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EXTENSIVITY OF CATEGORIES OF RELATIONAL STRUCTURES

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Abstract. We prove that the category of models of any relational Horn theory satisfying a mild syntactic condition is infinitely extensive. Central examples of such categories include the categories of preordered sets and partially ordered sets, and the categories of small $\mathscr{V}$-categories, (symmetric) pseudo-$\mathscr{V}$-metric spaces, and (symmetric) $\mathscr{V}$-metric spaces for a commutative unital quantale $\mathscr{V}$. We also explicitly characterize initial sources and final sinks in such categories, and in particular embeddings and quotients.

1. Introduction

A category $\mathscr{C}$ is *infinitely extensive* [3] if it has small coproducts and for any small family $(X_i)_{i\in I}$ of objects of $\mathscr{C}$, the canonical functor $\prod_i \mathscr{C}/X_i \to \mathscr{C}/(\coprod_i X_i)$ is an equivalence. Prominent and well-known examples of infinitely extensive categories include cocomplete elementary toposes (such as Grothendieck toposes), the category Cat of small categories and functors, and the category Top of topological spaces and continuous maps. Generalizing the last example, Mahmoudi-Schubert-Tholen showed in [10] that many of the categories studied in *monoidal topology* [7] (which are defined in terms of commutative unital quantales $\mathscr{V}$) are also infinitely extensive. Clementino [4] recently extended the results of [10] to a more general setting (where a commutative unital quantale $\mathscr{V}$ is replaced by a complete and cocomplete symmetric monoidal closed category $\mathscr{V}$).

As a further contribution to this line of work, we show in this article that if $\mathbb{T}$ is a *relational Horn theory* satisfying a mild syntactic condition (5.2), then the category $\mathbb{T}$-Mod of $\mathbb{T}$-models and their morphisms is infinitely extensive. As we show in Example 3.7, key examples of such categories include: the category Preord of preordered sets and monotone maps, and its full subcategory Pos of partially ordered sets; for a commutative unital quantale $\mathscr{V}$, the category $\mathscr{V}$-Cat of (small) $\mathscr{V}$-categories and $\mathscr{V}$-functors, the category $\mathsf{PMet}_{\mathscr{V}}$ of *(symmetric) pseudo-$\mathscr{V}$-metric spaces* and *$\mathscr{V}$-contractions*, and its full subcategory $\mathsf{Met}_{\mathscr{V}}$ of *(symmetric) $\mathscr{V}$-metric spaces*. The infinite extensivity of some of these categories (e.g. Preord and $\mathscr{V}$-Cat) is already known from the results of [10, 4]; but the categories studied in [10, 4] do not subsume all of the categories studied in the present article.[1] On the other hand, the categories considered herein are all *locally presentable* [2],

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[1] For instance, the categories studied in [10, 4] are all *topological* over Set, while not all categories

and there are certainly examples of non-locally-presentable categories studied in [10, 4] (e.g. Top) whose infinite extensivity is therefore not a consequence of the results of the present article. In summary, the results of the present article neither subsume nor are subsumed by the results of [10, 4].

We now provide an outline of the article. After recalling some relevant background on concrete and topological categories in Section 2, we begin Section 3 by defining the notion of a *relational signature* Π, which is a set of *relation symbols* equipped with an assignment to each relation symbol of a finite positive arity. We then define the concrete category $\mathsf{Str}(\Pi)$ of Π-*structures* and Π-*morphisms*. For a regular cardinal λ, we define the notion of a λ-*ary relational Horn theory* $\mathbb{T}$ over a relational signature Π, and we provide some central examples of such theories. In Section 4 we study some topological properties of the category $\mathbb{T}$-Mod for a relational Horn theory $\mathbb{T}$ *without equality*, and we provide an explicit characterization of initial sources (and hence embeddings) and final sinks (and hence quotients) in $\mathbb{T}$-Mod. We prove the main result of the paper in Section 5. We begin Section 5 by stating the mild syntactic condition (5.2) that we will impose on relational Horn theories to prove the infinite extensivity of their categories of models; this condition is satisfied by all of our examples, and many others. We then establish in Theorem 5.5 that if $\mathbb{T}$ is a relational Horn theory that satisfies the syntactic condition (5.2), then $\mathbb{T}$-Mod is infinitely extensive. We thank the referee for useful comments and suggestions that improved the content and presentation of the paper.

2. Notation and background

We first recall some background material on concrete and topological categories, which can be found (e.g.) in [1].

2.1. A *concrete category (over* Set*)* is simply a category $\mathscr{C}$ equipped with a faithful functor $|-| : \mathscr{C} \to \mathsf{Set}$ (which we will usually not mention explicitly). Concrete categories $(\mathscr{A}, |-|_{\mathscr{A}})$ and $(\mathscr{B}, |-|_{\mathscr{B}})$ are *concretely isomorphic* if there is an isomorphism of categories $F : \mathscr{A} \to \mathscr{B}$ satisfying $|-|_{\mathscr{B}} \circ F = |-|_{\mathscr{A}}$. A *source* in an arbitrary category $\mathscr{C}$ is a (possibly large) class of morphisms $(h_i : X \to X_i)_{i \in I}$ in $\mathscr{C}$ with the same domain, while a *sink* in $\mathscr{C}$ is a (possibly large) class of morphisms $(h_i : X_i \to X)_{i \in I}$ with the same codomain.

A source $(h_i : X \to X_i)_{i \in I}$ in a concrete category $\mathscr{C}$ is *initial* if for any $\mathscr{C}$-object Y and function $h : |Y| \to |X|$, the function h lifts to a $\mathscr{C}$-morphism $h : Y \to X$ iff the composite functions $h_i \circ h : |Y| \to |X_i|$ lift to $\mathscr{C}$-morphisms $h_i \circ h : Y \to X_i$ for all $i \in I$. In particular, a morphism of $\mathscr{C}$ is *initial* if the source consisting of just that morphism is initial. Dually, a sink $(h_i : X_i \to X)_{i \in I}$ in a concrete category $\mathscr{C}$ is *final* if for any $\mathscr{C}$-object Y and function $h : |X| \to |Y|$, the function h lifts to a $\mathscr{C}$-morphism $h : X \to Y$ iff the composite functions $h \circ h_i : |X_i| \to |Y|$ lift to $\mathscr{C}$-morphisms $h \circ h_i : X_i \to Y$ for all

studied herein (e.g. Pos) have this property. Moreover, it does not seem that the categories $\mathsf{PMet}_{\mathscr{V}}$ and $\mathsf{Met}_{\mathscr{V}}$ of *symmetric* (pseudo-)$\mathscr{V}$-metric spaces are examples that are (directly) captured by [10, 4].

$i \in I$. In particular, a morphism of $\mathscr{C}$ is *final* if the sink consisting of just that morphism is final.

If $\mathscr{C}$ is a concrete category, then a *structured source* is a (possibly large) class of functions $(f_i : S \to |X_i|)_{i \in I}$ with S a set and X_i a $\mathscr{C}$-object for each $i \in I$, while a *structured sink* is defined dually. A concrete category $\mathscr{C}$ is *topological (over* Set*)* if every structured source $(f_i : S \to |X_i|)_{i \in I}$ has an initial lift, meaning that there is a $\mathscr{C}$-object X with $|X| = S$ such that each $f_i : |X| = S \to |X_i|$ lifts to a $\mathscr{C}$-morphism $f_i : X \to X_i$, and the resulting source $(f_i : X \to X_i)_{i \in I}$ is initial. A topological category over Set also satisfies the dual property that every structured sink has a final lift (see e.g. [1, 21.9]).

2.2. Let $\mathscr{C}$ be a topological category over Set. We say that a morphism of $\mathscr{C}$ is *injective* (resp. *surjective, bijective*) if its underlying function is injective (resp. surjective, bijective). Since $|-| : \mathscr{C} \to \mathsf{Set}$ is faithful and preserves small limits and colimits (see [1, 21.15]), it follows that the monomorphisms (resp. epimorphisms) of $\mathscr{C}$ are precisely the injective (resp. surjective) morphisms.

An *embedding* of $\mathscr{C}$ is an injective morphism that is initial, while a *quotient (morphism)* of $\mathscr{C}$ is a surjective morphism that is final. The embeddings (resp. quotients) are precisely the strong monomorphisms (resp. strong epimorphisms) of $\mathscr{C}$, which in turn are precisely the regular monomorphisms (resp. regular epimorphisms) of $\mathscr{C}$ (see e.g. [1, 21.13]). It follows that the isomorphisms of $\mathscr{C}$ are precisely the bijective embeddings, or equivalently the bijective quotients.

3. Relational Horn theories

We begin by defining the notion of a *relational signature* and its structures.

3.1. Definition. A **relational signature** is a set Π of **relation symbols** together with an assignment to each relation symbol of a finite arity, i.e. a positive integer $n \geq 1$.

We will typically write R for a general relation symbol. We fix a relational signature Π for the remainder of Section 3. The next two definitions are essentially taken from [5, Definition 3.1].

3.2. Definition. A **Π-edge** in a set S is a pair $(R, (s_1, \ldots, s_n))$ consisting of a relation symbol $R \in \Pi$ (of arity $n \geq 1$) and an ordered n-tuple $(s_1, \ldots, s_n)$ of elements of S. A **Π-structure** X consists of a set $|X|$ together with a subset $R^X \subseteq |X|^n$ for each relation symbol $R \in \Pi$ (of arity $n \geq 1$). We may equivalently describe a Π-structure X as a set $|X|$ equipped with a set $\mathsf{E}(X)$ of Π-edges in $|X|$: if $R \in \Pi$ of arity $n \geq 1$, then $(x_1, \ldots, x_n) \in R^X$ iff $\mathsf{E}(X)$ contains the Π-edge $(R, (x_1, \ldots, x_n))$. We will pass between these equivalent descriptions of Π-structures without further comment. We will often write $X \models Rx_1 \ldots x_n$ to mean that $(x_1, \ldots, x_n) \in R^X$.

3.3. Definition. Let $h : S \to S'$ be a function from a set S to a set S', and let $e = (R, (s_1, \ldots, s_n))$ be a Π-edge in S. We write $h \cdot e = h \cdot (R, (s_1, \ldots, s_n))$ for the Π-edge $(R, (h(s_1), \ldots, h(s_n)))$ in S'. If E is a set of Π-edges in S, then we write $h \cdot E$ for the set of Π-edges $\{h \cdot e \mid e \in S\}$ in S'. If E' is a set of Π-edges in S', then we write $h^{-1}[E']$ for the set of Π-edges e in S such that $h \cdot e \in E'$.

Given Π-structures X and Y, a **(Π-)morphism** $h : X \to Y$ is a function $h : |X| \to |Y|$ such that $h \cdot \mathsf{E}(X) \subseteq \mathsf{E}(Y)$, or equivalently such that $\mathsf{E}(X) \subseteq h^{-1}[\mathsf{E}(Y)]$. We let $\mathsf{Str}(\Pi)$ be the concrete category of Π-structures and Π-morphisms.

We now describe the syntax of relational Horn theories.

3.4. Definition. Let λ be a regular cardinal, and let Var be a set of variables of cardinality λ. A **λ-ary[2] Horn formula (over Π)** is an expression of the form $\Phi \Longrightarrow \psi$, where Φ is a set of Π-edges in Var of cardinality $< \lambda$ and ψ is a $(\Pi \cup \{=\})$-edge in Var, where $=$ is a fresh binary relation symbol not in Π. If $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ is finite, then we write $\varphi_1, \ldots, \varphi_n \Longrightarrow \psi$, and if $\Phi = \varnothing$, then we write $\Longrightarrow \psi$. A **λ-ary Horn formula without equality (over Π)** is a λ-ary Horn formula $\Phi \Longrightarrow \psi$ (over Π) such that ψ is a Π-edge in Var, i.e. such that ψ does not contain the fresh binary relation symbol $=$.

3.5. Definition. Let λ be a regular cardinal. A **λ-ary relational Horn theory** $\mathbb{T}$ **(without equality)** is a set of λ-ary Horn formulas (without equality) over Π, which we call the *axioms* of $\mathbb{T}$. A **relational Horn theory (without equality)** is a λ-ary relational Horn theory (without equality) for some regular cardinal λ.

3.6. Definition. Let λ be a regular cardinal, and let X be a Π-structure. We let $\overline{X}$ be the $(\Pi \cup \{=\})$-structure defined by $\left|\overline{X}\right| := |X|$ and $\mathsf{E}\left(\overline{X}\right) := \mathsf{E}(X) \cup \{(=, (x, x)) \mid x \in |X|\}$. A **valuation in** X is a function $\kappa : \mathsf{Var} \to |X|$. We say that X **satisfies** a λ-ary Horn formula $\Phi \Longrightarrow \psi$ over Π if whenever κ is a valuation in X such that $X \models \kappa \cdot \varphi$ for each $\varphi \in \Phi$, then $\overline{X} \models \kappa \cdot \psi$. A Π-structure X is a **model** of a λ-ary relational Horn theory $\mathbb{T}$ if X satisfies every axiom of $\mathbb{T}$. We let $\mathbb{T}\text{-}\mathsf{Mod}$ be the full subcategory of $\mathsf{Str}(\Pi)$ consisting of the models of $\mathbb{T}$, which is a concrete category when equipped with the faithful functor $|-| : \mathbb{T}\text{-}\mathsf{Mod} \to \mathsf{Set}$ obtained by restricting the faithful functor $|-| : \mathsf{Str}(\Pi) \to \mathsf{Set}$.

3.7. Example. We provide the following central examples of relational Horn theories. Some further examples may be found in [5, Example 3.5].

1. Let $\mathbb{T}$ be the empty relational Horn theory over Π. Then of course $\mathbb{T}\text{-}\mathsf{Mod} = \mathsf{Str}(\Pi)$. In particular, if Π is empty, then $\mathbb{T}\text{-}\mathsf{Mod} = \mathsf{Set}$.

2. Let Π consist of a single binary relation symbol $\leq$, and let $\mathbb{T}$ be the finitary relational Horn theory without equality over Π that consists of the two axioms $\Longrightarrow x \leq x$ and $x \leq y, y \leq z \Longrightarrow x \leq z$. Then $\mathbb{T}\text{-}\mathsf{Mod}$ is the concrete category Preord of preordered sets and monotone maps. If one extends $\mathbb{T}$ by adding the further axiom $x \leq y, y \leq x \Longrightarrow x = y$, then the category of models of the resulting finitary relational Horn theory with equality is the concrete category Pos of posets and monotone maps.

[2] When $\lambda = \aleph_0$, we will say "finitary" rather than "$\aleph_0$-ary".

3. The following examples come from [14, Definition 2.2 and Remark 2.4(2)]. Let $(\mathscr{V}, \leq, \otimes, \mathsf{k})$ be a *commutative unital quantale* [11], meaning that $(\mathscr{V}, \leq)$ is a complete lattice and $(\mathscr{V}, \otimes, \mathsf{k})$ is a commutative monoid such that $\otimes$ distributes over arbitrary suprema in each variable. A *(small)* $\mathscr{V}$*-category* (X, d) (see also [8]) is a set X equipped with a function $d : X \times X \to \mathscr{V}$ satisfying the two conditions

$$d(x,x) \geq \mathsf{k}$$

$$d(x,z) \geq d(x,y) \otimes d(y,z)$$

for all $x, y, z \in X$. A *pseudo-*$\mathscr{V}$*-metric space* is a $\mathscr{V}$-category (X, d) such that $d : X \times X \to \mathscr{V}$ satisfies the further symmetry condition

$$d(x,y) = d(y,x)$$

for all $x, y \in X$. Finally, a $\mathscr{V}$*-metric space* is a pseudo-$\mathscr{V}$-metric space (X, d) satisfying the further "separation" condition

$$d(x,y) \geq \mathsf{k} \Longrightarrow x = y$$

for all $x, y \in X$.

If (X, d_X) and (Y, d_Y) are $\mathscr{V}$-categories, then a $\mathscr{V}$*-functor* or $\mathscr{V}$*-contraction* $h : (X, d_X) \to (Y, d_Y)$ is a function $h : X \to Y$ satisfying $d_X(x, x') \leq d_Y(h(x), h(x'))$ for all $x, x' \in X$. We let $\mathscr{V}$-Cat be the concrete category of $\mathscr{V}$-categories and $\mathscr{V}$-functors, we let $\mathsf{PMet}_{\mathscr{V}}$ be the full subcategory of $\mathscr{V}$-Cat consisting of the pseudo-$\mathscr{V}$-metric spaces, and we let $\mathsf{Met}_{\mathscr{V}}$ be the full subcategory of $\mathsf{PMet}_{\mathscr{V}}$ consisting of the $\mathscr{V}$-metric spaces. We regard $\mathsf{PMet}_{\mathscr{V}}$ and $\mathsf{Met}_{\mathscr{V}}$ as concrete categories by suitably restricting the faithful functor $|-| : \mathscr{V}\text{-}\mathsf{Cat} \to \mathsf{Set}$. As indicated in [14, Example 2.3], one has the following specific examples of $\mathscr{V}$-Cat, $\mathsf{PMet}_{\mathscr{V}}$, and $\mathsf{Met}_{\mathscr{V}}$ for suitable choices of $(\mathscr{V}, \leq, \otimes, \mathsf{k})$ (see [14, Example 2.1]):

- For the trivial quantale $\mathbb{1}$ with just one element, both $\mathbb{1}$-Cat and $\mathsf{PMet}_{\mathbb{1}}$ are equivalent to Set, while $\mathsf{Met}_{\mathbb{1}}$ is equivalent to the terminal category.
- For the Boolean 2-chain quantale $\mathbf{2}$, we have that $\mathbf{2}$-Cat is equivalent to the category Preord of preordered sets and monotone maps, while $\mathsf{PMet}_{\mathbf{2}}$ is equivalent to the category whose objects are sets equipped with an equivalence relation, and whose morphisms are functions preserving the equivalence relations. $\mathsf{Met}_{\mathbf{2}}$ is equivalent to Set.
- For the *Lawvere quantale* $\mathbb{R}_+ = ([0, \infty], \geq, +, 0)$ [9] given by the extended real half line with the reverse ordering, we have that $\mathbb{R}_+$-Cat is the category of *Lawvere metric spaces*, while $\mathsf{PMet}_{\mathbb{R}_+}$ is the category PMet of extended pseudo-metric spaces (i.e. pseudo-metric spaces where two points may have distance ∞) and contractions (i.e. non-expanding maps), and $\mathsf{Met}_{\mathbb{R}_+}$ is the category Met of extended metric spaces (i.e. extended pseudo-metric spaces (X, d) satisfying $d(x, y) = 0 \Longrightarrow x = y$) and contractions.

- For the quantale Δ of *distance distribution functions* (see e.g. [6, §3.1]), we have that Δ-Cat is equivalent to the category $\mathsf{ProbMet}$ of *probabilistic metric spaces* (see [6, §3.2] and [13]), while PMet_Δ is equivalent to the category of *symmetric* probabilistic metric spaces, and Met_Δ is equivalent to the category of *symmetric, separated* probabilistic metric spaces.

Let $\Pi_{\mathscr{V}}$ consist of just the binary relation symbols $\sim_v$ for each $v \in \mathscr{V}$. Let $\mathbb{T}_{\mathscr{V}\text{-}\mathsf{Cat}}$ be the relational Horn theory without equality over $\Pi_{\mathscr{V}}$ that consists of the following axioms, where v, v' range over $\mathscr{V}$:

$$\Longrightarrow x \sim_{\mathsf{k}} x$$

$$x \sim_v y, y \sim_{v'} z \Longrightarrow x \sim_{v \otimes v'} z$$

$$x \sim_v y \Longrightarrow x \sim_{v'} y \qquad (v \geq v')$$

$$\{x \sim_{v_i} y \mid i \in I\} \Longrightarrow x \sim_{\bigvee_i v_i} y$$

Then $\mathbb{T}_{\mathscr{V}\text{-}\mathsf{Cat}}$-$\mathsf{Mod}$ is concretely isomorphic (2.1) to $\mathscr{V}$-Cat, which we prove in the Appendix (Section 6). Let $\mathbb{T}_{\mathsf{PMet}_{\mathscr{V}}}$ be the relational Horn theory without equality over $\Pi_{\mathscr{V}}$ that extends $\mathbb{T}_{\mathscr{V}\text{-}\mathsf{Cat}}$ by adding the symmetry axioms

$$x \sim_v y \Longrightarrow y \sim_v x$$

for all $v \in \mathscr{V}$. Then $\mathbb{T}_{\mathsf{PMet}_{\mathscr{V}}}$-$\mathsf{Mod}$ is concretely isomorphic to $\mathsf{PMet}_{\mathscr{V}}$, which we also prove in the Appendix (Section 6). Finally, let $\mathbb{T}_{\mathsf{Met}_{\mathscr{V}}}$ be the relational Horn theory with equality over $\Pi_{\mathscr{V}}$ that extends $\mathbb{T}_{\mathsf{PMet}_{\mathscr{V}}}$ by adding the axiom

$$x \sim_{\mathsf{k}} y \Longrightarrow x = y.$$

Then we also show in the Appendix (Section 6) that $\mathbb{T}_{\mathsf{Met}_{\mathscr{V}}}$-$\mathsf{Mod}$ is concretely isomorphic to $\mathsf{Met}_{\mathscr{V}}$.

4. Topological properties of relational Horn theories without equality

Throughout Section 4 we fix a relational Horn theory $\mathbb{T}$ *without equality* over a relational signature Π (an assumption that we will occasionally repeat for emphasis). We first aim to characterize the initial sources and final sinks in the concrete category $\mathbb{T}$-Mod, which will allow us to prove that $\mathbb{T}$-Mod is topological over Set, which is in fact a special case of a result [12, Proposition 5.1] by Rosický. We will then characterize the embeddings and quotients in $\mathbb{T}$-Mod. We first require the following definition.

4.1. Definition. Let S be a set. A **$\mathbb{T}$-relation** on S is a set $\mathcal{E}$ of Π-edges in S satisfying the following condition: for any axiom $\Phi \Longrightarrow \psi$ of $\mathbb{T}$ and any valuation $\kappa : \mathsf{Var} \to S$ such that $\kappa \cdot \varphi \in \mathcal{E}$ for each $\varphi \in \Phi$, we have $\kappa \cdot \psi \in \mathcal{E}$. The set of all Π-edges in S is clearly a $\mathbb{T}$-relation on S, and the intersection of a small family of $\mathbb{T}$-relations on S is a $\mathbb{T}$-relation on S. So for any set $\mathcal{E}$ of Π-edges in S, we can define the **$\mathbb{T}$-closure** $\mathbb{T}(\mathcal{E})$ of $\mathcal{E}$ to be the smallest $\mathbb{T}$-relation on S that contains $\mathcal{E}$, i.e. the intersection of all $\mathbb{T}$-relations on S that contain $\mathcal{E}$. Note that a Π-structure X is a $\mathbb{T}$-model iff $\mathsf{E}(X)$ is a $\mathbb{T}$-relation on $|X|$.

The proof of the following lemma is elementary.

4.2. Lemma. *Let $h : S \to S'$ be a function from a set S to a set S', and let $\mathcal{E}'$ be a $\mathbb{T}$-relation on S'. Then $h^{-1}[\mathcal{E}']$ is a $\mathbb{T}$-relation on S.*

4.3. Proposition. *A source $(h_i : X \to X_i)_{i \in I}$ in $\mathbb{T}$-Mod is initial iff for each $R \in \Pi$ of arity $n \geq 1$ and all $x_1, \ldots, x_n \in |X|$, we have $X \models Rx_1 \ldots x_n$ iff $X_i \models Rh_i(x_1) \ldots h_i(x_n)$ for all $i \in I$. A sink $(h_i : X_i \to X)_{i \in I}$ in $\mathbb{T}$-Mod is final iff $\mathsf{E}(X)$ is the $\mathbb{T}$-closure of $\bigcup_{i \in I} h_i \cdot \mathsf{E}(X_i)$.*

Proof. For the first assertion, if $\mathsf{E}(X)$ has the stated characterization, then it readily follows that $(h_i)_i$ is initial. So assume that $(h_i)_i$ is initial; since each h_i $(i \in I)$ is a Π-morphism, we need only prove that $X \models Rx_1 \ldots x_n$ whenever $X_i \models Rh_i(x_1) \ldots h_i(x_n)$ for all $i \in I$. Assuming the latter condition, we define a Π-structure Y by setting $|Y| := |X|$ and letting $\mathsf{E}(Y)$ be the $\mathbb{T}$-closure of $\mathsf{E}(X) \cup \{(R, (x_1, \ldots, x_n))\}$. Then Y is a $\mathbb{T}$-model, and for each $i \in I$ we claim that the function $h_i : |Y| = |X| \to |X_i|$ is a Π-morphism $h_i : Y \to X_i$, i.e. that $\mathsf{E}(Y) \subseteq h_i^{-1}[\mathsf{E}(X_i)]$. By definition of $\mathsf{E}(Y)$, it suffices to show that $h_i^{-1}[\mathsf{E}(X_i)]$ is a $\mathbb{T}$-relation on $|Y| = |X|$ that contains $\mathsf{E}(X) \cup \{(R, (x_1, \ldots, x_n))\}$. Since $\mathsf{E}(X_i)$ is a $\mathbb{T}$-relation on $|X_i|$ (because X_i is a $\mathbb{T}$-model), we deduce from Lemma 4.2 that $h_i^{-1}[\mathsf{E}(X_i)]$ is a $\mathbb{T}$-relation on $|Y| = |X|$. Then because $h_i : X \to X_i$ is a Π-morphism and $X_i \models Rh_i(x_1) \ldots h_i(x_n)$ by hypothesis, it follows that $h_i^{-1}[\mathsf{E}(X_i)]$ contains $\mathsf{E}(X) \cup \{(R, (x_1, \ldots, x_n))\}$. So each $h_i : Y \to X_i$ $(i \in I)$ is a Π-morphism, and the initiality of the source $(h_i)_i$ then implies that the identity function $1_{|X|} : |Y| = |X| \to |X|$ is a Π-morphism $Y \to X$, which entails that $X \models Rx_1 \ldots x_n$, as desired.

For the second assertion, suppose first that $\mathsf{E}(X)$ is the $\mathbb{T}$-closure of $\bigcup_{i \in I} h_i \cdot \mathsf{E}(X_i)$. To show that the sink $(h_i)_i$ is final, let Y be a $\mathbb{T}$-model and let $h : |X| \to |Y|$ be a function such that the function $h \circ h_i : |X_i| \to |Y|$ is a Π-morphism $X_i \to Y$ for each $i \in I$. To show that h is a Π-morphism $X \to Y$, we must show that $\mathsf{E}(X) \subseteq h^{-1}[\mathsf{E}(Y)]$. By assumption on $\mathsf{E}(X)$, it suffices to show that $h^{-1}[\mathsf{E}(Y)]$ is a $\mathbb{T}$-relation on $|X|$ that contains $\bigcup_{i \in I} h_i \cdot \mathsf{E}(X_i)$. The first claim holds by Lemma 4.2 because $\mathsf{E}(Y)$ is a $\mathbb{T}$-relation on $|Y|$ (since Y is a $\mathbb{T}$-model), and the second claim holds because each $h \circ h_i$ $(i \in I)$ is a Π-morphism $X_i \to Y$.

Now suppose that the sink $(h_i : X_i \to X)_{i \in I}$ is final, and let us show that $\mathsf{E}(X)$ must be the $\mathbb{T}$-closure of $\bigcup_{i \in I} h_i \cdot \mathsf{E}(X_i)$. That is, we must show that $\mathsf{E}(X)$ is the smallest $\mathbb{T}$-relation on $|X|$ that contains $\bigcup_{i \in I} h_i \cdot \mathsf{E}(X_i)$. That $\mathsf{E}(X)$ is a $\mathbb{T}$-relation holds because X is a $\mathbb{T}$-model, and since each $h_i : X_i \to X$ $(i \in I)$ is a Π-morphism, it follows that $\mathsf{E}(X)$ contains $\bigcup_{i \in I} h_i \cdot \mathsf{E}(X_i)$. Now let $\mathcal{R}$ be any $\mathbb{T}$-relation on $|X|$ that contains $\bigcup_{i \in I} h_i \cdot \mathsf{E}(X_i)$, and let us show that $\mathsf{E}(X) \subseteq \mathcal{R}$. Let X' be the Π-structure defined by $|X'| := |X|$ and $\mathsf{E}(X') := \mathcal{R}$, so that X' is a $\mathbb{T}$-model because $\mathcal{R}$ is a $\mathbb{T}$-relation. Showing that $\mathsf{E}(X) \subseteq \mathcal{R}$ is equivalent to showing that the identity function $1_{|X|} : |X| \to |X| = |X'|$ is a Π-morphism $X \to X'$. By finality of the sink $(h_i)_{i \in I}$, it then suffices to show that each function $h_i : |X_i| \to |X| = |X'|$ is a Π-morphism $X_i \to X'$, which is true by assumption on $\mathcal{R}$. This proves the desired characterization of $\mathsf{E}(X)$. ■

The following result, whose proof we outline in (4.5) below, is a special case of [12, Prop. 5.1]:

4.4. Proposition. [Rosický [12]] *Let $\mathbb{T}$ be a relational Horn theory without equality. Then the concrete category $\mathbb{T}$-Mod is topological over Set.*

4.5. The initial lift of a structured source $(h_i : S \to |X_i|)_{i \in I}$ is the source $(h_i : X \to X_i)_{i \in I}$, where X is the Π-structure with $|X| := S$ and $X \models Rx_1 \ldots x_n$ iff $X_i \models Rh_i(x_1) \ldots h_i(x_n)$ for all $i \in I$ (for any $R \in \Pi$ of arity $n \geq 1$ and any $x_1, \ldots, x_n \in |X|$). Since $\mathbb{T}$ is a relational Horn theory *without equality* and each X_i $(i \in I)$ is a $\mathbb{T}$-model, it readily follows that X is a $\mathbb{T}$-model, and the source $(h_i)_i$ in $\mathbb{T}$-Mod is initial by Proposition 4.3. The final lift of a structured sink $(h_i : |X_i| \to S)_{i \in I}$ is the sink $(h_i : X_i \to X)_{i \in I}$, where X is the Π-structure with $|X| := S$ and $\mathsf{E}(X)$ the $\mathbb{T}$-closure of $\bigcup_{i \in I} h_i \cdot \mathsf{E}(X_i)$. Then X is a $\mathbb{T}$-model because $\mathsf{E}(X)$ is a $\mathbb{T}$-relation, and the sink $(h_i)_i$ is final by Proposition 4.3.

4.6. Given a small diagram $D : \mathscr{B} \to \mathbb{T}\text{-}\mathsf{Mod}$, the limit cone of D is the initial lift of the limit cone of $|-| \circ D$ in Set, while the colimit cocone of D is the final lift of the colimit cocone of $|-| \circ D$ in Set (see e.g. [1, 21.15]). In particular, the functor $|-| : \mathbb{T}\text{-}\mathsf{Mod} \to \mathsf{Set}$ strictly preserves small limits and colimits. If $\mathbb{T}$ is a relational Horn theory *with* equality, then $\mathbb{T}$-Mod is complete and cocomplete (e.g. because $\mathbb{T}$-Mod is locally presentable, by [2, Proposition 5.30]), and it is still true that $|-| : \mathbb{T}\text{-}\mathsf{Mod} \to \mathsf{Set}$ strictly preserves limits, since the inclusion $\mathbb{T}\text{-}\mathsf{Mod} \hookrightarrow \mathsf{Str}(\Pi)$ preserves limits.

4.7. Definition. Let $h : X \to Y$ be a morphism in $\mathsf{Str}(\Pi)$. Then h **reflects relations** if for each $R \in \Pi$ of arity $n \geq 1$ and any $x_1, \ldots, x_n \in |X|$, we have $X \models Rx_1 \ldots x_n$ if $Y \models Rh(x_1) \ldots h(x_n)$.

The characterizations of initial sources and final sinks in $\mathbb{T}$-Mod provided in Proposition 4.3 immediately entail that embeddings and quotients in $\mathbb{T}$-Mod have the following characterizations:

4.8. Proposition. *Let $h : X \to Y$ be a morphism in $\mathbb{T}$-Mod. Then h is an embedding iff h is injective and reflects relations, while h is a quotient morphism iff h is surjective and $\mathsf{E}(Y)$ is the $\mathbb{T}$-closure of $h \cdot \mathsf{E}(X)$.*

If $\mathscr{C}$ is a topological category over Set, then a bimorphism in $\mathscr{C}$ (i.e. a morphism that is both epic and monic) is precisely a bijective morphism in view of (2.2). As in [1, Definition 16.1], we say that a full replete[3] subcategory $\mathscr{B} \hookrightarrow \mathscr{C}$ is *bireflective* if every object of $\mathscr{C}$ has a $\mathscr{B}$-reflection morphism that is a bimorphism of $\mathscr{C}$, i.e. that is bijective. Since the concrete category $\mathsf{Str}(\Pi)$ is topological over Set in view of Example 3.7.1 and Proposition 4.4, we now have the following result (cf. [5, Proposition 3.6]):

4.9. Proposition. *Let $\mathbb{T}$ be a relational Horn theory without equality. Then the full replete subcategory $\mathbb{T}\text{-}\mathsf{Mod} \hookrightarrow \mathsf{Str}(\Pi)$ is bireflective.*

Proof. Let X be a Π-structure. We define a Π-structure X^* by setting $|X^*| := |X|$ and letting $\mathsf{E}(X^*)$ be the $\mathbb{T}$-closure of $\mathsf{E}(X)$. Then X^* is a $\mathbb{T}$-model, and the identity function $1_{|X|} : |X| \to |X| = |X^*|$ is a bijective Π-morphism, which we claim is a $\mathbb{T}$-Mod-reflection morphism for X. So let $h : X \to Y$ be a Π-morphism from X to a $\mathbb{T}$-model Y. Then the function $h : |X^*| = |X| \to |Y|$ is also a Π-morphism $h : X^* \to Y$, because $\mathsf{E}(X^*)$ is the smallest $\mathbb{T}$-relation on $|X|$ that contains $\mathsf{E}(X)$, and thus $\mathsf{E}(X^*) \subseteq h^{-1}[\mathsf{E}(Y)]$ because $h^{-1}[\mathsf{E}(Y)]$ is a $\mathbb{T}$-relation on $|X|$ (4.2) that contains $\mathsf{E}(X)$ (since $h : X \to Y$ is a Π-morphism). ■

5. Extensivity of $\mathbb{T}$-Mod for a relational Horn theory $\mathbb{T}$

Throughout Section 5, we fix a relational Horn theory $\mathbb{T}$ over a relational signature Π.

5.1. Definition. Let Var be a set of variables. For any $(\Pi \cup \{=\})$-edge φ in Var, we define the set of variables $\mathsf{Var}(\varphi)$ occurring in φ as follows: if $\varphi = (R, (v_1, \ldots, v_n))$ for some $R \in \Pi \cup \{=\}$ of arity $n \geq 1$ and some $v_1, \ldots, v_n \in \mathsf{Var}$, then $\mathsf{Var}(\varphi) := \{v_1, \ldots, v_n\}$. If Φ is a set of $(\Pi \cup \{=\})$-edges in Var, then we define $\mathsf{Var}(\Phi) := \bigcup_{\varphi \in \Phi} \mathsf{Var}(\varphi)$.

5.2. Condition. We suppose throughout Section 5 that $\mathbb{T}$ satisfies the following mild syntactic condition. For each axiom $\Phi \Longrightarrow \psi$ of $\mathbb{T}$, we require that:

1. Any two distinct elements of Φ share at least one variable in common; i.e. if $\varphi, \varphi' \in \Phi$ are distinct, then $\mathsf{Var}(\varphi) \cap \mathsf{Var}(\varphi') \neq \varnothing$.

2. If $\Phi \neq \varnothing$, then $\mathsf{Var}(\psi) \subseteq \mathsf{Var}(\Phi)$; and if $\Phi = \varnothing$, then $\mathsf{Var}(\psi)$ is a singleton.

This condition is certainly satisfied by all of our central examples (3.7) (and by the additional examples of [5, Example 3.5]), and will be used to conveniently characterize small coproducts in $\mathbb{T}$-Mod in 5.4 below. We shall provide some commentary on this condition in Remark 5.7 and Remark 5.8 below.

[3] Recall that a full subcategory $\mathscr{B} \hookrightarrow \mathscr{C}$ is *replete* if whenever C is an object of $\mathscr{C}$ that is isomorphic to an object of $\mathscr{B}$, then C is an object of $\mathscr{B}$.

5.3. In view of (4.6), the pullback $A \times_{f,g} B$ in $\mathbb{T}$-Mod of two $\mathbb{T}$-model morphisms $f : A \to C$ and $g : B \to C$ is formed by taking the initial lift of the pullback of the underlying functions $f : |A| \to |C|$ and $g : |B| \to |C|$ in Set. So we have

$$|A \times_{f,g} B| = |A| \times_{f,g} |B| = \{(a,b) \in |A| \times |B| \mid f(a) = g(b)\}$$

with

$$R^{A \times_{f,g} B} = \{((a_1,b_1),\ldots,(a_n,b_n)) \in |A \times_{f,g} B|^n \mid A \models Ra_1 \ldots a_n \text{ and } B \models Rb_1 \ldots b_n\}$$

for each $R \in \Pi$ of arity $n \geq 1$. The pullback projections $\pi_A : |A| \times_{f,g} |B| \to |A|$ and $\pi_B : |A| \times_{f,g} |B| \to |B|$ in Set then lift to pullback projections in $\mathbb{T}$-Mod.

5.4. Let $\mathbb{T}$ be a relational Horn theory *without equality*. In view of (4.6), the coproduct $\coprod_{i \in I} X_i$ of a small family of $\mathbb{T}$-models X_i $(i \in I)$ is formed by taking the final lift of the coproduct of the underlying sets $|X_i|$ $(i \in I)$ in Set. So we have $|\coprod_i X_i| = \coprod_i |X_i|$, and $\mathsf{E}\left(\coprod_i X_i\right)$ is the $\mathbb{T}$-closure of the set of all Π-edges in the images of all coproduct injections $s_i : |X_i| \to \coprod_i |X_i|$ $(i \in I)$. We now claim that in fact

$$\mathsf{E}\left(\coprod_i X_i\right) = \bigcup_{i \in I} s_i \cdot \mathsf{E}(X_i),$$

for which it suffices to show that $\bigcup_{i \in I} s_i \cdot \mathsf{E}(X_i)$ is a $\mathbb{T}$-relation on $\coprod_i |X_i|$.

So let $\Phi \Longrightarrow \psi$ be an axiom of $\mathbb{T}$, let $\kappa : \mathsf{Var} \to \coprod_i |X_i|$ be a valuation, and suppose that $\kappa \cdot \varphi \in \bigcup_{i \in I} s_i \cdot \mathsf{E}(X_i)$ for each $\varphi \in \Phi$; we must show that $\kappa \cdot \psi \in \bigcup_{i \in I} s_i \cdot \mathsf{E}(X_i)$. Since any two distinct elements of Φ share at least one variable in common (see Condition 5.2.1) and the union $\bigcup_{i \in I} s_i \cdot \mathsf{E}(X_i)$ is disjoint (since the coproduct injections s_i $(i \in I)$ have disjoint images), there must be some $i \in I$ such that $\kappa \cdot \varphi \in s_i \cdot \mathsf{E}(X_i)$ for all $\varphi \in \Phi$. In view of Condition 5.2.2, we may then assume without loss of generality that κ factors through $s_i : |X_i| \to \coprod_i |X_i|$ via a valuation $\kappa' : \mathsf{Var} \to |X_i|$, so that $\kappa = s_i \circ \kappa'$ and hence $s_i \cdot \kappa' \cdot \varphi \in s_i \cdot \mathsf{E}(X_i)$ for each $\varphi \in \Phi$. Since s_i is injective, it follows that $\kappa' \cdot \varphi \in \mathsf{E}(X_i)$ for each $\varphi \in \Phi$, so that $\kappa' \cdot \psi \in \mathsf{E}(X_i)$ because X_i is a $\mathbb{T}$-model. It then follows that $\kappa \cdot \psi \in \bigcup_{i \in I} s_i \cdot \mathsf{E}(X_i)$, as desired.

Now let $\mathbb{T}$ be an arbitrary relational Horn theory (possibly with equality), and let $\mathbb{T}^-$ be the relational Horn theory *without equality* obtained from $\mathbb{T}$ by removing all axioms $\Phi \Longrightarrow \psi$ of $\mathbb{T}$ that contain equality, i.e. where ψ is a $\{=\}$-edge in Var. We claim that the inclusion $\mathbb{T}$-$\mathsf{Mod} \hookrightarrow \mathbb{T}^-$-$\mathsf{Mod}$ preserves small coproducts. So let $(X_i)_{i \in I}$ be a small family of $\mathbb{T}$-models, and let us show that the coproduct $\coprod_i X_i$ in $\mathbb{T}^-$-Mod is a $\mathbb{T}$-model. Since $\coprod_i X_i$ is a $\mathbb{T}^-$-model, it just remains to show that $\coprod_i X_i$ satisfies each axiom $\Phi \Longrightarrow \psi$ of $\mathbb{T}$ where ψ is a $\{=\}$-edge $x = y$. So let $\kappa : \mathsf{Var} \to \coprod_i |X_i|$ be a valuation such that $\kappa \cdot \varphi \in \mathsf{E}\left(\coprod_i X_i\right) = \bigcup_i s_i \cdot \mathsf{E}(X_i)$ for each $\varphi \in \Phi$, and let us show that $\kappa(x) = \kappa(y)$. Since $\mathbb{T}$ satisfies Condition 5.2, we deduce exactly as in the previous paragraph that there must be some $i \in I$ such that κ factors through a valuation $\kappa' : \mathsf{Var} \to |X_i|$ (so that $s_i \circ \kappa' = \kappa$) and $\kappa' \cdot \varphi \in \mathsf{E}(X_i)$ for all $\varphi \in \Phi$. Because X_i is a $\mathbb{T}$-model, we then deduce that $\kappa'(x) = \kappa'(y)$, so that $\kappa(x) = \kappa(y)$ as desired. Since $|-| : \mathbb{T}^{-}\text{-}\mathsf{Mod} \to \mathsf{Set}$ preserves small coproducts, it follows that $|-| : \mathbb{T}\text{-}\mathsf{Mod} \to \mathsf{Set}$ preserves small coproducts.

Let $\mathbb{T}$ be a relational Horn theory and let $(X_i)_{i\in I}$ be a small family of $\mathbb{T}$-models. For each $R \in \Pi$ of arity $n \geq 1$ and any $(i_1, x_1), \ldots, (i_n, x_n) \in \coprod_i |X_i|$, we thus have $\coprod_i X_i \models R(i_1, x_1) \ldots (i_n, x_n)$ iff $i_1 = \ldots = i_n = i$ and $X_i \models Rx_1 \ldots x_n$. Therefore, each coproduct injection $s_i : X_i \to \coprod_i X_i$ $(i \in I)$ is an embedding[4] (4.8). The initial object of $\mathbb{T}$-Mod is the empty set equipped (of course) with the empty set of Π-edges.

We recall from [3] that a category $\mathscr{C}$ is said to be *infinitely extensive* if it has small coproducts and for any small family $(X_i)_{i\in I}$ of objects of $\mathscr{C}$, the canonical functor $\prod_{i\in I} \mathscr{C}/X_i \to \mathscr{C}/(\coprod_i X_i)$ is an equivalence. If $\mathscr{C}$ has small coproducts and pullbacks, recall that small coproducts in $\mathscr{C}$ are said to be *universal* (or *stable under pullback*) if for any small family $(X_i)_{i\in I}$ of objects of $\mathscr{C}$ with coproduct $(s_i : X_i \to \coprod_i X_i)_{i\in I}$ and any morphism $f : Y \to \coprod_i X_i$, if the following diagram is a pullback in $\mathscr{C}$ for each $i \in I$:

$$\begin{array}{ccc} P_i & \xrightarrow{\;t_i\;} & Y \\ {\scriptstyle \pi_i}\downarrow & & \downarrow{\scriptstyle f} \\ X_i & \xrightarrow[\;s_i\;]{} & \coprod_i X_i, \end{array}$$

then $(t_i : P_i \to Y)_{i\in I}$ is a coproduct diagram in $\mathscr{C}$. If $\mathscr{C}$ has small coproducts and pullbacks, then a coproduct $(s_i : X_i \to \coprod_i X_i)_{i\in I}$ of a small family $(X_i)_{i\in I}$ of objects of $\mathscr{C}$ is said to be *disjoint* if for any distinct indices $i, j \in I$, the pullback of s_i along s_j is isomorphic to the initial object of $\mathscr{C}$. If $\mathscr{C}$ has small coproducts and pullbacks, then by (the infinitary version of) [3, Proposition 2.14], we have that $\mathscr{C}$ is infinitely extensive iff small coproducts in $\mathscr{C}$ are universal and disjoint.

5.5. Theorem. *Let $\mathbb{T}$ be a relational Horn theory that satisfies Condition 5.2. Then $\mathbb{T}$-Mod is infinitely extensive.*

Proof. Since $\mathbb{T}$-Mod has small coproducts and pullbacks (4.6), it is equivalent to show that small coproducts in $\mathbb{T}$-Mod are universal and disjoint. For the first claim, let $(X_i)_{i\in I}$ be a small family of $\mathbb{T}$-models with coproduct $(s_i : X_i \to \coprod_i X_i)_{i\in I}$ in $\mathbb{T}$-Mod. Let $f : Y \to \coprod_i X_i$ be a morphism of $\mathbb{T}$-Mod, and for each $i \in I$ suppose that the following diagram is a pullback in $\mathbb{T}$-Mod:

$$\begin{array}{ccc} P_i & \xrightarrow{\;t_i\;} & Y \\ {\scriptstyle \pi_i}\downarrow & & \downarrow{\scriptstyle f} \\ X_i & \xrightarrow[\;s_i\;]{} & \coprod_i X_i. \end{array}$$

[4] Even if $\mathbb{T}$ is a relational Horn theory *with* equality, a morphism of $\mathbb{T}$-Mod that is injective and reflects relations is still an embedding.

We must show that $(t_i : P_i \to Y)_{i\in I}$ is a coproduct diagram in $\mathbb{T}$-Mod. Because Set is infinitely extensive and $|-| : \mathbb{T}\text{-}\mathsf{Mod} \to \mathsf{Set}$ preserves small coproducts and pullbacks (see 4.6 and 5.4), we deduce that $(t_i : |P_i| \to |Y|)_{i\in I}$ is a coproduct diagram in Set. Now let $(h_i : P_i \to Z)_{i\in I}$ be a small family of $\mathbb{T}$-model morphisms. Then there is a unique function $h : |Y| \to |Z|$ satisfying $h \circ t_i = h_i$ for each $i \in I$, so we just have to show that h is a Π-morphism $Y \to Z$. So let $R \in \Pi$ of arity $n \geq 1$ and suppose that $Y \models Ry_1 \dots y_n$. Since $f : Y \to \coprod_i X_i$ is a Π-morphism, we obtain $\coprod_i X_i \models Rf(y_1)\dots f(y_n)$. Then by 5.4, we deduce that there are some $i \in I$ and some $x_1, \dots, x_n \in |X_i|$ such that $s_i(x_k) = f(y_k)$ for each $1 \leq k \leq n$ and $X_i \models Rx_1 \dots x_n$. For each $1 \leq k \leq n$ we then have $(x_k, y_k) \in |P_i|$, and we have $P_i \models R(x_1, y_1)\dots(x_n, y_n)$ because $X_i \models Rx_1 \dots x_n$ and $Y \models Ry_1 \dots y_n$ (see 5.3). Since $h_i : P_i \to Z$ is a Π-morphism, we then obtain $Z \models Rh_i(x_1, y_1)\dots h_i(x_n, y_n)$, i.e. $Z \models Rh(t_i(x_1, y_1))\dots h(t_i(x_n, y_n))$, i.e. $Z \models Rh(y_1)\dots h(y_n)$, as desired. This proves that small coproducts are universal in $\mathbb{T}$-Mod.

It remains to show that the coproduct $(s_i : X_i \to \coprod_i X_i)_{i\in I}$ is disjoint. So let $i, j \in I$ be distinct. Since small coproducts are disjoint in Set and $|-| : \mathbb{T}\text{-}\mathsf{Mod} \to \mathsf{Set}$ preserves pullbacks and small coproducts, we deduce that the underlying set of the pullback of s_i along s_j in $\mathbb{T}$-Mod is $\varnothing$. But there is a unique $\mathbb{T}$-model with underlying set $\varnothing$, and this is the initial object of $\mathbb{T}$-Mod (5.4). ■

Recall from [3] that a category $\mathscr{C}$ is said to be *infinitely distributive* if it has finite products and small coproducts and for each $X \in \mathsf{ob}\mathscr{C}$, the functor $X \times (-) : \mathscr{C} \to \mathscr{C}$ preserves small coproducts. The following result now follows immediately from Theorem 5.5 and (the infinitary version of) [3, Proposition 4.5]:

5.6. Corollary. *Let $\mathbb{T}$ be a relational Horn theory that satisfies Condition 5.2. Then $\mathbb{T}$-Mod is infinitely distributive.*

5.7. Example. Let $\mathbb{T}$ be a relational Horn theory. Theorem 5.5 shows that the satisfaction of Condition 5.2 is sufficient for infinite extensivity of $\mathbb{T}$-Mod, but it is not *necessary*, as the following (rather trivial) examples show.

1. Let Π be any relational signature that contains at least one relation symbol of arity ≥ 2, and suppose that $\mathbb{T}$ consists of just the axioms $\Longrightarrow Rv_1 \dots v_n$ for each $R \in \Pi$ of arity $n \geq 1$, where $v_1, \dots, v_n$ are pairwise distinct variables. Then $\mathbb{T}$ clearly violates Condition 5.2.2. A Π-structure X is a $\mathbb{T}$-model iff $R^X = |X|^n$ for each $R \in \Pi$ of arity $n \geq 1$, and hence there is a unique $\mathbb{T}$-model structure on any set (the *indiscrete* Π-structure), and moreover $\mathbb{T}\text{-}\mathsf{Mod}(X, Y) = \mathsf{Set}(|X|, |Y|)$ for any $\mathbb{T}$-models X, Y. It follows that the forgetful functor $|-| : \mathbb{T}\text{-}\mathsf{Mod} \to \mathsf{Set}$ is an isomorphism, so that $\mathbb{T}$-Mod is infinitely extensive because Set is.

2. As another (trivial) example, let Π be the empty relational signature, and suppose that $\mathbb{T}$ consists of just the axiom $\Longrightarrow x = y$ for distinct variables x, y. Then $\mathbb{T}$ again violates Condition 5.2.2, but clearly $\mathbb{T}$-Mod is equivalent to the terminal category, which is (trivially) infinitely extensive.

We are not aware of any "natural" or well-studied examples of relational Horn theories that fail to satisfy Condition 5.2, and we also do not know whether there is an alternative syntactic condition that is both sufficient *and* necessary for infinite extensivity of $\mathbb{T}$-Mod.

5.8. REMARK. Let $\mathbb{T}$ be a relational Horn theory *without equality*, and let $R \in \Pi$ be a relation symbol of arity $n \geq 1$. We define a $\mathbb{T}$-model $R_{\mathbb{T}}$ by setting $|R_{\mathbb{T}}| := \{1, \ldots, n\}$ and letting $\mathsf{E}(R_{\mathbb{T}})$ be the $\mathbb{T}$-closure of the set consisting of the single Π-edge $(R, (1, \ldots, n))$ on $\{1, \ldots, n\}$; in other words, $R_{\mathbb{T}}$ is the free $\mathbb{T}$-model on the Π-structure with underlying set $\{1, \ldots, n\}$ and the unique Π-edge $(R, (1, \ldots, n))$ (see the proof of Proposition 4.9). For any $\mathbb{T}$-model X, it follows that Π-morphisms $R_{\mathbb{T}} \to X$ are in bijective correspondence with Π-edges in $\mathsf{E}(X)$ whose first component is R. The small full subcategory $\Pi_{\mathbb{T}}$ of $\mathbb{T}$-Mod consisting of the $\mathbb{T}$-models $R_{\mathbb{T}}$ $(R \in \Pi)$ is *finally dense* in $\mathbb{T}$-Mod (see [1, Definition 10.69]), which means that for any $\mathbb{T}$-model X, there is a final sink $(h_i : X_i \to X)_{i \in I}$ with codomain X and $X_i \in \Pi_{\mathbb{T}}$ for each $i \in I$. Specifically, one takes $I := \mathsf{E}(X)$, and for $e = (R, (x_1, \ldots, x_n)) \in \mathsf{E}(X)$, one takes $X_e := R_{\mathbb{T}}$ and $h_e : R_{\mathbb{T}} \to X$ to be the Π-morphism corresponding to the Π-edge e. Then the resulting sink $(h_e : X_e \to X)_{e \in \mathsf{E}(X)}$ is final by Proposition 4.3, because $\mathsf{E}(X)$ is clearly the $\mathbb{T}$-closure of $\bigcup_{e \in \mathsf{E}(X)} h_e \cdot \mathsf{E}(X_e)$.

Now if $\mathbb{T}$ satisfies Condition 5.2, then by 5.4 it readily follows that for each $R \in \Pi$, the $\mathbb{T}$-model $R_{\mathbb{T}}$ is a *connected* object of $\mathbb{T}$-Mod, meaning that the representable functor $\mathbb{T}\text{-}\mathsf{Mod}(R_{\mathbb{T}}, -) : \mathbb{T}\text{-}\mathsf{Mod} \to \mathsf{Set}$ preserves small coproducts. We conclude that if $\mathbb{T}$ is a relational Horn theory without equality that satisfies Condition 5.2, then $\mathbb{T}$-Mod has a small finally dense subcategory of connected objects. Given that $\mathbb{T}$-Mod is topological over Set (4.4) and infinitely extensive (5.5), one may now wonder about the following conjecture:

5.9. CONJECTURE. *Let $\mathscr{C}$ be a topological category over* Set *that has a finally dense subcategory of connected objects. Then $\mathscr{C}$ is infinitely extensive.*

We thank Rory Lucyshyn-Wright for discussions that led to the posing of this conjecture.

6. Appendix

In this Appendix, we prove the claims asserted at the end of Example 3.7.3. So let $(\mathscr{V}, \leq, \otimes, \mathsf{k})$ be a commutative unital quantale, and let $\mathbb{T} := \mathbb{T}_{\mathscr{V}\text{-}\mathsf{Cat}}$. We first show that $\mathbb{T}$-Mod is concretely isomorphic to $\mathscr{V}$-Cat. We first define a concrete functor $F : \mathbb{T}\text{-}\mathsf{Mod} \to \mathscr{V}\text{-}\mathsf{Cat}$, i.e. a functor that commutes with the faithful functors to Set. So let X be a $\mathbb{T}$-model. We define a $\mathscr{V}$-category $FX = (|X|, d_X)$ by setting $d_X(x, y) := \bigvee\{v \in \mathscr{V} \mid X \models x \sim_v y\}$ for any $x, y \in |X|$. For each $x \in |X|$ we have $d_X(x, x) \geq \mathsf{k}$ because $X \models x \sim_{\mathsf{k}} x$. Now let $x, y, z \in |X|$, and let us show that $d(x, z) \geq d(x, y) \otimes d(y, z)$, i.e. that

$$\begin{aligned}\bigvee\{v \in \mathscr{V} \mid X \models x \sim_v z\} &\geq \bigvee\{v' \in \mathscr{V} \mid X \models x \sim_{v'} y\} \otimes \bigvee\{v'' \in \mathscr{V} \mid X \models y \sim_{v''} z\} \\ &= \bigvee\{v' \otimes v'' \mid v', v'' \in \mathscr{V}, X \models x \sim_{v'} y \text{ and } X \models y \sim_{v''} z\},\end{aligned}$$

where the equality holds because $\otimes$ preserves arbitrary suprema in each variable separately. For any $v', v'' \in \mathscr{V}$ such that $X \models x \sim_{v'} y$ and $X \models y \sim_{v''} z$, we have $X \models x \sim_{v' \otimes v''} z$ and hence $\bigvee\{v \in \mathscr{V} \mid X \models x \sim_v z\} \geq v' \otimes v''$, which yields the desired inequality. This proves that $FX = (|X|, d_X)$ is a well-defined $\mathscr{V}$-category. If $h : X \to Y$ is a morphism of $\mathbb{T}$-models, then the function $h : |X| \to |Y|$ is a $\mathscr{V}$-functor $h : (|X|, d_X) \to (|Y|, d_Y)$ because for any $x, y \in |X|$ we have

$$d_Y(h(x), h(y)) = \bigvee\{v \in \mathscr{V} \mid Y \models h(x) \sim_v h(y)\} \geq \bigvee\{v \in \mathscr{V} \mid X \models x \sim_v y\} = d_X(x, y),$$

because $X \models x \sim_v y$ implies $Y \models h(x) \sim_v h(y)$. So we set $F(h) := h$, and $F : \mathbb{T}\text{-}\mathsf{Mod} \to \mathscr{V}\text{-}\mathsf{Cat}$ is then clearly functorial and commutes with the faithful functors to Set.

We now define a functor $G : \mathscr{V}\text{-}\mathsf{Cat} \to \mathbb{T}\text{-}\mathsf{Mod}$. So let (X, d) be a $\mathscr{V}$-category. We define a $\Pi_{\mathscr{V}}$-structure $G(X, d)$ by setting $|G(X, d)| := X$ and, for any $v \in \mathscr{V}$ and $x, y \in X$, by setting $G(X, d) \models x \sim_v y$ iff $d(x, y) \geq v$. It is essentially immediate that $G(X, d)$ is a $\mathbb{T}$-model. Now let $h : (X, d_X) \to (Y, d_Y)$ be a $\mathscr{V}$-functor. Then the function $h : X \to Y$ is a $\Pi_{\mathscr{V}}$-morphism $h : G(X, d_X) \to G(Y, d_Y)$, because for any $v \in \mathscr{V}$ and $x, y \in X$ we have the implications

$$G(X, d_X) \models x \sim_v y \iff d_X(x, y) \geq v \implies d_Y(h(x), h(y)) \geq v$$

$$\iff G(Y, d_Y) \models h(x) \sim_v h(y).$$

So we set $G(h) := h$, and $G : \mathscr{V}\text{-}\mathsf{Cat} \to \mathbb{T}\text{-}\mathsf{Mod}$ is then clearly functorial.

We now show that F and G are mutually inverse on objects, which will complete the proof. First let X be a $\mathbb{T}$-model, and let us show that $X = GFX = G(|X|, d_X)$, i.e. that for any $v \in \mathscr{V}$ and $x, y \in |X|$ we have $X \models x \sim_v y$ iff $d_X(x, y) = \bigvee\{v' \in \mathscr{V} \mid X \models x \sim_{v'} y\} \geq v$. The forward implication is immediate. Now suppose that $w := \bigvee\{v' \in \mathscr{V} \mid X \models x \sim_{v'} y\} \geq v$, and let us show that $X \models x \sim_v y$. Since X is a $\mathbb{T}$-model, we have $X \models x \sim_w y$. Then since $w \geq v$ (and X is a $\mathbb{T}$-model), we obtain $X \models x \sim_v y$, as desired.

Now let (X, d) be a $\mathscr{V}$-category, and let us show that $(X, d) = FG(X, d) = \big(X, d_{G(X,d)}\big)$. So for all $x, y \in X$, we must show that $d(x, y) = d_{G(X,d)}(x, y)$, i.e. that

$$d(x, y) = \bigvee\{v \in \mathscr{V} \mid G(X, d) \models x \sim_v y\} = \bigvee\{v \in \mathscr{V} \mid d(x, y) \geq v\},$$

which is immediate. This completes the proof that $\mathbb{T}\text{-}\mathsf{Mod} = \mathbb{T}_{\mathscr{V}\text{-}\mathsf{Cat}}\text{-}\mathsf{Mod}$ is concretely isomorphic to $\mathscr{V}\text{-}\mathsf{Cat}$. It is clear that if X is a model of $\mathbb{T}_{\mathscr{V}\text{-}\mathsf{Cat}}$, then X is a model of $\mathbb{T}_{\mathsf{PMet}_{\mathscr{V}}}$ iff the associated $\mathscr{V}$-category FX is a pseudo-$\mathscr{V}$-metric space, whence we obtain the further concrete isomorphism $\mathbb{T}_{\mathsf{PMet}_{\mathscr{V}}}\text{-}\mathsf{Mod} \cong \mathsf{PMet}_{\mathscr{V}}$. Finally, it is easy to verify that if X is a model of $\mathbb{T}_{\mathsf{PMet}_{\mathscr{V}}}$, then X is a model of $\mathbb{T}_{\mathsf{Met}_{\mathscr{V}}}$ iff the associated pseudo-$\mathscr{V}$-metric space FX is a $\mathscr{V}$-metric space, whence we obtain the concrete isomorphism $\mathbb{T}_{\mathsf{Met}_{\mathscr{V}}}\text{-}\mathsf{Mod} \cong \mathsf{Met}_{\mathscr{V}}$.

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