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QUANTALE-ENRICHED MULTICATEGORIES VIA ACTIONS

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ABSTRACT. In this communication, motivated by a classical result that relates cocomplete quantale-enriched categories to modules over a quantale, we prove a similar result for quantale-enriched multicategories.

1. Introduction

Lawvere, in his seminal paper [26], made the important observation that fundamental mathematical structures do not only constitute the objects of a category but are themselves categories. In fact, it has been known for a long time that ordered sets can be seen as categories enriched in the two element boolean algebra; moreover, monotone maps between them are exactly enriched functors. As a leading example, Lawvere explains how metric spaces fit into his thesis by showing how they are instances of enriched categories and how results from enriched category theory are able to capture important metric constructions.

We must point out that, although it is possible to develop enriched category theory in the more general setting in which the enrichment is taken in a closed symmetric monoidal category, in many cases it is sufficient to take the enrichment in a commutative quantale V, that is to say a monoid in the monoidal category of suplattices. This leads to the notion of quantale-enriched categories which can be seen as a generalization of the notion of ordered sets where one substitutes the ordered relation with a more general relation—called enriched structure—with values in the quantale V.

Since quantale-enriched categories are a generalization of ordered sets, it is natural to ask which relations there are between the two. The very first observation is that to

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every quantale-enriched category X we can associate an ordered set, called the underlying ordered set of X; its order relation relates elements of X whose value under the enriched structure of X is greater or equal than the unit of V. This construction is part of a right adjoint functor between the category of quantale-enriched categories and the category of ordered sets. Due to the form this functor has, any hope to recovery the structure of a quantale-enriched category (X, a) from its underlying ordered set is going to be disappointed. In order to maintain such hope, we we must add some structure to the category of ordered sets; a structure that must contain the information which gets lost in the discretization procedure: the values of the enriched relation at elements of X.

The solution to this problem is to consider ordered sets equipped with a suitable action of the base quantale subject to conditions that allow us to define a copowered enriched category, where the copower becomes the action itself. This association will give us an equivalence between the category of ordered sets equipped with such an action and the category of copowered categories (see [8] for the general construction). The aforementioned equivalence restricts to an equivalence between the category of cocomplete quantale-enriched categories and the category of cocomplete ordered sets equipped with an action of the base quantale, also called the category of modules (see [30] and [39]). These last two equivalences allow us to reason about enriched categories by using order theoretic arguments. An example where this is not only useful, but it has proven to be essential, is given by the results contained in [13], where, in order to obtain the duality between metric compact Hausdorff spaces and (suitably defined) finitely cocomplete categories enriched in the unit interval $[0, 1]$, the representation of the latter as ordered sets with an action of $[0, 1]$ is essential.

In [10] D. Hofmann and G. Gutierres proved that a similar result holds also for approach spaces. Approach spaces are particular examples of (T, V)-categories (see [14]) where the monad T is specialized to the ultrafilter monad U and V is specialized to the quantale $[0, \infty]^{\mathsf{op}}$. For these categories, Clementino, Hofmann and Tholen showed how it is possible to develop many constructions that come from enriched category theory in the more general context of (T, V)-categories (see [11, 5, 12]). In particular, in [11], Hofmann showed how algebras for a Kock-Zöberlein monad, which generalizes the presheaf monad, characterize cocomplete (T, V)-categories. By using the machinery of (T, V)-categories, D. Hofmann and G. Gutierres proved that separated (i.e. T_0) cocomplete approach spaces are equivalent to continuous lattices (cocomplete topological spaces in the $(U, 2)$ setting) equipped with an action of the quantale $[0, \infty]^{\mathsf{op}}$.

The aim of this paper is to prove that a similar result holds also for quantale-enriched multicategories. We also notice that quantale-enriched multicategories, from now on called (L, V)-categories, are particular examples of (T, V)-categories where the monad T is specialized to the list monad L. We prove that the category $\mathbf{CoCts}((L, V)\text{-}\mathbf{Cat}_{\mathsf{sep}})$ of cocomplete separated (L, V)-categories is equivalent to the category of quantales (ordered cocomplete multicategories) equipped with a suitable action of V and denoted $V\text{-}\mathbf{Mod}(\mathbf{Quant})$.

We must point out that, although approach spaces and quantale-enriched multicategories are both examples of (T, V)-categories, the strategy used to prove the main result of this paper bears little relationship to the one used in [10]; while the latter relies on a careful study of weighted $(U, [0, \infty]^{\mathsf{op}})$-colimits, the former essentially relies on the fact that we can internalize the notion of monoid in every monoidal category. The deep reason why approach spaces and quantale-enriched multicategories behave differently is an interesting open question the author wants to investigate; the hope is to provide a more general theory of "actions" for (T, V)-categories.

The structure of the paper is as follows:

- In the first section we introduce some background material on V-categories. We briefly sketch the equivalence between the category of cocomplete quantale-enriched categories and the category of modules:

$$\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}}) \simeq V\text{-}\mathbf{Mod}.$$

- The second section contains the first step towards our desired result. We analyze further the equivalence $\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}}) \simeq V\text{-}\mathbf{Mod}$. First we prove that both categories can be equipped with a monoidal structure, then we prove that the aforementioned equivalence extends to the corresponding categories of monoids:

$$\mathbf{Mon}(V\text{-}\mathbf{Mod}, \otimes_V, V) \simeq \mathbf{Mon}(\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}}), \otimes_V, V).$$

- In the third section we introduce (L, V)-categories. We show how many constructions that come from enriched category theory can be developed in the more general context of (L, V)-categories.

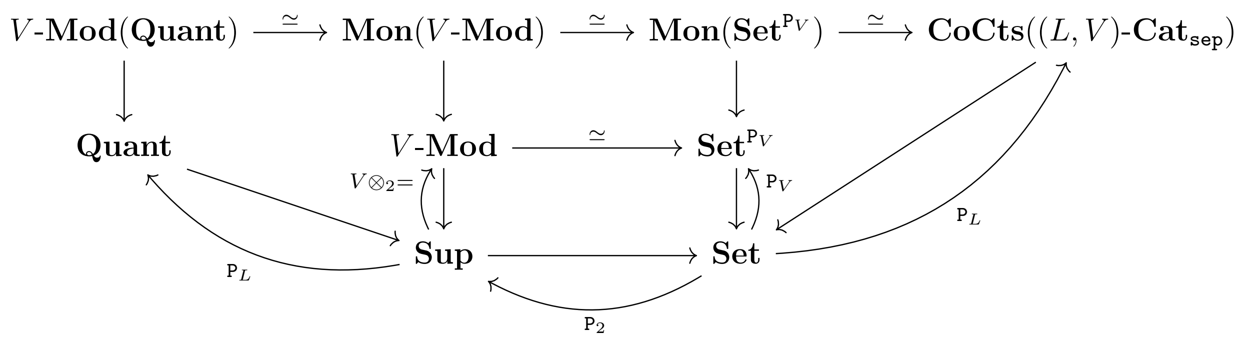
- In the fourth section we study further the category $\mathbf{Mon}(V\text{-}\mathbf{Mod}, \otimes_V, V)$. We prove that $\mathbf{Mon}(V\text{-}\mathbf{Mod}, \otimes_V, V)$ is equivalent to a particular subcategory of $V \downarrow \mathbf{Quant}$.

- In the fifth section we study further the category $\mathbf{Mon}(\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}}), \otimes_V, V)$. We prove that it is monadic over **Set** and that it is equivalent to the category of cocomplete separated (L, V)-categories, $\mathbf{CoCts}((L, V)\text{-}\mathbf{Cat}_{\mathsf{sep}})$.

- In the last section we collect everything together and prove that $\mathbf{CoCts}((L, V)\text{-}\mathbf{Cat}_{\mathsf{sep}})$ is equivalent to $V\text{-}\mathbf{Mod}(\mathbf{Quant})$, the category of quantales equipped with a suitable action of V.

- In the appendix we recall some useful materials from [16] about commutative monads we use across the paper.

  Here is a commutative diagram that summarizes all the categories involved; all the bended arrows are monadic adjunctions.



2. Preliminaries on Quantale-Enriched Categories

In this section we recall/introduce some basic notions of V-categories. Our point of view is slightly different from the more "standard" one contained in [19], it is more "relational": following [2, 6], we introduce the *quantaloid* of V-relations and we define V-categories starting from there. This might be seen as an overkill, but it will be clear in the section related to (L, V)-categories how this approach allows us to smoothly introduce some concepts also in the (L, V)-case.

2.1. V-Categories and V-Functors.

2.2. Definition. *A quantale $(V, \otimes, k)$ is a complete lattice endowed with a multiplication $\otimes : V \times V \to V$ that preserves suprema in each variable and for which $k \in V$ is the neutral element. If $k \neq \perp$, we call V non-trivial.*

2.3. Remark. *When we talk about quantale-enriched categories we always assume our base quantale V to be commutative.*

2.4. Remark. *In this paper we assume—unless explicitly stated—quantales in which we take the enrichment to be non-trivial.*

2.5. Remark. *By the adjoint functor theorem applied to ordered sets, it follows that $-\otimes =$ admits a right adjoint (in each variable) denoted by $[-, =]$ and called "internal hom".*

2.6. Examples.

1. *The two-element boolean algebra $\mathbf{2} = \{0, 1\}$ with $\wedge$ as multiplication and $\Rightarrow$ as internal hom is a quantale.*
2. *More generally, every frame becomes a quantale with the multiplication given by $\wedge$. In this case we have $k = \top$, where $\top$ is the top element of the frame.*
3. *$[0, \infty]^{\mathrm{op}}$ (with the opposite of the natural order) with $+$ as multiplication is a quantale. The internal hom is given by "truncated minus" defined as*

$$[u, v] = v \ominus u = \mathtt{max}(v - u, 0).$$

As we stated in the introduction of this section, we are going to present V-categories from a more "relational" point of view—which is a special case of the more general construction considered in [2]. The first step is to define the *quantaloid of V-relations* which is the enriched generalization of the category **Rel** of (ordinary) binary relations. For an account on quantaloids we refer to [40] for a brief overview and to [38] for a more in depth description.

The quantaloid V-**Rel** is the order-enriched category whose objects are sets, and an arrow $r : X \nrightarrow Y$ is given by a function

$$r : X \times Y \to V.$$

The composition of $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$ is given by "matrix multiplication" and it is defined pointwise as

$$s \circ r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

The identity arrow $Id : X \nrightarrow X$ is

$$Id(x_1, x_2) = \begin{cases} k & \text{if } x_1 = x_2, \\ \bot & \text{if } x_1 \neq x_2. \end{cases}$$

The complete order on V-**Rel**(X, Y) is the one induced (pointwise) by V, i.e.

$$r \leq r' \text{ in } V\text{-}\mathbf{Rel}(X, Y) \text{ whenever } r(x, y) \leq r'(x, y) \text{ in } V \text{ for all } x, y \in X, Y. \tag{1}$$

2.7. Remark. *When $V = \mathbf{2}$, $\mathbf{2}$-**Rel** is the quantaloid of relations, and the "matrix multiplication" defined previously becomes the "classical" relational composition.*

We also have an involution $(-)^\circ : V\text{-}\mathbf{Rel}^{\mathrm{op}} \to V\text{-}\mathbf{Rel}$ defined as $r^\circ(y, x) = r(x, y)$, which satisfies

$$(1_X)^\circ = 1_X, \quad (s \circ r)^\circ = r^\circ \circ s^\circ, \quad (r^\circ)^\circ = r.$$

2.8. Definition. *A V-category is a pair (X, a), where X is a set and $a : X \nrightarrow X$ is a V-relation that satisfies*

- $Id \leq a$;
- $a \circ a \leq a$.

2.9. Remark. *In this paper, when the V-structure is clear from the context, we will denote a V-category (X, a) simply as X.*

2.10. Definition. *Let (X, a) and (Y, b) be V-categories. A V-functor $f : (X, a) \to (Y, b)$ is a function between the underlying sets such that*

$$a \leq f^\circ \circ b \circ f,$$

which, in pointwise terms, means that, for all $x, y \in X$,

$$a(x, y) \leq b(f(x), f(y)).$$

If the equality holds, we call f fully faithful.

2.11. Definition. *A V-category (X, a) is called separated (see [15]) whenever $f \simeq g$ implies $f = g$, for all V-functors of the form $f, g : (Y, b) \to (X, a)$.*

2.12. Examples.

1. *For $V = \mathbf{2}$, a $\mathbf{2}$-category is an ordered set and a $\mathbf{2}$-functor is a monotone map. The order relation of a $\mathbf{2}$-category $(X, \leq_X)$ does not need to be antisymmetric. Separated $\mathbf{2}$-categories are partially ordered sets.*

2. *Categories enriched in the quantale $[0, \infty]^{\mathrm{op}}$, as first recognized by Lawvere in [26], are generalized metric spaces and $[0, \infty]^{\mathrm{op}}$-functors between them are non-expansive maps.*

3. *The quantale V defines a V-category with the V-structure given by its internal hom $[-, =]$.*

4. *By using the involution $(-)^\circ : V\text{-}\mathbf{Rel}^{\mathrm{op}} \to V\text{-}\mathbf{Rel}$, for every V-category (X, a), one can define its opposite category $X^{\mathrm{op}} = (X, a^\circ)$.*

5. *([19, Section 2.2]) Let (X, a) and (Y, b) be V-categories. We define the V-category formed by all V-functors $f : (X, a) \to (Y, b)$, denoted by $([X, Y], [X, Y](-, =))$, with the following V-structure:*

$$[X, Y](f, g) = \bigwedge_{x \in X} b(f(x), g(x)).$$

In particular we have two very important V-categories:

$$\mathbb{D}(X) = [X^{\mathrm{op}}, V], \text{ the category of presheaves,}$$

$$\mathbb{U}(X) = [X, V]^{\mathrm{op}}, \text{ the category of co-presheaves.}$$

These two categories are generalizations (for a general V) of the classical down(up)-closed subsets construction that corresponds to the case in which $V = \mathbf{2}$.

6. *([19, Section 2.4]) Given a V-category (X, a), there are two V-functors, called the Yoneda embedding and the co-Yoneda embedding:*

$$\mathbf{y}_X : X \to \mathbb{D}(X), \quad x \mapsto a(-, x),$$

$$\lambda_X : X \to \mathbb{U}(X), \quad x \mapsto a(x, =).$$

Moreover, one can prove that

$$\mathbb{U}(X)[\lambda_X(x), g] = g(x), \quad \mathbb{D}(X)[\mathbf{y}_X(x), g] = g(x).$$

The last two results are known as the co-Yoneda lemma and Yoneda lemma, respectively. For a general X, $\mathbf{y}_X$ and λ_X are not injective functions; they are injective iff X is separated (see [15, Proposition 1.5]).

7. *([19, Section 1.4]) Let (X, a) and (Y, b) be V-categories. We define their tensor product*
$$X \otimes Y = (X \times Y, a \otimes b),$$
where, for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, $a \otimes b(x_1 \otimes y_1, x_2 \otimes y_2) = a(x_1, x_2) \otimes b(y_1, y_2)$. In particular, one has: $X \otimes K \simeq X$ where K denotes the one-point V-category $(1, k)$.

For $V = \mathbf{2}$, the ordered structure on $X \otimes Y$ is the product order. This means that $(x_1, y_1) \leq_{X \otimes Y} (x_2, y_2)$ if and only if $x_1 \leq_X x_2$ and $y_1 \leq_Y y_2$.

For $V = [0, \infty]^{\mathsf{op}}$, the metric structure on $X \otimes Y$ is the taxicab metric, which is defined as:
$$d_{X \otimes Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

In this way we define V-**Cat** as the category whose objects are V-categories and whose arrows are V-functors. Moreover, V-**Cat** becomes an order-enriched category, if we define, for V-functors $f, g : (X, a) \to (Y, b)$,
$$f \leq g \text{ whenever } k \leq \bigwedge_{x \in X} b(f(x), g(x)).$$
With the tensor product previously defined, V-**Cat** becomes a symmetric closed monoidal category (see [19, Section 2.3]), since one can show that, for V-categories $(X, a), (Y, b), (Z, c)$, one has
$$V\text{-}\mathbf{Cat}(X \otimes Y, Z) \simeq V\text{-}\mathbf{Cat}(X, [Y, Z]) \simeq V\text{-}\mathbf{Cat}(Y, [X, Z]).$$
This allows us to define monoids with respect to such product, which we call monoidal V-categories.

2.13. Definition. *A monoidal V-category $(X, a, *, u_X)$ is a V-category (X, a) equipped with two V-functors: $* : X \otimes X \to X$ and $u_X : K \to X$, such that $(X, a, *, u_X)$ is a monoid (with respect to the monoidal structure $(\otimes, K)$).*

2.14. Remark. *Since V-**Cat**—with the product described before—is a symmetric closed monoidal category, to give a V-functor of the form $* : X \otimes X \to X$ is equivalent to give a set of V-functors, for all $x \in X$, of the form $- * x : X \to X$ (or equivalently of the form $x * =: X \to X$).*

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2.16. Definition. *Let $(X, a, *_X, u_X)$ and $(Y, b, *_Y, u_Y)$ be monoidal V-categories. A (strong) monoidal functor $f : (X, a, *_X, u_X) \to (Y, b, *_Y, u_Y)$ is a V-functor, such that, the following diagrams commute:*

$$\begin{array}{ccc} X \otimes X & \xrightarrow{f \otimes f} & Y \otimes Y \\ {\scriptstyle *_X}\downarrow & & \downarrow{\scriptstyle *_Y} \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{u_X} & X \\ & {\scriptstyle u_Y}\searrow & \downarrow{\scriptstyle f} \\ & & Y. \end{array}$$

That is to say, f is a (strict) monoid morphisms.

As before, we can define the category of monoidal enriched categories and denote it by $\mathbf{Mon}(V\text{-}\mathbf{Cat}, \otimes_V, V)$. With the 2-structure inherited from V-$\mathbf{Cat}$ (introduced in 2.1), $\mathbf{Mon}(V\text{-}\mathbf{Cat}, \otimes_V, V)$ becomes an order-enriched category.

2.17. Remark. *For $V = \mathbf{2}$, a monoidal ordered set is just an ordered monoid. That is to say it is a monoid endowed with an order relation which is compatible with the monoid structure.*
For $V = [0, \infty]^{\mathrm{op}}$, a monoid in $[0, \infty]^{\mathrm{op}}$-$\mathbf{Cat}$ is a metric space endowed with a monoid structure on its underlying set which is compatible with the metric.
Examples of monoidal metric spaces are the underlying additive groups of normed vector spaces. Let $(X, +, \underline{0}, ||\text{-}||)$ be a normed vector space. Let $\tilde{x}$ be an element of X, then for all $z, w \in X$, we have ([33], or any other book on functional analysis)

$$d(z + \tilde{x}, w + \tilde{x}) = ||z + \tilde{x} - w - \tilde{x}|| = d(z, w).$$

By Remark 2.15 it follows that $(X, d, +, \underline{0})$ is a monoidal $[0, \infty]^{\mathrm{op}}$-category.

2.18. The Presheaf Monad. The presheaf construction is part of a monad defined on V-$\mathbf{Cat}$ (see [41]). The underlying 2-functor is defined as

$$\mathbb{D}(-) : V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}, \quad f : X \to Y \mapsto \mathbb{D}(f) := - \circ f^* : \mathbb{D}(X) \to \mathbb{D}(Y),$$

where $f^* : Y \nrightarrow X$, $f^*(y, x) = b(x, f(x))$. The unit at a V-category X, is the Yoneda embedding

$$\mathbf{y}_X : X \to \mathbb{D}(X);$$

while the multiplication is given by

$$- \circ (\mathbf{y}_X)_* : \mathbb{D}(X)^2 \to \mathbb{D}(X),$$

where $(\mathbf{y}_X)_* : X \nrightarrow \mathbb{D}(X)$, $(\mathbf{y}_X)_*(x, g) = g(x)$.

2.19. Remark. *In the definition of the presheaf monad we implicitly used distributors (also called profunctors in the literature), see [1]. We could have explicitly written down the pointwise definition of* $\mathbb{D}(f)$ *and of* $-\circ(\mathbf{y}_X)_*$ *without having to define* f^* *and* $(\mathbf{y}_X)_*$. *The reason why we did not will become clear when we will introduce the presheaf monad for* (L,V)*-categories.*

From the Yoneda lemma it follows that the monad $(\mathbb{D}(-),\mathbf{y}_-,-\circ(\mathbf{y}_-)_*)$ is of *Kock–Zöberlein* type (see [22]).
The 2-category of pseudo-algebras for this monad is 2-equivalent to the 2-category formed by cocomplete V-categories and cocontinuous V-functors among them with the 2-structure inherited by the one on V-**Cat** (see 2.1), and denoted by **CoCts**(V-**Cat**). These last two observations, combined together, allow us to give a characterization of cocomplete V-categories.

2.20. Theorem. *[41, Proposition 5.2] Let* (X,a) *be a* V*-category. The following are equivalent:*

- (X,a) *is a cocomplete* V*-category;*
- *There exists a* V*-functor*
$$\mathtt{Sup}_X:\mathbb{D}(X)\to X,$$
such that, for every $x\in X$, $\mathtt{Sup}_X(\mathbf{y}_X(x))\simeq x$.

2.21. Remark. *Since* $(\mathbb{D}(-),\mathbf{y}_-,-\circ(\mathbf{y}_-)_*)$ *is of Kock-Zöberlein type,* $\mathtt{Sup}_X$ *(whenever it exists) is automatically the left adjoint to the Yoneda functor.*

Separated cocomplete V-categories are strict algebras for the presheaf monad

$$(\mathbb{D}(-),\mathbf{y}_-,-\circ(\mathbf{y}_-)_*).$$

2.22. Theorem. *[41, Proposition 5.2] Let* (X,a) *be a* V*-category. The following are equivalent:*

- (X,a) *is a separated cocomplete* V*-category;*
- *There exists a* V*-functor*
$$\mathtt{Sup}_X:\mathbb{D}(X)\to X,$$
such that, for every $x\in X$, $\mathtt{Sup}_X(\mathbf{y}_X(x))=x$.

Every set X can be endowed with the discrete V-structure given by

$$d_X(x_1,x_2)=\begin{cases}\bot & \text{if } x_1\neq x_2,\\ k & \text{if } x_1=x_2.\end{cases}$$

In this way we obtain a functor $d : \mathbf{Set} \to V\text{-}\mathbf{Cat}_{\mathsf{sep}}$, where the latter is the full subcategory of V-**Cat** formed by separated V-categories. Since presheaf categories are always separated, we can compose it with

$$\mathbb{D}(-) : V\text{-}\mathbf{Cat}_{\mathsf{sep}} \to \mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}}).$$

In this way we get a functor which is left adjoint to the forgetful functor

$$\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}}) \to \mathbf{Set}.$$

We have the well-known result (see [17, 30, 39]).

2.23. Theorem. *The forgetful functor* $G : \mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}}) \to \mathbf{Set}$ *is monadic.*

If we study the monad which arises from the adjunction, we see that the resulting monad is the V-powerset monad $(\mathtt{P}_V, u, n)$, the enriched generalization of the classical powerset monad, where $\mathtt{P}_V : \mathbf{Set} \to \mathbf{Set}$ is defined by $\mathtt{P}_V(X) = V^X$ and, for $f : X \to Y$ and $\phi \in V^X$

$$\mathtt{P}_V(f)(\phi)(y) = \bigvee_{x \in f^{-1}(y)} \phi(x);$$

and:

- $u_X : X \to V^X$ is the transpose of the diagonal $\triangle_X : X \times X \to V$;
- $n_X : \mathtt{P}_V(\mathtt{P}_V(X)) \to \mathtt{P}_V(X)$ is defined by $n_X(\Phi)(x) = \bigvee_{\phi \in V^X} \Phi(\phi) \otimes \phi(x)$.

In this way we have the equivalence (see *ibid.*)

$$\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}}) \simeq \mathbf{Set}^{\mathtt{P}_V},$$

which can be explicitly described as the one that sends the cocomplete V-category (X, a) to the algebra (X, α), where $\alpha(\psi) = \mathtt{Sup}_{\mathtt{X}}(\psi \circ a)$.

2.24. Remark. *The equivalence* $\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}}) \simeq \mathbf{Set}^{\mathtt{P}_V}$ *generalizes the well known equivalence*

$$\mathbf{Sup} \simeq \mathbf{Set}^{\mathtt{P}_\mathbf{2}},$$

where $\mathtt{P}_\mathbf{2}$ *is the powerset monad and* **Sup** *is the category of suplattices with suprema preserving maps.*

2.25. Remark. *This is a particular case of a more general result that holds for categories enriched in a quantaloid, see [32] for details.*

2.26. Enrichment via Actions. To every V-category (X, a) we can associate an ordered set $(X, \leq_a)$, where the order is defined as

$$x \leq_a y \iff k \leq a(x, y).$$

We call $(X, \leq_a)$ the underlying ordered set of the V-category (X, a). This defines a 2-functor

$$V\text{-}\mathbf{Cat} \to \mathbf{Ord}, \quad f : (X, a) \to (Y, b) \mapsto f : (X, \leq_a) \to (Y, \leq_b).$$

2.27. Remark. *The underlying ordered set of the V-category $(V, [-,=])$ is $(V, \leq)$, the underlying partially ordered set of the quantale V.*

2.28. Remark. *The 2-functor V-**Cat** $\to$ **Ord** restricts to a 2-functor*

$$V\text{-}\mathbf{Cat}_{\mathsf{sep}} \to \mathbf{Ord}_{\mathsf{sep}}.$$

Moreover, it is easy to see that $\mathbf{Ord}_{\mathsf{sep}} \simeq \mathbf{Pos}$, where the latter is the 2-category of partially ordered sets and monotone maps.

2.29. Remark. *The arguments we are going to use in this paper rely—mainly—on the monadicity over* **Set** *of certain categories. For this reason we restrict ourself to consider only separated categories. Take* **CoCts**(V-**Cat**)*, the category of cocomplete enriched categories with cocontinuous functors.* **CoCts**(V-**Cat**) *is not separated and, morover, it is not complete; thus it can not be monadic over* **Set***.*

In order to show that **CoCts**(V-**Cat**) *is not complete, consider the V-categories $(\{\bullet\}, \top)$ and $(\{\bullet_1, \bullet_2\}, \top)$; then the equalizer of the two evident maps*

$$(\{\bullet\}, \top) \rightrightarrows (\{\bullet_1, \bullet_2\}, \top)$$

is empty which is not complete.

2.30. Definition. *A V-category (X, a) is copowered if, for all $x \in X$, $a(x, =) : X \to V$ admits a left adjoint in V-**Cat** denoted by $- \odot x : V \to X$. That is to say*

$$a(u \odot x, y) = [u, a(x, y)],$$

for all $x, y \in X$ and $u \in V$. We say that a V-functor $f : (X, a) \to (Y, b)$ between copowered V-categories preserves copowers if, for all $x \in X$ and $u \in V$, $f(u \odot x) \simeq u \odot f(x)$.

In this way we can form the 2-category of copowered categories with copowers preserving V-functors among them, denoted as V-$\mathbf{Cat}^{\odot}$. In the same way, if we consider only separated V-categories, we obtain the category V-$\mathbf{Cat}^{\odot}_{\mathsf{sep}}$.

If we start with a separated copowered category (X, a) and we take its underlying ordered set, then $- \odot x : V \to X$ becomes a monotone map of the type

$$- \odot x : (V, \leq) \to (X, \leq_a).$$

Moreover, we have the following lemma.

2.31. Lemma. *Under the same hypothesis as above, the monotone map*

$$- \odot x : (V, \leq) \to (X, \leq_a),$$

enjoys the following properties, for all $x \in X$, $u, v \in V$:

- $k \odot x = x$;
- $v \odot (u \odot x) = (v \otimes u) \odot x$;
- $(\bigvee_i u_i) \odot x = \bigvee_i (u_i \odot x)$*, for every set $\{u_i,\ i \in I\}$ of elements of V, where either side existing if the other does.*

PROOF. First observe that $k \odot x = x$ follows from $[k, w] = w$.
Fix an $x \in X$. Then, for all $y \in X$, we have

$$\begin{aligned} a(v \odot (u \odot x), y) &= [v, a(u \odot x, y)] \\ &= [v, [u, a(x, y)]] \\ &= [v \otimes u, a(x, y)] \\ &= a((v \otimes u) \odot x, y), \end{aligned}$$

from which $v \odot (u \odot x) = (v \otimes u) \odot x$ follows.
Finally, the last property follows from the adjunction $- \odot x \dashv a(x, =)$.

■

2.32. REMARK. *Let X, Y be ordered sets. Then $X \otimes Y \simeq X \times Y$, that is to say the monoidal structure $\otimes$ in the category* **Ord** *coincides with the cartesian product $\times$.*

2.33. DEFINITION. *Let $\mathbf{Pos}_\vee^V$ be the category described as follows. An object of $\mathbf{Pos}_\vee^V$ is a poset $(X, \leq_X)$ equipped with a monotone map*

$$\rho : V \otimes X \to X,$$

such that, for all $x \in X$, $u, v \in V$:

- $\rho(k, x) = x$;
- $\rho(v, \rho(u, x)) = \rho(v \otimes u, x)$;
- $\rho(\bigvee_i u_i, x) = \bigvee_i \rho(u_i, x)$, *for every set $\{u_i,\ i \in I\}$ of elements of V, where either side existing if the other does.*

An arrow $f : (X, \leq_X, \rho) \to (Y, \leq_Y, \theta)$ in $\mathbf{Pos}_\vee^V$ is a monotone map between the underlying ordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$, such that the following diagram commutes

$$\begin{array}{ccc} V \otimes X & \xrightarrow{Id \otimes f} & V \otimes Y \\ \rho \downarrow & & \downarrow \theta \\ X & \xrightarrow{\quad f \quad} & Y. \end{array}$$

2.34. REMARK. *The underlying ordered set of the base quantale V, $(V, \leq)$, acts on itself via the multiplication $\otimes : V \times V \to V$. Moreover, since $\otimes$ preserves suprema, we also have $(\bigvee_i u_i) \otimes v = \bigvee_i (u_i \otimes v)$.*

2.35. PROPOSITION. *There exists a 2-functor $V\text{-}\mathbf{Cat}^{\odot}_{\mathsf{sep}} \to \mathbf{Pos}_\vee^V$ that associates to a copowered V-category (X, a) its underlying ordered sets $(X, \leq_a)$ with the action given by $-\odot =: V \otimes X \to V$.*

2.36. Remark. *From the adjoint functor theorem, it follows that*

$$\rho(\bigvee_i u_i, x) = \bigvee_i \rho(u_i, x)$$

is equivalent to the statement: $\rho_x = \rho(-, x) : V \to X$ *has a right adjoint for all* $x \in X$*. In particular, when we apply this to* $(V, \leq)$*, we get as a right adjoint the internal hom* $[x, =]$*. This crucial observation will allow us to define a* V*-structure starting from the action and it is a particularization to the quantale-enriched case of a the more general construction made for closed bicategories and described in [8] which we are going to briefly present.*

Let $(X, \leq_X, \rho)$ be an object of $\mathbf{Pos}^V_\vee$. By Remark 2.36, for all $x \in X$, there exists a monotone map $a(x, =) : X \to V$ which is right adjoint to $\rho_x : V \to X$. Thus we have, for all $x, y \in X$ and $v \in V$,

$$\rho(v, x) \leq y \iff v \leq a(x, y).$$

In this way we can define a V-relation $a : X \nrightarrow X$. As one might expect, this relation defines a V-structure on X; moreover, with such structure, (X, a) becomes a copowered category, with copowers given by ρ_x for all $x \in X$. In this way we can define a 2-functor

$$\mathbf{Pos}^V_\vee \to V\text{-}\mathbf{Cat}^\odot_{\mathsf{sep}}.$$

2.37. Theorem. *[8, Theorem 3.7] The two* 2*-functors*

$$V\text{-}\mathbf{Cat}^\odot_{\mathsf{sep}} \rightleftarrows \mathbf{Pos}^V_\vee$$

establish a 2*-equivalence between* $V\text{-}\mathbf{Cat}^\odot_{\mathsf{sep}}$ *and* $\mathbf{Pos}^V_\vee$*.*

Remember that a sufficient and necessary condition for a V-category (X, a) to be cocomplete is to be copowered and to have all conical suprema (see [19]). In the light of this result, it is natural to ask if we can restrict ♠ to a 2-equivalence of the form:

$$\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}}) \simeq \mathbf{CoCts}(\mathbf{Pos})^V_?,$$

where ? reflects the *a priori* unknown property (or properties) that we have to add in order to obtain an equivalence.

Before we dip further into our quest, let us spend a few words about $\mathbf{CoCts}(\mathbf{Pos})$. We know that

$$\mathbf{CoCts}(\mathbf{Pos}) \simeq \mathbf{Sup} \simeq \mathbf{Set}^{\mathsf{P}_\mathbf{2}}.$$

Here $\mathbf{Sup}$ denotes the 2-category of suplattices with suprema preserving monotone maps among them, while $\mathbf{Set}^{\mathsf{P}_\mathbf{2}}$ is the Eilenberg-Moore category for the powerset monad $\mathsf{P}_\mathbf{2}$. Since $\mathsf{P}_\mathbf{2}$ is a *commutative monad*, $\mathbf{Sup}$ becomes a closed symmetric monoidal category (see Appendix 8 or [17] for a more direct construction) $(\mathbf{Sup}, \otimes_\mathbf{2}, \mathbf{2})$ with the monoidal structure that classifies bimorphisms. Here a bimorphism in $\mathbf{Sup}$ is a monotone map of type $f : X \otimes Y \to Z$ such that f preserves suprema separately in both variables and where $\otimes$ is the tensor product we defined in Example 7 of Examples 2.12. We have the following lemma.

2.38. Lemma. *[39] Let (X, a) be a cocomplete separated V-category. Then its underlying ordered set $(X, \leq_a)$ is cocomplete. Moreover, if $f : (X, a) \to (Y, b)$ is a cocontinuous V-functor between cocomplete V-categories, then $f : (X, \leq_a) \to (Y, \leq_b)$ preserves suprema.*

In the light of what we wrote before, and because of the properties of arrows in $\mathbf{Pos}^V_\vee$, the copower of a cocomplete separated V-category (X, a) extends to a unique suprema preserving map

$$(V, \leq) \otimes_{\mathbf{2}} (X, \leq_a) \to (X, \leq_a).$$

This shows that we have a 2-functor

$$\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}}) \to V\text{-}\mathbf{Mod},$$

where the latter is the category whose objects are suplattices $(X, \leq_X)$ endowed with an action $\rho : V \otimes_{\mathbf{2}} X \to X$ and whose arrows are suprema preserving equivariant monotone maps.

As one might expect, we obtain the analogue of Theorem 2.37.

2.39. Theorem. *[39] The 2-equivalence*

$$V\text{-}\mathbf{Cat}^{\odot}_{\mathsf{sep}} \simeq \mathbf{Pos}^V_\vee$$

restricts to a 2-equivalence

$$\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}}) \simeq V\text{-}\mathbf{Mod}.$$

3. Refining V-$\mathbf{Mod} \simeq \mathbf{Set}^{\mathsf{P}_V}$

In Section 2.26 we recalled the equivalence

$$V\text{-}\mathbf{Mod} \simeq \mathbf{Set}^{\mathsf{P}_V}.$$

In this section, first we prove that both categories can be equipped with a monoidal structure, and then we prove that the aforementioned equivalence extends to the corresponding categories of monoids.

In Proposition 8.7 we recalled that $\mathbf{Set}^{\mathsf{P}_V}$ admits a closed symmetric monoidal structure $(\otimes_{\mathsf{P}_V}, V)$ for which the free functor

$$\mathsf{P}_V : \mathbf{Set} \to \mathbf{Set}^{\mathsf{P}_V}$$

becomes strong monoidal. This monoidal structure comes from the fact that the V-powerset monad (P_V, u, n) is commutative—as shown in [9]. Moreover, this monoidal structure has another interesting property: it classifies bimorphisms—as shown in [20]. This means that there exists a natural isomorphism (for all (X, α), (Y, β), $(Z, \theta) \in \mathbf{Set}^{\mathsf{P}_V}$)

$$\mathbf{Bim}_{\mathbf{Set}^{\mathsf{P}_V}}(X \times Y, Z) \simeq \mathbf{Set}^{\mathsf{P}_V}(X \otimes_{\mathsf{P}_V} Y, Z),$$

where

$$\mathbf{Bim}_{\mathbf{Set}^{\mathsf{P}_V}}(X \times Y, =) : \mathbf{Set}^{\mathsf{P}_V} \to \mathbf{Set}$$

is the functor that sends an algebra (Z, θ), to the set of bimorphisms of the form $f : X \times Y \to Z$. Here a function $f : X \times Y \to Z$, is a bimorphism if, for all $x \in X$, $y \in Y$,

$$f_x : Y \to Z,\ y \mapsto f(x,y) \text{ and } f_y : X \to Z,\ x \mapsto f(x,y)$$

are morphisms in $\mathbf{Set}^{\mathsf{P}_V}$ (see Definition 8.8 and Proposition 8.10).

3.1. Remark. *When $V = \mathbf{2}$ one obtains the well known monoidal structure on suplattices. See [17] for its description.*

In [17], Joyal and Tierney defined the tensor product $X \otimes_V Y$ of V-modules $(X, \leq_X, \rho)$ and $(Y, \leq_Y, \eta)$ as the coequalizer of

$$V \otimes_{\mathbf{2}} X \otimes_{\mathbf{2}} Y \underset{\rho \otimes_{\mathbf{2}} Id}{\overset{\tau_{V,X} \otimes_{\mathbf{2}} \eta}{\rightrightarrows}} X \otimes_{\mathbf{2}} Y. \qquad \text{(where } \tau_{V,X} : V \otimes_{\mathbf{2}} X \simeq X \otimes_{\mathbf{2}} V\text{)}$$

With this tensor product one can prove that V-**Mod** becomes a symmetric closed monoidal category. Moreover, $\otimes_V$ classifies bimorphisms, where a bimorphism $f : X \times Y \to Z$ between V-modules is function such that, for all $x \in X$, $y \in Y$,

$$f_x : Y \to Z,\ y \mapsto f(x,y) \text{ and } f_y : X \to Z,\ x \mapsto f(x,y)$$

are morphisms in V-**Mod**.

Let $f : X \times Y \to Z$, be a bimorphism between V-modules. Since for all $x \in X$, $y \in Y$,

$$f_x : Y \to Z,\ y \mapsto f(x,y) \text{ and } f_y : X \to Z,\ x \mapsto f(x,y)$$

are morphisms in V-**Mod**, by applying the forgetful functor V-**Mod** $\to$ **Sup**, we get two morphisms in **Sup**. Since the monoidal structure on **Sup** classifies bimorphisms too, we get a unique morphism $\overline{f} : X \otimes_{\mathbf{2}} Y \to Z$ in **Sup** that makes the following diagram commutes

$$\begin{array}{ccc} X \times Y & \longrightarrow & X \otimes_{\mathbf{2}} Y \\ & {}_{f}\searrow & \downarrow{\scriptstyle \overline{f}} \\ & & Z. \end{array}$$

This shows that we can define a bimorphism in V-**Mod** in two equivalent ways:

- As a function $f : X \times Y \to Z$, such that f is a morphism in V-**Mod** in each variable separately;
- A suprema preserving map $\overline{f} : X \otimes_{\mathbf{2}} Y \to Z$ such that its associate arrow $f : X \times Y \to Z$, is equivariant in each variable separately.

In other words, we have the following natural bijections:

$$\mathbf{Bim}_{V\text{-}\mathbf{Mod}}(X \times Y, Z) \simeq \mathbf{Bim}_{V\text{-}\mathbf{Mod}}(X \otimes_{\mathbf{2}} Y, Z) \simeq V\text{-}\mathbf{Mod}(X \otimes_V Y, Z).$$

3.2. Remark. *It is possible to prove that the forgetful functor* $U : V\text{-}\mathbf{Mod} \to \mathbf{Sup}$ *is monadic (see [36] for a more general perspective on the problem). The resulting monad is* $T = (V \otimes_{\mathbf{2}} =, \eta, \mu)$ *and it has as unit*

$$\eta_X : X \xrightarrow{\sim} 1 \otimes_{\mathbf{2}} X \to V \otimes_{\mathbf{2}} M,$$

and as multiplication

$$\mu_X : V \otimes_{\mathbf{2}} V \otimes_{\mathbf{2}} X \xrightarrow{(-\otimes=)\otimes_{\mathbf{2}} Id} V \otimes_{\mathbf{2}} X.$$

3.3. Remark. *The monoidal structure on* V-$\mathbf{Mod}$ *we have described is obtained from the monoidal structure on* $\mathbf{Sup}$ *via the commutative monad* $T = (V \otimes_{\mathbf{2}} =, \eta, \mu)$ *we introduced in Remark 3.2, where the strength is defined as*

$$t_{X,Y} : X \otimes_{\mathbf{2}} TY \xrightarrow{\tau_{X,V} \otimes_{\mathbf{2}} Id} T(X \otimes_{\mathbf{2}} Y).$$

See [36] for more details.

3.4. Remark. *From Remark 8.5, it follows that the left unitor* $l_M : V \otimes_V X \simeq X$*, of the monoidal structure on* V-$\mathbf{Mod}$*, is the unique morphism associated to the bimorphism* $\rho : V \otimes_{\mathbf{2}} X \to X$*, where* $(X, \leq_X, \rho)$ *is in* V-$\mathbf{Mod}$*.*

3.5. Proposition. *The equivalence*

$$V\text{-}\mathbf{Mod} \simeq \mathbf{Set}^{\mathsf{P}_V}$$

extends to an equivalence between the corresponding category of monoids

$$\mathbf{Mon}(V\text{-}\mathbf{Mod}, \otimes_V, V) \simeq \mathbf{Mon}(\mathbf{Set}^{\mathsf{P}_V}, \otimes_{\mathsf{P}_V}, V).$$

Proof. Let us call $[-] : V\text{-}\mathbf{Mod} \to \mathbf{Set}^{\mathsf{P}_V}$ the functor which realizes the equivalence $V\text{-}\mathbf{Mod} \simeq \mathbf{Set}^{\mathsf{P}_V}$ which is the composite of $V\text{-}\mathbf{Mod} \simeq \mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}})$ we obtained in Theorem 2.39 with $\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}}) \simeq \mathbf{Set}^{\mathsf{P}_V}$ we described in Subsection 2.18. Let $f : X \times Y \to Z$ be a bimorphism between V-modules. Since equivalences preserve products we have $[X \times Y] \simeq [X] \times [Y]$. Moreover, because for all $x, y \in X, Y$, f_x, f_y are morphisms in V-$\mathbf{Mod}$, we have $[f_x]$ and $[f_y]$ are morphisms in $\mathbf{Set}^{\mathsf{P}_V}$. From $[X \times Y] \simeq [X] \times [Y]$ it follows that $[f_x] = [f]_x$ and $[f_y] = [f]_y$. This implies that f defines a bimorphism $f : [X] \times [Y] \to [Z]$ in $\mathbf{Set}^{\mathsf{P}_V}$. This shows that we have a bijection

$$\mathbf{Bim}_{V\text{-}\mathbf{Mod}}(X \times Y, Z) \simeq \mathbf{Bim}_{\mathbf{Set}^{\mathsf{P}_V}}([X] \times [Y], [Z])$$

which can be easily seen to be natural (since $[-]$ is a functor and all the "change of base" components are given by pre-post composition). Now, from

$$V\text{-}\mathbf{Mod}(X \otimes_V Y, Z) \simeq \mathbf{Set}^{\mathsf{P}_V}([X \otimes_V Y], [Z])$$

and from

$$\mathbf{Bim}_{V\text{-}\mathbf{Mod}}(X \times Y, Z) \simeq \mathbf{Bim}_{\mathbf{Set}^{\mathtt{P}_V}}([X] \times [Y], [Z]), \tag{2}$$

it follows that

$$\mathbf{Set}^{\mathtt{P}_V}([X] \otimes_{\mathtt{P}_V} [Y], [Z]) \simeq \mathbf{Set}^{\mathtt{P}_V}([X \otimes_V Y], [Z]),$$

from which we can deduce that

$$[X] \otimes_{\mathtt{P}_V} [Y] \simeq [X \otimes_V Y].$$

The compatibility of $[-]$ and the associators follows from the bijection (2) and from the fact that both associators derive from the associator of the cartesian product **Set**. Moreover, since $[V] \simeq V$, and the unitors are compatible with $[-]$, the result follows (see the next remarks for further details).

■

3.6. Remark. *The $\mathtt{P}_V$-algebra V has the structure given by*

$$n_1 : \mathtt{P}_V(\mathtt{P}_V(1)) \to \mathtt{P}_V(1), \quad j \mapsto \bigvee_w j(w) \otimes w,$$

while the $\mathtt{P}_V$-algebra associated to the cocomplete V-category $(V, [-, =])$ is (V, α), where

$$\alpha(j) = \mathtt{Sup}_V(\bar{j}) = \bigvee_w \bar{j}(w) \otimes w, \text{ where } \bar{j}(w) = \bigvee_v [w, v] \otimes j(v).$$

In order to show that $[V] \simeq V$, we need to prove that the two structures are the same, that is to say

$$\bigvee_w j(w) \otimes w = \bigvee_w (\bigvee_v [w, v] \otimes j(v)) \otimes w.$$

But since $[-, v] = \mathbf{y}_V(v)$, we have $\mathtt{Sup}_V(\mathbf{y}_V(v)) = \bigvee_v [w, v] \otimes v = v$. Thus the result follows.

3.7. Remark. *The (left) unitor in the monoidal category $\mathbf{Set}^{\mathtt{P}_V}$, at an object (X, α), is the corresponding morphism to the bimorphism*

$$\mathtt{P}_V(1) \times X \to X, \quad (v, x) \mapsto \alpha(v \otimes e_X(x)).$$

*While the (left) unitor in the monoidal category V-**Mod**, at an object (X, ρ), is the corresponding morphism to the bimorphism*

$$V \times X \to X, \quad (v, x) \mapsto \rho(v, x).$$

The equivalence $[-]$ sends (X, ρ) to the cocomplete V-category (X, a) that has ρ as copower. The category (X, a) is then sent to the $\mathtt{P}_V$-algebra (X, α'), where $\alpha' = \mathtt{Sup}_X(- \circ a)$. In this way, we get that the unitor

$$V \times X \to X, \quad (v, x) \mapsto \rho(v, x),$$

becomes the copower of (X, a) *which is then sent to the map*

$$\mathtt{P}_V(1) \times X \to X, \quad (v, x) \mapsto v \odot_\rho x.$$

In order to conclude, we notice that

$$\begin{aligned} \mathtt{Sup}_X(v \otimes e_X(x) \circ a) &= \mathtt{Sup}_X(v \otimes \mathbf{y}_X(x)), \\ &= v \odot_\rho x. \end{aligned}$$

4. Quantale-Enriched Multicategories

(L, V)-categories are a special case of the more general (T, V)-categories, where the list monad L is considered. They are also the order-enriched version of *multicategories* (see [27] and [14] for an account on them, and [24] for a historical perspective). The basic idea is that, instead of having arrows with just a single object as the domain, we allow them to have as domain a list of objects.

In this section we introduce (L, V)-categories and some of their basic constructions, by mirroring what we have done in the previous section.

4.1. (L, V)-Categories and (L, V)-functors. Recall that the list monad is the monad whose underlying functor $L : \mathbf{Set} \to \mathbf{Set}$ is given by

$$f : X \to Y \mapsto Lf : \amalg_{n\geq 0} X^n \to \amalg_{m \geq 0} Y^m, \quad \underline{x} = (x_1, ..., x_n) \mapsto (f(x_1), ..., f(x_n)),$$

and whose unit and multiplication at a set X are defined as:

- $e_X : X \to L(X), \quad x \mapsto (x);$
- $m_X : L^2(X) \to L(X), \quad (\underline{x}_1, ..., \underline{x}_n) \mapsto (x_{11}, ..., x_{1k}, ..., x_{n1}, ..., x_{nl}).$

4.2. Remark. *Let* $\underline{x}$*,* $\underline{w}$ *be lists. In order to avoid possible confusion with the list of lists* $\underline{\underline{y}} = (\underline{x}, \underline{w})$*, we denote the list obtained by concatenating* $\underline{x}$ *and* $\underline{w}$ *as* $(\underline{x}; \underline{w})$*. Moreover, in the case in which one of the two is a single element list, we use the shortcut* $(\underline{x}; w)$ *instead of* $(\underline{x}; (w))$*.*

We can extend (in a functorial way) the list monad L to V-**Rel** by defining, for $r : X \nrightarrow Y$:

$$\tilde{L}r : L(X) \nrightarrow L(Y), \quad (\underline{x}, \underline{y}) \mapsto \begin{cases} r(x_1, y_1) \otimes ... \otimes r(x_n, y_n) & \text{if they have the same length,} \\ \bot & \text{otherwise.} \end{cases}$$

This particular extension—which is the Barr-Extension of the list monad (see [14])—defines a monad on V-**Rel** that, moreover, preserves the involution

$$(-)^\circ : V\text{-}\mathbf{Rel}^{\mathrm{op}} \to V\text{-}\mathbf{Rel}.$$

4.3. Remark. *From now on we will use L for both the ordinary list monad and its extension to V-**Rel**.*

4.4. Remark. *It is possible to prove that the extension of the list monad we have just described, extends to a 2-monad on V-**Cat** (see [3, Theorem 4.4]). Strict algebras for this monad are monoidal V-categories, where a monoidal V-category $(X, a, *, u_X)$ is sent to (X, a, α), where*

$$\alpha : L(X) \to X, \quad \underline{x} \mapsto x_1 * ... * x_n, \quad (-) \mapsto u_X;$$

algebra morphisms correspond to strict monoidal V-functors (2.16). In this way we have a 2-equivalence $V\text{-}\mathbf{Cat}^L \simeq \mathbf{Mon}(V\text{-}\mathbf{Cat}, \otimes, V)$, see [23].

This allows us to define the order-enriched category (L, V)-**Rel** in which a morphism $r : X \,{-\!\!\circ\!\!\rightharpoonup}\, Y$ is a V-relation of the form

$$r : L(X) \nrightarrow Y,$$

and in which composition is given by

$$s \bullet r = s \circ Lr \circ m_X^\circ,$$

and where the relation $e_X^\circ : X \,{-\!\!\circ\!\!\rightharpoonup}\, X$ is the identity morphism.

4.5. Remark. *Note that, due to the Kleisli-style composition we defined, $- \bullet r$ preserves suprema, but $s \bullet (=)$ does not in general.*

4.6. Definition. *An (L, V)-category is a pair (X, a), where X is a set and $a : X \,{-\!\!\circ\!\!\rightharpoonup}\, X$ is an (L, V)-relation that satisfies:*

- $e_X^\circ \leq a$;
- $a \bullet a \leq a$.

4.7. Remark. *When $V = \mathbf{2}$, the $(L, \mathbf{2})$-structure of an $(L, \mathbf{2})$-category (X, a) is a subset $a \subseteq L(X) \times X$ such that:*

- *for all $x \in X$, $((x), x) \in a$;*
- *given $(\underline{z}_1, ..., \underline{z}_n) \in L^2(X)$, $\underline{x} \in LX$, and $y \in X$, such that*

  $$((\underline{z}_1, ..., \underline{z}_n), \underline{x}) \in La, \text{ and } (\underline{x}, y) \in a,$$

  then

  $$((\underline{z}_1; ...; \underline{z}_n), y) \in a.$$

The relation La is the subset that corresponds to the relation $La : L^2(X) \times LX \to \mathbf{2}$ that one obtains by applying the extension of the list monad to the relation $a : LX \times X \to \mathbf{2}$.

4.8. Remark. *Let* $(M, *, \leq)$ *be an ordered monoid, that is to say, a monoid endowed with an order relation that is compatible with its multiplication. We can define a* $(L, \mathbf{2})$*-structure* a_M *on* M *in the following way, for* $x_1, ..., x_n, y \in M$,

$$a_M(x_1, ..., x_n, y) \iff x_1 * ... * x_n \leq y.$$

4.9. Definition. *Let* (X, a) *and* (Y, b) *be* (L, V)*-categories. An* (L, V)*-functor* $f : (X, a) \to (Y, b)$ *is a function between the underlying sets such that*

$$a \leq f^\circ \circ b \circ Lf,$$

which, in pointwise terms, means that, for all $\underline{x} \in LX$, $y \in X$,

$$a(\underline{x}, y) \leq b(Lf(\underline{x}), f(y)).$$

If the equality holds, we call f *fully faithful.*

4.10. Remark. *If* $V = \mathbf{2}$*, then an* $(L, \mathbf{2})$*-functor* $f : (X, a) \to (Y, b)$ *satisfies, for all* $\underline{x} \in LX$, $y \in X$,

$$(\underline{x}, y) \in a \text{ implies } (Lf(\underline{x}), f(y)) \in b.$$

Notice how this generalizes the classical monotonicity condition.

4.11. Remark. *A map* $f : (M, *_M, \leq_M) \to (N, *_N, \leq_N)$ *between ordered monoids is called submultiplicative (see [25]) if, for all* $x_1, ..., x_n, x \in M$,

$$x_1, ..., x_n \leq_M x \text{ implies } f(x_1), ..., f(x_n) \leq_N f(x).$$

With the $(L, \mathbf{2})$*-structures* a_M, a_N *defined in Remark 4.8,* $f : (M, a_M) \to (N, a_N)$ *becomes an* $(L, \mathbf{2})$*-functor.*

In this way we define (L, V)-**Cat** as the category whose objects are (L, V)-categories and whose arrows are (L, V)-functors, moreover, (L, V)-**Cat** becomes an order-enriched category if we define, for two (L, V)-functors $f, g : (X, a) \to (Y, b)$,

$$f \leq g \text{ whenever } k \leq \bigwedge_{x \in X} b(Lf((x)), g(x)).$$

4.12. Examples.

1. *Every set* X *defines an* (L, V)*-category with* e_X° *as* (L, V)*-structure. In particular, we define the one-point* (L, V)*-category* $E = (1, e_1^\circ)$.
2. *Every set* X *defines an* (L, V)*-category if we consider the free* L*-algebra on* X*,* (LX, m_X).
3. V *itself defines an* (L, V)*-category where* $[\underline{v}, w] = [v_1 \otimes ... \otimes v_n, w]$.

4. *Let (X, a) and (Y, b) be (L, V)-categories. We can form their tensor product $X \otimes Y = (X \times Y, a \otimes b)$, where*

$$a \otimes b(\gamma, (x, y)) = a(L\pi_1(\gamma), x) \otimes b(L\pi_2(\gamma), y).$$

Here $\gamma \in L(X \times Y)$ and π_1, π_2 are the obvious projections. Unluckily, in general it is not true that $X \otimes E \simeq X$.

5. *Quantum B-algebras [34] and their enriched counterpart [29] are examples of order-enriched multicategories and of quantale-enriched multicategories, respectively.*

6. *More generally, promonoidal categories enriched in a quantale are examples of quantale-enriched multicategories, see [29, Proposition 6.8] for a direct proof, or for a more general overview on the subject [27].*

In [3] it is shown that there exists a 2-functor

$$\texttt{Kmp} : V\text{-}\mathbf{Cat}^L \to (L, V)\text{-}\mathbf{Cat},$$

that sends an algebra (X, a, β) to the (L, V)-category $(X, \hat{a})$, where $\hat{a} = a \circ \beta$. This 2-functor has a left adjoint

$$M : (L, V)\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}^L,$$

that sends an (L, V)-category (X, a) to the V-$\mathbf{Cat}^L$ algebras $(LX, La \circ m_X^\circ, m_X)$ and an (L, V)-functor f to Lf [3, Theorem 5.4]. Using the aforementioned adjunction, we can extend the monad L to a Kock-Zöberlein monad on (L, V)-$\mathbf{Cat}$, denoted by $\hat{L}$, [3, Theorem 6.1]) . Moreover, one can prove (see [3, Theorem 7.2]) that there is a 2-equivalence

$$V\text{-}\mathbf{Cat}^L \simeq (L, V)\text{-}\mathbf{Cat}^{\hat{L}},$$

where the latter is the 2-category of pseudo-algebras for the monad $\hat{L}$, whose objects are called *representable* (L, V)-categories.

The 2-equivalence V-$\mathbf{Cat}^L \simeq (L, V)$-$\mathbf{Cat}^{\hat{L}}$, allows us to generalize Remarks 4.8, 4.11 for an arbitrary quantale V. In Remark 4.4 we mentioned that V-$\mathbf{Cat}^L$ is 2-equivalent to $\mathbf{Mon}(V\text{-}\mathbf{Cat}, \otimes, V)$, where a monoidal V-category $(X, a, *, u_X)$ is sent to the algebra (X, a, α). The 2-functor `Kmp` sends this algebra to the (L, V)-category that has the same underlying set and whose (L, V)-structure is given by $\hat{a} = a \circ \alpha$. This means that, for $\underline{x} \in LX$ and $y \in X$,

$$a(\underline{x}, y) = a(x_1 * ... * x_n, y),$$

which, when we consider $V = \mathbf{2}$, coincides with the structure we defined in Remark 4.8.

4.13. Remark. *As we showed in the examples, quantale-enriched multicategories provide a common roof for several categories studied in the literature. In particular [29] shows how they allow to unify several results that appeared in the literature regarding injective hulls, such as for example [25, 35].*

4.14. Remark. *A priori, due to the non-symmetric form of arrows in* (L,V)-**Rel**, *it is not clear how to define an* (L,V)*-category that seems to play the role of a dual. Luckily, we can use the adjunction* $\mathtt{Kmp} \dashv M$ *and the involution in* V-**Rel** *to define, for an* (L,V)*-category* (X,a)*, its opposite category as* $X^{\mathsf{op}} = (LX, m_X \circ La^\circ \circ m_X)$*. At first this might be seen as an ad hoc definition, but if we apply this construction to a* V*-category* (X,a)*, seen as an* (L,V)*-category* $(LX, e_X^\circ \circ a)$*, we get*

$$X^{\mathsf{op}} = \mathtt{Kmp}(LX, La^\circ),$$

where (LX, La°) *is the dual, as a* V*-category, of* (LX, La)*. See [4, 5, 3, 11] for further details.*

For any (L,V)-category (X,a) we can form the (L,V)-category $\mathbb{D}_L(X)[-,=]$ whose underlying set consists of all (L,V)-functors of the form: $f : X^{\mathsf{op}} \otimes E \to V$, where $\otimes$ was defined in Example 4, and whose (L,V)-structure is given by

$$\mathbb{D}_L(X)[\underline{f}, g] = \bigwedge_{(\underline{x}_1,\ldots,\underline{x}_n)\in LX^2} [(f_1(\underline{x}_1), \ldots, f_n(\underline{x}_n)), g(m_X((\underline{x}_1, \ldots, \underline{x}_n))))],$$

where $\underline{f} \in L(\mathbb{D}_L(X))$ and $g \in \mathbb{D}_L(X)$.

4.15. Remark. *We have a fully faithful functor, called the Yoneda embedding,*

$$\mathbf{y}_X : X \to \mathbb{D}_L(X), \quad x \mapsto a(-,x).$$

Moreover, it can be proved (see [4]) that

$$\mathbb{D}_L(X)[L\mathbf{y}_X(\underline{x}), g] = g(\underline{x}).$$

The last result is known as the Yoneda Lemma.

4.16. The Presheaf Monad. As in the V-case, the presheaf construction for quantale-enriched multicategories is part of a monad defined on (L,V)-**Cat** (see [11]). The underlying 2-functor of this monad is

$$\mathbb{D}_L(-) : (L,V)\text{-}\mathbf{Cat} \to (L,V)\text{-}\mathbf{Cat}, \quad f : X \to Y \mapsto \mathbb{D}_L(f) := - \bullet f^{\circledast} : \mathbb{D}_L(X) \to \mathbb{D}_L(Y),$$

where $f^{\circledast} : Y \,{-\!\!\circ\!\!\rightharpoonup}\, X$, $f^{\circledast}(\underline{y}, x) = b(\underline{y}, f(x))$. The unit at X is the Yoneda embedding,

$$\mathbf{y}_X : X \to \mathbb{D}_L(X);$$

while its multiplication is given by

$$- \bullet (\mathbf{y}_X)_{\circledast} : \mathbb{D}_L(X)^2 \to \mathbb{D}_L(X),$$

where $(\mathbf{y}_X)_{\circledast} : X \,{-\!\!\circ\!\!\rightharpoonup}\, \mathbb{D}_L(X)$, $(\mathbf{y}_X)_{\circledast}(\underline{x}, g)) = g(\underline{x})$. In particular, as in the V-case, from the Yoneda lemma it follows that $(\mathbb{D}_L(-), \mathbf{y}_X, - \bullet (\mathbf{y}_X)_{\circledast})$ is of Kock–Zöberlein type.

Similarly to what happens in the V-case, one can prove that (pseudo)-algebras for the monad $\mathbb{D}_L$ are exactly cocomplete categories (see *ibid.* for more details). Moreover, since $(\mathbb{D}_L(-), \mathbf{y}_-, - \bullet (\mathbf{y}_-)_{\circledast})$ is of Kock–Zöberlein type, we have the analogue of Theorem 2.20 (see *ibid.*).

4.17. THEOREM. *Let (X, a) be an (L, V)-category. The following are equivalent:*

- *(X, a) is a cocomplete (L, V)-category;*
- *There exists an (L, V)-functor*

$$\mathtt{Sup}_X : \mathbb{D}_L(X) \to X,$$

such that, for every $x \in X$, $\mathtt{Sup}_X(\mathbf{y}_X(x)) \simeq x$;

- *(X, a) is pseudo-injective with respect to fully faithful (L, V)-functors. That is to say, for every (L, V)-functor $f : (Y, b) \to (X, a)$ and for every fully faithful (L, V)-functor $i : (Y, b) \to (Z, c)$, there exists an extension $f' : (Z, c) \to (X, a)$ such that $f' \cdot i \simeq f$.*

4.18. MONADICITY OVER **Set**. As in the V-case, we restrict ourself to consider only separated (L, V)-categories. An (L, V)-category (X, a) is called separated (see [15]) whenever $f \simeq g$ implies $f = g$, for all (L, V)-functors of the form $f, g : (Y, b) \to (X, a)$. We have the analogue of Theorem 2.22.

4.19. THEOREM. *Let (X, a) be an (L, V)-category. The following are equivalent:*

- *(X, a) is a separated cocomplete (L, V)-category;*
- *There exists an (L, V)-functor*

$$\mathtt{Sup}_X : \mathbb{D}_L(X) \to X,$$

such that, for every $x \in X$, $\mathtt{Sup}_X(\mathbf{y}_X(x)) = x$;

- *(X, a) is injective with respect to fully faithful (L, V)-functors. That is to say, for every (L, V)-functor $f : (Y, b) \to (X, a)$ and for every fully faithful (L, V)-functor $i : (Y, b) \to (Z, c)$, there exists an extension $f' : (Z, c) \to (X, a)$ such that $f' \cdot i = f$.*

One can prove that the forgetful functor (L, V)-**Cat** $\to$ **Set** has a left adjoint given by

$$d : \mathbf{Set} \to (L, V)\text{-}\mathbf{Cat},\ X \mapsto (X, e_X^\circ).$$

Thus, the forgetful functor

$$\mathbf{CoCts}((L, V)\text{-}\mathbf{Cat}_{\mathsf{sep}}) \to \mathbf{Set},$$

where $\mathbf{CoCts}((L, V)\text{-}\mathbf{Cat}_{\mathsf{sep}})$ denotes the (2-)category formed by cocomplete separated (L, V)-category and cocontinuous (L, V)-functors among them, has a left adjoint which is given by the composite

$$\mathbf{Set} \xrightarrow{d} (L, V)\text{-}\mathbf{Cat}_{\mathsf{sep}} \xrightarrow{\mathbb{D}_L(-)} \mathbf{CoCts}((L, V)\text{-}\mathbf{Cat}_{\mathsf{sep}}).$$

As in the V-case we have (see Theorem 2.23 of [11]) that this functor is monadic.

4.20. THEOREM. *The forgetful functor* $G : \mathbf{CoCts}((L,V)\text{-}\mathbf{Cat}_{\mathtt{sep}}) \to \mathbf{Set}$ *is monadic.*

Define $\mathtt{P}_L = \mathbb{D}_L \cdot d$. Then the monad which arises from the previous theorem is the monad $(\mathtt{P}_L, e, n)$ whose unit and multiplication, at a set X, are given by

$$e_X : X \to \mathtt{P}_L(X),\ x \mapsto \mathbf{y}_X(x) = e_X^\circ(-, x),$$

$$n_X = - \bullet (\mathbf{y}_X)_\circledast : \mathtt{P}_L\mathtt{P}_L(X) \to \mathtt{P}_L(X).$$

In Section 6 we will study better this monad and its algebras.

5. First Interlude: Algebras and Modules

In this section we study further the category $\mathbf{Mon}(V\text{-}\mathbf{Mod}, \otimes_V, V)$. In commutative algebra it is well known that monoids in the category of modules over a commutative ring R are (associative and unital) R-algebras. In our case the quantale V plays the role of the base ring R, thus one might expect that a similar result holds also for $\mathbf{Mon}(V\text{-}\mathbf{Mod}, \otimes_V, V)$. The answer is positive but it requires us to restrict our attention to a particular subcategory of $V \downarrow \mathbf{Quant}$.

5.1. DEFINITION. *We define* $(V \downarrow \mathbf{Quant})_\spadesuit$ *to be the full subcategory of the coslice category* $V \downarrow \mathbf{Quant}$ *whose objects are morphisms of quantales* $f : V \to Q$ *such that, for all* $v \in V$, $u \in Q$, $f(v) *_Q u = u *_Q f(v)$.

5.2. REMARK. *At the time the author was writing this article he was not aware that the following result was already proven in [7] as pointed out by the referee. I decided to keep the original proof in order to be as self-contained as possible.*

5.3. PROPOSITION. *There is an equivalence of categories:*

$$\mathbf{Mon}(V\text{-}\mathbf{Mod}, \otimes_V, V) \simeq (V \downarrow \mathbf{Quant})_\spadesuit.$$

PROOF. Since the monoidal structure on V-**Mod** is the one induced by a commutative monad, as we explained in Remark 3.3, by Theorem 8.3, it follows that the functor

$$V \otimes_2 =: \mathbf{Sup} \to V\text{-}\mathbf{Mod}, \qquad X \mapsto V \otimes_2 X,$$

is strong monoidal. This implies, by doctrinal adjunction [18], that the forgetful functor

$$U : V\text{-}\mathbf{Mod} \to \mathbf{Sup}$$

is lax monoidal. Let X and Y be V-modules. Then the laxator

$$\pi : X \otimes_2 Y \to X \otimes_V Y,$$

is the universal bimorphism that "defines" $\otimes_V$, while

$$2 \to V$$

is the canonical inclusion. Moreover, since U is lax monoidal, it sends monoids in V-**Mod** to monoids in **Sup**.

Let $e : V \to X$, $m : X \otimes_V X \to X$, $\alpha : V \otimes_2 X \to X$ be an object of $\mathbf{Mon}(V\text{-}\mathbf{Mod}, \otimes_V, V)$. Then $X = (X, m \cdot \pi, e \cdot k)$ is a monoid in **Sup**, thus a quantale. In order to have an object in $(V \downarrow \mathbf{Quant})_\spadesuit$, we have to show that $e : V \to X$ is a quantale homomorphism and that it satisfies the condition $e(v) *_X x = x *_X e(v)$ (where $*_X$ is a shortcut for $m \cdot \pi$, the multiplication of X seen as a quantale).

The first condition follows from the fact that in every monoidal category $(C, \otimes, 1)$ the unit 1 is a monoid and, for every monoid (M, m, e) in C, $e : 1 \to M$ is a monoid homomorphism. The second condition is the pointwise expression of the image under U of the unit axioms for X

$$\begin{array}{ccccc} V \otimes_V X & \xrightarrow{e \otimes_V Id} & X & \xleftarrow{Id \otimes_V e} & X \otimes_V V \\ & \searrow & \downarrow{\scriptstyle m} & \swarrow & \\ & & X. & & \end{array}$$

Thus we have a functor

$$F : \mathbf{Mon}(V\text{-}\mathbf{Mod}, \otimes_V, V) \to (V \downarrow \mathbf{Quant})_\spadesuit.$$

This functor is clearly faithful, and with a little effort it is possible to show that F is also full. Let us prove that F is essentially surjective. Let $f : V \to Q$, $*_Q : Q \otimes_{\mathbf{2}} Q \to Q$, $e : \mathbf{2} \to Q$ be and object of $(V \downarrow \mathbf{Quant})_\spadesuit$. We have that

$$V \otimes_{\mathbf{2}} Q \xrightarrow{f \otimes_{\mathbf{2}} Id} Q \otimes_{\mathbf{2}} Q \xrightarrow{*_Q} Q$$

defines an action, call it ρ. The compatibility of ρ with the unit follows from the unital condition for the multiplication of Q, while the associativity condition follows from the fact that f is a quantale homomorphism and from the associativity of $*_Q$.

We can prove that $*_Q : Q \otimes_{\mathbf{2}} Q \to Q$ coequalizes the fork that defines $Q \otimes_V Q$, hence that there is a unique $\overline{*}_Q : Q \otimes_V Q \to Q$ that makes the following diagram commute

$$\begin{array}{ccc} Q \otimes_{\mathbf{2}} Q & \xrightarrow{*_Q} & Q. \\ {\scriptstyle \pi}\downarrow & \nearrow{\scriptstyle \overline{*}_Q} & \\ Q \otimes_V Q & & \end{array}$$

Let us prove this statement. In the fork that defines $Q \otimes_V Q$ the two arrows are defined as follows

$$V \otimes_{\mathbf{2}} Q \otimes_{\mathbf{2}} Q \xrightarrow{\rho \otimes_{\mathbf{2}} Id} V \otimes_{\mathbf{2}} Q$$

and

$$V \otimes_{\mathbf{2}} Q \otimes_{\mathbf{2}} Q \xrightarrow{\tau \otimes_{\mathbf{2}} Id} Q \otimes_{\mathbf{2}} V \otimes_{\mathbf{2}} Q \xrightarrow{Id \otimes_{\mathbf{2}} \rho} Q \otimes_{\mathbf{2}} Q.$$

But, since $\rho = *_Q \cdot (f \otimes_{\mathbf{2}} Id)$, and since the condition $f(v) *_Q u = u *_Q f(v)$ means

$$\begin{array}{ccccc} Q \otimes_{\mathbf{2}} Q & \xrightarrow{*_Q} & Q & \xleftarrow{*_Q} & Q \otimes_{\mathbf{2}} Q \\ {\scriptstyle f\otimes_{\mathbf{2}} Id}\uparrow & & & & \uparrow{\scriptstyle Id\otimes_{\mathbf{2}} f} \\ V \otimes_{\mathbf{2}} Q & & \xrightarrow{\tau_{V,Q}} & & Q \otimes_{\mathbf{2}} V \end{array}$$

by using $(Id \otimes_{\mathbf{2}} f) \otimes_{\mathbf{2}} Id = Id \otimes_{\mathbf{2}} (f \otimes_{\mathbf{2}} Id)$ and $*_Q \cdot (Id \otimes_{\mathbf{2}} *_Q) = *_Q \cdot (*_Q \otimes_{\mathbf{2}} Id)$, we have

$$\begin{aligned} *_Q \cdot (Id \otimes_{\mathbf{2}} \rho) \cdot (\tau \otimes_{\mathbf{2}} Id) &= *_Q \cdot (Id \otimes_{\mathbf{2}} *_Q) \cdot (Id \otimes_{\mathbf{2}} (f \otimes_{\mathbf{2}} Id)) \cdot (\tau \otimes_{\mathbf{2}} Id) \\ &= *_Q \cdot (*_Q \otimes_{\mathbf{2}} Id) \cdot ((Id \otimes_{\mathbf{2}} f) \otimes_{\mathbf{2}} Id) \cdot (\tau \otimes_{\mathbf{2}} Id) \\ &= *_Q \cdot ((*_Q \cdot (Id \otimes_{\mathbf{2}} f) \cdot \tau) \otimes_{\mathbf{2}} Id) \\ &= *_Q \cdot ((*_Q \cdot (f \otimes_{\mathbf{2}} Id)) \otimes_{\mathbf{2}} Id) \\ &= *_Q \cdot (\rho \otimes_{\mathbf{2}} Id). \end{aligned}$$

The associativity of $\overline{*}_Q$ follows from the associativity of $*_Q$, while the unit condition follows from f being a morphism of quantales.

From the fact that f is a morphism of quantales, and since the action on V its is multiplication, it follows that $f : V \to Q$ is equivariant. This ends the proof of the proposition.

■

5.4. Remark. *Note that in the case in which* $V = \mathbf{2}$, $(V \downarrow \mathbf{Quant})_\spadesuit \simeq \mathbf{Quant}$. *Every morphism of quantales* $f : \mathbf{2} \to Q$ *satisfies* $f(v) \cdot u = u \cdot f(v)$ *and:*

$$\mathbf{2} \downarrow \mathbf{Quant} \simeq \mathbf{Quant}.$$

6. Second Interlude: Injectives and Monoids

The enriched powerset monad P_V is commutative. This means (see Proposition 8.7) that the free functor

$$\mathsf{P}_V : \mathbf{Set} \to \mathbf{Set}^{\mathsf{P}_V}$$

is strong monoidal. Thus P_V extends to a functor, which we denote again as P_V, between the corresponding categories of monoids:

$$\mathsf{P}_V : \mathbf{Mon} \to \mathbf{Mon}(\mathbf{Set}^{\mathsf{P}_V}, \otimes_{\mathsf{P}_V}, V).$$

Since P_V is left adjoint to the forgetful functor

$$\mathbf{Set}^{\mathsf{P}_V} \to \mathbf{Set},$$

by doctrinal adjunction [18], we can conclude that

$$\mathsf{P}_V : \mathbf{Mon} \to \mathbf{Mon}(\mathbf{Set}^{\mathsf{P}_V}, \otimes_{\mathsf{P}_V}, V),$$

has a right adjoint. Hence

$$W : \mathbf{Mon}(\mathbf{Set}^{\mathtt{P}_V}, \otimes_{\mathtt{P}_V}, V) \to \mathbf{Set},$$

which is the functor obtained by composing the right adjoint of $\mathtt{P}_V$ with the forgetful functor $\mathbf{Mon} \to \mathbf{Set}$, it is the right adjoint of

$$\mathtt{P}_V L : \mathbf{Set} \to \mathbf{Mon}(\mathbf{Set}^{\mathtt{P}_V}, \otimes_{\mathtt{P}_V}, V).$$

We want to show that $W : \mathbf{Mon}(\mathbf{Set}^{\mathtt{P}_V}, \otimes_{\mathtt{P}_V}, V) \to \mathbf{Set}$ is monadic.

In order to do so, we use the strategy deployed in [31]. We introduce a category in which $\mathbf{Mon}(\mathbf{Set}^{\mathtt{P}_V}, \otimes_{\mathtt{P}_V}, V)$ is closed with respect to epimorphisms. Then we prove that such category satisfies the "hard" part of Beck's monadicity Theorem, namely that is equipped with a functor to $\mathbf{Set}$ that creates coequalizers of split pairs.

6.1. Definition. *$\mathbf{Alg}(T_+)$ is the category of algebras for the endofunctor*

$$T_+ : \mathbf{Set}^{\mathtt{P}_V} \to \mathbf{Set}^{\mathtt{P}_V}, \quad X \mapsto (X \otimes_{\mathtt{P}_V} X) \amalg V.$$

That is to say, the the category whose objects are of the form $m : (X \otimes_{\mathtt{P}_V} X) \amalg V \to X$ (here m is an arrow in $\mathbf{Set}^{\mathtt{P}_V}$) and whose arrows $f : (X, m) \to (Y, n)$ are those in $\mathbf{Set}^{\mathtt{P}_V}$ that make the following diagram commute

$$\begin{array}{ccc} (X \otimes_{\mathtt{P}_V} X) \amalg V & \xrightarrow{(f \otimes_{\mathtt{P}_V} f) \amalg Id} & (Y \otimes_{\mathtt{P}_V} Y) \amalg V \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} \\ X & \xrightarrow{\quad f \quad} & Y. \end{array}$$

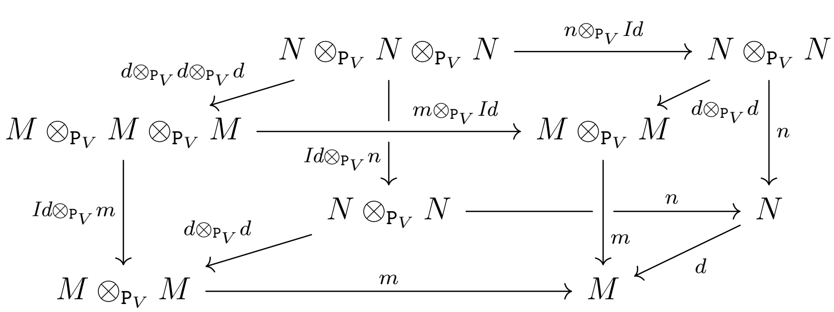
6.2. Remark. *Because W just forgets the structure, we can immediately define*

$$\overline{W} : \mathbf{Alg}(T_+) \to \mathbf{Set},$$

again as the forgetful functor; it is clear that its restriction to $\mathbf{Mon}(\mathbf{Set}^{\mathtt{P}_V}, \otimes_{\mathtt{P}_V}, V)$ is W.

6.3. Lemma. *$\mathbf{Mon}(\mathbf{Set}^{\mathtt{P}_V}, \otimes_{\mathtt{P}_V}, V)$ is closed in $\mathbf{Alg}(T_+)$ with respect to epimorphisms.*

Proof. (Based on [31], Proposition 2.6)
Let (N, n, e_n) be an object of $\mathbf{Mon}(\mathbf{Set}^{\mathtt{P}_V}, \otimes_{\mathtt{P}_V}, V)$ and let $d : (N, n, e_n) \to (M, m, e_m)$ be an epimorphism in $\mathbf{Alg}(T_+)$.
From diagram chasing over



we get

$$m \cdot (m \otimes_{\mathtt{P}_V} Id) \cdot (d \otimes_{\mathtt{P}_V} d \otimes_{\mathtt{P}_V} d) = m \cdot (Id \otimes_{\mathtt{P}_V} m) \cdot (d \otimes_{\mathtt{P}_V} d \otimes_{\mathtt{P}_V} d),$$

which implies, since $d \otimes_{\mathtt{P}_V} d \otimes_{\mathtt{P}_V} d$ is an epimorphism (being $\otimes_{\mathtt{P}_V}$ a closed structure), that

$$m \cdot (m \otimes_{\mathtt{P}_V} Id) = m \cdot (Id \otimes_{\mathtt{P}_V} m).$$

In a similar way one proves the corresponding equation for the unit from which it follows that (M, m, e_m) is an object of $\mathbf{Mon}(\mathbf{Set}^{\mathtt{P}_V}, \otimes_{\mathtt{P}_V}, V)$.

■

6.4. Lemma. *The functor* $\overline{W} : \mathbf{Alg}(T_+) \to \mathbf{Set}$ *creates coequalizers of* $\overline{W}$*-split pairs.*

Proof. Let

$$R \rightrightarrows X$$

be a $\overline{W}$-split pair. Let

$$R \underset{t}{\overset{}{\rightrightarrows}} X \xrightarrow[s]{\pi} Q,$$

be its (splitting) coequalizer in **Set**.
Since the monoidal structure in $\mathbf{Set}^{\mathtt{P}_V}$ classifies bimorphisms, we have that, associated to the "monoid structure without equations" of R and X: $\tilde{r} : R \otimes_{\mathtt{P}_V} R \to R$ and $\tilde{m} : X \otimes_{\mathtt{P}_V} X \to X$, there exist two unique bimorphisms $r : R \times R \to R$ and $m : X \times X \to X$. Because π is an epimorphism and $\times$ is closed, it follows that $\pi \times \pi$ is an epimorphism too, hence

$$R \times R \underset{t \times t}{\rightrightarrows} X \times X \xrightarrow[s \times s]{\pi \times \pi} Q \times Q$$

is again a split coequalizer in **Set**. Moreover, since we obtained $\overline{W}$ first by forgetting the "free monoid structure" and then by forgetting the $\mathtt{P}_V$-structure, and since $\mathbf{Set}^{\mathtt{P}_V}$ is monadic over **Set**, it follows that there exists a unique $\mathtt{P}_V$-structure on Q such that π becomes a $\mathtt{P}_V$-algebra morphism.

Since

$$R \rightrightarrows X$$

are both T_+-morphisms, by using the split and the universal property of coequalizers we get a unique function $n : Q \times Q \to Q$, as displayed in the following diagram

$$\begin{array}{ccccc} R \times R & \rightrightarrows & X \times X & \longrightarrow & Q \times Q \\ \downarrow & & \downarrow & & \downarrow n \\ R & \rightrightarrows & X & \longrightarrow & Q. \end{array} \tag{3}$$

Since $\pi \cdot m \cdot (\pi \times \pi)$ and $\pi \cdot m$ are both bimorphisms in $\mathbf{Set}^{\mathtt{P}_V}$, it follows that n is a bimorphism too. Indeed, fix $q \in Q$, then—in **Set**—we have that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Q \\ {\scriptstyle \langle x, Id\rangle}\downarrow & & \downarrow{\scriptstyle \langle q, Id\rangle} \\ X \times X & \longrightarrow & Q \times Q \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow{\pi} & Q \end{array}$$

commutes, where $\pi(x) = q$ and $\langle q, \mathrm{Id}\rangle(w) = (q, w)$. Because π is and epimorphism and

$$\pi \cdot m \cdot \langle x, \mathrm{Id}\rangle : X \to Q$$

is a $\mathtt{P}_V$-algebra morphism, it follows that $n \cdot \langle q, \mathrm{Id}\rangle$ is a $\mathtt{P}_V$-algebra morphism too. We can do the same for $\langle \mathrm{Id}, q\rangle$, thus showing that n is a bimorphism. This shows that there exists a unique arrow $\tilde{n} : Q \otimes_{\mathtt{P}_V} Q \to Q$, in $\mathbf{Set}^{\mathtt{P}_V}$ that makes the diagram

$$\begin{array}{ccc} Q \times Q & \xrightarrow{n} & Q \\ \downarrow & \nearrow{\scriptstyle \tilde{n}} & \\ Q \otimes_{\mathtt{P}_V} Q & & \end{array}$$

commute.

In order to have an object of $\mathbf{Alg}(T_+)$, we also need a $\mathtt{P}_V$-algebra morphism

$$V \to Q.$$

Since X is in $\mathbf{Alg}(T_+)$, we have $I_X : V \to X$; by taking the composite $I_Q := \pi \cdot I_X : V \to X \to Q$ we get our desired arrow. In order to show that $\pi : X \to Q$ is in $\mathbf{Alg}(T_+)$, we appeal to the following diagram

$$\begin{array}{ccccc} (R \otimes_{\mathtt{P}_V} R) \amalg V & \rightrightarrows & (X \otimes_{\mathtt{P}_V} X) \amalg V & \longrightarrow & (Q \otimes_{\mathtt{P}_V} Q) \amalg V \\ {\scriptstyle \tilde{r} \amalg I_R}\downarrow & & {\scriptstyle \tilde{m} \amalg I_X}\downarrow & & \downarrow{\scriptstyle \tilde{n} \amalg I_Q} \\ R & \rightrightarrows & X & \longrightarrow & Q \end{array}$$

whose commutativity follows from the universal property of bimorphisms and from Diagram 3.

In order to conclude our proof we are left to show that Q is the coequalizer of

$$R \rightrightarrows X$$

in $\mathbf{Alg}(T_+)$.

Since we have already noticed that Q is a coequalizer in $\mathbf{Set}^{\mathtt{P}_V}$, for every (appropriate) arrow $g : X \to E$ in $\mathbf{Alg}(T_+)$, we get a unique $\mathtt{P}_V$-algebra morphism such that the following diagram

$$\begin{array}{ccccc} R & \rightrightarrows & X & \xrightarrow{\pi} & Q \\ & & {\scriptstyle g}\downarrow & \swarrow{\scriptstyle f} & \\ & & E & & \end{array}$$

commutes. In this way we get a unique morphism $f \otimes_{\mathsf{P}_V} f : Q \otimes_{\mathsf{P}_V} Q \to E \otimes_{\mathsf{P}_V} E$. What we want to show is that $f : E \to Q$ is a morphism in $\mathbf{Alg}(T_+)$, thus that the following diagram commutes, where the vertical arrows are the multiplication on Q and E respectively,

$$\begin{array}{ccc} Q \otimes_{\mathsf{P}_V} Q & \xrightarrow{f \otimes_{\mathsf{P}_V} f} & E \otimes_{\mathsf{P}_V} E \\ {\scriptstyle \tilde{n}}\downarrow & & \downarrow{\scriptstyle \tilde{e}} \\ Q & \xrightarrow{\quad f \quad} & E \end{array}$$

which would then imply the commutativity of

$$\begin{array}{ccc} (Q \otimes_{\mathsf{P}_V} Q) \amalg V & \xrightarrow{(f \otimes_{\mathsf{P}_V} f) \amalg Id} & (E \otimes_{\mathsf{P}_V} E) \amalg V \\ \downarrow & & \downarrow \\ Q & \xrightarrow{\quad f \amalg Id \quad} & E. \end{array}$$

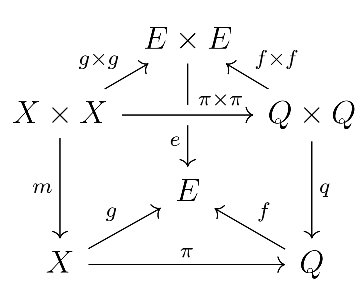
Since

$$R \times R \rightrightarrows X \times X \longrightarrow Q \times Q$$

is a coequalizer in **Set**, it follows that the following diagram commutes

$$\begin{array}{ccc} Q \times Q & \xrightarrow{f \times f} & E \times E \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle e} \\ Q & \xrightarrow{\quad f \quad} & E \end{array} \tag{4}$$

were $e : E \times E \to E$ is the unique bimorphism associates to $\tilde{e} : E \otimes_{\mathsf{P}_V} E \to E$. Indeed, from the following commutative diagram



it follows that

$$e \cdot (f \times f) \cdot (\pi \times \pi) = f \cdot q \cdot (\pi \times \pi),$$

which implies that

$$e \cdot (f \times f) = f \cdot q,$$

since $\pi \times \pi$ is an epimorphism.

This allows us to conclude. Indeed, from the commutativity of Diagram 4, by using the universal property of bimorphisms, we can deduce the commutativity of the following diagram

$$\begin{array}{ccc} Q \otimes_{\mathsf{P}_V} Q & \xrightarrow{f \otimes_{\mathsf{P}_V} f} & E \otimes_{\mathsf{P}_V} E \\ {\scriptstyle \tilde{n}}\downarrow & & \downarrow{\scriptstyle \tilde{e}} \\ Q & \xrightarrow{\quad f \quad} & E \end{array}$$

as required. This ends the proof of the lemma.

∎

6.5. Proposition. *The functor* $W : \mathbf{Mon}(\mathbf{Set}^{\mathsf{P}_V}, \otimes_{\mathsf{P}_V}, V) \to \mathbf{Set}$ *is monadic.*

Proof. Let

$$R \rightrightarrows X$$

be a W-split pair in $\mathbf{Mon}(\mathbf{Set}^{\mathsf{P}_V}, \otimes_{\mathsf{P}_V}, V)$. By Lemma 6.4 we know that there exists the coequalizer of following diagram in $\mathbf{Alg}(T_+)$

$$R \rightrightarrows X \xrightarrow{e} Q.$$

Since e is an epimorphism in $\mathbf{Alg}(T_+)$, by Lemma 6.3, it follows that Q is an object of $\mathbf{Mon}(\mathbf{Set}^{\mathsf{P}_V}, \otimes_{\mathsf{P}_V}, V)$. Moreover, since $\mathbf{Mon}(\mathbf{Set}^{\mathsf{P}_V}, \otimes_{\mathsf{P}_V}, V) \to \mathbf{Alg}(T_+)$ is fully faithful it follows that Q is the coequalizer of

$$R \rightrightarrows X.$$

Now, from $\overline{W}|_{\mathbf{Mon}(\mathbf{Set}^{\mathsf{P}_V}, \otimes_{\mathsf{P}_V}, V)} = W$, it also follows that Q is preserved by W. The fact that W reflects isomorphisms is straightforward.

∎

The last proposition shows that objects of $\mathbf{Mon}(\mathbf{Set}^{\mathsf{P}_V}, \otimes_{\mathsf{P}_V}, V)$ are algebras for the monad induced by the adjunction $\mathsf{P}_V L \dashv W$. In order to study better this monad, we use the following remark.

6.6. Remark. *Suppose that we have two adjunctions*

$$\mathbf{C} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathbf{D} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{E} \quad (F' \dashv G',\ F \dashv G)$$

with units and counits given by

$$\eta' : Id \Rightarrow G'F', \quad \eta : Id \Rightarrow GF, \quad \epsilon' : F'G' \Rightarrow Id, \quad \epsilon : FG \Rightarrow Id.$$

Then $FF' \dashv G'G$, with unit and counit given by

$$\overline{\eta} = G'\eta_{F'}\eta', \quad \overline{\epsilon} = \epsilon F(\epsilon').$$

If we apply Remark 6.6 to the adjunction

$$\mathsf{P}_V \dashv U : \mathbf{Mon} \rightleftarrows \mathbf{Mon}(\mathbf{Set}^{\mathsf{P}_V}, \otimes_{\mathsf{P}_V}, V)$$

whose unit and counit, at a monoid $(M, \cdot, 1_M)$ and at an object

$$(Q, \alpha, *_Q, k_Q) \text{ of } \mathbf{Mon}(\mathbf{Set}^{\mathsf{P}_V}, \otimes_{\mathsf{P}_V}, V),$$

are

$$\eta_M : M \xrightarrow{\ulcorner \Delta_M \urcorner} \mathsf{P}_V(M),$$

(where $\ulcorner \Delta_M \urcorner$ is the transpose of the diagonal $\Delta_M : M \times M \to V$) and

$$\epsilon_Q : \mathsf{P}_V(Q) \xrightarrow{\alpha} Q,$$

and to the adjunction

$$L \dashv U' : \mathbf{Set} \rightleftarrows \mathbf{Mon}$$

whose unit and counit, at a set X and at a monoid $(M, \cdot, 1_M)$, are

$$\eta'_X : X \to L(X), \ x \mapsto (x),$$

$$\epsilon'_M : L(M) \to M, \ (m_1, ..., m_n) \mapsto m_1 \cdot ... \cdot m_n,$$

we get that the unit and the counit of the adjunction $\mathsf{P}_V L \dashv W$ are

$$\overline{\eta}_X : X \to \mathsf{P}_V(L(X)), \ x \mapsto \ulcorner \Delta_X \urcorner((x)),$$

$$\overline{\epsilon}_Q : \mathsf{P}_V(L(Q)) \xrightarrow{\mathsf{P}_V(\epsilon'_Q)} \mathsf{P}_V(Q) \xrightarrow{\alpha} Q.$$

Hence the monad structure on $\mathsf{P}_V L$ is defined as

$$\eta_X = \overline{\eta}_X : X \to \mathsf{P}_V(L(X)), \ x \mapsto \ulcorner \Delta_X \urcorner((x)),$$

$$\mu_X = U'\overline{\epsilon}_{\mathtt{P}_V L} : (\mathtt{P}_V L)(\mathtt{P}_V L)(X) \to \mathtt{P}_V L(X).$$

Let us decompose a little bit more the multiplication. First of all, we notice that the $\mathtt{P}_V$-structure $\mathtt{P}_V L(X)$ possesses is the multiplication of the enriched powerset monad $\mathtt{P}_V$ at $L(X)$

$$n_X : \mathtt{P}_V\mathtt{P}_V(LX) \to \mathtt{P}_V(LX), \quad n_X(\Phi)(\underline{x}) = \bigvee_{\phi \in V^{L(X)}} \Phi(\phi) \otimes \phi(\underline{x}).$$

We can write $n_X(\Phi)(\underline{x})$ as the relational composite of Φ viewed as a V-relation

$$\Phi : \mathtt{P}_V L(X) \nrightarrow 1,$$

with the V-relation

$$\mathtt{ev}_{LX} : LX \nrightarrow \mathtt{P}_V L(X), \quad \mathtt{ev}_{LX}(\underline{x}, \phi) = \phi(\underline{x}).$$

Interestingly enough, we can also write $\mathtt{P}_V(\epsilon'_{\mathtt{P}_V L})$ as the composition of V-relations. Indeed, for an element $\psi \in \mathtt{P}_V(LX)$ seen as a V-relation

$$\psi : L(X) \nrightarrow 1,$$

we have that $\mathtt{P}_V(\epsilon'_{\mathtt{P}_V L})(\psi) = \epsilon'^{\circ}_{\mathtt{P}_V L} \circ \psi$.

In this way we can write the multiplication μ_X as the composite of

$$LX \xrightarrow{\mathtt{ev}_{LX}} \mathtt{P}_V LX \xrightarrow{\epsilon'^{\circ}_{\mathtt{P}_V L}} L\mathtt{P}_V LX \xrightarrow{(-)} 1.$$

In Section 4.18 we stated that

$$\mathbf{CoCts}((L,V)\text{-}\mathbf{Cat}_{\mathtt{sep}}) \simeq \mathbf{Set}^{\mathtt{P}_L},$$

where $\mathtt{P}_L$ is the **Set**-monad we obtained by composing the presheaf monad $\mathbb{D}_L : (L,V)$-**Cat** $\to (L,V)$-**Cat** with the "discrete" functor $d : \mathbf{Set} \to (L,V)$-**Cat**. We showed that its unit and multiplication at X are given by

$$e_X : X \to \mathtt{P}_L(X), \; x \mapsto \mathbf{y}_X(x) = e^{\circ}_X(-,x),$$

$$n_X = - \bullet \mathbf{y}_{\circledast} : \mathtt{P}_L\mathtt{P}_L(X) \to \mathtt{P}_L(X).$$

A brief calculation shows that

$$\mathtt{P}_L(X) = V\text{-}\mathbf{Rel}(L(X), 1) = \mathtt{P}_V(L(X)).$$

Notice that $e^{\circ}_X(-,x) = \ulcorner \Delta_X \urcorner((x))$ and, furthermore, the two V-relations $\mathtt{ev}_{LX}$ and $\mathbf{y}_{\circledast}$, by the Yoneda lemma, are the same.

6.7. Proposition. *There is an equivalence of categories:*

$$\mathbf{Mon}(\mathbf{Set}^{\mathtt{P}_V}, \otimes_{\mathtt{P}_V}, V) \simeq \mathbf{CoCts}((L,V)\text{-}\mathbf{Cat}_{\mathtt{sep}}).$$

Proof. If we prove that $\mathtt{P}_V L \simeq \mathtt{P}_L$ as monads, i.e. in the sense of [37], we are able to prove our proposition.

As we noticed, the unit of $\mathtt{P}_V L$ and the unit of $\mathtt{P}_L$ are the same. Hence, in order to conclude, we have to show that also the two multiplications are compatible.

If we decompose the two multiplications, we have that pointwise they are defined as

$$LX \xrightarrow{m_X^\circ} LLX \xrightarrow{L\mathbf{y}_\circledast} L\mathtt{P}_V LX \xrightarrow{(-)} 1 \quad \text{(multiplication of } \mathtt{P}_L)$$

and as

$$LX \xrightarrow{\mathtt{ev}_{LX}} \mathtt{P}_V LX \xrightarrow{\epsilon'^\circ_{\mathtt{P}_V L}} L\mathtt{P}_V LX \xrightarrow{(-)} 1. \quad \text{(multiplication of } \mathtt{P}_V L)$$

Because in V-**Rel** $\mathbf{y}_\circledast$ is the same as $\mathtt{ev}_{LX}$, once we have shown that in V-**Rel** the following diagram commutes

$$\begin{array}{ccc} LX & \xrightarrow{\mathbf{y}_\circledast} & \mathtt{P}_V LX \\ {\scriptstyle m_X^\circ}\downarrow & & \downarrow{\scriptstyle \epsilon'^\circ_{\mathtt{P}_V L}} \\ LLX & \xrightarrow{L\mathbf{y}_\circledast} & L\mathtt{P}_V LX \end{array} \tag{5}$$

we can conclude that the two monads are the same.

The monoid structure on $\mathtt{P}_V L(X)$ is defined as

$$\begin{aligned} \mathtt{P}_V(LX) \times \mathtt{P}_V(LX) &\longrightarrow \mathtt{P}_V(LX \times LX) \longrightarrow \mathtt{P}_V L(X) \\ (\psi, \phi) &\longmapsto \psi \otimes \phi \longmapsto m_2 \cdot (\psi \otimes \phi) \end{aligned}$$

where

$$\begin{aligned} m_2 : LX \times LX &\longrightarrow LX \\ (\underline{x}, \underline{y}) &\longmapsto m_X(\underline{x}, \underline{y}). \end{aligned}$$

Hence it follows that $\epsilon'_{\mathtt{P}_V L}$ is the composite

$$L(\mathtt{P}_V L(X)) \xrightarrow{\mathrm{II}\otimes^n} \mathtt{P}_V(LLX) \xrightarrow{\mathtt{P}_V(m_X)} \mathtt{P}_V L(X),$$

where, for a list $\underline{\phi} \in L(\mathtt{P}_V L(X))$, $(\mathrm{II}\otimes^n)(\underline{\phi}) = \phi_1 \otimes ... \otimes \phi_n$.
In this way we can decompose Diagram 5 as follows

$$\begin{array}{ccc} LX & \xrightarrow{\mathbf{y}_\circledast} & \mathtt{P}_V LX \\ {\scriptstyle m_X^\circ}\downarrow & & \downarrow{\scriptstyle \mathtt{P}_V(m_X)^\circ} \\ LLX & \xrightarrow{(\mathbf{y}_{LX})_\circledast} & \mathtt{P}_V LLX \\ & {\scriptstyle L\mathbf{y}_\circledast}\searrow & \downarrow{\scriptstyle (\mathrm{II}\otimes^n)^\circ} \\ & & L\mathtt{P}_V LX. \end{array}$$

We can easily prove that the two sub diagrams commute. Let $\underline{x} \in LX$ and $\phi \in \mathtt{P}_V LX$, then we have

$$
\begin{aligned}
\mathtt{P}_V(m_X)^\circ \bullet \mathbf{y}_\circledast(\underline{x}, \phi) &= \mathbf{y}_\circledast(\underline{x}, \mathtt{P}_V(m_X)(\phi)) \\
&= \mathtt{P}_V(m_X)(\phi)(\underline{x}) \\
&\text{(from the Yoneda lemma)} \\
&= \bigvee_{\{\underline{\underline{x}} \in LLX \ , \ m_X(\underline{\underline{x}}) = \underline{x}\}} \phi(\underline{\underline{x}}) \\
&\text{(by definition of } \mathtt{P}_V) \\
&= \bigvee_{\underline{\underline{x}} \in LLX} m_X^\circ(\underline{x}, \underline{\underline{x}}) \otimes \mathbf{y}_\circledast(\underline{\underline{x}}, \phi) \\
&\text{(from the Yoneda lemma applied to } \phi(\underline{x})) \\
&= ((\mathbf{y}_{LX})_\circledast \bullet m_X^\circ)(\underline{x}, \phi),
\end{aligned}
$$

which proves the commutativity of the upper square. If we fix $\underline{\underline{x}} \in LLX$ and $\underline{\phi} \in L\mathtt{P}_V LX$, we have

$$
\begin{aligned}
(\text{II}\otimes^n)^\circ \bullet (\mathbf{y}_{LX})_\circledast(\underline{\underline{x}}, \underline{\phi}) &= (\text{II}\otimes^n)(\underline{\phi})(\underline{\underline{x}}) \\
&= \phi_1(\underline{x}_1) \otimes ... \otimes \phi_n(\underline{x}_n) \\
&= L\mathbf{y}_\circledast(\underline{\underline{x}}, \underline{\phi}),
\end{aligned}
$$

which proves the commutativity of the lower triangle and concludes the proof of the proposition.

■

6.8. Remark. *There is another interesting and conceptual way to prove that*

$$
\begin{array}{ccc}
LX & \xrightarrow{\ \mathbf{y}_\circledast\ } & \mathtt{P}_V LX \\
{\scriptstyle m_X^\circ}\downarrow & & \downarrow{\scriptstyle \mathtt{P}_V(m_X)^\circ} \\
LLX & \xrightarrow{\ (\mathbf{y}_{LX})_\circledast\ } & \mathtt{P}_V LLX
\end{array}
$$

*commutes. Consider it as a diagram in V-**Dist**, the 2-category of distributors [1], with LX seen as a discrete V-category and use the fact that $\mathbf{y}_\circledast$ is the unit of a monad—hence a natural transformation in V-**Dist**. We have that $\mathtt{P}_V(m_X)^* = \mathtt{P}_V(m_X)^\circ$ and, because $m_X \dashv m_X^*$ in V-**Dist**, we get that $\mathtt{P}_V(m_X) \dashv \mathtt{P}_V(m_X^*)$ and $\mathtt{P}_V(m_X) \dashv \mathtt{P}_V(m_X)^*$; by unicity of adjoints, it follows that $\mathtt{P}_V(m_X^*) = \mathtt{P}_V(m_X)^*$. In this way, from the commutativity of*

$$
\begin{array}{ccc}
LLX & \xrightarrow{\ \mathbf{y}_\circledast\ } & \mathtt{P}_V LLX \\
{\scriptstyle m_X}\downarrow & & \downarrow{\scriptstyle \mathtt{P}_V(m_X)} \\
LX & \xrightarrow{\ (\mathbf{y}_{LX})_\circledast\ } & \mathtt{P}_V LX
\end{array}
$$

follows the commutativity of the desired one, since

$$
\mathtt{P}_V(m_X^\circ) = \mathtt{P}_V(m_X^*) = \mathtt{P}_V(m_X)^* = \mathtt{P}_V(m_X)^\circ.
$$

6.9. Corollary. $\mathbf{CoCts}((L, \mathbf{2})\text{-}\mathbf{Cat}_{\mathsf{sep}}) \simeq \mathbf{Quant}$.

Proof. Since we have just proven that

$$\mathbf{CoCts}((L, \mathbf{2})\text{-}\mathbf{Cat}_{\mathsf{sep}}) \simeq \mathbf{Mon}(\mathbf{Set}^{\mathsf{P}_2}, \otimes_{\mathbf{2}}, \mathbf{2}),$$

from $\mathbf{Set}^{\mathsf{P}_2} \simeq \mathbf{Sup}$, and since—by definition—quantales are monoids in the category of suplattices, the result follows.

■

6.10. Remark. *In [29] the author gave another proof of the characterization of cocomplete multicategories exposed in Proposition 6.7. The main difference is the approach used; in [29], the author obtained his result by using the machinery of (L, V)-colimits which are a generalization to the realm of (L, V)-categories of the notion of weighted colimits, while in Proposition 6.7 we compared two monads. The advantage of the latter is that it gives a more manageable description of the category of algebras as a generalization to the enriched case of the notion of quantales. We must point out that the proof of Proposition 6.7 came before [29, Theorem 6.19] and it was the guiding principle that led to the proof of [29, Theorem 6.19].*

7. Conclusions

We are now ready to conclude our tour de force and finally prove our desired result.

7.1. Definition. *Let V-$\mathbf{Mod}(\mathbf{Quant})$ be the category whose objects are quantales $(Q, *, k_Q)$ equipped with an action $\rho : V \otimes_2 Q \to Q$ that is a monoid homomorphism and whose arrows are equivariant morphisms of quantales.*

7.2. Remark. *Notice that the action functor $V \otimes_2 (=) : \mathbf{Sup} \to V\text{-}\mathbf{Mod}$ extends to a functor $V \otimes_2 (=) : \mathbf{Quant} \to V\text{-}\mathbf{Mod}(\mathbf{Quant})$; as in the case of suplattices, this extension is left adjoint to the forgetful functor $V\text{-}\mathbf{Mod}(\mathbf{Quant}) \to \mathbf{Quant}$.*

7.3. Remark. *To give an arrow $\rho : V \otimes_2 Q \to Q$ in $\mathbf{Sup}$, is equivalent to give an arrow*

$$\rho' : V \otimes Q \to Q$$

in $\mathbf{Ord}$ that preserves suprema in each variable. Moreover, ρ is an action iff ρ' is an action. It is also true that ρ is a monoid homomorphism iff ρ' is a monoid homomorphism.

7.4. Proposition. $V\text{-}\mathbf{Mod}(\mathbf{Quant}) \simeq (V \downarrow \mathbf{Quant})_{\spadesuit}$.

Proof. Let $f : V \to (Q, *_Q, k_Q)$ be an object of $(V \downarrow \mathbf{Quant})_{\spadesuit}$. Define the following function:

$$\rho'_f : V \otimes Q \to Q, \quad (v, q) \mapsto f(v) *_Q q.$$

Because f is a morphism of quantales and the multiplication of a quantale preserves suprema, it follows that ρ'_f defines a unique arrow

$$\rho_f : V \otimes_2 Q \to Q$$

in **Sup**. It is straightforward to show that ρ'_f is an action, hence ρ_f is an action too. By using the fact that $f : V \to Q$ is an object of $(V \downarrow \mathbf{Quant})_\spadesuit$, we can also prove that ρ'_f is a monoid homomorphism. Indeed, let $v_1, v_2 \in V$ and $q_1, q_2 \in Q$, then we have

$$\begin{aligned}
\rho'_f((v_1, q_1) *_{V\otimes Q} (v_2, q_2)) &= \rho'_f(v_1 \otimes v_2, q_1 *_Q q_2) \\
&= f(v_1 \otimes v_2) *_Q q_1 *_Q q_2 \\
&= f(v_1) *_Q f(v_2) *_Q q_1 *_Q q_2 \\
&= f(v_1) *_Q q_1 *_Q f(v_2) *_Q q_2 \\
&= \rho'_f(v_1, x_1) *_{V\otimes Q} \rho'_f(v_2, x_2).
\end{aligned}$$

Thus (Q, ρ) is an object of V-**Mod**(**Quant**). Let $h : Q \to W$, where $f : V \to Q$ and $g : V \to W$ are objects of $(V \downarrow \mathbf{Quant})_\spadesuit$, be a morphism in $(V \downarrow \mathbf{Quant})_\spadesuit$. It is straightforward to verify that $h : (Q, \rho_f) \to (W, \rho_g)$ is a morphism in V-**Mod**(**Quant**). Thus we have a functor

$$F : (V \downarrow \mathbf{Quant})_\spadesuit \to V\text{-}\mathbf{Mod}(\mathbf{Quant}).$$

Let $\rho : V \otimes_2 (Q, *, k_Q) \to (Q, *, k_Q)$ be an object of V-**Mod**(**Quant**) and let $\rho' : V \otimes Q \to Q$ be as in Remark 7.3. Define the following morphism of quantales

$$f_{\rho'} : V \to Q, \quad v \mapsto \rho'(v, k_Q).$$

We have, for $q \in Q$, $v \in V$,

$$f_{\rho'}(v) *_Q q = \rho'(v, k_Q) *_{V\otimes Q} \rho'(k, q) = \rho'(v, q) = \rho'(k, q) *_{V\otimes Q} \rho'(v, k_Q) = q *_Q f_{\rho'}(v).$$

If $h : (Q, \rho) \to (W, \theta)$ is an arrow in V-**Mod**(**Quant**), then $h \cdot f_{\rho'} = f_{\theta'}$. Thus we have a functor

$$G : V\text{-}\mathbf{Mod}(\mathbf{Quant}) \to (V \downarrow \mathbf{Quant})_\spadesuit.$$

Easy calculations show that F and G establish an equivalence between V-**Mod**(**Quant**) and $(V \downarrow \mathbf{Quant})_\spadesuit$.

■

7.5. REMARK. *If $(X, \rho, \leq_X)$ is an object in V-**Mod**, then the map $\rho(v, =) : X \to X$ defines a morphism in **Sup**. Let $\rho : V \otimes Q \to Q$ be an object of V-**Mod**(**Quant**). We might be tempted to see (or at least, the author was) if something similar holds. Unfortunately, $\rho(v, =) : Q \to Q$ does not define a morphism of quantales. Consider $q_1, q_2 \in Q$, then we would have*

$$\rho(v, q_1 *_Q q_2) = \rho(v, q_1) *_Q \rho(v, q_2)$$

which is not true in general. In the previous proposition we showed that every $\rho : V \otimes Q \to Q$ is "essentially" of the form $f(-)*_Q =$, for an object $f : V \to Q$ of $(V \downarrow \mathbf{Quant})_\spadesuit$. It is easy to see that

$$\rho(v, q_1 *_Q q_2) := f(v) *_Q q_1 *_Q q_2,$$

in general is not equal to

$$\rho(v, q_1) *_Q \rho(v, q_2) := f(v) *_Q q_1 * f(v) *_Q q_2.$$

For example, one can take $Q = [0, \infty]^{\mathrm{op}}$, $V = [0, \infty]^{\mathrm{op}}$ *and* $f = Id$.

The last proposition allows us to conclude:

7.6. THEOREM. $\mathbf{CoCts}((L, V)\text{-}\mathbf{Cat}_{\mathtt{sep}}) \simeq V\text{-}\mathbf{Mod}(\mathbf{Quant})$.

PROOF. We have the following chain of equivalences

$$\begin{aligned}
\mathbf{CoCts}((L, V)\text{-}\mathbf{Cat}_{\mathtt{sep}}) &\simeq \mathbf{Mon}(\mathbf{Set}^{\mathtt{P}_V}, \otimes_{\mathtt{P}_V}, V) && \text{(by Proposition 6.7)}\\
&\simeq \mathbf{Mon}(V\text{-}\mathbf{Mod}, \otimes_V, V) && \text{(by Proposition 3.5)}\\
&\simeq (V \downarrow \mathbf{Quant})_{\spadesuit} && \text{(by Proposition 5.3)}\\
&\simeq V\text{-}\mathbf{Mod}(\mathbf{Quant}). && \text{(by Proposition 7.4)}
\end{aligned}$$

■

As an immediate corollary we have.

7.7. COROLLARY. *The forgetful functor* $V\text{-}\mathbf{Mod}(\mathbf{Quant}) \to \mathbf{Set}$ *is monadic.*

7.8. REMARK. *From the previous theorem it follows that the forgetful functor*

$$V\text{-}\mathbf{Mod}(\mathbf{Quant}) \to \mathbf{Set}$$

is monadic. Moreover, since **Quant** *is monadic over* **Set** *too and since* $V \otimes_2 (=)$ *is left adjoint to* $V\text{-}\mathbf{Mod}(\mathbf{Quant}) \to \mathbf{Quant}$, *if we compose the two free functors,*

$$\mathtt{P}_L^{(2)} : \mathbf{Set} \to \mathbf{Quant} \text{ (see 6.9)},$$

$$V \otimes_2 (=) : V\text{-}\mathbf{Mod}(\mathbf{Quant}) \to \mathbf{Quant} \text{ (see 7.2)},$$

we obtain the left adjoint to $V\text{-}\mathbf{Mod}(\mathbf{Quant}) \to \mathbf{Set}$.

7.9. REMARK. *Theorem 7.6 has a nice consequence. The categories* $\mathbf{CoCts}((L, V)\text{-}\mathbf{Cat}_{\mathtt{sep}})$ *and* $V\text{-}\mathbf{Mod}(\mathbf{Quant})$ *are monadic over* **Set** *(see Theorem 4.20 and Remark 7.8), thus it follows there is an equivalence of monads:*

$$\mathtt{P}_L \simeq V \otimes_2 \mathtt{P}_L^{(2)}(=),$$

where the latter is the $\mathtt{P}_L$ *monad in the ordered case. Notice how this generalizes the well-known result (see [17, 36]) that relates the enriched powerset monad* $\mathtt{P}_V$ *to the "classical" powerset monad* $\mathtt{P}_V$, *namely the equivalence of monads:*

$$\mathtt{P}_V \simeq V \otimes_2 \mathtt{P}_2(=).$$

8. Appendix: Commutative Monads

In this appendix we present the main results of [16] about commutative monads and we apply them to $\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathtt{sep}})$, which is equivalent to the category of algebras for the V-powerset monad $(\mathtt{P}_V, u, n)$.

8.1. Definition. *Let* $(C, \otimes, 1)$ *be a monoidal category and* (T, e, m) *be a monad with* $T : C \to C$. *The monad* (T, e, m) *is called strong if it is equipped with a natural transformation, called strength, with components*

$$\mathtt{st}_{X,Y} : X \otimes TY \to T(X \otimes Y),$$

compatible with the monoidal structure defined on C, *as expressed in [16].*

If C *is a symmetic monoidal category, the monad* (T, e, m) *is called commutative if the following diagram commutes*

$$\begin{array}{ccccc}
TX \otimes TY & \xrightarrow{\mathtt{st}_{TX,Y}} & T(TX \otimes Y) & \xrightarrow{T\mathtt{st}'_{X,Y}} & T^2(X \otimes Y) \\
\big\downarrow{\scriptstyle \mathtt{st}'_{X,TY}} & & & & \big\downarrow{\scriptstyle m_{X\otimes Y}} \\
T(X \otimes TY) & \xrightarrow{T\mathtt{st}_{X,Y}} & T^2(X \otimes Y) & \xrightarrow{m_{X\otimes Y}} & T^2(X \otimes Y),
\end{array}$$

where

$$\mathtt{st}'_{X,Y} : TX \otimes Y \xrightarrow{\gamma_{TX,Y}} Y \otimes TX \xrightarrow{\mathtt{st}_{Y,X}} T(Y \otimes X) \xrightarrow{T\gamma_{X,Y}} T(X \otimes Y)$$

is called co-strength.

8.2. Definition. *Let* $(C, \otimes, 1)$ *be a symmetric monoidal category and let* (T, e, m) *be a strong monad defined on it. Suppose* $(X, \alpha), (Y, \beta), (Z, \gamma)$ *are* T*-algebras. An arrow (in* C*)* $f : X \otimes Y \to Z$ *is called a bimorphism if the following diagrams commute*

$$\begin{array}{ccccc}
X \otimes TY & \xrightarrow{\mathtt{st}_{X,Y}} & T(X \otimes Y) & \xrightarrow{Tf} & TZ \\
\big\downarrow{\scriptstyle Id\otimes\beta} & & & & \big\downarrow{\scriptstyle \gamma} \\
X \otimes Y & & \xrightarrow{\qquad f \qquad} & & Z
\end{array}
\qquad
\begin{array}{ccccc}
TX \otimes Y & \xrightarrow{\mathtt{st}'_{X,Y}} & T(X \otimes Y) & \xrightarrow{Tf} & TZ \\
\big\downarrow{\scriptstyle \alpha\otimes Id} & & & & \big\downarrow{\scriptstyle \gamma} \\
X \otimes Y & & \xrightarrow{\qquad f \qquad} & & Z.
\end{array}$$

In this way, for all $X, Y \in C^T$, *we define a functor*

$$\mathbf{Bim}(X \otimes Y, =) : C^T \to \mathbf{Set},$$

where $\mathbf{Bim}(X \otimes Y, Z)$ *denotes the sets of bimorphisms from* $X \otimes Y$ *to* Z.

The main result about strong monads we are interested in is contained in the following theorem.

8.3. Theorem. *[16, Lemmas 5.1-5.3] Let (T, e, m) be a strong monad on a symmetric monoidal category $(C, \otimes, 1)$ such that its associated category of algebras C^T has coequalizers of reflexive pairs. Then, for each algebras (X, α), (Y, β), $\mathbf{Bim}(X \otimes Y, =)$ is representable by an algebra $(X \otimes_T Y, \alpha \otimes_T \beta)$.*

If additionally (T, e, m) is commutative, then C^T becomes a symmetric monoidal category with $\otimes_T$ as tensor product and with the free algebra $(T1, m_1)$ as the unit; moreover, the free functor $F : C \to C^T$ becomes strong monoidal. If C has equalizers and its monoidal structure is closed, then also C^T becomes a closed monoidal category.

8.4. Remark. *Let (T, m, e) be a monad with $T : \mathbf{Set} \to \mathbf{Set}$. Then, if we assume the axiom of choice, $\mathbf{Set}^T$ is cocomplete (see [28]). Thus, Eilenberg-Moore categories for strong monads defined on $\mathbf{Set}$ always satisfy the hypothesis of Theorem 8.3.*

8.5. Remark. *We obtain the associator in C^T from the one in C by using the universal property of bimorphisms. Similarly, we can obtain the unitors in C^T by using the tensorial strength. As an example, the left unitor at an object (X, α) is the arrow associated to the bimorphism*

$$T1 \otimes X \xrightarrow{st'_{1,X}} T(1 \otimes X) \xrightarrow{\simeq} T(X) \xrightarrow{\alpha} X.$$

Consider $\mathbf{Set}$ as a monoidal category in the usual way, that is to say, with its cartesian structure and let $(\mathtt{P}_V, u, n)$ be the V-powerset monad. Consider the following function

$$\mathtt{st}_{X,Y} : X \times \mathtt{P}_V Y \to \mathtt{P}_V(X \times Y),\ (x, \phi) \mapsto (u_X(x) \otimes \phi),$$

where $(u_X(x) \otimes \phi)(\tilde{x}, y) = u_X(x)(\tilde{x}) \otimes \phi(y)$.

Long and boring computations show that this makes $(\mathtt{P}_V, u, n)$ into a strong monad.

8.6. Remark. *The strongness of $(\mathtt{P}_V, u, n)$ follows from the fact that every functor F : $\mathbf{Set} \to \mathbf{Set}$ is a $\mathbf{Set}$-functor (where the monoidal structure on $\mathbf{Set}$ is the usual one), and from the fact that to give a $\mathbf{Set}$-enrichment, for a $\mathbf{Set}$-monad (T, e, m), is equivalent to give a strength (see Propositions* 1.1, 1.2 *of [21]).*

Moreover, since we always assume our base quantale V to be commutative, it is easy to show that $(\mathtt{P}_V, u, n)$ is also commutative, with the co-strength $\mathtt{st}'$ given by

$$\mathtt{st'}_{X,Y} : \mathtt{P}_V X \times Y \to \mathtt{P}_V(X \times Y),\ (\psi, y) \mapsto \psi \otimes u_Y.$$

By applying Theorem 8.3 we get the following result (see [9, 20]).

8.7. Proposition. *The category $\mathbf{Set}^{\mathtt{P}_V}$ of algebras for the V-powerset monad $(\mathtt{P}_V, u, n)$ admits a symmetric closed monoidal structure $\otimes_{\mathtt{P}_V}$ with unit given by $\mathtt{P}_V(1) = V$ such that the free functor*

$$\mathtt{P}_V : \mathbf{Set} \to \mathbf{Set}^{\mathtt{P}_V},\ X \mapsto (\mathtt{P}_V(X), n_X)$$

becomes strong monoidal. Moreover, $\otimes_{\mathtt{P}_V}$ classifies bimorphisms in the sense of Definition 8.2.

The last thing we have to do is to tune a little bit more the notion of bimorphism in our particular case, in order to have a more manageable formulation. The notion of bimorphism in categories in which the notion of "point" resembles the one in **Set** seems to reduce to the "componentwise preserving structure" notion like in **Sup**. This motivates us to introduce the following definition.

8.8. Definition. *Suppose* $(X,\alpha),(Y,\beta),(Z,\gamma)$ *are in* $\mathbf{Set}^{\mathsf{P}_V}$. *A function* $f : X\times Y \to Z$ *is called a bimorphism if the following diagrams commute, for all* $x, y \in X, Y$,

$$\begin{array}{ccccccc}
\mathsf{P}_V(X) & \xrightarrow{\simeq} & \mathsf{P}_V(X)\times 1 & \xrightarrow{\mathsf{P}_V(Id\times y)} & \mathsf{P}_V(X\times Y) & \xrightarrow{\mathsf{P}_V f} & \mathsf{P}_V(Z) \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\alpha\times 1}\downarrow & & & & \downarrow{\scriptstyle\gamma} \\
X & \xrightarrow{\simeq} & X\times 1 & \xrightarrow{Id\times y} & X\times Y & \xrightarrow{f} & Z
\end{array}$$

$$\begin{array}{ccccccc}
\mathsf{P}_V(Y) & \xrightarrow{\simeq} & 1\times\mathsf{P}_V(Y) & \xrightarrow{\mathsf{P}_V(x\times Id)} & \mathsf{P}_V(X\times Y) & \xrightarrow{\mathsf{P}_V f} & \mathsf{P}_V(Z) \\
{\scriptstyle\beta}\downarrow & & {\scriptstyle 1\times\beta}\downarrow & & & & \downarrow{\scriptstyle\gamma} \\
Y & \xrightarrow{\simeq} & 1\times Y & \xrightarrow{x\times Id} & X\times Y & \xrightarrow{f} & Z.
\end{array}$$

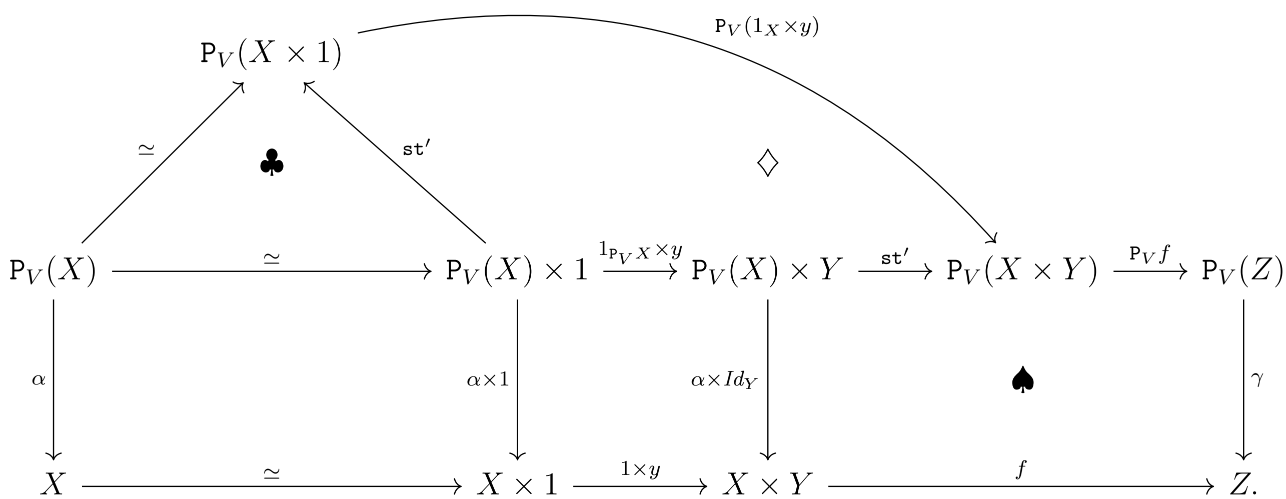
8.9. Remark. *Let* $f : (X,\alpha)\times(Y,\beta)\to(Z,\gamma)$ *be a bimorphism according to Definition 8.8. This is amount to say that, for all* $x, y \in X, Y$, f_y *and* f_x *are algebra morphisms, where*

$$f_x : Y \to Z,\quad y\mapsto f(x,y),\ \textit{and}\ f_y : X\to Z,\quad x\mapsto f(x,y).$$

Now we have not only one but two notions of bimorphism! Of course, as one might expect, the two notions coincide.

8.10. Proposition. *Suppose* $(X,\alpha),(Y,\beta),(Z,\gamma)$ *are in* $\mathbf{Set}^{\mathsf{P}_V}$. *A function* $f : X\times Y \to Z$ *is a bimorphism according to Definition 8.2 iff it is so according to Definition 8.8.*

PROOF. Let us do the case in which we "fix" $y \in Y$, the other one is similar.
The proof follows by contemplating the following diagram



Here ♣ commutes since $\mathtt{P}_V$ is a strong monad while ♢ commutes because $\mathtt{st}'$ is a natural transformation.

Suppose f is a bimorphism according to 8.2, then ♠ commutes, hence the outer diagram too. This implies that f is a bimorphism according to 8.8 too.

If f is a bimorphism according to 8.8, then the outer diagram commutes, hence, for all $y \in Y$, we have

$$\gamma \cdot \mathtt{P}_V f \cdot \mathtt{st}' \cdot 1_{\mathtt{P}_V X} \times y = f \cdot \alpha \times Id_Y \cdot 1_{\mathtt{P}_V X} \times y.$$

Since $(1_{\mathtt{P}_V X} \times y : \mathtt{P}_V(X) \times 1 \to \mathtt{P}_V(X) \times Y)_{y \in Y}$, is a jointly epic family, we can jointly cancel them in the previous equation. Thus we obtain the commutativity of ♠ which implies that f is a bimorphism according to 8.2 too.

■

8.11. REMARK. *In Theorem 2.23 we proved that the category of algebras for this monads is equivalent to* $\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathtt{sep}})$. *Hence the monoidal structure on* $\mathbf{Set}^{\mathtt{P}_V}$ *transfers to a monoidal structure on* $\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathtt{sep}})$.

In particular, from the previous proposition, and since the equivalence

$$\mathbf{Set}^{\mathtt{P}_V} \simeq \mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathtt{sep}})$$

changes only the corresponding structures (and it leaves the underlying sets and arrows unchanged), we have that a V*-functor* $f : (X, a) \otimes (Y, b) \to (Z, c)$ *is a bimorphism if, for all* $x, y \in X, Y$*, one has*

$$f_x : (Y, b) \to (Z, c),\ y \mapsto f(x, y),$$

$$f_y : (X, a) \to (Z, c),\ x \mapsto f(x, y),$$

are cocontinuous V-functor.

Since these kind of bimorphisms are classified by a monoidal structure on the category $\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}})$*, as described at the end of [19], by arguments similar to the one we used in Proposition 3.5, we get that the monoidal structure on* $\mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}})$*, induced by the equivalence* $\mathbf{Set}^{\mathsf{P}_V} \simeq \mathbf{CoCts}(V\text{-}\mathbf{Cat}_{\mathsf{sep}})$*, and the one studied in [19] coincide.*

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