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THE CLOSED MODEL STRUCTURE ON THE CATEGORY OF WEAKLY UNITAL DG CATEGORIES: AN ADDENDUM

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Abstract. This paper is an addendum to our paper [PS], where a closed model structure on the category $\mathbb{C}_{dgwu}(\Bbbk)$ of small (Kontsevich-Soibelman) weakly unital dg categories is constructed, $\Bbbk$ a field of any characteristic. In [PS], we referred to our earlier preprint for proofs of the following results: (A) small (co)completeness of $\mathbb{C}_{dgwu}(\Bbbk)$, and (B) the non-symmetric dg operad $\mathcal{O}'$ (which governs weakly unital dg categories acting on $\Bbbk$-quivers, we recall its definition in Section 2.12) is quasi-isomorphic to the operad of unital associative algebras $\mathcal{A}ssoc_+$, under a natural projection. Recall that the (co)completeness in (A) is the first axiom of a closed model category, while (B) was crucial in the proof in [PS, Th. 5.3] of the Quillen equivalence between $\mathbb{C}_{dgwu}(\Bbbk)$ (equipped with our model structure) and the category of small dg categories $\mathbb{C}_{dg}(\Bbbk)$ (equipped with the Tabuada model structure for which the weak equivalences are quasi-equivalences).

In this paper we collect our earlier proofs of (A) and (B), which serves as an addendum to [PS], and makes these two papers self-contained.

1. Introduction

This paper, served as addendum to [PS], provides proofs of two statements our treatment in loc.cit. relies on, and previously hidden in our earlier preprint.

Denote by $\mathbb{C}_{dgwu}(\Bbbk)$ the category of (Kontsevich-Soibelman) weakly unital small dg categories over a field $\Bbbk$ (of any characteristic), see Section 2.1. In [PS], we endow $\mathbb{C}_{dgwu}(\Bbbk)$ with a closed model structure, and construct a Quillen equivalence

$$L\colon \mathbb{C}_{dgwu}(\Bbbk) \rightleftarrows \mathbb{C}_{dg}(\Bbbk) : R \tag{1}$$

where $\mathbb{C}_{dg}(\Bbbk)$ is the category of small dg categories over $\Bbbk$ equipped with the Tabuada model structure [Tab] (whose weak equivalences are quasi-equivalences of dg categories).

The two statements mentioned above are:

(A) the category $\mathbb{C}_{dgwu}(\Bbbk)$ is small complete and small cocomplete;

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(B) the non-symmetric (non-Σ) dg operad $\mathcal{O}'$, which governs (Kontsevich-Soibelman) weakly unital dg categories and introduced in [PS, Section 2.2] (we recall its definition in Section 2.12), is quasi-equivalent to the operad $\mathcal{A}ssoc_+$ of unital associative algebras. There is an explicit map of dg operads $\mathcal{O}' \to \mathcal{A}ssoc_+$ of Theorem 2.13, which is proven to be a quasi-isomorphism.

Note that (B) is used in [PS] to prove that (1), which is a priori a Quillen pair, is in fact a Quillen equivalence.

The paper consists of two Sections. We prove (A) and introduce the dg operad $\mathcal{O}'$ in Section 1, and prove (B) and Theorem 2.13 in Section 2.

Here we briefly outline our methods.

In Section 1 we use monadic methods, and adopt to our problem the classical papers [Wo] and [Li], which provides an approach to small limits and colimits in (unital) enriched categories. The products, the coproducts, and the equalizers are constructed directly. The coequalizers are less trivial, and computing them requires the technique of monads. We construct a monad T on the category of dg graphs and prove in Theorem 2.24 that the categories of T-algebras and of weakly unital dg categories are equivalent. The coequalizers are constructed in Proposition 2.22. We also construct a non-symmetric dg operad $\mathcal{O}'$ such that $\mathcal{O}'$-algebras in small dg quivers are exactly small weakly unital dg categories (over a field $\Bbbk$).

The proof of Theorem 2.13 in Section 2 goes by a rather tricky computation with several spectral sequences, which step by step reduce $\mathcal{O}'$ to simpler ones. This sequence of reductions ends up with the operad $\mathcal{A}ssoc_+$.

2. Small (co)completeness of $\mathbb{C}_{dgwu}(\Bbbk)$

2.1. Weakly unital dg categories. Recall the definition of a weakly unital dg category [KS, 4.2].

Let A be a non unital dg category. Denote by $A \oplus \Bbbk_A$ the strictly unital dg category where $Ob(A \oplus \Bbbk_A) = Ob(A)$ and

$$\mathrm{Hom}_{A\oplus\Bbbk_A}(x,y) = \begin{cases} \mathrm{Hom}_A(x,y) & \text{if } x \neq y \\ \mathrm{Hom}_A(x,x) \oplus \Bbbk 1_x & \text{if } x = y. \end{cases}$$

One has a natural imbedding $i : A \to A \oplus \Bbbk_A$, sending x to x, and $f \in A(x,x)$ to the pair $(f, 0) \in (A \oplus \Bbbk_A)(x,x)$. We denote by 1_x the generator of $\Bbbk_x$.

2.2. Definition. *A weakly unital dg category A over $\Bbbk$ is a non-unital dg category A over $\Bbbk$ with a distinguished closed element $\mathrm{id}_x \in A^0(x,x)$ for any object x in A, such that there exists an A_∞-functor $p : A \oplus \Bbbk_A \to A$ which is the identity on the objects, such that $p \circ i = \mathrm{id}_A$, $p_1(1_x) = \mathrm{id}_x, \forall x \in Ob(A)$, and $p_n(f_1, \dots, f_n) = 0$ for $n \geq 2$ if f_i morphisms in the image $i(A)$.*

Note that this definition gives rise to the sequence of relations on the Taylor coefficients p_n, $n \geq 1$, of the A_∞ functor p. The first non-trivial relations read:

$$dp_2(f, 1_x) + p_2(df, 1_x) = f - f \circ \mathrm{id}_x, \quad dp_2(1_x, f) + p_2(1_x, df) = f - \mathrm{id}_x \circ f \tag{2}$$

The reader is referred to [PS, eq.(2.4)] for several next relations.

2.3. Definition. *Let A, C be two weakly unital dg categories, with the structure maps $p^A : A \oplus \Bbbk_A \to A$ and $p^C : C \oplus \Bbbk_C \to C$. A* weakly unital dg functor *$F : A \to C$ is a non unital dg functor $F\colon A \to C$ such that the following diagram commutes:*

$$\begin{tikzcd} A \oplus \Bbbk_A \arrow[r, "F \oplus \Bbbk_F"] \arrow[d, "p^A"'] & C \oplus \Bbbk_C \arrow[d, "p^C"] \\ A \arrow[r, "F"] & C \end{tikzcd} \tag{3}$$

In this way, we define the category $\mathbb{C}_{dgwu}(\Bbbk)$. Its full subcategory, for which $\mathrm{id}_x \circ \mathrm{id}_x = \mathrm{id}_x$ for any object x, is denoted by $\mathbb{C}^0_{dgwu}(\Bbbk)$.

It follows from the definition that:

$$\begin{aligned} F(\mathrm{id}_x) &= \mathrm{id}_{F(x)} & \forall x \in \mathrm{Ob}(A) \\ F(p^A_n(f_1, .., f_n)) &= p^{\mathcal{C}}_n(F(f_1), .., F(f_n)) & f_i \in A \oplus \Bbbk_A, i = 1 \dots n \end{aligned} \tag{4}$$

For a weakly unital dg category A, define $H^0(A)$ as an (a priori non-unital) $\Bbbk$-linear category, having the same objects, and having morphisms $(H^0(A))(X, Y) = H^0(A(X, Y))$.

2.4. Lemma. *Let A be a weakly unital dg category. Then the homotopy category $H^0(A)$ is a strictly unital $\Bbbk$-linear category.*

Proof. The map $[p_1]\colon H^0(A) \oplus \Bbbk_{H^0(A)} \to H^0(A)$, induced by the first Taylor component p_1 of the A_∞ functor p, is a dg functor. One has $[p_1](1_X) = \mathrm{id}_X$ and $[p_1] \circ [i] = \mathrm{id}$. It follows from (2) that $\mathrm{id}_X \circ f = f \circ \mathrm{id}_X = f$, for any $f \in H^0(A)(X, X)$. ■

2.5. Lemma. *Let $F\colon C \to D$ be a weakly unital dg functor between weakly unital dg categories. Then it defines a $\Bbbk$-linear functor $H^0(F)\colon H^0(C) \to H^0(D)$ of unital $\Bbbk$-linear categories.*

It is clear.

2.6. The products, coproducts, and equalizers in $\mathbb{C}_{dgwu}(\Bbbk)$. Our goal is to show that the dg category $\mathbb{C}_{dgwu}(\Bbbk)$ is small complete and small cocomplete. One constructs directly small *products* and small *coproducts*. The *equalizers* are also straightforward, as follows.

Let $F, G\colon C \to D$ be two morphisms. Define $\mathrm{Eq}(F, G)$ as the dg category whose objects are

$$\mathrm{Ob}(\mathrm{Eq}(F, G)) = \{X \in \mathrm{Ob}(C) | F(X) = G(X)\}$$

Let $X, Y \in \mathrm{Ob}(\mathrm{Eq}(F, G))$. Define

$$\mathrm{Eq}(F, G)(X, Y) = \{f \in C(X, Y) | F(f) = G(f)\}$$

It is clear that $\mathrm{Eq}(F, G)$ is a non-unital dg category. For any $X \in \mathrm{Ob}(\mathrm{Eq}(F, G))$, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $G(\mathrm{id}_X) = \mathrm{id}_{G(X)}$, therefore $\mathrm{id}_X \in \mathrm{Eq}(F, G)(X, X)$.

One has to construct an A_∞ functor $p\colon \mathrm{Eq}(F, G) \oplus \Bbbk_{\mathrm{Eq}(F,G)} \to \mathrm{Eq}(F, G)$ such that $p_1(1_X) = \mathrm{id}_X$, and $p \circ i = \mathrm{id}$. We define

$$p_n^{\mathrm{Eq}(F,G)}(f_1 \otimes \dots \otimes f_n) = p_n^C(f_1 \otimes \dots \otimes f_n)$$

One has to check that $p_n^{\mathrm{Eq}(F,G)}(f_1 \otimes \dots \otimes f_n)$ is a morphism in $\mathrm{Eq}(F, G)$, that is,

$$F(p_n^C(f_1 \otimes \dots \otimes f_n)) = G(p_n^C(f_1 \otimes \dots \otimes f_n)) \tag{5}$$

From (4) one gets

$$F(p_n^C(f_1 \otimes \dots \otimes f_n)) = p_n^D(F(f_1) \otimes \dots F(f_n))$$

and

$$G(p_n^C(f_1 \otimes \dots \otimes f_n)) = p_n^D(G(f_1) \otimes \dots \otimes G(f_n))$$

Now (5) follows from $F(f_i) = G(f_i)$ for all f_i, which holds because all f_i are morphisms in $\mathrm{Eq}(F, G)$. Thus, $\mathrm{Eq}(F, G)$ is a weakly unital dg category.

To construct the *coequalizers* is a harder task. For the category $\mathcal{V}-\mathrm{Cat}$ of small $\mathcal{V}$-enriched categories, the coequalizers were constructed in [Li] and [Wo], assuming $\mathcal{V}$ to be a symmetric monoidal closed and cocomplete, and were constructed in [BCSW] and [KL] in weaker assumptions on $\mathcal{V}$. All these proofs rely on the theory of monads. We associate a monad which governs the weakly unital dg categories in Section 2.12.

We adapt the approach of [Wo] for a proof of existence of the coequalizers in $\mathbb{C}_{dgwu}(\Bbbk)$. We also prove the corresponding monadicity theorem.

2.7. Reminder on monads. Here we recall definitions and some general facts on monads and algebras over monads. The reader is referred to [ML], [R] for more detail.

Let $\mathcal{C}$ be a category. Recall that a monad in $\mathcal{C}$ is given by an endofunctor

$$T\colon \mathcal{C} \to \mathcal{C}$$

and natural transformations

$$\eta\colon \mathrm{Id} \Rightarrow T \text{ and } \mu\colon T^2 \Rightarrow T$$

so that the following diagrams commute:

$$\begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "\mathrm{Id}"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, Rightarrow, "\mathrm{Id}"] \\ & T & \end{tikzcd}$$

A monad appears from a pair of adjoint functors. Assume we have an adjoint pair

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D} : U \tag{6}$$

with adjunction unit and counit $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow UF$ and $\varepsilon\colon FU \Rightarrow \mathrm{Id}_{\mathcal{D}}$.

It gives rise to a monad in $\mathcal{C}$, defined as:

$$T = UF, \quad \eta = \eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow T, \quad \mu = U\epsilon F\colon T^2 \Rightarrow T$$

An *algebra A over a monad T* is given by an object $A \in \mathcal{C}$ equipped with a morphism $a\colon TA \to A$ such that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{Id}_A"'] & TA \arrow[d, "a"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "Ta"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"] & A \end{tikzcd}$$

The morphisms of algebras over a monad T are defined as morphisms $f\colon A \to B$ in $\mathcal{C}$ such that the natural diagram commutes.

The category of T-algebras is denotes by $\mathcal{C}^T$.

There is an adjunction

$$F^T\colon \mathcal{C} \rightleftarrows \mathcal{C}^T : U^T$$

which by its own gives rise to a monad.

There is a functor $\Phi\colon \mathcal{D} \to \mathcal{C}^T$, sending an object Y of $\mathcal{D}$ to the T-algebra $A = UY$, with $a\colon TA = UFUY \to UY = A$ equal to $U\varepsilon_Y$. The functor Φ is called the *Eilenberg-Moore comparison functor*.

An adjunction (6) is called *monadic* if the functor $\Phi\colon \mathcal{D} \to \mathcal{C}^T$ is an equivalence.

There is a criterion when an adjunction is monadic, called the *Beck monadicity theorem*. We recall its statement below.

Recall that a *split coequalizer* in a category is a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend right=60, "s"'] \arrow[r, "h"] & C \arrow[l, bend right=60, "t"'] \end{tikzcd}$$

such that

(1) $f \circ s = \mathrm{id}_B$,

(2) $g \circ s = t \circ h$,

(3) $h \circ t = \mathrm{id}_C$,

(4) $h \circ f = h \circ g$

Recall

2.8. Lemma. *A split coequalizer is a coequalizer, and is an absolute coequalizer (that is, is preserved by any functor).*

Proof. It is enough to prove the first statement, because a split equalizer remains a split equalizer after application of any functor. See e.g. [R, Lemma 5.4.6] for detail. ∎

Given a pair

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

in a category $\mathcal{D}$, and a functor $U\colon \mathcal{D} \to \mathcal{C}$, we say that this pair is *U-split* if the pair

$$U(A) \underset{g}{\overset{f}{\rightrightarrows}} U(B)$$

in $\mathcal{C}$ can be extended to a split coequalizer.

2.9. Theorem. *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon U$ be a pair of adjoint functors, and let $T = UF$ be the corresponding monad. Consider the Eilenberg-MacLane comparison functor $\Phi\colon \mathcal{D} \to \mathcal{C}^T$. Then:*

(1) if $\mathcal{D}$ has coequalizers of all U-split pairs, the functor Φ has a left adjoint $\Psi\colon \mathcal{C}^T \to \mathcal{D}$,

(2) if, furthermore, U preserves coequalizers of all U-split pairs, the unit $\mathrm{Id}_{\mathcal{C}^T} \Rightarrow \Phi\Psi$ is an isomorphism,

(3) if, furthermore, U reflects isomorphisms (that is, $U(f)$ an isomorphism implies f an isomorphism), the counit $\Psi\Phi \Rightarrow \mathrm{Id}_D$ is also an isomorphism.

Therefore, if (1)-(3) hold, (U, F) is monadic. Conversely, if (U, F) is monadic, conditions (1)-(3) hold.

The reader is referred to [ML] or [R] for a proof.

There is another monadicity theorem, which gives sufficient but not necessary conditions for $\Phi\colon \mathcal{D} \to \mathcal{C}^T$ to be monadic.

It uses *reflexive pairs* in $\mathcal{D}$ instead of U-split pairs.

A pair of morphisms $f, g\colon A \to B$ in $\mathcal{D}$ is called *reflexive* if there is a morphism $h\colon B \to A$ which splits both f and g: $f \circ h = \mathrm{id}_B = g \circ h$.

We refer the reader to [MLM, Ch.IV.4, Th.2] for a proof of the following result, also known as the *crude monadicity Theorem*:

2.10. Theorem. *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon U$ be a pair of adjoint functors, and let $T = UF$ be the corresponding monad. Consider the Eilenberg-MacLane comparison functor $\Phi\colon \mathcal{D} \to \mathcal{C}^T$. Then:*

(1) if $\mathcal{D}$ has coequalizers of all reflexive pairs, the functor Φ admits a left adjoint $\Psi\colon \mathcal{C}^T \to \mathcal{D}$,

(2) *if, furthermore, U preserves these coequalizers, the unit of the adjunction* $\mathrm{Id}_{\mathcal{C}^T} \to \Phi \circ \Psi$ *is an isomorphism,*

(3) *if, furthermore, U reflects isomorphisms, the counit of the adjunction* $\Psi \circ \Phi \to \mathrm{Id}_D$ *is also an isomorphism.*

Therefore, if (1)-(3) hold, (U, F) is monadic.

Note that, unlike for Theorem 2.9, *the converse statement is not true.* That is, the conditions for monadicity, given in Theorem 2.10, are sufficient but not necessary.

The following construction is of fundamental importance for both monadicity theorems.

In the notations as above, let $A \in \mathcal{D}$. Consider two morphisms

$$FUFUA \underset{g}{\overset{f}{\rightrightarrows}} FUA \tag{7}$$

where $f = FU\varepsilon_A$ and $g = \varepsilon_{FUA}$. (Similarly, one defines such two maps for $A \in \mathcal{C}^T$).

One has *two different extensions* of this pair of arrows, which form a U-split coequalizer and a reflexive pair, correspondingly.

For the first case, consider

$$\begin{tikzcd} UFUFUA \arrow[r, shift left, "Uf"] \arrow[r, shift right, "Ug"'] & UFUA \arrow[l, bend right=40, "s_1"'] \arrow[r, "h"] & UA \arrow[l, bend right=40, "t"'] \end{tikzcd} \tag{8}$$

with $s_1 = \eta_{UFUA}$, $t = \eta_{UA}$, $h = U\varepsilon_A$.

For the second case, consider

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & FUA \arrow[l, bend right=40, "s_2"'] \end{tikzcd} \tag{9}$$

with $s_2 = F\eta_{UA}$.

The following lemma is proven by a direct check:

2.11. Lemma. *For any $A \in \mathcal{D}$ (or $A \in \mathcal{C}^T$), (8) is a split coequalizer in $\mathcal{C}$, whence (9) is a reflexive pair in $\mathcal{D}$ (corresp., in $\mathcal{C}^T$).*

Proof. ∎

Note that s_1 is *not* a U-image of a morphism in $\mathcal{D}$, though Uf and Ug are. On the other hand, s_2 is a morphism in $\mathcal{D}$ (corresp., in $\mathcal{C}^T$).

2.12. The dg operad $\mathcal{O}'$ and the monad of weakly unital dg categories. A dg quiver Γ over $\Bbbk$ is an oriented graph, given by a *set* V_Γ of vertices, and a complex $\Gamma(x,y)\in C^\bullet(\Bbbk)$ for any ordered pair $x,y\in V_\Gamma$. A morphism $F\colon \Gamma_1\to\Gamma_2$ is given by a map of sets $F_V\colon V_{\Gamma_1}\to V_{\Gamma_2}$, and by a map of complexes $F_E\colon \Gamma_1(x,y)\to\Gamma_2(F_V(x),F_V(y))$, for any $x,y\in V_{\Gamma_1}$. We denote by $\mathbb{G}_{dg}(\Bbbk)$ the category of dg quivers over $\Bbbk$.

A *unital* dg quiver Γ over $\Bbbk$ is an dg quiver over $\Bbbk$ such that there is an element $\mathrm{id}_x\in\Gamma(x,x)$, closed of degree 0, for any $x\in V_\Gamma$. A map of unital dg quivers is a map F of the underlying dg graphs such that $F(\mathrm{id}_x)=\mathrm{id}_{F(x)}$, for any $x\in V_\Gamma$. We denote by $\mathbb{G}_{dg}(\Bbbk)$ the category of unital dg quivers over $\Bbbk$.

There is a natural forgetful functor $U\colon \mathbb{C}_{dgwu}(\Bbbk)\to\mathbb{G}_{dg}(\Bbbk)$, where $U(C)$ is a quiver Γ with $V_\Gamma=\mathrm{Ob}(C)$, and $\Gamma(x,y)=C(x,y)$.

This functor admits a left adjoint $F\colon \mathbb{G}_{dg}(\Bbbk)\to\mathbb{C}_{dgwu}(\Bbbk)$. It is constructed via a dg operad $\mathcal{O}'$, see (14).

Define a non-Σ the dg operad $\mathcal{O}'$ as the quotient-operad of the free operad generated by the composition operations:

(a) the composition operation $m\in\mathcal{O}'(2)^0$

(b) $p_{n;i_1,\dots,i_k}\in\mathcal{O}'(n-k)^{-n+1}$, $0\le k\le n$, $1\le i_1<i_2<\dots<i_k\le n$, with the following meaning: For a weakly unital dg category C, the operation $p_{n;i_1,\dots,i_k}(f_1,\dots,f_{n-k})$ is defined as

$$p_n\big(f_1,\dots,f_{i_1-1},\underset{i_1}{1_{x_1}},f_{i_1},\dots,f_{i_2-2},\underset{i_2}{1_{x_2}},f_{i_2-1},\dots,f_{i_3-3},\underset{i_3}{1_{x_3}},\dots\dots,\underset{i_k}{1_{x_k}},f_{i_k-k+1},\dots,f_{n-k}\big)\tag{10}$$

where by 1_{x_i}s are denoted the morphisms $1_{x_i}\in\Bbbk_C$ for the corresponding objects $x_i\in C$.

by the following relations:

$$\begin{aligned}&(i)\text{ the associativity of } m,\text{ and } dm=0\\&(ii)\ p_{n;i_1,\dots,i_k}=0\text{ if }k=0\\&(iii)\ p_{1;-}=\mathrm{id}\\&(iv)\text{ the }A_\infty\text{ morphism relation for }dp_{n;i_1,\dots,i_k}\text{ see (12) below}\end{aligned}\tag{11}$$

We use the notation $j=p_{1,1}$, the degree zero 0-ary operation generating the weak unit. It follows from (iv) that $dj=0$. Note that relation (iv) expresses the relations like (2) and its higher analogues [PS, eq.(2.4)] in the operadic terms, using (10).

It remains to specify relation (iv):

$$\begin{gathered}dp_{n;i_1,\dots,i_k}=\sum_{1\le\ell\le n-1}\pm m\circ(p_{\ell;i_1,\dots,i_{s(\ell)}},p_{n-\ell;i_{s(\ell)+1},\dots,i_k})+\\ \sum_{r=1}^{n-1}\pm p_{n-1;j_1,\dots,j_{q(r)}}\circ(\mathrm{id},\dots,\mathrm{id},m(a(r),\underset{r}{a}(r+1)),\mathrm{id},\dots,\mathrm{id})\end{gathered}\tag{12}$$

with the notations explained below.

We have to explain notations in (12). By $s(\ell)$ is denoted the maximal s such that $i_s \le \ell$; $a(r)$ is equal to id if $r \notin \{i_1, \dots, i_k\}$ and is equal to j otherwise. Finally, $q(r) \in \{k, k-1, k-2\}$; $q(r) = k$ if neither $r, r+1$ are in $\{i_1, \dots, i_k\}$, and in this case $j_s = i_s$ for $i_s \le r$ and $j_s = i_s - 1$ for $i_s > r$; $q(r) = k-1$ if either r or $r+1$ are in $\{i_1, \dots, i_k\}$ but not both, in this case $j_s = i_s$ for $i_s < r$, and $j_s = i_{s+1} - 1$ for $i_{s+1} > r$; finally, if both $r, r+1$ are in $\{i_1, \dots, i_k\}$ one sets $q(r) = k-2$ and $j_s = i_s$ for $i_s < r$, and $j_s = i_{s+2} - 1$ for $i_{s+2} > r+1$.

Morally, the dg operad $\mathcal{O}'$ comprises all universal operations one can define on a weakly unital dg category.

Denote by $\mathcal{A}ssoc_+$ the operad of unital associative $\mathbb{k}$-algebras. In Section 3 we prove the following theorem:

2.13. THEOREM. *The natural map of dg operads $\mathcal{O}' \to \mathcal{A}ssoc_+$, sending all $p_{n;i_1,\dots,i_k}$, $n \ge 2$, to 0, sending $j = p_{1;1}$ to the 0-ary unit generating operation, and sending m to m, is a quasi-isomorphism.*

The proof is a rather long and tricky computation with several spectral sequences.

PROOF. ∎

The left adjoint functor $F\colon \mathbb{G}_{dg}(\mathbb{k}) \to \mathbb{C}_{dgwu}(\mathbb{k})$ is defined in two steps, as follows. Given a unital dg quiver Γ, consider the free $\mathcal{O}'$-algebra $T_{\mathcal{O}'}(\Gamma)$, generated by Γ. It is defined as follows:

We define a chain of length n in Γ as an ordered set $x_0, x_1, \dots, x_n$. Denote by Γ_n the set of all chains of length n in Γ. For $c \in \Gamma_n$, set

$$\Gamma(c) := \Gamma(x_0, x_1)_+ \otimes \Gamma(x_1, x_2)_+ \otimes \cdots \otimes \Gamma(x_{n-1}, x_n)_+$$

and

$$\Gamma(n)(x,y) := \sum_{\substack{c \in \Gamma_n \\ x_0(c)=x, x_n(c)=y}} \Gamma(c)$$

(for $n = 0$ we set $\Gamma(0)(x,x) = \mathbb{k}\,\mathrm{id}_x$ and $\Gamma(0)(x,y) = 0$ for $x \ne y$). Set

$$\Gamma_{\mathcal{O}'}(x,y) := \sum_{n \ge 0} \mathcal{O}'(n) \otimes \Gamma(n)(x,y) \tag{13}$$

It is a weakly unital dg category with objects V_Γ. The 0-ary operation j generates an element $j_x \in T_{\mathcal{O}'}(x,x)$, for any $x \in V_\Gamma$.

After that, define $F(\Gamma)$ as the dg quotient-category

$$F(\Gamma) = T_{\mathcal{O}'}(\Gamma)/(j_x - \mathrm{id}_x, x \in V_\Gamma) \tag{14}$$

In this way, we identify $\mathrm{id}_x \in \Gamma(x,x)$ with the "weak unit" j_x generated by $\mathcal{O}'$.

One has:

2.14. Proposition. *There is an adjunction:*

$$\mathbb{C}_{dgwu}(F(\Gamma), C) \simeq \mathbb{G}_{dg}(\Gamma, U(C)) \tag{15}$$

Proof. ■

Note that for Γ a (non-unital) dg quiver, one defines a unital dg quiver Γ_+, formally adding $\Bbbk_x$ to $\Gamma(x,x)$, for any $x \in V_\Gamma$. Then

$$F(\Gamma_+) \simeq T_{\mathcal{O}'}(\Gamma)$$

2.15. The coequalizers in $\mathbb{G}_{dg}(\Bbbk)$. It is standard that coequalizers, and, therefore, all small colimits exist in $\mathbb{G}_{dg}(k)$.

Recall the construction.

Let

$$\Gamma_1 \overset{f}{\underset{g}{\rightrightarrows}} \Gamma_2 \tag{16}$$

be a pair of morphisms in $\mathbb{G}_{dg}(\Bbbk)$.

Define its coequalizer $\Gamma_{f,g}$ as a small quiver in $\mathbb{G}_{dg}(\Bbbk)$ whose set of objects is the coequalizer of the corresponding maps of the sets of objects

$$\mathrm{Ob}(\Gamma_1) \overset{f}{\underset{g}{\rightrightarrows}} \mathrm{Ob}(\Gamma_2)$$

It is the quotient set of $\mathrm{Ob}(\Gamma_2)$ by the equivalence relation generated by the binary relation $f(x)\mathrm{R}g(x)$, $x \in \mathrm{Ob}(\Gamma_1)$.

Let $[x]$ and $[y]$ be two equivalence classes. Define a complex $\Gamma_{f,g}([x],[y])$ as the coequalizer in $\mathcal{V}ect_{dg}(\Bbbk)$ of

$$\bigoplus_{\substack{w,z \in \mathrm{Ob}(\Gamma_1) \\ [f(w)]=[g(w)]=[x] \\ [f(z)]=[g(z)]=[y]}} \Gamma_1(w,z) \overset{f_*}{\underset{g_*}{\rightrightarrows}} \bigoplus_{\substack{a,b \in \mathrm{Ob}(\Gamma_2) \\ [a]=[x],[b]=[y]}} \Gamma_2(a,b) \tag{17}$$

where f_* maps $\phi \in \Gamma_1(w,z)$ to $f(\phi)$, and g_* maps it to $g(\phi)$. If at least one class of $[x],[y]$ is not in the image of f (which is the same that the image of g), we define source complex in (17) as 0.

It is easy to check that the constructed dg quiver $\Gamma_{f,g}$ is a coequalizer of (16).

2.16. The coequalizers in $\mathbb{C}_{dgwu}(\Bbbk)$, I. Consider a pair of maps of weakly unital dg categories

$$A \overset{F}{\underset{G}{\rightrightarrows}} B \tag{18}$$

It is not straightforward to find (or to prove existence of) its coequalizer.

However, one always can find the coequalizer of the maps of graphs

$$U(A) \underset{U(G)}{\overset{U(F)}{\rightrightarrows}} U(B) \xrightarrow{\ell} \mathrm{Coeq}(U(F), U(G)) \tag{19}$$

as in Section 2.15. For some special diagrams (18), the functor U creates coequalizers, see below. Afterwards, we reduce the general coequalizers (18) to these special ones, in Section 2.20.

2.17. Definition. *We say that the diagram* (18) *is good if* $\mathrm{Ob}(A) = \mathrm{Ob}(B)$, *and both* F *and* G *are identity maps on the sets objects.*

Assume that (18) is good. Then the quiver $\mathrm{Coeq}(U(F), U(G))$, which is a particular case of general coequalizers (16) in $\mathbb{G}_{dg}(\Bbbk)$, is especially simple. It has the set of vertices equal to $\mathrm{Ob}(A) = \mathrm{Ob}(B)$, and its morphisms are the quotient-complexes

$$\mathrm{Coeq}(U(F), U(G))(X, Y) = B(X, Y)/(F(f) - G(f))_{f \in A(X,Y)}$$

2.18. Lemma. *Suppose we are given a diagram* (18) *which is good. Then a weakly unital dg category structure* Q *and a map of weakly unital dg categories* $L \colon B \to Q$ *such that*

$$A \underset{G}{\overset{F}{\rightrightarrows}} B \xrightarrow{L} Q$$

is a coequalizer, and $U(Q) = \mathrm{Coeq}(U(F), U(G))$, $U(L) = \ell$, *exist if and only if the following two conditions hold:*

(1) the subcomplexes $(F(f) - G(f))_{f \in A(X,Y)}$, $X, Y \in \mathrm{Ob}(A)$, *form a two-sided ideal in* B*:*

$$\ell(g \circ (F(f) - G(f)) \circ g') = 0 \tag{20}$$

for any morphism f *in* A *and any morphisms* g, g' *in* B *(such that the compositions are defined),*

(2)

$$\ell(p_n^B(g_1 \otimes \dots g_k \otimes (g \circ (F(f) - G(f)) \circ g') \otimes g_{k+1} \otimes \dots \otimes g_{n-1})) = 0 \tag{21}$$

for $n \geq 2$, *and any morphism* f *in* A *(some of* g_i *are elements of* $\Bbbk_B$*).*

In particular, the weakly unital dg category Q, *if it exists, is uniquely defined (which means that in this case* U *strictly creates the coequalizer).*

Proof. It is clear. ∎

Recall that diagram (18) is called reflexive if there exists $H\colon B\to A$ such that $FH=GH=\mathrm{id}_B$.

2.19. Proposition. *Assume we are given a good and reflexive diagram (18). Then conditions (1) and (2) of Lemma 2.18 are fulfilled. Consequently, the functor U strictly creates the coequalizer.*

Proof. Prove that (1) holds. One has:

$$\begin{aligned}&\ell(g\circ(F(f)-G(f))\circ g')=\ell(g\circ F(f)\circ g')-\ell(g\circ G(f)\circ g')=\\&\ell(FH(g)\circ F(f)\circ FH(g'))-\ell(GH(g)\circ G(f)\circ GH(g'))=\\&\ell(F(H(g)\circ f\circ H(g'))-\ell(G(H(g)\circ f\circ H(g'))=0\end{aligned}\tag{22}$$

Prove that (2) holds. One has:

$$\begin{aligned}&\ell(p_n^B(g_1\otimes\dots\otimes(g\circ(F(f)-G(f))\circ g')\otimes\dots\otimes g_{n-1}))=\\&\ell(p_n^B(g_1\otimes\dots\otimes(g\circ F(f)\circ g')\otimes\dots\otimes g_{n-1}))-\ell(p_n^B(g_1\otimes\dots\otimes(g\circ G(f)\circ g')\otimes\dots\otimes g_{n-1}))=\\&\ell(p_n^B(FH(g_1)\otimes\dots\otimes(FH(g)\circ F(f)\circ FH(g'))\otimes\dots\otimes FH(g_{n-1}))-\\&\ell(p_n^B(GH(g_1)\otimes\dots\otimes(GH(g)\circ G(f)\circ GH(g'))\otimes\dots\otimes GH(g_{n-1})))=\\&\ell(p_n^B(FH(g_1)\otimes\dots\otimes(F(H(g)\circ f\circ H(g'))\otimes\dots\otimes FH(g_{n-1}))-\\&\ell(p_n^B(GH(g_1)\otimes\dots\otimes(G(H(g)\circ f\circ H(g')))\otimes\dots\otimes GH(g_{n-1})))\overset{*}{=}\\&\ell(Fp_n^A(H(g_1)\otimes\dots\otimes(H(g)\circ f\circ H(g'))\otimes\dots\otimes H(g_{n-1})))-\\&\ell(Gp_n^A(H(g_1)\otimes\dots\otimes(H(g)\circ f\circ H(g'))\otimes\dots\otimes H(g_{n-1})))=0\end{aligned}\tag{23}$$

where the equality marked by * follows from the fact that F,G are functors of weakly unital dg categories and (4). ■

2.20. The coequalizers in $\mathbb{C}_{dgwu}(\Bbbk)$, II. In this Section, we closely follow the arguments in [Wo, Prop. 2.11]. We reproduce them here for completeness.

We make use of the following lemma, due to [Li, pp. 77-78], and known as the 3x3-lemma.

2.21. Lemma. *Consider the following diagram in a category*

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] \arrow[d, shift right, "\alpha_1"'] \arrow[d, shift left, "\alpha_2"] & B_1 \arrow[r, "h_3"] \arrow[d, shift right, "\beta_1"'] \arrow[d, shift left, "\beta_2"] & C_1 \arrow[d, shift right, "\gamma_1"'] \arrow[d, shift left, "\gamma_2"] \\
A_2 \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d, "\alpha_3"'] & B_2 \arrow[r, "g_3"] \arrow[d, "\beta_3"'] \arrow[dr, phantom, "*"] & C_2 \arrow[d, "\gamma_3"] \\
A_3 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & B_3 \arrow[r, "f_3"] & C_3
\end{tikzcd}\tag{24}$$

in which the top and the middle rows are coequalizers, the leftmost and the middle columns are coequalizers, and all squares commute: $g_i\alpha_i = \beta_i h_i$, $f_i\alpha_3 = \beta_3 g_i$, $g_3\beta_i = \gamma_i h_3$, $f_3\beta_3 = \gamma_3 g_3$, $i = 1, 2$. *Then the following statements are equivalent:*

(1) the bottom row is a coequalizer,

(2) the rightmost column is a coequalizer,

(3) the square in the lower right corner (marked by $*$*) is a push-out.*

Proof. ■

2.22. Proposition. *The category* $\mathbb{C}_{dgwu}(\Bbbk)$ *has all coequalizers.*

Proof. Let

$$\begin{tikzcd} A \arrow[r, shift left, "H_1"] \arrow[r, shift right, "H_2"'] & B \end{tikzcd} \tag{25}$$

be two arrows in $\mathbb{C}_{dgwu}(\Bbbk)$ coequalizer of which we'd like to compute. Embed it to the following solid arrow diagram, where (F, U) is the adjoint pair of functors from Proposition 2.14.

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FUFU(H_1)"] \arrow[r, shift right, "FUFU(H_2)"'] \arrow[d, shift right, "\epsilon_{FUA}"'] \arrow[d, shift left, "FU\epsilon_A"] & FUFUB \arrow[r, "F(L')"] \arrow[d, shift right, "\epsilon_{FUB}"'] \arrow[d, shift left, "FU\epsilon_B"] & FE' \arrow[d, dotted, shift right, "\alpha_1"'] \arrow[d, dotted, shift left, "\alpha_2"] \\ FUA \arrow[r, shift left, "FU(H_1)"] \arrow[r, shift right, "FU(H_2)"'] \arrow[d, "\epsilon_A"'] & FUB \arrow[r, "F(L)"] \arrow[d, "\epsilon_B"'] & FE \arrow[d, dotted, "p"'] \\ A \arrow[r, shift left, "H_1"] \arrow[r, shift right, "H_2"'] & B \arrow[r, dotted, "q"] & X \end{tikzcd} \tag{26}$$

The upper and the middle rows are obtained from (25) by application of $FUFU$ and FU, correspondingly. Denote by E the coequalizer of (UH_1, UH_2) in $\mathbb{G}_{dg}(\Bbbk)$, and by E' the coequalizer of $(UFUH_1, UFUH_2)$ in $\mathbb{G}_{dg}(\Bbbk)$. As F is left adjoint, FE and FE' are the coequalizers of (FUH_1, FUH_2) and $(FUFUH_1, FUFUH_2)$ in $\mathbb{C}_{dgwu}(\Bbbk)$, correspondingly. Therefore, the upper and the middle rows of (26) are coequalizers.

The leftmost and the middle columns fulfil the assumptions of Proposition 2.19. Indeed, the upper pairs of arrows are reflexive, by the second case of Lemma 2.11, see (9). Therefore, these columns are coequalizers, by Proposition 2.19.

The dotted arrows α_1, α_2 are constructed as follows. For α_1, consider the map

$$F(L) \circ \epsilon_{FUB} \colon FUFUB \to FE$$

The two compositions

$$FUFUA \underset{FUFUH_2}{\overset{FUFUH_1}{\rightrightarrows}} FUFUB \xrightarrow{F(L)\circ\epsilon_{FUB}} FE$$

are equal, which gives rise to a unique map $\alpha_1\colon FE' \to FE$.

Similarly, taking $FU\epsilon_B$ instead of ϵ_{FUB}, one gets a unique map $\alpha_2\colon FE' \to FE$, which coequalizes the corresponding two arrows.

We claim that the pair (α_1, α_2) is reflexive. We construct $\varkappa_E\colon FE \to FE'$ such that $\alpha_1 \circ \varkappa_E = \alpha_2 \circ \varkappa_E = \mathrm{id}_{FE}$.

Recall $\varkappa_A\colon FUA \to FUFUA$ and $\varkappa_B\colon FUB \to FUFUB$ given as in (9):

$$\varkappa_A = F\eta_{UA}, \quad \varkappa_B = F\eta_{UB}$$

These maps are sections of the corresponding pairs of maps, which make them reflexive pairs, see Lemma 2.11. Consider

$$F(L') \circ \varkappa_B\colon FUB \to FE'$$

The two maps

$$FUA \rightrightarrows FUB \xrightarrow{F(L')\circ\varkappa_B} FE'$$

are equal, which gives rise to a unique map

$$\varkappa_E\colon FE \to FE'$$

A simple diagram chasing shows that $\alpha_1 \circ \varkappa_E = \alpha_2 \circ \varkappa_E = \mathrm{id}_{FE}$.

One has $\mathrm{Ob}(FE) = \mathrm{Ob}(FE')$, and Proposition 2.22 is applied. We get an arrow $p\colon FE \to X$ which is a coequalizer of (α_1, α_2).

Finally, we have to construct an arrow $q\colon B \to X$ making the square in the lower right corner commutative. To this end, consider $p \circ F(L)\colon FUB \to X$. The two compositions

$$FUFUB \rightrightarrows FUB \xrightarrow{p\circ F(L)} X$$

are equal, which gives a unique map $q\colon B \to X$. One checks that the lower right square commutes.

One makes use of Lemma 2.21 to conclude that the bottom row is a coequalizer.

∎

We have already seen in Section 2.6 that the products, the coproducts, and the equalizers in $\mathbb{C}_{dgwu}(\Bbbk)$ are constructed straightforwardly. Then Proposition 2.22, and the classic result [R, Th. 3.4.11] give:

2.23. Theorem. *The category $\mathbb{C}_{dgwu}(\Bbbk)$ is small complete and small cocomplete.*

Proof. ∎

2.23.1. The monadicity. Although we will not be using the following result in this paper, it may have an independent interest. The argument is close to [Wo, Th. 2.13].

2.24. Theorem. *The adjunction*

$$F\colon \mathbb{G}_{dg}(\Bbbk) \rightleftarrows \mathbb{C}_{dgwu}(\Bbbk)\colon U$$

is monadic.

Proof. We deduce the statement from the Beck Monadicity Theorem 2.9, for which we have to prove that the assumptions in (1)-(3) in Theorem 2.9 hold.

(1) has been proven in Proposition 2.22, by which $\mathbb{C}_{dgwu}(\Bbbk)$ has all coequalizers, and (3) is clear. One has to prove (2), that is, that the functor $U\colon \mathbb{C}_{dgwu}(\Bbbk)\to\mathbb{G}_{dg}(\Bbbk)$ preserves all U-split coequalizers. We make use of Lemma 2.21, once again.

Let a pair of arrows in $\mathbb{C}_{dgwu}(\Bbbk)$

$$A \underset{H_2}{\overset{H_1}{\rightrightarrows}} B \tag{27}$$

be U-split. Then

$$UA \underset{UH_2}{\overset{UH_1}{\rightrightarrows}} UB \xrightarrow{L} E \tag{28}$$

is a split coequalizer, for some L and E. The upper and the middle rows in (26) are defined now as the result of application of FUF and F, correspondingly, to (28). (In particular, now $E'=UF(E)$, $L'=UF(L)$). Therefore, the upper and the middle rows are split, and, therefore, *absolute* coequalizers, by Lemma 2.8.

Then we get the dotted arrows in (26), and construct X, as in the proof of Proposition 2.22. In particular, we get a coequalizer

$$A \underset{H_2}{\overset{H_1}{\rightrightarrows}} B \xrightarrow{q} X \tag{29}$$

at the bottom row of (26). One has to prove that $UX\simeq E$.

In the obtained diagram all columns and two upper rows are split coequalizers, but the bottom row is also a coequalizer but possibly not split. Now apply the functor U to the whole diagram. As split coequalizers are absolute, by Lemma 2.8, the upper two rows and all three columns remain coequalizers. Therefore, by Lemma 2.21, the bottom row also remains a coequalizer, after application of the functor U. ∎

3. Cohomology of the dg operad $\mathcal{O}'$

Here we prove Theorem 2.13.

PROOF. Let $\omega \in \mathcal{O}'$. Then ω is a linear combination of labelled "trees", where each vertex (excluding the leaves) is labelled either by $p_{n;n_1,\dots,n_k}$ or by m. We say that $p_{n;n_1,\dots,n_k}$ has $n-k$ *operadic arguments* (the remaining k arguments are 1's). We use notation $\sharp(p_{n;n_1,\dots,n_k}) = n-k$. Given a tree T in which a vertex v is labelled by $p_{n;n_1,\dots,n_k}$, we write $\sharp(v) = n-k$. We extend $\sharp(-)$ to all vertices of T, by setting $\sharp(v) = 0$ if v is labelled by m. Denote by V_T the set of all vertices of T excluding the leaves.

For a given tree T, denote

$$\sharp(T) = \sum_{v \in V_T} \sharp(v)$$

We also denote by $\sharp_p(T)$ the total number of vertices with $p_{\dots}$, *excluding* $p_1(1), p_2(1,1), \dots$.

Define a descending filtration $F_{\bullet}$ on $\mathcal{O}'$, as follows. Its $(-\ell)$-th term $F_{-\ell}$ is formed by linear combinations of labelled trees T for which

$$\sharp(T) - \sharp_p(T) \leq \ell$$

Note that for any tree T one has $\sharp(T) - \sharp_p(T) \geq 0$.

One has:

$$\dots \supset F_{-3} \supset F_{-2} \supset F_{-1} \supset F_0 \supset 0$$

Note that $dF_{-\ell} \subset F_{-\ell}$, and any component of the differential on $\mathcal{O}'$ either preserves $\sharp(T) - \sharp_p(T)$ or decreases it by 1.

We get a similar filtration $F_{\bullet}$ on the component $\mathcal{O}'(N)$ of the airity N operations.

We compute cohomology of $\mathcal{O}'(N)$ using the spectral sequence associated with filtration $F_{\bullet}$ on $\mathcal{O}'(N)$. The spectral sequence lives in the quadrant $\{x \leq 0, y \leq 0\}$, the differential d_0 is horizontal. One easily sees that the spectral sequence converges. In fact, we show the spectral sequence collapses at the term E_1.

3.1. LEMMA. *Consider the filtration $F_{\bullet}$ on $\mathcal{O}'(N)$. One has:*

$$E_1^{-\ell,m} = \begin{cases} \mathcal{A}ssoc_+(N) & \ell = 0, m = 0 \\ 0 & \textit{otherwise} \end{cases}$$

In particular, the spectral sequence collapses at the term E_1.

PROOF. We write $p_{n;n_1,\dots,n_k}$ as $p_n(f_1, f_2, \dots, 1, \dots, f_{n-k})$ where $f_1, \dots, f_{n-k}$ are operadic arguments, and 1s stand on the places $n_1, n_2, \dots, n_k$. In these notations, describe the differential in $E_0^{-\ell,\bullet} = F_{-\ell}/F_{-\ell+1}$.

It has components of the following three types, which we refer to as Type I, Type II and Type III components.

Type I components: a component of Type I acts on a group of consecutive 1s, surrounded by operadic arguments from both sides, such as

$$p_n(\dots, f_s, \underbrace{1, 1, \dots, 1}_{\text{a group of } i \text{ consecutive 1s}}, f_{s+1}, \dots)$$

For such a group, the component of d_0 is a sum of expressions, each summand of which corresponds to either a product $1 \cdot 1$ of two consecutive 1s, or to extreme products $f_s \cdot 1$ or $1 \cdot f_{s+1}$, taken with alternated signs. It is clear that totally the component d_0^S corresponding to such a group S is equal to

$$d_0^S(p_n(\dots, f_s, \underbrace{1, \dots, 1}_{i \text{ of 1s in the group } S}, f_{s+1}, \dots)) = \begin{cases} \pm p_n(\dots, f_s, \underbrace{1, 1, \dots, 1}_{i-1 \text{ of 1s}}, f_{s+1}, \dots) & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

Type II components: a component of Type II acts on the groups of leftmost (corresp., rightmost) 1s, such as $p_n(1, 1, \dots, 1, f_1, \dots)$ or $p_n(\dots, f_{n-k}, 1, 1, \dots, 1)$, surrounded by an operadic argument from one side. There should be ≥ 1 of 1s in the group for a non-zero result, and by assumption $p_n(\dots)$ contains at least one operadic argument.

The corresponding component d_0^S of the differential is a sum of two subcomponents: $d_0^S = d_0^{S,1} + d_0^{S,2}$.

The first subcomponent $d_0^{S,1} = d_0^{S,1,-} \pm d_0^{S,1,+}$, where

$$d_0^{S,1,-}(p_n(\underbrace{1, \dots, 1}_{i \text{ of 1s}}, f_1, \dots)) =$$
$$p_n(1 \cdot 1, 1, \dots, 1, f_1, \dots) - p_n(1, 1 \cdot 1, \dots, f_1, \dots) + \dots + (-1)^{i-1} p_n(1, \dots, 1, 1 \cdot f_1, \dots)$$

and similarly for $d_0^{S,1,+}$ for the group of rightmost 1s.

One has

$$d_0^{S,1,-}(p_n(\underbrace{1, \dots, 1}_{i \text{ of 1s}}, f_1, \dots)) = \begin{cases} p_n(\underbrace{1, \dots, 1}_{i-1 \text{ of 1s}}, f_1, \dots) & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}$$

and similarly for $d_0^{S,1,+}$.

The second subcomponent $d_0^{S,2} = d_0^{S,2,-} \pm d_0^{S,2,+}$, where

$$d_0^{S,2,-}(p_n(\underbrace{1, \dots, 1}_{i \text{ of 1s}}, f_1, \dots)) =$$
$$p_1(1) \cdot p_{n-1}(1, \underset{i-1}{\dots}, 1, f_1, \dots) - p_2(1, 1) \cdot p_{n-2}(1, \underset{i-2}{\dots}, 1, f_1, \dots) + \dots + (-1)^{i-1} p_i(1, 1, \dots, 1) \cdot p_{n-i}(f_1, \dots)$$

and similarly for $d_0^{S,2,+}$ for the rightmost group of 1s.

One checks that all other components of the differential d on $\mathcal{O}'$ decrease $\sharp(T) - \sharp_p(T)$ by 1.

Type III components: Here we have d_0 acting on $p_n(\underset{n \text{ of 1s}}{1, 1, \dots, 1})$.

One has:

$$\begin{aligned} &d_0(p_n(1, 1, \dots, 1)) = \\ &p_{n-1}(1 \cdot 1, 1, \dots, 1) - p_{n-1}(1, 1 \cdot 1, 1, \dots, 1) + \dots + (-1)^{i-1} p_{n-1}(1, 1, \dots, 1 \cdot 1) + \\ &\pm \sum_{1 \leq i \leq n-1} (-1)^{i-1} p_i(1, 1, \dots, 1) \cdot p_{n-i}(1, 1, \dots, 1) + \end{aligned} \tag{30}$$

Denote the first summand by $d_0^{S,1}$ and the second summand by $d_0^{S,2}$ One sees that

$$d_0^{S,1}(p_n(1,1,\dots,1)) = \begin{cases} p_{n-1}(1,1,\dots,1) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

The computation of cohomology of the complex $(E_0^{-\ell,\bullet}, d_0)$ is reduced to the computation of the cohomology of a tensor product of complexes (the factors are labelled by combinatorial data of the labelled tree T), corresponding to different components S as listed above:

$$E_0^{-\ell,\bullet} = \bigotimes_{S,T} K_S^\bullet \tag{31}$$

The complexes K_S corresponding to Type I components are isomorphic to

$$K^\bullet = \{\dots \xrightarrow{0} \underset{i=4}{\Bbbk} \xrightarrow{\mathrm{id}} \underset{i=3}{\Bbbk} \xrightarrow{0} \underset{i=2}{\Bbbk} \xrightarrow{\mathrm{id}} \underset{\substack{i=1\\ \deg=-1}}{\Bbbk} \to 0\} \tag{32}$$

The complex $K^\bullet$ is acyclic in all degrees. It implies that the complex $(E_0^{-\ell,\bullet}, d_0)$ is quasi-isomorphic to its subcomplex which is formed by the trees in which any p is of the type $p_n(1,1,\dots,1,f_1,\dots,f_{n-k},1,\dots,1)$, where all $n-k$ operadic arguments stand successively, without 1s between them.

It remains to treat the Type II and Type III cases.

The complexes whose cohomology we need to compute are of two types. They are formed either by linear combinations of

$$p_{n_1}(1,1,\dots,1)\cdot p_{n_2}(1,1,\dots,1)\dots p_{n_k}(1,1,\dots,1)\cdot p_n(1,1,\dots,1,f_1,\dots)$$

or by all linear combinations of

$$p_{n_1}(1,1,\dots,1)\cdot p_{n_2}(1,1,\dots,1)\dots p_{n_k}(1,1,\dots,1)$$

Denote them by $K_1^\bullet$ and $K_2^\bullet$.

Their cohomology are computed similarly, we consider the case of $K_2^\bullet$, leaving the case of $K_1^\bullet$ to the reader.

Denote $p_\ell = p_\ell(1,1,\dots,1)$ and by P_ℓ the 1-dimensional vector space $\Bbbk p_\ell(1,1,\dots,1) = \Bbbk p_\ell$, $\ell \geq 1$.

One has:

$$K_2^{-n} = \bigoplus_{k\geq 1,\ n_1+\dots+n_k-k=n} P_{n_1}\otimes P_{n_2}\otimes\dots\otimes P_{n_k}$$

We denote the differential d_0 on $K_2^\bullet$, see (30), by d.

3.2. Lemma. *The complex $(K_2^\bullet, d)$ is quasi-isomorphic to $P_1[0]$.*

PROOF. Consider on $K_2^\bullet$ the following descending filtration $\Phi_\bullet$, where

$$\Phi_{-\ell} = \bigoplus_{n_1+n_2+\dots+n_k\le \ell} P_{n_1}\otimes P_{n_2}\otimes\dots\otimes P_{n_k}$$

One has

$$\dots\supset\Phi_{-3}\supset\Phi_{-2}\supset\Phi_{-1}\supset\Phi_0=0$$
$$d\Phi_{-\ell}\subset\Phi_{-\ell}$$

Denote by $d_{0,\Phi}$ the differential in $E_{0,\Phi}^{-\ell,\bullet}=\Phi_{-\ell}/\Phi_{-\ell+1}$. It is given by

$$d_{0,\Phi}(p_{n_1}\otimes p_{n_2}\otimes\dots\otimes p_{n_k})=\sum_{i=1}^k(-1)^{n_1+\dots+n_{i-1}-i+1}p_{n_1}\otimes\dots\otimes d_{0,\Phi}(p_{n_i})\otimes\dots\otimes p_{n_k} \quad (33)$$

where

$$d_0(p_n)=\sum_{1\le i\le n-1}(-1)^{i-1}p_i\otimes p_{n-i} \quad (34)$$

It is well-known that the complex $E_{0,\Phi}^{-\ell,\bullet}$ is acyclic when $\ell\ge 2$, and is quasi-isomorphic to $P_1[0]$ when $\ell=1$.

We can identify $P_n\simeq(\Bbbk[1])^{\otimes n}$, then $\oplus_{n\ge1}\Bbbk[1]^{\otimes n}=P$ becomes the (non-unital) cofree coalgebra cogenerated by $\Bbbk[1]$. The complex (33), (34) is identified with the cobar-complex of the cofree coalgebra P. It is standard that its cohomology is equal to $\Bbbk[1][-1]\simeq\Bbbk$.

Therefore, the spectral sequence collapses at the term E_1 by dimensional reasons.

It completes the proof of Lemma 3.2. ■

Similarly we prove that $K_1^\bullet$ is acyclic in all degrees.

In this way we see that any cohomology class in $E_0^{-\ell,\bullet}$ can be represented by a linear combination of trees which do not contain p_ns with $n\ge 2$.

It follows that any cohomology class can be represented by a linear combination of trees containing only m and $p(1)$, and all such trees have cohomological degree 0.

It completes the proof. ■

Theorem 2.13 follows from Lemma 3.1. ■

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