CATEGORICAL-ALGEBRAIC PROPERTIES OF LATTICE-ORDERED GROUPS

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ABSTRACT. We study the categorical-algebraic properties of the semi-abelian variety $\ell\mathbb{G}rp$ of lattice-ordered groups. In particular, we show that this category is fiber-wise algebraically cartesian closed, arithmetical, and strongly protomodular. Moreover, we observe that $\ell\mathbb{G}rp$ is not action accessible, despite the good behaviour of centralizers of internal equivalence relations. Finally, we restrict our attention to the subvariety $\ell\mathbb{A}b$ of lattice-ordered abelian groups, showing that it is algebraically coherent; this provides an example of an algebraically coherent category which is not action accessible.

1. Introduction

A *lattice-ordered group* is a set endowed with both a group structure and a lattice structure such that the underlying order relation is invariant under translations. In other words, a lattice-ordered group can be defined as an algebraic structure of signature $\{\cdot, e, {}^{-1}, \vee, \wedge\}$ satisfying the axioms of groups, the axioms of lattices, and the axioms related to the distributivity of the group product over both the lattice operations. Therefore, the category of lattice-ordered groups (denoted by $\ell\mathbb{G}rp$) can be presented as the variety of models associated with the equational theory just described.

Recently, lattice-ordered groups have emerged in many areas of mathematics. For instance, in the study of many-valued logic (as shown in [Mundici, 1986], the category of lattice-ordered abelian groups with a distinguished order-unit is equivalent to the one of MV-algebras, which provides algebraic semantics for Łukasiewicz many-valued propositional logic [Cignoli et al., 2013]), in the theory of Bézout domains, in complex intuitionistic fuzzy soft set theory, and in varietal questions in universal algebra.

Although the notion of lattice-ordered groups is as natural as that of rings or partially ordered groups (it suffices to say that examples of lattice-ordered groups include the set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$, and the set of real numbers $\mathbb{R}$ with the usual group sum and the usual order structure), there are currently no studies about this variety from a categorical point of view. The purpose of this work is precisely to explore these

aspects.

A first observation is that the category of lattice-ordered groups is semi-abelian. In a similar way to how abelian categories describe the properties of the categories of abelian groups and of modules over a ring, the notion of semi-abelian category is aimed at capturing the main algebraic properties of the category of groups. Briefly, a *semi-abelian category* [Janelidze et al., 2002] is a pointed finitely cocomplete category which is Barr-exact [Barr, 1971] and protomodular [Bourn, 1991] (protomodularity, in this context, is equivalent to the Split Short Five Lemma). Examples of semi-abelian categories include, for instance, groups, rings without unit, loops, Lie algebras, Heyting semilattices, etc. However, the notion of semi-abelian category is not as efficient in capturing the properties of groups as the one of abelian category is with respect to abelian groups and modules. Therefore, additional categorical-algebraic conditions have been introduced over the years to get closer to a characterization of the structural properties of the category of groups; among these, one can mention representability of actions [Borceux et al., 2005], algebraic coherence [Cigoli et al., 2015a], and strong protomodularity [Bourn, 2000]. This paper is aimed at studying which of these properties hold in the category of lattice-ordered groups.

In Section 2 we recall some classical facts about lattice-ordered groups and we focus on the notion of semi-direct product in $\ell\mathbb{G}rp$.

In Section 3 we study the nature of commutators in $\ell\mathbb{G}rp$ and we show that every subobject admits a centralizer, which coincides with the classical notion of polar; moreover, we prove that $\ell\mathbb{G}rp$ is algebraically cartesian closed.

In Section 4 we give an alternative proof of the known fact that $\ell\mathbb{G}rp$ is arithmetical using the observation that the only internal group object is the trivial one.

In Section 5 we show that $\ell\mathbb{G}rp$ is strongly protomodular; this property implies that, among other things, the commutativity of internal equivalence relations in the Smith-Pedicchio sense [Pedicchio, 1995] is equivalent to the commutativity in the Huq sense [Huq, 1968] of their associated ideals. Moreover, we observe that in $\ell\mathbb{G}rp$ every internal equivalence relation admits a centralizer and we provide a description of it.

Section 6 is devoted to the study of action accessibility, a property related to the existence of centralizers of internal equivalence relations; here we observe that, despite $\ell\mathbb{G}rp$ is not action accessible, there is a construction of centralizers which is very close to the one developed in [Bourn and Janelidze, 2009] for the action accessible category of rings without unit.

Section 7 is aimed at proving that the category of lattice-ordered groups is fiber-wise algebraically cartesian closed (i.e. each category of points in $\ell\mathbb{G}rp$ is algebraically cartesian closed); in detail, we show that in the categories of points in $\ell\mathbb{G}rp$ every subobject admits a centralizer, and we provide a description of it.

In Section 8 we study the properties of the Higgins commutator in $\ell\mathbb{G}rp$; in particular, we prove that $\ell\mathbb{G}rp$ satisfies the condition of normality of the Higgins commutators showing that the Huq commutator of a pair of ideals (i.e. kernels of some arrows) is nothing more than the intersection of the two subobjects.

Finally, in Section 9, we focus our attention on the study of the categorical-algebraic

properties of the variety of lattice-ordered abelian groups (denoted by $\ell\mathbb{A}b$). To be precise, we show that $\ell\mathbb{A}b$ is algebraically coherent; this condition implies several algebraic properties (such as, for example, strong protomodularity, normality of the Higgins commutator, and the so-called "Smith is Huq" condition). Furthermore, we observe that the category of lattice-ordered abelian groups provides an example of an algebraically coherent category that is not action accessible (thus partially solving the Open Problem 6.28 presented in [Cigoli et al., 2015a]).

2. Preliminaries

In this section, we recall the notion of *lattice-ordered group*. Roughly speaking, a lattice-ordered group is a set endowed with a group structure and a lattice structure such that the group operation is distributive with respect to the lattice operations.

2.1. DEFINITION. *A* lattice-ordered group *is an algebra* $(X, \cdot, e, (-)^{-1}, \vee)$ *where:*

LG1) $(X, \cdot, e, (-)^{-1})$ *is a group;*

LG2) $(X, \vee)$ *is a semilattice (i.e.* $\vee$ *is a binary, associative, commutative and idempotent operation on X);*

LG3) for every $x, y, z \in X$ *the following equalities hold*

$$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z) \ and$$
$$(x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z).$$

A morphism between two lattice-ordered groups $(X, \cdot, e, (-)^{-1}, \vee)$ *and* $(Y, \cdot, e, (-)^{-1}, \vee)$ *is a map* $f \colon X \to Y$ *such that* f *is both a group homomorphism between* $(X, \cdot, e, (-)^{-1})$ *and* $(Y, \cdot, e, (-)^{-1})$ *and a semilattice homomorphism between* $(X, \vee)$ *and* $(Y, \vee)$.
The category $\ell\mathbb{G}rp$ *is the category whose objects are the lattice-ordered groups and whose arrows are the morphisms between them.*

Many fundamental algebraic structures naturally admit the structure of lattice-ordered groups. In particular, the set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$, and the set of real numbers $\mathbb{R}$ with the usual group sum and the usual order structure are lattice-ordered groups. Moreover, given a totally ordered set Γ we can provide a lattice-ordered group structure on the set of order automorphisms $\mathrm{Aut}(\Gamma)$: for every $f, g \in \mathrm{Aut}(\Gamma)$ the group product is defined as the composition $f \circ g$, and $(f \vee g)(x) \coloneqq \max(f(x), g(x))$ for all $x \in \Gamma$. For these and further examples see e.g. [Kopytov and Medvedev, 2013] and [Anderson and Feil, 2012].

Given a lattice-ordered group X, we typically denote an algebra by its underlying set. In our notation, we will use xy instead of $x \cdot y$ as usual. Furthermore, we will assume that

the product operation precedes the lattice operation. Therefore, we will write $xy \vee z$ to mean $(xy) \vee z$.

In the literature, lattice-ordered groups are usually presented as algebras on the set of operations $\{\cdot, e, (-)^{-1}, \vee, \wedge\}$ satisfying the group axioms, the lattice axioms and the axioms related to the left and right distributivity of the group operation over both lattice operations. However, in this paper, we have preferred a presentation that does not directly involve the meet operation in order to reduce the number of operations to deal with. In fact, starting from Definition 2.1 it is always possible to define, in a unique way, the meet operation. To show this, we are about to cite and prove several well-known results, that can be found, for example, in [Birkhoff, 1942], [Kopytov and Medvedev, 2013], and [Anderson and Feil, 2012]. Given a lattice-ordered group $(X, \cdot, e, (-)^{-1}, \vee)$, since $(-)^{-1}$ is an isomorphism of groups between $(X, \cdot, e, (-)^{-1})$ and $(X, \cdot^{\mathrm{op}}, e, (-)^{-1})$, it immediately follows that defining

$$x \wedge y := (x^{-1} \vee y^{-1})^{-1}$$

makes $(X, \cdot^{\mathrm{op}}, e, (-)^{-1}, \wedge)$ a lattice-ordered group, which implies that also $(X, \cdot, e, (-)^{-1}, \wedge)$ is a lattice-ordered group. Moreover, $(X, \vee, \wedge)$ is a distributive lattice. To prove this, we start by observing that the identity

$$a(x \wedge y)^{-1}b = (ax^{-1}b) \vee (ay^{-1}b) \tag{2.1}$$

holds. This is immediate since $a(x \wedge y)^{-1}b = a(x^{-1} \vee y^{-1})b = (ax^{-1}b) \vee (ay^{-1}b)$. The identity (2.1) implies (setting $a = x$ and $b = y$) that

$$x(x \wedge y)^{-1}y = x \vee y, \tag{2.2}$$

$$x = (x \vee y)y^{-1}(x \wedge y). \tag{2.3}$$

To prove that $(X, \vee, \wedge)$ is a lattice, we observe that $a \vee b = b$ if and only if $a \wedge b = a$. In fact, note that if $a \vee b = b$, then $a = (a \vee b)b^{-1}(a \wedge b) = b(b^{-1}(a \wedge b)) = a \wedge b$, and if $a \wedge b = a$, then $b = (b \vee a)a^{-1}(b \wedge a) = (b \vee a)a^{-1}a = b \vee a$. Finally, to show the distributivity, we recall the well-known fact that a lattice is distributive if and only if the implication

$$[x \wedge b = y \wedge b \text{ and } x \vee b = y \vee b] \implies x = y$$

holds. This is true because if $x \wedge b = y \wedge b$ and $x \vee b = y \vee b$, then, by the identity (2.3), we get $x = (x \vee b)b^{-1}(x \wedge b) = (y \vee b)b^{-1}(y \wedge b) = y$.

Moreover, when $\cdot$ is commutative, identity (2.2) implies $xy = (x \wedge y)(x \vee y)$. This fact will be crucial in the last part of this paper.

2.2. DEFINITION. [Kopytov and Medvedev, 2013, Anderson and Feil, 2012] *Let X be an object of $\ell\mathbb{G}rp$. For every $x \in X$ we define*

$$x^{+} := x \vee e, \ x^{-} := x \wedge e \ \text{and} \ |x| := x \vee x^{-1};$$

x^{+} *is called the* positive part *of x, x^{-} the* negative part *of x, and $|x|$ the* absolute value

of x.

The previous definition is useful in order to see that, in a lattice-ordered group, every element can be written as the product of a positive element and a negative one. Specifically, given an object X of $\ell\mathbb{G}rp$, we define the positive cone of X as

$$P := \{x \in X \mid x \geq e\}.$$

It is a well-known fact that X is generated by its positive cone (i.e. $X = PP^{-1}$). We offer a proof of this fact by essentially following the approach outlined in [Birkhoff, 1942], [Kopytov and Medvedev, 2013], and [Anderson and Feil, 2012]. To do this, we show that

$$x = x^+ x^-, \tag{2.4}$$

for every $x \in X$. Clearly, (2.4) follows by (2.3) setting $y = e$. Another well-known and interesting fact regarding the behavior of elements in a lattice-ordered group is that

$$|x| = x^+(x^-)^{-1}. \tag{2.5}$$

We prove this identity, too (once again, the proof follows what is presented in [Birkhoff, 1942], [Kopytov and Medvedev, 2013], and [Anderson and Feil, 2012]). Note that $x \vee x^{-1} \geq x \wedge x^{-1}$, hence $(x \vee x^{-1})(x \wedge x^{-1})^{-1} \geq e$, and so $(x \vee x^{-1})^2 \geq e$. Now, since $(a \wedge e)^n = (a^n \wedge e) \wedge (a \wedge e)^{n-1}$ (to see this, it suffices to expand and observe that $(a \wedge e)^n = a^n \wedge a^{n-1} \wedge \cdots \wedge e)$, it follows that $a^n \geq e$ implies $(a \wedge e)^n = (a \wedge e)^{n-1}$, and so $a \geq e$. Therefore, $x \vee x^{-1} \geq e$. To conclude, we compute $x^+(x^-)^{-1} = (x \vee e)(x^{-1} \vee e) = e \vee x \vee x^{-1} \vee e = x \vee x^{-1}$.

Furthermore, the notion of positive part is extremely useful in order to characterize group homomorphisms between lattice-ordered groups which are, in addition, morphisms of $\ell\mathbb{G}rp$. In fact, the following holds:

2.3. LEMMA. *Let X, Y be two objects of $\ell\mathbb{G}rp$. A map $f\colon X \to Y$ is a morphism of $\ell\mathbb{G}rp$ if and only if f preserves the group product and*

$$f(x \vee e) = f(x) \vee e, \text{ for all } x \in X.$$

PROOF. *One implication is trivial. So, let us suppose that f preserves the group product (i.e. is a group homomorphism) and $f(x \vee e) = f(x) \vee e$ for every $x \in X$. We have to prove that*

$$f(x \vee y) = f(x) \vee f(y), \text{ for all } x, y \in X.$$

We have $x \vee y = (xy^{-1} \vee e)y$, hence $f(x \vee y) = f(xy^{-1} \vee e)f(y)$ and, by assumption,

$$f(xy^{-1} \vee e)f(y) = (f(xy^{-1}) \vee e)f(y) = (f(x)f(y)^{-1} \vee e)f(y) = f(x) \vee f(y). \qquad \blacksquare$$

This last result follows essentially from Theorem 9 of [Birkhoff, 1942].

Now, we want to provide a description of the ideals in the variety $\ell\mathbb{G}rp$. In a category where it makes sense to speak of a kernel of a morphism (for example a pointed finitely complete category), a subobject of X is called an *ideal* (or *normal subobject*) if it is the kernel of some morphism. A detailed study of the notion of an ideal in the variety $\ell\mathbb{G}rp$ can be found in [Kopytov and Medvedev, 2013] and [Anderson and Feil, 2012].
First of all, we have to recall the definition of a *convex* subset. Given an object X of $\ell\mathbb{G}rp$ and a subset $S \subseteq X$, S is said to be convex if for every $a, b \in S$ and every $x \in X$, if $a \leq x \leq b$ then $x \in S$.
A subobject $A \leq X$ is an ideal if and only if it is normal (in the classical sense) as a subgroup and it is a convex subset.

The aim of the following proposition is to describe the notion of convexity only with terms. This characterization will be crucial for the purpose of working with semi-direct products.

2.4. PROPOSITION. *Let X be an object of $\ell\mathbb{G}rp$ and $A \leq X$ a subalgebra. A is convex if and only if for every $a_1, a_2 \in A$ and $x, y \in X$ one has*

$$(a_1 x \vee a_2 y)(x \vee y)^{-1} \in A.$$

PROOF. Let us suppose that A is convex. We consider the following inequalities:

$$((a_1 \wedge a_2)x) \vee ((a_1 \wedge a_2)y) \leq a_1 x \vee a_2 y \leq ((a_1 \vee a_2)x) \vee ((a_1 \vee a_2)y),$$

hence

$$(a_1 \wedge a_2)(x \vee y) \leq a_1 x \vee a_2 y \leq (a_1 \vee a_2)(x \vee y),$$

and so

$$(a_1 \wedge a_2) \leq (a_1 x \vee a_2 y)(x \vee y)^{-1} \leq (a_1 \vee a_2).$$

Thus, since A is convex, we deduce $(a_1 x \vee a_2 y)(x \vee y)^{-1} \in A$.
Conversely, let us suppose that for every $a_1, a_2 \in A$ and $x, y \in X$ one has $(a_1 x \vee a_2 y)(x \vee y)^{-1} \in A$. Let us take an element $x \in X$ and two elements $a_1, a_2 \in A$ such that $a_1 \leq x \leq a_2$. We want to prove that x belongs to A, We observe that $a_1 \vee x = x$ and $a_2 \wedge x = x$. Therefore,

$$\begin{aligned} x &= (a_1 \vee x)x^{-1}(x \wedge a_2) = (a_1 \vee x)(e \wedge x^{-1}a_2) = (a_1 \vee x)(e \vee a_2^{-1}x)^{-1} \\ &= (a_1 e \vee a_2 a_2^{-1}x)(e \vee a_2^{-1}x)^{-1} \in A; \end{aligned}$$

where the last term belongs to A by assumption. ∎

This means that a subobject $A \leq X$ in $\ell\mathbb{G}rp$ is normal if and only if for every $a_1, a_2, a \in A$ and $x, y, z \in X$ one has $(a_1 x \vee a_2 y)(x \vee y)^{-1} \in A$ and $z^{-1}az \in A$.

In the following part of this section, we will deal with the notion of a semi-abelian

category. The concept of a semi-abelian category aims to capture some of the common algebraic properties of the category of groups; among the examples of semi-abelian category we can find those of groups, rings without unit, Lie algebras, and Heyting semilattices.

2.5. DEFINITION. [Janelidze et al., 2002] *A pointed category (i.e. a category with a zero object) $\mathbb{C}$ is* semi-abelian *if:*

- *it is Barr-exact [Barr, 1971] (which means that $\mathbb{C}$ is a regular category in which every internal equivalence relation is a kernel pair);*

- *it has finite coproducts;*

- *it is protomodular [Bourn, 1991] (in this context, this is equivalent to the Split Short Five Lemma holding in $\mathbb{C}$).*

In [Bourn and Janelidze, 2003], Theorem 1.1 the authors provided, in the case of a variety $\mathbb{V}$ of universal algebras, a characterization for protomodularity depending on terms. In fact, the authors proved that a variety $\mathbb{V}$ is protomodular if and only if it has 0-ary terms $e_1, \ldots, e_n$, binary terms $t_1, \ldots, t_n$ and an $(n+1)$-ary term t satisfying the identities

$$t(x, t_1(x, y), \ldots, t_n(x, y)) = y \text{ and } t_i(x, x) = e_i$$

for all $i = 1, \ldots, n$.

Since every variety of Ω-groups is semi-abelian (see Example 2.6 of [Janelidze et al., 2002]) we obtain:

2.6. PROPOSITION. *$\ell\mathbb{G}rp$ is a semi-abelian category.*

As shown in [Bourn and Janelidze, 1998], in every semi-abelian category there exist semi-direct products in a categorical sense. In the category of groups, the categorical semi-direct product coincides with the classical one. Now we can describe semi-direct products in the category $\ell\mathbb{G}rp$. In order to do this, we will apply, in the next proposition, the results provided in [Clementino et al., 2015].

2.7. PROPOSITION. *Let $p\colon A \to B$ be a split epimorphism in $\ell\mathbb{G}rp$ with fixed section $s\colon B \to A$, and $k\colon K \to A$ a kernel of p. Without loss of generality let us suppose that K, B are subalgebras of A and k, s are the inclusions of subalgebras. Then A is isomorphic (as a lattice-ordered group) to the set $K \times B$ endowed with the operations*

- $(k_1, b_1)(k_2, b_2) = (k_1 b_1 k_2 b_1^{-1}, b_1 b_2),$

- $(k_1, b_1) \vee (k_2, b_2) = ((k_1 b_1 \vee k_2 b_2)(b_1 \vee b_2)^{-1}, b_1 \vee b_2),$

(which takes the name of semi-direct product *and will be denoted by $K \rtimes B$) via the morphism*

$$\varphi\colon K \rtimes B \to A$$
$$(k, b) \mapsto kb.$$

Moreover, considering the following diagram in $\ell\mathbb{G}rp$:

$$
\begin{array}{ccccc}
K & \xrightarrow{i_K} & K \rtimes B & \underset{i_B}{\overset{p_B}{\rightleftarrows}} & B \\
\| & & \varphi\downarrow & & \| \\
K & \xrightarrow{k} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B
\end{array}
$$

where $i_K(k) = (k,e)$, $i_B(b) = (e,b)$ and $p_B(k,b) = b$, we have $\varphi i_K = k$, $p\varphi = p_B$, and $\varphi i_B = s$.

3. Centralizers and Algebraic Cartesian Closedness

In this section we study, from a categorical point of view, the commutativity of subobjects in the variety $\ell\mathbb{G}rp$.

In order to introduce the topic, we mention some known results related to the category of groups. Given a group G and two subgroups $A, B \leq G$, the condition that, for every $a \in A$ and $b \in B$, $ab = ba$ can be reformulated in the following equivalent way: there exists a group homomorphism $\varphi \colon A \times B \to G$ making the following diagram commutative:

$$
\begin{array}{ccc}
A & & B \\
& \searrow^{(id_A,0)} \quad \swarrow^{(0,id_B)} & \\
& A \times B & \\
& \downarrow{\varphi} & \\
& G. &
\end{array}
$$

Moreover, it is easy to show that φ must be the group product and, therefore, it is necessarily unique. Hence, with the aim of generalising the notion of commutativity, we must place ourselves in a context in which a morphism φ of this type is unique. This reasoning justifies the following definition:

3.1. DEFINITION. [Bourn, 1996] *A pointed category $\mathbb{C}$ with finite products is* unital *if, for X and Y objects of $\mathbb{C}$, the pair of morphisms $(id_X, 0) \colon X \to X \times Y$, $(0, id_Y) \colon Y \to X \times Y$ is jointly extremally epimorphic.*

To be more explicit, a pair of arrows $f \colon A \to B$ and $g \colon C \to B$ of a category $\mathbb{C}$ is said to be jointly extremally epimorphic when, for every commutative diagram

$$
\begin{array}{ccc}
 & M & \\
\nearrow^{f'} & \downarrow{m} & \nwarrow^{g'} \\
A \xrightarrow{f} & B \xleftarrow{g} & C,
\end{array}
$$

if m is a monomorphism, then m is an isomorphism.

It has been shown in [Borceux and Bourn, 2004] that every semi-abelian category is unital.

We are ready to mention the generalized notion of commutativity between subobjects.

3.2. DEFINITION. [Huq, 1968] *Let $\mathbb{C}$ be a unital category. Two subobjects $a \colon A \rightarrowtail X$ and $b \colon B \rightarrowtail X$ of X are said to* cooperate *(or* commute in the sense of Huq, *and we write $[a, b] = 0$) if there exists a (necessarily unique) morphism $\varphi \colon A \times B \to X$ (called* cooperator*) such that the following diagram commutes:*

$$
\begin{array}{ccc}
A & & B \\
& A \times B & \\
a & \exists\varphi & b \\
& X. &
\end{array}
$$
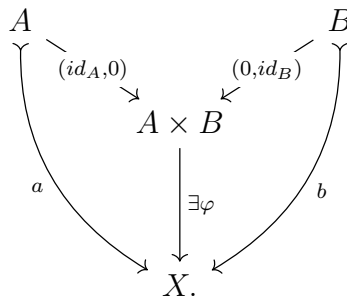
Given a subobject $a \colon A \rightarrowtail X$, the centralizer *of a in X, if it exists, is the greatest subobject of X that cooperates with a.*

Now, let us recall the definition of *orthogonal* subobjects of a lattice-ordered group. This concept will be essential in order to study the condition of cooperation.

3.3. DEFINITION. [Birkhoff, 1942] *Let X be an object of $\ell\mathbb{G}rp$. Two elements $a, b \in X$ are called* orthogonal *if*

$$|a| \wedge |b| = e.$$

Two subsets $A, B \subseteq X$ are called orthogonal *(and one writes $A \perp B$) if, for every $a \in A$ and for every $b \in B$, a and b are orthogonal as elements.*

It is a known fact that two orthogonal subobjects of a lattice-ordered group commute as subgroups. More generally, if a and b are orthogonal then $ab = ba$. A proof of this can be found, for instance, in Proposition 2.2.10 of [Kopytov and Medvedev, 2013].

3.4. PROPOSITION. *Let X be an object of $\ell\mathbb{G}rp$ and $A, B \leq X$ two subobjects. Then A and B cooperate if and only if $A \perp B$.*

PROOF. ($\Rightarrow$) The cooperator $\varphi \colon A \times B \to X$ is given by $\varphi(a, b) = ab$. In fact, since φ preserves the group operation, we have

$$\varphi(a, b) = \varphi(a, e)\varphi(e, b) = \varphi(i_A(a))\varphi(i_B(b)) = ab.$$

We observe that, for every $a \in A$ and $b \in B$, $(|a|, e) \wedge (e, |b|) = (e, e)$ holds. So, since φ preserves the lattice operations, we get

$$e = \varphi(e, e) = \varphi((|a|, e) \wedge (e, |b|)) = \varphi(|a|, e) \wedge \varphi(e, |b|) = |a| \wedge |b|.$$

($\Leftarrow$) If a cooperator φ exists then it must be the group multiplication because of what we observed at the beginning of the proof. In fact, in order to guarantee the existence of a cooperator, we only have to prove

$$ab = ba \text{ and } (ab)^+ = a^+ b^+ \tag{3.1}$$

for all $a \in A$ and $b \in B$. Indeed, if $ab = ba$ for every $a \in A$ and $b \in B$, then $\varphi((a,b)(c,d)) = \varphi(ac, bd) = acbd = abcd = \varphi(a,b)\varphi(c,d)$ since, by assumption, $cb = bc$; furthermore, if $(ab)^+ = a^+ b^+$ for every $a \in A$ and $b \in B$, then $\varphi((a,b)^+) = \varphi((a,b) \vee (e,e)) = \varphi(a^+, b^+) = a^+ b^+ = (ab)^+ = \varphi(a,b)^+ = \varphi(a,b) \vee e$, and thus we can apply Lemma 2.3 to say that φ is a morphism of lattice-ordered groups. As recalled above, the first equality of (3.1) holds since $A \perp B$. For the second one, we observe that $a^+ b^+ = (a \vee e)(b \vee e) = ab \vee a \vee b \vee e$ and $(ab)^+ = ab \vee e$; so we have to prove $a \vee b \leq ab \vee e$. Since $|a| \wedge |b| = e$, we have $(a \vee a^{-1}) \wedge (b \vee b^{-1}) = e$ and, by distributivity, we get

$$(a \wedge b) \vee (a \wedge b^{-1}) \vee (a^{-1} \wedge b) \vee (a^{-1} \wedge b^{-1}) = e.$$

Hence, $a^{-1} \wedge b \leq e$ implies $a \vee b^{-1} \geq e$ and, multiplying by b on the right, we obtain $ab \vee e \geq b$; with a similar argument we get $ab \vee e \geq a$. Finally, considering the last two inequalities, we conclude that $ab \vee e \geq a \vee b$. ∎

As a direct consequence of this proof we have the following:

3.5. COROLLARY. *Let X be an object of $\ell\mathbb{G}rp$ and $A, B \leq X$ two subobjects. Then $A \perp B$ if and only if, for every $a \in A$ and $b \in B$, the following equalities hold:*

$$ab = ba \text{ and } (ab)^+ = a^+ b^+.$$

The previous result will be of fundamental importance in the next sections.

We recall the notion of *polar* of a subset S of a lattice-ordered group (i.e. the set of elements orthogonal to each element of S). We will show that the polar of a subobject is nothing but the centralizer of the subobject. Hence, we will exhibit some properties of centralizers related to being ideals.

3.6. PROPOSITION. [Kopytov and Medvedev, 2013], Proposition 1.2.6 *Let X be an object of $\ell\mathbb{G}rp$ and $S \subseteq X$ a non-empty subset. Then the set $S^\perp := \{x \in X \mid \text{for each } s \in S \; |x| \wedge |s| = e\}$ (called the polar of S) is a convex subalgebra of X.*

3.7. LEMMA. *Let X be an object of $\ell\mathbb{G}rp$ and $S \subseteq X$ a non-empty subset of X closed under conjugation. Then $S^\perp$ is an ideal of X.*

PROOF. First of all we observe that, for all $x, y \in X$, one has

$$|x^{-1}yx| = x^{-1}yx \vee x^{-1}y^{-1}x = x^{-1}(y \vee y^{-1})x = x^{-1}|y|x.$$

We want to show that $|x^{-1}yx| \wedge |s| = e$, for every $x \in X$, $y \in S^\perp$, and $s \in S$. We observe that $|x^{-1}yx| \wedge |s| = x^{-1}|y|x \wedge |s| = x^{-1}(|y| \wedge x|s|x^{-1})x = x^{-1}(|y| \wedge |xsx^{-1}|)x = x^{-1}(|y| \wedge |\overline{s}|)x = e$, where $xsx^{-1} = \overline{s} \in S$ because S is closed under conjugation, and $|y| \wedge |\overline{s}| = e$ (since $y \in S^\perp$). ∎

3.8. COROLLARY. *Let X be an object of $\ell\mathbb{G}rp$ and $A, B \leq X$ two subobjects. A and B cooperate if and only if $B \subseteq A^\perp$. Therefore, $A^\perp \leq X$ is the centralizer of $A \leq X$. Moreover, if A is an ideal of X, then $A^\perp$ is an ideal of X, too.*

Finally, we recall a property that is strictly related to the existence of centralizers. It is well known that a category $\mathbb{E}$ with finite products is cartesian closed if and only if for every object Y of $\mathbb{E}$ the change-of-base functor $\tau_Y^* \colon \mathbb{E} \to \mathbb{E}/Y$ along the terminal arrow $\tau_Y \colon Y \to \mathbf{1}$ has a right adjoint. For algebraic categories, such adjoints rarely exist, but it turns out to be of interest to consider a variation of this notion; this leads to:

3.9. DEFINITION. [Bourn and Gray, 2012]
    A category $\mathbb{C}$ is algebraically cartesian closed (a.c.c.) *if for every object X of $\mathbb{C}$ the change-of-base functor $\tau_X^* \colon Pt_{\mathbf{1}}\mathbb{C} \to Pt_X\mathbb{C}$ has a right adjoint, where $\tau_X \colon X \to \mathbf{1}$ is the unique arrow from X to the terminal object.*

In [Bourn and Gray, 2012] the authors show that the existence of such adjoints is related to the existence of cofree structures for the split epimorphisms $p_Y \colon Y \times X \to Y$ in $Pt_Y\mathbb{E}$ with fixed section (id_Y, u), where $u \colon Y \to X$ can be chosen to be a monomorphism.

3.10. PROPOSITION. [Bourn and Gray, 2012], Proposition 1.2 *A unital category $\mathbb{C}$ is algebraically cartesian closed if and only if, for every X object of $\mathbb{C}$, each subobject of X has a centralizer.*

3.11. COROLLARY. *The category $\ell\mathbb{G}rp$ is algebraically cartesian closed.*

4. Congruence Distributivity and Arithmeticity

It is a widely known fact that in the category $\ell\mathbb{G}rp$ the lattice of congruences on any object is distributive (see Theorem 21 of [Birkhoff, 1942]). In this section we provide an alternative proof of this fact based on categorical tools. We recall that a category is a *Mal'tsev category* [Carboni et al., 1991] if it is finitely complete and if every internal reflexive relation is an internal equivalence relation. If the category is regular, this notion is equivalent to the following property: for every object X and for every pair of internal equivalence relations $(s_1, s_2) \colon S \rightarrowtail X \times X$ and $(r_1, r_2) \colon R \rightarrowtail X \times X$ one has $R \circ S = S \circ R$; in detail, $R \circ S \colon \rightarrowtail X \times X$ is defined as the regular image of (p_1, p_3), where (p_1, p_3) is

given by the following diagram:

$$
\begin{array}{ccc}
R \times_X S & \xrightarrow{\ \pi_S\ } & S \xrightarrow{\ s_2\ } X \\
\Big\downarrow{\scriptstyle \pi_R} & & \Big\downarrow{\scriptstyle s_1} \\
R & \xrightarrow{\ r_2\ } & X \\
\Big\downarrow{\scriptstyle r_1} & & \\
X. & &
\end{array}
$$

The composite $S \circ R$ can be defined in a similar way. Moreover, if the category is a variety of universal algebras, the property of being a Mal'tsev category is equivalent to the existence of a ternary term $p(x, y, z)$ (called *Mal'tsev term*) satisfying the axioms

$$p(x, x, z) = z \text{ and } p(x, y, y) = x,$$

for every object X and for every $x, y, z \in X$. Therefore, if the theory contains a group operation, the associated variety is a Mal'tsev category: in fact, a Mal'tsev ternary term is $p(x, y, z) := xy^{-1}z$.

Then, we immediately get the following result:

4.1. COROLLARY. *The category $\ell\mathrm{Grp}$ is a Mal'tsev category.*

If $\mathbb{C}$ is a Barr-exact category with coequalizers then, for every object X of $\mathbb{C}$, the set $\mathrm{Eq}(X)$ of internal equivalence relations on X is a lattice; given two internal equivalence relations $(s_1, s_2) \colon S \rightarrowtail X \times X$ and $(r_1, r_2) \colon R \rightarrowtail X \times X$, the meet $S \wedge R$ is defined as the meet of subobjects of $X \times X$, and the join $(t_1, t_2) \colon S \vee R \rightarrowtail X \times X$ is defined as the kernel pair of $q = \mathrm{coeq}(v_1, v_2)$, where $(v_1, v_2) \colon V \rightarrowtail X \times X$ is the join of S and R as subobjects of $X \times X$ (we recall that the join, as subobjects, of two internal equivalence relations is not, in general, an internal equivalence relation). Thanks to the previous observations, the classical notion of arithmetical variety of universal algebras can be extended to a categorical context as follows:

4.2. DEFINITION. [Pedicchio, 1996] *A Barr-exact category with coequalizers $\mathbb{C}$ is* arithmetical *if it is a Mal'tsev category and, for any object X of $\mathbb{C}$, the lattice $\mathrm{Eq}(X)$ of internal equivalence relations on X is distributive.*

It is a known fact (a proof of this can be found in [Borceux and Bourn, 2004]) that the property of being an arithmetical category is related to the absence of non-trivial *internal group objects* in the category. In fact, the following holds:+

4.3. PROPOSITION. [Borceux and Bourn, 2004], Proposition 2.9.9 *Let $\mathbb{C}$ be a semi-abelian category. If in $\mathbb{C}$ the only internal group object is the zero object then $\mathbb{C}$ is arithmetical.*

4.4. PROPOSITION. *The only internal group object in $\ell\mathrm{Grp}$ is the zero object.*

PROOF. Given an internal group X in $\ell\mathbb{G}rp$, with multiplication $\mu\colon X \times X \to X$ and neutral element $\eta\colon \{*\} \to X$, we want to show that $X = \{e\}$. It is not difficult to see that $\eta(*) = e$ and $\mu(x,y) = xy$ for all $x, y \in X$. This determines a cooperator between X and itself. Hence, we get $X \perp X$ and so $x^+ = x^+ \wedge x^+ = |x^+| \wedge |x^+| = e$, for every $x \in X$. With a similar argument we can prove $x^- = e$, for every $x \in X$. Finally, recalling that $x = x^+x^-$, we obtain $x = e$ for every $x \in X$ (i.e. $X = \{e\}$).

∎

4.5. COROLLARY. $\ell\mathbb{G}rp$ *is an arithmetical category.*

5. Strong Protomodularity

Given a category $\mathbb{C}$, we denote by $\mathrm{Pt}(\mathbb{C})$ the category whose objects are the diagrams in $\mathbb{C}$ of the form

$$A \underset{s}{\overset{p}{\rightleftarrows}} B$$

where $ps = id_B$, and whose arrows are the pairs (f, g) of arrows of $\mathbb{C}$

$$
\begin{array}{ccc}
A & \underset{s}{\overset{p}{\rightleftarrows}} & B \\
f \downarrow & & \downarrow g \\
C & \underset{r}{\overset{q}{\rightleftarrows}} & D
\end{array}
$$

such that $qf = gp$ and $fs = rg$. We denote by $\pi\colon \mathrm{Pt}(\mathbb{C}) \to \mathbb{C}$ the functor that associates to every split extension (i.e. an object of $\mathrm{Pt}(\mathbb{C})$) (p, s) the codomain of p.

5.1. DEFINITION. [Bourn, 2000] *A finitely complete category $\mathbb{C}$ is* strongly protomodular *when all the change-of-base functors of $\pi\colon \mathrm{Pt}(\mathbb{C}) \to \mathbb{C}$ reflect both isomorphisms and normal monomorphisms (in the semi-abelian case a monomorphism is normal if and only if it is the kernel of some arrow).*

In [Bourn, 2000] the author shows that, if $\mathbb{C}$ is a pointed protomodular category, there is a characterization of strong protomodularity related to the stability of kernels. Let us consider a diagram in $\mathbb{C}$ of the form

$$
\begin{array}{ccccc}
X & \overset{k}{\longrightarrow} & A & \underset{p}{\overset{s}{\leftrightarrows}} & B \\
m \downarrow & & f \downarrow & & \| \\
Y & \underset{l}{\longrightarrow} & C & \underset{q}{\overset{r}{\leftrightarrows}} & B
\end{array}
$$

where $k = \ker(p)$, $ps = id_B$, $l = \ker(q)$, $qr = id_B$, m is a normal monomorphism and the right-rightward square, the right-leftward square, and the left square commute. Then $\mathbb{C}$ is strongly protomodular if and only if the composite lm is a normal monomorphism, for every diagram of this form.

5.2. PROPOSITION. *$\ell\mathbb{G}rp$ is a strongly protomodular category.*

PROOF. Let us consider the following commutative diagram (without loss of generality we can assume that the monomorphisms are inclusions):

$$
\begin{array}{ccc}
X \lhook\joinrel\longrightarrow A & \xleftarrow[p]{\ \ s\ \ } & B \\
\Big\downarrow \quad\quad f\Big\downarrow & & \Big\| \\
Y \lhook\joinrel\longrightarrow C & \xleftarrow[q]{\ \ r\ \ } & B
\end{array}
$$

where $ps = id_B$, $qr = id_B$, X is the ideal of A determined by $\ker(p)$, Y is the ideal of C determined by $\ker(q)$, X is an ideal of Y and the right-rightward square, the right-leftward square, and the left square commute. We want to show that X is an ideal of C, too.

- X is a normal subgroup of C. This immediately follows from the fact that the category of groups is strongly protomodular, as established in [Bourn, 2000].

- X is a convex subset of C. We know that, for every $y \in Y$, if $x_1 \le y \le x_2$, with $x_1, x_2 \in X$, then $y \in X$ and, for every $c \in C$, if $y_1 \le c \le y_2$, with $y_1, y_2 \in Y$, then $c \in Y$. So, given an element $c \in C$ such that $x_1 \le c \le x_2$, with $x_1, x_2 \in X$, we have $c \in Y$ (since $x_1, x_2 \in Y$) and thus $c \in X$.   ∎

In the final part of this section we study, in the case of $\ell\mathbb{G}rp$, the consequences of strong protomodularity relatively to the commutativity, in the Smith-Pedicchio sense, of internal equivalence relations. In particular, we show that every internal equivalence relation admits a centralizer. Let us begin by recalling the necessary notions to deal with this subject.

5.3. DEFINITION. [Pedicchio, 1995], [Bourn and Gran, 2002a]
   Let $\mathbb{C}$ be a Mal'tsev category and

$$
R \xrightarrow[\quad r_2\quad]{\overset{r_1}{\longrightarrow}} \overset{\delta_R}{\longleftarrow} X, \qquad S \xrightarrow[\quad s_2\quad]{\overset{s_1}{\longrightarrow}} \overset{\delta_S}{\longleftarrow} X
$$

a pair of internal equivalence relations on an object X of $\mathbb{C}$. We say that (R, r_1, r_2) and (S, s_1, s_2) commute in the Smith-Pedicchio sense (and we write $[R, S] = 0$) if, given the following diagram:

$$
\begin{array}{ccc}
R \times_X S & \xrightleftharpoons[\tau_S]{\pi_S} & S \\
\pi_R \big\uparrow\big\downarrow \tau_R \quad \ulcorner & & s_1 \big\uparrow\big\downarrow \delta_S \\
R & \xrightleftharpoons[\delta_R]{r_2} & X
\end{array}
$$

where $R \times_X S$ is the pullback of r_2 through s_1, $\tau_R = (id_R, \delta_S r_2)$ and $\tau_S = (\delta_R s_1, id_S)$ are induced by the universal property, there exists a unique morphism $p\colon R \times_X S \to X$

(called connector *between R and S) such that $p\tau_S = s_2$ and $p\tau_R = r_1$. The* centralizer *of an internal equivalence relation (R, r_1, r_2) on X, if it exists, is the greatest internal equivalence relation on X which commutes with (R, r_1, r_2).*

It has been shown in Proposition 3.2 of [Bourn and Gran, 2002b] that, in a pointed Mal'tsev category, if two internal equivalence relations (R, r_1, r_2) and (S, s_1, s_2) commute in the Smith-Pedicchio sense, then necessarily their associated normal subobjects j_R and j_S commute in the Huq sense, where $j_R := \ker(q_R)$ and $j_S := \ker(q_S)$, with $q_R := \operatorname{coeq}(r_1, r_2)$ and $q_S := \operatorname{coeq}(s_1, s_2)$. Briefly, $[R, S] = 0$ implies $[j_R, j_S] = 0$. The converse is not true, in general. We say that a pointed Mal'tsev category satisfies the so-called *Smith is Huq* condition (SH) if $[j_R, j_S] = 0$ implies $[R, S] = 0$. It has been proved in Theorem 6.1 of [Bourn and Gran, 2002b] that in every pointed strongly protomodular category the Smith is Huq condition holds.

In [Smith, 2006], page 39, it is shown that, in every Mal'tsev variety, internal equivalence relations have centralizers. We provide an explicit description of these centralizers in $\ell\mathbb{G}rp$.

Since $\ell\mathbb{G}rp$ is a semi-abelian category, we have, for every object X, an order-preserving bijection φ between $\operatorname{Eq}(X)$ and the lattice $\operatorname{Ideals}(X)$ of ideals of X, where $\varphi(R) := I_R$, with $I_R := \{x \in X \mid (x, e) \in R\}$. Given two internal equivalence relations $R \leq X \times X$ and $S \leq X \times X$, we know that $[R, S] = 0$ if and only if $[\varphi(R), \varphi(S)] = 0$ ($\ell\mathbb{G}rp$ is a strongly protomodular category, hence (SH) holds). Moreover, given an internal equivalence relation R on X, we recall from Lemma 3.7 that the centralizer $I_R^\perp \leq X$ of the ideal I_R associated with R is an ideal. Therefore, since φ is an order-preserving bijection and $\ell\mathbb{G}rp$ satisfies (SH), the centralizer of R in X is $\varphi^{-1}(I_R^\perp)$.

6. Action Accessibility

To approach the topic covered in this section, let us recall a fundamental concept in the category $\mathbb{G}rp$ of groups: the notion of *split extension*.

6.1. DEFINITION. *Let $\mathbb{C}$ be a pointed protomodular category. A* split extension *of $\mathbb{C}$ is a diagram of the form*

$$X \xrightarrow{\ k\ } A \underset{p}{\overset{s}{\longleftarrow}} B,$$

where $k = \ker(p)$ and $ps = id_B$.
We denote by $\mathbb{S}pl\mathbb{E}xt_\mathbb{C}(X)$ the category whose objects are the split extensions of $\mathbb{C}$ with the same fixed kernel object X and whose arrows are the pairs (g, f) of arrows in $\mathbb{C}$

$$
\begin{array}{ccccc}
X & \xrightarrow{\ k\ } & A & \underset{p}{\overset{s}{\longleftarrow}} & B \\
\Big\| & & \downarrow{g} & & \downarrow{f} \\
X & \xrightarrow[l]{} & C & \underset{q}{\overset{r}{\longleftarrow}} & D
\end{array}
$$

such that $gk = l, fp = qg$ *and* $gs = rf$.

Given a group X, we can define a functor

$$\mathbb{G}rp^{op} \xrightarrow{\quad \mathrm{SplExt}(-,X) \quad} \mathbb{S}et$$

$$
\begin{array}{ccc}
B & \longmapsto & \{X \to A \leftrightarrows B\}/\sim \\
f\downarrow & & \uparrow \mathrm{SplExt}(f,X) \\
B' & \longmapsto & \{X \to A' \leftrightarrows B'\}/\sim
\end{array}
$$

where $\{X \to A \leftrightarrows B\}$ is the set of split extensions with fixed kernel object X and fixed quotient object B; two split extensions $X \to A \leftrightarrows B$ and $X \to \overline{A} \leftrightarrows B$ are equivalent (under the equivalence relation $\sim$) if there exists an arrow $g \colon A \to \overline{A}$ such that $gk = \overline{k}$, $gs = \overline{s}$ and $\overline{p}g = p$

$$
\begin{array}{ccccc}
X & \xrightarrow{\ k\ } & A & \overset{s}{\underset{p}{\leftrightarrows}} & B \\
\| & & g\downarrow & & \| \\
X & \xrightarrow[\ \overline{k}\ ]{} & \overline{A} & \overset{\overline{s}}{\underset{\overline{p}}{\leftrightarrows}} & B
\end{array}
$$

(hence, thanks to the Split Short Five Lemma, g is an isomorphism). Finally, $\mathrm{SplExt}(f, X)$ sends the class of a split extension $X \to A' \leftrightarrows B'$ to the class of the split extension defined via the following diagram, where the right-rightward square is a pullback:

$$
\begin{array}{ccccc}
X & \xrightarrow{(k',0)} & A' \times_{B'} B & \overset{(s'f,id_B)}{\underset{\pi_B}{\leftrightarrows}} & B \\
\| & & \pi_{A'}\downarrow & \lrcorner & \downarrow f \\
X & \xrightarrow[\ k'\ ]{} & A' & \overset{s'}{\underset{p'}{\leftrightarrows}} & B'.
\end{array}
$$

It is a known fact that, in $\mathbb{G}rp$, there is a one-to-one correspondence between the set $\{X \to A \leftrightarrows B\}/\sim$ and the set of group homomorphisms with domain B and codomain the group $\mathrm{Aut}(X)$ of automorphisms of X. In fact, it turns out that the functor $\mathrm{SplExt}(-, X)$ is representable and a representing object is $\mathrm{Aut}(X)$. A pointed protomodular category in which the functor $\mathrm{SplExt}(-, X)$ is representable for every object X is called *action representable* [Borceux et al., 2005]. It can be seen that this condition is extremely strong. In fact, it emerged that many of categories of interest in classical algebra are not action representable, but do satisfy a weaker related condition we will describe now. In order to do this, it is easy to observe that, in $\mathbb{G}rp$, the property of being an action representable category can be restated in the following way: the split extension

$$X \xrightarrow{(id_X,0)} X \rtimes \mathrm{Aut}(X) \overset{(0,id_{\mathrm{Aut}(X)})}{\underset{\pi_{\mathrm{Aut}(X)}}{\leftarrow}} \mathrm{Aut}(X), \tag{6.1}$$

corresponding to the action $id_{\mathrm{Aut}(X)}\colon \mathrm{Aut}(X) \to \mathrm{Aut}(X)$, is a terminal object of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{G}rp}(X)$. Therefore, in $\mathbb{G}rp$, for every object of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{G}rp}(X)$ there exists a unique morphism into (6.1).

In light of this, the authors of [Bourn and Janelidze, 2009] have weakened the notion of action representable category in the following way:

6.2. DEFINITION. [Bourn and Janelidze, 2009] *An object F of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{C}}(X)$ is said to be* faithful *if for each object E of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{C}}(X)$ there is at most one arrow from E to F.*

6.3. DEFINITION. [Bourn and Janelidze, 2009] *Let $\mathbb{C}$ be a pointed protomodular category. An object in $\mathbb{S}pl\mathbb{E}xt_{\mathbb{C}}(X)$ is said to be* accessible *if it admits a morphism into a faithful object. We say that $\mathbb{C}$ is* action accessible *if, for every object X of $\mathbb{C}$, every object in $\mathbb{S}pl\mathbb{E}xt_{\mathbb{C}}(X)$ is accessible.*

As mentioned above, the notion of action accessible category appears as a generalization of the one of action representable category: if there is a terminal object T of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{C}}(X)$, this object is also faithful and each object of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{C}}(X)$ admits a unique morphism into T. Examples of action accessible categories include, for instance, not necessarily unitary rings (as shown in [Bourn and Janelidze, 2009]) and all *categories of interest* in the sense of [Orzech, 1972] (as shown in [Montoli, 2010]).

One of the interesting properties, among other things, implied by action accessibility is that centralizers of internal equivalence relations exist and they have a simple description (see Theorem 4.1 in [Bourn and Janelidze, 2009]). Moreover, action accessibility implies (SH) (Theorem 5.4 in [Bourn and Janelidze, 2009]); hence, in an action accessible category centralizers of normal subobjects exist and they are normal. In the next part of this section we show that the category $\ell\mathbb{G}rp$ is not action accessible despite the fact that centralizers of internal equivalence relations exist and they have the same simple description as in action accessible categories (as shown at the end of previous section).

6.4. PROPOSITION. *$\ell\mathbb{G}rp$ is not action accessible.*

PROOF. Consider the lexicographic product $\mathbb{Z}\overleftarrow{\times}\mathbb{Z}$ of the group of integers $\mathbb{Z}$ (with the usual order) with itself. The underlying set is the product, and in terms of structure the group operations are defined component-wise, while the order is defined as follows: $(a,b) \leq (c,d)$ if and only if $b < d$, or $b = d$ and $a \leq c$. We consider the following split extension

$$\mathbb{Z} \xrightarrow{\ i_1\ } \mathbb{Z}\overleftarrow{\times}\mathbb{Z} \underset{p_2}{\overset{i_2}{\longleftrightarrow}} \mathbb{Z}, \tag{6.2}$$

where $i_1 = (id_{\mathbb{Z}},0)$, $i_2 = (0,id_{\mathbb{Z}})$ and p_2 is the projection on the second component. Now, for every $n \in \mathbb{N}_{>0}$ we can consider the morphism of lattice-ordered groups $f_n\colon \mathbb{Z} \to \mathbb{Z}$

given by $f_n(x) := nx$. This morphism induces the following morphism in $\mathbb{S}pl\mathbb{E}xt_{\ell\mathbb{G}rp}(\mathbb{Z})$:

$$
\begin{array}{ccccc}
\mathbb{Z} & \xrightarrow{i_1} & \mathbb{Z}\overleftarrow{\times}\mathbb{Z} & \underset{p_2}{\overset{i_2}{\longleftarrow}} & \mathbb{Z} \\
\| & & \downarrow{g_n} & & \downarrow{f_n} \\
\mathbb{Z} & \xrightarrow{i_1} & \mathbb{Z}\overleftarrow{\times}\mathbb{Z} & \underset{p_2}{\overset{i_2}{\longleftarrow}} & \mathbb{Z}
\end{array}
$$

where $g_n(x,y) := (x,ny)$. Thus we can deduce that (6.2) is not faithful. So, if $\ell\mathbb{G}rp$ were action accessible then there should exist a faithful object

$$
X \xrightarrow{k} A \underset{p}{\overset{s}{\longleftarrow}} B
$$

and a morphism

$$
\begin{array}{ccccc}
\mathbb{Z} & \xrightarrow{i_1} & \mathbb{Z}\overleftarrow{\times}\mathbb{Z} & \underset{p_2}{\overset{i_2}{\longleftarrow}} & \mathbb{Z} \\
\| & & \downarrow{g} & & \downarrow{f} \\
\mathbb{Z} & \xrightarrow{k} & A & \underset{p}{\overset{s}{\longleftarrow}} & B.
\end{array}
$$

Then, if we consider the (regular epimorphism, monomorphism)-factorization of (g,f) we get:

$$
\tag{6.3}
\begin{array}{ccccc}
\mathbb{Z} & \xrightarrow{i_1} & \mathbb{Z}\overleftarrow{\times}\mathbb{Z} & \underset{p_2}{\overset{i_2}{\longleftarrow}} & \mathbb{Z} \\
\| & {}^{g}\nearrow\,\downarrow & & & \downarrow\,\searrow{}^{f} \\
\mathbb{Z} & \xrightarrow{\overline{k}} & \mathrm{Im}(g) \underset{\overline{p}}{\overset{\overline{s}}{\longleftarrow}} \mathrm{Im}(f) & & \\
\| & \downarrow & & & \downarrow \\
\mathbb{Z} & \xrightarrow{k} & A & \underset{p}{\overset{s}{\longleftarrow}} & B.
\end{array}
$$

Therefore, $\mathrm{Im}(f)$ is a quotient (in $\ell\mathbb{G}rp$) of $\mathbb{Z}$. However, $\mathbb{Z}$ has only two ideals: $\{0\}$ and $\mathbb{Z}$. Hence we have two possibilities: $\mathrm{Im}(f) \cong \mathbb{Z}$ or $\mathrm{Im}(f) \cong \{e\}$. If $\mathrm{Im}(f) \cong \mathbb{Z}$ then f is injective and so the split extension

$$
\mathbb{Z} \xrightarrow{i_1} \mathbb{Z}\overleftarrow{\times}\mathbb{Z} \underset{p_2}{\overset{i_2}{\longleftarrow}} \mathbb{Z}
$$

has to be a faithful object, and this is a contradiction. Alternatively, if $\mathrm{Im}(f) \cong \{e\}$, f has to be the trivial morphism, and so $\mathrm{Im}(g) \cong \mathbb{Z}$. Therefore, recalling that the top right-rightward square of (6.3) is a pullback, we get a contradiction since $\mathbb{Z}\overleftarrow{\times}\mathbb{Z}$ is not isomorphic, as a lattice-ordered group, to $\mathbb{Z} \times \mathbb{Z}$. ∎

In [Bourn and Janelidze, 2009] the authors show that, in the case of the variety $\mathbb{R}ng$ of not necessarily unitary rings, given a split extension there is a procedure, based on centralizers of subobjects, to build a morphism from it into a faithful split extension.

This same argument has also been extended in [Montoli, 2010] to categories of interest in the sense of [Orzech, 1972]. We recall here a sketch of the proof presented in [Bourn and Janelidze, 2009].

Given an object A of $\mathbb{R}ng$, two subobjects $X, Y \leq A$ cooperate if and only if, for every $x \in X$ and for every $y \in Y$,

$$xy = 0 = yx.$$

Hence, it can be shown that the centralizer of X in A is the subobject

$$Z_A(X) := \{a \in A \mid ax = 0 = xa \text{ for all } x \in X\}.$$

Given an object of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{R}ng}(X)$

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B, \tag{6.4}$$

they define $I := \{b \in B \mid s(b)k(x) = 0 = k(x)s(b) \text{ for all } x \in X\}$; they prove that I is an ideal of B and $s(I) = Z_A(k(X)) \cap s(B)$ is an ideal of A. Thus, they show that the split extension

$$X \xrightarrow{\bar{k}} A/s(I) \underset{\bar{s}}{\overset{\bar{p}}{\rightleftarrows}} B/I, \tag{6.5}$$

where the morphisms are induced by the universal property of the quotient, is a faithful object of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{R}ng}(X)$ and the pair $(\pi_{s(I)}, \pi_I)$, obtained by the quotient projections $\pi_{s(I)} \colon A \to A/s(I)$ and $\pi_I \colon B \to B/I$, is a morphism between (6.4) and (6.5). Therefore, in the case of rings, there is a canonical way to construct an arrow of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{R}ng}(X)$ into a faithful object making use of the notion of centralizer.

Although the category $\ell\mathbb{G}rp$ is not action accessible, it is possible to emulate the previous construction in this case. This shows that in $\ell\mathbb{G}rp$ centralizers of subobjects have a good behaviour even though the category is not action accessible.

Let us fix a split extension in $\ell\mathbb{G}rp$:

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B.$$

We want to show that the intersection between $s(B)$ and the centralizer of $k(X)$ in A is an ideal of A. In other words, we need to prove that $k(X)^{\perp} \cap S(B)$ is convex and closed under conjugation in A.

- Convexity: let us consider $s(b_1) \leq a \leq s(b_2)$ where $s(b_1), s(b_2) \in k(X)^{\perp} \cap s(B)$ and $a \in A$. We recall that for all $a \in A$ there exist $x \in k(X)$ and $b \in B$ such that $a = k(x)s(b)$ (see Proposition 2.7). Then, applying p to the inequalities, we obtain $b_1 \leq b \leq b_2$ and thus

$$s(b_1) \leq s(b) \leq s(b_2).$$

Therefore, since $k(X)^{\perp}$ is a convex subobject of A, we get $s(b) \in k(X)^{\perp} \cap s(B)$.

Hence, from $s(b_1) \leq a \leq s(b_2)$ multiplying on the right by $s(b^{-1})$, we obtain

$$s(b_1 b^{-1}) \leq k(x) \leq s(b_2 b^{-1}).$$

So, since $s(b_1 b^{-1}), s(b_2 b^{-1}) \in k(X)^\perp \cap s(B)$ (because $k(X)^\perp \cap s(B)$ is a subalgebra of A and $s(b), s(b_1), s(b_2) \in k(X)^\perp \cap s(B)$), we get $k(x) \in k(X)^\perp$ ($k(X)^\perp$ is a convex subalgebra). Therefore, $k(x) = e$ and then we obtain $a = k(x)s(b) = s(b) \in k(X)^\perp \cap s(B)$.

- Closedness under conjugation: let us consider $s(c) \in k(X)^\perp \cap s(B)$ and $a = k(x)s(b) \in A$. Then, we have

$$as(c)a^{-1} = k(x)s(b)s(c)s(b^{-1})k(x)^{-1} = k(x)s(d)k(x)^{-1}$$

where $s(d) = s(b)s(c)s(b^{-1}) \in k(X)^\perp \cap s(B)$ since $k(X)^\perp$ is closed under conjugation and, clearly, $s(d) \in s(B)$. Therefore, we get

$$as(c)a^{-1} = k(x)s(d)k(x)^{-1} = s(d)$$

since $s(d) \in k(X)^\perp$ and the elements of $k(X)^\perp$ commute with the ones of $k(X)$.

7. Fiber-wise Algebraic Cartesian Closedness

In this section we deal with a stronger version of the notion of algebraically cartesian closed category. We propose an equivalent version of the definition presented in [Bourn and Gray, 2012]:

7.1. DEFINITION. [Bourn and Gray, 2012] *A category* $\mathbb{C}$ *is* fiber-wise algebraically cartesian closed *if for every split epimorphism*

$$A \underset{s}{\overset{p}{\rightleftarrows}} B$$

the change-of-base functor

$$p^* \colon Pt_B\mathbb{C} \to Pt_A\mathbb{C}$$

has a right adjoint.

As established in [Bourn and Gray, 2012], it is not difficult to see that this condition holds for a category $\mathbb{C}$ if and only if every category of points over $\mathbb{C}$ is algebraically cartesian closed. First of all, we observe that the category $Pt_{A \rightleftarrows B}Pt_B\mathbb{C}$ is isomorphic to

$Pt_A\mathbb{C}$: an object of $Pt_{A\leftrightarrows B}Pt_B\mathbb{C}$ can be seen as a diagram of type

$$
\begin{array}{ccc}
C & \xleftarrow{\;\;r\;\;} & B \\[-2pt]
 & \xrightarrow[q]{} & \\
h \downarrow\uparrow t & & \| \\[-2pt]
A & \xleftarrow{\;\;s\;\;} & B, \\[-2pt]
 & \xrightarrow[p]{} &
\end{array}
$$

where $ps = id_B$, $qr = id_B$, $ht = id_A$, $ph = q$, and $ts = r$; therefore, q and r are uniquely determined by h and t. Hence, each category of points is algebraically cartesian closed if and only if the functor

$$
\tau^* \colon Pt_{B=B}Pt_B\mathbb{C} \longrightarrow Pt_{A\leftrightarrows B}Pt_B\mathbb{C}
$$

has a right adjoint; thanks to the isomorphism shown above between $Pt_A\mathbb{C}$ and $Pt_{A\leftrightarrows B}Pt_B\mathbb{C}$ and recalling that $\tau = p$, we get that τ^* has a right adjoint if and only if p^* has a right adjoint. Therefore, $\mathbb{C}$ is fiber-wise algebraically cartesian closed if and only if every category of points over $\mathbb{C}$ is algebraically cartesian closed.

Our aim is to show, thanks to the previous observations, that $\ell\mathbb{G}rp$ is fiber-wise algebraically cartesian closed. In order to do this we will prove that, in every category of points over $\ell\mathbb{G}rp$, subobjects have centralizers. As a preliminary remark, we recall that an arrow $(A \leftrightarrows B) \xrightarrow{f} (C \leftrightarrows B)$ of $Pt_B\mathbb{C}$ is a monomorphism if and only if $f \colon A \to C$ is a monomorphism of $\mathbb{C}$.

We are ready to show the existence of centralizers in every category of points over $\ell\mathbb{G}rp$ and to provide an explicit description of them.

7.2. DEFINITION. *Let X be an object of $\ell\mathbb{G}rp$ and B a subalgebra of X. A subalgebra L of X is* closed under the action of B *if*

$$
blb^{-1} \in L \text{ and } (l_1 b_1 \vee l_2 b_2)(b_1 \vee b_2)^{-1} \in L
$$

for every $l, l_1, l_2 \in L$ and $b, b_1, b_2 \in B$.

7.3. PROPOSITION. *Let B be an object of $\ell\mathbb{G}rp$. In the category $Pt_B\ell\mathbb{G}rp$ subobjects have centralizers.*

PROOF. Let us consider an object (A, p, s) of $Pt_B\ell\mathbb{G}rp$, i.e. a diagram of the form

$$
K \xrightarrow{\;\;k\;\;} A \xleftarrow[p]{\;\;s\;\;} B
$$

where $ps = id_B$ and $k = \ker(p)$. Given two subobjects $(X, p_{|X}, r_X)$ and $(Y, p_{|Y}, r_Y)$ of (A, p, s) and the product between them in $Pt_B\ell\mathbb{G}rp$, we need to describe the arrows $i_X \colon X \to X \times_B Y$ and $i_Y \colon Y \to X \times_B Y$ induced by the universal property (we recall that the product in $Pt_B\ell\mathbb{G}rp$ is given by the pullback of $p_{|X}$ along $p_{|Y}$ in $\ell\mathbb{G}rp$). Then, if

we consider the following diagram
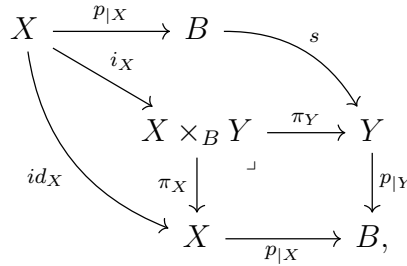
we get $i_X(x) = (x, sp(x))$; in a similar way $i_Y \colon Y \to X \times_B Y$ is given by $i_Y(y) = (sp(y), y)$. We know that $(x, y) \in X \times_B Y$ if and only if $p(x) = p(y)$. Moreover, given $(x, y) \in X \times_B Y$, we have $(x, y) = (x, sp(x))(sp(y), sp(x))^{-1}(sp(y), y)$ where $(x, sp(x)) \in X \times_B Y$, $(sp(y), y) \in X \times_B Y$ and, since $psp(x) = p(x) = p(y) = psp(y)$, we get $(sp(y), sp(x)) = (sp(x), sp(x)) = (sp(y), sp(y)) \in X \times_B Y$.

Hence, if there exists a cooperator $\varphi \colon X \times_B Y \to A$, then

$$\varphi(x, y) = \varphi(x, sp(x))\varphi(sp(y), sp(x))^{-1}\varphi(sp(y), y) = xsp(x)^{-1}y = xsp(y)^{-1}y,$$

since

$$\varphi(x, sp(x))\varphi(sp(y), sp(x))^{-1}\varphi(sp(y), y) = \varphi(i_X(x))\varphi(i_X(sp(x)))^{-1}\varphi(i_Y(y))$$
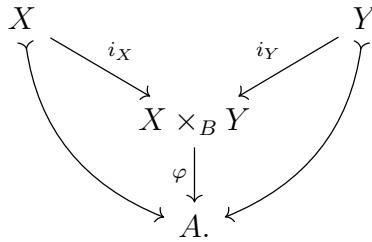
It is well known that the category of points over B, in the case of groups, is equivalent to the category of groups with an action of B. Therefore, thanks to Proposition 2.7, we can extend this to the case of lattice-ordered groups and obtain that A is isomorphic to $K \rtimes B$ with join operation defined by

$$(k_1, b_1) \vee (k_2, b_2) = ((k_1 b_1 \vee k_2 b_2)(b_1 \vee b_2)^{-1}, b_1 \vee b_2);$$

more specifically, the split extension (A, p, s) is isomorphic to the split extension

$$K \rtimes B \xleftarrow[p_B]{i_B} B.$$

Hence, a subobject (X, q, r) of (A, p, s) in $Pt_B\ell\mathbb{G}rp$ can be seen, modulo isomorphisms, as a subalgebra $X \leq K \rtimes B$ in $\ell\mathbb{G}rp$ such that, referring to the diagram

$$
\begin{array}{c}
X \\
\downarrow \qquad \nwarrow q \\
\qquad r \\
\qquad p_B \\
K \rtimes B \; \underset{i_B}{\overset{}{\rightleftarrows}} \; B,
\end{array}
$$

q is the restriction of p_B to X and, for every $b \in B$, $r(b) = (e, b)$ (in particular $\{e\} \times B \leq X$).

Given a subobject (X, q, r) of $(K \rtimes B, p_B, i_B)$ we define

$$\overline{X} := \{\overline{x} \in K \mid \text{there exists } b \in B \text{ s.t. } (\overline{x}, b) \in X\}.$$

We show that $X = \overline{X} \times B$ as sets. Clearly $X \subseteq \overline{X} \times B$. Conversely, fix an element $(k, b) \in \overline{X} \times B$. Then $k \in \overline{X}$, and so there exists $b_1 \in B$ such that $(k, b_1) \in X$; but $(e, b), (e, b_1) \in X$, therefore $(k, b) = (k, b_1)(e, b_1)^{-1}(e, b) \in X$. In general, we have a one-to-one correspondence between the subobjects of (A, p, s) and the subalgebras of K closed under the action of B.

Therefore, given two subobjects $(X, p_{|X}, r_X)$ and $(Y, p_{|Y}, r_Y)$ of (A, p, s) we can suppose $X = \overline{X} \times B$ and $Y = \overline{Y} \times B$. Thus,

$$X \times_B Y = \{((\overline{x}, b_1), (\overline{y}, b_2)) \in X \times Y \mid b_1 = b_2\}$$

and $\varphi \colon X \times_B Y \to K \rtimes B$ is such that

$$\varphi((\overline{x}, b), (\overline{y}, b)) = (\overline{x}, b)(e, b)^{-1}(\overline{y}, b) = (\overline{x}\,\overline{y}, b).$$

We start by showing that

$$\varphi \text{ is a group homomorphism if and only if } xy = yx \text{ for all } x \in \overline{X}, y \in \overline{Y}.$$

Given an element $((x, b), (y, b)), ((z, c), (w, c)) \in (\overline{X} \times B) \times_B (\overline{Y} \times B)$ one has

$$
\begin{aligned}
((x, b), (y, b))((z, c), (w, c)) &= ((x, b)(z, c), (y, b)(w, c)) \\
&= ((xbzb^{-1}, bc), (ybwb^{-1}, bc)).
\end{aligned}
$$

Hence

$$\varphi((x, b), (y, b))\varphi((z, c), (w, c)) = (xy, b)(zw, c) = (xybzwb^{-1}, bc)$$

and, since $((x, b), (y, b))((z, c), (w, c)) = ((xbzb^{-1}, bc), (ybwb^{-1}, bc))$, we get

$$\varphi((xbzb^{-1}, bc), (ybwb^{-1}, bc)) = (xbzb^{-1}ybwb^{-1}, bc).$$

So, φ is a group homomorphism if and only if $bzb^{-1}y = ybzb^{-1}$ for all $z \in \overline{X}$, $y \in \overline{Y}$ and $b \in B$. Then, setting $b = e$, we obtain $zy = yz$ for all $z \in \overline{X}$ and $y \in \overline{Y}$. Moreover, since $\overline{X}$ is closed under the action of B and the conjugation is a bijection, we get that every element of $\overline{X}$ can be seen as bzb^{-1}, for appropriate $b \in B$ and $z \in \overline{X}$; thus, if $zy = yz$ for all $z \in \overline{X}$ and $y \in \overline{Y}$ then $bzb^{-1}y = ybzb^{-1}$ for all $z \in \overline{X}$, $y \in \overline{Y}$ and $b \in B$. Now, let us deal with the order structure. We know that φ is a morphism of lattice-ordered groups if and only if φ is a group homomorphism and for all $((x, b), (y, b)) \in (\overline{X} \times B) \times_B (\overline{Y} \times B)$

$$\varphi(((x, b), (y, b)) \vee ((e, e), (e, e))) = \varphi(((x, b), (y, b))) \vee (e, e). \tag{7.1}$$

Observing that

$$
\begin{aligned}
((x, b), (y, b)) \vee ((e, e), (e, e)) &= ((x, b) \vee (e, e), (y, b) \vee (e, e)) \\
&= (((xb \vee e)(b \vee e)^{-1}, b \vee e), ((yb \vee e)(b \vee e)^{-1}, b \vee e)) \\
&= (((xb)^+(b^+)^{-1}, b^+), ((yb)^+(b^+)^{-1}, b^+)),
\end{aligned}
$$

we have

$$\varphi(((x, b), (y, b)) \vee ((e, e), (e, e))) = ((xb)^+(b^+)^{-1}(yb)^+(b^+)^{-1}, b^+).$$

Considering the right term of (7.1), we obtain

$$\varphi(((x, b), (y, b))) \vee (e, e) = (xy, b) \vee (e, e) = ((xyb)^+(b^+)^{-1}, b^+).$$

We prove that

$$\varphi \text{ is a lattice-ordered group morphism if and only if } \overline{X} \perp \overline{Y}.$$

If φ is a lattice-ordered group morphism, then

$$(xb)^+(b^+)^{-1}(yb)^+(b^+)^{-1} = (xyb)^+(b^+)^{-1}$$

for each $x \in \overline{X}$, $y \in \overline{Y}$ and $b \in B$; therefore, setting $b = e$, we get

$$(xy)^+ = x^+y^+ \text{ for every } x \in \overline{X} \text{ and } y \in \overline{Y},$$

and so, thanks to Corollary 3.5, $\overline{X}$ and $\overline{Y}$ are orthogonal.
Conversely, let us suppose $\overline{X} \perp \overline{Y}$. We want to show that, for all $x \in \overline{X}, y \in \overline{Y}$ and $b \in B$,

$$(xb)^+(b^+)^{-1}(yb)^+(b^+)^{-1} = (xyb)^+(b^+)^{-1}$$

or, equivalently, that $(xb \vee e)(b \vee e)^{-1}(yb \vee e) = xyb \vee e$. We start with the term on the left:

$$(xb \vee e)(b \vee e)^{-1}(yb \vee e) = (xb(b^{-1} \wedge e) \vee (b^{-1} \wedge e))(yb \vee e)$$
$$= xb(b^{-1} \wedge e)(yb \vee e) \vee (b^{-1} \wedge e)(yb \vee e)$$
$$= xb(b^{-1} \wedge e)yb \vee xb(b^{-1} \wedge e) \vee (b^{-1} \wedge e)yb \vee (b^{-1} \wedge e).$$

Now, we know that there exists an element $y_1 \in \overline{Y}$ such that $yb = by_1$ (since $\overline{Y}$ is closed under the action of B and the conjugation is an automorphism of $\overline{Y}$), hence the last term is equal to

$$xb(b^{-1} \wedge e)by_1 \vee xb(b^{-1} \wedge e) \vee (b^{-1} \wedge e)by_1 \vee (b^{-1} \wedge e)$$
$$= x(b \wedge b^2)y_1 \vee x(b \wedge e) \vee (b \wedge e)y_1 \vee (b^{-1} \wedge e).$$

Moreover, we observe that $(b \wedge e)y_1 = y_2(b \wedge e)$ for an appropriate element $y_2 \in \overline{Y}$, thus

$$x(b \wedge e) \vee (b \wedge e)y_1 = x(b \wedge e) \vee y_2(b \wedge e) = (x \vee y_2)(b \wedge e)$$
$$= (xy_2 \vee e)(b \wedge e) = xy_2(b \wedge e) \vee (b \wedge e)$$
$$= x(b \wedge e)y_1 \vee (b \wedge e)$$

(by Corollary 3.5 we know that $x \vee y_2 = xy_2 \vee e$, since $\overline{X} \perp \overline{Y}$). Then, one has

$$x(b \wedge b^2)y_1 \vee x(b \wedge e) \vee (b \wedge e)y_1 \vee (b^{-1} \wedge e)$$
$$= x(b \wedge b^2)y_1 \vee x(b \wedge e)y_1 \vee (b \wedge e) \vee (b^{-1} \wedge e).$$

We recall that $(b \wedge e) \vee (b^{-1} \wedge e) = (b \vee b^{-1}) \wedge e = |b| \wedge e = e$, so we finally get

$$x(b \wedge b^2)y_1 \vee x(b \wedge e)y_1 \vee (b \wedge e) \vee (b^{-1} \wedge e) = x(b \wedge b^2)y_1 \vee x(b \wedge e)y_1 \vee e$$
$$= x[(b \wedge b^2) \vee (b \wedge e)]y_1 \vee e$$
$$= xby_1 \vee e = xyb \vee e,$$

observing that $(b \wedge b^2) \vee (b \wedge e) = b(b \wedge e) \vee e(b \wedge e) = (b \vee e)(b \wedge e) = b^+b^- = b$.
To conclude, we have to prove that, for every subalgebra $\overline{X} \leq K$ closed under the action of B, then $\overline{X}^{\perp} \leq K$ is closed under the action of B (and so the centralizer of $X = \overline{X} \times B$ is $\overline{X}^{\perp} \times B$ endowed with the structure induced by the semi-direct product). Fix an element $y \in \overline{X}^{\perp}$ and an element $b \in B$; we show that $byb^{-1} \in \overline{X}^{\perp}$. We know that

$$byb^{-1} \in \overline{X}^{\perp} \iff |byb^{-1}| \wedge |x| = e \text{ for all } x \in \overline{X} \iff$$
$$|y| \wedge |b^{-1}xb| = e \text{ for all } x \in \overline{X} \iff |y| \wedge |x| = e \text{ for all } x \in \overline{X};$$

observe that the last assertion holds since $y \in \overline{X}^{\perp}$. We recall that $\overline{X}^{\perp}$ is a convex subalgebra of K. For every $y_1, y_2 \in \overline{X}^{\perp}$ and $b_1, b_2 \in B$ we have $y_1 b_1 \leq (y_1 \vee y_2)(b_1 \vee b_2)$ and $y_2 b_2 \leq (y_1 \vee y_2)(b_1 \vee b_2)$; we also observe that $(y_1 \wedge y_2) b_1 \leq y_1 b_1$ and $(y_1 \wedge y_2) b_2 \leq y_2 b_2$. So one has

$$y_1 b_1 \vee y_2 b_2 \leq (y_1 \vee y_2)(b_1 \vee b_2)$$

and

$$(y_1 \wedge y_2)(b_1 \vee b_2) = (y_1 \wedge y_2) b_1 \vee (y_1 \wedge y_2) b_2 \leq y_1 b_1 \vee y_2 b_2.$$

Therefore, for all $y_1, y_2 \in \overline{X}^{\perp}$ and $b_1, b_2 \in B$, we obtain

$$y_1 \wedge y_2 \leq (y_1 b_1 \vee y_2 b_2)(b_1 \vee b_2)^{-1} \leq y_1 \vee y_2;$$

then, since $\overline{X}^{\perp}$ is convex in K, we get $(y_1 b_1 \vee y_2 b_2)(b_1 \vee b_2)^{-1} \in \overline{X}^{\perp}$ for all $y_1, y_2 \in \overline{X}^{\perp}$ and $b_1, b_2 \in B$. $\blacksquare$

7.4. COROLLARY. *For every object B of $\ell\mathbb{G}rp$, in the category $Pt_B\ell\mathbb{G}rp$ subobjects have centralizers, therefore $Pt_B\ell\mathbb{G}rp$ is algebraically cartesian closed. Hence, the category $\ell\mathbb{G}rp$ is fiber-wise algebraically cartesian closed.*

8. Normality of the Higgins Commutator

The aim of this section is to propose a further study regarding the properties of commutators in the category of lattice-ordered groups.

We recall a first notion of categorical commutator strongly linked to the concept of cooperation.

8.1. DEFINITION. [Huq, 1968], [Bourn, 2002] *Let $\mathbb{C}$ be a unital category. For a pair of subobjects $a\colon A \rightarrowtail X$ and $b\colon B \rightarrowtail X$ of an object X in $\mathbb{C}$, the* Huq commutator *is the smallest normal subobject $[A, B]_X \rightarrowtail X$ such that the images of a and b cooperate in the quotient $X/[A, B]_X$.*

Next we recall here the notion of *Higgins commutator*. In a pointed category $\mathbb{C}$ with binary products and coproducts, for every pair of objects H and K, we have the following canonical arrows:

$$H \xrightarrow{(id_H, 0)} H \times K \xleftarrow{(0, id_K)} K$$

$$H \xleftarrow{[id_H, 0]} H + K \xrightarrow{[0, id_K]} K;$$

combining them we get a canonical arrow

$$\Sigma = \begin{pmatrix} id_H & 0 \\ 0 & id_K \end{pmatrix} \colon H + K \to H \times K.$$

In other words, Σ is the unique morphism making either the diagram

$$H \xrightarrow{\iota_K} H + K \xleftarrow{\iota_H} K$$

with $(id_H, 0)$, Σ, $(0, id_K)$ to $H \times K$

or equivalently the diagram

$$H + K$$

with $[id_H, 0]$, Σ, $[0, id_K]$ to $H \xleftarrow{\pi_H} H \times K \xrightarrow{\pi_K} K$

commute. For instance, in the case of the variety of groups, the morphism Σ associates to each word $h_1 k_1 h_2 k_2 \dots h_n k_n$, where $h_i \in H$ and $k_i \in K$ for $i = 1, \dots, n$, the pair $(h_1 h_2 \dots h_n, k_1 k_2 \dots k_n) \in H \times K$. It is easy to see that a category with binary products and coproducts is unital if and only if, for every pair of objects H and K, Σ is a strong epimorphism (see e.g. [Borceux and Bourn, 2004]). Hence, again in the case of groups, the kernel of Σ, denoted by $H \diamond K$ and called the *cosmash product* of H and K, can be described as the subgroup of $H + K$ generated by the elements of the form $hkh^{-1}k^{-1}$ with $h \in H$ and $k \in K$. In the light of the above, we are ready to recall the following:

8.2. DEFINITION. [Mantovani and Metere, 2010] *Let $\mathbb{C}$ be a semi-abelian category. Given a pair of subobjects $a\colon A \rightarrowtail X$ and $b\colon B \rightarrowtail X$ of an object X in $\mathbb{C}$, the Higgins commutator of A and B is the subobject $[A, B] \rightarrowtail X$ constructed, via the (regular epimorphism, monomorphism)-factorization, as in diagram*

$$A \diamond B \xrightarrow{k_{A,B}} A + B$$

with vertical maps down to $[A, B] \rightarrowtail X$, and right vertical map $[a, b]$,

where $k_{A,B}$ is the kernel of $\Sigma = \begin{pmatrix} id_A & 0 \\ 0 & id_B \end{pmatrix}\colon A + B \to A \times B$.

In general, the Higgins commutator of two normal subobjects is not normal. Therefore, it makes sense to mention the following definition:

8.3. DEFINITION. [Cigoli et al., 2015b] *A semi-abelian category $\mathbb{C}$ satisfies the condition of normality of Higgins commutators (NH) when, for every pair of normal subobjects $H \rightarrowtail X$, $K \rightarrowtail X$ where X is an object of $\mathbb{C}$, the Higgins commutator $[H, K] \rightarrowtail X$ is a normal subobject of X.*

We have everything we need to prove that that the category of lattice-ordered groups satisfies (NH).

8.4. LEMMA. *Let $H, K \leq X$ be two convex subalgebras of X in $\ell\mathbb{G}rp$. Then H and K cooperate if and only if $H \cap K = \{e\}$.*

PROOF. ($\Rightarrow$) Trivial since $H \perp K$ (thanks to Proposition 3.4).
($\Leftarrow$) We want to show that $H \perp K$: let us consider two elements $h \in H$ and $k \in K$; then $e \leq |h| \wedge |k| \leq |h|$ and $e \leq |h| \wedge |k| \leq |k|$. Therefore, since H and K are convex, we have $|h| \wedge |k| \in H \cap K = \{e\}$. ∎

8.5. NOTATION. *We will write $[H, K]$ for the Higgins commutator of $H \rightarrowtail X$ and $K \rightarrowtail X$, and $[H, K]_Y$ for the Huq commutator of $H \rightarrowtail X$ and $K \rightarrowtail X$ in the subobject Y of X, where H and K are subobjects of Y.*

8.6. PROPOSITION. *Let X be an object of $\ell\mathbb{G}rp$ and H, K ideals of X. Then, $[H, K]_X = H \cap K$.*

PROOF. Let us prove the inclusion $H \cap K \subseteq [H, K]_X$. Consider the following diagram:
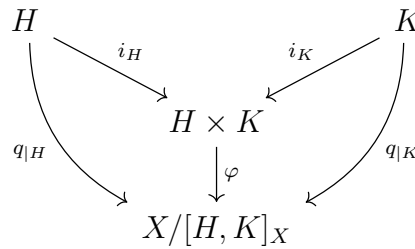
where $q \colon X \twoheadrightarrow X/[H, K]_X$ is the canonical projection. Then, by Lemma 8.4, we know that $q(H) \cap q(K) = \{e\}$. So, since $q(H \cap K) \subseteq q(H) \cap q(K) = \{e\}$ we get $H \cap K \subseteq [H, K]_X$. The other inclusion holds in every semi-abelian category (see Theorem 3.9 in [Everaert and Van der Linden, 2012]). ∎

8.7. PROPOSITION. *The category $\ell\mathbb{G}rp$ satisfies (NH).*

PROOF. Thanks to Theorem 2.8 of [Cigoli et al., 2015b], it suffices to prove that, given an ideal H of X and an ideal K of Y such that $H, K \leq Y$, then $[H, K]_X = [H, K]_Y$. Thus the statement follows from the previous proposition, since $[H, K]_X = [H, K]_Y = H \cap K$. ∎

9. Algebraic Coherence for $\ell\mathbb{A}b$

In the last part of the paper we focus on the notion of *algebraically coherent category*. This concept has an important algebraic meaning: an algebraically coherent category satisfies a large set of properties related to the good behaviour of commutators (such as, for example, strong protomodularity); moreover, in the case of a variety of universal algebras, the property of being fiber-wise algebraically cartesian closed is implied by algebraic coherence.

9.1. DEFINITION. [Cigoli et al., 2015a] *A category* $\mathbb{C}$ *with finite limits is* algebraically coherent *if, for every morphism* $f\colon X \to Y$ *in* $\mathbb{C}$*, the change-of-base functor*

$$f^*\colon Pt_Y\mathbb{C} \to Pt_X\mathbb{C}$$

is coherent*: a functor between categories with finite limits is* coherent *if it preserves finite limits and jointly extremally epimorphic pairs.*

Since in the semi-abelian case the split extensions with fixed splitting can be totally described in terms of semi-direct products, the authors in [Cigoli et al., 2015a] proved the following result:

9.2. PROPOSITION. [Cigoli et al., 2015a], Theorem 3.21 *Suppose* $\mathbb{C}$ *is a semi-abelian category. The following are equivalent:*

- $\mathbb{C}$ *is algebraically coherent;*

- *given* $K \rightarrowtail X$ *and* $H \rightarrowtail X$ *in* $\mathbb{C}$*, any action* $\xi\colon B\flat X \to X$ *which restricts to* K *and* H *also restricts to* $K \vee H$*.*

Let us now try to understand how this result can be interpreted in the category of lattice-ordered groups. We know (see [Bourn and Janelidze, 1998]) that, in the semi-abelian case, for each internal action $\xi\colon B\flat X \to X$ there exists a unique (up to isomorphism) split extension

$$X \xrightarrow{\ k\ } A \underset{p}{\overset{s}{\longleftrightarrow}} B$$

making, in the following diagram, the right-rightward square, the right-leftward square and the left square commute:

$$
\begin{array}{ccccc}
B\flat X & \xrightarrow{\ k_0\ } & X + B & \underset{[0,id_B]}{\overset{\iota_B}{\longleftrightarrow}} & B \\
{\scriptstyle \xi}\downarrow & & {\scriptstyle [k,s]}\downarrow & & \| \\
X & \xrightarrow[\ k\ ]{} & A & \underset{p}{\overset{s}{\longleftrightarrow}} & B.
\end{array}
$$

Therefore, in $\ell\mathbb{G}rp$, ξ restricts to a subalgebra $L \leq X$ if and only if L is closed under the corresponding action of B (in the sense of Definition 7.2).

In the next proposition we will deal with the category of *lattice-ordered abelian groups*. A lattice-ordered abelian group is a lattice-ordered group in which the group operation is commutative. The category $\ell\mathbb{A}b$ is the full subcategory of $\ell\mathbb{G}rp$ whose objects are lattice-ordered abelian groups.

9.3. PROPOSITION. $\ell\mathbb{A}b$ *is algebraically coherent.*

PROOF. Let us consider an object (A, p, s) of $Pt_B \ell \mathbb{A} b$ i.e. a diagram of the form

$$X \xrightarrow{\ k\ } A \underset{p}{\overset{s}{\longleftarrow}} B$$

where $ps = id_B$ and $k = \ker(p)$. For simplicity, let us suppose that k is the inclusion of $X \leq A$ and s is the inclusion of $B \leq A$. Given two subalgebras $K, H \leq X$ closed under the action of B, we want to show that also $K \vee H$ is closed under the action of B. First of all, let us observe that, given a subalgebra $L \leq X$, the following equality holds for every $l_1, l_2 \in L$ and $b_1, b_2 \in B$:

$$(l_1 b_1 \vee l_2 b_2)(b_1 \vee b_2)^{-1} = l_2(l_2^{-1} l_1 b_1 b_2^{-1} \vee e)(b_1 b_2^{-1} \vee e)^{-1}.$$

Therefore, L is closed under the action of B if and only if, for all $l \in L$ and $b \in B$,

$$(lb \vee e)(b \vee e)^{-1} \text{ belongs to } L.$$

We recall, as proved in Section 2, that in a lattice-ordered abelian group A the equation

$$xy = (x \vee y)(x \wedge y)$$

holds for all $x, y \in A$. Finally, it is easy to see that every element of $K \vee H$ can be written as

$$\bigvee_{i \in I} \bigwedge_{j \in J} k_{i,j} h_{i,j}$$

where I, J are finite sets of indices and $k_{i,j} \in K, h_{i,j} \in H$ for all $i \in I, j \in J$. This statement can be proved by iteratively applying the distributive properties of the lattice operations, the distributivity property of the group product over the lattice operations, and the commutative property of the group product. Therefore, given an element $b \in B$, one has

$$\left(\left(\bigvee_{i \in I} \bigwedge_{j \in J} k_{i,j} h_{i,j}\right) b \vee e\right) = \left(\bigvee_{i \in I} \bigwedge_{j \in J} k_{i,j} h_{i,j} b\right) \vee e$$

$$= \bigvee_{i \in I} \left(\bigwedge_{j \in J} k_{i,j} h_{i,j} b \vee e\right) = \bigvee_{i \in I} \bigwedge_{j \in J} (k_{i,j} h_{i,j} b \vee e)$$

where the first equality holds thanks to the distributivity of the group operation over the lattice operations, the second thanks to the idempotence of the join, and the third thanks to the distributivity of the join over the meet. Therefore

$$\left(\left(\bigvee_{i \in I} \bigwedge_{j \in J} k_{i,j} h_{i,j}\right) b \vee e\right)(b \vee e)^{-1} = \bigvee_{i \in I} \bigwedge_{j \in J} (k_{i,j} h_{i,j} b \vee e)(b \vee e)^{-1}$$

and hence, in order to prove that if K and H are closed under the action of B then also $K \vee H$ is closed under the action of B, it suffices to prove that

$$(khb \vee e)(b \vee e)^{-1} \in K \vee H,$$

for every $k \in K$, $h \in H$ and $b \in B$. We need to take care of an intermediate step: we want to show that, for all $k \in K$, $h \in H$ and $b \in B$,

$$((k \vee h)b \vee (k \wedge h)^{-1})(b \vee e)^{-1} \text{ belongs to } K \vee H.$$

To do this we observe that

$$
\begin{aligned}
((k \vee h)b \vee (k \wedge h)^{-1})(b \vee e)^{-1} &= (kb \vee hb \vee k^{-1} \vee h^{-1})(b \vee e)^{-1} \\
&= (kb \vee k^{-1})(b \vee e)^{-1} \vee (hb \vee h^{-1})(b \vee e)^{-1} \\
&= k^{-1}(k^2 b \vee e)(b \vee e)^{-1} \vee h^{-1}(h^2 b \vee e)(b \vee e)^{-1}.
\end{aligned}
$$

So, since K is closed under the action of B, we have $(k^2 b \vee e)(b \vee e)^{-1} \in K$ and then we obtain $k^{-1}(k^2 b \vee e)(b \vee e)^{-1} \in K$; similarly $h^{-1}(h^2 b \vee e)(b \vee e)^{-1} \in H$. Therefore, taking the join of these two terms, we get $((k \vee h)b \vee (k \wedge h)^{-1})(b \vee e)^{-1} \in K \vee H$. To conclude, since $k \wedge h \in K \vee H$, the product

$$(k \wedge h)((k \vee h)b \vee (k \wedge h)^{-1})(b \vee e)^{-1} \text{ belongs to } K \vee H;$$

then, applying the distributivity property of the group product over the lattice join and recalling that $(k \wedge h)(k \vee h) = kh$, we deduce

$$(khb \vee e)(b \vee e)^{-1} \in H \vee K. \qquad \blacksquare$$

This conclusive result partially answers the Open Problem 6.28 presented in [Cigoli et al., 2015a]. In fact, the category $\ell\mathbb{A}b$ is algebraically coherent, as just shown, however it is not action accessible: the example provided in Proposition 6.4 exclusively involves lattice-ordered groups whose group operations are commutative.

References

Anderson, M. and Feil, T. (2012). *Lattice-ordered groups: an introduction*, volume 4. Springer Science & Business Media.

Barr, M. (1971). Exact categories. *Exact categories and categories of sheaves*, 236:1–120.

Birkhoff, G. (1942). Lattice-ordered groups. *Annals of Mathematics*, pages 298–331.

Borceux, F. and Bourn, D. (2004). *Mal'cev, protomodular, homological and semi-abelian categories*, volume 566. Springer Science & Business Media.

Borceux, F., Janelidze, G., and Kelly, G. M. (2005). On the representability of actions in a semi-abelian category. *Theory Appl. Categ*, 14(11):244–286.

Bourn, D. (1991). Normalization equivalence, kernel equivalence and affine categories. In *Category theory*, volume 1488, pages 43–62. Springer Lecture Notes in Mathematics.

Bourn, D. (1996). Mal'cev categories and fibration of pointed objects. *Applied categorical structures*, 4(2):307–327.

Bourn, D. (2000). Normal functors and strong protomodularity. *Theory and Applications of Categories*, 7(9):206–218.

Bourn, D. (2002). Intrinsic centrality and associated classifying properties. *Journal of Algebra*, 256(1):126–145.

Bourn, D. and Gran, M. (2002a). Centrality and connectors in maltsev categories. *Algebra Universalis*, 48(3):309–331.

Bourn, D. and Gran, M. (2002b). Centrality and normality in protomodular categories. *Theory Appl. Categ*, 9(8):151–165.

Bourn, D. and Gray, J. R. A. (2012). Aspects of algebraic exponentiation. *Bulletin of the Belgian Mathematical Society-Simon Stevin*, 19(5):821–844.

Bourn, D. and Janelidze, G. (1998). Protomodularity, descent, and semidirect products. *Theory Appl. Categ*, 4(2):37–46.

Bourn, D. and Janelidze, G. (2003). Characterization of protomodular varieties of universal algebras. *Theory and Applications of categories*, 11(6):143–447.

Bourn, D. and Janelidze, G. (2009). Centralizers in action accessible categories. *Cahiers de topologie et géométrie différentielle catégoriques*, 50(3):211–232.

Carboni, A., Lambek, J., and Pedicchio, M. C. (1991). Diagram chasing in Mal'cev categories. *Journal of Pure and Applied Algebra*, 69(3):271–284.

Cignoli, R. L., d'Ottaviano, I. M., and Mundici, D. (2013). *Algebraic foundations of many-valued reasoning*, volume 7. Springer Science & Business Media.

Cigoli, A. S., Gray, J. R. A., and Van der Linden, T. (2015a). Algebraically coherent categories. *Theory and Applications of Categories*, 30(54):1864–1905.

Cigoli, A. S., Gray, J. R. A., and Van der Linden, T. (2015b). On the normality of Higgins commutators. *Journal of Pure and Applied Algebra*, 219(4):897–912.

Clementino, M. M., Montoli, A., and Sousa, L. (2015). Semidirect products of (topological) semi-abelian algebras. *Journal of Pure and Applied Algebra*, 219(1):183–197.

Everaert, T. and Van der Linden, T. (2012). Relative commutator theory in semi-abelian categories. *Journal of Pure and Applied Algebra*, 216(8-9):1791–1806.

Huq, S. A. (1968). Commutator, nilpotency and solvability in categories. *Q. J. Math.*, 19(2):363–389.

Janelidze, G., Márki, L., and Tholen, W. (2002). Semi-abelian categories. *Journal of Pure and Applied Algebra*, 168(2-3):367–386.

Kopytov, V. and Medvedev, N. (2013). *The theory of lattice-ordered groups*, volume 307. Springer Science & Business Media.

Mantovani, S. and Metere, G. (2010). Normalities and commutators. *Journal of Algebra*, 324(9):2568–2588.

Montoli, A. (2010). Action accessibility for categories of interest. *Theory Appl. Categ*, 23(1):7–21.

Mundici, D. (1986). Interpretation of AF C*-algebras in łukasiewicz sentential calculus. *Journal of Functional Analysis*, 65(1):15–63.

Orzech, G. (1972). Obstruction theory in algebraic categories, I. *Journal of Pure and Applied Algebra*, 2(4):287–314.

Pedicchio, M. C. (1995). A categorical approach to commutator theory. *Journal of Algebra*, 177(3):647–657.

Pedicchio, M. C. (1996). Arithmetical categories and commutator theory. *Applied Categorical Structures*, 4:297–305.

Smith, J. D. H. (2006). *Mal'cev varieties*, volume 554. Springer.

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