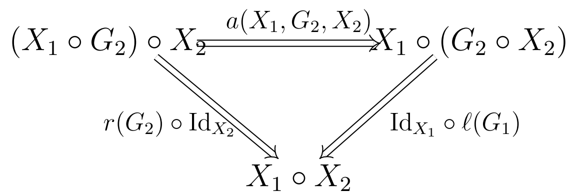
$$[[[x_1, x_2], x_3], x_4] \xmapsto{a(X_1, X_2, X_3) \times \mathrm{Id}_{X_4}} [[x_1, [x_2, x_3]], x_4]$$
$$\xmapsto{a(X_1, X_{23}, X_4)} [x_1, [[x_2, x_3], x_4]] \xmapsto{\mathrm{Id}_{X_1} \times a(X_2, X_3, X_4)} [x_1, [x_2, [x_3, x_4]]].$$

And, on the other way,

$$[[[x_1, x_2], x_3], x_4] \quad \xmapsto{a(X_{12}, X_3, X_4)} \quad [[x_1, x_2], [x_3, x_4]] \quad \xmapsto{a(X_1, X_2, X_{34})} \quad [x_1, [x_2, [x_3, x_4]]].$$

Thus the figure commutes.

viii) *Identity coherence*: let (X_i, λ_i) be topological correspondences from (G_i, α_i) to (G_{i+1}, α_{i+1}) for $i = 1, 2$. Let $(G_1 \circ X_1, \alpha_1^{-1} \circ \lambda_2)$ be a composite of the identity correspondence at (G_1, α_1) and (X_1, λ_1); let $(X_1 \circ G_2, \lambda_1 \circ \alpha_2^{-1})$ be a composite of (X_1, λ_1) and the identity correspondence at (G_2, α_2); and let (X_{12}, λ_{12}) be a composite of (X_1, λ_1) and (X_2, λ_2). Then we need to show that following diagram is commutative for identity coherence commutes:



Let $[[x_1, \gamma], x_2] \in (X_1 \circ G_2) \circ X_2$. Then

$$\begin{aligned}\mathrm{Id}_{X_1} \circ \ell(G_1)\,(a(X_1, G_2, X_2)\,([[x_1, \gamma], x_2])) &= [x_1, \gamma x_2] \\ &= [x_1\gamma, x_2] = r(G_2) \circ \mathrm{Id}_{X_2}([[x_1, \gamma], x_2]).\end{aligned}$$

This proves all the axioms and hence the theorem. ∎

From here up to Remark 3.22 on page 881 is the proof of the claim of consistency of measures in the associativity isomorphism in Theorem 3.14.

Firstly, we draw the commutating diagram in Figure 7 on page 875. The gist of this discussion is as follows: for each $u \in G_4^{(0)}$, the measure $\lambda_1 \times \lambda_2 \times \lambda_{3u}$ on $T^{(0)}$ is $(G_2 \times G_3, \alpha_2 \times \alpha_3)$-quasi-invariant. We multiply this measure by appropriate 0-cochains on the groupoid T so that the resultant measure is invariant. This invariant measure then agrees a disintegration along the map π_{123} with respect to a family of measures α_{123} along π_{123}—α_{123} is basically the averaging by $\alpha_2 \times \alpha_3$—to produce measure μ_{123u} on X_{123} (central vertical arrows in Figure 7). On the other hand, similar process happens twice along the two left slanting arrows—$\pi_{(12)} \times \mathrm{Id}_{X_3}$ and $\pi'_{(12)3}$—of Figure 7. By abusing the language a bit, one can say that an appropriate function-multiple of $\lambda_1 \times \lambda_2 \times \lambda_3$ produces a family of measures on $X_{12} \times_{G^{(0)}} X_3$; an appropriate function-multiple this produced family of measures, in turn, induces a family of measures $\mu_{(12)3}$ on $X_{(12)3}$. Now the critical, technical issue is to relate the 0-cocycles involved in the production of μ_{123} and $\mu_{(12)3}$. We relate these 0-cocycles desirably, and finally Lemma 3.20 shows that the families of measures ${a'_*}^{-1}(\mu_{(12)3})$ and μ_{123} on X_{123} are equivalent. Similar arguments imply that on $X_{1(23)}$, $a''_*(\mu_{123}) \sim \mu_{1(23)}$. Finally, in Proposition 3.21, an application of the Chain rule shows that $a''_*({a'_*}^{-1}(\mu_{(12)3})) \sim \mu_{1(23)}$—the result we are seeking for.

While proving above fact, we also compute the Radon–Nikodym derivatives which implement the equivalence of measures; these Radon–Nikodym derivatives are, basically, given by the combinations of the 0-cocycles. This leads to the conclusion that the possibility of *choosing* a 0-cocycle in the composition of topological correspondences is the source of the bicategorical nature of topological correspondences.

In the beginning, from Lemma 3.16 to 3.20, we discuss the measures residing on the spaces and various families of measures along the maps in the left-half of Figure 7.

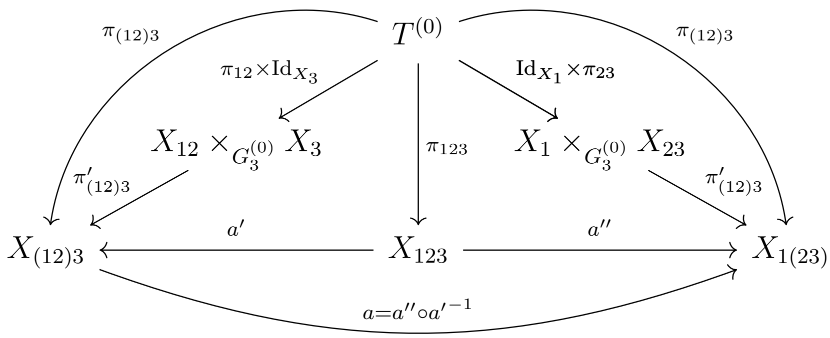


Figure 7

Let $[\alpha_2]\times\delta_{X_3}=\{[\alpha_2]\times\delta_{X_3([x,y],z)}\}_{([x,y],z)\in X_{12}\times_{G_3^{(0)}}X_3}$ be the family of measures along $\pi_{12}\times\mathrm{Id}_{X_3}$ defined as

$$\int_{T^{(0)}} f(t)\,\mathrm{d}[\alpha_2]\times\delta_{X_3([x,y],z)}(t)=\int_{G_2} f(x\gamma,\gamma^{-1}y,z)\,\mathrm{d}\alpha_2(\gamma)$$

for $f\in\mathrm{C_c}(T^{(0)})$; let $[\alpha_3']=\{\alpha_3{}'_{[[x,y],z]}\}_{[[x,y],z]\in X_{(12)3}}$ be the one along $\pi'_{(12)3}$ given by

$$\int_{X_{12}\times_{G_3^{(0)}}X_3} g(t)\,\mathrm{d}[\alpha_3']_{[[x,y],z]}(t)=\int_{G_3} g([x,y]\eta,\eta^{-1}z)\,\mathrm{d}\alpha_3(\eta)$$

where $g\in\mathrm{C_c}(X_{12}\times_{G_3^{(0)}}X_3)$. The composite $[\alpha_3']\circ([\alpha_2]\times\delta_{X_3}):=\alpha_{(12)3}$ is a family of measures along $\pi_{(12)3}$. One the other hand, define families of measures

(i) $\delta_{X_1}\times[\alpha_3]'$ along $\mathrm{Id}_{X_1}\times\pi_{23}$,

(ii) $[\alpha_2']$ along π'_{123} and

(iii) $\alpha_{1(23)}$ along $\pi_{1(23)}$

analogous to $[\alpha_2]\times\delta_{X_3},[\alpha_3']$ and $\pi_{(12)3}$, respectively. Finally, let α_{123} be the family of measures along π_{123} which is averaging by $\alpha_1\times\alpha_2$ as in Lemma 2.1.

3.15. Definition. *Define the following functions*

$$\begin{aligned}
A_{123}\colon \mathrm{C_c}(T^{(0)})&\longrightarrow\mathrm{C_c}(X_{123}), && A_{123}(f)(w):=\alpha_{123w}(f);\\
A_{12}\colon \mathrm{C_c}(T^{(0)})&\longrightarrow\mathrm{C_c}(X_{12}\times_{G_3^{(0)}}X_3), && A_{12}(f)(p):=([\alpha_2]\times\delta_{X_3})_p(f);\\
A'_{(12)3}\colon \mathrm{C_c}(X_{12}\times_{G_3^{(0)}}X_3)&\longrightarrow\mathrm{C_c}(X_{(12)3}), && A'_{(12)3}(h)(q):=[\alpha_3']_q(h);\\
A_{(12)3}&=A'_{(12)3}\circ A_{12}, && A_{(12)3}(f)(q),:=\alpha_{(12)3_q}(f);\\
A'_*\colon \mathrm{C_c}(X_{123})&\longrightarrow\mathrm{C_c}(X_{(12)3}), && A'_*(k)(w):=k\circ a'^{-1}(w)
\end{aligned}$$

for $f \in \mathrm{C_c}(T^{(0)}), h \in \mathrm{C_c}(X_{12} \times_{G_3^{(0)}} X_3), k \in \mathrm{C_c}(X_{123}), p \in X_{12} \times_{G_3^{(0)}} X_3, q \in X_{(12)3}$ *and* $w \in X_{(12)3}$*. As well, define* $A_{23}, A'_{1(23)}, A_{1(23)}$ *and* A''_* *analogously using the families of measures* $\delta_{X_1} \times \alpha_{23}, [\alpha_2]', \alpha_{1(23)}$ *and the homeomorphism* a''*.*

The functions in Definition 3.15, constitute Figure 8 on page 876.

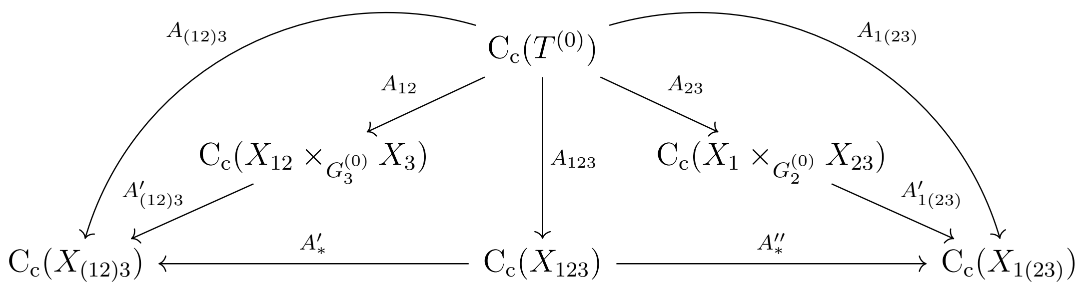


Figure 8

3.16. Lemma. *The maps* A'_* *and* A''_* *are isomorphisms of complex vector spaces. And* $A_{(12)3} = A'_* \circ A_{123}$ *and* $A_{1(23)} = A''_* \circ A_{123}$*.*

Proof. Since a' and a'' are homeomorphisms, A'_* and A''_* are isomorphisms of complex vector spaces. Next, let $f \in \mathrm{C_c}(T^{(0)})$ and $[x, y, z] \in X_{123}$. Then

$$\begin{aligned} A_{(12)3}(f)([[x,y],z]) &= \int_{G_3} A'_{(12)3}(f)([x,y]\eta, \eta^{-1}z)\, \mathrm{d}\alpha_3^{r_{X_3}}(\eta) \\ &= \int_{G_3}\int_{G_2} f(x\gamma, \gamma^{-1}y\eta, \eta^{-1}z)\, \mathrm{d}\alpha_2^{r_{X_2}(y)}(\gamma)\, \mathrm{d}\alpha_3^{r_{X_3}}(\eta) \\ &= \int_{G_3\times G_2} f(x\gamma, \gamma^{-1}y\eta, \eta^{-1}z)\, \mathrm{d}(\alpha_2\times\alpha_3)^{(r_{X_2}(y), r_{X_3}(z))}(\gamma,\eta) \\ &= A_{123}(f)([x,y,z]) = A'_*(A_{123}(f))([[x,y],z]). \end{aligned}$$

The third equality above is due to Fubini's theorem. This shows that $A_{(12)3} = A'_* \circ A_{123}$. The other follows from a similar computation. ∎

Lemma 3.16 makes the Figure 8 commutative.

Now we discuss the (families of) measures on the spaces involved in the left-half of Figure 7. Let $i = 1, 2$ and fix $u_i \in G_{i+2}^{(0)}$. Recall from the discussion about Equation (5) on page 856 that the measure $\lambda_i \times \lambda_{i+1_{u_i}}$ on $X_i \times_{G_{i+1}^{(0)}} X_{i+1}$ is (G_{i+1}, α_{i+1})-quasi-invariant; the 1-cocycle D_i on the transformation groupoid $(X_i \times_{G_{i+1}^{(0)}} X_{i+1}) \rtimes G_{i+1}$ that implements the quasi-invariance and is given by Equation (5); in this case it is

$$D_{i+1}(x_i, x_{i+1}, \gamma) = \Delta_{i+1}(\gamma^{-1}, x_{i+1})$$

where $(x_i, x_{i+1}, \gamma) \in (X_i \times_{G_{i+1}^{(0)}} X_{i+1}) \rtimes G_{i+1}$ and Δ_{i+1} is the adjoining function of the correspondence (X_{i+1}, λ_{i+1}).

We know that $T^{(0)}$ is a proper G_{i+1}-space for an appropriate diagonal action. Now we notice that the family of measures $\{\lambda_1 \times \lambda_2 \times \lambda_{3v}\}_{v \in G_4^{(0)}}$ on $T^{(0)}$ is (G_{i+1}, α_{i+1})-quasi-invariant with the function

$$(x_1, x_2, x_3, \gamma) \mapsto D_{i+1}(x_i, x_{i+1}, \gamma) = \Delta_{i+1}(\gamma^{-1}, x_{i+1}), \quad T^{(0)} \rtimes G_{i+1} \longrightarrow \mathbb{R}^+ \tag{11}$$

as the modular function. Moreover, if we focus on the case $i = 1$,

$$b_{12} \times \mathrm{Id}_{X_3} \colon T^{(0)} \longrightarrow \mathbb{R}^+$$

is the 0-cochain for which the measure $b_{12} \times \mathrm{Id}_{X_3} \cdot (\lambda_1 \times \lambda_2 \times \lambda_{3u})$ is G_2-invariant where $u \in G_4^{(0)}$. This invariant measure induces the measure $\mu_{12} \times \lambda_{3u}$ on $X_{12} \times_{G_2^{(0)}} X_3$. The action of G_3 on $T^{(0)}$ induces a proper diagonal action on $X_{12} \times_{G_3^{(0)}} X_3$. This is a direct computation. Now the measure $\mu_{12} \times \lambda_{3u}$ is, in turn, (G_3, α_3)-quasi-invariant with the function

$$D'_{(12)3} \colon ([x, y], z, \eta) \mapsto \Delta_3(\eta^{-1}, z), \quad (X_{12} \times_{G_3^{(0)}} X_3) \rtimes G_3 \longrightarrow \mathbb{R}^+ \tag{12}$$

as the modular function; this is a direct computation that uses (G_3, α_3)-quasi-invariance of the family of measures λ_3 on X_3.

Let $b'_{(12)3}$ be a 0-cochain on the transformation groupoid $(X_{12} \times_{G_3^{(0)}} X_3) \rtimes G_3$ such that

$$d^0(b'_{(12)3}) = D'_{(12)3}$$

Then $b'_{(12)3}(\mu_2 \times \lambda_3)$ is a G_3-invariant family of measures. This family of measures induces the family of measures $\mu'_{(12)3}$ on $X_{(12)3}$ that so that $(X_{(12)3}, \mu'_{(12)3}) \colon (G_1, \alpha_1) \longrightarrow (G_4, \alpha_4)$ is a topological correspondence. In the next lemma and what succeeds it, we discuss the measures on X_{123} using the map π_{123}.

3.17. Lemma. [For associativity isomorphism]

(i) For $u \in G_4^{(0)}$, the measure $\lambda_1 \times \lambda_2 \times \lambda_{3u}$ on $T^{(0)}$ is $(T, \alpha_2 \times \alpha_3)$-quasi-invariant. The 1-cocycle D on T that implements the quasi-invariance is given by

$$D(x, y, z, \gamma^{-1}, \eta^{-1}) := D_2(x, y, \gamma^{-1}) D'_{(12)3}([x, y], z, \eta^{-1})$$

where $D'_{(12)3}$ and D_2 are defined in above discussion.

(ii) The map $B' \colon T^{(0)} \longrightarrow \mathbb{R}^+$ given by $B' \colon (x, y, z) \mapsto b_{12}(x, y) b'_{(12)3}([x, y], z)$ is a 0-cochain on T with $d^0(B') = D$.

PROOF. (i): Fix $u \in G_4^{(0)}$, and let $f \in \mathrm{C_c}(T)$. Then

$$\int_{T^{(0)}} \int_T f(x, y, z, \gamma^{-1}, \eta^{-1}) \,\mathrm{d}(\alpha_2 \times \alpha_3)^{(x,y,z)}(\gamma, \eta) \,\mathrm{d}(\lambda_1 \times \lambda_2 \times \lambda_{3u})(x, y, z)$$
$$= \int_{X_3} \int_{X_2} \int_{X_1} \int_{G_3} \int_{G_2} f(x, y, z, \gamma^{-1}, \eta^{-1}) \,\mathrm{d}{\alpha_2}^{r_{X_2}(y)}(\gamma)\mathrm{d}{\alpha_3}^{r_{X_3}(z)}(\eta)) \,\mathrm{d}\lambda_1^{r_{X_2}(y)}(x) \,\mathrm{d}\lambda_2^{r_{X_3}(z)}(y) \,\mathrm{d}\lambda_3^u(z)$$

Changing the variable $(x, y, z, \gamma^{-1}, \eta^{-1}) \mapsto (x, y, z, \gamma^{-1}, \eta^{-1})^{-1} = (x\gamma, \gamma^{-1}y\eta, \eta^{-1}z, \gamma, \eta)$ in the last term above. Then an appropriate repeated use of Fubini's theorem shows that the last term above equals

$$\int_{X_3} \int_{X_2} \int_{X_1} \int_{G_3} \int_{G_2} f(x\gamma, \gamma^{-1}y\eta, \eta^{-1}z, \gamma, \eta) \ \Delta_2(\gamma, \gamma^{-1}y\eta)\Delta_3(\eta, \eta^{-1}z) \,\mathrm{d}{\alpha_2}^{r_{X_2}(y)}(\gamma)\mathrm{d}{\alpha_3}^{r_{X_3}(z)}(\eta)) \,\mathrm{d}\lambda_1^{r_{X_2}(y)}(x)\lambda_2^{r_{X_3}(z)}(y)\lambda_3^u(z).$$

Replace Δ_2 and Δ_3 by D_2 and $D'_{(12)3}$, respectively, in above term using Equations (11) and (12):

$$\int_{T^{(0)}} \int_T f(x\gamma, \gamma^{-1}y\eta, \eta^{-1}z, \gamma, \eta) \, D_2(x\gamma, \gamma^{-1}y\eta, \gamma^{-1}) D'_{(12)3}([x, y]\eta, \eta^{-1}z, \eta^{-1}) \,\mathrm{d}(\alpha_2 \times {\alpha_3}^{(x,y,z)}(\gamma, \eta)) \,\mathrm{d}(\lambda_1 \times \lambda_2 \times \lambda_3)(x, y, z)$$
$$= \int_{T^{(0)}} \int_T f(x\gamma, \gamma^{-1}y\eta, \eta^{-1}z, \gamma, \eta) \, D(x\gamma, \gamma^{-1}y\eta, \eta^{-1}z, \gamma, \eta) \,\mathrm{d}(\alpha_2 \times {\alpha_3}^{(x,y,z)}(\gamma, \eta)) \,\mathrm{d}(\lambda_1 \times \lambda_2 \times \lambda_3)(x, y, z).$$

Which implies that $D(x\gamma, \gamma^{-1}y\eta, \eta^{-1}z, \gamma, \eta) = D_2(x\gamma, \gamma^{-1}y\eta, \gamma^{-1}) D'_{(12)3}([x, y]\eta, \eta^{-1}z, \eta^{-1})$ almost everywhere, equivalently $D(x, y, z, \gamma^{-1}, \eta^{-1}) := D_2(x, y, \gamma^{-1}) D'_{(12)3}([x, y], z, \eta^{-1})$ almost everywhere. But due to continuity of all the functions involved, the equalities hold everywhere.

(ii): This claim follows from a direct computation: For $(x, y, z, \gamma^{-1}, \eta^{-1}) \in T$,

$$d^0(B')(x, y, z, \gamma^{-1}, \eta^{-1}) := \frac{B' \circ s_T(x, y, z, \gamma^{-1}, \eta^{-1})}{B' \circ r_T(x, y, z, \gamma^{-1}, \eta^{-1})} = \frac{B'(x\gamma, \gamma^{-1}y\eta, \eta^{-1}z)}{B'(x, y, z)}$$
$$= \frac{b_{12}(x\gamma, \gamma^{-1}y\eta)}{b_{12}(x, y)} \ \frac{b'_{(12)3}([x\gamma, \gamma^{-1}y\eta], \eta^{-1}z)}{b'_{(12)3}([x, y], z)} = \frac{b_{12}(x\gamma, \gamma^{-1}y)}{b_{12}(x, y)} \ \frac{b'_{(12)3}([x, y]\eta, \eta^{-1}z)}{b'_{(12)3}([x, y], z)}$$
$$= \frac{D_2 \circ s_{(X_1 \times_{G_2^{(0)}} X_2) \rtimes G_2}(x, y, \gamma)}{D_2 \circ r_{(X_1 \times_{G_2^{(0)}} X_2) \rtimes G_2}(x, y, \gamma)} \ \frac{D'_{(12)3} \circ s_{(X_{12} \times_{G_3^{(0)}} X_3) \rtimes G_3}([x, y], z, \eta)}{D'_{(12)3} \circ r_{(X_{12} \times_{G_3^{(0)}} X_3) \rtimes G_3}([x, y], z, \eta)}$$
$$= \Delta_2(\gamma^{-1}, y)\Delta_{(12)3}([x, y], z\eta) = D(x, y, z, \gamma^{-1}, \eta^{-1}).$$

We use the G_3-invariance of b_{12} (or the G_2-invariance of b_{23}) to get the fourth equality. ∎

To see what does the analogue of above lemma say regarding the right-half of Figure 7, we continue the discussion that follows Equation (11) for $i = 2$. In this case,

$$\mathrm{Id}_{X_1} \times b_{23} \colon T^{(0)} \longrightarrow \mathbb{R}^+$$

is the 0-cochain for which the measure $(\mathrm{Id}_{X_1} \times b_{23}) \cdot (\lambda_1 \times \lambda_2 \times \lambda_{3_u})$ is G_2-invariant where $u \in G_4^{(0)}$. This invariant measure induces the measure $\lambda_1 \times \mu_{23_u}$ on $X_1 \times_{G_2^{(0)}} X_{23}$. The action of G_2 on $T^{(0)}$ induces a proper diagonal action on $X_1 \times_{G_2^{(0)}} X_{23}$. The measure $\lambda_1 \times \mu_{12_u}$ is, in turn, (G_2, α_2)-quasi-invariant with the function

$$D''_{1(23)} \colon (x, [y, z], \eta) \mapsto \Delta_2(\eta^{-1}, y), \quad (X_1 \times_{G_2^{(0)}} X_{23}) \rtimes G_2 \longrightarrow \mathbb{R}^+$$

as the modular function; recall from a remark following Definition 2.5 that the adjoining function Δ_2 is G_3-invariant due to which $D''_{1(23)}$ is well-defined. Now fix a 0-cochain $b''_{1(23)}$ on the transformation groupoid $(X_1 \times_{G_2^{(0)}} X_{23}) \rtimes G_2$ with the property that

$$d^0(b''_{1(23)}) = D'_{1(23)}.$$

Then $b''_{1(23)}(\lambda_1 \times \mu_2)$ is a G_2-invariant family of measures. This family of measures induces the family of measures $\mu'_{1(23)}$ on $X_{1(23)}$ that so that $(X_{1(23)}, \mu'_{1(23)}) \colon (G_1, \alpha_1) \longrightarrow (G_4, \alpha_4)$ is a topological correspondence. Now the analogue of Lemma 3.17 can be stated as

3.18. Lemma.

(i) For $u \in G_4^{(0)}$, the measure $\lambda_1 \times \lambda_2 \times \lambda_{3_u}$ on $T^{(0)}$ is $(T, \alpha_2 \times \alpha_3)$-quasi-invariant. The 1-cocycle D_R on T that implements the quasi-invariance is given by

$$D_R(x, y, z, \gamma^{-1}, \eta^{-1}) := D_{1(23)}(x, [y, z], \gamma) D_3(y, z, \eta).$$

(ii) The map $B'' \colon T^{(0)} \longrightarrow \mathbb{R}^+$ given by $B'' \colon (x, y, z) \mapsto b_{23}(y, z) b''_{1(23)}(x, [y, z])$ is a 0-cochain on T with $d^0(B'') = D_R$.

Proof. Similar to that of Lemma 3.17. ∎

3.19. Remark. Let $u \in G_4^{(0)}$. Since the cocycles D and D_R in Lemmas 3.17 and 3.18 are modular functions of the measure $\lambda_1 \times \lambda_2 \times \lambda_{3_u}$ on the unit space of the locally compact groupoid T equipped with the Haar system (induced by $\alpha_1 \times \alpha_2$), D and D_R are $\lambda_1 \times \lambda_2 \times \lambda_3$-almost everywhere on $T^{(0)}$. But both D and D_R are continuous. Therefore, $D = D_R$. Hereon, we shall dente this cocycle by D itself. In fact, one can show that $D(x, y, z, \gamma^{-1}, \eta^{-1}) = D_2(x, y, \gamma) D_3(y, z, \eta)$ and $P \colon T^{(0)} \longrightarrow \mathbb{R}^+$ defined by $P(x, y, z) = b_{12}(x, y) b_{23}(y, z)$ is a 0-cochain with $d^0(P) = D$.

Now we fix a 0-cochain B on T with the property that $d^0(B) = D$; use the same cochain while working with the right-half of Figure 7. Then $B \cdot (\lambda_1 \times \lambda_2 \times \lambda_3)$ is a T-invariant measure on $T^{(0)}$. Now Lemma 2.14(ii) gives us a family of measures $\mu_{123} := \{\mu_{123_u}\}_{u\in G_4^{(0)}}$ with the property that

$$\mu_{123_u}(A_{123}(f)) = (B\cdot(\lambda_1\times\lambda_2\times\lambda_{3_u}))\,(f) \tag{13}$$

for all $u \in G_4^{(0)}$. Additionally, since $d^0(B') = d^0(B) = D$, Remark 2.20 says that the function B'/B (or B''/B) is constant on the T-orbits of $T^{(0)}$. Thus B'/B (or B''/B) induces a continuous function on $T^{(0)}\backslash T = X_{123}$ which we denote by $[B'/B]$ (or $[B''/B]$, respectively).

3.20. LEMMA. *The families of measures $a'^{-1}_*(\mu_{(12)3})$ and μ_{123} on $X_{123} = T^{(0)}/T$ are equivalent, and $[B'/B]$ is the Radon–Nikodym derivative $\mathrm{d}a'^{-1}_*\mu_{(12)3_u}/\mathrm{d}\mu_{123_u}$ where $u \in G_4^{(0)}$.*

PROOF. Assume that the measure $a'^{-1}_*(\mu_{(12)3_u})$ on X_{123} disintegrates the measure $B'(\lambda_1\times\lambda_2\times\lambda_{3_u})$ as

$$B'(\lambda_1\times\lambda_2\times\lambda_{3_u}) = a'^{-1}_*(\mu_{(12)3_u})\circ A_{123}. \tag{14}$$

Therefore, $B'(\lambda_1\times\lambda_2\times\lambda_{3_u})$ is a $(T,\alpha_2\times\alpha_3)$-invariant measure on $T^{(0)}$ due to Lemma 2.14(i). On the other hand, from Equation (13) and the discussion preceding it, we already know that $B(\lambda_1\times\lambda_2\times\lambda_{3_u})$ is also an invariant measure on $T^{(0)}$ with

$$B(\lambda_1\times\lambda_2\times\lambda_{3_u}) = \mu_{123_u}\circ A_{123}.$$

But then, since B and B' are $\mathbb{R}^+$-valued cochains, $B'(\lambda_1\times\lambda_2\times\lambda_{3_u})$ and $B(\lambda_1\times\lambda_2\times\lambda_{3_u})$ are equivalent measures on $T^{(0)}$; the Radon–Nikodym derivative

$$\frac{\mathrm{d}\,B'(\lambda_1\times\lambda_2\times\lambda_{3_u})}{\mathrm{d}\,B(\lambda_1\times\lambda_2\times\lambda_{3_u})} = \frac{B'}{B}.$$

Now we apply Lemma 3.7(iii) to these two invariant measures on $T^{(0)}$ to infer the claim of the present lemma.

Following is the proof for Equation (14): let $f\in \mathrm{C_c}(T^{(0)})$ and $u\in G_4^{(0)}$. Then

$$\begin{aligned} a'^{-1}_*(\mu_{(12)3_u})(A_{123}(f)) &:= \mu_{(12)3_u}(A_{123}(f)\circ a'^{-1}) \\ &= \mu_{(12)3_u}(A'_*\circ A_{123}(f)) = \mu_{(12)3_u}(A_{(12)3}(f)) = \mu_{(12)3_u}\left(A'_{(12)3}\left(A_{12}(f)\right)\right) \end{aligned}$$

where the first equality is the definition of pushforward measure, second one is the definition A'_*, the third one follows due to Lemma 3.16 and last one follows from the definition of $A_{(12)3}$. We use the disintegration $b'_{(12)3}(\mu_{12}\times\lambda_{3_u}) = \mu_{(12)3_u}\circ A_{(12)3}$ of measures along $\pi'_{(12)3}$ to see that the last term above equals

$$\begin{aligned} &b'_{(12)3}\cdot\mu_{12}\times\lambda_{3_u}\,(A_{12}(f)) \\ &= \int_{X_3}\int_{X_{12}} A_{12}(f)([x,y],z)b'_{(12)3}([x,y],z)\,\mathrm{d}\mu_{12_{r_{X_3}(z)}}([x,y])\,\mathrm{d}\lambda_{3_u}(z) \\ &= \int_{X_3} (\mu_{12})_{r_{X_3}(z)}\left(A_{12}(f)b'_{(12)3}\right)\,\mathrm{d}\lambda_{3_u}(z). \end{aligned}$$

Now we use the disintegration of families of measures $\mu_{12} \circ A_{12} = b_{12}(\lambda_1 \times \lambda_2)$ to see that the last term in above equation equals

$$\int_{X_3} \int_{X_1} \int_{X_2} f(x,y,z) b_{12}(x,y) b'_{(12)3}([x,y],z)\, \mathrm{d}\lambda_{1 r_{X_2}(y)}(x)\, \mathrm{d}\lambda_{2 r_{X_3}(z)}(y)\, \mathrm{d}\lambda_{3u}(z)$$
$$= \int_{X_1} \int_{X_2} \int_{X_3} f(x,y,z) B'(x,y,z)\, \mathrm{d}\lambda_{1 r_{X_2}(y)}(x)\, \mathrm{d}\lambda_{2 r_{X_3}(z)}(y)\, \mathrm{d}\lambda_{3u}(z)$$

where B' is the 0-cochain on T in Lemma 3.17. ∎

3.21. Proposition. *In the discussion of the associativity isomorphism on page 872, the families of measures* $a_*(\mu_{(12)3})$ *and* $\mu_{1(23)}$ *on* $X_{1(23)}$ *are equivalent, and the Radon–Nikodym derivative* $\mathrm{d}a_*\mu_{(12)3}/\mathrm{d}\mu_{1(23)} = [B'/B''] \circ a''^{-1}$.

This proposition follows from the Chain rule and the fact that being equivalent via a homeomorphism is an equivalence relation on the families of measures.

Proof Proof of Proposition 3.21. Lemma 3.20 shows that $a'^{-1}_*(\mu_{(12)3}) \sim \mu_{123}$ on X_{123} and the Radon–Nikodym derivative $\mathrm{d}a'^{-1}_*(\mu_{(12)3_u})/\mathrm{d}\mu_{123_u} = [B'/B]$ for $u \in G_4^{(0)}$. For the right-half of Figure 7, one may prove—on the similar lines as Lemma 3.20—that $a''^{-1}_*(\mu_{1(23)}) \sim \mu_{123}$ on X_{123} and the Radon–Nikodym derivative $\mathrm{d}a''^{-1}_*(\mu_{1(23)_u})/\mathrm{d}\mu_{123_u} = [B''/B]$ for $u \in G_4^{(0)}$.

Now the transitivity of equivalence of measures implies that $a'^{-1}_*(\mu_{(12)3}) \sim a''^{-1}_*(\mu_{1(23)})$ on X_{123}, and the function $[B'/B'']$ implements the equivalence. Using Lemma 3.3, we see that

$$a''_* a'^{-1}_*(\mu_{(12)3}) \sim a''_* a''^{-1}_*(\mu_{1(23)}) \quad \text{on } X_{1(23)}. \tag{15}$$

Lemma 3.4, implies that $a''_* a'^{-1}_*(\mu_{(12)3}) = (a'' \circ a'^{-1})_*(\mu_{(12)3}) = a_*(\mu_{(12)3})$, and, similarly, $a''_* a''^{-1}_*(\mu_{1(23)}) = \mu_{(12)3}$. Therefore, Equation (15) says that $a_*(\mu_{(12)3}) \sim \mu_{1(23)}$. Moreover, due to the Chain rule, for each $u \in G_4^{(0)}$ the Radon–Nikodym derivative

$$\mathrm{d}a_*\mu_{(12)3_u}/\mathrm{d}\mu_{1(23)_u} = [B'/B''] \circ a''^{-1}.$$

∎

3.22. Remark. Since B', B'' and a'' are G_4-invariant, so are B'/B'', $[B'/B'']$ and $[B'/B''] \circ a''^{-1}$. The 0-cochains B and B' on $T^{(0)}$ induces well-defined function $[B'/B'']$ on X_{123}.

3.23. The C*-bifunctor. Recall the bicategory $\mathfrak{Corr}(2)$ of C*-correspondences Buss, Meyer and Zhu introduce in [Buss-Meyer-Zhu, 2013, Section 2.2]. In current article, we denote the bicategory $\mathfrak{Corr}(2)$ by $\mathfrak{C}$. The objects of this bicategory are C*-algebras, 1-arrows the C*-correspondence and 2-arrows the unitary isomorphisms of the C*-correspondences which intertwine the left action (Definition 2.19). The horizontal composition is given by the interior tensor product and the vertical one is the usual composition of linear functions.

In the present section, we show that a topological correspondence going to a C*-one is a bifunctor $\mathfrak{T} \longrightarrow \mathfrak{C}$. The first main task is to prove Proposition 3.25 which says that the C*-functor maps an isomorphism of topological correspondences to an isomorphism of the corresponding C*-correspondences. Recall from Definition 2.19, that an isomorphism of C*-correspondences is a unitary isomorphism of corresponding Hilbert C*-modules which also intertwines the representation of the left C*-algebra.

3.24. LEMMA. *Let (X, λ) and (Y, τ) be topological correspondences, with Δ_X and Δ_Y as the adjoining functions, respectively, from (G, α) to (H, β). Let $\mathrm{t}\colon X \to Y$ be an isomorphism of topological correspondences. Let $G \ltimes X$ denote the transformation groupoid for the left action of G on X. Let $M\colon Y \longrightarrow \mathbb{R}^+$ denote the continuous Radon–Nikodym derivative $M(y) = \frac{\mathrm{d}\tau_{s_Y(y)}}{\mathrm{d}\mathrm{t}_*(\lambda_{s_Y(y)})}(y)$.*

i) M is H-invariant.

ii) $\Delta_X = \left(\frac{M \circ \mathrm{t} \circ r_{G\ltimes X}}{M \circ \mathrm{t} \circ s_{G\ltimes X}}\right) \cdot \Delta_Y \circ (\mathrm{Id} \times \mathrm{t})$.

PROOF. (i): Fix $u \in H^{(0)}$. By definition τ_u is an invariant measures on the space of units of the transformation groupoid $X \rtimes H$ when the transformation groupoid is equipped with the Haar system induced by β; similar claim holds for λ_u. Since t is a G-H-equivariant homeomorphism, $\mathrm{t}_*(\lambda_u)$ is also a $Y \rtimes H$-invariant measure on Y. Also, since t is an isomorphism of topological correspondences $\mathrm{t}_*(\lambda_u) \sim \tau_u$ for all $u \in H^{(0)}$. Now Lemma 3.7(ii) gives us the desired result.

(ii): The isomorphism t induces the H-equivariant isomorphism $\mathrm{Id} \times \mathrm{t}\colon G \ltimes X \longrightarrow G \ltimes Y$ of topological groupoids. Let $f \in \mathrm{C_c}(G \ltimes X)$ and $u \in H^{(0)}$. Then we may write

$$\begin{aligned} &\int_X \int_G f(\gamma^{-1}, x)\, \mathrm{d}\alpha^{r_X(x)}(\gamma)\, \mathrm{d}\lambda_u(x) \\ = &\int_X \int_G f(\gamma^{-1}, t^{-1}(y))\, \mathrm{d}\alpha^{r_Y(y)}(\gamma)\, \mathrm{d}\lambda_u(\mathrm{t}^{-1}(y)) \\ = &\int_X \int_G f \circ (\mathrm{Id} \times t)^{-1}(\gamma^{-1}, y)\, \mathrm{d}\alpha^{r_Y(y)}(\gamma)\, \mathrm{d}\mathrm{t}_*(\lambda_u)(y) \end{aligned}$$

as $r_Y(y) = r_X(x)$. Change the measures $\mathrm{t}_*(\lambda_u)$ to τ_u so that the above term becomes

$$\int_X \int_G f \circ (\mathrm{Id} \times t)^{-1}(\gamma^{-1}, y)\, \frac{\mathrm{d}\mathrm{t}_*(\lambda_u)}{\mathrm{d}\tau_u}(y)\, \mathrm{d}\alpha^{r_Y(y)}(\gamma)\, \mathrm{d}\tau_u(y).$$

Using the (G, α)-quasi-invariance of τ, we see that above term equals

$$\int_X \int_G f \circ (\mathrm{Id} \times t)^{-1}(\gamma, \gamma^{-1}y)\, \Delta_Y(\gamma, \gamma^{-1}y)\, \frac{\mathrm{d}\mathrm{t}_*(\lambda_u)}{\mathrm{d}\tau_u}(y)\, \mathrm{d}\alpha^{r_Y(y)}(\gamma)\, \mathrm{d}\tau_u(y).$$

Now change the measures τ_u to $\mathrm{t}_*(\lambda_u)$ to see that above term is

$$\int_X \int_G f \circ (\mathrm{Id} \times t)^{-1}(\gamma, \gamma^{-1}y)\, \Delta_Y(\gamma, \gamma^{-1}y)\, \frac{\mathrm{d}\mathrm{t}_*(\lambda_u)}{\mathrm{d}\tau_u}(y) \frac{\mathrm{d}\tau_u}{\mathrm{d}\mathrm{t}_*(\lambda_u)}(\gamma^{-1}y)\, \mathrm{d}\alpha^{r_Y(y)}(\gamma)\, \mathrm{d}\mathrm{t}_*(\lambda_u)(y).$$

Now note that, since $\mathrm{t}_*(\lambda_u) \sim \tau_u$, $\mathrm{d}\tau_u/\mathrm{d}\mathrm{t}_*(\lambda_u) = (\mathrm{d}\mathrm{t}_*(\lambda_u)/\mathrm{d}\tau_u)^{-1}$. Therefore, the above term equals

$$\int_X \int_G f(\gamma, \gamma^{-1}x)\, \Delta_Y(\gamma, \gamma^{-1}\mathrm{t}(x))\, \frac{\mathrm{d}\mathrm{t}_*(\lambda_u)}{\mathrm{d}\tau_u}(y) \left(\frac{\mathrm{d}\mathrm{t}_*(\lambda_u)}{\mathrm{d}\tau_u}(\gamma^{-1}y)\right)^{-1} \mathrm{d}\alpha^{r_X(x)}(\gamma)\, \lambda_u(x). \quad (16)$$

Comparing Equations (16) and (16) with the definition of the adjoining function Δ_X, we infer that

$$\begin{aligned}\Delta_X(\gamma^{-1}, x) &= \Delta_Y(\gamma^{-1}, \mathrm{t}(x))\, \frac{\mathrm{d}\mathrm{t}_*(\lambda_u)}{\mathrm{d}\tau_u}(\gamma^{-1}y) \Big/ \frac{\mathrm{d}\mathrm{t}_*(\lambda_u)}{\mathrm{d}\tau_u}(y) \\ &= \left(\frac{M \circ \mathrm{t} \circ r_{G\ltimes X}}{M \circ \mathrm{t} \circ s_{G\ltimes X}}(\gamma^{-1}, x)\right) \Delta_Y \circ (\mathrm{Id} \times \mathrm{t})(\gamma^{-1}, x)\end{aligned}$$

$\lambda_u \circ \alpha$-almost everywhere on $G \times_{G^{(0)}} X_u$. But $\Delta_X, \Delta_Y, \mathrm{t}$ and M_u are continuous functions, therefore, the equality holds everywhere on $G \times_{G^{(0)}} X$. ∎

3.25. Proposition. *With the same data and hypothesis as Lemma 3.24, define the mapping of complex vector spaces*

$$\mathrm{T}\colon \mathrm{C_c}(X) \to \mathrm{C_c}(Y), \quad \mathrm{T}(f) = f \circ \mathrm{t}^{-1} \cdot M^{1/2}$$

where $f \in \mathrm{C_c}(X)$. Then T extends to an isomorphism $\mathrm{T}\colon \mathcal{H}(X,\lambda) \longrightarrow \mathcal{H}(Y,\tau)$ of C^*-correspondences.

Proof. The mapping T is clearly $\mathbb{C}$-linear. We first prove that T extends to a unitary operator of Hilbert $\mathrm{C}^*(H,\beta)$-modules $\mathcal{H}(X,\lambda) \longrightarrow \mathcal{H}(Y,\tau)$. Let $\psi \in \mathrm{C_c}(H)$ and $f, g \in \mathrm{C_c}(X)$. Then

$$\begin{aligned}\mathrm{T}(f\psi)(y) = (f\psi)(\mathrm{t}^{-1}(y))\, M^{1/2}(y) &= \int_H f \circ \mathrm{t}^{-1}(y\eta)\psi(\eta^{-1})\, M^{1/2}(y\eta)\, \mathrm{d}\beta^{s_X(x)}(\eta) \\ &= \int_H \mathrm{T}(f)(y\eta)\psi(\eta^{-1})\, \beta^{s_Y(y)}(\eta) = T(f)\psi(\eta).\end{aligned}$$

To get the second equality above, we use the H-equivariance of t^{-1} and the H-invariant of M (cf. 3.24(i)). Thus T is a linear map of pre-Hilbert modules over the pre-C*-algebra $\mathrm{C_c}(H)$.

Now we show that T is also adjointable. To figure out the adjoint of T, first apply chain rule to the composites of the functions $(X, \lambda_u) \xrightarrow{\mathrm{t}} (Y, \tau_u) \xrightarrow{\mathrm{t}^{-1}} (X, \lambda_u)$ of measures spaces, where $u \in H^{(0)}$; then a small computation shows that

$$\frac{\mathrm{d}\mathrm{t}_*^{-1}(\tau_u)}{\mathrm{d}\lambda_u} = \frac{\mathrm{d}\tau_u}{\mathrm{d}\mathrm{t}_*(\lambda_u)} \circ \mathrm{t} = \frac{1}{M} \circ \mathrm{t}. \quad (17)$$

This observation motivates us to define the adjoint of T

$$\mathrm{T}^*\colon \mathrm{C_c}(Y) \to \mathrm{C_c}(X) \quad \text{as} \quad \mathrm{T}^*(g) = g \circ \mathrm{t} \cdot \sqrt{\frac{\mathrm{d}\tau_u}{\mathrm{d}\mathrm{t}_*(\lambda_u)}} \circ \mathrm{t} = g \circ \mathrm{t} \cdot \frac{1}{\sqrt{M}} \circ \mathrm{t}$$

for $g \in \mathrm{C_c}(Y)$. To verify this guess, let $f \in \mathrm{C_c}(X), g \in \mathrm{C_c}(Y)$ and $\eta \in H$. Then Equation (2) gives

$$\langle \mathrm{T}f\,,\, g\rangle\,(\eta) = \int_X \overline{f \circ t^{-1}(y)} M^{1/2}(y) g(y\eta)\,\mathrm{d}\tau_{r_H(\eta)}(y).$$

Now change the measure $\tau_{r_H(\eta)}$ to the equivalent measure $\mathrm{t}_*(\lambda_{r_H(\eta)})$ in the last term, so that it becomes

$$\int_X \overline{f \circ t^{-1}(y)} \sqrt{\frac{\mathrm{d}\mathrm{t}_*(\lambda_{r_H(\eta)})}{\mathrm{d}\tau_{r_H(\eta)}}(y)} \frac{\mathrm{d}\tau_{r_H(\eta)}}{\mathrm{d}\mathrm{t}_*(\lambda_{r_H(\eta)})}(y) g(y\eta)\,\mathrm{d}\mathrm{t}_*(\lambda_{r_H(\eta)})(y)$$
$$= \int_X \overline{f \circ t^{-1}(y)} \sqrt{\frac{\mathrm{d}\tau_{r_H(\eta)}}{\mathrm{d}\mathrm{t}_*(\lambda_{r_H(\eta)})}}(y) g(y\eta)\,\mathrm{d}\mathrm{t}_*(\lambda_{r_H(\eta)})(y).$$

Now change the measure $\mathrm{t}_*(\lambda_{r_H(\eta)})$ to $\mathrm{t}_*^{-1}(\mathrm{t}_*(\lambda_{r_H(\eta)})) = \lambda_{r_H(\eta)}$. Then $y \mapsto \mathrm{t}(x)$, and the last term above becomes

$$\int_X \overline{f(x)} \sqrt{\frac{\mathrm{d}\tau_{r_H(\eta)}}{\mathrm{d}\mathrm{t}_*(\lambda_{r_H(\eta)})}} \circ \mathrm{t}(x) g \circ \mathrm{t}(x\eta)\,\mathrm{d}\lambda_{r_H(\eta)}(x)$$
$$= \int_X \overline{f(x)}\,\mathrm{T}^*(g)(x\eta)\,\mathrm{d}\lambda_{r_H(\eta)}(x) = \langle f\,,\, T^*(g)\rangle\,(\eta);$$

here we assumed that $t(x) = y$. Moreover, a straightforward computation shows that $\mathrm{Id}_{\mathrm{C_c}(X)} = \mathrm{T}^* \circ \mathrm{T}$ and $\mathrm{Id}_{\mathrm{C_c}(Y)} = \mathrm{T} \circ \mathrm{T}^*$. This proves that $\mathrm{T}\colon \mathrm{C_c}(X) \longrightarrow \mathrm{C_c}(Y)$ is a unitary isomorphism of pre-Hilbert modules over the pre-C*-algebra $\mathrm{C_c}(H, \beta)$. Therefore T extends to a unitary isomorphism the of corresponding Hilbert $\mathrm{C}^*(H, \beta)$-modules; we denote the extension by T itself. Moreover, the adjoint of T is the extension of T^* which is also denoted by T^*.

In the rest of the part of the proof, we show that T intertwines the representations of $\mathrm{C}^*(G, \alpha)$ on $\mathcal{H}(X, \lambda)$ and $\mathcal{H}(Y, \tau)$ which will prove that T is an isomorphism of C*-correspondences.

Let $\pi_1\colon \mathrm{C}^*(G, \alpha) \longrightarrow \mathbb{B}(\mathcal{H}(X, \lambda))$ and $\pi_2\colon \mathrm{C}^*(G, \alpha) \longrightarrow \mathbb{B}(\mathcal{H}(Y, \tau))$ denote the representations in the C*-correspondences $\mathcal{H}(X, \lambda)$ and $\mathcal{H}(Y, \tau)$, respectively; Equation (1)(ii) gives the formulae of these representations for the function spaces. Due to density of $\mathrm{C_c}(X)$ and $\mathrm{C_c}(Y)$ in $\mathcal{H}(X, \lambda)$ and $\mathcal{H}(Y, \tau)$, respectively, it suffices to show that $\mathrm{T} \circ \pi_1(\psi)(f) = \pi_2(\psi) \circ \mathrm{T}(f)$ for $\psi \in \mathrm{C_c}(G)$ and $f \in \mathrm{C_c}(X)$. The following computation proves this required claim:

$$\begin{aligned}\pi_2(\psi)(\mathrm{T}(f))(y) &= \int_G \psi(\gamma)\mathrm{T}(f)(\gamma^{-1}y)\,\Delta_Y^{1/2}(\gamma,\gamma^{-1}y)\,\mathrm{d}\alpha^{r_Y(y)}(\gamma)\\ &= \int_G \psi(\gamma) f\circ \mathrm{t}^{-1}(\gamma^{-1}y)\,M^{-1/2}(\gamma^{-1}y)\,\Delta_Y(\gamma,\gamma^{-1}y)^{1/2}\,\mathrm{d}\alpha^{r_Y(y)}(\gamma)\end{aligned}$$

where $y \in Y$. The above term can also be written as

$$\int_G \psi(\gamma) f\circ \mathrm{t}^{-1}(\gamma^{-1}y)\frac{M^{1/2}(y)}{M^{1/2}(\gamma^{-1}y)}\,\Delta_Y(\gamma,\gamma^{-1}y)^{1/2}M^{-1/2}(y)\,\mathrm{d}\alpha^{r_Y(y)}(\gamma).$$

Now change the variable $y \mapsto t(x)$, and use the G-equivariance of t^{-1} so the above term can be written as

$$\int_G \psi(\gamma) f(\gamma^{-1}x)\sqrt{\left(\frac{M\circ \mathrm{t}(x)}{M\circ \mathrm{t}(\gamma^{-1}x)}\,\Delta_Y\circ(\mathrm{Id}_G\times \mathrm{t})(\gamma,\gamma^{-1}x)\right)}M^{-1/2}(\mathrm{t}(x))\,\mathrm{d}\alpha^{r_X(x)}(\gamma).$$

Using Lemma 3.24(ii), we can write the above term as

$$\begin{aligned}&\int_G \psi(\gamma) f(\gamma^{-1}x)\sqrt{\Delta_X(\gamma,\gamma^{-1}x)}M^{-1/2}(\mathrm{t}(x))\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\\ &= M^{-1/2}(\mathrm{t}(x))\int_G \psi(\gamma) f(\gamma^{-1}x)\Delta_X^{1/2}(\gamma,\gamma^{-1}x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\\ &= M^{-1/2}(\mathrm{t}(x))\cdot \pi_2(\psi)(f)(x) = M^{-1/2}(y)\cdot(\pi_1(\psi)f)\circ \mathrm{t}^{-1}(y) = T\circ\pi_1(\psi)(f)(y)\end{aligned}$$

where $y = t(x)$. ∎

Note that in Proposition 3.25, the identity map on X induces the identity isomorphism on $\mathcal{H}(X,\lambda)$.

3.26. Corollary. *Along with the same data and hypothesis as Lemma 3.24, assume that (Z,κ) is another topological correspondence from $(G,\alpha)\longrightarrow(H,\beta)$, and $\mathrm{l}\colon Y\longrightarrow Z$ is an isomorphism of correspondences. If $\mathrm{L}\colon \mathcal{H}(Y,\tau)\longrightarrow\mathcal{H}(Z,\kappa)$ is the isomorphism of C^*-correspondences that l induces, and T the one induced by t, then $\mathrm{T}\circ\mathrm{L}\colon \mathcal{H}(X,\lambda)\longrightarrow\mathcal{H}(Z,\kappa)$ is the isomorphism of C^*-correspondences that the composite $l\circ t$ induces.*

Proof. Follows from the definitions of T and L and the chain rule for measures. ∎

3.27. Corollary. *Let $(X,\alpha)\colon(G_1,\lambda_1)\to(G_2,\lambda_2)$ and $(Y,\beta)\colon(G_2,\lambda_2)\to(G_3,\lambda_3)$ be correspondences and let $(\Omega,\mu),(\Omega,\mu')\colon(G_1,\lambda_1)\to(G_3,\lambda_3)$ be two composites of them. Then $\mathcal{H}(\Omega,\mu)$ and $\mathcal{H}(\Omega,\mu')$ are isomorphic C^*-correspondences.*

Proof. Follows directly from Proposition 3.10 and Proposition 3.25. ∎

Compare Corollary 3.27 with Theorem 2.18.

Denote the bicategory of topological correspondences by $\mathfrak{T}$ and the one of C*-correspondences by $\mathfrak{C}$.

3.28. Theorem. *Let $(G,\alpha),(H,\beta)$ objects in $\mathfrak{T}$ and $(X,\lambda)\colon(G,\alpha)\longrightarrow(H,\beta)$ a 1-arrow. Then the assignments $(G,\alpha)\mapsto C^*(G,\alpha)$ and $(X,\lambda)\mapsto\mathcal{H}(X,\lambda)$ define a bifunctor from $\mathfrak{T}$ to $\mathfrak{C}$.*

Proof. Recall the definition of a bifunctor (Definition 2.27). We define the bifunctor $\mathfrak{F}=(F,\phi)\colon\mathfrak{T}\longrightarrow\mathfrak{C}$ by the following assignments of objects, 1-arrows, 2-arrows, identity morphisms and natural transformations in $\mathfrak{T}$ to that of $\mathfrak{C}$. The note in parentheses at the end of each item indicates what data in the definition of bifunctor it referes to.

Object $F((G,\alpha))=\mathrm{C}^*(G,\alpha)$ (Data (i) in Definition 2.27)

1-arrow map a 1-arrow $(X,\lambda)\colon(G,\alpha)\longrightarrow(H,\beta)$ to the 1-arrow $F((X,\lambda))=\mathcal{H}(X,\lambda)$ in $\mathfrak{C}(\mathrm{C}^*(G,\alpha),\mathrm{C}^*(H,\beta))$ (Data (ii) in Definition 2.27).

2-arrow map a 2-arrow t in $\mathfrak{T}((G,\alpha),(H,\beta))$ to the isomorphism of C*-correspondences $F(\mathrm{t}):=\mathrm{T}$ in $\mathfrak{C}(\mathrm{C}^*(G,\alpha),\mathrm{C}^*(H,\beta)$ as in Proposition 3.25. Note that F is a functor from $\mathfrak{T}((G,\alpha),(H,\beta))$ to $\mathfrak{C}(\mathrm{C}^*(G,\alpha),\mathrm{C}^*(H,\beta))$; this follows from Corollary 3.26 and the remark above it (Data (ii) in Definition 2.27).

Identity 2-morphism In $\mathfrak{T}$ the identity 1-arrow at (G,α) is the identity correspondence (G,α^{-1}), and the one at $\mathrm{C}^*(G,\alpha)$ in $\mathfrak{C}$ is the identity correspondence $\mathrm{C}^*(G,\alpha)$. The identity 2-arrow $\mathrm{C}^*(G,\alpha)\longrightarrow\mathcal{H}(G,\alpha^{-1})$ is the identity isomorphism $\mathrm{Id}_{\mathrm{C}^*(G,\alpha)}$ of C*-correspondences as discussed in Example 2.11 (Data (iii) in Definition 2.27).

Natural transformation between composites Let $(X,\lambda)\colon(G,\alpha)\longrightarrow(H,\beta)$ and $(Y,\mu)\colon(H,\beta)\longrightarrow(K,\nu)$ be 1-arrows. Then the natural transformation $\phi((G,\alpha),(H,\beta),(K,\nu))$ between the composites in $\mathfrak{C}$ and $\mathfrak{T}$ is the (unitary) isomorphism of C*-correspondences $\mathcal{H}(X,\lambda)\otimes_{\mathrm{C}^*(H,\beta)}\mathcal{H}(Y,\mu)\longrightarrow\mathcal{H}(X\circ Y,\lambda\circ\mu)$ defined in Theorem 2.18.

We briefly recall definition of $\phi((G,\alpha),(H,\beta),(K,\nu))$; let's write simply ϕ for time being. For $f\in\mathrm{C_c}(X)$ and $g\in\mathrm{C_c}(Y)$, $\phi(f\otimes g)\in\mathrm{C_c}((X\times_{H^{(0)}}Y)/H)$ is the function

$$\phi(f\otimes g)([x,y]) \\ =\int_H f(x\eta)g(\eta^{-1}y)b^{-1/2}(x\eta,\eta^{-1}y)\,\mathrm{d}\beta^{r_Y(y)}(\eta):=[(f\otimes g)b^{1/2}]([x,y]) \quad (18)$$

where $[x,y]$ is the equivalence class of $(x,y)\in X\times_{H^{(0)}}Y$ in $(X\times_{H^{(0)}}Y)/H$, and b is the 0-cocyle used to construct the family of measures on $(X\times_{H^{(0)}}Y)/H$. For more details, reader may refer to proof of Theorem 3.14 in [Holkar-2, 2017], particularly, page 110 there. For understanding the rest of the discussion in this article, the definition of ϕ in Equation (18) is sufficient.

Now we prove that the pair $(F, \phi) = \mathfrak{F}$ is a morphism from the bicategory $\mathfrak{T}$ to the bicategory $\mathfrak{C}$. For this, we need to check that associativity coherence for transformations (Figures 2) and the coherence of (left and right) identities (Figure 3) hold. For this purpose, one can check the commutativity on the elementary tensors in the dense pre-Hilbert C*-modules consisting of $\mathrm{C_c}$ functions. Then the result can be extended to Hilbert C*-modules by using the linearity of maps and standard density arguments.

Checking the commutativity of Figure 2 is not so hard; it follows from a direct computation that, basically, uses the associativity of multiplication in the *usual* function spaces. Let $i \in \{1, 2, 3\}$, and let $(X_i, \lambda_i): (G_i, \alpha_i) \longrightarrow (G_{i+1}, \alpha_{i+1})$ be three 1-arrows in $\mathfrak{T}$. Let $f_i \in \mathrm{C_c}(X_i, \lambda_i)$ where $i = 1, 2, 3$. We draw Figure 9 which shows how the function $(f_1 \otimes f_2) \otimes f_3$ travels across Figure 2. From Figure 9, one can see that the desired results clearly holds. In this figure, we write $F(X_i)$ instead of $F((X_i, \lambda_i))$ for simplicity of writing where $i = 1, 2, 3$.

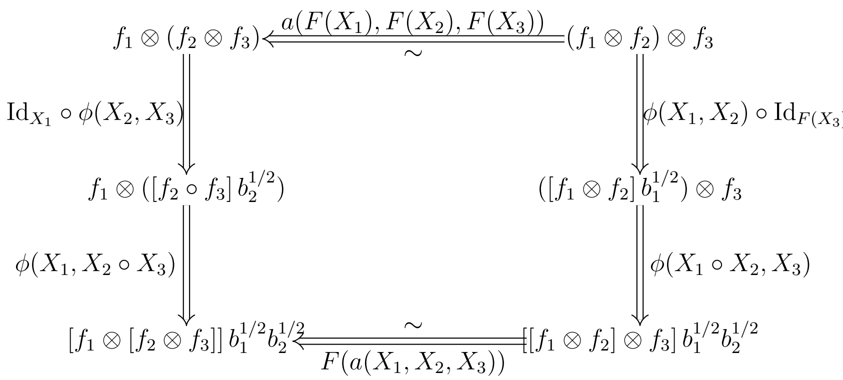


Figure 9

The identity isomorphisms: Let (X, λ) be a correspondence from (G, α) to (H, β) with Δ as the adjoining function. First we check the left identity coherence. Let $(G \circ X, \mu)$ denote a composite of the identity correspondence (G, α^{-1}) at (G, α) and (X, λ). Let b be a 0-cochain on the transformation groupoid $Q = (G \ltimes_{G^{(0)}} X) \rtimes G$ that is used to create μ. Let Δ_1 denote the 1-cocycle on Q such that $d^0(b) = \Delta_1$ (cf. Equation (8)). For the left identity coherence we need to check that Figure 10 commutes. In this figure,

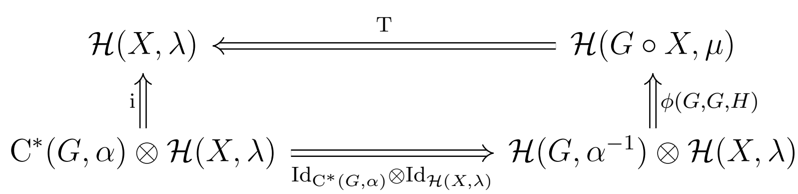


Figure 10

the map $\mathrm{Id}_{\mathrm{C}^*(G,\alpha)}$ in the bottom horizontal arrow the identity isomorphism discussed in

Example 3.9; this isomorphism of Hilbert $\mathrm{C}^*(G,\alpha)$-modules is induced by the identity map on $\mathrm{C_c}(G)$. Thus $\mathrm{Id}_{\mathrm{C}^*(G,\alpha)}\otimes\mathrm{Id}_{\mathcal{H}(X,\lambda)}$ is the identity map. The right vertical map $\phi(G,G,H)$ is the assignment ϕ of 2-arrows; we write simply ϕ instead of $\phi(G,G,H)$ in what follows. The top horizontal map T is given in Proposition 3.25. And i is the isomorphism of Hilbert modules which is defined on the elementary tensors $a\otimes b\in \mathrm{C}^*(G,\alpha)\otimes\mathcal{H}(X,\lambda)$ by $a\otimes b\mapsto ab$. Let $f\in \mathrm{C_c}(G)\subseteq \mathrm{C}^*(G,\alpha)$ and $g\in\mathrm{C_c}(X)\subseteq\mathcal{H}(X,\lambda)$. Then for the elementary tensor $f\otimes g\in \mathrm{C}^*(G,\alpha)\otimes\mathcal{H}(X,\lambda)$,

$$\phi\left(\mathrm{Id}_{\mathrm{C}^*(G,\alpha)}\otimes\mathrm{Id}_{\mathcal{H}(X,\lambda)}(f\otimes g)\right)=[(f\otimes g)b^{1/2}]$$

as in Equation (18).

Regarding T, firstly note that $G\circ X=(G\times_{G^{(0)}}X)/G$ and X are homeomorphic. Moreover, Example 3.9 shows that this homeomorphism establishes an equivalence between the families of measures μ and λ; Equation 9 shows that the Radon–Nikodym derivative $\mathrm{d}\mu/\mathrm{d}\lambda(\gamma^{-1},x)=b(r_G(\gamma^{-1}),\gamma^{-1}x)$ for $(\gamma^{-1},x)\in G\ltimes X$. Here we are abusing notation by identifying $G\circ X$ with X and removing T in the Radon–Nikodym derivative. With this in mind, we write

$$\begin{aligned}\mathrm{T}([(f\otimes g)b^{1/2}])(x)=&[(f\otimes g)b^{1/2}][r_X(x),x]\cdot\frac{\mathrm{d}\mu_u}{\mathrm{d}\lambda_u}([r_X(x),x])^{1/2}\\ =&b^{1/2}(r_X(x),x)\int_G f(\gamma)g(\gamma^{-1}x)b^{-1/2}(\gamma,\gamma^{-1}x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\\ =&\int_G f(\gamma)g(\gamma^{-1}x)\sqrt{\frac{b(r_X(x),x)}{b(\gamma,\gamma^{-1}x)}}\,\mathrm{d}\alpha^{r_X(x)}(\gamma).\end{aligned}$$

Note that $r_X(x)=r_G(\gamma)$, and that for $(\gamma,\gamma^{-1}x,\gamma^{-1})\in Q$,

$$s_Q(\gamma,\gamma^{-1}x,\gamma^{-1})=(r_G(\gamma),\gamma^{-1}x)=(r_X(x),\gamma^{-1}x)$$

and $r_Q(\gamma,\gamma^{-1}x,\gamma^{-1})=(\gamma,\gamma^{-1}x)$. With this we re-write the last term in above computation as follows and compute further:

$$\begin{aligned}&\int_G f(\gamma)g(\gamma^{-1}x)\sqrt{\frac{b\circ s_Q(\gamma,\gamma^{-1}x,\gamma^{-1})}{b\circ r_Q(\gamma,\gamma^{-1}x,\gamma^{-1})}}\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\\ =&\int_G f(\gamma)g(\gamma^{-1}x)\sqrt{\Delta_1(\gamma,\gamma^{-1}x,\gamma^{-1})}\,\mathrm{d}\alpha^{r_X(x)}(\gamma).\end{aligned}$$

But $\Delta_1(\gamma,\gamma^{-1}x,\gamma^{-1})=\Delta(\gamma,\gamma^{-1}x)$, cf. Equation (8). Therefore, the above term equals

$$\int_G f(\gamma)g(\gamma^{-1}x)\Delta^{1/2}(\gamma,\gamma^{-1}x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)=f\cdot g(x)=\mathrm{i}(f\otimes g)(x)$$

where $f\cdot g$ is the action of $f\in\mathrm{C_c}(G)$ on $g\in\mathrm{C_c}(X)$. This shows that Figure 10 commutes.

With the help of Example 3.9, one may verify the coherence for the right identity in a similar fashion as above. ∎

Finally, we give three illustrations describing the C*-(bi)functor: the first one involving spaces, second one involving topological quivers and the last one involving topological groups.

3.29. Illustration. [The C*-functor on the category of spaces] *Let $\mathfrak{S}$ denote the category of locally compact spaces with continuous functions a morphisms. Examples 2.7 and 2.21 show that $\mathfrak{S}$ is a subcategory of $\mathfrak{T}$. Theorem 3.28 reduces to the well-known functoriality of the C^*-functor in Gelfand's characterisation of abelian C^*-algebras.*

3.30. Illustration. [The C*-functor for groups] *Example 2.8 and 2.22 show that the category of locally compact groups is a subcategory of $\mathfrak{T}$. The C^*-functor assigns a group G its C^*-algebra, a group homomorphism $\phi\colon G\longrightarrow H$ the C^*-correspondence $C^*(H,\beta^{-1})\colon C^*(G)\longrightarrow C^*(H)$.*

3.31. Illustration. [The (bi)category of quivers] *Muhly and Tomford [Muhly-Tomforde, 2005, Definition 3.17] define Topological quivers. Example 2.12 shows that a topological quiver is a topological correspondence. The composite of two topological quivers is again a topological quiver as remarked in Example 2.21 (we leave this easy proof for the readers). Thus topological quivers form a sub-bicategory of $\mathfrak{T}$; the 2-arrows here are homeomorphisms commuting with the backward and forward maps. Then Theorem 3.28 explains the C^*-functor for quivers. This functoriality can be checked more easily for quivers.*

3.32. Invertible 1-arrows. It has been observed that the groupoids equivalences (due to Muhly, Renault and Williams [Muhly-Renault-Williams, 1987]) play the role of *invertible arrows* in various categories of groupoids, see [Mrcun, 1999], [Moerdijk-Mrcun2005], and [Hilsum-Skandalis, 1987]. Following these works, it is natural to expect that the groupoid equivalences characterise the invertible 1-arrows in $\mathfrak{T}$. We could not prove this result for $\mathfrak{T}$, but we could prove it for the sub-bicategory of it consisting of the Macho-Stalder–O'uchi correspondences, Theorem 3.33. The last remark of this section briefly discusses the issue(s) we faced and possible ideas for proving this result for $\mathfrak{T}$ itself. Before proving Theorem 3.33, we start describing the sub-bicategory of Macho-Stadler–O'uchi correspondences.

Recall Example 2.9 of the correspondences of Macho-Stadler and O'uchi which are the topological versions of the Hilsum–Skandalis morphisms. In [Tu, 2004], Jean-Louis Tu defines a version of these morphisms for non-Hausdorff groupoids. The correspondences of Tu, and Macho-Stadler and O'uchi differ in the fact that Tu needs the left action to be free.

Tu observes [Tu, 2004, Proposition 7.5] that his topological correspondences constitute a category. However, Macho-Stadler and O'uchi do not mention similar result in [Stadler-Ouchi, 1999]. Nonetheless, essentially, Tu's proof can be carried over to show that the Macho-Stadler–O'uchi correspondences also form a category. Following is the proof of this fact.

Since the identity correspondences are basically Macho-Stadler–O'uchi correspondences (see Example 2.11) the main claim that needs a proof here is that a composite of Macho-

Stadler–O'uchi correspondences is a correspondence of the same type. This can be proved as follows: let $(G_1, \alpha_1) \xrightarrow{X} (G_2, \alpha_2) \xrightarrow{Y} (G_3, \alpha_3)$ be Macho-Stadler–O'uchi correspondences of locally compact Hausdorff second countable groupoids with Haar systems. Assume that $X \circ Y\colon (G_1, \alpha_1) \longrightarrow (G_3, \alpha_3)$ the composite of these correspondences. Then $X \circ Y := (X \times_{s_X, G_2^{(0)}, r_Y} Y)/G_2$ clearly satisfies the first two conditions of Macho-Stadle–O'uchi correspondences in 2.9. The not-so-obvious fact is that it also satisfies the last condition, namely, the momentum map for the G_3-action induces a homeomorphism $G_1 \backslash (X \circ Y) \longrightarrow G_3^{(0)}$. However, this follows easily from Lemma 2.31 in [Tu, 2004].

By adding equivariant homeomorphisms of bispaces, we enrich the category of Macho-Stadler–O'uchi correspondences to a sub-bicategory of $\mathfrak{T}$. The next result characterises the (weak) isomorphisms in the bicategory of Macho-Stadler–O'uchi correspondences as Morita equivalences of groupoids.

We call two objects a, b in a bicategory $\mathfrak{B}$ isomorphic if there are 1-arrows $a \xrightarrow{f} b \xrightarrow{g} a$ such that $g \circ f$ is isomorphic to the identity arrow at a, and $f \circ g$ is isomorphic to the identity arrow at b. Thus an isomorphism $a \longrightarrow b$ of objects in $\mathfrak{B}$ is a quintuple (f, g, ϕ, γ) such that $a \underset{g}{\overset{f}{\leftrightarrows}} b$ are 1-arrows and $\phi\colon g \circ f \longrightarrow 1_a$ and $\gamma\colon f \circ g \longrightarrow 1_b$ are isomorphism 2-arrows; in this case, g is called an inverse of f. Weak isomorphism is a more appropriate terminology here. However, we drop the adjective *weak* since this is the only notion of isomorphism in bicategories that we shall use in this article.

3.33. THEOREM. *The Macho-Stadler–O'uchi correspondences constitute a sub-bicategory $\mathfrak{SO}$ of $\mathfrak{T}$. In $\mathfrak{SO}$, two groupoids with Haar systems (G, α) and (H, β) are isomorphic if and only if they are equivalent in the sense of Muhly, Renault and Williams [Muhly-Renault-Williams, 1987].*

Last theorem is a consequence of following proposition.

3.34. LEMMA. *Let $(G, \alpha) \xrightarrow{X} (H, \beta)$ be an isomorphism in $\mathfrak{SO}$ whose inverse is $(H, \beta) \xrightarrow{Y} (G, \alpha)$, $\phi_G\colon X \circ Y \longrightarrow G$ and $\phi_H\colon Y \circ X \longrightarrow H$ are the given 2-arrows in the isomorphism. Then the following hold.*

(i) The actions of G and H on X and Y are free.

(ii) Given $x \in X$, $\phi_G([x, y]) = \phi_G([x, z])$ if and only if $y = z$.

(iii) Given $x \in X$ there is unique $y' \in Y$ such that $\phi_G([x, y']) \in G^{(0)}$.

(iv) Given $y \in Y$ there is unique $x \in X$ such that $\phi_G([x, y]) \in G^{(0)}$.

(v) The assignment $\phi\colon X \longrightarrow Y, x \mapsto y$ such that $\phi_G([x, y]) \in G^{(0)}$ is a homeomorphism such that $\phi(\gamma x \eta^{-1}) = \eta \phi(x) \gamma^{-1}$ for all composable pairs $(\gamma, x) \in G \times X$ and $(x, \eta^{-1}) \in X \times H$.

We mention two quick observation before proving this lemma: for the first one, assume that G is a groupoid and $k\colon P \longrightarrow Q$ a map of G-spaces. If the action of G on Q is free, then the action of G on P is also free. The proof is simple: assume $\gamma \in G$ is a stabiliser of $p \in P$. Then γ is also a stabiliser of $k(p) \in Q$. Since the action of G on Q is free, $\gamma \in G^{(0)}$.

The second observation is that the right momentum map s_X of X is open surjection: openness is built in the definition of a Macho-Stadler–O'uchi correspondence, Example 2.9. For the surjectivity of s_X, note that s_X and $s_{Y\circ X}$ have same ranges. Next, as $\phi_H\colon Y\circ X \longrightarrow H$ is an H-equivariant homeomorphism, the range of $s_{Y\circ X}$ is same as that of s_H which all of $H^{(0)}$. Similarly, s_Y is also an open surjection.

Proof Proof of Lemma 3.34. (i): We are given that $X\circ Y$ is isomorphic to the identity correspondence on (G,α) via ϕ_G, and $Y\circ X$ is isomorphic to the identity correspondence on (H,β) via ϕ_H. Since ϕ_G is a G-G-equivariant homeomorphism, and the left and right multiplication actions of G on itself are free, so are the left and right actions of G on $X\circ Y$. Now, we know that the quotient map $X\times_{s_X,H^{(0)},r_X} Y \longrightarrow X\circ Y$ is G-G-equivariant. The observation just before this proof implies that the G-actions on the fibre product $X\times_{s_X,H^{(0)},r_X} Y$ are free. Which, in turn, imply that the left and right G-actions on X and Y, respectively, are free. Similarly, the H-actions on Y and X are free.

Before proving the next claims, note that the rest of the proof uses merely the freeness and properness of the groupoid actions; to be precise, we wish to draw reader's attention to the fact that the proof does not use Condition (iii) in Example 2.9.

(ii): Assume that $\phi_G([x,y]) = \phi_G([x,z]) \in G$. Since ϕ_G is injective, $[x,y]=[x,z]$ in $X\circ Y$. That is, $x = x\eta$ and $y=\eta^{-1}z$ for some $\eta\in H$. Since the right action of H on X is free, $\eta\in H^{(0)}$. Therefore, $y=z$.

(iii): Let $x\in X$. Let $y\in Y$ such that $(x,y)\in X\times_{s_X,H^{(0)},r_X} Y$. Write $\gamma=\phi_G([x,y])\in G$. Then $r_G(\gamma)=\phi_G([x,y])\gamma^{-1}=\phi_G([x,y\gamma^{-1}])\in G^{(0)}$ as ϕ_G is G-equivariant. Thus given $x\in X$, there is $y'\in Y$ such that $\phi_G([x,y'])\in G^{(0)}$. Next we prove that this y' is unique.

Assume that $\phi_G([x,y]),\phi_G([x,z])\in G^{(0)}$. Then, since ϕ_G is G-equivariant, $\phi_G([x,y])$ is the unit $r_X(x)=s_Y(y)$, and $\phi_G([x,z])$ is the unit $r_X(x)=s_X(z)$ in $G^{(0)}$. Thus $\phi_G([x,y])=\phi_G([x,z])\in G^{(0)}\subseteq G$. Therefore, by (ii) above, we have $y=z$.

The last observation allows us to define $\phi\colon X\longrightarrow Y$ as follows: for $x\in X$, $\phi(x)\in Y$ is the unique element such that $\phi_G([x,\phi(x)])\in G^{(0)}$. One may also give a *formula* for ϕ as:

$$\phi(x) = y\phi_G([x,y]^{-1}) \tag{19}$$

where $y\in Y^{s_X(x)}$. Note that $\phi(x)$ does not depend on the choice of y.

(iv): This claim shows that ϕ is a bijection. The sketch of the proof is as follows: first prove an analogue of (ii) above, namely, show that given $y\in Y$, $\phi_G([x,y])=\phi_G([z,y])$ iff $x=z$ using the freeness of the action of H on Y. Then a similar argument as in (iii) above proves this claim.

(v): We first prove that ϕ is continuous. Let

$$X \xleftarrow{\pi_1} X\times_{s_X,H^{(0)},r_Y} Y \xrightarrow{F} X\times_{s_X,H^{(0)},r_Y} Y\times_{s_Y,G^{(0)},r_G} G \xrightarrow{A} Y$$

be the maps: π_1 is the projection on X;

$$F(x,y) = (x, y, \phi_G([x,y]^{-1})) \qquad \text{for } (x,y) \in X \times_{s_X, H^{(0)}, r_Y} Y;$$
$$A(x,y,\gamma) = (x, y\gamma) \qquad \text{for } (x,y,\gamma) \in X \times_{s_X, H^{(0)}, r_Y} Y \times_{s_Y, G^{(0)}, r_G} G.$$

The facts to observe here are that π_1 is open since s_X and r_X are open surjections; F and A are continuous because ϕ_G and the action of G on Y are continuous. Then we note that for $U \subseteq Y$, $\phi^{-1}(U) = \pi_1(F^{-1}(A^{-1}(U)))$ where the latter set is open. Hence ϕ is continuous.

To show that ϕ is a homeomorphism we prove that ϕ^{-1} is continuous. Using (iv) of the present lemma, define $\phi^{-1}\colon Y \longrightarrow X$ as

$$\phi^{-1}(y) = \phi_G([x,y])^{-1}x$$

for $x \in X_{r_y(y)}$; reader may check that this is indeed the inverse of ϕ. Notice the symmetry of the hypotheses of the lemma. Then the similar symmetric arguments used to prove that ϕ is a well-defined continuous map also show that ϕ^{-1} is a well-defined continuous map.

At the end, we check the (*anti-*)equivariance of ϕ. Let $(\gamma, x, \eta) \in G \times_{s_G, G^{(0)}, r_X} X \times_{s_Y, G^{(0)}, r_G} H$. And let $\phi(x) = y\phi_G([x,y])^{-1}$. Then

$$\eta\phi(x) = \eta y \phi_G([x,y])^{-1} = (\eta y)\phi_G([x\eta^{-1}, \eta y])^{-1} = \phi(x\eta^{-1}).$$

And

$$\phi(x)\gamma^{-1} = y\phi_G([x,y])^{-1}\gamma^{-1} = y(\gamma\phi_G([x,y]))^{-1} = y(\phi_G([\gamma x, y]))^{-1} = \phi(\gamma x).$$

∎

The second remark after Lemma 3.34 and the fact that $\phi\colon X \longrightarrow Y$ is the anti-G-H-homeomorphism implies that r_X is an open surjection as $r_X = s_Y \circ \phi$. Similarly, r_Y is also an open surjection. Thus the momentum maps of all actions are open surjections.

Lemma 3.34(v) implies that the transformation groupoids $G \ltimes X$ and $Y \rtimes G$, and $X \rtimes H$ and $H \ltimes Y$ are isomorphic.

Proof Proof of Theorem 3.33. It is clear that $\mathfrak{GD}$ is a sub-bicategory of $\mathfrak{T}$. Let (G,α) and (H,β) be two isomorphic objects in $\mathfrak{GD}$, and let $(G,\alpha) \underset{Y}{\overset{X}{\rightleftarrows}} (H,\beta)$, and $\phi_G\colon X \circ Y \longrightarrow G$ and $\phi_H\colon Y \circ X \longrightarrow H$ be the isomorphism. Then we show that X is a G-H-equivalence.

Note that, just before this proof we have observed that all momentum maps of X and Y are open surjections. Then Lemma 3.34(i) implies that X is a free G- and H-space. Condition (i) in the definition of Macho-Stadler–O'uchi correspondences (see Example 2.9) makes sure that these actions are also proper.

Now we need to only check that $[s_X]\colon G\backslash X \longrightarrow H^{(0)}$ and $[r_X]\colon X/H \longrightarrow G^{(0)}$ are homeomorphisms, see [Muhly-Renault-Williams, 1987, Definition 2.1]. The first homeomorphism clear since $s_X\colon X \longrightarrow H^{(0)}$ is an open surjection and $[s_X]$ is a bijections by hypothesis, cf. Example 2.9(iii).

For the second homeomorphism, we note that because $[s_X]\colon H\backslash Y \longrightarrow G^{(0)}$ is a bijection, Lemma 3.34(v) implies that $[r_X]\colon X/H \longrightarrow G^{(0)}$ is a bijection. Finally, since the momentum map $r_X\colon X \longrightarrow G^{(0)}$ is an open surjection, $[r_X]$ is a homeomorphism. This verifies that X is a G-H-equivalence.

Though we don't need it here, a remark is that one may prove that Y is an H-G-equivalence in a similar fashion.

Conversely, let X be a G-H-equivalence. Then, it is well-known that X^{op} is an H-G-equivalence, see [Sims-Williams, 2012] for details. We show that X^{op} is an inverse of X in $\mathfrak{GD}$. To justify the claim, we need to find the 2-arrows that implement the isomorphisms $X \circ X^{\text{op}} \simeq G$ and $X^{\text{op}} \circ X \simeq H$ of correspondences. We show the first isomorphism and the other can be constructed similarly. We write x' for the representative of x in X^{op}.

The space X being a G-H-equivalence, we know that the mapping

$$m\colon G \times_{s_G, G^{(0)}, r_X} X \longrightarrow X \times_{s_X, H^{(0)}, r_{X^{\text{op}}}} X^{\text{op}}, \quad (\gamma, x) \mapsto (\gamma x, x')$$

is a homeomorphism[3]. Equip the domain and codomain of m with G-G-actions as follows: for $(\xi, x) \in G \times_{s_G, G^{(0)}, r_X} X$ and appropriate $\gamma, \zeta \in G$, define $\gamma(\xi, x)\zeta = (\gamma\xi\zeta^{-1}, \zeta x)$. And for $(x, y') \in X \times_{s_X, H^{(0)}, r_{X^{\text{op}}}} X^{\text{op}}$ and appropriate $\gamma, \zeta \in G$, define $\gamma(x, y')\zeta = (\gamma x, y'\zeta^{-1})$. Then m is a G-G-equivariant homeomorphism.

Now we define proper actions of H on the domains and codomains of m so that m is also an H-equivariant map. And, the induce map $[m]$ of the quotients is a homeomorphism of quotients.

Let H act on $G \times_{s_G, G^{(0)}, r_X} X$ by $(\xi, x)\eta = (\xi, x\eta)$ for $(\xi, x) \in G \times_{s_G, G^{(0)}, r_X} X$ and appropriate $\eta \in H$; and let H act on $X \times_{s_X, H^{(0)}, r_{X^{\text{op}}}} X^{\text{op}}$ as $(x, y')\eta = (x\eta, \eta^{-1}y')$ for $(x, y') \in X \times_{s_X, H^{(0)}, r_{X^{\text{op}}}} X^{\text{op}}$ and appropriate $\eta \in H$. Both these actions are proper. Moreover, m is an H-equivariant homeomorphisms. Therefore, m induces a G-G-equivariant homeomorphism

$$[m]\colon (G \times_{s_G, G^{(0)}, r_X} X)/H \longrightarrow X \circ X^{\text{op}}.$$

Finally, the identification $(G \times_{s_G, G^{(0)}, r_X} X)/H \approx G \times_{s_G, G^{(0)}, [r_X]} (X/H) \approx G$ produces the desired 2-arrow. In the last identification, the first isomorphism uses Lemma 2.31 in [Tu, 2004], and the second one uses the fact that $X/H \xrightarrow[\approx]{[r_X]} G^{(0)}$. ∎

Above theorem is a topological analogue of [Landsman, 2000, Proposition 4.7]. This theorem is known in various special cases (e.g. [Tu, 2004]). However, we did not find an explicit proof of the theorem for Macho-Stadler–O'uchi correspondences. Therefore, we took this opportunity to spell out the proof. The converse part in Theorem 3.33 is well-known; for example, Sims and Williams mention an alternate proof of it in [Sims-Williams, 2012] immediately after Definition 1.4.

3.35. Remark. Lemma 3.34(v) says that, essentially, X^{op} is the inverse of an isomorphism 1-arrow X in $\mathfrak{GD}$.

[3] Because $m'\colon G \times_{s_G, G^{(0)}, r_X} X \longrightarrow X \times_{s_X, H^{(0)}, s_X} X$ defined by $(\gamma, x) \mapsto (\gamma x, x)$ is a homeomorphism.

3.36. Remark. It is *very* legitimate expectation that Theorem 3.33 holds for $\mathfrak{T}$ itself. Assume that an invertible 1-arrow $(X,\lambda)\colon (G,\alpha)\longrightarrow(H,\beta)$ in $\mathfrak{T}$ is given; and an inverse of it—consisting of the 1-arrow $(Y,\mu)\colon (H,\beta)\longrightarrow(G,\alpha)$, and the 2-arrows $\phi_G\colon X\circ Y\longrightarrow G$ and $\phi_H\colon Y\circ X\longrightarrow H$—is given. Then the first remark after Lemma 3.34 implies that the groupoid actions on X and Y are free.

Secondly, since the action of G on the quotient $X\circ Y\approx G$ is proper, so is its action on $X\times_{s_X,G^{(0)},r_Y} Y$—one may take this as a small exercise that can be proved using [Tu, 2004, Proposition 2.10(iii)]. This, in turn implies that G acts properly on X. Similarly, all other actions can be shown to be proper.

Same as in the case of the Macho-Stadler–O'uchi correspondences, the momentum maps for all the actions can be shown to be open surjections.

Now notice that Lemma 3.34 holds for the bispaces X and Y as the lemma uses freeness and properness of the actions; earlier arguments have proven that all actions on X and Y have these virtues. Thus, we may identify Y with X^{op} through the equivariant homeomorphism ϕ in Lemma 3.34. At this point, a tricky computation involving ϕ shows that Condition (iii) in Example 2.9 holds for the momentum map s_X. Then by the symmetry of situations, this condition holds for all momentum maps. In short, we have shown that the spaces X and Y are equivalences of groupoids.

At this point, we *expect* that the families of measures on these equivalences (as in Example 2.9) are equivalent to the given families of measures λ and μ. If it happens, then one has shown that *up to isomorphism* the groupoid equivalences are the invertible 1-arrows in $\mathfrak{T}$. Thus this equivalence of families of measures is basically the missing piece in proving Theorem 3.33 for topological correspondences in general. We expect this result to hold.

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