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# ERRATUM TO "THE MONOTONE-LIGHT FACTORIZATION FOR 2-CATEGORIES VIA 2-PREORDERS"

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Abstract. In this note we correct a proposition from the paper "The monotone-light factorization for 2-categories via 2-preorders". Moreover, we clarify what is meant by a 2-preorder generated by a 2-relation.

## 1. Introduction

In the article [3], we show that the reflection from the category of all 2-categories $2Cat$ into the category of all 2-preorders $2Preord$ determines a monotone-light factorization on $2Cat$. In order to do so, we have given a characterization of effective descent morphisms in $2Cat$ in Proposition 4.1 in [3]. However, we subsequently realized that this characterization is wrong. There is no real harm done, because no results in [3] depend upon the incorrect characterization.

In this erratum we give another statement for Proposition 4.1 in [3], where a certain class of 2-functors is said to be contained in the class of effective descent morphisms (also called monadic extensions in categorical Galois theory) in $2Cat$. It is open if this inclusion is proper or not.

There is also the need to adapt Example 4.2 in [3] to the true effective descent morphisms. No other change is required, so that all other results in [3] hold. Nevertheless, we did not resist to add another relevant comment in Remark 2.3, specifying what we mean when saying that a diagram *generates a 2-category*.

## 2. Effective descent morphisms in 2Cat

2.1. Proposition. *A 2-functor* $2p : 2\mathbf{E} \to 2\mathbf{B}$ *is an e.d.m. in the category of all 2-categories* $2Cat$ *if it is surjective both on*

- *vertically composable triples of horizontally composable pairs of 2-cells, and on*
- *horizontally composable triples of vertically composable pairs of 2-cells.*

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Proof. Please confer the following Example 2.2, for the exact meaning of the statement.

Let $2p : 2\mathbf{E} \to 2\mathbf{B}$ be surjective on vertically/horizontally composable triples of horizontally/vertically composable pairs of 2-cells. Then, $2p$ is an e.d.m. in $2\hat{\mathbf{P}} = Set^{2\mathbf{P}}$ (cf. section 2 in [3]), since the effective descent morphisms in a category of presheaves are simply those surjective pointwise (which, of course, is implied by either surjectivity on triples of composable pairs of 2-cells). Hence, the following instance of [1, Corollary 3.9] can be applied:

if $2p : 2\mathbf{E} \to 2\mathbf{B}$ in $2Cat$ is an e.d.m. in $2\hat{\mathbf{P}} = Set^{2\mathbf{P}}$ then $2p$ is an e.d.m. in $2Cat$ if and only if, for every pullback square

$$\begin{array}{ccc} 2\mathbf{D} & \longrightarrow & 2A \\ \downarrow & & \downarrow \\ 2\mathbf{E} & \xrightarrow{2p} & 2\mathbf{B} \end{array}$$

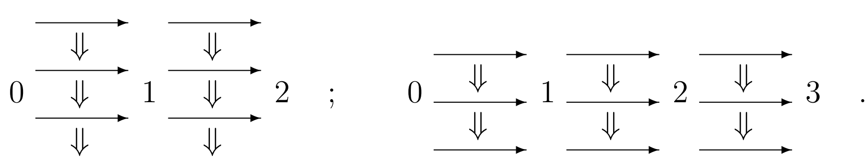
in $2\hat{\mathbf{P}} = Set^{2\mathbf{P}}$ such that $2\mathbf{D}$ is in $2Cat$, then also $2A$ is in $2Cat$.

Since the pullback square just above is calculated pointwise (cf. Corollary 3.3 in [3]), it induces six other pullback squares in $\hat{\mathbf{P}} = Set^{\mathbf{P}}$, corresponding to the three rows $P$, $hP$ and $hvP$, and the three columns $vhP$, $vP$ and $P_0$, in the 2-precategory diagram (2.1) in [3].

The fact that $2p$ is surjective on vertically/horizontally composable triples of horizontally/vertically composable pairs of 2-cells, implies that its six restrictions (to the six rows and columns $2\mathbf{E}(P)$, $2\mathbf{E}(hP)$, $2\mathbf{E}(hvP)$, $2\mathbf{E}(vhP)$, $2\mathbf{E}(vP)$ and $2\mathbf{E}(P_0)$) are surjective on triples of composable morphisms in $Cat$, as is easy to check. Hence, these six restrictions are effective descent morphisms in $Cat$. Therefore, $2A$ must always be a 2-category, provided so is $2\mathbf{D}$. ■

2.2. Example. It is obvious that the coproduct of 2-categories is just the disjoint union, as for categories.

Let $vh\mathbf{4}$ and $hv\mathbf{4}$ be the 2-categories generated by the following two diagrams, respectively:



Consider, for each 2-category $2\mathbf{B}$, the 2-category

$$2\mathbf{E} = (\coprod_{i\in I} vh\mathbf{4}) + (\coprod_{j\in J} hv\mathbf{4}),$$

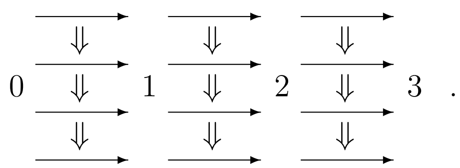
where $I$ is the set of all vertically composable triples of horizontally composable pairs of 2-cells in $2\mathbf{B}$, and $J$ is the set of all horizontally composable triples of vertically composable pairs of 2-cells in $2\mathbf{B}$.

Then, there is an e.d.m. $2p : 2\mathbf{E} \to 2\mathbf{B}$ which projects the corresponding copy of $vh\mathbf{4}$ and $hv\mathbf{4}$ to every $i \in I$ and every $j \in J$, respectively.

As another option, let

$$2\mathbf{E} = \coprod_{k \in I \cup J} hvh\mathbf{4},$$

with $hvh\mathbf{4}$ the 2-category generated by the following diagram,



Let us now specify, in the following remark, what we mean when saying that a diagram *generates a 2-category*.

2.3. Remark. The 2-categories $vh\mathbf{4}$, $hv\mathbf{4}$ and $hvh\mathbf{4}$ are really 2-preorders, that is, objects in the category $2Preord$, as defined at the beginning of section 5 in [3]. In fact, it will be shown in this remark that they are free 2-preorders generated by 2-relations.

Let's first recall the well known adjunction from graphs into categories $Grph \to Cat$, where $Grph$ is the presheaves category $Set^{\mathbf{G}}$ with $\mathbf{G}$ the subcategory of the category $\mathbf{P}$ determined by $P_0, P_1$, $d$ and $c$ (cf. the beginning of section 2 in [3] and Theorem 1 in [2][II.7]); more explicitly, $\mathbf{G}$ is the category generated by the diagram
$P_1 \underset{c}{\overset{d}{\rightrightarrows}} P_0$.

Consider also the subcategory $2\mathbf{G}$ of the category $2\mathbf{P}$, determined by $2P_1$, $P_1, P_0$, $vd$, $vc$, $d$ and $c$ (cf. section 2 in [3]); more explicitly, $2\mathbf{G}$ is the category generated by the diagram $2P_1 \underset{vc}{\overset{vd}{\rightrightarrows}} P_1 \underset{c}{\overset{d}{\rightrightarrows}} P_0$, in which $d \circ vc = d \circ vd$ and $c \circ vc = c \circ vd$.

The category of all 2-relations $2Rel$ is defined as the full subcategory of the presheaves category $Set^{2\mathbf{G}}$, determined by those presheaves $G : 2\mathbf{G} \to Set$ such that $Gvc$ and $Gvd$ are jointly monic. Hence, a 2-relation $R \in 2Rel$ may be described as a graph

$$R(P_1) \underset{Rc}{\overset{Rd}{\rightrightarrows}} R(P_0), \quad \text{plus a relation} \quad R(2P_1) \underset{Rvc}{\overset{Rvd}{\rightrightarrows}} R(P_1)$$

which only relates 1-cells (arrows) with the same initial and terminal 0-cell (node), since $Rd \circ Rvc = Rd \circ Rvd$ and $Rc \circ Rvc = Rc \circ Rvd$.

There is an adjunction from $2Rel$ into $2Preord$, where the right adjoint $U : 2Preord \to 2Rel$ is the forgetful functor, and the image of a 2-relation $R$ (for instance, any of the three diagrams in Example 2.2 just above) by the left adjoint $L : 2Rel \to 2Preord$ is built as follows:

- first, one adds the $1Cat = Cat$ structure to the 0 and 1-cells using the adjunction between graphs and categories, so that 0-cells in $L(R)$ are the same as in $R$, and

1-cells in $L(R)$ are the finite strings of 1-cells in $R$ as described in the proof of Theorem 1 in [2][II.7];

- then, the intersection of all those relations on such finite strings of 1-cells in $R$, which constitute a 2-preorder[1] and which include the original relation in $R$ on the 1-cells (now the strings with just one arrow), gives the new 2-preorder structure in $L(R)$ (notice that there exists one such structure, relating each pair of strings with the same initial and terminal 0-cells, so that the intersection makes sense).

It is easy to check that the inclusion of $R$ in $UL(R)$ is the unit morphism $\eta_R$ for the adjunction $L \dashv U$. If $h = (2h_1, h_1, h_0) : R \to U(A)$ is a morphism in $2Rel$,[2] with $A$ a 2-preorder, then there is a unique $h' = (2h'_1, h'_1, h'_0) : L(R) \to A$ such that $Uh' \circ \eta_R = h$. The uniqueness follows from:

- the existence of a unique morphism of categories $(h'_1, h'_0)$ corresponding to the morphism of graphs $(h_1, h_0)$, for the adjunction $Grph \to Cat$;[3] and

- the unique $2h'_1$ corresponding to $2h_1$, so that the restriction of $2h'_1$ to the 2-cells in $R$ is the same as $2h_1$.

## References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[1]Both a preorder vertically relating the 1-cells in $L(R)$, and horizontally relating the 0-cells.

[2]$2h_1$, $h_1$ and $h_0$ are respectively functions on 2-cells, 1-cells and 0-cells, the components of the morphism $h$ of presheaves.

[3]Therefore, $h'_0 = h_0$, and the restriction of $h'_1$ to the strings with one 1-cell (arrow) is the same as $h_1$.

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