

## ERRATUM TO “THE MONOTONE-LIGHT FACTORIZATION FOR 2-CATEGORIES VIA 2-PREORDERS”

JOÃO J. XAREZ

ABSTRACT. In this note we correct a proposition from the paper “The monotone-light factorization for 2-categories via 2-preorders”. Moreover, we clarify what is meant by a 2-preorder generated by a 2-relation.

### 1. Introduction

In the article [3], we show that the reflection from the category of all 2-categories  $2Cat$  into the category of all 2-preorders  $2Preord$  determines a monotone-light factorization on  $2Cat$ . In order to do so, we have given a characterization of effective descent morphisms in  $2Cat$  in Proposition 4.1 in [3]. However, we subsequently realized that this characterization is wrong. There is no real harm done, because no results in [3] depend upon the incorrect characterization.

In this erratum we give another statement for Proposition 4.1 in [3], where a certain class of 2-functors is said to be contained in the class of effective descent morphisms (also called monadic extensions in categorical Galois theory) in  $2Cat$ . It is open if this inclusion is proper or not.

There is also the need to adapt Example 4.2 in [3] to the true effective descent morphisms. No other change is required, so that all other results in [3] hold. Nevertheless, we did not resist to add another relevant comment in Remark 2.3, specifying what we mean when saying that a diagram *generates a 2-category*.

### 2. Effective descent morphisms in $2Cat$

2.1. PROPOSITION. *A 2-functor  $2p : 2\mathbf{E} \rightarrow 2\mathbf{B}$  is an e.d.m. in the category of all 2-categories  $2Cat$  if it is surjective both on*

- *vertically composable triples of horizontally composable pairs of 2-cells, and on*
- *horizontally composable triples of vertically composable pairs of 2-cells.*

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Received by the editors 2023-09-20 and, in final form, 2023-12-06.

Transmitted by Tim Van der Linden. Published on 2023-12-08.

2020 Mathematics Subject Classification: 18A32, 18E50, 18N10.

Key words and phrases: Monotone-light factorization, 2-categories.

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PROOF. Please confer the following Example 2.2, for the exact meaning of the statement.

Let  $2p : 2\mathbf{E} \rightarrow 2\mathbf{B}$  be surjective on vertically/horizontally composable triples of horizontally/vertically composable pairs of 2-cells. Then,  $2p$  is an e.d.m. in  $2\hat{\mathbf{P}} = \text{Set}^{2\mathbf{P}}$  (cf. section 2 in [3]), since the effective descent morphisms in a category of presheaves are simply those surjective pointwise (which, of course, is implied by either surjectivity on triples of composable pairs of 2-cells). Hence, the following instance of [1, Corollary 3.9] can be applied:

if  $2p : 2\mathbf{E} \rightarrow 2\mathbf{B}$  in  $2\text{Cat}$  is an e.d.m. in  $2\hat{\mathbf{P}} = \text{Set}^{2\mathbf{P}}$  then  $2p$  is an e.d.m. in  $2\text{Cat}$  if and only if, for every pullback square

$$\begin{array}{ccc}
 2\mathbf{D} & \longrightarrow & 2\mathbf{A} \\
 \downarrow & & \downarrow \\
 2\mathbf{E} & \xrightarrow{2p} & 2\mathbf{B}
 \end{array}$$

in  $2\hat{\mathbf{P}} = \text{Set}^{2\mathbf{P}}$  such that  $2\mathbf{D}$  is in  $2\text{Cat}$ , then also  $2\mathbf{A}$  is in  $2\text{Cat}$ .

Since the pullback square just above is calculated pointwise (cf. Corollary 3.3 in [3]), it induces six other pullback squares in  $\hat{\mathbf{P}} = \text{Set}^{\mathbf{P}}$ , corresponding to the three rows  $P$ ,  $hP$  and  $hvP$ , and the three columns  $vhP$ ,  $vP$  and  $P_0$ , in the 2-precategory diagram (2.1) in [3].

The fact that  $2p$  is surjective on vertically/horizontally composable triples of horizontally/vertically composable pairs of 2-cells, implies that its six restrictions (to the six rows and columns  $2\mathbf{E}(P)$ ,  $2\mathbf{E}(hP)$ ,  $2\mathbf{E}(hvP)$ ,  $2\mathbf{E}(vhP)$ ,  $2\mathbf{E}(vP)$  and  $2\mathbf{E}(P_0)$ ) are surjective on triples of composable morphisms in  $\text{Cat}$ , as is easy to check. Hence, these six restrictions are effective descent morphisms in  $\text{Cat}$ . Therefore,  $2\mathbf{A}$  must always be a 2-category, provided so is  $2\mathbf{D}$ . ■

2.2. EXAMPLE. It is obvious that the coproduct of 2-categories is just the disjoint union, as for categories.

Let  $vh4$  and  $hv4$  be the 2-categories generated by the following two diagrams, respectively:

$$\begin{array}{ccc}
 \longrightarrow & \longrightarrow & \\
 \Downarrow & \Downarrow & \\
 0 \longrightarrow & 1 \longrightarrow & 2 \\
 \Downarrow & \Downarrow & \\
 \longrightarrow & \longrightarrow & \\
 \Downarrow & \Downarrow &
 \end{array}
 \quad ; \quad
 \begin{array}{cccc}
 \longrightarrow & \longrightarrow & \longrightarrow & \\
 \Downarrow & \Downarrow & \Downarrow & \\
 0 \longrightarrow & 1 \longrightarrow & 2 \longrightarrow & 3 \\
 \Downarrow & \Downarrow & \Downarrow & \\
 \longrightarrow & \longrightarrow & \longrightarrow &
 \end{array}
 .$$

Consider, for each 2-category  $2\mathbf{B}$ , the 2-category

$$2\mathbf{E} = \left( \coprod_{i \in I} vh4 \right) + \left( \coprod_{j \in J} hv4 \right),$$

where  $I$  is the set of all vertically composable triples of horizontally composable pairs of 2-cells in  $2\mathbf{B}$ , and  $J$  is the set of all horizontally composable triples of vertically composable

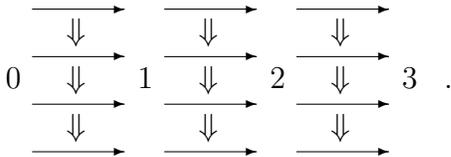
pairs of 2-cells in  $2\mathbf{B}$ .

Then, there is an e.d.m.  $2p : 2\mathbf{E} \rightarrow 2\mathbf{B}$  which projects the corresponding copy of  $vh4$  and  $hv4$  to every  $i \in I$  and every  $j \in J$ , respectively.

As another option, let

$$2\mathbf{E} = \coprod_{k \in I \cup J} hvh4,$$

with  $hvh4$  the 2-category generated by the following diagram,



Let us now specify, in the following remark, what we mean when saying that a diagram generates a 2-category.

**2.3. REMARK.** The 2-categories  $vh4$ ,  $hv4$  and  $hvh4$  are really 2-preorders, that is, objects in the category  $2Preord$ , as defined at the beginning of section 5 in [3]. In fact, it will be shown in this remark that they are free 2-preorders generated by 2-relations.

Let's first recall the well known adjunction from graphs into categories  $Grph \rightarrow Cat$ , where  $Grph$  is the presheaves category  $Set^{\mathbf{G}}$  with  $\mathbf{G}$  the subcategory of the category  $\mathbf{P}$  determined by  $P_0, P_1, d$  and  $c$  (cf. the beginning of section 2 in [3] and Theorem 1 in [2][II.7]); more explicitly,  $\mathbf{G}$  is the category generated by the diagram

$$P_1 \begin{array}{c} \xrightarrow{d} \\ \xrightarrow{c} \end{array} P_0.$$

Consider also the subcategory  $2\mathbf{G}$  of the category  $2\mathbf{P}$ , determined by  $2P_1, P_1, P_0, vd, vc, d$  and  $c$  (cf. section 2 in [3]); more explicitly,  $2\mathbf{G}$  is the category generated by the diagram  $2P_1 \begin{array}{c} \xrightarrow{vd} \\ \xrightarrow{vc} \end{array} P_1 \xrightarrow{d} P_0$ , in which  $d \circ vc = d \circ vd$  and  $c \circ vc = c \circ vd$ .

The category of all 2-relations  $2Rel$  is defined as the full subcategory of the presheaves category  $Set^{2\mathbf{G}}$ , determined by those presheaves  $G : 2\mathbf{G} \rightarrow Set$  such that  $Gvc$  and  $Gvd$  are jointly monic. Hence, a 2-relation  $R \in 2Rel$  may be described as a graph

$$R(P_1) \begin{array}{c} \xrightarrow{Rd} \\ \xrightarrow{Rc} \end{array} R(P_0), \quad \text{plus a relation} \quad R(2P_1) \begin{array}{c} \xrightarrow{Rvd} \\ \xrightarrow{Rvc} \end{array} R(P_1)$$

which only relates 1-cells (arrows) with the same initial and terminal 0-cell (node), since  $Rd \circ Rvc = Rd \circ Rvd$  and  $Rc \circ Rvc = Rc \circ Rvd$ .

There is an adjunction from  $2Rel$  into  $2Preord$ , where the right adjoint  $U : 2Preord \rightarrow 2Rel$  is the forgetful functor, and the image of a 2-relation  $R$  (for instance, any of the three diagrams in Example 2.2 just above) by the left adjoint  $L : 2Rel \rightarrow 2Preord$  is built as follows:

- first, one adds the  $1Cat = Cat$  structure to the 0 and 1-cells using the adjunction between graphs and categories, so that 0-cells in  $L(R)$  are the same as in  $R$ , and

1-cells in  $L(R)$  are the finite strings of 1-cells in  $R$  as described in the proof of Theorem 1 in [2][II.7];

- then, the intersection of all those relations on such finite strings of 1-cells in  $R$ , which constitute a 2-preorder<sup>1</sup> and which include the original relation in  $R$  on the 1-cells (now the strings with just one arrow), gives the new 2-preorder structure in  $L(R)$  (notice that there exists one such structure, relating each pair of strings with the same initial and terminal 0-cells, so that the intersection makes sense).

It is easy to check that the inclusion of  $R$  in  $UL(R)$  is the unit morphism  $\eta_R$  for the adjunction  $L \dashv U$ . If  $h = (2h_1, h_1, h_0) : R \rightarrow U(A)$  is a morphism in  $2Rel$ ,<sup>2</sup> with  $A$  a 2-preorder, then there is a unique  $h' = (2h'_1, h'_1, h'_0) : L(R) \rightarrow A$  such that  $Uh' \circ \eta_R = h$ . The uniqueness follows from:

- the existence of a unique morphism of categories  $(h'_1, h'_0)$  corresponding to the morphism of graphs  $(h_1, h_0)$ , for the adjunction  $Grph \rightarrow Cat$ ,<sup>3</sup> and
- the unique  $2h'_1$  corresponding to  $2h_1$ , so that the restriction of  $2h'_1$  to the 2-cells in  $R$  is the same as  $2h_1$ .

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*CIDMA - Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Portugal.*

Email: [xarez@ua.pt](mailto:xarez@ua.pt)

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<sup>1</sup>Both a preorder vertically relating the 1-cells in  $L(R)$ , and horizontally relating the 0-cells.

<sup>2</sup> $2h_1$ ,  $h_1$  and  $h_0$  are respectively functions on 2-cells, 1-cells and 0-cells, the components of the morphism  $h$  of presheaves.

<sup>3</sup>Therefore,  $h'_0 = h_0$ , and the restriction of  $h'_1$  to the strings with one 1-cell (arrow) is the same as  $h_1$ .

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Matias Menni, Conicet and Universidad Nacional de La Plata, Argentina: [matias.menni@gmail.com](mailto:matias.menni@gmail.com)

Giuseppe Metere, Università degli Studi di Palermo: [giuseppe.metere@unipa.it](mailto:giuseppe.metere@unipa.it)

Kate Ponto, University of Kentucky: [kate.ponto@uky.edu](mailto:kate.ponto@uky.edu)

Robert Rosebrugh, Mount Allison University: [rrosebrugh@mta.ca](mailto:rrosebrugh@mta.ca)

Jiri Rosický, Masaryk University: [rosicky@math.muni.cz](mailto:rosicky@math.muni.cz)

Giuseppe Rosolini, Università di Genova: [rosolini@unige.it](mailto:rosolini@unige.it)

Michael Shulman, University of San Diego: [shulman@sandiego.edu](mailto:shulman@sandiego.edu)

Alex Simpson, University of Ljubljana: [Alex.Simpson@fmf.uni-lj.si](mailto:Alex.Simpson@fmf.uni-lj.si)

James Stasheff, University of North Carolina: [jds@math.upenn.edu](mailto:jds@math.upenn.edu)

Tim Van der Linden, Université catholique de Louvain: [tim.vanderlinden@uclouvain.be](mailto:tim.vanderlinden@uclouvain.be)

Christina Vasilakopoulou, National Technical University of Athens: [cvasilak@math.ntua.gr](mailto:cvasilak@math.ntua.gr)