

COLIMITS IN ENRICHED ∞ -CATEGORIES AND DAY CONVOLUTION

VLADIMIR HINICH

ABSTRACT. Let \mathcal{M} be a monoidal ∞ -category with colimits. In this paper we study colimits of \mathcal{M} -functors $\mathcal{A} \rightarrow \mathcal{B}$ where \mathcal{B} is left-tensored over \mathcal{M} and \mathcal{A} is an \mathcal{M} -enriched category. We prove that the enriched Yoneda embedding $Y : \mathcal{A} \rightarrow P_{\mathcal{M}}(\mathcal{A})$ yields a universal \mathcal{M} -functor. In case when \mathcal{A} has a certain monoidal structure, the category of enriched presheaves $P_{\mathcal{M}}(\mathcal{A})$ inherits the same monoidal structure and the enriched Yoneda embedding acquires the structure of universal monoidal \mathcal{M} -functor.

1. Introduction

In this paper we use the word *category* to denote an ∞ -category and the word *operad* to denote an ∞ -operad in the sense of Lurie [L.HA], Section 2. On the contrary, if we want to stress that a certain ∞ -category is a category in the classical sense, we call it a conventional category.

1.1. Throughout the paper we assume that \mathcal{M} is a monoidal category with colimits, such that the tensor product in \mathcal{M} preserves colimits in both arguments. This means that $\mathcal{M} \in \mathbf{Alg}_{\mathbf{Ass}}(\mathbf{Cat}^L)$, where \mathbf{Cat}^L denotes the category of categories with colimits, the arrows being the colimit-preserving functors. We denote by $\mathbf{LMod}_{\mathcal{M}}$ the category of left \mathcal{M} -modules in \mathbf{Cat}^L . In [H.EY] we constructed a Yoneda embedding $Y : \mathcal{A} \rightarrow P_{\mathcal{M}}(\mathcal{A})$ of an \mathcal{M} -enriched category \mathcal{A} into the category of enriched presheaves. In this paper we prove that Y is universal among the functors to a left \mathcal{M} -module \mathcal{B} with colimits: one has a natural equivalence

$$Y^* : \mathbf{Fun}_{\mathbf{LMod}_{\mathcal{M}}}(P_{\mathcal{M}}(\mathcal{A}), \mathcal{B}) \rightarrow \mathbf{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}). \quad (1)$$

1.2. The existence of the functor (1) results from the functoriality of the assignment $\mathcal{B} \mapsto \mathbf{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B})$. There is a functor in the opposite direction that can be described in two ways: as an operadic left Kan extension, or using the notion of weighted colimit.¹

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¹One can think of colimits for functors of two kinds: functors from one \mathcal{M} -enriched category to another, and functors from an \mathcal{M} -enriched category to a category left tensored over \mathcal{M} . In this paper we deal with this second kind of functors.

Given an \mathcal{M} -functor $f : \mathcal{A} \rightarrow \mathcal{B}$, where \mathcal{B} is a category with colimits left-tensored over \mathcal{M} , the weighted colimit $\text{colim}(f) : P_{\mathcal{M}}(\mathcal{A}) \rightarrow \mathcal{B}$ is defined. This gives a functor

$$\text{colim} : \text{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}) \rightarrow \text{Fun}_{\text{LMod}_{\mathcal{M}}}(P_{\mathcal{M}}(\mathcal{A}), \mathcal{B}) \quad (2)$$

quasi-inverse to Y^* .

1.3. In the case when \mathcal{M} is an \mathcal{O} -algebra in the category of monoidal categories (that is, \mathcal{M} is a $\mathcal{O} \otimes \text{Ass}$ -monoidal category), one can define \mathcal{O} -monoidal enriched \mathcal{M} -categories, as well as \mathcal{O} -monoidal left-tensored categories over \mathcal{M} . In this case, if \mathcal{A} is an \mathcal{O} -monoidal \mathcal{M} -enriched category, $P_{\mathcal{M}}(\mathcal{A})$ inherits an \mathcal{O} -monoidal structure, Yoneda embedding becomes an \mathcal{O} -monoidal \mathcal{M} -functor (see 8.3.3), and (1) induces an equivalence

$$\text{Fun}_{\text{LMod}_{\mathcal{M}}}^{\mathcal{O}}(P_{\mathcal{M}}(\mathcal{A}), \mathcal{B}) \rightarrow \text{Fun}_{\mathcal{M}}^{\mathcal{O}}(\mathcal{A}, \mathcal{B}) \quad (3)$$

of the corresponding categories of \mathcal{O} -monoidal functors.

The \mathcal{O} -monoidal structure on $P_{\mathcal{M}}(\mathcal{A})$ is an enriched version of the Day convolution defining a monoidal structure on the presheaves on a monoidal category.

1.4. The paper was started with the aim to prove universality of enriched Yoneda embedding constructed in [H.EY]. At first the task seemed very easy: given an \mathcal{M} -enriched category \mathcal{A} and an \mathcal{M} -functor $f : \mathcal{A} \rightarrow \mathcal{B}$, where \mathcal{B} is a left \mathcal{M} -module with colimits, one can define the functor $\text{colim } f : P_{\mathcal{M}}(\mathcal{A}) \rightarrow \mathcal{B}$ as a weighted colimit, using Lurie's general machinery [L.HA] of relative tensor product (this is now explained in the beginning of Section 6). One can easily prove that any map $F : P_{\mathcal{M}}(\mathcal{A}) \rightarrow \mathcal{B}$ in $\text{LMod}_{\mathcal{M}}$ is equivalent to the colimit of its composition with the Yoneda embedding. But we have not found an easy argument to show that for any $f \in \text{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B})$ the composition of $\text{colim } f$ with the Yoneda embedding gives back f .

This is why we had to add Section 3 comparing our working definition of enriched categories with the one given by Lurie in [L.HA], 4.2.1.28. Now universality of Y follows from the description of colimit preserving left \mathcal{M} -module maps $P_{\mathcal{M}}(\mathcal{A}) \rightarrow \mathcal{B}$ as operadic left Kan extensions of their restriction to $\bar{\mathcal{A}} \subset P_{\mathcal{M}}(\mathcal{A})$, the essential image of the Yoneda embedding.

An \mathcal{O} -monoidal version of the universality easily follows from the interpretation of this operadic Kan extension in terms of weighted colimits.

Our work has a very considerable overlap with the recent manuscript by Hadrian Heine [HH]. In it the category of enriched \mathcal{M} -categories is proven to be equivalent to the one defined by Lurie (we only prove that the functor $\mathcal{A} \mapsto \bar{\mathcal{A}}$ is fully faithful). Heine also proves universality of the Yoneda embedding. It seems, however, that his methods are insufficient to deduce the \mathcal{O} -monoidal version of the universality.

1.5. In Section 2 we provide a digest of the theory of enriched categories and enriched Yoneda lemma. The notion of enriched category used here is the one presented in [H.EY], Sect. 3. Our definition of enriched categories is practically equivalent to the earlier definition of [GH].

J. Lurie defines in [L.HA], 4.2.1.28 another notion of \mathcal{M} -enriched category, as a category weakly tensored over \mathcal{M} , and satisfying some properties.

In Section 3 we compare the notion of enriched categories used in this paper with the one defined by Lurie. We construct a fully faithful functor from the category $\mathbf{Cat}(\mathcal{M})$ of categories enriched over a monoidal category \mathcal{M} with colimits, to the category of Lurie enriched \mathcal{M} -categories². What is more important to us, we interpret \mathcal{M} -functors $f : \mathcal{A} \rightarrow \mathcal{B}$ from an enriched category \mathcal{A} to a left- tensored category \mathcal{B} as functors between the categories weakly tensored over \mathcal{M} .

In Section 4 we review the theory of relative tensor products [L.HA], 3.1 and 4.6.

In Section 5 we study bar resolutions for enriched presheaves. This is a technical section whose result is only needed in the characterization 6.4.2 of morphisms of \mathcal{M} -modules $P_{\mathcal{M}}(\mathcal{A}) \rightarrow \mathcal{B}$ as operadic left Kan extensions.

The notion of relative tensor product allows us to define in Section 6 the weighted colimits. In Section 7 we study the functoriality of the construction of Section 6. This allows one to deduce the multiplicative version of the main universality result in Section 8.

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2. Enriched categories and enriched Yoneda: digest

In this section we recall some important constructions of [H.EY]. The notion of operadic left Kan extension is reviewed in 2.5.

2.1. Throughout the paper we denote by \mathcal{S} the category of spaces and by \mathbf{Cat} the category of small categories. We denote (very seldom) by \mathbf{CAT} the category of big categories. Categories with small colimits and colimit-preserving functors form a category $\mathbf{Cat}^L \subset \mathbf{CAT}$.

2.2. The category of operads \mathbf{Op} is a subcategory of $\mathbf{Cat}/_{\mathbf{Fin}_*}$, where \mathbf{Fin}_* is the category of finite pointed sets. If \mathcal{O} is an operad, we denote $\mathbf{Op}_{/\mathcal{O}}$ or just $\mathbf{Op}_{\mathcal{O}}$ the category of \mathcal{O} -operads, that is operads endowed with a morphism to \mathcal{O} . The terminal object in \mathbf{Op} is the operad for commutative algebras. We denote it \mathbf{Com} or \mathbf{Fin}_* .

The operad \mathbf{Ass}^{\otimes} governs associative algebras, and $\mathbf{Op}_{\mathbf{Ass}}$ is the category of planar operads. We denote by $\mathbf{LM}^{\otimes}, \mathbf{BM}^{\otimes}$ the operads governing the left modules and the bimodules, respectively. Thus, the operad \mathbf{BM}^{\otimes} has three colors, so that the \mathbf{BM}^{\otimes} -algebras are the triples (A, M, B) consisting of two associative algebras A and B acting from the left and

²H. Heine recently proved that these two notion of enrichment are equivalent, see [HH].

from the right on M . Similarly, a \mathbf{BM}^\otimes -operad has three components, two planar operads A and B , and a category M , with two compatible weak actions of A and of B on M .

Following [L.HA], 2.3.3 and [H.EY], 2.7.1, we often replace operads with their strong approximations. In particular, we use the approximation \mathbf{Ass} , \mathbf{LM} and \mathbf{BM} of \mathbf{Ass}^\otimes , \mathbf{LM}^\otimes and \mathbf{BM}^\otimes as defined in [H.EY], 2.9.

2.3. QUIVERS. The notion of enriched category, as presented in [H.EY], is based on a functor

$$\mathbf{Quiv} : \mathbf{Cat}^{\text{op}} \times \mathbf{Op}_{\mathbf{Ass}} \rightarrow \mathbf{Op}_{\mathbf{Ass}} \tag{4}$$

carrying a pair (X, \mathcal{M}) to a planar operad $\mathbf{Quiv}_X(\mathcal{M})$ whose colors are \mathcal{M} -quivers, that is functors $A : X^{\text{op}} \times X \rightarrow \mathcal{M}$.

The functor (4) has two variations. The first is a functor

$$\mathbf{Quiv}^{\mathbf{LM}} : \mathbf{Cat}^{\text{op}} \times \mathbf{Op}_{\mathbf{LM}} \rightarrow \mathbf{Op}_{\mathbf{LM}}, \tag{5}$$

and the second is

$$\mathbf{Quiv}^{\mathbf{BM}} : \mathbf{Cat}^{\text{op}} \times \mathbf{Op}_{\mathbf{BM}} \rightarrow \mathbf{Op}_{\mathbf{BM}}. \tag{6}$$

The functors are compatible: the \mathbf{Ass} -component of $\mathbf{Quiv}_X^{\mathbf{LM}}(\mathcal{M}, \mathcal{B})$ is $\mathbf{Quiv}_X(\mathcal{M})$, and so on.

In good cases, the functors \mathbf{Quiv} applied to monoidal categories with enough colimits, produce a monoidal category.

2.3.1. MORE DETAILS. In Section 5 we will need a more detailed information about the functor (5)³.

In what follows $\Delta_{/\mathbf{LM}}$ denotes the category of simplices in \mathbf{LM} . For a fixed $X \in \mathbf{Cat}$, one defines an \mathbf{LM} -operad \mathbf{LM}_X by a presheaf

$$(\Delta_{/\mathbf{LM}})^{\text{op}} \rightarrow \mathcal{S}$$

given by the formula

$$\mathbf{LM}_X(\sigma) = \text{Map}(\mathcal{F}(\sigma), X),$$

where $\mathcal{F} : \Delta_{/\mathbf{LM}} \rightarrow \mathbf{Cat}$ is a functor with values in conventional categories combinatorially defined in [H.EY], 3.2. The \mathbf{LM} -operad \mathbf{LM}_X is always flat, [H.EY], 3.3. This means that the functor $\mathbf{Op}_{\mathbf{LM}} \rightarrow \mathbf{Op}_{\mathbf{LM}}$ given by product with \mathbf{LM}_X , admits a right adjoint, which is denoted $\text{Funop}_{\mathbf{LM}}(\mathbf{LM}_X, -)$. Finally, given $\mathcal{M} = (\mathcal{M}_a, \mathcal{M}_m) \in \mathbf{Op}_{\mathbf{LM}}$, one defines the \mathbf{LM} -operad $\mathbf{Quiv}_X^{\mathbf{LM}}(\mathcal{M})$ as $\text{Funop}_{\mathbf{LM}}(\mathbf{LM}_X, \mathcal{M})$. Two other variations of the category of quivers, \mathbf{Quiv} and $\mathbf{Quiv}^{\mathbf{BM}}$, have a similar description. Given $\mathcal{M} = (\mathcal{M}_a, \mathcal{M}_m, \mathcal{M}_b) \in \mathbf{Op}_{\mathbf{BM}}$, the \mathbf{BM} -operad $\mathbf{Quiv}_X^{\mathbf{BM}}(\mathcal{M})$ has components $(\mathbf{Quiv}_X(\mathcal{M}_a), \text{Fun}(X, \mathcal{M}_m), \mathcal{M}_b)$.

³The functors (4), (6) have a similar description.

2.3.2. **Cat-ENRICHMENT.** Let \mathcal{O} be an operad or a strong approximation of an operad. The category of \mathcal{O} -operads $\mathbf{Op}_{\mathcal{O}}$ has a **Cat**-enrichment that assigns to $\mathcal{P}, \mathcal{Q} \in \mathbf{Op}_{\mathcal{O}}$ the category $\mathbf{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{Q})$.

The functor $\mathbf{Quiv}_X^{\mathbf{LM}} : \mathbf{Op}_{\mathbf{LM}} \rightarrow \mathbf{Op}_{\mathbf{LM}}$ respects this enrichment. This means that, given $\mathcal{P}, \mathcal{Q} \in \mathbf{Op}_{\mathbf{LM}}$, one has a functor

$$\mathbf{Alg}_{\mathcal{P}/\mathbf{LM}}(\mathcal{Q}) \rightarrow \mathbf{Alg}_{\mathbf{Quiv}_X^{\mathbf{LM}}(\mathcal{P})/\mathbf{LM}}(\mathbf{Quiv}_X^{\mathbf{LM}}(\mathcal{Q})) \quad (7)$$

extending the map

$$\mathbf{Map}_{\mathbf{Op}_{\mathbf{LM}}}(\mathcal{P}, \mathcal{Q}) \rightarrow \mathbf{Map}_{\mathbf{Op}_{\mathbf{LM}}}(\mathbf{Quiv}_X^{\mathbf{LM}}(\mathcal{P}), \mathbf{Quiv}_X^{\mathbf{LM}}(\mathcal{Q})).$$

The map (7) is defined as follows. Its target is naturally equivalent, according to [H.EY], 2.8.6, to $\mathbf{Alg}_{\mathbf{Quiv}_X^{\mathbf{LM}}(\mathcal{P}) \times_{\mathbf{LM}} \mathbf{LM}_X}(\mathcal{Q})$. The map (7) can therefore be defined as the one induced by the canonical evaluation map

$$\mathbf{Quiv}_X^{\mathbf{LM}}(\mathcal{P}) \times_{\mathbf{LM}} \mathbf{LM}_X = \mathbf{Funop}_{\mathbf{LM}}(\mathbf{LM}_X, \mathcal{P}) \times_{\mathbf{LM}} \mathbf{LM}_X \rightarrow \mathcal{P}.$$

2.4. **ALGEBRAS IN QUIVERS.** Fixing the second (operadic) argument, we will look at the functors (4)–(6) as cartesian families of (planar, LM or BM) operads. Let us describe our interpretation for the categories of algebras in various operads of quivers.

2.4.1. **ENRICHED PRECATEGORIES.** For a fixed planar operad \mathcal{M} with colimits, an associative algebra in the family $\mathbf{Quiv}(\mathcal{M})$ is called \mathcal{M} -enriched precategory. We denote $\mathbf{PCat}(\mathcal{M}) = \mathbf{Alg}_{\mathbf{Ass}}(\mathbf{Quiv}(\mathcal{M}))$ the category of \mathcal{M} -enriched precategories.

An enriched precategory \mathcal{A} has a category X of objects, and an associative multiplication in the underlying quiver $\mathcal{A} : X^{\mathrm{op}} \times X \rightarrow \mathcal{M}$.

2.4.2. **\mathcal{M} -FUNCTORS.** Fix an LM-operad, consisting of a planar operad \mathcal{M} and a category \mathcal{B} weakly tensored over \mathcal{M} . For fixed $X \in \mathbf{Cat}$, the LM-operad $\mathbf{Quiv}_X^{\mathbf{LM}}(\mathcal{M}, \mathcal{B})$ consists of the planar operad $\mathbf{Quiv}_X(\mathcal{M})$ and a category $\mathbf{Fun}(X, \mathcal{B})$, weakly tensored over $\mathbf{Quiv}_X(\mathcal{M})$.

The LM-operads $\mathbf{Quiv}_X^{\mathbf{LM}}(\mathcal{M}, \mathcal{B})$ form a family $\mathbf{Quiv}^{\mathbf{LM}}(\mathcal{M}, \mathcal{B})$.

An LM-algebra in it consists of a pair (\mathcal{A}, f) where \mathcal{A} is an \mathcal{M} -enriched precategory, and f is an \mathcal{A} -module in $\mathbf{Fun}(X, \mathcal{B})$.

We denote $\mathbf{PCat}^{\mathbf{LM}}(\mathcal{M}, \mathcal{B}) = \mathbf{Alg}_{\mathbf{LM}}(\mathbf{Quiv}^{\mathbf{LM}}(\mathcal{M}, \mathcal{B}))$.

We interpret \mathcal{A} -modules in $\mathbf{Fun}(X, \mathcal{B})$ as \mathcal{M} -functors from \mathcal{A} to \mathcal{B} , whence the notation

$$\mathbf{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}) = \mathbf{LMod}_{\mathcal{A}}(\mathbf{Fun}(X, \mathcal{B})), \quad (8)$$

the category of left \mathcal{A} -modules in $\mathbf{Fun}(X, \mathcal{B})$.

2.4.3. Assume now that \mathcal{M} is a monoidal category with colimits. Applying the above to $\mathcal{B} := \mathcal{M}$ considered as a right \mathcal{M} -module (which is the same as left $\mathcal{M}^{\mathrm{rev}}$ -module), we can define the category of enriched presheaves $P_{\mathcal{M}}(\mathcal{A}) = \mathbf{Fun}_{\mathcal{M}^{\mathrm{rev}}}(\mathcal{A}^{\mathrm{op}}, \mathcal{M})$. It is left-tensored over \mathcal{M} and has colimits.

Yoneda embedding is an \mathcal{M} -functor $Y : \mathcal{A} \rightarrow P_{\mathcal{M}}(\mathcal{A})$, defined by \mathcal{A} -bimodule structure on \mathcal{A} , see details in [H.EY], Section 6.

2.4.4. ENRICHED CATEGORIES. An \mathcal{M} -enriched category is an enriched precategory satisfying a certain completeness condition. The full embedding $\mathbf{Cat}(\mathcal{M}) \subset \mathbf{PCat}(\mathcal{M})$ is right adjoint to a localization functor $L : \mathbf{PCat}(\mathcal{M}) \rightarrow \mathbf{Cat}(\mathcal{M})$ which can be described as follows. Given $\mathcal{A} \in \mathbf{Alg}_{\mathbf{Ass}}(\mathbf{Quiv}(\mathcal{M}))$, we define X as the subspace of $P_{\mathcal{M}}(\mathcal{A})^{eq}$ spanned by the representable functors, and define $L(\mathcal{A})$ as the endomorphism object in $\mathbf{Quiv}_X(\mathcal{M})$ of the tautological embedding $X \rightarrow P_{\mathcal{M}}(\mathcal{A})$, see [H.EY], 7.2.

2.4.5. RESTRICTION OF SCALARS. Given a cartesian family $p : \mathcal{Q} \rightarrow B \times \mathbf{LM}$ of LM-operads, the embedding $\mathbf{Ass} \rightarrow \mathbf{LM}$ induces a functor

$$\mathbf{Alg}_{\mathbf{LM}}(\mathcal{Q}) \rightarrow \mathbf{Alg}_{\mathbf{Ass}}(\mathcal{Q})$$

which is a cartesian fibration. This result can be found in [L.HA], 4.2.3.2 or [H.EY], 2.11.

2.4.6. Let $(\mathcal{M}, \mathcal{B})$ be an LM-operad. The assignment $\mathcal{A} \mapsto \mathbf{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B})$ is contravariant in \mathcal{A} . This is a special case of a general setup presented in 2.4.5.

Thus, a map $f : \mathcal{A} \rightarrow \mathcal{A}'$ of \mathcal{M} -enriched precategories gives rise to a functor $f^* : \mathbf{Fun}_{\mathcal{M}}(\mathcal{A}', \mathcal{B}) \rightarrow \mathbf{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B})$. The definition of f^* allows one to compose a map of \mathcal{M} -enriched precategories $f : \mathcal{A} \rightarrow \mathcal{A}'$ with an \mathcal{M} -functor $\mathcal{A}' \rightarrow \mathcal{B}$.

2.5. OPERADIC LEFT KAN EXTENSIONS. Operadic colimits and operadic left Kan extensions defined in Lurie’s [L.HA], Section 3.1, are a part of the construction of a free operad algebra. In this subsection we present necessary details connected to these notions.

Let \mathcal{O} be an operad and let $\mathcal{C} \in \mathbf{Alg}_{\mathcal{O}}(\mathbf{Cat}^L)$ be an \mathcal{O} -monoidal category with colimits. Given a morphism $f : \mathcal{P} \rightarrow \mathcal{Q}$ of \mathcal{O} -operads, one has a forgetful functor $f^* : \mathbf{Alg}_{\mathcal{Q}}(\mathcal{C}) \rightarrow \mathbf{Alg}_{\mathcal{P}}(\mathcal{C})$. In this context the operadic left Kan extension always exists and defines a functor

$$f_! : \mathbf{Alg}_{\mathcal{P}}(\mathcal{C}) \rightarrow \mathbf{Alg}_{\mathcal{Q}}(\mathcal{C})$$

left adjoint to f^* .

2.5.1. OPERADIC COLIMIT, SEE [L.HA], 3.1.1. For $K \in \mathbf{Cat}$ we denote $K^\triangleright = (K \times [1]) \sqcup^K [0]$, the category obtained by adding to K the terminal object $*$.

Let $p : \mathcal{C} \rightarrow \mathcal{O}$ be an \mathcal{O} -operad. For any functor $f : K \rightarrow \mathcal{C}^{\text{act}}$ to the subcategory of \mathcal{C} spanned by the active arrows we denote

$$\mathcal{C}_{f/}^{\text{act}} = \{f\} \times_{\mathbf{Fun}(K, \mathcal{C}^{\text{act}})} \mathbf{Fun}(K^\triangleright, \mathcal{C}^{\text{act}}) \times_{\mathcal{C}^{\text{act}}} \mathcal{C}_{\langle 1 \rangle}, \tag{9}$$

where the arrows from $\mathbf{Fun}(K^\triangleright, \mathcal{C}^{\text{act}})$ are defined by the embeddings $K \rightarrow K^\triangleright$ and $* \in K^\triangleright$.

A functor $\bar{f} : K^\triangleright \rightarrow \mathcal{C}^{\text{act}}$ with $f := \bar{f}|_K$ is called a weak operadic colimit diagram if the natural map

$$\mathcal{C}_{\bar{f}/}^{\text{act}} \rightarrow \mathcal{C}_{f/}^{\text{act}} \times_{\mathcal{O}_{q \circ f}^{\text{act}}} \mathcal{O}_{p \circ \bar{f}}^{\text{act}}$$

is an equivalence.

One says that \bar{f} is an operadic colimit diagram if for any $C \in \mathcal{C}$ the composition

$$K^\triangleright \xrightarrow{\bar{f}} \mathcal{C}^{\text{act}} \xrightarrow{\oplus_C} \mathcal{C}^{\text{act}}$$

is a weak operadic colimit diagram.

Let $f : K \rightarrow \mathcal{C}^{\text{act}}$ be a diagram in an \mathcal{O} -operad \mathcal{C} and let $\bar{g} : K^\triangleright \rightarrow \mathcal{O}$ be an extension of $g = p \circ f$. We say that the diagram f has an operadic colimit over \bar{g} if there exists \bar{f} over \bar{g} that is an operadic colimit diagram.

In the case when \mathcal{C} is an \mathcal{O} -monoidal category with colimits, such that the monoidal structure is compatible with the colimits, any diagram $f : K \rightarrow \mathcal{C}^{\text{act}}$ has an operadic colimit.

2.5.2. OPERADIC LEFT KAN EXTENSIONS. We present below a definition of operadic left Kan extension due to Lurie, [L.HA], 3.1.2. We restrict ourselves to a special case of what Lurie calls “free algebra”, see [L.HA], 3.1.3.1.

Given $f : \mathcal{P} \rightarrow \mathcal{Q}$ a morphism of \mathcal{O} -operads and an \mathcal{O} -operad \mathcal{C} , one has an obvious functor

$$f^* : \mathbf{Alg}_{\mathcal{Q}}(\mathcal{C}) \rightarrow \mathbf{Alg}_{\mathcal{P}}(\mathcal{C}).$$

For any $q \in \mathcal{Q}$ we define $K_q = \mathcal{P} \times_{\mathcal{Q}} \mathcal{Q}_{/q}^{\text{act}}$.

Given $A \in \mathbf{Alg}_{\mathcal{P}}(\mathcal{C})$ and $B \in \mathbf{Alg}_{\mathcal{Q}}(\mathcal{C})$, a morphism $j : A \rightarrow f^*(B)$ in $\mathbf{Alg}_{\mathcal{P}}(\mathcal{C})$ determines a morphism of functors $\alpha_q \rightarrow \text{const}_{B(q)} : K_q \rightarrow \mathcal{C}^{\text{act}}$, with α_q being the composition of A with the projection $K_q \rightarrow \mathcal{P}$ and $\text{const}_{B(q)}$ being the constant functor with the value $B(q)$. Equivalently, this translates into a functor

$$\bar{\alpha}_q : K_q^\triangleright \rightarrow \mathcal{C}^{\text{act}}. \tag{10}$$

A morphism $j : A \rightarrow f^*(B)$ is called an operadic left Kan extension of A with respect to f if for any $q \in \mathcal{Q}$ the functor $\bar{\alpha}_q : K_q^\triangleright \rightarrow \mathcal{C}^{\text{act}}$ is an operadic colimit diagram.

In the case when $\mathcal{C} \in \mathbf{Alg}_{\mathcal{O}}(\mathbf{Cat}^L)$, the operadic left Kan extension exists and defines a functor $f_! : \mathbf{Alg}_{\mathcal{P}}(\mathcal{C}) \rightarrow \mathbf{Alg}_{\mathcal{Q}}(\mathcal{C})$ left adjoint to f^* .

3. Lurie’s enriched categories

In this section we compare the notion of \mathcal{M} -enriched category and of an \mathcal{M} -enriched functor, as presented in 2.4.4 and 2.4.2, with the similar (but simpler) notions of Lurie, [L.HA], 4.2.1.

3.1. LM-OPERADS AND LURIE’S ENRICHED CATEGORIES. Probably, the simplest way to define an enriched ∞ -category over a monoidal category \mathcal{M} is presented in Lurie’s [L.HA], 4.2.1.28.

3.1.1. The map $a : \mathbf{Ass} \rightarrow \mathbf{LM}$ of operads induces a base change functor

$$a^* : \mathbf{Op}_{\mathbf{LM}} \rightarrow \mathbf{Op}_{\mathbf{Ass}} \tag{11}$$

assigning to each LM-operad \mathcal{O} its planar component $\mathcal{O}_a \rightarrow \mathbf{Ass}$. The fiber \mathcal{O}_m at $m \in \mathbf{LM}$ is a category that is called *weakly enriched* over \mathcal{O}_a .

We denote by $\mathbf{LMod}_{\mathcal{M}}^w$ the fiber of a^* at $\mathcal{M} \in \mathbf{Op}_{\mathbf{Ass}}$. This is the category of categories weakly enriched over \mathcal{M} .

An object of $\mathbf{LMod}_{\mathcal{M}}^w$ is an LM-operad \mathcal{O} together with an equivalence $\mathcal{M} = \mathbf{Ass} \times_{\mathbf{LM}} \mathcal{O}$. We will sometimes denote it as a pair $(\mathcal{M}, \mathcal{A})$, where $\mathcal{A} = \{m\} \times_{\mathbf{LM}} \mathcal{O}$, or (when \mathcal{M} is fixed) as \mathcal{A} .

3.1.2. For $\mathcal{O}, \mathcal{O}' \in \mathbf{LMod}_{\mathcal{M}}^w$ we define $\mathbf{Fun}_{\mathbf{LMod}_{\mathcal{M}}^w}(\mathcal{O}, \mathcal{O}')$ as the fiber of the map

$$\mathbf{Alg}_{\mathcal{O}/\mathbf{LM}}(\mathcal{O}') \rightarrow \mathbf{Alg}_{\mathcal{M}/\mathbf{Ass}}(\mathcal{M})$$

at $\mathrm{id}_{\mathcal{M}}$.

Let now \mathcal{M} be a monoidal category. Here is the definition of Lurie’s \mathcal{M} -enriched category.

3.1.3. DEFINITION. Let \mathcal{M} be a monoidal category. A Lurie \mathcal{M} -enriched category \mathcal{A} is an LM-operad \mathcal{O} with the equivalences $\mathcal{M} = \mathbf{Ass} \times_{\mathbf{LM}} \mathcal{O}$, $\mathcal{A} = \{m\} \times_{\mathbf{LM}} \mathcal{O}$, satisfying the following properties.

1. The natural map $\oplus m_i \rightarrow \otimes m_i$ induces an equivalence $\mathrm{Map}((\otimes m_i) \oplus a, b) \rightarrow \mathrm{Map}(\oplus m_i \oplus a, b)$ for any $m_i \in \mathcal{M}$ and $a, b \in \mathcal{A}$. Here we use the sign \oplus as in [L.HA], 2.1.1.15⁴.
2. For any $a, b \in \mathcal{A}$ the weak enrichment functor $\mathrm{hom}_{\mathcal{A}}(a, b) : \mathcal{M}^{\mathrm{op}} \rightarrow \mathcal{S}$, defined by the formula

$$\mathrm{hom}_{\mathcal{A}}(a, b)(m) = \mathrm{Map}(m \oplus a, b),$$

is representable.

Lurie \mathcal{M} -enriched categories form a category denoted $\mathbf{Cat}^{\mathrm{Lur}}(\mathcal{M})$. This is a full subcategory of $\mathbf{LMod}_{\mathcal{M}}^w$ spanned by the Lurie \mathcal{M} -enriched categories.

In this section we assign to any \mathcal{M} -enriched category \mathcal{A} a Lurie \mathcal{M} -enriched category $\bar{\mathcal{A}}$. We prove that the category $\mathbf{Cat}(\mathcal{M})$ defined in 2.4.4 is equivalent to a full subcategory of $\mathbf{Cat}^{\mathrm{Lur}}(\mathcal{M})$, see 3.4.1. Note that H. Heine has recently proven [HH] that the two notions are equivalent.

3.1.4. Let \mathcal{M} be a monoidal category with colimits.

For any \mathcal{M} -enriched category \mathcal{A} we define $\bar{\mathcal{A}} \subset P_{\mathcal{M}}(\mathcal{A})$, the full subcategory of $P_{\mathcal{M}}(\mathcal{A})$ spanned by the representable presheaves. Obviously, $\bar{\mathcal{A}}$ is an \mathcal{M} -enriched category in the sense of Lurie. By [H.EY], 6.1.4, this defines a functor

$$\lambda : \mathbf{Cat}(\mathcal{M}) \rightarrow \mathbf{Cat}^{\mathrm{Lur}}(\mathcal{M}).$$

Corollary 3.4.1 asserts that this functor is fully faithful. As H. Heine shows in [HH], the functor λ is actually an equivalence, see Remark 3.4.2.

⁴The first property is a *pseudo-enrichment* in Lurie’s terminology. see [L.HA], 4.2.1.25. The condition makes sense for the number of factors $n \geq 0$.

3.2. BABY YONEDA FUNCTOR. In what follows we denote by $(\mathcal{M}, \bar{\mathcal{A}})^\otimes$ the LM-operad formed by the category $\bar{\mathcal{A}}$ weakly tensored over a monoidal category \mathcal{M} .

Let \mathcal{M} be a monoidal category with colimits, \mathcal{A} be an \mathcal{M} -enriched category and \mathcal{B} be a category with colimits left-tensored over \mathcal{M} .

Let $(\mathcal{M}, \bar{\mathcal{A}})$ be the corresponding Lurie enriched category.

Our aim is to construct an equivalence

$$\text{Fun}_{\text{LMod}_{\mathcal{M}}^w}(\bar{\mathcal{A}}, \mathcal{B}) \rightarrow \text{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}). \tag{12}$$

3.2.1. Functoriality of the assignment

$$\mathcal{B} \mapsto \text{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}),$$

as defined by the formula (8), and the preservation of Cat -enrichment by Quiv^{LM} , see 2.3.2, yields a canonical functor

$$\text{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}') \times \text{Fun}_{\text{LMod}_{\mathcal{M}}^w}(\mathcal{B}', \mathcal{B}) \rightarrow \text{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}). \tag{13}$$

In particular, for $\mathcal{B}' := \bar{\mathcal{A}}$, we get

$$\text{Fun}_{\mathcal{M}}(\mathcal{A}, \bar{\mathcal{A}}) \times \text{Fun}_{\text{LMod}_{\mathcal{M}}^w}(\bar{\mathcal{A}}, \mathcal{B}) \rightarrow \text{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}). \tag{14}$$

We claim that the Yoneda embedding $Y : \mathcal{A} \rightarrow P_{\mathcal{M}}(\mathcal{A})$ factors uniquely through the full embedding $\bar{Y} : \bar{\mathcal{A}} \rightarrow P_{\mathcal{M}}(\mathcal{A})$; the natural \mathcal{M} -functor $y : \mathcal{A} \rightarrow \bar{\mathcal{A}}$ so defined will be denoted y and called *the baby Yoneda functor*. The existence (and uniqueness) of the baby Yoneda immediately follows from the lemma below.

3.2.2. LEMMA. *Let \mathcal{B} be a full subcategory in \mathcal{C} that is weakly enriched over a monoidal category \mathcal{M} . Then, for an associative algebra \mathcal{A} in \mathcal{M} , one has*

$$\text{LMod}_{\mathcal{A}}(\mathcal{B}) = \text{LMod}_{\mathcal{A}}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{B}.$$

Note another obvious property of the \mathcal{M} -functor $y : \mathcal{A} \rightarrow \bar{\mathcal{A}}$.

3.2.3. LEMMA. *The forgetful functor $\text{Fun}_{\mathcal{M}}(\mathcal{A}, \bar{\mathcal{A}}) \rightarrow \text{Fun}(X, \bar{\mathcal{A}})$ carries y to a map $i : X \rightarrow \bar{\mathcal{A}}$ identifying X with the maximal subspace of $\bar{\mathcal{A}}$.*

3.3. THE FUNCTOR (12) IS AN EQUIVALENCE. In Proposition 3.3.7 we prove that the functor (12) is an equivalence. The idea is to present the source and the target of the map by a monad on $\text{Fun}(X, \mathcal{B})$ and to verify the equivalence of the monads.

3.3.1. Recall that $(\mathcal{M}, \mathcal{B})$ is an LM-monoidal category with colimits. Let $\mathcal{A} \in \text{Alg}(\text{Quiv}_X(\mathcal{M}))$. Let $\Phi : X \rightarrow \mathcal{B}$ be an \mathcal{A} -module in $\text{Fun}(X, \mathcal{B})$ and let $a : \phi \rightarrow \Phi$ be an arrow in $\text{Fun}(X, \mathcal{B})$. In Lemma 3.3.5 below we formulate the condition for a to represent Φ as a free \mathcal{A} -module generated by ϕ .

The \mathcal{A} -module structure on Φ defines an active arrow $(\mathcal{A}, \Phi) \rightarrow \Phi$ in $\text{Quiv}_X^{\text{LM}}(\mathcal{M}, \mathcal{B})$ that is explicitly described in [H.EY] 4.2.1 and 4.3.1, case (w2) with $n = 2, k = 1$.

Here is the description. The active arrow above is given by a map $A := [1] \times_{\mathbf{LM}} \mathbf{LM}_X \rightarrow (\mathcal{M}, \mathcal{B})^{\text{act}}$ where

$$A = A_0 \sqcup^C (C \times [1]) \sqcup^C A_1,$$

with $A_0 = X \times X^{\text{op}} \times X$, $A_1 = X$, $C = \mathbf{Tw}(X)^{\text{op}} \times X$, the map $C \rightarrow A_0$ is given by the projection $\mathbf{Tw}(X)^{\text{op}} \rightarrow X \times X^{\text{op}}$, whereas $C \rightarrow A_1$ is the projection to the last factor.

This yields, for any $x \in X$, a functor

$$\bar{\theta}_x^\Phi : (\mathbf{Tw}(X)^{\text{op}})^\triangleright \rightarrow (\mathcal{M}, \mathcal{B})^{\text{act}} \tag{15}$$

carrying the terminal object $* \in (\mathbf{Tw}(X)^{\text{op}})^\triangleright$ to $\Phi(x)$ and the arrow $\alpha : z \rightarrow y$ from $\mathbf{Tw}(X)$ to the pair $(\mathcal{A}(y, x), \Phi(z))$.

We denote by θ_x^Φ the restriction of $\bar{\theta}_x^\Phi$ to $\mathbf{Tw}(X)^{\text{op}}$. The functor $\theta_x^\phi : \mathbf{Tw}(X)^{\text{op}} \rightarrow (\mathcal{M}, \mathcal{B})^{\text{act}}$ is define in the same way and the map $a : \phi \rightarrow \Phi$ gives rise to a map of functors $a : \theta_x^\phi \rightarrow \theta_x^\Phi$.

3.3.2. A functor $f : K^\triangleright \rightarrow \mathcal{C}$ can be uniquely presented by a map $f|_K \rightarrow \text{const}_{f(*)}$ in $\text{Fun}(K, \mathcal{C})$, where $\text{const}_{f(*)}$ is the constant functor with the value $f(*) \in \mathcal{C}$. In particular, given f as above and an arrow $\alpha : f_1 \rightarrow f|_K$ in $\text{Fun}(K, \mathcal{C})$, we get a canonically defined functor $f' : K^\triangleright \rightarrow \mathcal{C}$ with $f'|_K = f_1$ and $f'(*) = f(*)$.

This allows one to define

$$\bar{\theta}_x^a : (\mathbf{Tw}(X)^{\text{op}})^\triangleright \rightarrow (\mathcal{M}, \mathcal{B})^{\text{act}} \tag{16}$$

as the functor induced by $a : \phi \rightarrow \Phi$ whose restriction to $\mathbf{Tw}(X)^{\text{op}}$ is θ_x^ϕ and the value at the terminal object is $\Phi(x)$.

3.3.3. COCARTESIAN SHIFT. Let $p : \mathcal{C} \rightarrow B$ be a cocartesian fibration and let, as above, $f : K^\triangleright \rightarrow \mathcal{C}$ be a functor.

As above, f gives rise to a map $f|_K \rightarrow \text{const}_{f(*)}$ in $\text{Fun}(K, \mathcal{C})$, as well as to its image $p \circ f|_K \rightarrow \text{const}_{p \circ f(*)}$ in $\text{Fun}(K, B)$. Since the map $\text{Fun}(K, p) : \text{Fun}(K, \mathcal{C}) \rightarrow \text{Fun}(K, B)$ is a cocartesian fibration, we get a map of functors

$$f \rightarrow \mathbf{Sh}(f) : K^\triangleright \rightarrow \mathcal{C},$$

such that $p \circ \mathbf{Sh}(f) = \text{const}_{p \circ f(*)}$ and for each $x \in K$ the arrow $f(x) \rightarrow \mathbf{Sh}(f)(x)$ is p -cocartesian. In this case we will say that $\mathbf{Sh}(f)$ is obtained from f by a *cocartesian shift*.

3.3.4. Applying the cocartesian shift to (16), we get

$$\mathbf{Sh}(\bar{\theta}_x^a) : (\mathbf{Tw}(X)^{\text{op}})^\triangleright \rightarrow \mathcal{B}.$$

One has the following.

3.3.5. LEMMA. *A map $a : \phi \rightarrow \Phi$ in $\text{Fun}(X, \mathcal{B})$ presents Φ as a free \mathcal{A} -module if and only if for any $x \in X$ the diagram $\bar{\theta}_x^a$ is an operadic colimit diagram (or, equivalently, if $\mathbf{Sh}(\bar{\theta}_x^a)$ is a colimit diagram).*

PROOF. The map $a : \phi \rightarrow \Phi$ presents Φ as a free \mathcal{A} -module generated by ϕ if a induces a cocartesian arrow $(\mathcal{A}, \phi) \rightarrow \Phi$ in $\text{Quiv}_X^{\mathbf{LM}}(\mathcal{M}, \mathcal{B})$. This easily translates to our condition. ■

The evaluation of (14) at y defines the canonical functor

$$y^* : \text{Fun}_{\text{LMod}_{\mathcal{M}}}^w(\bar{\mathcal{A}}, \mathcal{B}) \rightarrow \text{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}).$$

Lemma 3.2.3 asserts that $G \circ y^* = G'$ where $G : \text{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}) \rightarrow \text{Fun}(X, \mathcal{B})$ is the forgetful functor and $G' : \text{Fun}_{\text{LMod}_{\mathcal{M}}}^w(\bar{\mathcal{A}}, \mathcal{B}) \rightarrow \text{Fun}(X, \mathcal{B})$ is given by the composition with $X \rightarrow \bar{\mathcal{A}}$.

3.3.6. We will now prove that y^* is an equivalence. Our proof will use the description of the source and the target of y^* by monads on $\text{Fun}(X, \mathcal{B})$.

Let us consider the following diagram.

$$\begin{array}{ccc}
 \text{Fun}_{\text{LMod}_{\mathcal{M}}}^w(\bar{\mathcal{A}}, \mathcal{B}) & \xrightarrow{y^*} & \text{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}) \\
 \swarrow G' & & \searrow F \\
 & \text{Fun}(X, \mathcal{B}) & \\
 \nwarrow F' & & \nearrow G
 \end{array} \tag{17}$$

Here the functor F , left adjoint to G , is the free \mathcal{A} -module functor. The functor F' , left adjoint to G' , is defined by the operadic left Kan extension with respect to the map of LM-operads

$$\epsilon : \mathcal{M} \sqcup X \rightarrow (\mathcal{M}, \bar{\mathcal{A}}).$$

The equivalence $G' = G \circ y^*$ gives rise to a morphism of functors $\eta : F \rightarrow y^* \circ F'$. Here is the main result of this section.

3.3.7. PROPOSITION. *The functor y^* defined above is an equivalence.*

PROOF. According to [L.HA], 4.7.3.16, we have to verify the following conditions.

1. The functors G' and G preserve geometric realizations.
2. The functors G and G' are conservative.
3. $\eta(\phi)$ is an equivalence for any $\phi \in \text{Fun}(X, \mathcal{B})$.

The functor G is conservative by [L.HA], 3.2.2.6 and preserves colimits by [L.HA], 4.2.3.5. The functor G' is conservative (by [L.HA], 3.2.2.6) and preserves geometric realizations by [L.HA], 3.2.3.1.

It remains to verify that $\eta(\phi) : F(\phi) \rightarrow y^* \circ F'(\phi)$ is an equivalence for any $\phi : X \rightarrow \mathcal{B}$. We will do so by verifying that the unit of the adjunction

$$a : \phi \rightarrow G' \circ F'(\phi) \tag{18}$$

satisfies the condition of 3.3.5 with $\Phi = G' \circ F'(\phi)$.

The map (18) defines, for any $x \in X$, $\Phi(x)$ as an operadic colimit, see [L.HA], 3.1.1.20, 3.1.3.5, which we are now going to describe.

To shorten the notation, we denote $\mathfrak{X} = \mathcal{M} \sqcup X$ and $\mathfrak{A} = (\mathcal{M}, \bar{\mathcal{A}})$, both considered as LM-operads. Similarly, \mathfrak{P} will denote the LM-operad $(\mathcal{M}, P_{\mathcal{M}}(\mathcal{A}))$. We denote $\mathcal{F}_x = \mathfrak{X} \times_{\mathfrak{A}} \mathfrak{A}_{/x}^{\text{act}}$ and define the functor

$$\bar{\phi}_x : \mathcal{F}_x \rightarrow (\mathcal{M}, \mathcal{B})$$

as the composition of the projection $\mathcal{F}_x \rightarrow \mathfrak{X}$ and the map $\mathfrak{X} \rightarrow (\mathcal{M}, \mathcal{B})$ induced by ϕ .

The map (18) defines $\Phi(x)$ as the operadic colimit of $\bar{\phi}_x$.

In 3.3.9 and 3.3.10 below we define a functor $\tau_x : \mathbf{Tw}(X)^{\text{op}} \rightarrow \mathcal{F}_x$ and prove that $\theta_x^\phi : \mathbf{Tw}(X)^{\text{op}} \rightarrow (\mathcal{M}, \mathcal{B})$ factors as $\theta_x^\phi = \bar{\phi}_x \circ \tau_x$.

Then we prove (see 3.3.11) that τ_x is cofinal. This implies that the diagram $\bar{\theta}_x^a$ is an operadic colimit diagram. This, by 3.3.5, proves the assertion. ■

3.3.8. CONSTRUCTION OF τ_x , A GENERAL IDEA. Here is how τ_x looks like. An object $f \in \mathcal{F}_x$ is given by a collection $(m_1, \dots, m_n, z, \beta)$ where $m_i \in \mathcal{M}$, $z \in X$ and $\beta : (m_1, \dots, m_n, z) \rightarrow x$ is an arrow in \mathcal{Q} over an active arrow $(a^n m) \rightarrow m$ in LM. Note that an arrow β can be equivalently described (by the Yoneda lemma, see [H.EY], Sect. 6) by an arrow $\otimes m_i \rightarrow \mathcal{A}(z, x)$ in \mathcal{M} .

The functor $\tau_x : \mathbf{Tw}(X)^{\text{op}} \rightarrow \mathcal{F}_x$ will carry an arrow $\alpha : z \rightarrow y$ to

$$\tau_x(\alpha) = (\mathcal{A}(y, x), z, \alpha^* : \mathcal{A}(y, x) \rightarrow \mathcal{A}(z, x)).$$

We present below a more accurate description of τ_x .

3.3.9. CONSTRUCTION OF τ_x . The functor τ_x is defined by its components $\tau^{\mathfrak{X}} : \mathbf{Tw}(X)^{\text{op}} \rightarrow \mathfrak{X}$ and $\tau^{\mathfrak{A}} : \mathbf{Tw}(X)^{\text{op}} \rightarrow \mathfrak{A}_{/x}^{\text{act}}$ and an equivalence of their compositions $\mathbf{Tw}(X)^{\text{op}} \rightarrow \mathfrak{A}$. The functor $\tau^{\mathfrak{X}}$ is the composition

$$\mathbf{Tw}(X)^{\text{op}} \rightarrow X \times X^{\text{op}} \rightarrow \mathcal{M} \times X, \tag{19}$$

where the second map carries (z, y) to $(\mathcal{A}(y, x), z)$.

Since $\bar{\mathcal{A}}$ is a full subcategory of $P := P_{\mathcal{M}}(\mathcal{A})$, $\mathfrak{A}_{/x}^{\text{act}}$ is a full subcategory of $\mathfrak{P}_{/Y(x)}^{\text{act}}$. The functor $\tau^{\mathfrak{A}}$ is therefore uniquely defined by a functor $\mathbf{Tw}(X)^{\text{op}} \rightarrow \mathfrak{P}_{/Y(x)}^{\text{act}}$ whose composition with the forgetful functor to \mathfrak{P} is given by (19).

Since \mathfrak{P} is LM-monoidal, $\tau^{\mathfrak{A}}$ is determined by a functor $\mathbf{Tw}(X)^{\text{op}} \rightarrow P_{/Y(x)}$ assigning to $\alpha : z \rightarrow y$ an arrow $\mathcal{A}(y, x) \otimes Y(z) \rightarrow Y(x)$ in P .

The right fibration $p : \mathbf{Tw}(X)^{\text{op}} \rightarrow X^{\text{op}} \times X$ carrying $\alpha : z \rightarrow y$ to the pair (y, z) , is classified by the functor $h : X \times X^{\text{op}} \rightarrow \mathcal{S}$ given by the formula $h(z, y) = \text{Map}_X(z, y)$.

One has a functor $X^{\text{op}} \times X \rightarrow P$ carrying (y, z) to $\mathcal{A}(y, x) \otimes Y(z)$. In order to lift it to a functor $\mathbf{Tw}(X)^{\text{op}} \rightarrow P_{/Y(x)}$, it is enough to present a morphism of functors $\text{Map}_X(z, y) \rightarrow \text{Map}_{\mathcal{M}}(\mathcal{A}(y, x), \mathcal{A}(z, x))$. This latter comes from functoriality of \mathcal{A} .

We have the following.

3.3.10. LEMMA. $\theta_x^\phi = \bar{\phi}_x \circ \tau_x$.

PROOF. The functor $\theta_x^\phi : \mathbf{Tw}(X)^{\text{op}} \rightarrow (\mathcal{M}, \mathcal{B})$ factors as $\theta_x^\phi = \bar{\phi}_x \circ \tau_x$ as $\bar{\phi}_x$ factors through \mathfrak{X} , so that the composition $\bar{\phi}_x \circ \tau_x$ can be expressed through $\tau^{\mathfrak{X}}$ given by the formula (19). ■

3.3.11. LEMMA. *The functor $\tau_x : \mathbf{Tw}(X)^{\text{op}} \rightarrow \mathcal{F}_x$ is cofinal.*

PROOF. We use Quillen’s Theorem A, see [L.T], 4.1.3.1.

We claim that the comma category

$$\mathbf{Tw}(X)^{\text{op}} \times_{\mathcal{F}_x} (\mathcal{F}_x)_f/ \tag{20}$$

has a terminal object for any $f \in \mathcal{F}_x$. In fact, let $f = (m_1, \dots, m_n, z, u)$ where $u : \otimes m_i \rightarrow \mathcal{A}(z, x)$ is an arrow in \mathcal{M} . The terminal object of (20) is given by $\tau_x(\text{id}_z) = (\mathcal{A}(z, x), z, \text{id}_{\mathcal{A}(z, x)})$. ■

3.4. ENRICHED CATEGORIES AND LURIE ENRICHED CATEGORIES. As an easy consequence of the above, we have the following.

3.4.1. COROLLARY. *The functor $\lambda : \mathbf{Cat}(\mathcal{M}) \rightarrow \mathbf{Cat}^{\text{Lur}}(\mathcal{M})$ is fully faithful.*

PROOF. Let $\mathcal{A}, \mathcal{A}' \in \mathbf{Cat}(\mathcal{M})$. The map

$$\text{Map}_{\mathbf{Cat}(\mathcal{M})}(\mathcal{A}, \mathcal{A}') \rightarrow \text{Fun}_{\mathcal{M}}(\mathcal{A}, P_{\mathcal{M}}(\mathcal{A}'))^{eq}$$

is embedding, identifying the left-hand side with the subspace of the right-hand side consisting of $f : \mathcal{A} \rightarrow P_{\mathcal{M}}(\mathcal{A}')$ with representable images. In other words, it induces an equivalence

$$\text{Map}_{\mathbf{Cat}(\mathcal{M})}(\mathcal{A}, \mathcal{A}') \rightarrow \text{Fun}_{\mathcal{M}}(\mathcal{A}, \bar{\mathcal{A}}')^{eq}.$$

According to the theorem, one has an equivalence

$$\text{Fun}_{\mathcal{M}}(\mathcal{A}, P_{\mathcal{M}}(\mathcal{A}')) \rightarrow \text{Fun}_{\text{LMod}_{\mathcal{M}}^w}(\mathcal{A}, P_{\mathcal{M}}(\mathcal{A}'))$$

which identifies $\text{Map}_{\mathbf{Cat}(\mathcal{M})}(\mathcal{A}, \mathcal{A}')$ with $\text{Map}_{\text{LMod}_{\mathcal{M}}^w}(\bar{\mathcal{A}}, \bar{\mathcal{A}}')$. ■

Note that H. Heine has recently proven [HH] that λ is an equivalence.

3.4.2. REMARK. Let \mathcal{B} be weakly enriched over a monoidal category \mathcal{M} . Here is a reasonable way to assign to \mathcal{B} an \mathcal{M} -enriched category \mathcal{A} . Let $X = \mathcal{B}^{eq}$ and let $i : X \rightarrow \mathcal{B}$ be the natural embedding. The LM-operad $\text{Quiv}_X^{\text{LM}}(\mathcal{M}, \mathcal{B})$ defines a weak enrichment of $\text{Fun}(X, \mathcal{B})$ over $\text{Quiv}_X(\mathcal{M})$.

The \mathcal{M} -enriched category \mathcal{A} can now be defined as the endomorphism object of $i \in \text{Fun}(X, \mathcal{B})$ (if it exists). Apparently, this is precisely how [HH] proves that λ is an equivalence.

. The forgetful functor $p : \mathbf{Alg}_{\text{LM}}(\mathbf{Cat}) \rightarrow \mathbf{Alg}_{\text{Ass}}(\mathbf{Cat})$ is a cartesian fibration. In particular, given a monoidal functor $f : \mathcal{A} \rightarrow \mathcal{B}$ and a category \mathcal{X} left-tensored over \mathcal{B} , we have an LM-monoidal cartesian lifting $f^! : (\mathcal{A}, \mathcal{X}) \rightarrow (\mathcal{B}, \mathcal{X})$ in $\mathbf{Alg}_{\text{LM}}(\mathbf{Cat})$.

3.4.3. LEMMA. *The arrow $f^! : (\mathcal{A}, \mathcal{X}) \rightarrow (\mathcal{B}, \mathcal{X})$ is also p' -cartesian, where*

$$p' : \mathbf{Op}_{\text{LM}} \rightarrow \mathbf{Op}_{\text{Ass}}$$

is the forgetful functor.

PROOF. The embedding $\mathbf{Alg}_{\mathbf{LM}}(\mathbf{Cat}) \rightarrow \mathbf{Op}_{\mathbf{LM}}$ has a left adjoint functor denoted $P_{\mathbf{LM}}$ (monoidal envelope functor). Similarly, $P_{\mathbf{Ass}} : \mathbf{Op}_{\mathbf{Ass}} \rightarrow \mathbf{Alg}_{\mathbf{Ass}}(\mathbf{Cat})$ is left adjoint to the embedding. The lemma immediately follows from the equivalence

$$P_{\mathbf{LM}}(\mathcal{O})_a = P_{\mathbf{Ass}}(\mathcal{O}_a)$$

valid for any $\mathcal{O} \in \mathbf{Op}_{\mathbf{LM}}$. ■

4. Relative tensor product and duality

4.1. INTRODUCTION. This section is mostly an exposition of (parts of) Lurie's [L.HA], 4.6 and 3.1. In it we do the following.

1. Starting with a monoidal category \mathcal{C} with geometric realizations, we construct a 2-category $\mathbf{BMOD}(\mathcal{C})$ ⁵ called the Morita 2-category of \mathcal{C} , whose objects are associative algebras in \mathcal{C} , so that the category of morphisms $\mathbf{Fun}(A, B)$ is the category of A - B -bimodules. Composition of arrows in $\mathbf{BMOD}(\mathcal{C})$ is given by the relative tensor product of bimodules. The description of $\mathbf{BMOD}(\mathcal{C})$ is based on a study of an operad \mathbf{Tens} over $\mathbf{Ass}_{\Delta^{\text{op}}}$, see [H.EY], 2.10.5 (3)⁶ describing collections of bimodules and multilinear maps between them.
2. Duality for bimodules describes adjunction between morphisms in $\mathbf{BMOD}(\mathcal{C})$. The special case, when the unit and the counit of the adjunction are equivalences, describes a Morita equivalence between the corresponding associative algebras in \mathcal{C} .
3. A more general type of the relative tensor product of bimodules, with different bimodules belonging to different categories, is described using the same operad \mathbf{Tens} and the ones obtained from it by a base change.

4.2. MORITA 2-CATEGORY AND THE OPERAD \mathbf{Tens} . We now present a construction of the Morita 2-category $\mathbf{BMOD}(\mathcal{C})$ for a monoidal category \mathcal{C} with geometric realizations. We describe $\mathbf{BMOD}(\mathcal{C})$ as a Segal simplicial object in \mathbf{Cat} , carrying $[n]$ to a category $\mathbf{BMOD}_n(\mathcal{C})$. The category $\mathbf{BMOD}_n(\mathcal{C})$ can be described as the category of algebras in \mathcal{C} over a certain planar operad (in sets).

We will denote this operad \mathbf{Tens}_n . Algebras over it are collections (A_0, \dots, A_n) of associative algebras in \mathcal{C} , together with a collection of A_{i-1} - A_i -bimodules M_i for $i = 1, \dots, n$. So, $\mathbf{BMOD}_n(\mathcal{C}) = \mathbf{Alg}_{\mathbf{Tens}_n}(\mathcal{C})$. To define the simplicial object $\mathbf{BMOD}_{\bullet}(\mathcal{C})$, we have to provide a compatible collection of functors $s^* : \mathbf{BMOD}_n(\mathcal{C}) \rightarrow \mathbf{BMOD}_m(\mathcal{C})$ defined for each $s : [m] \rightarrow [n]$ in Δ , together with the coherence data.

For a map $s : [m] \rightarrow [n]$ the functor $s^* : \mathbf{BMOD}_n(\mathcal{C}) \rightarrow \mathbf{BMOD}_m(\mathcal{C})$ comes from a correspondence between the operads \mathbf{Tens}_m and \mathbf{Tens}_n .

⁵More precisely, a category object in \mathbf{Cat} , that is, a simplicial object in \mathbf{Cat} satisfying the Segal condition.

⁶Lurie [L.HA] describes it as a family of operads based on Δ^{op} .

We will present an operad \mathbf{Tens}_s over $\mathbf{Ass}_{[1]}^\otimes$ with fibers \mathbf{Tens}_m and \mathbf{Tens}_n over 0 and 1 respectively.⁷

The functors $i_0^* : \mathbf{Alg}_{\mathbf{Tens}_s}(\mathcal{C}) \rightarrow \mathbf{BMOD}_m(\mathcal{C})$ and $i_1^* : \mathbf{Alg}_{\mathbf{Tens}_s}(\mathcal{C}) \rightarrow \mathbf{BMOD}_n(\mathcal{C})$ are defined by the embeddings $i_0 : \mathbf{Tens}_m \rightarrow \mathbf{Tens}_s$ and $i_1 : \mathbf{Tens}_n \rightarrow \mathbf{Tens}_s$.

The map $s^* : \mathbf{BMOD}_n(\mathcal{C}) \rightarrow \mathbf{BMOD}_m(\mathcal{C})$ will be defined as the composition $i_0^* \circ i_1!$ where $i_1!$ is left adjoint to i_1^* .

In order to describe the compatibility of s^* with respect to composition, we will define an operad \mathbf{Tens} over $\mathbf{Ass}_{\Delta^{\text{op}}}^\otimes = \mathbf{Ass}^\otimes \times \mathbf{Com}_{\Delta^{\text{op}}}$. We will have $\mathbf{Tens}_s = \mathbf{Tens} \times_{\mathbf{Ass}_{\Delta^{\text{op}}}^\otimes} \mathbf{Ass}_{[1]}^\otimes$ where $s : [1] \rightarrow \Delta^{\text{op}}$ defines the map $\mathbf{Ass}_{[1]}^\otimes \rightarrow \mathbf{Ass}_{\Delta^{\text{op}}}^\otimes$.

4.2.1. THE OPERAD \mathbf{Tens} . \mathbf{Tens} is the operad in sets governing the following collection of data.

1. For each $n \geq 0$ the collection of monoids $A_{0,n}, \dots, A_{n,n}$ and A_{i-1} - A_i bimodules $M_{i,n}$ for $i = 1, \dots, n$.
2. For each map $s : [m] \rightarrow [n]$ in Δ the collection of arrows:
 - a. Morphism of algebras $A_{s(i),n} \rightarrow A_{i,m}$, for $i = 0, \dots, m$.
 - b. Multilinear morphisms (see remark below)

$$M_{s(i-1)+1,n} \times \dots \times M_{s(i),n} \rightarrow M_{i,m}$$

of $A_{s(i-1),n}$ - $A_{s(i),n}$ -bimodules.

3. The collections of arrows defined in (2) for each $s : [m] \rightarrow [n]$ compose in an obvious way.

4.2.2. REMARK. Multilinearity in the last sentence means that, in case $s(i-1)+1 < s(i)$, the map is compatible with the actions of all intermediate $A_{j,n}$, $j = s(i-1)+2, \dots, s(i)-1$; it means nothing if $s(i) = s(i-1)+1$; and it means an $A_{s(i),n}$ -bimodule map $A_{s(i),n} \rightarrow M_{i,m}$ if $s(i-1) = s(i)$.

4.2.3. THE MAP TO $\mathbf{Ass}_{\Delta^{\text{op}}}^\otimes$. An $\mathbf{Ass}_{\Delta^{\text{op}}}^\otimes$ -algebra in an \mathbf{Ass}^\otimes -operad \mathcal{C} is given by a functor $A : \Delta^{\text{op}} \rightarrow \mathbf{Alg}_{\mathbf{Ass}}(\mathcal{C})$. This functor defines a canonical \mathbf{Tens} -algebra defined by the formulas $A_{i,n} = A([n]) = M_{i,n}$. This gives a functor $\pi^* : \mathbf{Alg}_{\mathbf{Ass}_{\Delta^{\text{op}}}}(\mathcal{C}) \rightarrow \mathbf{Alg}_{\mathbf{Tens}}(\mathcal{C})$ realized as the inverse image with respect to the map

$$\pi : \mathbf{Tens} \rightarrow \mathbf{Ass}_{\Delta^{\text{op}}}^\otimes.$$

⁷Recall that \mathbf{Ass}^\otimes denotes the operad governing associative algebras. The operad \mathbf{Ass}_K^\otimes for $K \in \mathbf{Cat}$ governs K -diagrams of associative algebras.

4.2.4. For any $\phi : S \rightarrow \Delta^{\text{op}}$ one defines \mathbf{Tens}_S (or \mathbf{Tens}_ϕ) as $\mathbf{Com}_S \times_{\mathbf{Com}_{\Delta^{\text{op}}}} \mathbf{Tens}$.

One defines $p : \mathbf{BMOD}(\mathcal{C}) \rightarrow \Delta^{\text{op}}$ as a category over Δ^{op} representing the functor

$$\mathbf{Fun}_{\Delta^{\text{op}}}(B, \mathbf{BMOD}(\mathcal{C})) = \mathbf{Alg}_{\mathbf{Tens}_B}(\mathcal{C}). \tag{21}$$

In the case when \mathcal{C} has geometric realizations and a monoidal structure preserving geometric realizations, $\mathbf{BMOD}(\mathcal{C})$ is a cocartesian fibration over Δ^{op} , so it defines a simplicial object $\mathbf{BMOD}_\bullet(\mathcal{C})$ in \mathbf{Cat} , see [L.HA], 4.4.3.12. It satisfies the Segal condition by [L.HA], 4.4.3.11.

4.2.5. REMARK. Note that $\mathbf{BMOD}(\mathcal{C})$ is not complete. The zero component $\mathbf{BMOD}_0(\mathcal{C})$ is the category of algebras in \mathcal{C} which is not a space. An equivalence defined by $\mathbf{BMOD}_1(\mathcal{C})$ is a Morita equivalence which is not equivalence in $\mathbf{BMOD}_0(\mathcal{C})$.

4.3. DUALITY. In this subsection we apply the general notion of adjunction in a 2-category to the Morita 2-category described in the previous subsection.

4.3.1. DEFINITION. (see [L.HA], 4.6.2.3) Let \mathcal{C} be a monoidal category with geometric realizations. Let A, B be two associative algebras in \mathcal{C} , $M \in {}_A \mathbf{BMod}_B(\mathcal{C})$ and $N \in {}_B \mathbf{BMod}_A(\mathcal{C})$. A map $c : B \rightarrow N \otimes_A M$ is said to exhibit N as left dual of M (or M as a right dual of N) if there exists $e : M \otimes_B N \rightarrow A$ in ${}_A \mathbf{BMod}_A(\mathcal{C})$ such that the compositions

$$M = M \otimes_B B \xrightarrow{\text{id}_M \otimes c} M \otimes_B N \otimes_A M \xrightarrow{e \otimes \text{id}_M} M$$

and

$$N = B \otimes_B N \xrightarrow{c \otimes \text{id}_N} N \otimes_A M \otimes_B N \xrightarrow{\text{id}_N \otimes e} N$$

are equivalent to id_M and id_N , respectively.

Let \mathcal{M} be a left \mathcal{C} -tensorial category with geometric realizations. A dual pair of bimodules $M \in {}_A \mathbf{BMod}_B(\mathcal{C})$ and $N \in {}_B \mathbf{BMod}_A(\mathcal{C})$ determines an adjunction (see [L.HA], 4.6.2.1)

$$F : \mathbf{LMod}_B(\mathcal{M}) \xrightarrow{\leftarrow} \mathbf{LMod}_A(\mathcal{M}) : G \tag{22}$$

given by the formulas $F(X) = M \otimes_B X$ and $G(Y) = N \otimes_A Y$. This adjunction deserves the name *Morita adjunction*.

A Morita adjunction is called a Morita equivalence if the arrows c and e are equivalences.

Two properties of Morita adjunction are listed below. The first one, Proposition 4.3.2, describes a good behavior of Morita adjunctions under composition. The second one, Proposition 4.3.3, claims that the left dualizability of $M \in {}_A \mathbf{BMod}_B(\mathcal{C})$ is independent of the algebra B .

4.3.2. PROPOSITION. (see [L.HA], 4.6.2.6) let \mathcal{C} be a monoidal category with geometric realizations, A, B, C three associative algebras in \mathcal{C} . If $c : B \rightarrow N \otimes_A M$ exhibits N as a left dual to $M \in {}_A \mathbf{BMod}_B$ and $c' : C \rightarrow N' \otimes_B M'$ exhibits N' as a left dual to $M' \in {}_B \mathbf{BMod}_C$ then the composition

$$C \xrightarrow{c'} N' \otimes_B M' = N' \otimes_B B \otimes_B M' \xrightarrow{c} N' \otimes_B N \otimes_A M \otimes_B M'$$

exhibits $N' \otimes_B N$ as a left dual to $M \otimes_B M'$.

4.3.3. PROPOSITION. (see [L.HA], 4.6.2.12, 4.6.2.13) Let \mathcal{C} be a monoidal category with geometric realizations. A bimodule $M \in {}_A\mathbf{BMod}_B(\mathcal{C})$ is left dualizable if and only if its image M' in

$\mathbf{LMod}_A(\mathcal{C}) = {}_A\mathbf{BMod}_1(\mathcal{C})$ is left dualizable. Moreover, if $N \in {}_B\mathbf{BMod}_A(\mathcal{C})$ is left dual to M , its image in $\mathbf{RMod}_A(\mathcal{C})$ is a left dual of $M' \in \mathbf{LMod}_A(\mathcal{C})$.

4.3.4. REMARK. In the classical context of associative rings, an A - B -bimodule N is right-dualizable iff it is finitely generated projective as a right A -module. This property is independent of B and right dualizability of N is sufficient to have an adjunction between the categories of left A and B -modules. This adjunction is an equivalence, for $B = \text{End}_A(N)$, if N is a generator in \mathbf{RMod}_A . It would be very nice to describe in our general context a condition on a right dualizable module $N \in \mathbf{RMod}_A$ leading to Morita equivalence.

4.3.5. We can fix a right-dualizable A -module $N \in \mathbf{RMod}_A(\mathcal{C})$ and try to reconstruct a would-be Morita equivalence.

Let $M \in \mathbf{LMod}_A(\mathcal{C})$ be the right dual of N .

The category $\mathbf{RMod}_A(\mathcal{C})$ is left-tensored over \mathcal{C} . So, given $N \in \mathbf{RMod}_A(\mathcal{C})$, one can define an endomorphism object $\text{End}_A(N)$ which, if it exists, acquires an associative algebra structure. Since N is right dualizable, this object does exist, as one has a canonical equivalence

$$\text{Map}_{\mathcal{C}}(X, N \otimes_A M) = \text{Map}_{\mathbf{RMod}_A(\mathcal{C})}(X \otimes N, N) \quad (23)$$

by [L.HA], 4.6.2.1 (3), so that $\text{End}_A(N) = N \otimes_A M$ as an object of \mathcal{C} .

4.3.6. COROLLARY. Let \mathcal{C} be a monoidal category with geometric realizations, A an associative algebra in \mathcal{C} , $N \in \mathbf{RMod}_A(\mathcal{C})$ a right dualizable A -module. Then $M \in \mathbf{LMod}_A(\mathcal{C})$, the right dual of N , has a canonical structure of A - $\text{End}_A(N)$ -bimodule and the pair (M, N) defines a Morita adjunction

$$F : \mathbf{LMod}_{\text{End}_A(N)}(\mathcal{C}) \rightleftarrows \mathbf{LMod}_A(\mathcal{C}),$$

with $F(X) = M \otimes_{\text{End}_A(N)} X$ and $G(Y) = N \otimes_A Y$, for which the coevaluation

$$c : \text{End}_A(N) \rightarrow N \otimes_A M$$

is an equivalence.

PROOF. See [L.HA], 4.6.2.1 (2). ■

Note that this construction produces the A - A -bimodule evaluation map

$$e : M \otimes_{\text{End}_A(N)} N \rightarrow A. \quad (24)$$

4.3.7. REMARK. The algebra $B = \text{End}_A(N)$ can be also described in terms of the left A -module M . In fact, the category $\mathbf{LMod}_A(\mathcal{C})$ is right-tensored over \mathcal{C} , so it is left-tensored over the reversed monoidal category \mathcal{C}^{rev} . The endomorphism object of $M \in \mathbf{LMod}_A(\mathcal{C})$ in \mathcal{C}^{rev} exists, and it coincides with the algebra B^{op} .

4.4. RELATIVE TENSOR PRODUCT. The relative tensor product of bimodules with values in a monoidal category \mathcal{C} is encoded in the composition of arrows of $\mathbf{BMOD}(\mathcal{C})$. There exists a slightly more general relative tensor product, for the bimodules having values in different categories.

Let now $\mathcal{C} \in \mathbf{Alg}_{\mathbf{Tens}_S}(\mathbf{Cat}^L)$ where, as before, \mathbf{Cat}^L denotes the category of categories with small colimits, with the arrows being the colimit preserving functors.

We wish to study tensor product of bimodules with values in \mathcal{C} .

We define, slightly generalizing 4.2.4, $p : \mathbf{BMOD}^\phi(\mathcal{C}) \rightarrow S$ as a category over S representing the functor

$$\mathbf{Fun}_S(B, \mathbf{BMOD}^\phi(\mathcal{C})) = \mathbf{Alg}_{\mathbf{Tens}_B/\mathbf{Tens}_S}(\mathcal{C}).$$

One has

4.4.1. PROPOSITION.

1. The map $p : \mathbf{BMOD}^\phi(\mathcal{C}) \rightarrow S$ is a cocartesian fibration.
2. An arrow $\tilde{\alpha}$ in $\mathbf{BMOD}^\phi(\mathcal{C})$ over $\alpha : x \rightarrow y$ in S is p -cocartesian iff the corresponding $F_\alpha \in \mathbf{Alg}_{\mathbf{Tens}_{\phi(\alpha)}/\mathbf{Tens}_S}(\mathcal{C})$ is an operadic left Kan extension of its restriction $F_{\phi(x)} : \mathbf{Tens}_{\phi(x)} \rightarrow \mathcal{C}$.
3. Let $f : \mathcal{C} \rightarrow \mathcal{D}$ be a \mathbf{Tens}_ϕ -monoidal functor preserving geometric realizations. Then the induced map $\mathbf{BMOD}^\phi(f) : \mathbf{BMOD}^\phi(\mathcal{C}) \rightarrow \mathbf{BMOD}^\phi(\mathcal{D})$ preserves cocartesian arrows.

PROOF. The first two claims are just [L.HA], Corollary 4.4.3.2, with $\mathcal{O} = \mathbf{Tens}_S$. The condition (*) is fulfilled as \mathcal{C} is \mathbf{Tens}_S -monoidal category with geometric realizations, commuting with the tensor product. Claim 3 follows from Claim 2. ■

4.4.2. Here is an important example of the above construction.

Let $\succ : [1] \rightarrow \Delta^{\text{op}}$ be defined by the arrow $\partial^1 : [1] \rightarrow [2]$ in Δ . We have then $\mathbf{Tens}_\succ = \mathbf{Com}_{[1]} \times_{\mathbf{Com}_{\Delta^{\text{op}}}} \mathbf{Tens}$.

One has natural embeddings $i_1 : \mathbf{Tens}_1 \rightarrow \mathbf{Tens}_\succ$ and $i_2 : \mathbf{Tens}_2 \rightarrow \mathbf{Tens}_\succ$ induced by the embedding of the ends $\{1\} \rightarrow [1]$ and $\{0\} \rightarrow [1]$.

Note that $\mathbf{Tens}_1 = \mathbf{BM}^\otimes$ and $\mathbf{Tens}_2 = \mathbf{BM}^\otimes \sqcup^{\mathbf{Ass}^\otimes} \mathbf{BM}^\otimes$.

4.4.3. Let \mathcal{C} be a \mathbf{Tens}_\succ -monoidal category. Up to equivalence, \mathcal{C} is uniquely described by a collection of five monoidal categories $\mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c, \mathcal{C}_{a'}, \mathcal{C}_{c'}$, three bimodule categories $\mathcal{C}_m \in_{\mathcal{C}_a} \mathbf{BMod}_{\mathcal{C}_b}(\mathbf{Cat})$, $\mathcal{C}_n \in_{\mathcal{C}_b} \mathbf{BMod}_{\mathcal{C}_c}(\mathbf{Cat})$, $\mathcal{C}_k \in_{\mathcal{C}_{a'}} \mathbf{BMod}_{\mathcal{C}_{c'}}(\mathbf{Cat})$, monoidal functors $\phi_a : \mathcal{C}_a \rightarrow \mathcal{C}_{a'}$ and $\phi_c : \mathcal{C}_c \rightarrow \mathcal{C}_{c'}$, and a \mathcal{C}_b -bilinear functor

$$\mathcal{C}_m \times \mathcal{C}_n \rightarrow \mathcal{C}_k$$

of \mathcal{C}_a - \mathcal{C}_c -bimodule categories.

The embedding $i_2 : \mathbf{Tens}_2 \rightarrow \mathbf{Tens}_\succ$ induces

$$i_2^* : \mathbf{Alg}_{\mathbf{Tens}_\succ}(\mathcal{C}) \rightarrow \mathbf{Alg}_{\mathbf{Tens}_2/\mathbf{Tens}_\succ}(\mathcal{C}).$$

The relative tensor product functor

$$\text{RT} : \text{Alg}_{\mathbf{Tens}_2/\mathbf{Tens}_\succ}(\mathcal{C}) \rightarrow \text{Alg}_{\mathbf{Tens}_\succ}(\mathcal{C}) \tag{25}$$

is defined as the functor left adjoint to i_2^* .

The functor RT exists if $\mathcal{C} \in \text{Alg}_{\mathbf{Tens}_\succ}(\text{Cat}^L)$.

It makes sense to fix associative algebras $A \in \text{Alg}_{\text{Ass}}(\mathcal{C}_a)$, $B \in \text{Alg}_{\text{Ass}}(\mathcal{C}_b)$, $C \in \text{Alg}_{\text{Ass}}(\mathcal{C}_c)$, and restrict (25) to \mathbf{Tens}_2 -algebras in \mathcal{C} having algebra-components A, B and C . If $A' = \phi_a(A) \in \text{Alg}_{\text{Ass}}(\mathcal{C}_{a'})$ and $C' = \phi_c(C) \in \text{Alg}_{\text{Ass}}(\mathcal{C}_{c'})$, this gives

$$\text{RT}_{A,B,C} : {}_A\text{BMod}_B(\mathcal{C}_m) \times {}_B\text{BMod}_C(\mathcal{C}_n) \rightarrow {}_{A'}\text{BMod}_{C'}(\mathcal{C}_k). \tag{26}$$

4.4.4. TWO-SIDED BAR CONSTRUCTION. The following explicit formula for the calculation of relative tensor product explains why does it exist for categories with geometric realizations.

Recall that \mathbf{Tens}_2 governs 5-tuples of objects, (A, M, B, N, C) , where A, B, C are associative monoids, M is an A - B -bimodule and N is a B - C -bimodule. We denote the colors of \mathbf{Tens}_2 by a, m, b, n, c . The operad \mathbf{Tens}_1 has colors a', k, c' .

Define a functor $u : \Delta^{\text{op}} \rightarrow \mathbf{Tens}_2$ carrying $[i]$ to $mb^i n \in \mathbf{Tens}_2$, where the action of u on the arrows is defined as follows.

- Faces correspond to the action maps $mb \rightarrow m, bb \rightarrow b$ or $bn \rightarrow n$.
- Degeneracies correspond to the unit maps $1 \rightarrow b$.

We extend the map $u : \Delta^{\text{op}} \rightarrow \mathbf{Tens}_2$ to $u_+ : \Delta_+^{\text{op}} \rightarrow \mathbf{Tens}_\succ$ carrying the terminal object of Δ_+^{op} to $k \in \mathbf{Tens}_1$.

Let $q : \mathcal{C} \rightarrow \mathbf{Tens}_\succ$ present a \mathbf{Tens}_\succ -monoidal category. The map $\text{Fun}(\Delta^{\text{op}}, q) : \text{Fun}(\Delta^{\text{op}}, \mathcal{C}) \rightarrow \text{Fun}(\Delta^{\text{op}}, \mathbf{Tens}_\succ)$ is a cocartesian fibration. The functor u_+ defines an arrow $\beta : u \rightarrow u_*$ in $\text{Fun}(\Delta^{\text{op}}, \mathbf{Tens}_\succ)$, u_* being the constant functor with the value $k \in \mathbf{Tens}_\succ$. Therefore, any $\phi \in \text{Alg}_{\mathbf{Tens}_2/\mathbf{Tens}_\succ}(\mathcal{C})$ gives rise to a unique lift $\beta_! : \phi \circ u \rightarrow X$, where X is a simplicial object in \mathcal{C}_k . We will denote $X = \text{Sh}(\phi \circ u)$ and call it the two-sided bar construction, $\text{Sh}(\phi \circ u) = \text{Bar}(\phi)$.

The following explicit description of relative tensor product is a reformulation of [L.HA], 4.4.2.8.

4.4.5. PROPOSITION. *Let \mathcal{C} be a \mathbf{Tens}_\succ -monoidal category with geometric realizations and the tensor structure commuting with the geometric realizations, and let $q : \mathcal{C} \rightarrow \mathbf{Tens}_\succ$ be the corresponding cocartesian fibration. Given a commutative diagram*

$$\begin{array}{ccc} \mathbf{Tens}_2 & \xrightarrow{\phi} & \mathcal{C} \\ \downarrow & \searrow \Phi & \downarrow q \\ \mathbf{Tens}_\succ & \xlongequal{\quad} & \mathbf{Tens}_\succ \end{array} \tag{27}$$

of marked categories, with ϕ corresponding to a pair of bimodules $M \in {}_A\mathbf{BMod}_B$ and $N \in {}_B\mathbf{BMod}_C$ with values in \mathcal{C} . Then there exists Φ presenting a relative tensor product of M and N . Vice versa, any extension Φ of ϕ presents a relative tensor product of M with N if and only if the following conditions are fulfilled.

- Φ carries the maps $A_{02} \rightarrow A_{01}$ and $A_{22} \rightarrow A_{11}$ to q -cocartesian arrows in \mathcal{C} .
- The functor Φ induces an equivalence

$$|\mathbf{Bar}(\phi)| \rightarrow \Phi(k).$$

4.4.6. ASSOCIATIVITY. To formulate associativity, we need to use Proposition 4.4.1 applied to the family $\phi : S \rightarrow \Delta^{\text{op}}$ defined by the commutative square

$$\begin{array}{ccc} [1] & \xrightarrow{\partial^1} & [2] \\ \downarrow \partial^1 & & \downarrow \partial^2 \\ [2] & \xrightarrow{\partial^1} & [3] \end{array} \tag{28}$$

in Δ , and a \mathbf{Tens}_ϕ -monoidal category \mathcal{C} with geometric realizations, see [L.HA], 4.4.3.14.

4.5. VARIANTS. The tensor product of bimodules (26) commutes with the functor forgetting the left A -module structure and the right C -module structure.

We would like to formulate this observation as follows. Let \mathbf{Tens}_\succ be the full suboperad of \mathbf{Tens}_\succ spanned by the colors $a, b, a', m, n, k \in [\mathbf{Tens}_\succ]$. There is an obvious embedding $i : \mathbf{Tens}_\succ \rightarrow \mathbf{Tens}_\succ$ and the functor $i^* : \mathbf{Alg}_{\mathbf{Tens}_\succ} \rightarrow \mathbf{Alg}_{\mathbf{Tens}_\succ}$ forgets the right module structure on the bimodules described by the colors n and k .

Similarly, it makes sense to describe a yet smaller suboperad \mathbf{En}_\succ spanned by the colors $b, m, n, k \in [\mathbf{Tens}_\succ]$. We denote $j : \mathbf{En}_\succ \rightarrow \mathbf{Tens}_\succ$ the obvious embedding that forgets both the right module structure on bimodules described by n, k and the left module structure on bimodules described by m, k .

We define \mathbf{Tens}_2 and \mathbf{En}_2 as for \mathbf{Tens}_\succ ; this yields the functors i_2^* and the left adjoints \mathbf{RT} exactly as for \mathbf{Tens}_\succ -monoidal categories. One has

4.5.1. PROPOSITION. *The forgetful functors i^* and j^* commute with the relative tensor product.*

PROOF. The commutative square

$$\begin{array}{ccc} \mathbf{Alg}_{\mathbf{Tens}_\succ}(\mathcal{C}) & \xrightarrow{i_2^*} & \mathbf{Alg}_{\mathbf{Tens}_2/\mathbf{Tens}_\succ}(\mathcal{C}) \\ \downarrow i^* & & \downarrow i^* \\ \mathbf{Alg}_{\mathbf{Tens}_\succ/\mathbf{Tens}_\succ}(\mathcal{C}) & \xrightarrow{i_2^*} & \mathbf{Alg}_{\mathbf{Tens}_2/\mathbf{Tens}_\succ}(\mathcal{C}) \end{array} \tag{29}$$

defines a morphism of functors

$$\mathbf{RT} \circ i^* \rightarrow i^* \circ \mathbf{RT}.$$

To prove that this functor is an equivalence, we use the description of \mathbf{RT} in terms of the two-sided bar construction. The functor $u_+ : \Delta^{\text{op}} \rightarrow \mathbf{Tens}_{\succ}$ factors through $i : \mathbf{Ten}_{\succ} \rightarrow \mathbf{Tens}_{\succ}$, so the bar construction used to calculate \mathbf{RT} as a colimit, is the same for both setups.

The version for $j : \mathbf{En}_{\succ} \rightarrow \mathbf{Tens}_{\succ}$ is proven in the same way. ■

4.6. REDUCTION. An A - B -bimodule in \mathcal{C} can be equivalently described as a left A -module in the category $\mathbf{RMod}_B(\mathcal{C})$.

We present below a similar transformation of \mathbf{Tens}_{\succ} -monoidal categories compatible with the formation of the weighted colimit.

The construction is based on the notion of bilinear map of operads and their tensor product as presented in [H.EY], 2.10.

4.6.1. We define a map $p : \mathbf{Tens}_{\succ} \rightarrow \mathbf{BM}^{\otimes}$ as the obvious map carrying the colors a, a', b, m to $a \in [\mathbf{BM}]$, n, k to $m \in [\mathbf{BM}]$ and c, c' to $b \in [\mathbf{BM}]$. We have $\mathbf{Ten}_{\succ} = \mathbf{LM}^{\otimes} \times_{\mathbf{BM}^{\otimes}} \mathbf{Tens}_{\succ}$.

One has a standard bilinear map $\text{Pr} : \mathbf{LM}^{\otimes} \times \mathbf{RM}^{\otimes} \rightarrow \mathbf{BM}^{\otimes}$ defined in [L.HA], 4.3.2.1 and [H.EY], 2.10.7. There is a lifting of Pr to a bilinear map

$$\mu : \mathbf{Ten}_{\succ} \times \mathbf{RM}^{\otimes} \rightarrow \mathbf{Tens}_{\succ} \tag{30}$$

uniquely defined by its action on the colors.

- $\mu(*, m) = *$ where $*$ is any color of \mathbf{Ten}_{\succ} .
- $\mu(n, b) = c, \mu(k, b) = c'$.

4.6.2. Let \mathcal{C} be a \mathbf{Tens}_{\succ} -operad. Following a general pattern [H.EY], 2.10.1, we define a \mathbf{Ten}_{\succ} -operad $\mathcal{C}^{\text{red}} := \mathbf{Alg}_{\mathbf{RM}/\mathbf{Tens}_{\succ}}^{\mu}(\mathcal{C})$ as the one representing the functor

$$K \in \mathbf{Cat}/_{\mathbf{Ten}_{\succ}} \mapsto \mathbf{Map}_{\mathbf{Cat}^+ /_{\mathbf{Tens}_{\succ}^{\natural}}} (K^{\flat} \times \mathbf{RM}^{\natural}, \mathcal{C}^{\natural}).$$

We call \mathcal{C}^{red} the reduction of \mathcal{C} .

Here is a more convenient description of \mathcal{C}^{red} in the case when \mathcal{C} is a \mathbf{Tens}_{\succ} -monoidal category. In this case \mathcal{C} is classified by a lax cartesian structure

$$\tilde{\mathcal{C}} : \mathbf{Tens}_{\succ} \rightarrow \mathbf{Cat}.$$

Composing it with μ , we get a functor

$$\tilde{\mathcal{C}} \circ \mu : \mathbf{Ten}_{\succ} \times \mathbf{RM}^{\otimes} \rightarrow \mathbf{Cat},$$

defining

$$\bar{\mathcal{C}} : \mathbf{Ten}_{\succ} \rightarrow \mathbf{Fun}^{\text{lax}}(\mathbf{RM}^{\otimes}, \mathbf{Cat}), \tag{31}$$

that is, a functor with the values in \mathbf{RM} -monoidal categories. Composing it with the functor $\mathbf{Alg}_{\mathbf{RM}}$, we get a functor $\bar{\mathcal{C}}^{\text{red}} : \mathbf{Ten}_{\succ} \rightarrow \mathbf{Cat}$ classifying \mathcal{C}^{red} .

Here is a more detailed information about the functor $\bar{\mathcal{C}}$. For $x = a, a', m, b$, $\bar{\mathcal{C}}(x)$ is the RM-monoidal category $(\mathcal{C}_x, [0])$ describing the trivial action of the trivial monoidal category on \mathcal{C}_x . Thus, one has $\mathcal{C}_x^{red} = \mathcal{C}_x$ for these values of x . Furthermore, $\bar{\mathcal{C}}(n) = (\mathcal{C}_n, \mathcal{C}_c)^\otimes$ and $\bar{\mathcal{C}}(k) = (\mathcal{C}_n, \mathcal{C}_{c'})^\otimes$, so that $\mathcal{C}_n^{red} = \mathbf{Alg}_{\mathbf{RM}}(\mathcal{C}_n, \mathcal{C}_c)$ and $\mathcal{C}_k^{red} = \mathbf{Alg}_{\mathbf{RM}}(\mathcal{C}_k, \mathcal{C}_{c'})$.

The standard embedding $i : \mathbf{Tens}_> \rightarrow \mathbf{Tens}_>$ identifies the m -component of $\bar{\mathcal{C}}$ with $i^*(\mathcal{C})$.

This defines a functor $G : \mathcal{C}^{red} \rightarrow i^*(\mathcal{C})$ forgetting the right module structure on the components $\mathcal{C}_n^{red}, \mathcal{C}_k^{red}$.

The restriction with respect to μ (30) defines a natural map

$$\theta : \mathbf{Alg}_{\mathbf{Tens}_>}(\mathcal{C}) \rightarrow \mathbf{Alg}_{\mathbf{Tens}_>}(\mathcal{C}^{red}), \tag{32}$$

whose composition with the map induced by G is the obvious restriction

$$\mathbf{Alg}_{\mathbf{Tens}_>}(\mathcal{C}) \rightarrow \mathbf{Alg}_{\mathbf{Tens}_>/\mathbf{Tens}_>}(\mathcal{C}). \tag{33}$$

We believe that the map (32) is an equivalence, that is that μ presents $\mathbf{Tens}_>$ as a tensor product.

We will actually verify a somewhat weaker statement Proposition 4.6.4 that will be used in Section 6.

4.6.3. LEMMA. *The bilinear map*

$$\mu_2 : \mathbf{Ten}_2 \times \mathbf{RM}^\otimes \rightarrow \mathbf{Tens}_2$$

obtained by restriction of μ , presents \mathbf{Tens}_2 as a tensor product of \mathbf{Ten}_2 with \mathbf{BM}^\otimes .

PROOF. We compose μ_2 with the standard strong approximations $\mathbf{RM} \rightarrow \mathbf{RM}^\otimes, \mathbf{ten}_2 \rightarrow \mathbf{Ten}_2$ as described in [H.EY], 2.9, with $\mathbf{ten}_2 = \mathbf{BM} \sqcup^{\mathbf{Ass}} \mathbf{LM}$. We get a bilinear map $\mathbf{ten}_2 \times \mathbf{RM} \rightarrow \mathbf{Tens}_2$ that is easily seen to be a strong approximation. ■

4.6.4. PROPOSITION. *Let \mathcal{C} be a $\mathbf{Tens}_>$ -monoidal category with colimits. Then μ induces a commutative diagram*

$$\begin{array}{ccc} \mathbf{Alg}_{\mathbf{Tens}_2/\mathbf{Tens}_>}(\mathcal{C}) & \xrightarrow{\mathbf{RT}} & \mathbf{Alg}_{\mathbf{Tens}_>}(\mathcal{C}) \\ \theta_2 \downarrow \sim & & \downarrow \theta \\ \mathbf{Alg}_{\mathbf{Ten}_2/\mathbf{Tens}_>}(\mathcal{C}^{red}) & \xrightarrow{\mathbf{RT}^{red}} & \mathbf{Alg}_{\mathbf{Tens}_>}(\mathcal{C}^{red}), \end{array} \tag{34}$$

where \mathbf{RT}^{red} is the relative tensor product defined for the $\mathbf{Tens}_>$ -monoidal category \mathcal{C}^{red} .

PROOF. If \mathcal{C} is a $\mathbf{Tens}_>$ -monoidal category with colimits, \mathcal{C}^{red} is a $\mathbf{Tens}_>$ -monoidal category with colimits. This implies that \mathbf{RT}^{red} is defined as the functor left adjoint to the restriction $i_2^{red*} : \mathbf{Alg}_{\mathbf{Tens}_>}(\mathcal{C}^{red}) \rightarrow \mathbf{Alg}_{\mathbf{Ten}_2/\mathbf{Tens}_>}(\mathcal{C}^{red})$. The equivalence θ_2 is defined by the universal

property of $\mathcal{C}^{red} = \mathbf{Alg}_{\mathbf{RM}/\mathbf{Tens}_{\succ}}^{\mu}(\mathcal{C})$ and Lemma 4.6.3. The equivalence $\theta_2 \circ i_2^* = i_2^{red*} \circ \theta$ induces a morphism of functors

$$\mathbf{RT}^{red} \circ \theta_2 \rightarrow \theta \circ \mathbf{RT}. \tag{35}$$

We claim that this morphism is an equivalence.

Let $\phi \in \mathbf{Alg}_{\mathbf{Tens}_2/\mathbf{Tens}_{\succ}}(\mathcal{C})$ be given by a pair of bimodules $M \in {}_A\mathbf{BMod}_B(\mathcal{C}_m)$, $N \in {}_B\mathbf{BMod}_C(\mathcal{C}_n)$ and let $\Phi = \mathbf{RT}(\phi)$. By 4.4.5, the composition $\Phi \circ u_+ : \Delta_+^{\text{op}} \rightarrow \mathcal{C}$ is an operadic colimit diagram. This is equivalent to saying that the cocartesian shift $\mathbf{Sh}(\Phi \circ u_+) : \Delta_+^{\text{op}} \rightarrow \mathcal{C}_k$ is a colimit diagram.

The map $u_+ : \Delta_+^{\text{op}} \rightarrow \mathbf{Tens}_{\succ}$ factors through $i : \mathbf{Ten}_{\succ} \rightarrow \mathbf{Tens}_{\succ}$. By 4.4.5, the claim of Proposition 4.6.4 will be proven once we verify that $\theta(\Phi) \circ u_+$ is an operadic colimit diagram in \mathcal{C}^{red} , or, equivalently, that the cocartesian shift $\mathbf{Sh}(\theta(\Phi) \circ u_+) : \Delta_+^{\text{op}} \rightarrow \mathcal{C}_k^{red}$ is a colimit diagram.

The composition $G \circ \mathbf{Sh}(\theta(\Phi) \circ u_+) : \Delta_+^{\text{op}} \rightarrow \mathcal{C}_k$ is a colimit diagram as the composition $G \circ \theta$ is the restriction (33). According to [L.HA], 3.2.3.1, G creates colimits. This proves the proposition. ■

4.6.5. REMARK. Let $C \in \mathbf{Alg}_{\mathbf{Ass}}(\mathcal{C}_c)$ be the c -component of ϕ . We define a \mathbf{Ten}_{\succ} -monoidal subcategory \mathcal{C}_C^{red} of \mathcal{C}^{red} as follows. The restriction $\mathbf{Alg}_{\mathbf{RM}} \rightarrow \mathbf{Alg}_{\mathbf{Ass}/\mathbf{RM}}$ applied to the functor $\bar{\mathcal{C}}$ (31) yields a morphism of functors

$$\tilde{\mathcal{C}}^{red} \rightarrow \mathbf{Alg}_{\mathbf{Ass}/\mathbf{RM}} \circ \bar{\mathcal{C}}.$$

The \mathbf{Ten}_{\succ} -monoidal subcategory \mathcal{C}_C^{red} is defined as the fiber of this functor at the object of $\mathbf{Alg}_{\mathbf{Ass}/\mathbf{RM}} \circ \bar{\mathcal{C}}$ determined by the cocartesian arrow $C \rightarrow \phi_c(C)$, in the notation of 4.4.3. The functor $\theta(\Phi) \circ u_+ : \Delta_+^{\text{op}} \rightarrow \mathcal{C}^{red}$ canonically factors through \mathcal{C}_C^{red} . By [L.HA], 3.2.3.1, the functor $\Delta_+^{\text{op}} \rightarrow \mathcal{C}_C^{red}$ so defined is also an operadic colimit diagram.

5. Bar resolutions for enriched presheaves

The aim of this very technical section is to construct a certain operadic colimit diagram, see 5.3.1, used later in the proof of the important result 6.4.2.

Given a monadic adjunction $\mathcal{C} \rightleftarrows \mathbf{LMod}_A(\mathcal{C})$, any A -module M acquires a standard resolution $\mathbf{Bar}_{\bullet}(A, M)$ (sometimes called *bar resolution*). This is a simplicial resolution of M consisting of free A -modules. If one forgets the A -module structure on $\mathbf{Bar}_{\bullet}(A, M)$, one will get a special case of the bar construction described in 4.4.4. It should not surprise us that the A -module structure that we have just forgotten, can be reconstructed from a monadic action.

Let $\mathcal{C} = (\mathcal{C}_a, \mathcal{C}_m)$ be an LM-monoidal category with colimits, let A be an associative algebra in \mathcal{C}_a and M be an A -module in \mathcal{C}_m . The pair (A, M) is given by a map of operads $\gamma : \mathbf{LM} \rightarrow \mathcal{C}$. Its composition with a functor $u_+ : \Delta_+^{\text{op}} \rightarrow \mathbf{LM}$ (a variant of the u_+ defined in 4.4.4) defines an operadic colimit diagram; its cocartesian shift $\mathbf{Sh}(\gamma \circ u_+)$ is equivalent to

$G(\mathbf{Bar}_\bullet(A, M))$, where $\mathbf{Bar}(A, M)$ is the G -split simplicial objects defined by the monad $G \circ F$ on \mathcal{C} and $G : \mathbf{LMod}_A(\mathcal{C}_m) \rightarrow \mathcal{C}_m$ is the forgetful functor.

In Subsection 5.1 we explain how to reconstruct $\mathbf{Bar}_\bullet(A, M)$ from $\mathbf{Sh}(\gamma \circ u_+)$, in terms an action of the monad corresponding to A .

We use a similar reasoning to describe the bar resolution for enriched presheaves in Subsection 5.2. Here we are able to say more than for general modules. In general, we have no chance to introduce the monad action on $\gamma \circ u_+$ instead of $\mathbf{Sh}(\gamma \circ u_+)$, as the category $\mathbf{LMod}_A(\mathcal{C}_m)$ is not left-tensored over \mathcal{C}_a .

As for the enriched presheaves that are defined as $\mathbf{LMod}_{A^{\text{op}}}(\text{Fun}(X^{\text{op}}, \mathcal{M}))$, they have a left \mathcal{M} -module structure. This allows us to encode the bar resolution for a presheaf $f \in P_{\mathcal{M}}(\mathcal{A})$ into an operadic colimit diagram

$$K^\triangleright \rightarrow (\mathcal{M}, P_{\mathcal{M}}(\mathcal{A}))^\otimes, \tag{36}$$

with an appropriate choice of a category K , see 5.2.6 for the precise formulation.

5.1. BAR RESOLUTION OF A MODULE. Let \mathcal{O} be an LM-operad and let $\gamma : \mathbf{LM} \rightarrow \mathcal{O}$ be an LM-algebra in \mathcal{O} defined by a pair (A, M) , where A is an associative algebra in the planar operad \mathcal{O}_a and $M \in \mathcal{O}_m$ is a left A -module.

A very special case of the two-sided bar construction 4.4.4 gives the following simplicial resolution of a module.

We define the functor $u_+ : \Delta_+^{\text{op}} \rightarrow \mathbf{LM}$ by the formula

$$u_+([n]) = a^{n+1}m, \quad n \geq -1,$$

with the face maps defined by the multiplication in a and by its action on m . Note that the image of u_+ belongs to \mathbf{LM}^{act} , the active part of \mathbf{LM} .

5.1.1. LEMMA. *The composition*

$$\gamma \circ u_+ : \Delta_+^{\text{op}} \rightarrow \mathcal{O}$$

is an operadic colimit diagram. In the case when \mathcal{O} is a monoidal LM-category with geometric realizations, it induces an equivalence $|\mathbf{Sh}(\gamma \circ u)| \rightarrow M$ in \mathcal{O}_m .

PROOF. This is a direct consequence of [L.HA], 4.4.2.5, 4.4.2.8, applied to the tensor product $A \otimes_A M = M$. ■

5.1.2. The functor $\mathbf{Sh}(\gamma \circ u_+)$ has a canonical lifting to an augmented simplicial object in $\mathbf{LMod}_A(\mathcal{O}_m)$ that we call *the bar resolution of A -module M* and denote $\mathbf{Bar}_\bullet(A, M)$. In 5.1.2—5.1.5 we show how this canonical lifting can be described in terms of a monadic action.

5.1.3. AN ACTION OF \mathcal{O}_a ON $\text{Fun}(K, \mathcal{O}_m)$. An LM-monoidal category $\mathcal{O} = (\mathcal{O}_a, \mathcal{O}_m)$ encodes an action of a monoidal category \mathcal{O}_a on a category \mathcal{O}_m , or, in other words, a monoidal functor $\mathcal{O}_a \rightarrow \text{End}_{\text{Cat}}(\mathcal{O}_m)$.

Fix $K \in \text{Cat}$. The functor $C \mapsto \text{Fun}(K, C)$ defines a monoidal functor $\text{End}(C) \rightarrow \text{End}(\text{Fun}(K, C))$. Thus, any LM-monoidal category \mathcal{O} defines a monoidal functor $\mathcal{O}_a \rightarrow \text{End}(\mathcal{O}_m) \rightarrow \text{End}(\text{Fun}(K, \mathcal{O}_m))$, that is an LM-monoidal category $(\mathcal{O}_a, \text{Fun}(K, \mathcal{O}_m))$.

We wish to present two more constructions of the LM-monoidal category $(\mathcal{O}_a, \text{Fun}(K, \mathcal{O}_m))$.

1. Applying the functor $\text{Fun}^{\text{LM}}(K, _)$ defined in [H.EY], 6.1.6, we get an LM-monoidal category with the a -component $\text{Fun}^{\text{LM}}(K, \mathcal{O})_a = \text{Fun}(K, \mathcal{O}_a)$ and $\text{Fun}^{\text{LM}}(K, \mathcal{O})_m = \text{Fun}(K, \mathcal{O}_m)$. The forgetful functor $\text{Alg}_{\text{LM}}(\text{Cat}) \rightarrow \text{Alg}_{\text{Ass}}(\text{Cat})$ being a cartesian fibration, the LM-monoidal category $(\mathcal{O}_a, \text{Fun}(K, \mathcal{O}_m))$ described above, is equivalent to $i^*(\text{Fun}^{\text{LM}}(K, \mathcal{O}))$, where $i : \mathcal{O}_a \rightarrow \text{Fun}(K, \mathcal{O}_a)$ is induced by the map $K \rightarrow [0]$.

2. Let $K \in \text{Cat}$. Denote by K^{LM} the LM-monoidal category describing the action of the terminal monoidal category $[0]$ on K . We claim that $\text{Fun}_{\text{p-LM}}(K^{\text{LM}}, \mathcal{O})$ gives yet another presentation of $(\mathcal{O}_a, \text{Fun}(K, \mathcal{O}_m))$.

We start with the map of cocartesian fibrations over LM

$$q : K \times \text{LM} \rightarrow K^{\text{LM}} \tag{37}$$

constructed as an obvious natural transformation of the classifying functors $\text{LM} \rightarrow \text{Cat}$. The map q induces

$$K \times \text{Fun}_{\text{LM}}(K^{\text{LM}}, \mathcal{O}) \rightarrow K^{\text{LM}} \times_{\text{LM}} \text{Fun}_{\text{LM}}(K^{\text{LM}}, \mathcal{O}) \rightarrow \mathcal{O}, \tag{38}$$

and, therefore, an LM-operad map

$$Q : \text{Fun}_{\text{LM}}(K^{\text{LM}}, \mathcal{O}) \rightarrow \text{Fun}^{\text{LM}}(K, \mathcal{O}).$$

The monoidal component of Q is the monoidal functor $i : \mathcal{O}_a \rightarrow \text{Fun}(K, \mathcal{O}_a)$ mentioned above. Therefore, the map Q factors through

$$Q' : \text{Fun}_{\text{LM}}(K^{\text{LM}}, \mathcal{O}) \rightarrow i^*(\text{Fun}^{\text{LM}}(K, \mathcal{O})).$$

One can easily see that Q' is an equivalence.

5.1.4. COROLLARY. Let $\mathcal{O} = (\mathcal{O}_a, \mathcal{O}_m)$ be LM-monoidal category, $A \in \text{Alg}_{\text{Ass}}(\mathcal{O}_a)$, $f : K \rightarrow \mathcal{O}_m$ a functor. Let

$$F : \mathcal{O}_m \rightleftarrows \text{LMod}_A(\mathcal{O}_m) : G$$

be the adjunction. There is an equivalence between decompositions $f = G \circ f'$, $f' : K \rightarrow \text{LMod}_A(\mathcal{O}_m)$, and A -module structures on f .

5.1.5. We apply Corollary 5.1.4 to the functor $\mathbf{Sh}(\gamma \circ u_+) : \Delta_+^{\text{op}} \rightarrow \mathcal{O}_m$.

By 5.1.3 and 5.1.4, we have to produce a map of operads $\mathbf{LM} \rightarrow \mathbf{Funop}_{\mathbf{LM}}((\Delta_+^{\text{op}})^{\mathbf{LM}}, \mathcal{O})$ or, equivalently, an LM-operad map $(\Delta_+^{\text{op}})^{\mathbf{LM}} \rightarrow \mathcal{O}$, whose a -component is given by A and m -component is $\gamma \circ u_+ : \Delta_+^{\text{op}} \rightarrow \mathcal{O}_m$.

Let $P_{\mathbf{LM}}$ be the monoidal envelope of \mathbf{LM} . The monoidal part of $P_{\mathbf{LM}}$ has objects a^n , $n \geq 0$, and the arrows are generated by the unit $\mathbb{1} \rightarrow a$, the product $aa \rightarrow a$, subject to the standard identities. The module part of $P_{\mathbf{LM}}$ consists of the objects a^nm with the obvious action of the monoidal part.

One has an LM-monoidal functor $i : (\Delta_+^{\text{op}})^{\mathbf{LM}} \rightarrow P_{\mathbf{LM}}$ that is the unit on the monoidal part, and $u_+ : \Delta_+^{\text{op}} \rightarrow P_{\mathbf{LM}}$ carrying $[n - 1]$ to a^nm .

Since \mathcal{O} is LM-monoidal, the map $\gamma : \mathbf{LM} \rightarrow \mathcal{O}$ uniquely extends to $\Gamma : P_{\mathbf{LM}} \rightarrow \mathcal{O}$. The composition with i yields the required LM-operad map $(\Delta_+^{\text{op}})^{\mathbf{LM}} \rightarrow \mathcal{O}$.

5.2. BAR RESOLUTION FOR PRESHEAVES . We apply the reasoning of the previous subsection to enriched presheaves. Given a monoidal category \mathcal{M} and an \mathcal{M} -enriched category \mathcal{A} with the space of objects X , we want to describe a bar resolution for $f \in P_{\mathcal{M}}(\mathcal{A}) = \mathbf{Fun}_{\mathcal{M}^{\text{rev}}}(\mathcal{A}^{\text{op}}, \mathcal{M})$.

The pair $\gamma = (\mathcal{A}^{\text{op}}, f)$ is given by the functor

$$\gamma : \mathbf{LM} \rightarrow \mathbf{Quiv}_{X^{\text{op}}}^{\mathbf{LM}}(\mathcal{M}^{\text{rev}}, \mathcal{M}). \tag{39}$$

The information we need about the bar resolution of f is contained in the composition

$$\gamma \circ u_+ : \Delta_+^{\text{op}} \rightarrow \mathbf{LM} \rightarrow \mathbf{Quiv}_{X^{\text{op}}}^{\mathbf{LM}}(\mathcal{M}^{\text{rev}}, \mathcal{M}).$$

Since

$$\mathbf{Quiv}_{X^{\text{op}}}^{\mathbf{LM}}(\mathcal{M}^{\text{rev}}, \mathcal{M}) = \mathbf{Funop}_{\mathbf{LM}}(\mathbf{LM}_{X^{\text{op}}}, (\mathcal{M}^{\text{rev}}, \mathcal{M})),$$

the functor γ defines (and is uniquely defined by)

$$\gamma' : \mathbf{LM}_{X^{\text{op}}} \rightarrow (\mathcal{M}^{\text{rev}}, \mathcal{M}), \tag{40}$$

where the LM-operad $\mathbf{LM}_{X^{\text{op}}}$ is the one discussed in 2.3.1.

5.2.1. We will need to know more about the LM-operad \mathbf{LM}_X and its base change $\mathbf{LM}_X^{\circ} := \Delta_+^{\text{op}} \times_{\mathbf{LM}} \mathbf{LM}_X$.

The explicit description of \mathbf{LM}_X is given in [H.EY], 3.2. According to this description, \mathbf{LM}_X is presented by a functor

$$(\Delta_{/\mathbf{LM}})^{\text{op}} \rightarrow \mathcal{S}$$

carrying $\sigma : [n] \rightarrow \mathbf{LM}$ to $\mathbf{Map}(\mathcal{F}(\sigma), X)$ where $\mathcal{F} : \Delta_{/\mathbf{LM}} \rightarrow \mathbf{Cat}$ has values in conventional categories described by certain diagrams, see [H.EY], 3.2, especially the diagrams (51), (55), (60).

The base change $\mathbf{LM}_X^{\circ} = \Delta_+^{\text{op}} \times_{\mathbf{LM}} \mathbf{LM}_X$ is described by the collection of $\mathcal{F}(\sigma)$ for $\sigma : [n] \rightarrow \mathbf{LM}$ that factor through $u_+ : \Delta_+^{\text{op}} \rightarrow \mathbf{LM}$.

The categories $\mathcal{F}(\sigma)$ for these values of σ canonically decompose $\mathcal{F}(\sigma) = \mathcal{F}^-(\sigma) \sqcup [n]$, where $[n]$ appears as the rightmost component of $\mathcal{F}(\sigma)$ in the graphic presentation [H.EY], (55), (60), the component containing the vertex y_1 , see *op. cit.*, diagram (51).

This means that $\mathbf{LM}_X^\circ = \mathbf{LM}_X^{\circ,-} \times X$, so that the canonical projection $\mathbf{LM}_X^\circ \rightarrow \Delta_+^{\text{op}}$ factors through the projection to $\mathbf{LM}_X^{\circ,-}$.

5.2.2. The restriction of (40) to $\mathbf{LM}_{X^{\text{op}}}^\circ$ gives therefore

$$\gamma^\circ : \mathbf{LM}_{X^{\text{op}}}^{\circ,-} \times X^{\text{op}} \rightarrow u_+^*(\mathcal{M}^{\text{rev}}, \mathcal{M}), \quad (41)$$

where $u_+^*(\mathcal{M}^{\text{rev}}, \mathcal{M})$ is the base change of $(\mathcal{M}^{\text{rev}}, \mathcal{M})$ considered as a category over \mathbf{LM} .

We compose γ° with the equivalence $\text{op} : u_+^*(\mathcal{M}^{\text{rev}}, \mathcal{M}) \rightarrow u_+^*(\mathcal{M}, \mathcal{M})$ ⁸.

We get a functor

$$\gamma^{\circ,-} : \mathbf{LM}_{X^{\text{op}}}^{\circ,-} \rightarrow \text{Fun}(X^{\text{op}}, u_+^*(\mathcal{M}, \mathcal{M})).$$

Since the projection $\mathbf{LM}_{X^{\text{op}}}^\circ \rightarrow \Delta_+^{\text{op}}$ factors through $\mathbf{LM}_{X^{\text{op}}}^{\circ,-} \rightarrow \mathbf{LM}$, the map $\gamma^{\circ,-}$ defines a map

$$\gamma^{\circ,-} : \mathbf{LM}_{X^{\text{op}}}^{\circ,-} \rightarrow u_+^*(\text{Fun}^{\mathbf{LM}}(X^{\text{op}}, (\mathcal{M}, \mathcal{M}))), \quad (42)$$

where we use the notation of [H.EY], 6.1.6 to define the target of the map.

5.2.3. The right-hand side of (42) has, as **Ass**-component, the monoidal category $\text{Fun}(X^{\text{op}}, \mathcal{M})$. There is a monoidal functor

$$c : \mathcal{M} \rightarrow \text{Fun}(X^{\text{op}}, \mathcal{M})$$

carrying $m \in \mathcal{M}$ to the corresponding constant functor. The arrow

$$c^\dagger : (\mathcal{M}, \text{Fun}(X^{\text{op}}, \mathcal{M})) \rightarrow \text{Fun}^{\mathbf{LM}}(X^{\text{op}}, (\mathcal{M}, \mathcal{M})). \quad (43)$$

induced by c is cartesian in $\mathbf{Op}_{\mathbf{LM}}$ by Lemma 3.4.3. Since the **Ass**-component of $\gamma^{\circ,-}$ factors through the map $\mathbf{Ass}_{X^{\text{op}}} \rightarrow \mathcal{M}$ defining \mathcal{A}^{op} , the map (42) factors through c^\dagger giving the map that we denote by the same letter

$$\gamma^{\circ,-} : \mathbf{LM}_{X^{\text{op}}}^{\circ,-} \rightarrow u_+^*(\mathcal{M}, \text{Fun}(X^{\text{op}}, \mathcal{M})). \quad (44)$$

5.2.4. The category $\mathbf{LM}_{X^{\text{op}}}^{\circ,-}$ has one object over the terminal object $[-1]$ of Δ_+^{op} . We will denote this object by $*$ (note that it is not a terminal object). The functor $\gamma^{\circ,-}$ applied to $*$ gives $G(f) \in \text{Fun}(X^{\text{op}}, \mathcal{M})$ (once more, G is the forgetful functor $P_{\mathcal{M}}(\mathcal{A}) \rightarrow \text{Fun}(X^{\text{op}}, \mathcal{M})$).

An object of $\mathbf{LM}_{X^{\text{op}}}^{\circ,-}$ over $[n-1]$, $n \geq 1$, is given by a collection of objects $(y, x_n, y_n, \dots, x_1)$ of X .

The functor $\gamma^{\circ,-}$ carries $(y, x_n, y_n, \dots, x_1)$ to

$$(f(y), \mathcal{A}(y_n, x_n), \dots, \mathcal{A}(y_2, x_2), Y(x_1)) \in \mathcal{M}^n \times \text{Fun}(X^{\text{op}}, \mathcal{M}),$$

where Y is the Yoneda embedding.

⁸Note that this is an equivalence over $\text{op} : \Delta_+^{\text{op}} \rightarrow \Delta_+^{\text{op}}$.

5.2.5. It is interesting to see what does $\gamma^{\circ,-}$ do with the arrows.

An arrow α in $\mathbf{LM}_{X^{\text{op}}}^{\circ,-}$ from $(y, x_n, y_n, \dots, x_1)$ to $*$ is given by a collection of maps $\alpha_i : x_i \rightarrow y_{i+1}$ (or $\alpha_n : x_n \rightarrow y$).

The functor $\gamma^{\circ,-}$ carries α to the arrow

$$(f(y), \mathcal{A}(y_n, x_n), \dots, \mathcal{A}(y_2, x_2), Y(x_1)) \rightarrow f$$

defined by the map

$$f(y) \otimes \mathcal{A}(y_n, x_n) \otimes \dots \otimes \mathcal{A}(y_2, x_2) \rightarrow f(x_1),$$

defined by the \mathcal{A}^{op} -module structure on $P_{\mathcal{M}}(\mathcal{A}) \ni f$.

We are now ready to formulate the enriched presheaf analog of Lemma 5.1.1.

5.2.6. LEMMA. *The functor*

$$(\mathbf{LM}_{X^{\text{op}}}^{\circ,-})_{/*} \rightarrow (\mathcal{M}, \text{Fun}(X^{\text{op}}, \mathcal{M})) \tag{45}$$

induced by $\gamma^{\circ,-}$, is an operadic colimit diagram.

PROOF. Note that the source of the functor (45) has form K^{\triangleright} , where $K = \Delta^{\text{op}} \times_{\Delta_+^{\text{op}}} (\mathbf{LM}_{X^{\text{op}}}^{\circ,-})_{/*}$. The evaluation map $e_x : \text{Fun}(X^{\text{op}}, \mathcal{M}) \rightarrow \mathcal{M}$, $x \in X^{\text{op}}$, commutes with the left \mathcal{M} -module structure, so, in order to prove the lemma, it is sufficient to verify that for any $x \in X^{\text{op}}$ the composition of (45) with the evaluation map is an operadic colimit diagram $K^{\triangleright} \rightarrow (\mathcal{M}, \mathcal{M})$.

We know that the composition $\gamma \circ u_+ : \Delta_+^{\text{op}} \rightarrow \mathbf{Quiv}_{X^{\text{op}}}^{\text{LM}}(\mathcal{M}^{\text{rev}}, \mathcal{M})$ is an operadic colimit diagram. Since \mathcal{M} is a monoidal category with colimits, the restriction functor

$$\text{Fun}_{\text{LM}}(\Delta_+^{\text{op}}, \mathbf{Quiv}_{X^{\text{op}}}^{\text{LM}}(\mathcal{M}^{\text{rev}}, \mathcal{M})) \rightarrow \text{Fun}_{\text{LM}}(\Delta^{\text{op}}, \mathbf{Quiv}_{X^{\text{op}}}^{\text{LM}}(\mathcal{M}^{\text{rev}}, \mathcal{M})) \tag{46}$$

has a left adjoint carrying $\gamma \circ u$ to $\gamma \circ u_+$. Since, by adjunction, the arrow (46) can be rewritten as

$$\text{Fun}_{\text{LM}}(\mathbf{LM}_{X^{\text{op}}}^{\circ}, (\mathcal{M}, \mathcal{M})) \rightarrow \text{Fun}_{\text{LM}}(\Delta^{\text{op}} \times_{\Delta_+^{\text{op}}} \mathbf{LM}_{X^{\text{op}}}^{\circ}, (\mathcal{M}, \mathcal{M})), \tag{47}$$

this implies that the functor γ° is an operadic left Kan extension of its restriction to $\Delta^{\text{op}} \times_{\Delta_+^{\text{op}}} \mathbf{LM}_{X^{\text{op}}}^{\circ}$. This easily implies the claim. ■

5.2.7. We now intend to show that (44) canonically factors through the forgetful functor

$$(\mathcal{M}, P_{\mathcal{M}}(\mathcal{A})) \xrightarrow{G} (\mathcal{M}, \text{Fun}(X^{\text{op}}, \mathcal{M})). \tag{48}$$

This implies the main result of this section, Proposition 5.3.1.

To deduce the factorization, we need, according to 5.1.4, to present the action of the monad $G \circ F$ defined by \mathcal{A}^{op} on the functor (44). As a first step, we will describe a left $\mathbf{Quiv}_{X^{\text{op}}}(\mathcal{M}^{\text{rev}})$ -tensoring structure on the target of (44), the category $u_+^*(\mathcal{M}, \text{Fun}(X^{\text{op}}, \mathcal{M}))$.

Here is a general construction in the context of BM-monoidal categories.

5.2.8. **CONDENSATION.** Let $\mathcal{O} = (\mathcal{O}_a, \mathcal{O}_m, \mathcal{O}_b)$ be a **BM**-monoidal category. We can look at \mathcal{O}_m as an object of $\mathbf{RMod}_{\mathcal{O}_b}(\mathbf{LMod}_{\mathcal{O}_a}(\mathbf{Cat}))$. Its bar construction gives an augmented simplicial object in $\mathbf{LMod}_{\mathcal{O}_a}(\mathbf{Cat})$ that classifies a cocartesian fibration over Δ_+^{op} . The total category of this cocartesian fibration in $v_+^*(\mathcal{O}_m, \mathcal{O}_b)$ where the functor $v_+ : \Delta^{\text{op}} \rightarrow \mathbf{RM}$ is defined as the composition of u_+ with $\text{op} : \mathbf{LM} \rightarrow \mathbf{RM}$. The resulting **LM**-monoidal category will be called *the condensation* of \mathcal{O} and denoted by $\text{cond}(\mathcal{O})$.

Applying the condensation functor to $\mathcal{O} = \mathbf{Quiv}_{X^{\text{op}}}^{\mathbf{BM}}(\mathcal{M}^{\text{rev}})$, we get an **LM**-monoidal category whose monoidal part is $\mathbf{Quiv}_{X^{\text{op}}}(\mathcal{M}^{\text{rev}})$ and whose left-tensored part is

$$v_+^*(\text{Fun}(X^{\text{op}}, \mathcal{M}), \mathcal{M}^{\text{rev}}) = u_+^*(M, \text{Fun}(X^{\text{op}}, \mathcal{M})).$$

5.2.9. Therefore, in order to show that (44) canonically factors through (48), we have to extend $\gamma^{\circ,-}$ to a map of operads

$$\Gamma : (\mathbf{LM}_{X^{\text{op}}}^{\circ,-})^{\mathbf{LM}} \rightarrow \text{cond}(\mathbf{Quiv}_{X^{\text{op}}}^{\mathbf{BM}}(\mathcal{M}^{\text{rev}})). \tag{49}$$

5.2.10. We will construct (49) using the presentation of **LM**-operads by simplicial spaces over **LM**, or, equivalently, by presheaves on $\Delta_{/\mathbf{LM}}$. We will describe the functors corresponding to the source and the target of Γ , and we will show that (39) allows one to construct a map of these presheaves.

We will now describe the source of (49). First of all, $\mathbf{LM}_{X^{\text{op}}}^{\circ,-}$ is a category over Δ_+^{op} , so the space of its k -simplices decomposes

$$(\mathbf{LM}_{X^{\text{op}}}^{\circ,-})_k = \coprod_{\tau: [k] \rightarrow \Delta_+^{\text{op}}} \mathbf{LM}_{X^{\text{op}}}^{\circ,-}(u_+ \circ \tau),$$

where $\mathbf{LM}_{X^{\text{op}}}^{\circ,-}(u_+ \circ \tau) = \text{Map}(\mathcal{F}^-(u_+ \circ \tau), X^{\text{op}})$, see the notation of 5.2.1.

Let $\sigma : [n] \rightarrow \mathbf{LM}$ be an object of $\Delta_{/\mathbf{LM}}$ presented by a sequence

$$\sigma : a^{d_0}m \rightarrow \dots a^{d_k}m \rightarrow a^{d_{k+1}} \rightarrow \dots a^{d_n}.$$

We will assume $k = -1$ if σ factors through $\mathbf{Ass} \rightarrow \mathbf{LM}$.

Then the source of (49) is described by the following formula.

$$(\mathbf{LM}_{X^{\text{op}}}^{\circ,-})^{\mathbf{LM}}(\sigma) = \begin{cases} [0], & \text{if } k = -1, \\ (\mathbf{LM}_{X^{\text{op}}}^{\circ,-})_k = \coprod_{\tau: [k] \rightarrow \Delta_+^{\text{op}}} \mathbf{LM}_{X^{\text{op}}}^{\circ,-}(u_+ \circ \tau) & \text{otherwise.} \end{cases} \tag{50}$$

The description of the target of (49) will be presented in 5.2.15, after a certain digression.

5.2.11. **INTERNAL MAPPING OPERAD, REFORMULATED.** We need some detail on internal operad objects, [H.EY], 2.8.

The direct product in the category $P(C)$ of presheaves has a right adjoint assigning to a pair $F, G \in P(C)$ a presheaf $\text{Fun}_{P(C)}(F, G)$ whose value at $c \in C$ is calculated as the limit

$$\text{Fun}_{P(C)}(F, G)(c) = \lim_{a \rightarrow b \rightarrow c} \text{Map}(F(b), G(a)). \tag{51}$$

Fix a category B and let us look for a similar description of the internal Hom in $\mathbf{Cat}/_B$. The latter is a full subcategory of $P(\Delta/_B)$, so we can try to use the formula (51) with $C = \Delta/_B$. Given $F, G \in \mathbf{Cat}/_B$, we are looking for an object $\text{Funcat}/_B(F, G)$ furnishing an equivalence

$$\text{Map}_{\mathbf{Cat}/_B}(H, \text{Funcat}/_B(F, G)) = \text{Map}_{\mathbf{Cat}/_B}(H \times F, G). \tag{52}$$

Since $\mathbf{Cat}/_B$ is a full subcategory of $P(\Delta/_B)$, and since the representable presheaves belong to $\mathbf{Cat}/_B$, the object $\text{Funcat}/_B(F, G)$, if it exists, is equivalent to the presheaf $\text{Fun}_{P(\Delta/_B)}(F, G)$. This provides a very easy criterion for the existence of $\text{Funcat}/_B(F, G)$: it exists if and only if $\text{Fun}_{P(\Delta/_B)}(F, G)$ is a category over B . Note that, by definition, for $b \in B$, the fiber $\text{Funcat}/_B(F, G)_b$ identifies with $\text{Fun}(F_b, G_b)$.

Let us now apply the above reasoning to operads. If \mathcal{P} is a flat \mathcal{O} -operad, one defines a marked category $\pi : \mathcal{P}' \rightarrow \mathcal{P}$ over \mathcal{P} by the formulas

$$\mathcal{P}' = \text{Fun}^{in}([1], \mathcal{O}) \times_{\mathcal{O}} \mathcal{P}, \tag{53}$$

where $\text{Fun}^{in}([1], \mathcal{O})$ denotes the category of inert arrows in \mathcal{O} . Let $s, t : \text{Fun}^{in}([1], \mathcal{O}) \rightarrow \mathcal{O}$ be the standard projections. An arrow in \mathcal{P}' is marked iff its projections to \mathcal{P} and to \mathcal{O} (via s) are inert. \mathcal{P}' considered as a category over \mathcal{O} is flat. One defines $\text{Fun}_{\mathcal{O}}^{\sharp}(\mathcal{P}', \mathcal{Q})$ as the full subcategory of $\text{Funcat}/_{\mathcal{O}}(\mathcal{P}', \mathcal{Q})$ spanned by the arrows $\alpha : \mathcal{P}'_o \rightarrow \mathcal{Q}_o$, for some $o \in \mathcal{O}$, carrying the marked arrows in \mathcal{P}'_o to equivalences.

Proposition 2.8.3 of [H.EY] claims that, for any \mathcal{O} -operad \mathcal{Q} , $\text{Fun}_{\mathcal{O}}^{\sharp}(\mathcal{P}', \mathcal{Q})$ is an \mathcal{O} -operad representing $\text{Funop}_{\mathcal{O}}(\mathcal{P}, \mathcal{Q})$.

In particular, for $s : [n] \rightarrow \mathcal{O}$,

$$\text{Funop}_{\mathcal{O}}(\mathcal{P}, \mathcal{Q})(s) \subset \lim_{u \rightarrow v \rightarrow s} \text{Map}(\mathcal{P}'(v), \mathcal{Q}(u)),$$

consists of collections whose each component corresponding to $u = v : [1] \rightarrow [0] \xrightarrow{k} [n] \xrightarrow{s} \mathcal{O}$ carries the marked arrows of $\mathcal{P}'_{s(k)}$ to equivalences in $\mathcal{Q}_{s(k)}$.

5.2.12. REMARK. Note that a map $\mathcal{P}'_x \rightarrow \mathcal{Q}_x$ factoring through $\pi : \mathcal{P}'_x \rightarrow \mathcal{P}_x$ automatically carries the marked arrows to equivalences.

5.2.13. CONDENSATION, FOR BM-OPERADS. We need more explicit formulas for the condensation of a BM-monoidal category. This operation can be defined in the greater generality of BM-operads.

Given a BM-operad $p : \mathcal{O} \rightarrow \mathbf{BM}$, we will define its “condensation” $q : \mathcal{O}' = \text{cond}(\mathcal{O}) \rightarrow \mathbf{LM}$ so that

- $\mathcal{O}' \times_{\mathbf{LM}} \mathbf{Ass} = \mathcal{O} \times_{\mathbf{BM}} \mathbf{Ass}_-$.
- $\mathcal{O}'_m = v_+^*(\mathcal{O} \times_{\mathbf{BM}} \mathbf{RM})$,

where $v_+ : \Delta_+^{\text{op}} \rightarrow \mathbf{RM}$ is defined as the composition $v_+ = \text{op} \circ u_+$.

We will be using a presentation of operads by presheaves on $\Delta_{/\mathbf{BM}}$ and $\Delta_{/\mathbf{LM}}$.

Recall that $\mathbf{BM} = (\Delta_{/[1]})^{\text{op}}$; its objects are arrows $s : [n] \rightarrow [1]$ and an arrow from $s : [n] \rightarrow [1]$ to $t : [m] \rightarrow [1]$ is given by $f : [m] \rightarrow [n]$ such that $s \circ f = t$.

An object $s : [n] \rightarrow [1]$ of \mathbf{BM} defined by the formulas $s(i) = 0, i = 0, \dots, k, s(i) = 1, i > k$, is otherwise denoted $a^k m b^{n-k-1}$, see [H.EY], 2.9.2.

We denote by \mathbf{BM}^0 the full subcategory of \mathbf{BM} spanned by $s : [n] \rightarrow [1]$ with $s(0) = 0$. The category \mathbf{LM} is the full subcategory of \mathbf{BM}^0 spanned by the arrows $s : [n] \rightarrow [1]$ having at most one value of 1. The subcategories \mathbf{BM}^1 and \mathbf{RM} are defined as images of \mathbf{BM}^0 and \mathbf{LM} under $\text{op} : \mathbf{BM} \rightarrow \mathbf{BM}$.

The full embeddings $\mathbf{LM} \rightarrow \mathbf{BM}^0$ and $\mathbf{RM} \rightarrow \mathbf{BM}^1$ have left adjoint functors $\ell : \mathbf{BM}^0 \rightarrow \mathbf{LM}$ and $r : \mathbf{BM}^1 \rightarrow \mathbf{RM}$ erasing superfluous values of 1 and 0 respectively. The functors ℓ and r induce functors $\Delta_{/\mathbf{BM}^0} \rightarrow \Delta_{/\mathbf{LM}}$ and $\Delta_{/\mathbf{BM}^1} \rightarrow \Delta_{/\mathbf{RM}}$. We will denote by $\Delta_{/\mathbf{RM}}^{\text{act}}$ the category of simplices in \mathbf{RM} whose all arrows are active.

For $\tau : [n] \rightarrow \mathbf{BM}^0$ given by

$$\tau : a^{c_0} m b^{d_0} \rightarrow \dots \rightarrow a^{c_k} m b^{d_k} \rightarrow a^{c_{k+1}} \rightarrow \dots \rightarrow a^{c_n}$$

or

$$\tau : a^{c_0} m b^{d_0} \rightarrow \dots \rightarrow a^{c_k} m b^{d_k} \rightarrow b^{d_{k+1}} \rightarrow \dots \rightarrow b^{d_n},$$

we denote by τ_- the k -simplex

$$a^{c_0} m b^{d_0} \rightarrow \dots \rightarrow a^{c_k} m b^{d_k}$$

(if τ is a simplex in $\mathbf{Ass}_- \subset \mathbf{BM}^0$, we put $k = -1$ and τ_- is empty in this case).

For $\sigma \in \Delta_{/\mathbf{LM}}$ we define

$$\Pi(\sigma) = \{ \tau \in \Delta_{/\mathbf{BM}^0} \mid \ell\tau = \sigma, r(\tau_-) \in \Delta_{/\mathbf{RM}}^{\text{act}} \}. \tag{54}$$

Note that for $\sigma \in \Delta_{/\mathbf{Ass}_-}$ $k = -1$ and $\Pi(\sigma) = \{ \sigma \}$.

The assignment $\sigma \mapsto \Pi(\sigma)$ is obviously functorial, with a map $\alpha : [m] \rightarrow [n]$ defining $\Pi(\sigma) \rightarrow \Pi(\sigma \circ \alpha)$ carrying τ to $\tau \circ \alpha$.

Let a \mathbf{BM} -operad \mathcal{O} be described by a presheaf $F \in P(\Delta_{/\mathbf{BM}})$. We define a presheaf $F' \in P(\Delta_{/\mathbf{LM}})$ describing $\text{cond}(\mathcal{O})$ by the formula

$$F'(\sigma) = \coprod_{\tau \in \Pi(\sigma)} F(\tau). \tag{55}$$

5.2.14. LEMMA. *The presheaf F' defined above represents an \mathbf{LM} -operad.*

PROOF. 1. Segal condition follows from the definition of $\Pi(\sigma)$. To verify completeness, we can fix $w = a^k m \in \mathbf{LM}$ and study the simplicial space $n \mapsto \coprod_{\tau \in \Pi(w_n)} F(\tau)$, where w_n is the degenerate n -simplex determined by $w \in \mathbf{LM}$. This simplicial space is equivalent to the product of the fibers $\{a^k\} \times_{\mathbf{BM}} \mathcal{O}$ and $v_+^*(\mathcal{O})$, which is, of course, complete as a simplicial space. Thus, F' represents a category over \mathbf{LM} which we will denote by $\text{cond}(\mathcal{O})$.

2. It remains to prove that $\text{cond}(\mathcal{O})$ is fibrous.

Let us describe cocartesian liftings of the inerts in \mathbf{LM} . These are of two types

- Erasing m . Such inert has form $\alpha : a^n m \rightarrow a^l$.
- Not erasing m : either $\alpha : a^n \rightarrow a^l$ or $\alpha : a^n m \rightarrow a^l m$.

In the first case the cocartesian lifting of α in $F'(\alpha)$ having a source in $F(t)$ for $t = a^n m b^k$ is an inert arrow in $F(\tau)$, where $\tau \in \Pi(\alpha)$ is (the only) inert arrow from $a^n m b^k$ to a^l such that $\ell(\tau) = \alpha$.

In the second case, for $\alpha : a^n m \rightarrow a^l m$, with the source $t = a^n m b^k$, is the inert arrow in $F(\tau)$ where τ is defined by the conditions $\ell(\tau) = \alpha$, $r(\tau) = \text{id}_{m b^k}$.

The rest of the fibrousness conditions [L.HA], 2.3.3.28 or [H.EY], 2.6.3 easily follow from the above description. ■

5.2.15. We will now present the target of (49) by a presheaf on $\Delta_{/LM}$.

For $\sigma : [n] \rightarrow LM$ we have

$$\begin{aligned} \text{cond}(\text{Quiv}_{X^{\text{op}}}^{\text{BM}}(\mathcal{M}^{\text{rev}}))(\sigma) &= \coprod_{\hat{\sigma} \in \Pi(\sigma)} \text{Quiv}_{X^{\text{op}}}^{\text{BM}}(\mathcal{M}^{\text{rev}})(\hat{\sigma}) \\ &\subset \coprod_{\hat{\sigma} \in \Pi(\sigma)} \lim_{u \rightarrow v \rightarrow \hat{\sigma}} \text{Map}(\text{BM}'_{X^{\text{op}}}(v), \mathcal{M}(\text{op} \circ u)), \end{aligned} \tag{56}$$

where the inclusion means that we have to choose the connected components preserving the inerts, and $\text{BM}'_{X^{\text{op}}}$ is defined by the formula (53).

5.2.16. We are now ready to construct Γ . It consists of a compatible collection of maps

$$\Gamma_{\sigma, \tau, s, t} : \text{LM}_{X^{\text{op}}}^{\circ, -}(u_+ \circ \tau) \times \text{BM}'_{X^{\text{op}}}(t) \rightarrow \mathcal{M}(\pi \circ \text{op} \circ s) \tag{57}$$

for each $\sigma \in \Delta_{LM}$, $\hat{\sigma} = \sigma * (v_+ \circ \text{op} \circ \tau) \in \Pi(\sigma)$ and $s \rightarrow t \rightarrow \hat{\sigma}$ in $\Delta_{/BM}$.

Equivalently, this can be described by a compatible collection

$$\Gamma_{\sigma, \tau} : \text{LM}_{X^{\text{op}}}^{\circ, -}(u_+ \circ \tau) \times \text{BM}'_{X^{\text{op}}}(\hat{\sigma}) \rightarrow \mathcal{M}(\pi \circ \text{op} \circ \hat{\sigma}) \tag{58}$$

for each $\sigma \in \Delta_{LM}$ and $\hat{\sigma} = \sigma * (v_+ \circ \text{op} \circ \tau) \in \Pi(\sigma)$.

5.2.17. The collection of maps (58) we are going to present will factor through the natural projections $\text{BM}'_{X^{\text{op}}} \rightarrow \text{BM}_{X^{\text{op}}}$, so, by Remark 5.2.12, it will induce a map to $\text{Fun}^\sharp(\text{BM}'_{X^{\text{op}}}, \mathcal{M}^{\text{rev}})$ as needed.

The explicit formulas for BM_X show that $\text{BM}_{X^{\text{op}}}(\hat{\sigma}) = \text{BM}_{X^{\text{op}}}(\sigma)$, so we will rewrite the source of (58) as

$$\begin{aligned} \text{Map}(\mathcal{F}^-(u_+ \circ \tau), X^{\text{op}}) \times \text{Map}(\mathcal{F}(\sigma), X^{\text{op}}) &= \text{Map}(\mathcal{F}^-(u_+ \circ \tau) \sqcup \mathcal{F}(\sigma), X^{\text{op}}) = \\ \text{Map}(\mathcal{F}(u_+ \circ \tau) \sqcup^{\mathcal{F}(\sigma_k)} \mathcal{F}(\sigma), X^{\text{op}}) &= \text{Map}(\mathcal{F}(\sigma), X^{\text{op}}) \times_{\text{Map}(\mathcal{F}(\sigma_k), X^{\text{op}})} \text{Map}(\mathcal{F}(\tau), X^{\text{op}}), \end{aligned}$$

where σ_k denotes the k -simplex in LM with the constant value $m \in LM$. Similarly, one has a canonical presentation

$$\mathcal{M}(\pi \circ \text{op} \circ \hat{\sigma}) = \mathcal{M}(\pi \circ \text{op} \circ \sigma) \times_{\mathcal{M}(\pi \circ \sigma_n)} \mathcal{M}(\pi \circ \text{op} \circ \tau). \tag{59}$$

5.2.18. The map γ' (40) is described by a map of presheaves, given, for each $\sigma : [n] \rightarrow \mathbf{LM}$, by a map

$$\gamma_\sigma : \text{Map}(\mathcal{F}(\sigma), X^{\text{op}}) \rightarrow M(\pi \circ \text{op} \circ \sigma). \tag{60}$$

The collection of maps $\Gamma_{\sigma,\tau}$ is defined as the fiber product of γ_σ and γ_τ .

5.3. CONCLUSION. We have just constructed the functor Γ (49) defined by the collection of maps $\Gamma_{\sigma,\tau}$ (58) obtained as the fiber product of γ_σ and $\gamma_{u+\tau}$ (60). This implies the main result of this section.

5.3.1. PROPOSITION. *The functor Γ (49) defines an operadic colimit diagram*

$$(\mathbf{LM}_{X^{\text{op}}}^{\circ,-})_{/*} \rightarrow (\mathcal{M}, P_{\mathcal{M}}(\mathcal{A})). \tag{61}$$

6. Weighted colimits

In this paper we study weighted colimits of \mathcal{M} -functors.

Given an \mathcal{M} -functor $f : \mathcal{A} \rightarrow \mathcal{B}$ where \mathcal{A} is \mathcal{M} -enriched category and \mathcal{B} is a left \mathcal{M} -module, and an enriched presheaf $W \in P_{\mathcal{M}}(\mathcal{A})$, we will define a weighted colimit $\text{colim}_W(f) \in \mathcal{B}$. The construction is functorial in W , so that $\text{colim}_W(f)$ is the evaluation at W of a certain colimit preserving functor $\text{colim}(f) : P_{\mathcal{M}}(\mathcal{A}) \rightarrow \mathcal{B}$ preserving the left \mathcal{M} -module structure. The composition $\text{colim}(f) \circ Y$ with the enriched Yoneda embedding yields $f : \mathcal{A} \rightarrow \mathcal{B}$.

The construction of weighted colimit is also functorial in $f : \mathcal{A} \rightarrow \mathcal{B}$. This will imply Theorem 6.4.4 claiming that the Yoneda embedding $Y : \mathcal{A} \rightarrow P_{\mathcal{M}}(\mathcal{A})$ is a universal \mathcal{M} -functor.

6.1. INTERNAL HOM. We keep the notation of 4.4.3. Let $\mathcal{C} = \mathbf{Cat}^L$.

Given A, B, C associative algebras in \mathcal{C} , we have a functor

$$\text{RT}_{A,B,C} : {}_A \mathbf{BMod}_B(\mathcal{C}) \times {}_B \mathbf{BMod}_C(\mathcal{C}) \rightarrow {}_A \mathbf{BMod}_C(\mathcal{C})$$

defined by the relative tensor product. This functor has a right adjoint

$$\text{Fun}_A^L : {}_A \mathbf{BMod}_B(\mathcal{C})^{\text{op}} \times {}_A \mathbf{BMod}_C(\mathcal{C}) \rightarrow {}_B \mathbf{BMod}_C(\mathcal{C})$$

which we are now going to describe.

6.1.1. The formula

$$\text{Map}_{{}_B \mathbf{BMod}_C} (N, \text{Fun}_{A,B,C}^L(M, K)) = \text{Map}_{{}_A \mathbf{BMod}_C} (M \otimes_B N, K)$$

determines $\text{Fun}_{A,B,C}^L$ ⁹ as presheaf. In the case when M is a free A - B -bimodule $M = A \otimes X \otimes B$, this presheaf is represented by the B - C -bimodule $\text{Fun}^L(X \otimes B, K)$; since any bimodule is a colimit of free bimodules and since the Yoneda embedding commutes with the limits, this proves the existence of $\text{Fun}_{A,B,C}^L(M, K)$ in general¹⁰.

⁹This is a temporary notation.

¹⁰The forgetful functor $\mathbf{Cat}^L \rightarrow \mathbf{CAT}$ to the category of big categories creates the limits.

6.1.2. The functor $\text{Fun}_{A,B,C}^L$, as defined above, depends on three algebras A, B, C . We will omit B and C from the notation for the following reason.

Let $b : B \rightarrow B'$ and $c : C \rightarrow C'$ be algebra maps. Then the $B'-C'$ -bimodule $\text{Fun}_{A,B',C'}^L(M, K)$ identifies with the restriction of scalars of the $B-C$ -bimodule $\text{Fun}_{A,B,C}^L(M, K)$.

6.1.3. Thus, we now assume that $B = C = \mathbb{1}$. For $M, K \in \text{LMod}_A(\mathcal{C})$, we can define $\text{Fun}_A^L(M, K)$ as the full subcategory of $\text{Fun}_{\text{LMod}_A^w}(M, K)$, see 3.1.2, spanned by the lax LM-monoidal functors $f = (\text{id}_A, f_m) : (A, M) \rightarrow (A, K)$ satisfying two extra properties:

- f is LM-monoidal.
- f_m preserves small colimits.

As we explained above, in the case when M is an A - B -bimodule and K is an A - C -bimodule, $\text{Fun}_A^L(M, K)$ acquires a natural B - C -bimodule structure.

6.2. WEIGHTED COLIMIT. We apply the notion of tensor product described in 4.4 to the following context.

Fix a monoidal category $\mathcal{M} \in \text{Alg}_{\text{Ass}}(\text{Cat}^L)$ and a $X \in \text{Cat}$. Let \mathcal{B} be a left \mathcal{M} -module. Let $N = \text{Fun}(X, \mathcal{M})$. This is a right \mathcal{M} -module.

6.2.1. LEMMA. *N is right dualizable. Its right dual is $M = \text{Fun}(X^{\text{op}}, \mathcal{M})$ considered as a left \mathcal{M} -module.*

PROOF. We deduce the duality between M and N from the special case $\mathcal{M} = \mathcal{S}$. In this case $N = P(X^{\text{op}})$, $M = P(X)$ and the duality is given by the maps

$$c : \mathcal{S} \rightarrow P(X^{\text{op}}) \otimes P(X) = P(X^{\text{op}} \times X)$$

defined as the colimit-preserving map preserving the terminal objects, and

$$e : P(X) \otimes P(X^{\text{op}}) \rightarrow \mathcal{S}$$

extending the Yoneda map $X^{\text{op}} \times X \rightarrow \mathcal{S}$ to preserve colimits.

To get the duality for arbitrary \mathcal{M} , we use Proposition 4.3.2. We have three associative algebras in Cat^L , $A = B = \mathcal{S}$ and $C = \mathcal{M}$, the adjoint pair $(P(X^{\text{op}}), P(X))$ that we just constructed, and the one defined by $(\mathcal{M}, \mathcal{M})$. The composition of these gives an adjunction for (N, M) . ■

6.2.2. Let us now apply Corollary 4.3.6 to $A = \mathcal{M}$ and $N = \text{Fun}(X, \mathcal{M})$. We get the right dual module $M = \text{Fun}(X^{\text{op}}, \mathcal{M})$ and the endomorphism ring $B = \text{Quiv}_X(\mathcal{M})$, see [H.EY], 4.5.3. We also get a canonical \mathcal{M} - \mathcal{M} -bimodule map

$$e : \text{Fun}(X^{\text{op}}, \mathcal{M}) \otimes_{\text{Quiv}_X(\mathcal{M})} \text{Fun}(X, \mathcal{M}) \rightarrow \mathcal{M}. \tag{62}^{11}$$

¹¹In fact e is an equivalence. We do not use this fact.

We would like to comment about the right $\mathbf{Quiv}_X(\mathcal{M})$ -module structure on $\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{M})$.

According to Remark 4.3.7, the reverse monoidal category $\mathbf{Quiv}_X(\mathcal{M})^{\mathrm{rev}}$ identifies with the endomorphism object of the left \mathcal{M} -module $\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{M})$, that is, of the right $\mathcal{M}^{\mathrm{rev}}$ -module $\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{M})$. This means that the right $\mathbf{Quiv}_X(\mathcal{M})$ -action on $\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{M})$ is defined by the same construction as the left action on $\mathrm{Fun}(X, \mathcal{M})$, but with $\mathcal{M}^{\mathrm{rev}}$ replacing \mathcal{M} and X^{op} replacing X .

Tensoring (62) with \mathcal{B} over \mathcal{M} and using associativity of the relative tensor product 4.4.6, we get a map of left \mathcal{M} -modules

$$e_{\mathcal{B}} : \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{M}) \otimes_{\mathbf{Quiv}_X(\mathcal{M})} \mathrm{Fun}(X, \mathcal{B}) \rightarrow \mathcal{B}. \quad (63)$$

This gives a \mathbf{Ten}_{\succ} -algebra $\mathcal{Q} = \mathcal{Q}_{X, \mathcal{M}, \mathcal{B}}$ in \mathbf{Cat}^L consisting of the following categories and operations between them described above.

- Monoidal categories $\mathcal{Q}_a = \mathcal{Q}_{a'} = \mathcal{M}, \mathcal{Q}_b = \mathbf{Quiv}_X(\mathcal{M})$,
- A bimodule category $\mathcal{Q}_m = \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{M})$, a left $\mathbf{Quiv}_X(\mathcal{M})$ -module $\mathcal{Q}_n = \mathrm{Fun}(X, \mathcal{B})$ and a left \mathcal{M} -module $\mathcal{Q}_k = \mathcal{B}$.

6.2.3. Fix $X \in \mathbf{Cat}$, $(\mathcal{M}, \mathcal{B}) \in \mathbf{Alg}_{\mathbf{LM}}(\mathbf{Cat}^L)$. We have a relative tensor product functor

$$\mathbf{RT} : \mathbf{Alg}_{\mathbf{Ten}_2/\mathbf{Ten}_{\succ}}(\mathcal{Q}_{X, \mathcal{M}, \mathcal{B}}) \rightarrow \mathbf{Alg}_{\mathbf{Ten}_{\succ}}(\mathcal{Q}_{X, \mathcal{M}, \mathcal{B}}). \quad (64)$$

Let $\mathcal{A} \in \mathbf{Alg}_{\mathbf{Ass}}(\mathbf{Quiv}_X(\mathcal{M}))$ be an \mathcal{M} -enriched precategory with space of objects X . The restriction $\mathbf{RT}_{\mathbb{1}, \mathcal{A}}$ of (64) is a functor

$$\mathbf{RT}_{\mathbb{1}, \mathcal{A}} : \mathbf{RMod}_{\mathcal{A}}(\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{M})) \times \mathbf{LMod}_{\mathcal{A}}(\mathrm{Fun}(X, \mathcal{B})) \rightarrow \mathcal{B}. \quad (65)$$

Taking into account that $\mathbf{RMod}_{\mathcal{A}}(\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{M})) = P_{\mathcal{M}}(\mathcal{A})$ and $\mathbf{LMod}_{\mathcal{A}}(\mathrm{Fun}(X, \mathcal{B})) = \mathrm{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B})$, we finally get the functor called *weighted colimit*,

$$\mathrm{colim} : P_{\mathcal{M}}(\mathcal{A}) \times \mathrm{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{B}, \quad (66)$$

carrying a pair $(W \in P_{\mathcal{M}}(\mathcal{A}), f : \mathcal{A} \rightarrow \mathcal{B})$ to $\mathrm{colim}_W(f) := f \otimes W \in \mathcal{B}$. This functor preserves colimits separately in both arguments, as well as left \mathcal{M} -tensored structure in the first argument.

In particular, for a fixed $f : \mathcal{A} \rightarrow \mathcal{B}$ the functor $\mathrm{colim}(f) : P_{\mathcal{M}}(\mathcal{A}) \rightarrow \mathcal{B}$ preserving the colimits and the left \mathcal{M} -tensored structure, is defined.

Since the bifunctor (66) is a special case of the relative tensor product, it defines, using the notation (6.1), a canonical functor (that we also denote as colim)

$$\mathrm{colim} : \mathrm{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}) \rightarrow \mathrm{Fun}_{\mathbf{LMod}_{\mathcal{M}}}(P_{\mathcal{M}}(\mathcal{A}), \mathcal{B}) \quad (67)$$

6.3. PROPERTIES OF THE WEIGHTED COLIMIT.

6.3.1. LEMMA. *Let $Y : \mathcal{A} \rightarrow P_{\mathcal{M}}(\mathcal{A})$ be the Yoneda embedding. Then $\mathrm{colim}(Y) = \mathrm{id}_{P_{\mathcal{M}}(\mathcal{A})}$.*

PROOF. Look at the \mathbf{Tens}_{\succ} -monoidal category \mathcal{F} in \mathbf{Cat}^L having the following components.

- Monoidal categories $\mathcal{F}_a = \mathcal{F}_{a'} = \mathcal{M}$, $\mathcal{F}_b = \mathcal{F}_c = \mathcal{F}_{c'} = \mathbf{Quiv}_X(\mathcal{M})$,
- $\mathcal{F}_m = \mathcal{F}_k = \mathbf{Fun}(X^{\text{op}}, \mathcal{M})$, $\mathcal{F}_n = \mathbf{Quiv}_X(\mathcal{M})$,

with the standard \mathcal{M} - $\mathbf{Quiv}_X(\mathcal{M})$ -bimodule structure on $\mathbf{Fun}(X^{\text{op}}, \mathcal{M})$ and the unit $\mathbf{Quiv}_X(\mathcal{M})$ - $\mathbf{Quiv}_X(\mathcal{M})$ -bimodule structure on $\mathbf{Quiv}_X(\mathcal{M})$.

We will study the relative tensor product defined by \mathcal{F} . Let \mathcal{A} be an associative algebra in $\mathbf{Quiv}_X(\mathcal{M})$. The relative tensor product with \mathcal{A} - \mathcal{A} -bimodule \mathcal{A} defines the identity functor on $\mathbf{RMod}_{\mathcal{A}}(\mathbf{Fun}(X^{\text{op}}, \mathcal{M})) = P_{\mathcal{M}}(\mathcal{A})$.

We will now show that the calculation of $\text{colim}(Y)$ has the same answer.

We apply the reduction procedure described in 4.6. Let $\mathcal{F}' = \mathcal{F}'_{\mathcal{A}}{}^{\text{red}}$ be a \mathbf{Tens}_{\succ} -monoidal category obtained from \mathcal{F} by reduction, see 4.6.2 and 4.6.5. It has the following components.

- Monoidal categories $\mathcal{F}'_a = \mathcal{F}'_{a'} = \mathcal{M}$, $\mathcal{F}'_b = \mathbf{Quiv}_X(\mathcal{M})$.
- $\mathcal{F}'_m = \mathbf{Fun}(X^{\text{op}}, \mathcal{M})$, $\mathcal{F}'_n = \mathbf{RMod}_{\mathcal{A}}(\mathbf{Quiv}_X(\mathcal{M}))$, $\mathcal{F}'_k = \mathbf{RMod}_{\mathcal{A}}(\mathbf{Fun}(X^{\text{op}}, \mathcal{M}))$.

The category $\mathbf{Fun}(X, \mathbf{Fun}(X^{\text{op}}, \mathcal{M}))$ has a structure of $\mathbf{Quiv}_X(\mathcal{M})$ - $\mathbf{Quiv}_X(\mathcal{M})$ -bimodule described by the BM-monoidal category

$$\mathbf{Quiv}_X^{\text{BM}}(\mathbf{Quiv}_{X^{\text{op}}}^{\text{BM}}(\mathcal{M}^{\text{rev}})^{\text{rev}}),$$

as in [H.EY], 6.1.7. This bimodule identifies with $\mathbf{Quiv}_X(\mathcal{M})$. This implies that \mathcal{F}'_n as a left $\mathbf{Quiv}_X(\mathcal{M})$ -module identifies with $\mathbf{Fun}(X, P_{\mathcal{M}}(\mathcal{A}))$, so that \mathcal{F}' is equivalent to the category $\mathcal{Q}_{X, \mathcal{M}, \mathcal{B}}$ with $\mathcal{B} = P_{\mathcal{M}}(\mathcal{A})$. ■

. The construction $(X, \mathcal{M}, \mathcal{B}) \mapsto \mathcal{Q}_{X, \mathcal{M}, \mathcal{B}}$ described in 6.2.2 is functorial in $(X, \mathcal{M}, \mathcal{B}) \in \mathbf{Cat} \times \mathbf{Alg}_{\text{LM}}(\mathbf{Cat}^L)$. The precise expression of this functoriality is given in Section 7. We now need only a small (and obvious) fragment of it.

6.3.2. LEMMA. *A map $g : \mathcal{B} \rightarrow \mathcal{B}'$ of \mathcal{M} -modules in \mathbf{Cat}^L induces a \mathbf{Tens}_{\succ} -monoidal colimit-preserving functor $\mathcal{Q}_{X, \mathcal{M}, \mathcal{B}} \rightarrow \mathcal{Q}_{X, \mathcal{M}, \mathcal{B}'}$.*

By Proposition 4.4.5 this induces a map preserving weighted colimits. This implies that the colimit functor defined by formula (66) is functorial in \mathcal{B} .

6.3.3. COROLLARY. *For $g : \mathcal{B} \rightarrow \mathcal{B}'$ an arrow in $\mathbf{LMod}_{\mathcal{M}}$, the following diagram*

$$\begin{array}{ccc} P_{\mathcal{M}}(\mathcal{A}) \times \mathbf{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}) & \xrightarrow{\text{colim}} & \mathcal{B} \\ \downarrow & & \downarrow \\ P_{\mathcal{M}}(\mathcal{A}) \times \mathbf{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}') & \xrightarrow{\text{colim}} & \mathcal{B}' \end{array}, \quad (68)$$

with the vertical arrows defined by g , is commutative.

6.3.4. COROLLARY. *Let \mathcal{A} be a \mathcal{M} -enriched category and let \mathcal{B} be a left \mathcal{M} -modules with colimits. For any colimit-preserving map $F : P_{\mathcal{M}}(\mathcal{A}) \rightarrow \mathcal{B}$ of left \mathcal{M} -modules there is a natural equivalence*

$$F \xrightarrow{\sim} \operatorname{colim}(F \circ Y).$$

PROOF. Follows from 6.3.1 and 6.3.3. ■

6.4. UNIVERSALITY. We define the map

$$Y^* : \operatorname{Fun}_{\mathcal{M}}^L(P_{\mathcal{M}}(\mathcal{A}), \mathcal{B}) \rightarrow \operatorname{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}) \quad (69)$$

as the composition with the Yoneda embedding $Y : \mathcal{A} \rightarrow P_{\mathcal{M}}(\mathcal{A})$.

In this subsection we will show that Y^* is an equivalence.

6.4.1. The weighted colimit defines a map (67) in the opposite direction. According to Corollary 6.3.4, the composition $\operatorname{colim} \circ Y^*$ is equivalent to identity.

We deduce that the other composition is also equivalent to identity providing an interpretation of the weighted colimit as an operadic left Kan extension.

Recall 3.2 that $\bar{\mathcal{A}} \subset P_{\mathcal{M}}(\mathcal{A})$ is the full subcategory spanned by the representable functors.

In what follows we denote

$$P = P_{\mathcal{M}}(\mathcal{A}), \quad \mathfrak{P} = (\mathcal{M}, P) \in \operatorname{Alg}_{\operatorname{LM}}(\operatorname{Cat}^L), \quad \mathfrak{A} = (\mathcal{M}, \bar{\mathcal{A}}) \subset \mathfrak{P}, \quad \mathfrak{B} = (\mathcal{M}, \mathcal{B}).$$

Thus, \mathfrak{A} is an LM-suboperad of \mathfrak{P} . For $F \in \operatorname{Fun}_{\operatorname{LMod}_{\mathcal{M}}}(P, \mathcal{B})$ we will denote by the same letter $F : \mathfrak{P} \rightarrow \mathfrak{B}$ the corresponding LM-monoidal functor.

We claim the following.

6.4.2. PROPOSITION. *Any $F \in \operatorname{Fun}_{\operatorname{LMod}_{\mathcal{M}}}(P, \mathcal{B})$ is an operadic left Kan extension of $\bar{F} = F|_{\mathfrak{A}}$.*

6.4.3. PROPOSITION. *Let $\bar{F} : \mathfrak{A} \rightarrow \mathfrak{B}$ be a map of LM-operads. Then the map $F : \mathfrak{P} \rightarrow \mathfrak{B}$ defined as an operadic left Kan extension of \bar{F} preserves operadic colimits.*

6.4.4. THEOREM. *The map (69) is an equivalence.*

PROOF. Let $\bar{f} : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism in $\operatorname{LMod}_{\mathcal{M}}^w$ defined by $f : \mathcal{A} \rightarrow \mathcal{B}$. By 6.4.3, its operadic left Kan extension $F : \mathfrak{P} \rightarrow \mathfrak{B}$ preserves operadic colimits, and, therefore, defines an arrow in $\operatorname{LMod}_{\mathcal{M}}$. Therefore, by 6.4.2, $F = \operatorname{colim}(f)$. This implies that the composition $Y^* \circ \operatorname{colim}$ is equivalent to identity. ■

The rest of this subsection is devoted to proving 6.4.2 and 6.4.3.

6.4.5. PROOF OF PROPOSITION 6.4.2. Let $f \in P$. Denote $\mathcal{F}_f = \mathfrak{A} \times_{\mathfrak{P}} \mathfrak{P}_{/f}^{\text{act}}$.

We have to verify that the composition $\mathcal{F}_f \rightarrow \mathfrak{A} \xrightarrow{\bar{F}} \mathcal{B}$ extends to an operadic colimit diagram

$$\mathcal{F}_f^{\triangleright} \rightarrow \mathcal{B}$$

carrying the terminal object of $\mathcal{F}_f^{\triangleright}$ to $F(f)$.

The general case is immediately reduced to the case $\mathcal{B} = P$ and $F = \text{id}_P$. Thus, we have to verify that any $f \in P$ is an operadic colimit of the functor $\bar{y} : \mathcal{F}_f \rightarrow \mathfrak{P}$ defined as the composition of the projection $\mathcal{F}_f \rightarrow \mathfrak{A}$ with the embedding to \mathfrak{P} .

The plan of our proof is as follows.

Following to 5.3.1, the pair $\gamma = (\mathcal{A}^{\text{op}}, f)$ gives rise to a factorization of the functor $\gamma^{\circ, -}$, see (44),

$$\text{LM}_{X^{\text{op}}}^{\circ, -} \xrightarrow{\Gamma} \mathfrak{P} \xrightarrow{G} (\mathcal{M}, \text{Fun}(X^{\text{op}}, \mathcal{M})) \tag{70}$$

through the forgetful functor G , so that $\Gamma(*) = f$ where $*$ is a (unique) object of $\text{LM}_{X^{\text{op}}}^{\circ, -}$ over $[-1] \in \Delta_+^{\text{op}}$. The restriction of Γ to $\Delta^{\text{op}} \times_{\Delta_+^{\text{op}}} \text{LM}_{X^{\text{op}}}^{\circ, -}$ factors through the full embedding $\mathfrak{A} \rightarrow \mathfrak{P}$.

Recall that

$$K = \Delta^{\text{op}} \times_{\Delta_+^{\text{op}}} (\text{LM}_{X^{\text{op}}}^{\circ, -})_{/ *}$$

induces an equivalence

$$K^{\triangleright} = (\text{LM}_{X^{\text{op}}}^{\circ, -})_{/ *}$$

This yields a functor

$$\tau : K \rightarrow (\text{LM}_{X^{\text{op}}}^{\circ, -})_{/ *} \rightarrow \mathcal{F}_f \tag{71}$$

and an operadic colimit diagram (by Proposition 5.3.1)

$$\Gamma : K^{\triangleright} \rightarrow \mathfrak{P} \tag{72}$$

extending the composition of (71) with the projection $\mathcal{F}_f \rightarrow \mathfrak{P}$.

It remains to verify that τ is cofinal.

6.4.6. COFINALITY OF τ . To prove that τ is cofinal, we use Quillen's Theorem A in the form of [L.T], 4.1.3.1. We have to verify that, for any $\phi \in \mathcal{F}_f$, the comma category

$$K_{\phi} = K \times_{\mathcal{F}_f} (\mathcal{F}_f)_{\phi/}$$

is weakly contractible.

The proof goes as follows. In 6.4.8 below we present an object t_{ϕ} of K_{ϕ} and a functor $F : K_{\phi} \rightarrow \text{Fun}(\Lambda_0^2, K_{\phi})$ whose evaluation at $1 \in \Lambda_0^2$ is $\text{id}_{K_{\phi}}$ and at $2 \in \Lambda_0^2$ is the composition $K_{\phi} \rightarrow \{t_{\phi}\} \hookrightarrow K_{\phi}$. The functor F provides a null-homotopy for the identity map on K_{ϕ} . This proves cofinality of τ and, finally, Proposition 6.4.2.

6.4.7. We denote by $p : K \rightarrow \Delta^{\text{op}}$ the obvious projection. We also denote $p : K_\phi \rightarrow \Delta^{\text{op}}$ the composition $K_\phi \rightarrow K \rightarrow \Delta^{\text{op}}$. The functor $F : K_\phi \rightarrow \text{Fun}(\Lambda_0^2, K_\phi)$ will be defined as the one assigning to $t \in K_\phi$ a p -product diagram $t \leftarrow t' \rightarrow t_\phi$ in the sense of [L.T], 4.3.1.1. The latter means that for any $x \in K_\phi$ the diagram

$$\begin{array}{ccc} \text{Map}_{K_\phi}(x, t') & \longrightarrow & \text{Map}_{K_\phi}(x, t) \times \text{Map}_{K_\phi}(x, t_\phi) \\ \downarrow & & \downarrow \\ \text{Map}_{\Delta^{\text{op}}}(p(x), p(t')) & \longrightarrow & \text{Map}_{\Delta^{\text{op}}}(p(x), p(t)) \times \text{Map}_{\Delta^{\text{op}}}(p(x), p(t_\phi)) \end{array}$$

is cartesian. To construct F , we first construct a functor $\bar{F} : \Delta^{\text{op}} \rightarrow \text{Fun}(\Lambda_0^2, \Delta^{\text{op}})$ (this is very easy), and then prove that for any t there exists a p -product diagram $t \leftarrow t' \rightarrow t_\phi$ whose image under p is $\bar{F}(p(t))$. By the uniqueness of relative product diagrams, the functor \bar{F} lifts to a functor $F : K_\phi \rightarrow \text{Fun}(\Lambda_0^2, K_\phi)$.

6.4.8. We define $q : \mathcal{F}_f \rightarrow \Delta_+^{\text{op}}$ so that the composition $u_+ \circ \text{op} \circ q$ is the projection to $\mathcal{F}_f \rightarrow \mathfrak{P}_{/f}^{\text{act}} \rightarrow \text{LM}$. This condition uniquely determines q as any object in $\mathfrak{P}_{/f}^{\text{act}}$ has image in $u_+(\Delta_+^{\text{op}})$. The definition of q is chosen so that for $t \in K$ the equality $p(t) = q(\tau(k))$ holds.

Let $\phi \in \mathcal{F}_f$ be defined by the collection $(m_1, \dots, m_n, z, \beta)$ with $m_i \in \mathcal{M}$, $z \in X^{\text{op}}$ and $\beta : (m_1, \dots, m_n, z) \rightarrow f$ defined by a map $\otimes m_i \rightarrow f(z)$ (we will also denote it by β).

Given $d \in K$ defined by a sequence $(y, x_k, y_k, \dots, x_2, y_2, x_1)$ together with arrows $\alpha_i : x_i \rightarrow y_{i+1}$ ($\alpha_k : x_k \rightarrow y$), an object $t : \phi \rightarrow \tau(d)$ of K_ϕ is defined by a collection of maps

$$\begin{aligned} \otimes_{i=1}^{r_k} m_i &\rightarrow f(y); \\ \otimes_{i=r_j+1}^{r_{j-1}} m_i &\rightarrow \mathcal{A}(y_j, x_j), j = k, \dots, 2; \\ \otimes_{i=r_1+1}^n m_i &\rightarrow \mathcal{A}(z, x_1), \end{aligned} \tag{73}$$

for a certain sequence of numbers $1 \leq r_k \leq \dots \leq r_1 \leq n$ defining the arrow $a^n m \rightarrow a^k m$ in LM that is the image of t .

We define an object t_ϕ of K_ϕ by the arrow $t_\phi : \phi \rightarrow \tau(d_\phi)$ where $d_\phi = (z, z, \text{id}_z) \in K$, so that $\tau(d_\phi)$ is given by $(f(z), z, \text{id}_{f(z)})$, and $t_\phi : \phi \rightarrow \tau(d_\phi)$, an arrow in \mathcal{F}_f over the map $a^n m \rightarrow a m$ induced by $a^n \rightarrow a$ in LM , that is the identity on z and $\beta : \otimes m_i \rightarrow f(z)$ on the a -component.

Note that $q(t_\phi)$ is the map $\langle n-1 \rangle \rightarrow \langle 0 \rangle$ corresponding to $\{0\} \in [n-1]$.

We can now define $\bar{F} : \Delta^{\text{op}} \rightarrow \text{Fun}(\Lambda_0^2, \Delta^{\text{op}})$. Its opposite carries $[k-1] \in \Delta$ to the diagram $[k-1] \rightarrow [k] \leftarrow [0]$ in Δ with the arrows $\partial^0 : [k-1] \rightarrow [k]$ and $[0] \rightarrow \{0\} \in [k]$.

We claim that for any $t \in K_\phi$ with $p(t) = \langle k-1 \rangle$, there is a p -product diagram $t \leftarrow t' \rightarrow t_\phi$ over $\bar{F}(\langle k \rangle)$.

We will now define an arrow $d' \rightarrow d$ in K with a decomposition of $t : \phi \rightarrow \tau(d)$ via $\tau(d') \rightarrow \tau(d)$, as well as a map $d' \rightarrow d_\phi$ decomposing t_ϕ .

We put $d' = (y, x_k, y_k, \dots, x_1, z, z)$ together with $\alpha_i : x_i \rightarrow y_{i+1}$ and $\text{id}_z : z \rightarrow z$. We have $\tau(d') = (f(y), \mathcal{A}(y_k, x_k), \dots, \mathcal{A}(y_2, x_2), \mathcal{A}(z, x_1), z)$, so that the collection of maps (73), together with the unit $\mathbb{1} \rightarrow \mathcal{A}(z, z)$, yields a map that we denote $t' : \phi \rightarrow \tau(d')$.

The arrow $d' \rightarrow d$ in K is given by the commutative diagram

$$\begin{array}{ccc}
 \begin{array}{ccc} y & \xleftarrow{\alpha_k} & x_k \\ \uparrow & & \downarrow \\ y & \xleftarrow{\alpha_k} & x_k \end{array} &
 \begin{array}{ccc} y_k & \xleftarrow{\alpha_{k-1}} \dots \xleftarrow{\alpha_1} & x_1 \\ \uparrow & & \downarrow \\ y_k & \xleftarrow{\alpha_{k-1}} \dots \xleftarrow{\alpha_1} & x_1 \end{array} &
 \begin{array}{ccc} z & \xleftarrow{\quad} & z \\ \uparrow & & \downarrow \\ z & \xleftarrow{\quad} & z \end{array} \\
 \end{array} \tag{74}$$

where all unnamed arrows appearing in the diagram are the identity maps. The arrow $d' \rightarrow d_\phi$ is given by the commutative diagram

$$\begin{array}{ccc}
 \begin{array}{ccc} y & \xleftarrow{\alpha_k} & x_k \\ \uparrow & & \downarrow \\ y & \xleftarrow{\alpha_k} & x_k \end{array} &
 \begin{array}{ccc} y_k & \xleftarrow{\alpha_{k-1}} \dots \xleftarrow{\alpha_1} & x_1 \\ \uparrow & & \downarrow \\ y_k & \xleftarrow{\alpha_{k-1}} \dots \xleftarrow{\alpha_1} & x_1 \end{array} &
 \begin{array}{ccc} z & \xleftarrow{\quad} & z \\ \uparrow & & \downarrow \\ z & \xleftarrow{\quad} & z \end{array} \\
 \end{array} \tag{75}$$

where, once more, all unnamed arrows are the identity maps.

Thus, for a fixed map $p(d'') \rightarrow p(d')$ in Δ^{op} , its lifting in $d'' \rightarrow d'$ in K is described by the same collection of data as a pair of maps $d'' \leftarrow d$ and $d'' \rightarrow d_\phi$. Therefore, the diagram $d \leftarrow d' \rightarrow d_\phi$ is a p -product diagram in K . For the same reason the diagram $t \leftarrow t' \rightarrow t_\phi$ is a p -product diagram.

This proves the cofinality of $\tau : K \rightarrow \mathcal{F}_f$, and, therefore, Proposition 6.4.2.

6.4.9. PROOF OF PROPOSITION 6.4.3. We prove the claim in two steps.

1. The first step is to prove that $F : \mathfrak{P} \rightarrow \mathfrak{B}$ is a map of LM-monoidal categories, that is, that it preserves cocartesian arrows.

Let $X = m \oplus x \in \mathcal{M} \times P = \mathfrak{P}_{lm}$ and let $y = m \otimes x \in P$. We want to show that F carries the cocartesian arrow $\alpha : X \rightarrow y$ to a cocartesian arrow in \mathfrak{B} . By definition of F , $F(x)$ is the operadic colimit of the composition

$$\mathfrak{A}_{/x}^{\text{act}} \rightarrow \mathfrak{A} \xrightarrow{f} \mathfrak{B},$$

that is $m \oplus F(x)$ is the operadic colimit of the composition

$$\mathfrak{A}_{/x}^{\text{act}} \rightarrow \mathfrak{A} \xrightarrow{m \oplus -} \mathfrak{A} \xrightarrow{f} \mathfrak{B}.$$

One has an equivalence $\mathfrak{A}_{/X}^{\text{act}} \rightarrow \mathfrak{A}_{/m}^{\text{act}} \times \mathfrak{A}_{/x}^{\text{act}} = \mathcal{M}_{/m}^{\text{act}} \times \mathfrak{A}_{/x}^{\text{act}}$ (see Lemma 6.4.10 below) defined by the decomposition $X = m \oplus x$, so that the map $\mathfrak{A}_{/x} \xrightarrow{m \oplus -} \mathfrak{A}_{/X}$ is cofinal.

This implies that $m \oplus F(x)$ is the operadic colimit of the composition

$$\mathfrak{A}_{/X}^{\text{act}} \rightarrow \mathfrak{A} \xrightarrow{f} \mathfrak{B},$$

that implies that the arrow $F(X) \rightarrow F(y)$ is cocartesian.

2. Let $\bar{p} : K^\triangleright \rightarrow \mathfrak{P}^{\text{act}}$ be an operadic colimit diagram. We keep in mind that \mathfrak{P} is a LM-monoidal category with colimits, so operadic colimits can be expressed in terms of colimits. Since we already know that F preserves cocartesian arrows, it is sufficient to verify the claim in the case p factors through $P \subset \mathcal{P}^{\text{act}}$. Thus, from now on we can assume that $\bar{p} : K^\triangleright \rightarrow P$, is a colimit diagram; we have to verify that its composition with F remains a colimit diagram in \mathfrak{B} . Here we follow the proof of the similar claim in [L.T], 5.1.5.5.

For any $p : K \rightarrow P$ denote $\mathcal{E}(p) = \mathfrak{A}^{\text{act}} \times_{\mathfrak{P}^{\text{act}}} \mathfrak{P}^{\text{act}[1]} \times_{\mathfrak{P}^{\text{act}}} K$. We have two projections $s_p : \mathcal{E}(p) \rightarrow \mathfrak{A}^{\text{act}}$ and $d_p : \mathcal{E}(p) \rightarrow K$. One has a canonical map $u_p : Y \circ s_p \rightarrow p \circ d_p$ induced by the projection $\mathcal{E}(p) \rightarrow \mathfrak{P}^{\text{act}[1]}$. We assert that the composition $K \xrightarrow{p} P \xrightarrow{F} \mathfrak{B}^{\text{act}}$ is a left Kan extension of the composition $f \circ s_p : \mathcal{E}(p) \rightarrow \mathfrak{A}^{\text{act}} \rightarrow \mathfrak{B}^{\text{act}}$ along d . In fact, the map $d_p : \mathcal{E}(p) \rightarrow K$ is a cocartesian fibration so, by [L.T], 4.3.3.10, applied to $d_p : \mathcal{E}(p) \rightarrow K$ over K , the assertion can be verified fiberwise for each $x \in K$ where it follows from the definition of F as the left Kan extension of f . This establishes a canonical equivalence $\text{colim}(F \circ p) = \text{colim}(f \circ s_p)$.

Note that, by 6.4.2, $\text{id}_{\mathfrak{P}}$ is an operadic left Kan extension of $Y : \mathfrak{A} \rightarrow \mathfrak{P}$, so everything we said above is valid, in particular, for $F = \text{id}_{\mathfrak{P}}$.

Let $\bar{p} : K^\triangleright \rightarrow P$ be a colimit diagram and $p = \bar{p}|_K$. We have to prove that $F \circ \bar{p}$ is a colimit diagram in \mathfrak{B}_m . In what follows we denote $\mathcal{E} = \mathcal{E}(p)$ and $\bar{\mathcal{E}} = \mathcal{E}(\bar{p})$, as well as $s = s_p$, $\bar{s} = s_{\bar{p}}$, $d = d_p$, $\bar{d} = d_{\bar{p}}$, $u = u_p$ and $\bar{u} = u_{\bar{p}}$.

The map $s : \mathcal{E}(p) \rightarrow \mathfrak{A}^{\text{act}}$ is a cartesian fibration, so, in particular, the map $s^{\text{op}} : \mathcal{E}(p)^{\text{op}} \rightarrow (\mathfrak{A}^{\text{act}})^{\text{op}}$, is a smooth map.

Given $a \in \mathfrak{A}^{\text{act}}$, the fiber $\mathcal{E}(p)_a = \mathfrak{P}_{Y(a)/}^{\text{act}} \times_{\mathfrak{P}^{\text{act}}} K$ is a cocartesian fibration over K classified by the composition $K \xrightarrow{p} \mathfrak{P}^{\text{act}} \xrightarrow{\text{ev}_{Y(a)}} \mathfrak{S}$.

For any $a \in \mathfrak{A}$ the induced map $\mathcal{E}_a \rightarrow \bar{\mathcal{E}}_a$ is a weak homotopy equivalence by [L.T], 3.3.4.5. Therefore, [L.T], 4.1.2.18 implies that the map $\mathcal{E} \rightarrow \bar{\mathcal{E}}$ is a contravariant equivalence over $\mathfrak{A}^{\text{act}}$.

We will now apply [L.T], 5.1.5.4 to deduce the required result. We have a map of LM-operads $f : \mathfrak{A} \rightarrow \mathfrak{B}$ and a contravariant equivalence $\mathcal{E} \rightarrow \bar{\mathcal{E}}$ over $\mathfrak{A}^{\text{act}}$.

The morphism of functors $\bar{u} : Y \circ \bar{s} \rightarrow \bar{p} \circ \bar{d}$ gives rise to a composition

$$Y \circ \bar{s} \xrightarrow{\bar{u}} \bar{p} \circ \bar{d} \rightarrow \bar{p}(\ast)$$

to the constant functor with the value at $\bar{p}(\ast) \in \mathfrak{P}$, that is a functor $v : \bar{\mathcal{E}}^\triangleright \rightarrow \mathfrak{P}^{\text{act}}$ carrying the terminal object to $\bar{p}(\ast)$. Let us look at the composition $F \circ v : \bar{\mathcal{E}}^\triangleright \rightarrow \mathfrak{B}$.

This is a colimit diagram as $\text{colim}(f \circ \bar{s} : \bar{\mathcal{E}} \rightarrow \mathfrak{B}) = \text{colim}(F \circ \bar{p} : K^\triangleright \rightarrow \mathfrak{B})$ that is obviously $F(\bar{p}(\ast))$.

Therefore, the restriction of $F \circ v$ to $\mathcal{E}^\triangleright$ is colimit diagram. This means that $\text{colim}(F \circ p : K \rightarrow \mathfrak{B}^{\text{act}}) = F(\ast)$. This completes the proof of 6.4.3.

6.4.10. LEMMA. *Let $\mathcal{P} \rightarrow \mathcal{Q}$ be a morphism of operads and let $x, y \in \mathcal{Q}_1$. Then one has an equivalence*

$$\mathcal{P}_{/x \oplus y}^{\text{act}} = \mathcal{P}_{/x}^{\text{act}} \times \mathcal{P}_{/y}^{\text{act}}.$$

PROOF. The category \mathcal{P}^{act} has a symmetric monoidal structure, see [L.HA], 2.2.4.3. One has a symmetric monoidal functor $\mathcal{P}^{\text{act}} \rightarrow \mathcal{Q}^{\text{act}}$ so that the monoidal operation induces a functor $\otimes : \mathcal{P}_{/x}^{\text{act}} \times \mathcal{P}_{/y}^{\text{act}} \rightarrow \mathcal{P}_{/x \oplus y}^{\text{act}}$. This is clearly an equivalence. ■

7. Functoriality of $\mathcal{Q}_{X, \mathcal{M}, \mathcal{B}}$

In Section 8 we present a monoidal version of the equivalence (69). This requires a better understanding of the functorial properties of this equivalence. Our aim is to present the collection of \mathbf{Ten}_{\succ} -monoidal categories $\mathcal{Q}_{X, \mathcal{M}, \mathcal{B}}$ defined in 6.2.2 as a symmetric monoidal functor from $\mathbf{Cat} \times \mathbf{Alg}_{\mathbf{LM}}(\mathbf{Cat}^L)$ to a certain category of \mathbf{Ten}_{\succ} -monoidal categories, see 7.5.1. The rest of the section is devoted to justifying Corollary 7.4.6 and Lemma 7.4.7 used in the proof of 7.5.1. The proof of Proposition 7.4.1 was suggested to us by the referee.

7.1. FAMILIES OF MONOIDAL CATEGORIES.

7.1.1. Let \mathcal{P} be an operad. Recall that the category $\mathbf{Mon}_{\mathcal{P}}^{\text{ lax}}$ is defined as the full subcategory of the category of \mathcal{P} -operads $\mathbf{Op}_{\mathcal{P}}$ spanned by the \mathcal{P} -monoidal categories. The category $\mathbf{Mon}_{\mathcal{P}}^{\text{ colax}}$ of \mathcal{P} -monoidal categories and colax monoidal functors can be formally defined as follows. First of all, one defines $\mathbf{Coop}_{\mathcal{P}^{\text{op}}}$, the category of \mathcal{P}^{op} -cooperads, as the category of functors $p : C \rightarrow \mathcal{P}^{\text{op}}$ such that $p^{\text{op}} : C^{\text{op}} \rightarrow \mathcal{P}$ is a \mathcal{P} -operad. The categories $\mathbf{Coop}_{\mathcal{P}^{\text{op}}}$ and $\mathbf{Op}_{\mathcal{P}}$ are obviously equivalent but any \mathcal{P} -monoidal category has both operadic and cooperadic realization that are intertwined by the passage from a monoidal category to its opposite. If M is a \mathcal{P} -monoidal category, its operadic realization is a \mathcal{P} -operad $M^{\otimes} \rightarrow \mathcal{P}$ that is a cocartesian fibration obtained by Grothendieck construction from the functor $\mathcal{P} \rightarrow \mathbf{Cat}$ describing the \mathcal{P} -monoidal structure on M . The equivalence between cocartesian fibrations over \mathcal{P} and the cartesian fibrations over \mathcal{P}^{op} carries M^{\otimes} to the cooperadic realization ${}^{\otimes}M \rightarrow \mathcal{P}^{\text{op}}$ of M . The embeddings $\mathbf{Mon}_{\mathcal{P}}^{\text{ lax}} \rightarrow \mathbf{Cat}_{/\mathcal{P}}$ and $\mathbf{Mon}_{\mathcal{P}}^{\text{ colax}} \rightarrow \mathbf{Cat}_{/\mathcal{P}^{\text{op}}}$ preserve products.

The category $\mathbf{Fun}(B^{\text{op}}, \mathbf{Mon}_{\mathcal{P}}^{\text{ lax}})$ is equivalent, by Grothendieck construction, to the category of arrows $p : X \rightarrow B \times \mathcal{P}$, with components $p_B : X \rightarrow B$ and $p_{\mathcal{P}} : X \rightarrow \mathcal{P}$, satisfying the following properties.

1. p_B is a cartesian fibration, $p_{\mathcal{P}}$ is a cocartesian fibration.
2. p is a map of cartesian fibrations over B and of cocartesian fibrations over \mathcal{P} .
3. For any $b \in B$ the fiber $X_b = p_B^{-1}(b) \rightarrow \mathcal{P}$ is a \mathcal{P} -operad.
4. For any $\beta : b \rightarrow b'$ in B the cartesian lifting $\beta^! : X_{b'} \rightarrow X_b$ is a map of \mathcal{P} -operads.

The first two properties mean that $p : X \rightarrow B \times \mathcal{P}$ is a lax bifibration in the sense of [H.D], 3.1.2.

Recall that $\mathbf{Cat}_{/B}^{\text{coc}}$ (resp., $\mathbf{Cat}_{/B}^{\text{cart}}$) denotes the full subcategory of $\mathbf{Cat}_{/B}$ spanned by the cocartesian (resp., cartesian) fibrations.

One has

7.1.2. PROPOSITION. (see [H.EY], 2.11.3) *One has an equivalence $\text{Fun}(B^{\text{op}}, \mathbf{Mon}_{\mathcal{P}}^{\text{lax}}) = \mathbf{Alg}_{\mathcal{P}}(\mathbf{Cat}_{/B}^{\text{cart}})$.*

There is a similar description of $\text{Fun}(B, \mathbf{Mon}_{\mathcal{P}}^{\text{colax}})$. This category is equivalent to the category of arrows $q : X \rightarrow \mathcal{P}^{\text{op}} \times B$ such that

1. q_B is a cocartesian fibration, $q_{\mathcal{P}^{\text{op}}}$ is a cartesian fibration.
2. q is a map of cocartesian fibrations over B and of cartesian fibrations over \mathcal{P}^{op} .
3. For any $b \in B$ the fiber $X_b = q_B^{-1}(b) \rightarrow \mathcal{P}^{\text{op}}$ is a \mathcal{P}^{op} -cooperad.
4. For any $\beta : b \rightarrow b'$ in B the cocartesian lifting $\beta_! : X_b \rightarrow X_{b'}$ is a map of \mathcal{P} -cooperads.

The following result is an immediate consequence of 7.1.2.

7.1.3. PROPOSITION. *One has an equivalence $\text{Fun}(B, \mathbf{Mon}_{\mathcal{P}}^{\text{colax}}) = \mathbf{Alg}_{\mathcal{P}}(\mathbf{Cat}_{/B}^{\text{coc}})$.*

Let now \mathcal{P} and \mathcal{Q} be two operads. In the lemma below the categories $\mathbf{Mon}_{\mathcal{P}}^{\text{lax}}$ and $\mathbf{Mon}_{\mathcal{Q}}^{\text{colax}}$ are endowed with the cartesian symmetric monoidal structure.

7.1.4. LEMMA. *There is an equivalence of categories*

$$\mathbf{Alg}_{\mathcal{Q}}(\mathbf{Mon}_{\mathcal{P}}^{\text{lax}}) = \mathbf{Alg}_{\mathcal{P}}(\mathbf{Mon}_{\mathcal{Q}}^{\text{colax}}).$$

PROOF. The category $\mathbf{Alg}_{\mathcal{P}}(\mathbf{Mon}_{\mathcal{Q}}^{\text{colax}})$ identifies with the full subcategory of functors $f : \mathcal{P} \rightarrow \mathbf{Cat}_{/\mathcal{Q}^{\text{op}}}$ satisfying the Segal condition and such that for any $p \in \mathcal{P}$ $f(p)$ is a (cooperad presentation of a) \mathcal{P} -monoidal category. This can be equivalently described as the category of $(\mathcal{Q}^{\text{op}}, \mathcal{P})$ -lax bifibrations [H.D] that is maps $q : X \rightarrow \mathcal{Q}^{\text{op}} \times \mathcal{P}$ that are simultaneously map of cartesian fibrations over \mathcal{Q}^{op} and of cocartesian fibrations over \mathcal{P} , satisfying the listed above properties. Equivalently, this can be rewritten in terms of $(\mathcal{Q}, \mathcal{P})$ -Gray fibrations [HHLN] $p : X \rightarrow \mathcal{P} \times \mathcal{Q}$ satisfying the properties

- for any $q \in \mathcal{Q}$ the fiber $X_q \rightarrow \mathcal{P}$ is a \mathcal{P} -monoidal category.
- Any decomposition $q = q_1 \oplus q_2$ gives rise to a cartesian diagram

$$\begin{array}{ccc} X_q & \longrightarrow & X_{q_1} \\ \downarrow & & \downarrow \\ X_{q_2} & \longrightarrow & \mathcal{P} \end{array}$$

Rewriting these as functors $\mathcal{Q} \rightarrow \mathbf{Cat}_{/\mathcal{P}}$, we get the subcategory $\mathbf{Alg}_{\mathcal{Q}}(\mathbf{Mon}_{\mathcal{P}}^{\text{lax}})$. ■

7.2. MONOIDAL STRUCTURE ON $\mathbf{Mon}_p^{\text{lax}/\text{colax},L}$. The category $\mathbf{Mon}_p^L = \mathbf{Alg}_p(\mathbf{Cat}^L)$ has a symmetric monoidal structure inherited from that on \mathbf{Cat}^L . In this subsection we show that the lax and the colax versions of this category also have a symmetric monoidal structure.

We denote by $\mathbf{Mon}_p^{\text{lax},\times}$ the cocartesian fibration over Fin_* defined by the cartesian symmetric monoidal structure on $\mathbf{Mon}_p^{\text{lax}}$ ¹². The subcategory $\mathbf{Mon}_p^{\text{lax},L,\otimes}$ of $\mathbf{Mon}_p^{\text{lax},\times}$ is defined as follows. Its objects are the collections of \mathcal{P} -monoidal categories with colimits. An arrow $C_1 \times \dots \times C_n \rightarrow C$ in $\mathbf{Mon}_p^{\text{lax},\times}$ belongs to $\mathbf{Mon}_p^{\text{lax},L,\otimes}$ if it preserves colimits in each argument. The composition $p : \mathbf{Mon}_p^{\text{lax},L,\otimes} \rightarrow \mathbf{Mon}_p^{\text{lax},\times} \rightarrow \mathit{Fin}_*$ is an operad.

7.2.1. PROPOSITION. *p is a cocartesian fibration. This yields a structure of SM category on $\mathbf{Mon}_p^{\text{lax},L}$. The embedding $\mathbf{Mon}_p^L \rightarrow \mathbf{Mon}_p^{\text{lax},L}$ is a symmetric monoidal functor.*

PROOF. Given A_1, \dots, A_n in $\mathbf{Mon}_p^{\text{lax},L}$ let $\otimes A_i$ denote their tensor product in \mathbf{Mon}_p^L . We have to verify that the map $\oplus A_i \rightarrow \otimes A_i$ is a cocartesian lifting of the active arrow $\langle n \rangle \rightarrow \langle 1 \rangle$, that is, that the natural map

$$j : \text{Map}_{\mathbf{Mon}_p^{\text{lax},L}}(\otimes A_i, B) \rightarrow \text{Map}_{\mathbf{Mon}_p^{\text{lax},L,\otimes}}(\oplus A_i, B) \tag{76}$$

is an equivalence for any $B \in \mathbf{Mon}_p^{\text{lax},L}$. The source of j is a subspace of $\text{Map}_{\mathbf{Op}_p}(\otimes A_i, B) = \text{Map}_p(\mathcal{P}, \text{Funop}_p(\otimes A_i, B))$. We will use the following properties of Funop_p , see [H.EY], 2.8.9. Let A, B be \mathcal{P} -monoidal categories. Then $\text{Funop}_p(A, B)$ is a \mathcal{P} -operad whose objects over $p \in \mathcal{P}_1$ are the functors $A_p \rightarrow B_p$; for an active arrow $\oplus p_j \rightarrow q$ in \mathcal{P} let $f_j : A_{p_j} \rightarrow B_{p_j}$ and $g : A_q \rightarrow B_q$. Then $\text{Map}_{\text{Funop}_p(A,B)}(\oplus f_j, g)$ identifies with the space

$$\text{Map}_{\text{Fun}(\prod A_{p_j}, B_q)}(g \circ \alpha_1^A, \alpha_1^B \circ \oplus f_j), \tag{77}$$

where the notation is explained by the diagram

$$\begin{array}{ccc} \prod A_{p_j} & \xrightarrow{\alpha_1^A} & A_q \\ \downarrow \oplus f_j & & \downarrow g \\ \prod B_{p_j} & \xrightarrow{\alpha_1^B} & B_q \end{array}$$

We define $\text{Funop}'_p(\otimes A_i, B) \subset \text{Funop}_p(\otimes A_i, B)$ as the full suboperad spanned by the collections of functors $\otimes (A_i)_p \rightarrow B_p$ preserving the colimits. Then the source of (76) identifies with $\text{Map}_p(\mathcal{P}, \text{Funop}'_p(\otimes A_i, B))$.

Similarly, we define $\text{Funop}'_p(\prod A_i, B) \subset \text{Funop}_p(\prod A_i, B)$ as the full suboperad spanned by the collections of functors $\prod (A_i)_p \rightarrow B_p$ preserving the colimits in each argument. Then the target of (76) identifies with $\text{Map}_p(\mathcal{P}, \text{Funop}'_p(\prod A_i, B))$.

This implies that, in order to prove our assertion, it is sufficient to prove that the map of operads

$$\text{Funop}'_p(\otimes A_i, B) \rightarrow \text{Funop}'_p(\prod A_i, B)$$

¹²Here $\mathbf{Mon}_p^{\text{lax}}$ consists of big \mathcal{P} -monoidal categories and lax \mathcal{P} -monoidal functors.

is an equivalence. It is sufficient to prove that this map defines an equivalence of the spaces of colors (this is obvious) and of the spaces of operations $\text{Map}(\oplus f_j, g)$. The verification is straightforward, based on the description (77). ■

A similar symmetric monoidal structure can be defined on the category $\text{Mon}_{\mathcal{P}}^{\text{colax}, L}$ of \mathcal{P} -monoidal categories with colimits and colax monoidal functors. We define $\text{Mon}_{\mathcal{P}}^{\text{colax}, L, \otimes}$ as the suboperad of $\text{Mon}_{\mathcal{P}}^{\text{colax}, \times}$ whose objects are collections of \mathcal{P} -monoidal categories with colimits and arrows preserving colimits in each argument.

7.2.2. PROPOSITION. $\text{Mon}_{\mathcal{P}}^{\text{colax}, L, \otimes}$ is a symmetric monoidal category so that the embedding $\text{Mon}_{\mathcal{P}}^L \rightarrow \text{Mon}_{\mathcal{P}}^{\text{colax}, L}$ becomes a symmetric monoidal functor.

PROOF. Passing to the opposite monoidal categories, we are back to a suboperad of $\text{Mon}_{\mathcal{P}}^{\text{lax}, \times}$, but this time working with limits instead of colimits. The proof of 7.2.1 now works without change. ■

7.3. THE CATEGORY $\text{Cat}_{/B}^{\text{coc}, L}$. Here we define the category of cocartesian fibrations with the fibers having colimits.

7.3.1. Let B be a category. We denote by $\text{Cat}_{/B}^{\text{coc}}$ the full subcategory of $\text{Cat}_{/B}$ spanned by the cocartesian fibrations $X \rightarrow B$.

The subcategory $\text{Cat}_{/B}^{\text{coc}, L}$ of $\text{Cat}_{/B}^{\text{coc}}$ is spanned by the cocartesian fibrations $p : X \rightarrow B$ classified by a functor $B \rightarrow \text{Cat}^L \subset \text{CAT}$. The arrows in $\text{Cat}_{/B}^{\text{coc}, L}$ are those preserving the colimits on the fibers.

7.3.2. RELATIVE PRESHEAVES. A category X over B is called flat if the functor $\times_B X$ preserves colimits. In this case the right adjoint functor $Y \mapsto \text{Fun}^B(X, Y)$ is defined. Any cocartesian fibration is flat. One has

7.3.3. LEMMA. Let X be a cocartesian fibration over B classified by a functor $\mathcal{X} : B \rightarrow \text{Cat}$. Denote by X° the cocartesian fibration classified by the composition $\text{op} \circ \mathcal{X} : B \rightarrow \text{Cat} \rightarrow \text{Cat}$. Then $\text{Fun}^B(X^\circ, B \times \mathcal{S})$ is a cocartesian fibration classified by the composition

$$B \xrightarrow{\mathcal{X}} \text{Cat} \xrightarrow{P} \text{Cat}^L. \tag{78}$$

Having in mind this description, we will denote by $P_B(X)$ the obtained cocartesian fibration. This is the category of relative presheaves in X .

PROOF. $\text{Fun}^B(X^\circ, B \times \mathcal{S})$ is a cartesian fibration over B classified by the composition

$$\text{Fun}(-, \mathcal{S}) \circ \mathcal{X}^{\text{op}} : B^{\text{op}} \rightarrow \text{Cat}^{\text{op}} \rightarrow \text{Cat}$$

by [GHN], Proposition 7.3. This cartesian fibration is also cocartesian and as such it is classified by (78). ■

7.3.4. LEMMA. *The functor $P_B : X \mapsto P_B(X)$ is left adjoint to the forgetful functor $\mathbf{Cat}_{/B}^{\text{coc},L} \rightarrow \mathbf{Cat}_{/B}^{\text{coc}}$.*

PROOF. Let $X \in \mathbf{Cat}_{/B}^{\text{coc}}$ and $Y \in \mathbf{Cat}_{/B}^{\text{coc},L}$. We have to verify that the composition with the Yoneda embedding induces an equivalence

$$\text{Map}_{\mathbf{Cat}_{/B}^{\text{coc},L}}(P_B(X), Y) \rightarrow \text{Map}_{\mathbf{Cat}_{/B}^{\text{coc}}}(X, Y).$$

A standard reasoning of [L.T], 5.1.5.5 proves that a map $P_B(X) \rightarrow Y$ over B preserves vertical colimits iff it is a relative left Kan extension of its restriction to X . Since, by [L.T], 4.3.2.14, any functor $X \rightarrow Y$ over B has a relative left Kan extension, this implies the result. ■

7.4. SM STRUCTURE ON $\mathbf{Cat}_{/B}^{\text{coc},L}$. The category $\mathbf{Cat}_{/B}^{\text{coc}}$ has a cartesian SM structure. We denote by $\mathbf{Cat}_{/B}^{\text{coc},\times}$ the corresponding category over Fin_* .

Denote by $\mathbf{Cat}_{/B}^{\text{coc},L,\otimes}$ the subcategory of $\mathbf{Cat}_{/B}^{\text{coc},\times}$ whose objects over $\langle n \rangle \in \mathit{Fin}_*$ are collections $(X_i \rightarrow B)_{i=1,\dots,n}$ of objects in $\mathbf{Cat}_{/B}^{\text{coc},L}$ and morphisms are those preserving colimits in each argument. The composition $p : \mathbf{Cat}_{/B}^{\text{coc},L,\otimes} \rightarrow \mathit{Fin}_*$ is obviously an operad.

We have

7.4.1. PROPOSITION.

1. p is a cocartesian fibration. This yields a structure of SM category on $\mathbf{Cat}_{/B}^{\text{coc},L}$.
2. For $X_1, \dots, X_n \in \mathbf{Cat}_{/B}^{\text{coc},L}$ and $b \in B$ one has

$$(X_1 \otimes \dots \otimes X_n)_b = (X_1)_b \otimes \dots \otimes (X_n)_b.$$

PROOF. To prove the first claim, we present for each collection $X_1, \dots, X_n \in \mathbf{Cat}_{/B}^{\text{coc},L}$ a universal arrow

$$\alpha : \bigoplus_{i=1}^n X_i \rightarrow X$$

in $\mathbf{Cat}_{/B}^{\text{coc},L,\otimes}$. If $X_i \rightarrow X$ is classified by a functor $F_i : B_i \rightarrow \mathbf{Cat}^L$, we define $p : X \rightarrow B$ as the cocartesian fibration classified by the functor $F : B \rightarrow \mathbf{Cat}^L$ given by the formula $F(b) = F_1(b) \otimes \dots \otimes F_n(b)$. The arrow $\bigoplus X_i \rightarrow X$ is obviously defined. We will now show that it is cocartesian in $\mathbf{Cat}_{/B}^{\text{coc},L,\otimes}$, that is that for any $Y \in \mathbf{Cat}_{/B}^{\text{coc},L}$ α induces an equivalence

$$\alpha^* : \text{Map}_{\mathbf{Cat}_{/B}^{\text{coc},L}}(X, Y) \rightarrow \text{Map}_{\mathbf{Cat}_{/B}^{\text{coc},L,\otimes}}^{\text{act}}(\bigoplus X_i, Y). \quad (79)$$

Let us first assume that $Y \rightarrow B$ is both cocartesian and cartesian fibration¹³. In this case we will be able to present the source and the target of (79) as subspaces of the space of sections of cocartesian fibrations over B . The target space is embedded into

$$\text{Map}_{\mathbf{Cat}_{/B}^{\text{coc},\otimes}}^{\text{act}}(\bigoplus X_i, Y)$$

¹³We thank the referee for suggesting this idea of the proof of 7.4.1.

that is the space of sections of $\text{Fun}^B(\prod X_i, Y) \rightarrow B$ (here \prod denotes the product over B) which is a cocartesian fibration by [L.T], 3.2.2.13 classified by the functor carrying $b \in B$ to $\text{Fun}(\prod (X_i)_b, Y_b)$. Similarly, the source space is embedded into $\text{Map}_{\text{Cat}/B}^{\text{coc}}(X, Y)$ that is the space of sections of the cocartesian fibration $\text{Fun}^B(X, Y) \rightarrow B$ classified by the functor $b \mapsto \text{Fun}(\otimes (X_i)_b, Y_b)$. One readily sees that the source and the target of (79) both identify with the space of sections of the cocartesian fibration classified by the functor $b \mapsto \text{Fun}^L(\otimes (X_i)_b, Y_b)$. This proves the assertion in the case when $q : Y \rightarrow B$ is a cartesian fibration.

To prove that (79) is an equivalence for general q , we embed Y into $P_B(Y)$ as in 7.3.2. This is a full embedding. This allows one to identify the left and the right hand sides of (79) with the subcategories of the similar expressions for $P_B(Y)$. ■

7.4.2. LEMMA. *The functor $P_B : \text{Cat}/B^{\text{coc}} \rightarrow \text{Cat}/B^{\text{coc}, L}$ is symmetric monoidal.*

PROOF. We define \mathcal{C} as the full subcategory of $\text{Fun}([1], \text{Cat}/B)$ spanned by the arrows $X \rightarrow P$ where $X \in \text{Cat}/B^{\text{coc}}$ and $P \in \text{Cat}/B^{\text{coc}, L}$. Denote by $p_0 : \mathcal{C} \rightarrow \text{Cat}$ and $p_1 : \mathcal{C} \rightarrow \text{Cat}^L$ the obvious projections. \mathcal{C} is closed under finite products and we denote by \mathcal{C}^\times the corresponding category over Fin_* . We denote by $\mathcal{D}^\otimes \subset \mathcal{C}^\times$ the subcategory whose objects are collections of the arrows $f : X \rightarrow P$ inducing an equivalence $P_B(X) \rightarrow P$. Arrows in \mathcal{D}^\otimes are defined by the diagrams

$$\begin{array}{ccc} \prod X_i & \xrightarrow{a} & X \\ \downarrow \prod f_i & & \downarrow f \\ \prod P_i & \xrightarrow{b} & P \end{array}$$

where b preserves colimits in each argument. One has obvious functors $p_0 : \mathcal{D}^\otimes \rightarrow \text{Cat}/B^{\text{coc}, \times}$ and $p_1 : \mathcal{D}^\otimes \rightarrow \text{Cat}/B^{\text{coc}, L, \otimes}$. We will now show that \mathcal{D}^\otimes is a SM category. Given a collection $f_i : X_i \rightarrow P_i$, we define a map $u : \oplus f_i \rightarrow f$ in \mathcal{D}^\otimes lifting the active arrow $\langle n \rangle \rightarrow \langle 1 \rangle$ in Fin_* as follows. We put $X = \prod X_i$ and $a = \text{id}_X$. We put $P = \otimes P_i$ with the canonical map $b : \prod P_i \rightarrow \otimes P_i$. The map u so defined is a cocartesian lifting; this can be verified by induction in n , using universality of P_B proven in 7.3.4. Now the functor $p_0 : \mathcal{D}^\otimes \rightarrow \text{Cat}^\times$ is a SM functor that is an equivalence of categories. Therefore it is an equivalence of SM categories. Finally p_1 is also symmetric monoidal, and this proves the assertion. ■

7.4.3. LEMMA. *Let $X \in \text{Cat}/B^{\text{coc}}$. Then $P_B(X)$ is dualizable in $\text{Cat}/B^{\text{coc}, L}$ with dual $P_B(X^{\text{op}})$.*

PROOF. Yoneda embedding $Y : X \rightarrow P_B(X)$ gives rise to a morphism $X \times_B X^{\text{op}} \rightarrow \mathcal{S}$ in $\text{Cat}/B^{\text{coc}}$. This induces a map $e : P_B(X \times_B X^{\text{op}}) = P_B(X^{\text{op}}) \otimes P_B(X) \rightarrow \mathcal{S}$ in $\text{Cat}/B^{\text{coc}, L}$. The map $c : \mathcal{S} \rightarrow P_B(X^{\text{op}}) \otimes P_B(X) = P_B(X^{\text{op}} \times X)$ is uniquely defined by the final section of $P_B(X^{\text{op}} \times X)$. The maps c and e are a unit and a counit of adjunction. ■

7.4.4. $\mathbf{Cat}_{/B}^{\text{cart},L}$. Everything said about the category of cocartesian fibrations with colimits holds also for the category of cartesian fibrations with colimits. Passing to opposite categories establishes an equivalence of $\mathbf{Cat}_{/B}^{\text{cart}}$ with $\mathbf{Cat}_{/B^{\text{op}}}^{\text{coc}}$. This equivalence carries cartesian fibrations with colimits to cocartesian fibrations with limits. Fortunately, the proof of Proposition 7.4.1 is valid when one replaces the colimits with the limits.

We will now deduce from 7.1.2 the version for the categories with colimits as follows. The equivalence 7.1.2 restricts to the following.

7.4.5. COROLLARY. *The equivalence 7.1.2 induces an equivalence $\text{Fun}(B^{\text{op}}, \mathbf{Mon}_{\mathcal{P}}^{\text{lax},L}) = \mathbf{Alg}_{\mathcal{P}}(\mathbf{Cat}_{/B}^{\text{cart},L})$.*

PROOF. The left-hand side is a subcategory of $\text{Fun}(B^{\text{op}}, \mathbf{Mon}_{\mathcal{P}}^{\text{lax}})$, with the objects consisting of \mathcal{P} -monoidal categories with colimits and arrows preserving these colimits. The embedding $\mathbf{Cat}_{/\mathcal{P}}^{\text{cart},L,\otimes} \rightarrow \mathbf{Cat}_{/\mathcal{P}}^{\text{cart},\times}$ defines $\mathbf{Alg}_{\mathcal{P}}(\mathbf{Cat}_{/B}^{\text{cart},L})$ as a the same subcategory of $\mathbf{Alg}_{\mathcal{P}}(\mathbf{Cat}_{/B}^{\text{cart}})$. ■

Similarly, one has

7.4.6. COROLLARY. *The equivalence 7.1.3 induces an equivalence $\text{Fun}(B, \mathbf{Mon}_{\mathcal{P}}^{\text{colax},L}) = \mathbf{Alg}_{\mathcal{P}}(\mathbf{Cat}_{/B}^{\text{coc},L})$.*

Lemma 7.1.4 has also a version for categories with colimits.

7.4.7. LEMMA. *The equivalence 7.1.4 induces an equivalence*

$$\mathbf{Alg}_{\mathcal{Q}}(\mathbf{Mon}_{\mathcal{P}}^{\text{lax},L}) = \mathbf{Alg}_{\mathcal{P}}(\mathbf{Mon}_{\mathcal{Q}}^{\text{colax},L}). \quad (80)$$

PROOF. Both the left and the right hand side are subcategories of

$$\mathbf{Alg}_{\mathcal{Q}}(\mathbf{Mon}_{\mathcal{P}}^{\text{lax}}) = \mathbf{Alg}_{\mathcal{P}}(\mathbf{Mon}_{\mathcal{Q}}^{\text{colax}}).$$

A \mathcal{Q} -algebra in $\mathbf{Mon}_{\mathcal{P}}^{\text{lax}}$ represented by a functor $f : \mathcal{Q} \rightarrow \mathbf{Mon}_{\mathcal{P}}^{\text{lax}}$ belongs to $\mathbf{Alg}_{\mathcal{Q}}(\mathbf{Mon}_{\mathcal{P}}^{\text{lax},L})$ if the following three properties are satisfied.

1. For any $p \in \mathcal{P}_1$, $q \in \mathcal{Q}_1$ one has $f(q)_p \in \mathbf{Cat}^L$.
2. For $p = p_1 \oplus \cdots \oplus p_n$ with $r, p_i \in \mathcal{P}_1$, an active arrow $\alpha : p \rightarrow r$ in \mathcal{P} , the composition

$$\prod_i f(q)_{p_i} \xrightarrow{\rho^{-1}} f(q)_p \xrightarrow{\alpha_1} f(q)_r$$

preserves colimits in each argument. Here $\rho : f(q)_p \rightarrow \prod f(q)_{p_i}$ is an equivalence since $f(q)$ satisfies the Segal condition in p .

3. For $q = q_1 \oplus \cdots \oplus q_n$, $s, q_i \in \mathcal{Q}_1$, $p \in \mathcal{P}_1$ and an active arrow $\beta : q \rightarrow s$ in \mathcal{Q} , the composition

$$\prod_i f(q_i)_p \xrightarrow{\rho^{-1}} f(q)_p \xrightarrow{\alpha_1} f(s)_p$$

preserves colimits in each argument. Here $\rho : f(q)_p \rightarrow \prod f(q_i)_p$ is an equivalence since f satisfies the Segal condition in q .

The same three conditions define the subcategory $\mathbf{Alg}_{\mathcal{P}}(\mathbf{Mon}_{\mathcal{Q}}^{\text{colax},L})$ of $\mathbf{Alg}_{\mathcal{P}}(\mathbf{Mon}_{\mathcal{Q}}^{\text{colax}})$. ■

7.5. THE FUNCTOR $\mathcal{Q} : \mathbf{Cat} \times \mathbf{Alg}_{\mathbf{LM}}(\mathbf{Cat}^L) \rightarrow \mathbf{Mon}_{\mathbf{Ten}_{\succ}}^{\text{colax}}$. In this subsection we prove the following

7.5.1. PROPOSITION. *The assignment $(X, \mathcal{M}, \mathcal{B}) \mapsto \mathcal{Q}_{X, \mathcal{M}, \mathcal{B}}$ gives rise to a symmetric monoidal functor $\mathcal{Q} : \mathbf{Cat} \times \mathbf{Alg}_{\mathbf{LM}}(\mathbf{Cat}^L) \rightarrow \mathbf{Mon}_{\mathbf{Ten}_{\succ}}^{\text{colax},L}$.*

Note that the SM structure on $\mathbf{Alg}_{\mathbf{LM}}(\mathbf{Cat}^L)$ is induced from that on \mathbf{Cat}^L and the structure on $\mathbf{Mon}_{\mathbf{Ten}_{\succ}}^{\text{colax},L}$ is defined as in 7.2. In the rest of this section we use the notation $\mathcal{T} = \mathbf{Ten}_{\succ}$.

PROOF. Recall 6.2.2 that the \mathcal{T} -monoidal category $\mathcal{Q}_{X, \mathcal{M}, \mathcal{B}}$ is constructed in three steps, the first one assigning to $(X, \mathcal{M}, \mathcal{B})$ the dualizable right \mathcal{M} -module $\text{Fun}(X, \mathcal{M})$, the second assigning to it the corresponding counit diagram (62) and, finally (63), tensoring it with \mathcal{B} . We will follow 6.2.2 and 7.4.6 and present a \mathcal{T} -algebra object in $\mathcal{C} := \mathbf{Cat}_{/B}^{\text{coc},L}$ with $B = \mathbf{Cat} \times \mathbf{Alg}_{\mathbf{LM}}(\mathbf{Cat}^L)$. We denote by $X \in \mathbf{Cat}_{/B}^{\text{coc}}$ the tautological family defined by the projection $B \rightarrow \mathbf{Cat}$, use $P_B(X)$ and $P_B(X^{\text{op}})$ instead of $P(X)$ and $P(X^{\text{op}})$ and define \mathcal{M} and \mathcal{B} to be the cocartesian fibrations classified by the projections of B to the components of $\mathbf{Alg}_{\mathbf{LM}}(\mathbf{Cat}^L)$. Lemma 7.4.3, gives us a \mathcal{T} -algebra object in $\mathbf{Cat}_{/B}^{\text{coc},L}$, that is, by 7.4.6, a functor

$$\mathcal{Q} : \mathbf{Cat} \times \mathbf{Alg}_{\mathbf{LM}}(\mathbf{Cat}^L) \rightarrow \mathbf{Mon}_{\mathcal{T}}^{\text{colax},L} \quad (81)$$

carrying the triple $(X, \mathcal{M}, \mathcal{B})$ to the \mathcal{T} -monoidal category $\mathcal{Q}_{X, \mathcal{M}, \mathcal{B}}$.

We will now show that the functor (81) canonically extends to a symmetric monoidal functor. We will present \mathcal{Q} as a tensor product of two symmetric monoidal functors, $\mathcal{Q}_0 : \mathbf{Cat} \rightarrow \mathbf{Mon}_{\mathcal{T}}^{\text{colax},L}$ and $\Pi : \mathbf{Alg}_{\mathbf{LM}}(\mathbf{Cat}^L) \rightarrow \mathbf{Mon}_{\mathcal{T}}^{\text{colax},L}$. The functor Π is defined by the map of operads $\pi : \mathcal{T} \rightarrow \mathbf{LM}$ carrying the colors a, a', m, b to $A \in [\mathbf{LM}]$ and n, k to $M \in [\mathbf{LM}]$. The functor Π is the composition of $\pi^* : \mathbf{Alg}_{\mathbf{LM}}(\mathbf{Cat}^L) \rightarrow \mathbf{Alg}_{\mathcal{T}}(\mathbf{Cat}^L)$ and the obvious SM embedding $\mathbf{Alg}_{\mathcal{T}}(\mathbf{Cat}^L) \rightarrow \mathbf{Mon}_{\mathcal{T}}^{\text{colax},L}$.

It remains to construct the SM functor \mathcal{Q}_0 . \mathcal{T} -monoidal category $\mathcal{Q}_0(X)$ is defined by the formulas

$$A = A' = \mathcal{S}, \quad M = P(X), \quad N = P(X^{\text{op}}), \quad B = \text{Quiv}_X(\mathcal{S}), \quad K = \mathcal{S}.$$

Our aim is to canonically extend this to a SM functor. We construct a new category \mathcal{C} as follows. Its objects are collections consisting of a category X , a \mathcal{T} -monoidal category

(M, B, N, K) , a functor $i : X \rightarrow M$ and an arrow $k : [0] \rightarrow K$ (that is, an object of K). The category \mathcal{C} has a cartesian SM structure.

We now define a subcategory \mathcal{D}^\otimes of \mathcal{C}^\times as follows. The objects of \mathcal{D}^\otimes are the (collection of) objects of \mathcal{C} satisfying the following properties.

- (M, B, N, K) is a \mathcal{T} -monoidal category with colimits.
- $i : X \rightarrow M$ presents M as $P(X)$ and $k \in K$ induces an equivalence $\kappa : \mathcal{S} \rightarrow K$.
- The right action of B on M determines an equivalence $B^{\text{op}} = \text{End}(M)$.
- The composition $M \times N \rightarrow K \xrightarrow{\kappa^{-1}} \mathcal{S}$ establishes N as right dual to M .

The morphisms are the maps $C_1 \times \dots \times C_n \rightarrow C$ in \mathcal{C}^\times with C, C_i satisfying the properties as above and such that the corresponding maps $\coprod M_i \rightarrow M$, etc., $\coprod K_i \rightarrow K$, preserve colimits in each argument.

One can easily see that the forgetful functor $\mathcal{D}^\otimes \rightarrow \mathbf{Cat}^\times$ is an equivalence. The composition $\mathcal{D}^\otimes \rightarrow \mathcal{C}^\times \rightarrow \mathbf{Mon}_{\mathcal{T}}^{\text{colax}, \times}$ induces a SM functor $\mathcal{D}^\otimes \rightarrow \mathbf{Mon}_{\mathcal{T}}^{\text{colax}, L, \otimes}$.

■

8. Multiplicative structures

In this section we present a monoidal version of the universality Theorem 6.4.4.

Let \mathcal{O} be an operad. Assume that \mathcal{M} is an \mathcal{O} -algebra object in $\mathbf{Alg}_{\text{Ass}}(\mathbf{Cat}^L)$. Equivalently, we assume that $\mathcal{M} \in \mathbf{Alg}_{\mathcal{O} \otimes \text{Ass}}(\mathbf{Cat}^L)$. In this case many objects mentioned above acquire a structure of \mathcal{O} -algebra¹⁴. For instance, the notion of \mathcal{O} -monoidal \mathcal{M} -enriched category makes sense, as well as the notion of \mathcal{O} -monoidal left \mathcal{M} -module. We prove that if \mathcal{A} is an \mathcal{O} -monoidal \mathcal{M} -enriched category, the enriched presheaves $P_{\mathcal{M}}(\mathcal{A})$ form an \mathcal{O} -monoidal left \mathcal{M} -module category and Yoneda embedding is an \mathcal{O} -monoidal \mathcal{M} -functor universal among such functors to \mathcal{O} -monoidal left \mathcal{M} -modules with colimits.

8.1. FROM ASSOCIATIVE ALGEBRAS TO LEFT MODULES.

8.1.1. Recall [L.HA], 2.10, that, given a bilinear map of operads $\mu : \mathcal{P} \times \mathcal{Q} \rightarrow \mathcal{R}$, and an \mathcal{R} -operad \mathcal{X} , the \mathcal{P} -operad $p : \mathbf{Alg}_{\mathcal{Q}/\mathcal{R}}^\mu(\mathcal{X}) \rightarrow \mathcal{P}$ is defined as the object representing the functor

$$K \in \mathbf{Cat}_{/\mathcal{P}} \mapsto \text{Map}_{\mathbf{Cat}_{/\mathcal{R}^\natural}}^+(K^\flat \times \mathcal{Q}^\natural, \mathcal{X}^\natural),$$

see [H.EY], 2.10. In the case \mathcal{X} is \mathcal{R} -monoidal, $\mathbf{Alg}_{\mathcal{Q}/\mathcal{R}}^\mu(\mathcal{X})$ is \mathcal{P} -monoidal. In the case μ represents \mathcal{R} as a tensor product of \mathcal{P} and \mathcal{Q} , one has an equivalence

$$\mathbf{Alg}_{\mathcal{P}}(\mathbf{Alg}_{\mathcal{Q}/\mathcal{R}}^\mu(\mathcal{X})) = \mathbf{Alg}_{\mathcal{R}}(\mathcal{X}).$$

We will suppress the letter μ from the notation if it is clear from the context.

We will need the following general claim about cocartesian fibrations.

¹⁴This sentence becomes slightly imprecise if \mathcal{O} is not monochrome: an \mathcal{O} -algebra consists of more than one object.

8.1.2. LEMMA. *Let*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{f} & \mathcal{Q} \\ & \searrow p & \swarrow q \\ & & B \end{array}$$

be a map of cocartesian fibrations over B . Assume that

- (1) *For any $b \in B$ $f_b : \mathcal{P}_b \rightarrow \mathcal{Q}_b$ is a cocartesian fibration.*
- (2) *For any $\alpha : b \rightarrow b'$ in B the functor*

$$\alpha_! : \mathcal{P}_b \rightarrow \mathcal{P}_{b'}$$

carries f_b -cocartesian arrows to $f_{b'}$ -cocartesian arrows.

Then f is a cocartesian fibration.

PROOF. By [L.T], 2.4.2.11, f is a locally cocartesian fibration, with locally cocartesian arrows of the form $u = u'' \circ u'$ where u' is p -cocartesian and $u'' \in \mathcal{P}_{b'}$ $f_{b'}$ -cocartesian, with $p(u) : b \rightarrow b'$. Condition (2) ensures that the composition of locally cocartesian arrows is locally cocartesian. This implies the claim. ■

Let \mathcal{O} be an operad and let $\mathcal{C} \in \mathbf{Alg}_{\mathcal{O} \otimes \mathbf{LM}}(\mathbf{Cat}^L)$.

8.1.3. PROPOSITION. *The forgetful functor*

$$\mathbf{Alg}_{\mathbf{LM}/\mathcal{O} \otimes \mathbf{LM}}(\mathcal{C}) \rightarrow \mathbf{Alg}_{\mathbf{Ass}/\mathcal{O} \otimes \mathbf{LM}}(\mathcal{C}) \tag{82}$$

is an \mathcal{O} -monoidal cocartesian fibration.

PROOF. This result is very close to [L.HA], 4.5.3. We will apply Lemma 8.1.2 to the forgetful functor $f : \mathbf{Alg}_{\mathbf{LM}/\mathcal{O} \otimes \mathbf{LM}}(\mathcal{C}) \rightarrow \mathbf{Alg}_{\mathbf{Ass}/\mathcal{O} \otimes \mathbf{LM}}(\mathcal{C})$ over $B := \mathcal{O}$.

For $o \in \mathcal{O}$ we denote \mathcal{C}_o the LM-monoidal category obtained from \mathcal{C} by the base change along $\mathbf{LM} \xrightarrow{o} \mathcal{O} \otimes \mathbf{LM}$.

The fiber f_o of f at $o \in \mathcal{O}_1$ is a cartesian fibration as it is a forgetful functor $\mathbf{Alg}_{\mathbf{LM}}(\mathcal{C}_o) \rightarrow \mathbf{Alg}_{\mathbf{Ass}}(\mathcal{C}_o)$. Since these are categories with geometric realizations, f_o is also a cocartesian fibration. The same is true for f_o at any $o \in \mathcal{O}$ as cocartesian fibrations are closed under products. This proves the condition (1) of Lemma 8.1.2. Let us verify the condition (2). Given $\alpha : o \rightarrow o'$ in \mathcal{O} , the functor

$$\alpha_! : \mathbf{Alg}_{\mathbf{LM}}(\mathcal{C}_o) \rightarrow \mathbf{Alg}_{\mathbf{LM}}(\mathcal{C}_{o'})$$

is induced by the colimit preserving LM-monoidal functor $\mathcal{C}_\alpha : \mathcal{C}_o \rightarrow \mathcal{C}_{o'}$ induced by α . An arrow $(A, M) \rightarrow (B, N)$ in $\mathbf{Alg}_{\mathbf{LM}}(\mathcal{C}_o)$ is f_o -cocartesian if it induces an equivalence $B \otimes_A M \rightarrow N$. Thus, $\alpha_!$ preserves this property. ■

The forgetful functor (82) is classified by a lax \mathcal{O} -monoidal functor

$$\mathbf{LMod} : \mathbf{Alg}_{\mathbf{Ass}/\mathcal{O} \otimes \mathbf{LM}}(\mathcal{C}) \rightarrow \mathbf{Cat},$$

see Prop. A.2.1. of [H.R]. In particular, we have the following.

8.1.4. **COROLLARY.** *The functor \mathbf{LMod} defined as above assigns an \mathcal{O} -monoidal category of left modules $\mathbf{LMod}_A(\mathcal{C})$ to an \mathcal{O} -algebra object A in $\mathbf{Alg}_{\mathbf{Ass}/\mathcal{O} \otimes \mathbf{LM}}(\mathcal{C})$. A morphism of \mathcal{O} -algebra objects $A \rightarrow A'$ gives rise to an \mathcal{O} -monoidal functor $\mathbf{LMod}_A(\mathcal{C}) \rightarrow \mathbf{LMod}_{A'}(\mathcal{C})$.*

8.1.5. **REMARK.** It is worthwhile, in order to keep track of what we are doing, to repeat the above construction in down-to-earth terms.

An \mathcal{O} -algebra object in $\mathbf{Alg}_{\mathbf{Ass}/\mathcal{O} \otimes \mathbf{LM}}(\mathcal{C})$ consists of a collection of associative algebra objects A_o in \mathbf{LM} -monoidal categories \mathcal{C}_o , with operations $\otimes_{A_{o_i}} \rightarrow A_o$ corresponding to each operation $\alpha : (o_1, \dots, o_n) \rightarrow o$ in \mathcal{O} . The category $\mathbf{LMod}_A(\mathcal{C})$ has components $\mathbf{LMod}_A(\mathcal{C})_o = \mathbf{LMod}_{A_o}(\mathcal{C}_o)$, and the operation $\alpha : (o_1, \dots, o_n) \rightarrow o$ in \mathcal{O} assigns to a collection $M_i \in \mathbf{LMod}_{A_{o_i}}(\mathcal{C}_{o_i})$ the pushforward of $\otimes_{A_{o_i}}$ -module $\otimes M_i$ along $\otimes_{A_{o_i}} \rightarrow A_o$.

We apply Corollary 8.1.4 a number of times.

8.2. \mathcal{O} -MONOIDAL LEFT-TENSORED CATEGORIES.

8.2.1. For $\mathcal{M} \in \mathbf{Alg}_{\mathcal{O} \otimes \mathbf{Ass}}(\mathbf{Cat}^L)$ the category $\mathbf{LMod}_{\mathcal{M}}$ is \mathcal{O} -monoidal. An \mathcal{O} -monoidal left \mathcal{M} -module \mathcal{B} is defined as an \mathcal{O} -algebra in $\mathbf{LMod}_{\mathcal{M}}$.

An \mathcal{O} -monoidal left \mathcal{M} -module \mathcal{B} defines an $\mathcal{O} \otimes \mathbf{LM}$ -operad $(\mathcal{M}, \mathcal{B})$. It makes sense to talk about \mathcal{O} -monoidal and lax \mathcal{O} -monoidal functors between \mathcal{O} -monoidal left \mathcal{M} -modules.

8.2.2. **DEFINITION.** For $\mathcal{B}, \mathcal{B}' \in \mathbf{Alg}_{\mathcal{O}}(\mathbf{LMod}_{\mathcal{M}})$ we define $\mathbf{Fun}_{\mathbf{LMod}_{\mathcal{M}}}^{\mathcal{O}, \text{lax}}(\mathcal{B}, \mathcal{B}')$ as the full subcategory of the fiber of the forgetful functor

$$\mathbf{Alg}_{(\mathcal{M}, \mathcal{B})/\mathcal{O} \otimes \mathbf{LM}}(\mathcal{M}, \mathcal{B}') \rightarrow \mathbf{Alg}_{\mathcal{M}/\mathcal{O} \otimes \mathbf{Ass}}(\mathcal{M})$$

at $\text{id}_{\mathcal{M}}$, spanned by the maps of $\mathcal{O} \otimes \mathbf{LM}$ -operads $f = (\text{id}_{\mathcal{M}}, f_m) : (\mathcal{M}, \mathcal{B}) \rightarrow (\mathcal{M}, \mathcal{B}')$ satisfying the conditions

- f preserves cocartesian liftings of the arrows in \mathbf{LM} .
- f_m preserves colimits.

8.2.3. **DEFINITION.** A lax \mathcal{O} -monoidal morphism of left \mathcal{M} -modules $f : (\mathcal{M}, \mathcal{B}) \rightarrow (\mathcal{M}, \mathcal{B}')$ is called \mathcal{O} -monoidal if it preserves cocartesian liftings of all arrows in $\mathcal{O} \otimes \mathbf{LM}$.

The full subcategory of $\mathbf{Fun}_{\mathbf{LMod}_{\mathcal{M}}}^{\mathcal{O}, \text{lax}}(\mathcal{B}, \mathcal{B}')$ spanned by \mathcal{O} -monoidal arrows is denoted by $\mathbf{Fun}_{\mathbf{LMod}_{\mathcal{M}}}^{\mathcal{O}}(\mathcal{B}, \mathcal{B}')$.

8.3. \mathcal{O} -MONOIDAL ENRICHED CATEGORIES.

8.3.1. A lax SM functor

$$\mathbf{quiv} : \mathbf{Op}_{\mathbf{Ass}} \rightarrow \mathbf{Fam}^{\mathbf{cart}} \mathbf{Op}_{\mathbf{Ass}},$$

as well as its relatives $\mathbf{quiv}^{\mathbf{LM}} : \mathbf{Op}_{\mathbf{LM}} \rightarrow \mathbf{Fam}^{\mathbf{cart}} \mathbf{Op}_{\mathbf{LM}}$ and $\mathbf{quiv}^{\mathbf{BM}} : \mathbf{Op}_{\mathbf{BM}} \rightarrow \mathbf{Fam}^{\mathbf{cart}} \mathbf{Op}_{\mathbf{BM}}$, have been constructed in [H.EY], 3.5.2. Since the functor $\mathbf{Alg}_{\mathbf{Ass}} : \mathbf{Fam}^{\mathbf{cart}} \mathbf{Op}_{\mathbf{Ass}} \rightarrow \mathbf{Cat}$ preserves the limits, the composition $\mathbf{Alg}_{\mathbf{Ass}} \circ \mathbf{quiv} : \mathbf{Op}_{\mathbf{Ass}} \rightarrow \mathbf{Cat}$ carrying $\mathcal{M} \in \mathbf{Op}_{\mathbf{Ass}}$ to the category $\mathbf{PCat}(\mathcal{M})$ of \mathcal{M} -enriched precategories, is lax symmetric monoidal. The same holds for the compositions $\mathbf{Alg}_{\mathbf{LM}} \circ \mathbf{quiv}^{\mathbf{LM}} : \mathbf{Op}_{\mathbf{LM}} \rightarrow \mathbf{Cat}$ and $\mathbf{Alg}_{\mathbf{BM}} \circ \mathbf{quiv}^{\mathbf{BM}} : \mathbf{Op}_{\mathbf{BM}} \rightarrow \mathbf{Cat}$.

Fix an operad \mathcal{O} . Any \mathcal{O} -algebra \mathcal{M} in $\mathbf{Op}_{\mathbf{Ass}}$ (for instance, $\mathcal{M} \in \mathbf{Alg}_{\mathcal{O} \otimes \mathbf{Ass}}(\mathbf{Cat}^L)$), gives rise to an \mathcal{O} -monoidal category $\mathbf{PCat}(\mathcal{M})$.

Similarly, an \mathcal{O} -algebra \mathcal{M} in $\mathbf{Op}_{\mathbf{LM}}$ (for instance, $\mathcal{M} \in \mathbf{Alg}_{\mathcal{O} \otimes \mathbf{LM}}(\mathbf{Cat}^L)$), gives rise to an \mathcal{O} -monoidal category $\mathbf{PCat}^{\mathbf{LM}}(\mathcal{M})$.

8.3.2. DEFINITION. Let $\mathcal{M} \in \mathbf{Alg}_{\mathcal{O} \otimes \mathbf{Ass}}(\mathbf{Cat}^L)$. An \mathcal{O} -monoidal \mathcal{M} -enriched precategory is an \mathcal{O} -algebra object in $\mathbf{PCat}(\mathcal{M})$.

Note that a precategory $\mathcal{A} \in \mathbf{PCat}(\mathcal{M})$ is an associative algebra in the family of planar operads $\mathbf{Quiv}_X(\mathcal{M})$ parametrized by $X \in \mathbf{Cat}$. Thus, an \mathcal{O} -algebra in $\mathbf{PCat}(\mathcal{M})$ has automatically an \mathcal{O} -monoidal category X of objects.

8.3.3. DAY CONVOLUTION. Let $(\mathcal{M}, \mathcal{B}) \in \mathbf{Alg}_{\mathcal{O} \otimes \mathbf{LM}}(\mathbf{Cat}^L)$. This means that \mathcal{M} is an $\mathcal{O} \otimes \mathbf{Ass}$ -monoidal category with colimits and \mathcal{B} is an \mathcal{O} -monoidal category with colimits, left-tensored over \mathcal{M} .

By 8.3.1, the category $\mathbf{PCat}^{\mathbf{LM}}(\mathcal{M}, \mathcal{B}) = \mathbf{Alg}_{\mathbf{LM}}(\mathbf{Quiv}^{\mathbf{LM}}(\mathcal{M}, \mathcal{B}))$ is \mathcal{O} -monoidal.

By 8.1.3, the forgetful functor

$$\mathbf{PCat}^{\mathbf{LM}}(\mathcal{M}, \mathcal{B}) \rightarrow \mathbf{PCat}(\mathcal{M}) \tag{83}$$

is an \mathcal{O} -monoidal cocartesian fibration. Therefore, it is classified by the lax \mathcal{O} -monoidal functor $\mathbf{PCat}(\mathcal{M}) \rightarrow \mathbf{Cat}$ carrying $\mathcal{A} \in \mathbf{PCat}(\mathcal{M})$ to $\mathbf{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B})$.

In particular, for any \mathcal{O} -monoidal \mathcal{M} -enriched category \mathcal{A} the category $\mathbf{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B})$ is \mathcal{O} -monoidal. This is an enriched form of the Day convolution [L.HA], 2.6.

Let us repeat that if \mathcal{O} is not monochrome, \mathcal{M} is actually a collection of monoidal categories, $\mathbf{PCat}(\mathcal{M})$ is actually a collection of categories, etc.

8.3.4. Let $X \in \mathbf{Cat}$ and let $\mathbb{1}_X$ denote the unit of the monoidal category $\mathbf{Quiv}_X(\mathcal{M})$, see [H.EY], 4.7.3. By definition, $\mathbf{Fun}_{\mathcal{M}}(\mathbb{1}_X, \mathcal{B}) = \mathbf{Fun}(X, \mathcal{B})$. If now $\mathcal{M} \in \mathbf{Alg}_{\mathcal{O} \otimes \mathbf{Ass}}(\mathbf{Cat}^L)$, any \mathcal{O} -monoidal category X gives rise to an \mathcal{O} -monoidal \mathcal{M} -category $\mathbb{1}_X$. The enriched Day convolution defines an \mathcal{O} -monoidal structure on $\mathbf{Fun}_{\mathcal{M}}(\mathbb{1}_X, \mathcal{B})$. The internal Hom in operads [L.HA], 2.6 defines an \mathcal{O} -monoidal structure on $\mathbf{Fun}(X, \mathcal{B})$. These two \mathcal{O} -monoidal structures coincide, according to the following lemma.

8.3.5. LEMMA. *The forgetful functor $\mathbf{Fun}_{\mathcal{M}}(\mathbb{1}_X, \mathcal{B}) \rightarrow \mathbf{Fun}(X, \mathcal{B})$ is an equivalence of \mathcal{O} -monoidal categories.*

PROOF. The \mathcal{O} -monoidal structure on $\mathrm{Fun}_{\mathcal{M}}(\mathbb{1}_X, \mathcal{B})$ is induced from the identification

$$\mathrm{Fun}_{\mathcal{M}}(\mathbb{1}_X, \mathcal{B}) = \mathrm{Alg}_{\mathrm{LM}}(\mathrm{Quiv}_X^{\mathrm{LM}}(\mathcal{M}, \mathcal{B})) \times_{\mathrm{Alg}_{\mathrm{Ass}}(\mathrm{Quiv}_X(\mathcal{M}))} \{\mathbb{1}_X\}$$

and an \mathcal{O} -algebra structure on

$$\mathrm{Quiv}_X^{\mathrm{LM}}(\mathcal{M}, \mathcal{B}) = (\mathrm{Quiv}_X(\mathcal{M}), \mathrm{Fun}(X, \mathcal{B})).$$

This immediately implies that the forgetful functor $\mathrm{Fun}_{\mathcal{M}}(\mathbb{1}_X, \mathcal{B}) \rightarrow \mathrm{Fun}(X, \mathcal{B})$ is \mathcal{O} -monoidal. Since it is an equivalence, it is an \mathcal{O} -monoidal equivalence. ■

8.3.6. DEFINITION. Given $(\mathcal{M}, \mathcal{B}) \in \mathrm{Alg}_{\mathcal{O} \otimes \mathrm{LM}}(\mathrm{Cat}^L)$ and an \mathcal{O} -monoidal \mathcal{M} -enriched category \mathcal{A} , a lax \mathcal{O} -monoidal \mathcal{M} -functor $f : \mathcal{A} \rightarrow \mathcal{B}$ is an \mathcal{O} -algebra in $\mathrm{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B})$.

We denote by $\mathrm{Fun}_{\mathcal{M}}^{\mathcal{O}, \mathrm{lax}}(\mathcal{A}, \mathcal{B}) = \mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}))$ the category of lax \mathcal{O} -monoidal \mathcal{M} -functors from \mathcal{A} to \mathcal{B} .

8.3.7. \mathcal{O} -MONOIDAL \mathcal{M} -FUNCTORS. Let $(\mathcal{M}, \mathcal{B}) \in \mathrm{Alg}_{\mathcal{O} \otimes \mathrm{LM}}(\mathrm{Cat}^L)$ and let \mathcal{A} be an \mathcal{O} -monoidal \mathcal{M} -enriched category.

One has a canonical \mathcal{O} -monoidal functor $1 : \mathbb{1}_X \rightarrow \mathcal{A}$ for any \mathcal{O} -monoidal enriched precategory with the \mathcal{O} -monoidal category of objects X . It induces a forgetful functor $\mathrm{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}) \xrightarrow{1^*} \mathrm{Fun}_{\mathcal{M}}(\mathbb{1}_X, \mathcal{B}) = \mathrm{Fun}(X, \mathcal{B})$. It is automatically lax \mathcal{O} -monoidal since it is right adjoint to an \mathcal{O} -monoidal free module functor defined by $1 : \mathbb{1}_X \rightarrow \mathcal{A}$, see 8.1.4. \mathcal{O} -algebras in $\mathrm{Fun}(X, \mathcal{B})$ are lax \mathcal{O} -monoidal functors from X to \mathcal{B} .

DEFINITION. A lax \mathcal{O} -monoidal \mathcal{M} -functor $f : \mathcal{A} \rightarrow \mathcal{B}$ is called \mathcal{O} -monoidal if the lax \mathcal{O} -monoidal functor $1^*(f) : X \rightarrow \mathcal{B}$ obtained from f by forgetting the \mathcal{A} -module structure, is \mathcal{O} -monoidal. We denote by $\mathrm{Fun}_{\mathcal{M}}^{\mathcal{O}}(\mathcal{A}, \mathcal{B})$ the category of \mathcal{O} -monoidal \mathcal{M} -functors from \mathcal{A} to \mathcal{B} .

8.4. \mathcal{O} -MONOIDAL YONEDA EMBEDDING.

8.4.1. Assume \mathcal{M} is $\mathcal{O} \otimes \mathrm{Ass}$ -monoidal category in Cat^L and let \mathcal{A} be an \mathcal{O} -monoidal \mathcal{M} -enriched precategory, that is, an \mathcal{O} -algebra in $\mathrm{PCat}(\mathcal{M})$.

We will now repeat the construction of Yoneda embedding for \mathcal{A} .

Denote $\pi : \mathrm{BM} \rightarrow \mathrm{Ass}$ the canonical map of (planar) operads. For any planar operad \mathcal{C} (or a family of planar operads) the functor $\pi^* : \mathrm{Alg}_{\mathrm{Ass}}(\mathcal{C}) \rightarrow \mathrm{Alg}_{\mathrm{BM}}(\mathcal{C})$ carries an associative algebra A to the A - A -bimodule A .

The folding functor $\phi : \mathrm{Op}_{\mathrm{BM}} \rightarrow \mathrm{Op}_{\mathrm{LM}}$ defined in [H.EY], 3.6, preserves limits, so the functor $\phi \circ \pi^* : \mathrm{Alg}_{\mathrm{Ass}}(\mathcal{C}) \rightarrow \mathrm{Alg}_{\mathrm{LM}}(\phi \circ \pi^*(\mathcal{C}))$ also preserves limits, and, therefore, carries \mathcal{O} -algebras to \mathcal{O} -algebras.

We apply this to $\mathcal{C} := \mathrm{Quiv}(\mathcal{M})$ and an \mathcal{O} -algebra \mathcal{A} in $\mathrm{PCat}(\mathcal{M})$. We get an \mathcal{O} -algebra in $\mathrm{PCat}^{\mathrm{LM}}(\mathcal{M} \otimes \mathcal{M}^{\mathrm{rev}}, \mathcal{M})$.

According to 8.3.3, this defines a lax \mathcal{O} -monoidal $\mathcal{M} \otimes \mathcal{M}^{\mathrm{rev}}$ -functor $\tilde{Y} : \mathcal{A} \boxtimes \mathcal{A}^{\mathrm{op}} \rightarrow \mathcal{M}$.

. One uses the adjoint associativity equivalence to deduce Yoneda embedding from the functor $\mathcal{A} \boxtimes \mathcal{A}^{\text{op}} \rightarrow \mathcal{M}$.

Recall [H.EY], 6.1.7. Below \mathcal{B} is a left $\mathcal{M} \otimes \mathcal{M}'$ -module, $\mathcal{A} \in \text{PCat}(\mathcal{M})$ and $\mathcal{A}' \in \text{PCat}(\mathcal{M}')$. There is a canonical equivalence

$$\text{Fun}_{\mathcal{M} \otimes \mathcal{M}'}(\mathcal{A} \boxtimes \mathcal{A}', \mathcal{B}) = \text{Fun}_{\mathcal{M}}(\mathcal{A}, \text{Fun}_{\mathcal{M}'}(\mathcal{A}', \mathcal{B})). \quad (84)$$

Let now \mathcal{O} be an operad, \mathcal{M} and \mathcal{M}' be in $\text{Alg}_{\mathcal{O} \otimes \text{Ass}}(\text{Cat}^L)$ and \mathcal{B} be an \mathcal{O} -monoidal category left-tensored over $\mathcal{M} \otimes \mathcal{M}'$. Then the left and the right side of (84) are \mathcal{O} -monoidal categories by (8.3.3)

8.4.2. PROPOSITION. *Under these assumptions (84) is an equivalence of \mathcal{O} -monoidal categories.*

PROOF. Let $\mathcal{T} = (\mathcal{M}, \mathcal{B}, \mathcal{M}'^{\text{rev}})$ be the BM -monoidal category defined by the left $\mathcal{M} \otimes \mathcal{M}'$ -module \mathcal{B} . The equivalence (84) is constructed in [H.EY], 6.1.7, from the equivalence

$$\text{Funop}_{\text{BM}}(\text{BM}_X \times \text{BM}_{X'}^{\text{rev}}, \mathcal{T}) = \text{Quiv}_X^{\text{BM}}(\text{Quiv}_{X'}^{\text{BM}}(\mathcal{T}^{\text{rev}})^{\text{rev}}). \quad (85)$$

Both left and right hand sides, considered as functors of \mathcal{T} , preserve limits. Therefore, in case \mathcal{T} is \mathcal{O} -monoidal, (85) is an equivalence of $\mathcal{O} \otimes \text{BM}$ -monoidal categories. ■

Proposition 8.4.2 and the fact that the functor $\tilde{Y} : \mathcal{A} \boxtimes \mathcal{A}^{\text{op}} \rightarrow \mathcal{M}$ is lax \mathcal{O} -monoidal, immediately imply that the Yoneda embedding $Y : \mathcal{A} \rightarrow P_{\mathcal{M}}(\mathcal{A})$ is lax \mathcal{O} -monoidal.

8.4.3. PROPOSITION. *$Y : \mathcal{A} \rightarrow P_{\mathcal{M}}(\mathcal{A})$ is \mathcal{O} -monoidal.*

PROOF. We have to verify that the lax \mathcal{O} -monoidal functor

$$Y : X \rightarrow P_{\mathcal{M}}(\mathcal{A}) = \text{LMod}_{\mathcal{A}^{\text{op}}}(\text{Fun}(X^{\text{op}}, \mathcal{M}))$$

is \mathcal{O} -monoidal.

The unit $1 : \mathbb{1}_X \rightarrow \mathcal{A}$ gives rise to a decomposition of Y into

$$X \xrightarrow{Y_{\mathbb{1}}} P_{\mathcal{M}}(\mathbb{1}_X) = \text{Fun}(X^{\text{op}}, \mathcal{M}) \xrightarrow{F} P_{\mathcal{M}}(\mathcal{A}),$$

where F is the free \mathcal{A}^{op} -module functor, see [H.EY], 6.2.6. The free module functor F is \mathcal{O} -monoidal by 8.1.4, whereas $Y_{\mathbb{1}}$ is the composition of the usual ∞ -categorical Yoneda embedding $Y_X : X \rightarrow P(X)$ and the functor $P(X) \rightarrow P_{\mathcal{M}}(\mathbb{1}_X) = P(X) \otimes \mathcal{M}$, so it is also \mathcal{O} -monoidal. ■

8.5. UNIVERSALITY OF YONEDA EMBEDDING IN THE MONOIDAL SETTING. Recall that the composition with the Yoneda embedding defines an equivalence (69)

$$Y^* : \text{Fun}_{\mathcal{M}}^L(P_{\mathcal{M}}(\mathcal{A}), \mathcal{B}) \rightarrow \text{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B})$$

for any left \mathcal{M} -tensoring category with colimits \mathcal{B} . In this subsection we present two \mathcal{O} -monoidal versions of this equivalence.

8.5.1. Let \mathcal{A} be an \mathcal{O} -monoidal \mathcal{M} -enriched category.

In this case $P_{\mathcal{M}}(\mathcal{A})$ is also \mathcal{O} -monoidal (as a left \mathcal{M} -module) and the Yoneda embedding is an \mathcal{O} -monoidal \mathcal{M} -functor, see 8.4.3. In Theorem 8.5.3 below we show that the equivalence (69) induces the equivalences

$$Y^{\mathcal{O}, \text{lax}^*} : \text{Fun}_{\text{LMod}_{\mathcal{M}}}^{\mathcal{O}, \text{lax}}(P_{\mathcal{M}}(\mathcal{A}), \mathcal{B}) \rightarrow \text{Fun}_{\mathcal{M}}^{\mathcal{O}, \text{lax}}(\mathcal{A}, \mathcal{B}) \quad (86)$$

and

$$Y^{\mathcal{O}^*} : \text{Fun}_{\text{LMod}_{\mathcal{M}}}^{\mathcal{O}}(P_{\mathcal{M}}(\mathcal{A}), \mathcal{B}) \rightarrow \text{Fun}_{\mathcal{M}}^{\mathcal{O}}(\mathcal{A}, \mathcal{B}). \quad (87)$$

8.5.2. The assignment

$$\mathcal{B} \mapsto \text{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}),$$

as defined by the formula (8), defines a functor $\text{Alg}_{\mathcal{O}}(\text{LMod}_{\mathcal{M}}) \rightarrow \text{Alg}_{\mathcal{O}}(\text{Cat})$ which yields a canonical functor

$$\text{Fun}_{\mathcal{M}}^{\mathcal{O}, \text{lax}}(\mathcal{A}, P_{\mathcal{M}}(\mathcal{A})) \times \text{Fun}_{\text{LMod}_{\mathcal{M}}}^{\mathcal{O}, \text{lax}}(P_{\mathcal{M}}(\mathcal{A}), \mathcal{B}) \rightarrow \text{Fun}_{\mathcal{M}}^{\mathcal{O}, \text{lax}}(\mathcal{A}, \mathcal{B}). \quad (88)$$

Evaluating it at the \mathcal{O} -monoidal Yoneda embedding $Y : \mathcal{A} \rightarrow P_{\mathcal{M}}(\mathcal{A})$, we get a map (86).

The map (87) is its restriction.

8.5.3. THEOREM. *Let \mathcal{O} be an operad, $\mathcal{M} \in \text{Alg}_{\mathcal{O} \otimes \text{Ass}}(\text{Cat}^L)$ be an $\mathcal{O} \otimes \text{Ass}$ -monoidal category with colimits. Let, furthermore, \mathcal{A} be an \mathcal{O} -monoidal \mathcal{M} -enriched category and let $\mathcal{B} \in \text{Alg}_{\mathcal{O}}(\text{LMod}_{\mathcal{M}})$ be an \mathcal{O} -monoidal category with colimits left-tensored over \mathcal{M} . Then the functors (86) and (87) are equivalences.*

PROOF. By Proposition 7.5.1 $Q_{X, \mathcal{M}, \mathcal{B}}$ is an \mathcal{O} -algebra object in $\text{Mon}_{\text{Ten}_{\mathcal{O}}}^{\text{colax}, L}$. By Lemma 7.1.4 it gives rise to a $\text{Ten}_{\mathcal{O}}$ -algebra object $Q_{X, \mathcal{M}, \mathcal{B}}$ in $\text{Mon}_{\mathcal{O}}^{\text{lax}, L}$. This yields, for an \mathcal{O} -monoidal \mathcal{M} -enriched category \mathcal{A} , similarly to (66), a lax \mathcal{O} -monoidal functor

$$\text{colim} : P_{\mathcal{M}}(\mathcal{A}) \times \text{Fun}_{\mathcal{M}}(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{B}.$$

By 8.4.3, this induces a functor

$$\text{colim} : \text{Fun}_{\mathcal{M}}^{\mathcal{O}, \text{lax}}(\mathcal{A}, \mathcal{B}) \rightarrow \text{Fun}_{\text{LMod}_{\mathcal{M}}}^{\mathcal{O}, \text{lax}}(P_{\mathcal{M}}(\mathcal{A}), \mathcal{B}). \quad (89)$$

Let $[\mathcal{O}] = \mathcal{O}_1^{eq}$ be the space of colors of the operad \mathcal{O} . We have a forgetful functor $G : \text{Alg}_{\mathcal{O}}(\text{Cat}^{\mathcal{K}}) \rightarrow (\text{Cat}^{\mathcal{K}})^{[\mathcal{O}]}$ that commutes with sifted colimits, see [LHA], 3.2.3.1.

By Proposition 4.4.1(3), the forgetful functor G commutes with the weighted colimits.

Choose $o \in [\mathcal{O}]$ and look at the following diagram

$$\begin{array}{ccc}
 \mathrm{Fun}_{\mathrm{LMod}_{\mathcal{M}}}^{\mathcal{O}, \mathrm{lax}}(P_{\mathcal{M}}(\mathcal{A}), \mathcal{B}) & \begin{array}{c} \xrightarrow{Y^{\mathcal{O}*}} \\ \xleftarrow{\mathrm{colim}} \end{array} & \mathrm{Fun}_{\mathcal{M}}^{\mathcal{O}, \mathrm{lax}}(\mathcal{A}, \mathcal{B}) \\
 \downarrow G_a & & \downarrow G_a \\
 \mathrm{Fun}_{\mathrm{LMod}_{\mathcal{M}}}(P_{\mathcal{M}_o}(\mathcal{A}_o), \mathcal{B}_o) & \begin{array}{c} \xrightarrow{Y_o^*} \\ \xleftarrow{\mathrm{colim}} \end{array} & \mathrm{Fun}_{\mathcal{M}_o}(\mathcal{A}_o, \mathcal{B}_o),
 \end{array} \tag{90}$$

with Y_o^* induced by the Yoneda embedding for \mathcal{A}_o and G_a denoting the a -component of the functor forgetting the \mathcal{O} -algebra structure. Both squares of the diagram are homotopy commutative.

Since for all $o \in [\mathcal{O}]$ the arrows Y_o^* and colim are homotopy inverse, and since the forgetful functor G is conservative, $Y^{\mathcal{O}*}$ and colim are also homotopy inverse.

This proves that (86) is an equivalence. Let us prove that the map (87) is also an equivalence. This amounts to verifying that a lax \mathcal{O} -monoidal functor

$$\Phi : P_{\mathcal{M}}(\mathcal{A}) \rightarrow \mathcal{B}$$

is \mathcal{O} -monoidal whenever its composition ϕ with the embedding $Y : X \subset P_{\mathcal{M}}(\mathcal{A})$ is \mathcal{O} -monoidal. The embedding Y is a composition

$$X \xrightarrow{h} P(X) \xrightarrow{i} \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{M}) \xrightarrow{F} P_{\mathcal{M}}(\mathcal{A})$$

where h is the (non-enriched) Yoneda embedding, i is induced by the unit $\mathcal{S} \rightarrow \mathcal{M}$ and F is the free \mathcal{A} -module functor, see [H.EY], 6.2.6. If $\phi : X \rightarrow \mathcal{B}$ is \mathcal{O} -monoidal, the composition $\Phi \circ F \circ i : P(X) \rightarrow \mathcal{B}$ is \mathcal{O} -monoidal by Lemma 8.5.4 below. Then the composition $\Phi \circ F : \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{M}) \rightarrow \mathcal{B}$ is \mathcal{O} -monoidal as $\Phi \circ F$ can be reconstructed from $\Phi \circ F \circ i$ as the composition

$$\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{M}) = \mathcal{M} \otimes P(X) \xrightarrow{\mathrm{id}_{\mathcal{M}} \otimes (\Phi \circ F \circ i)} \mathcal{M} \otimes \mathcal{B} \rightarrow \mathcal{B}.$$

Finally, since any $\Phi \in P_{\mathcal{M}}(\mathcal{A})$ is a colimit of free \mathcal{A} -modules, Lemma 8.5.4 implies that Φ is \mathcal{O} -monoidal. ■

8.5.4. LEMMA. *Let $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ be an arrow in $\mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}, L}$, $i : \mathcal{C}_0 \rightarrow \mathcal{C}$ in $\mathrm{Mon}_{\mathcal{O}}$ so that $\Phi \circ i : \mathcal{C}_0 \rightarrow \mathcal{D}$ is also in $\mathrm{Mon}_{\mathcal{O}}$. Assume that \mathcal{C}_0 generates \mathcal{C} by colimits. Then Φ is in $\mathrm{Mon}_{\mathcal{O}}^L$.*

PROOF. Let $\alpha : x \rightarrow y$ be an arrow in \mathcal{O} . We have to verify that Φ preserves cocartesian liftings of α . Let $c \in \mathcal{C}$ be over x . We have $c = \mathrm{colim}\{i : I \rightarrow (\mathcal{C}_0)_x \rightarrow \mathcal{C}_x\}$. Since $\alpha_! : \mathcal{C}_x \rightarrow \mathcal{C}_y$ preserves colimits, $\alpha_!(c) = \mathrm{colim}\{\alpha_! \circ i\}$. Since $\Phi \circ i$ is in $\mathrm{Mon}_{\mathcal{O}}$, the arrows $\Phi(c_i) \rightarrow \Phi(\alpha_!(c_i))$ are cocartesian liftings of α . Since the functor $\alpha_! : \mathcal{D}_x \rightarrow \mathcal{D}_y$ preserves colimits, the arrow $\mathrm{colim}(\Phi(c_i)) \rightarrow \mathrm{colim}(\Phi(\alpha_!(c_i)))$ is a cocartesian lifting of α . ■

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Department of Mathematics, University of Haifa, Mount Carmel, Haifa 3498838, Israel
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