Theory and Applications of Categories, Vol. 39, No. 4, 2023, pp. 98–149.

ENHANCED TWISTED ARROW CATEGORIES

FERNANDO ABELLÁN GARCÍA AND WALKER H. STERN

Abstract. Given an ∞-bicategory $\mathbb{D}$ with underlying ∞-category $\mathcal{D}$, we construct a Cartesian fibration $\mathrm{Tw}(\mathbb{D}) \to \mathcal{D} \times \mathcal{D}^{\mathrm{op}}$, which we call the *enhanced twisted arrow* ∞-category, classifying the restricted mapping category functor $\mathrm{Map}_{\mathbb{D}} : \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{C}\mathrm{at}_\infty$. With the aid of this new construction, we provide a description of the ∞-category of natural transformations $\mathrm{Nat}(F, G)$ as an end for any functors F and G from an ∞-category to an ∞-bicategory. As an application of our results, we demonstrate that the definition of weighted colimits studied by Gepner-Haugseng-Nikolaus satisfies the expected 2-dimensional universal property.

Contents

Introduction

Of the many tools belonging to the study of categories, perhaps the most key is the Yoneda lemma. The fully faithfulness of the functor

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathrm{Set}_{\mathcal{C}} \\ x &\longmapsto h_x := \mathrm{Hom}_{\mathcal{C}}(-, x) \end{aligned}$$

means, in particular, that we can view functors $f : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ as *universal properties*, and thereby uniquely specify an object x by requiring $h_x \cong f$.

In the higher-categorical realm, the good news is that this result still holds. The $(\infty, 1)$-categorical Yoneda embedding

$$\mathcal{Y} : \mathcal{C} \to \mathcal{S}_{\mathcal{C}}$$

Received by the editors 2021-05-31 and, in final form, 2023-01-22.
Transmitted by Ieke Moerdijk. Published on 2023-01-26.
2020 Mathematics Subject Classification: 18N10, 18N60, 18N65.
Key words and phrases: $(\infty, 2)$-category, twisted arrow category, natural transformation, weighted colimit.

is fully faithful (c.f. e.g. [Lur09, 5.1.3.1]). While this is auspicious for the study of universal properties as described above, it comes with a significant complication. The standard presentation of the target category $\mathcal{S}_{\mathcal{C}}$ (which is also written variously as $\mathcal{P}(\mathcal{C})$ or $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$) is in terms of a model structure on the category $\mathrm{Fun}(\mathfrak{C}[\mathcal{C}], \mathrm{Set}_{\Delta})$ of simplicially enriched functors.

The model $\mathrm{Fun}(\mathfrak{C}[\mathcal{C}], \mathrm{Set}_{\Delta})$ is extremely useful in relating the underlying ∞-category to other ∞-categories — for example in the proof of the ∞-categorical Yoneda lemma. The problem arises in that it is often extremely difficult to write down explicit simplicially-enriched functors, and explicit simplicially-enriched natural transformations between them. When the initial definition of $\mathcal{C}$ is as a quasi-category, it can even be difficult to write down $\mathfrak{C}[\mathcal{C}]$ explicitly.

As in so many parts of higher category theory, the way out of this dilemma is the Grothendieck construction. We can proceed according to the

> Slogan: *Cartesian fibrations and maps between them are easier to work with than enriched functors and natural transformations between them.*

From this perspective, if we want to study representable functors and universal properties, we need first to classify the Yoneda embedding by a fibration.

The twisted arrow category. The canonical solution to this problem is the *twisted arrow (∞-)category.* In e.g. [Lur11] and [Cis19], it is shown that for each ∞-category $\mathcal{C}$, there is a right fibration[1]

$$\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$$

which classifies the functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Cat}_{\infty}$.

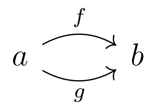
The uses of the twisted arrow category are manifold. It appears, as suggested above, in the analysis of questions of representability throughout the higher categorical literature — e.g. in [Lur11, Lur17]. In addition, it is used to explore $\mathbb{E}_k$-monoidal ∞-categories in [Lur17]. In a completely different direction, there is a fundamental connection between twisted arrow categories and ∞-categories of spans/correspondences as described in, e.g. [DK19, Ch. 10],[Bar17], and [BGN18]. Moreover, this approach has been used to tackle questions related to K-theory in [Bar17].

The 1-simplices of $\mathrm{Tw}(\mathcal{C})$ over a pair (α, β) comprising a 1-simplex in $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ take the form of coherent diagrams

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle\alpha}\downarrow & & \uparrow{\scriptstyle\beta} \\ a' & \xrightarrow[g]{} & b' \end{array}$$

[1] It is worth commenting that Cisinski and several other authors tend to work with the *left* fibration associated to the same functor. The difference between the two definitions amounts to an "op", in the definition of the simplices of $\mathrm{Tw}(\mathcal{C})$. Throughout the paper, we will only use the Cartesian/right fibration convention, and will omit any further mention of coCartesian/left fibrations.

in $\mathcal{C}$. In practice, this means that that the fibers have 1-simplices consisting of diagrams

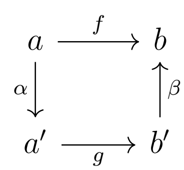


that commute up to a chosen 2-cell, i.e. the morphisms in the fiber can be easily interpreted as two-cells $f \overset{\sim}{\Longrightarrow} g$ in $\mathcal{C}$. More generally, the n-simplices of $\mathrm{Tw}(\mathcal{C})$ are given by maps $\Delta^n \star (\Delta^n)^{\mathrm{op}} \to \mathcal{C}$, and the projections to $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are induced by the inclusions $\Delta^n \longrightarrow \Delta^n \star (\Delta^n)^{\mathrm{op}} \longleftarrow (\Delta^n)^{\mathrm{op}}$.

Towards an enhanced twisted arrow category. Given an ∞-bicategory $\mathbb{C}$ with underlying $(\infty,1)$-category $\mathcal{C}$, presented as a fibrant scaled simplicial set, our aim will be to construct an ∞-category $\mathrm{Tw}(\mathbb{C})$ together with a Cartesian fibration $\mathrm{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ which classifies the composite functor

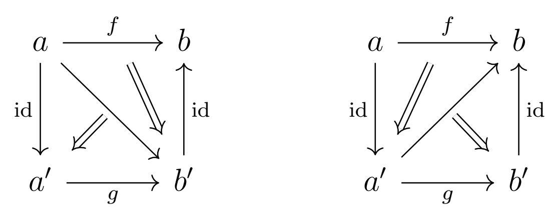
$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathfrak{Cat}_\infty,$$

which sends a pair of objects to the mapping $(\infty,1)$-category between them. Here $\mathfrak{Cat}_\infty$ is the $(\infty,2)$-category of ∞-categories. The first step towards this construction is to decide what the 1-simplices of $\mathrm{Tw}(\mathbb{C})$ should be. We would still like these to be something like diagrams

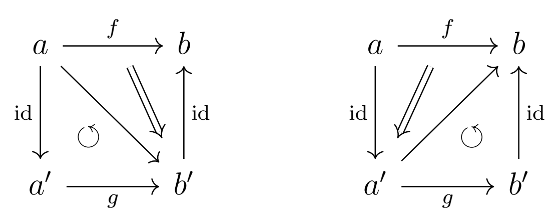


in $\mathbb{C}$, e.g. 3-simplices.

When α and β are identities, we would like these 3-simplices to encode precisely the choice of a 2-morphism $f \Longrightarrow g$. However, heuristically such a 3-simplex should, in fact, encode two factorizations:



together with the 3-simplex itself, which indicates that the composites — 2-morphisms $f \Longrightarrow g$ — of both factorizations are equivalent. Fortunately, in the realm of scaled simplicial sets, we can declare certain 2-simplices to be 'thin' — i.e., declare the corresponding 2-morphisms to be invertible. With this in mind, we can force half of each factorization to be invertible



In this case, we obtain two 2-morphisms $f \Longrightarrow g$ and a 3-simplex showing that they are equivalent — precisely the data that we would like.[2]

This suggests a trial definition for the twisted arrow category of an ∞-bicategory.

> The twisted arrow ∞-bicategory $\mathbb{T}\mathrm{w}(\mathbb{C})$ should have n-simplices
>
> $$\mathbb{T}\mathrm{w}(\mathbb{C})_n := \mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathrm{sc}}}((\Delta^n \star (\Delta^n)^{\mathrm{op}}, T), \mathbb{C})$$
>
> where T is the scaling given by requiring that, under the identification $\Delta^n \star (\Delta^n)^{\mathrm{op}} \cong \Delta^{2n+1}$, the simplices $\{i, j, 2n+1-j\}$ and $\{j, 2n+1-j, 2n+1-i\}$ are thin for $i < j$.

We would expect such a construction to yield a fibration over the $(\infty,2)$-category $\mathbb{C} \times \mathbb{C}^{\mathrm{op}}$, but it turns out that such an $(\infty,2)$-categorical fibration requires a more involved construction (see the next section for more details). There are also some technical difficulties to such a definition.

When this paper was first completed, a Grothendieck construction for $(\infty,2)$-categories fibred in $(\infty,1)$-categories which had the correct variance for our purposes had not yet appeared, though one of the four possible variances had been treated in [Lur09a]. Since then, Gagna, Harpaz, and Lanari used the Grothendieck construction appearing in [Lur09a] to prove that the desired equivalences hold for all four variances, in [GHL20]. However, their construction proceeds by applying various dualization functors on the category of marked simplicial set-enriched categories, and thus does not provide a computationally tractable version of the Grothendieck construction for other variances.

In the interim since this paper's first submission, the authors of the present paper have provided an explicit, computationally-minded version of the Grothendieck construction for ∞-bicategories fibred in ∞-bicategories, in [AGS22].

We expect that a substantially more technically involved variant of the arguments given here will provide a "full" ∞-bicategorical twisted arrow category construction. However, we defer such considerations to a later work, as the arguments here are sufficient to prove several noteworthy corollaries, as discussed below. While we expect a modification of the definition above to yield a genuine $(\infty,2)$ twisted arrow category, we will here restrict ourselves to the examination of the induced functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Cat}_\infty$

To restrict to the fibration classifying this functor, we use the base-change properties of the $(\infty,2)$-categorical Grothendieck construction from [AGS22]. To wit, we define $\mathrm{Tw}(\mathbb{C})$ to be the pullback

$$\begin{array}{ccc}
\mathrm{Tw}(\mathbb{C}) & \longrightarrow & \mathbb{T}\mathrm{w}(\mathbb{C}) \\
\big\downarrow & \lrcorner & \big\downarrow \\
\mathcal{C} \times \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathbb{C} \times \mathbb{C}^{\mathrm{op}}
\end{array}$$

[2]One thing we are glossing over is why we choose the "lower" 2-simplices as thin, rather than the "upper" ones. In a nutshell, the reason is that the lower 2-simplices will encode composites, and thus be unique up to contractible choice.

In terms of the scaling on $\Delta^n \star (\Delta^n)^{\mathrm{op}}$, This pullback simply amounts to requiring that every 2-simplex contained within Δ^n and every 2-simplex contained within $(\Delta^n)^{\mathrm{op}}$ is thin. Using pushouts by scaled anodyne morphisms of the kind described in [GHL19, Rmk. 1.18], we can extend this scaling to consider a cosimplicial object $Q(n) := (\Delta^n \star (\Delta^n)^{\mathrm{op}}, T)$ in scaled simplicial sets, where the non-degenerate thin simplices of T are:

- 2-simplices which factor through Δ^n or $(\Delta^n)^{\mathrm{op}}$.
- 2-simplices $\Delta^{\{i,j,2n+1-k\}}$ and $\Delta^{\{k,2n+1-j,2n+1-i\}}$ for $0 \leqslant i \leqslant j \leqslant k \leqslant n$.

This is the definition of $\mathrm{Tw}(\mathbb{C})$ we adopt throughout the present paper, which is justified by the following result.

0.1. Theorem. *Let $\mathbb{C}$ be an ∞-bicategory. Then $\mathrm{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ is a Cartesian fibration classifying the restricted mapping category functor*

$$\mathrm{Map}_{\mathbb{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathfrak{C}\mathrm{at}_{\infty}$$

This is an amalgam of 2.6 and 3.4 from the text.

Restricted functoriality. While our twisted arrow construction is wholly sufficient for our desired application to natural transformations, there is a fly in the ointment: the construction presented here does not encode the functoriality of $\mathbb{C}(x, y)$ in the 2-cells of $\mathbb{C}$. There are several reasons we restrict the functoriality in this fashion. Firstly, the considerations of Grothendieck constructions discussed above posed significant technical challenges until the completion of our recent paper [AGS22].

The more important reason to restrict the functoriality, though, is that the construction described above is slightly too naïve. A full ∞-bicategorical twisted arrow category construction needs to be based on a more sophisticated cosimplicial object than $\Delta^n \star (\Delta^n)^{\mathrm{op}}$.

As a heuristic for why a more complex construction is necessary, let us consider an ∞-bicategory $\mathbb{C}$, and the mapping ∞-category functor

$$\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \longrightarrow \mathfrak{C}\mathrm{at}_{\infty}\,.$$

Under an ∞-bicategorical Grothendieck construction, this corresponds to a fibration

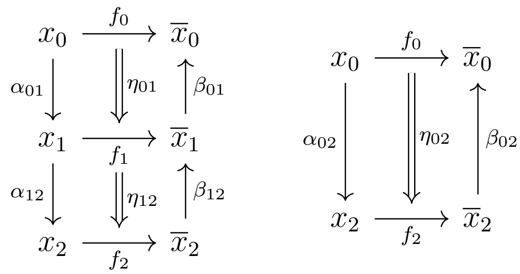
$$\mathbb{E} \longrightarrow \mathbb{C} \times \mathbb{C}^{\mathrm{op}}.$$

Unwinding the definitions, we find that, heuristically, a 1-simplex of $\mathbb{E}$ should consist of a diagram

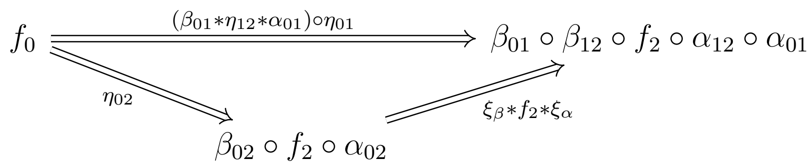
$$\begin{array}{ccc} x_0 & \xrightarrow{f_0} & \overline{x}_0 \\ {\scriptstyle \alpha_{01}}\downarrow & & \uparrow{\scriptstyle \beta_{01}} \\ x_1 & \xrightarrow[f_1]{} & \overline{x}_1 \end{array}$$

in $\mathbb{C}$ filled by a 2-morphism $\eta_{0,1} : f_0 \Rightarrow \beta_{01} \circ f_1 \circ \alpha_{01}$. Following the strategy described above, we might then encode this data as a functor $\Delta^1 \star (\Delta^1)^{\mathrm{op}} \to \mathbb{C}$, such that the triangles $0 \to 1 \to \overline{1}$ and $1 \to \overline{1} \to \overline{0}$ are sent to thin triangles.

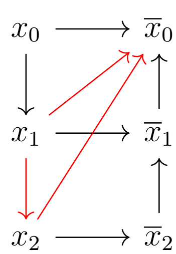
While this is all well and good, a problem arises as soon as we consider 2-simplices. In principle, these should be diagrams



together with 2-morphisms $\xi_\alpha : \alpha_{02} \Rightarrow \alpha_{12} \circ \alpha_{01}$ and $\xi_\beta : \beta_{02} \Rightarrow \beta_{01} \circ \beta_{12}$, such that the diagram



However, an issue arises when we try to represent this as a map $\Delta^2 \star (\Delta^2)^{\mathrm{op}} \to \mathbb{C}$. If we consider the red 2-simplex highlighted below



Morally, this should represent the 2-morphism

$$\beta_{01} \star \eta_{12} : \beta_{01} \circ f_1 \Longrightarrow \beta_{01} \circ \beta_{12} \circ f_2 \circ \alpha_{12}.$$

However, the necessary scalings inherited from the 1-morphisms mean that, in fact, this triangle will represent a 2-morphism

$$\beta_{01} \circ f_1 \Longrightarrow \beta_{02} \circ f_2 \circ \alpha_{12}$$

which is not part of the expected data coming from the Grothendieck construction.

This heuristic serves to illustrate one of the difficulties inherent in defining a full $(\infty, 2)$-categorical twisted arrow category. The simplicial set $\Delta^n \star (\Delta^n)^{\mathrm{op}}$ is too simple to encode the necessary data accurately. We expect a variant on the construction presented here, using more complicated cosimplicial objects in place of $\Delta^n \star (\Delta^n)^{\mathrm{op}}$, to provide an $(\infty, 2)$-categorical twisted arrow category, and this is a topic of ongoing work.

The fact that it is necessary to consider more complicated cosimplicial objects serves as a second motivation for considering the restricted functoriality encoded in the construction presented in this paper. Since the restricted version considered here is sufficient for many applications, and the full $(\infty, 2)$-categorical version is likely to be substantially more technically difficult to define and work with, it is advantageous to have the simpler version available, for those applications which do not require an examination of the 2-morphism functoriality.

Applications: The category of natural transformations as an end. Once verified that our definition enjoys the desired properties we turn into our main motivation for this paper: understanding the category of natural transformations $\mathrm{Nat}(F, G)$ between functors from an ∞-category to an ∞-bicategory. To do so, we obtain that expected description of the category of natural transformations as an end.

0.2. Theorem. *Let $\mathcal{C}$ be a ∞-category and $\mathbb{D}$ an ∞-bicategory. Then for every pair of functors $F, G : \mathcal{C} \to \mathbb{D}$ there exists a equivalence of ∞-categories*

$$\mathrm{Nat}_{\mathcal{C}}(F, G) \to \lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Map}_{\mathbb{D}}(F(-), G(-))$$

which is natural in each variable.

This result allows us to analyze in greater detail the theory of weighted colimits of $\mathcal{C}\mathrm{at}_\infty$-valued functors exposed in [GHN15], showing that this definition coincides with the definition provided by the first author in [AG20]. The proof of this fact together with the results of [AG20] constitute a partial answer to a series of conjectures involving ∞-bicategorical colimits and a categorified theory of cofinality introduced by the authors in [AGS20].

Structure of the paper. The paper will be laid out as follows. We begin with a preliminary section, which lays out the notational conventions we follow, and explains several technical constructions and lemmata which we use throughout the paper. In particular, we give basic definitions for cosimplicial objects, state and prove a general lemma on subsets $K \subset \Delta^n_\dagger$ of a scaled n-simplex such that $K \to \Delta^n_\dagger$ is scaled anodyne, and define a structure on a poset sufficient for us to give a clean description of the simplicial mapping spaces in a quotient of its nerve.

From there, the work starts in earnest. In 2, we give the formal definition of $\mathrm{Tw}(\mathbb{C})$, and prove that $\mathrm{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ is a Cartesian fibration, making use of the aforementioned lemma on simplicial subsets of scaled n-simplices. We then turn to 3, in which we prove that this Cartesian fibration classifies precisely the enhanced mapping functor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathfrak{C}\mathrm{at}_\infty .$$

This proof is highly technical, and freely uses results from [Lur09a] and [GHL19].

In 4, our attention then turns to the true aim of the paper, a proof of the proposition that, given two functors $F, G : \mathcal{C} \to \mathbb{D}$ from an ∞-category to an $(\infty, 2)$-category, the ∞-category of natural transformations between them can be expressed as a limit

$$\mathrm{Nat}(F, G) \simeq \lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Map}_{\mathbb{D}}(F(-), G(-)),$$

i.e., an end. Once again the proof is highly technical, making use of a wide variety of techniques native to the contexts of scaled simplicial sets and marked simplicial sets. In particular, the proof relies heavily on a sort of dévissage — one in which we reduce from the case of a general ∞-category (indeed, simplicial set) $\mathcal{C}$ to the cases $\mathcal{C} = \Delta^0$ and $\mathcal{C} = \Delta^1$.

We conclude with applications of this theorem, where we upgrade several results appearing in [GHN15].

Acknowledgments. We are grateful to an anonymous referee for the very careful reading which helped improve the article.

F.A.G. would like to acknowledge the support of the VolkswagenStiftung through the Lichtenberg Professorship Programme while he conducted this research. W.H.S was supported by Universität Hamburg during the early stages of this work, and by the NSF Research Training Group at the University of Virginia (grant number DMS-1839968) during the later stages.

1. Preliminaries

We begin by presenting some background information necessary for the paper, and proving some general lemmata which will help simplify the technical arguments in later sections. We will not, in general, recapitulate material from [Lur09] and [Lur09a], as doing so would greatly extend the length of the present document for dubious benefit. In particular, we will assume that the reader is familiar with the theories of quasi-categories, Cartesian fibrations, and scaled simplicial sets, as well as the attendant model structures. We will, however, briefly collect the notations and conventions we will use for these before embarking on the preliminaries proper.

1.1. Notation. [Model categories] We denote by Set_Δ the category of simplicial sets, Set_Δ^+ the category of marked simplicial sets, and $\mathrm{Set}_\Delta^{\mathrm{sc}}$ the category of scaled simplicial sets. We consider these to be equipped with the Joyal, Cartesian, and bicategorical model structures, respectively. Where context clarifies the meaning, an unadorned Latin capital — e.g. X — may be used to denote an object of any of these categories. When it is necessary to specify a marking or a scaling on $X \in \mathrm{Set}_\Delta$, we do so by writing a superscript — e.g. $X^\dagger$ — for a marking, and a subscript — e.g. $X_\dagger$ — for a scaling. In particular, the subscripts $\sharp$ and $\flat$ will denote the maximal and minimal scalings, respectively.

1.2. Notation. [Rigidification] We denote by Cat_Δ the category of simplicial set enriched categories, and by $\mathrm{Cat}_{\mathrm{Set}_\Delta^+}$ the category of marked simplicial set enriched categories. We denote by $\mathfrak{C} : \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$ the rigidification functor, and by $\mathfrak{C}^{\mathrm{sc}} : \mathrm{Set}_\Delta^{\mathrm{sc}} \to \mathrm{Cat}_{\mathrm{Set}_\Delta^+}$ its scaled variant. In the presence of the sub- and superscript convention above, we will conventionally denote

$$\mathfrak{C}[X](x,y)^\dagger := \mathfrak{C}^{\mathrm{sc}}[X_\dagger](x,y)$$

for any $x, y \in X$.

1.3. Notation. [Nerves] We will, in general denote both the nerve of 1-categories and the homotopy-coherent nerve of simplicially enriched categories by N. Since we can consider Cat as a full subcategory of Cat_Δ, and these two nerves agree on 1-categories, this notational convention is unambiguous.

1.4. Convention. [Fibrant objects] By an ∞-category, we will mean an $(\infty,1)$-category, presented as either a quasi-category or a fibrant marked simplicial set. We will, wherever possible, use calligraphic capitals — e.g. $\mathcal{C}$ — for ∞-categories.

By an ∞-bicategory, we will mean an $(\infty,2)$-category presented as a fibrant scaled simplicial set.[3] Where possible, we will denote ∞-bicategories by blackboard-bold capitals — e.g. $\mathbb{C}$.

Cosimplicial objects.

1.5. Definition. *Let C be an ordinary 1-category. A functor $F : \Delta \to C$ will be called a cosimplicial object in C.*

1.6. Notation. Given $[n] \in \Delta$ we will denote its image under F by $F(n)$.

In the following sections, we will make extensive use of cosimplicial objects with target a cocomplete category C. Namely, those that can be "freely extendend" by colimits. Indeed by taking the left Kan extension along the Yoneda embedding $\mathcal{Y} : \Delta \to \mathrm{Set}_\Delta$ we can produce a pair of adjoint functors

$$\mathcal{Y}_! F : \mathrm{Set}_\Delta \rightleftarrows C : F^*$$

where for every $c \in C$ the n-simplices of $F^*(c)$ are given by maps $F(n) \to c$.

1.7. Example. Let $C = \mathrm{Set}_\Delta$ and let $X \in \mathrm{Set}_\Delta$. We define a cosimplicial object

$$\begin{aligned} (-) \times X : \Delta &\longrightarrow \mathrm{Set}_\Delta \\ [n] &\longmapsto \Delta^n \times X. \end{aligned}$$

The restriction of the contravariant representable functor $\mathrm{Hom}_{\mathrm{Set}_\Delta}(-, Y)$ along this cosimplicial object sends any ∞-category Y to the functor ∞-category $\mathrm{Fun}(X,Y)$.[4]

[3] The potential for confusion between ∞-bicategories and weak ∞-bicategories created by the terminology of [Lur09a] is obviated by [GHL19, Thm. 5.1].

[4] More generally, the restriction gives the internal hom of Set_Δ.

1.8. Notation. Let C be a cocomplete category and F a cosimplicial object on C. We set the following notation

$$\partial F^n = \underset{\Delta^I \to \partial\Delta^n}{\operatorname{colim}} F(I).$$

This colimit is indexed over the category of simplices $\Delta \downarrow \partial\Delta^n$, however, since every non-degenerate simplex $\Delta^I \to \partial\Delta^n$ is a monomorphism, the inclusion of the poset of non-degenerate simplices into $\Delta \downarrow \partial\Delta^n$ is cofinal. Indeed, it is even homotopy cofinal (see, for example, the first paragraph of [Lur09, Variant 4.2.3.16]). Since this is the case, we can consider ∂F^n as the colimit

$$\partial F^n = \underset{\varnothing \neq I \subsetneq [n]}{\operatorname{colim}} F(I)$$

over the poset of non-degenerate simplices. This is the colimit we will most often work with in practice.

Scaled anodyne maps from dull subsets.

1.9. Definition. *Let* $\mathbf{P}(n)$ *denote the power set of* $[n]$ *with* $n \geqslant 2$. *We say that* $\mathcal{A} \subsetneq \mathbf{P}(n)$ *is* dull *if the following conditions are satisfied:*

1. *There exists* $0 < i < n$ *and a nonempty subset* $K \subset [n] \setminus \{i\}$ *such that* $\mathcal{A}$ *is a partition of* K.
2. *There are singletons* $\{u\}, \{v\} \in \mathcal{A}$ *such that* $u < i < v$.

We will call the element i *in condition (1),* the pivot point.

1.10. Remark. We can to spell out the conditions of Definition 1.9 more in detail. Equivalently, a dull subset is $\mathcal{A} \subset \mathbf{P}(n)$ such that

1. It does not contain the empty set, $\varnothing \notin \mathcal{A}$
2. There exists $0 < i < n$ such that $i \notin S$ for every $S \in \mathcal{A}$.
3. It contains a pair of singletons $\{u\}, \{v\} \in \mathcal{A}$ such that $u < i < v$.
4. For every $S, T \in \mathcal{A}$, we have $S \cap T = \varnothing$.

Conditions 1, 2, and 4 in the above list ensure that $\mathcal{A}$ is a partition of a subset of some $K \subset [n] \setminus \{i\}$, and condition 3 above is precisely condition 2 from the definition.

1.11. Definition. *Let* $\mathcal{A} \subsetneq \mathbf{P}(n)$ *be a dull subset. Given a scaled* n*-simplex* $\Delta^n_\dagger$*, we define*

$$\mathcal{S}^{\mathcal{A}} = \bigcup_{S \in \mathcal{A}} \Delta^{[n]\setminus S} \subsetneq \Delta^n$$

and denote $\mathcal{S}$ *equipped with the induced scaling by* $\mathcal{S}^{\mathcal{A}}_\dagger$. *When the choice of dull subset is clear, we will use the abusive notation* $\mathcal{S}_\dagger$.

1.12. Definition. *Let* $\mathcal{A} \subsetneq \mathbf{P}(n)$ *be a dull subset. We call* $X \in \mathbf{P}(n)$ *an* $\mathcal{A}$-basal set *if it contains precisely one element from each* $S \in \mathcal{A}$. *We denote the set of all* $\mathcal{A}$*-basal sets by* $\mathrm{Bas}(\mathcal{A})$.

1.13. Remark. Note that our definitions guarantee both that $\mathrm{Bas}(\mathcal{A}) \neq \varnothing$, and that all $\mathcal{A}$-basal sets have the same cardinality.

1.14. Definition. *Given a dull subset* $\mathcal{A}$, *we define* $\mathcal{M}_{\mathcal{A}}$ *to be the set of subsets* $X \in \mathbf{P}(n)$ *satisfying the following conditions:*

A1) *X contains the pivot point,* $i \in X$.

A2) *The simplex* $\sigma_X : \Delta^X \to \Delta^n$ *does not factor through* $\mathcal{S}$.

We set $\kappa_{\mathcal{A}} := \min\{|X| | X \in \mathcal{M}_{\mathcal{A}}\}$ *and define, for every* $\kappa_{\mathcal{A}} \leqslant j \leqslant n$, *the subset* $\mathcal{M}^j_{\mathcal{A}} \subset \mathcal{M}_{\mathcal{A}}$ *consisting of those sets of cardinality at most* j.

1.15. Remark. To ease the notation, when the choice of dull subset is clear we will drop the subscript $\mathcal{A}$ in $\mathcal{M}_{\mathcal{A}}$ and $\kappa_{\mathcal{A}}$.

1.16. Lemma. *Let* $\mathcal{A}$ *be a dull subset of* $\mathbf{P}(n)$ *with pivot point* i. *Then it follows that*

$$\mathcal{M}^{\kappa} = \{X_0 \cup \{i\} \ | X_0 \in \mathrm{Bas}(\mathcal{A})\}.$$

Proof. Left to the reader. ∎

1.17. Notation. Let $\mathcal{A} \subsetneq \mathbf{P}(n)$ be a dull subset with pivot point i. Given an $\mathcal{A}$-basal set X, we will denote by $\ell^X_{i-1}\, \ell^X_i$ the pair of consecutive elements in X such that $\ell^X_{i-1} < i < \ell^X_i$.

1.18. Lemma. [The pivot trick] *Let* $\mathcal{A} \subsetneq \mathbf{P}(n)$ *be a dull subset with pivot point* i, *and let* $\Delta^n_\dagger$ *be a scaled simplex. For* $Z \in \mathrm{Bas}(\mathcal{A})$ *suppose that the following condition holds.*

- *For every* $r, s \in [n]$ *such that* $\ell^Z_{i-1} \leqslant r < i < s \leqslant \ell^Z_i$ *the simplex* $\{r, i, s\}$ *is scaled in* $\Delta^n_\dagger$.

Then $\mathcal{S}_\dagger \to \Delta^n_\dagger$ *is scaled anodyne.*

Proof. To simplify notation we will drop the subscript denoting the scaling in this proof, assuming that all simplicial subsets are equipped with the scaling inherited from $\Delta^n_\dagger$. We define for $\kappa \leqslant j \leqslant n$

$$Y_j = Y_{j-1} \cup \bigcup_{X \in \mathcal{M}^j} \sigma_X,$$

where we set $Y_{\kappa-1} = \mathcal{S}$. This yields a filtration

$$\mathcal{S} \to Y_\kappa \to \cdots \to Y_{n-1} \to \Lambda^n_i \to \Delta^n.$$

where $\Lambda^n_i = Y_n$. We will show that each step of this factorization is scaled anodyne.

Let $X \in \mathcal{M}^j$ with $\kappa \leqslant j \leqslant n-1$. Let us note that as a consequence of 1.16 we obtain a pullback diagram

$$\begin{array}{ccc} \Lambda^X_i & \longrightarrow & \Delta^X \\ \big\downarrow & \lrcorner & \big\downarrow \sigma_X \\ Y_{j-1} & \longrightarrow & Y_j \end{array}$$

Additionally, the condition of the lemma guarantees that i together with its neighboring elements in Δ^X form a scaled 2-simplex. Thus, the map $\Lambda^X_i \to \Delta^X$ is scaled anodyne, allowing us to add Δ^X. It also follows from our definitions that given $X, Y \in \mathcal{M}^j$ such that $X \neq Y$ then $\sigma_X \cap \sigma_Y \in Y_{j-1}$, so that we can add the j-simplices Δ^X to Y_{j-1} irrespective of their order. This shows that $Y_{j-1} \to Y_j$ is scaled anodyne. ■

1.19. Remark. It is worth noting that the procedure outlined in 1.18 only makes use of a special subset of the scaled anodyne maps: that generated by the inner horn inclusions

$$\Lambda^n_i \to \Delta^n$$

where $\Delta^{\{i-1,i,i+1\}}$ is scaled. Significantly, while the class of scaled anodyne maps is not, in general, self-dual (i.e. $f^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$ need not be scaled anodyne when $f : X \to Y$ is), the class generated by these scaled inner horn inclusions is. We will make use of this property to further simplify applications of 1.18.

Poset partitions. In 3 it will be necessary for us to consider mapping spaces in quotients of nerves of posets, as well as their scaled analogues. While these mapping spaces are quite straightforward to describe, we here collect a number of descriptions and notations so as to better facilitate the flow of the later sections of the paper.

1.20. Definition. *Let J be a finite poset, and denote by $\mathfrak{J}$ its nerve. We call a pair of subsets J_0, J_1 an* ordered partition *of J if the following three conditions are satisfied.*

- $J_0 \cup J_1 = J$.
- $J_0 \cap J_1 = \varnothing$.
- *For every $x \in J_0$ and every $y \in J_1$, we have either $x < y$ or x and y are incomparable.*

For such an ordered partition, we denote by $\mathfrak{J}^R$ the quotient

$$\mathfrak{J}^R := \mathfrak{J} \coprod_{\mathrm{N}(J_1)} \Delta^0,$$

and by $\widetilde{\mathfrak{J}}$ the quotient

$$\widetilde{\mathfrak{J}} := \Delta^0 \coprod_{\mathrm{N}(J_0)} \mathfrak{J} \coprod_{\mathrm{N}(J_1)} \Delta^0.$$

We denote the two objects of $\widetilde{\mathfrak{J}}$ by $*_0$ and $*_1$, and denote the 'collapse point' of $\mathfrak{J}^R$ by $*_1$.

1.21. Remark. Note that the definition of an ordered partition is symmetric — the opposite of an ordered partition is still an ordered partition. It is for this reason that we only consider the quotient $\mathcal{J}^R$ and not some analogous $\mathcal{J}^L$ as well.

1.22. Example. In the sequel we will make extensive use of a cosimplicial object $Q(n) := \Delta^n \star (\Delta^n)^{\mathrm{op}}$. Each level of this cosimplicial object admits a canonical ordered partition. Under the identification $Q(n) \cong \Delta^{2n+1} = \mathrm{N}([2n+1])$, this ordered partition is given by $J_0 = [n]$ and $J_1 = \{n+1, \ldots, 2n+1\}$. We will abusively denote each of these ordered partitions by (J_0^Q, J_1^Q).

1.23. Construction. Given a finite poset J and an ordered partition (J_0, J_1), we construct a poset $P_{\mathcal{J}}$ as follows. The objects of $P_{\mathcal{J}}$ are totally ordered subsets $S \subset J$ such that $\min(S) \in J_0$ and $\max(S) \in J_1$, ordered by inclusion. We will denote the nerve by $\mathcal{P}_{\mathcal{J}} := \mathrm{N}(P_{\mathcal{J}})$.

Let $\underline{S} := (S_0 \subset \cdots \subset S_k)$ be a k-simplex of $\mathcal{P}_{\mathcal{J}}$. Set $s_0^R := \min(S_0 \cap J_1)$. We define the *right truncation* of $\underline{S}$ to be the simplex

$$\underline{S}^R := (S_0^R \subset \cdots \subset S_k^R)$$

where $S_\ell^R := \{s \in S_\ell \mid s \leqslant s_0^R\}$ for $0 \leqslant \ell \leqslant k$. We similarly define $s_0^L := \max(S_0 \cap J_0)$ and its corresponding *left truncation* $\underline{S}^L$ where $S_\ell^L := \{s \in S_\ell | s \geqslant s_0^L\}$. The *ambidextrous truncation* $\underline{S}^A$ is obtained by taking both the left and right truncation of $\underline{S}$. We can then define two equivalence relations on $\mathcal{P}_{\mathcal{J}}$.

1. We say that k-simplices $\underline{S}$ and $\underline{T}$ are *right equivalent*, and we write

$$\underline{S} \sim_R \underline{T},$$

   when $\underline{S}^R = \underline{T}^R$.

2. We say that k-simplices $\underline{S}$ and $\underline{T}$ are *ambi-equivalent*, and we write

$$\underline{S} \sim_A \underline{T},$$

   when $\underline{S}^A = \underline{T}^A$.

Note that both of these equivalence relations respect the face and degeneracy maps, so that the quotients of $\mathcal{P}_{\mathcal{J}}$ by $\sim_R$ and $\sim_A$ are simplicial sets.

Finally, for any $j \in J_0$, we define $P_{\mathcal{J}}^j \subset P_{\mathcal{J}}$ to be the full subposet on those sets S with $\min(S) = j$. Note that $\sim_R$ induces to an equivalence relation on $\mathcal{P}_{\mathcal{J}}^j$.

We can then characterize the desired mapping spaces of $\mathfrak{C}[\mathcal{J}]$ in terms of the above posets.

1.24. Lemma. *Let J be a finite poset, and (J_0, J_1) and ordered partition of J. Then*

1. *for every $j \in J_0$ there is an isomorphism*

$$\mathfrak{C}[\mathcal{J}^R](j, *_1) \cong (\mathcal{P}_{\mathcal{J}}^j)_{/\sim_R}.$$

2. *There is an isomorphism*

$$\mathfrak{C}[\widetilde{\mathcal{J}}](*_0, *_1) \cong (\mathcal{P}_{\mathcal{J}})_{/\sim_A}.$$

Proof. This follows from the necklace characterization of [DS11] by unwinding the definitions. We will prove the first statement, leaving the analogous second statement to the reader. For elements $s < t$ in a totally ordered set S, we will use the notation $[s, t] \subset S$ to indicate the set

$$[s, t] = \{r \in S \mid s \leqslant r \leqslant t\}.$$

We use the description of the mapping spaces given in [DS11, Corollary 4.4]. An element, $\underline{S} = \{S_0 \subset \cdots \subset S_k\}$ of $\mathcal{P}_J$ with $S_0 = \{s_0^1, \ldots s_0^\ell\}$ can, equivalently, be considered as the data of

- A map
$$f : \mathcal{N} := \Delta^{[s_0^1, s_0^2]} \vee \cdots \vee \Delta^{[s_0^{\ell-1}, s_0^\ell]} \to \mathcal{J}$$
from a necklace into $\mathcal{J}$, sending the last joint into $\mathcal{J}_1$ and the first to j.
- A flanked flag
$$J_{\mathcal{N}} = S_0 \subset \cdots \subset S_k = V_{\mathcal{N}}$$
in the necklace $\mathcal{N}$.

This is, equivalently, a flanked flagged necklace in $\mathcal{J}$.

We then note that, given a flanked flagged necklace in $\mathcal{J}$, we can obtain an element of $\mathcal{P}^j_{\mathcal{J}}$ by forgetting all of the information except the flag. This provides a bijection between flanked flagged necklaces in $\mathcal{J}$ and simplices of $\mathcal{P}^j_{\mathcal{J}}$. It is immediate that this bijection respects the face and degeneracy maps.

We denote the composite of f with the quotient map under the right-equivalence relation by

$$\tilde{f} : \mathcal{N} \xrightarrow{f} \mathcal{J} \longrightarrow \mathcal{J}^R$$

This gives a flanked flagged necklace $[\mathcal{N}, \tilde{f}, \underline{S}]$ in $\mathcal{J}^R$, which, under the isomorphism of [DS11, Cor. 4.4], represents a simplex in $\mathfrak{C}[\mathcal{J}^R](j, *_1)$. Note that passing to the right truncation of $\underline{S}$ yields another flanked flagged necklace in $\mathcal{J}^R$, which is equivalence to $[\mathcal{N}, \tilde{f}, \underline{S}]$ under the equivalence relation of [DS11, Cor. 4.4]. This is because there are canonical morphisms of necklaces

$$\Delta^{[s_0^1, s_0^2]} \vee \cdots \vee \Delta^{[s_0^{\ell-1}, s_0^\ell]} \longrightarrow \Delta^{[s_0^1, s_0^2]} \vee \cdots \Delta^{[s_0^{R-1}, s_0^R]} \vee \Delta^{[s_0^R, s_0^\ell]}$$

and

$$\Delta^{[s_0^1, s_0^2]} \vee \cdots \Delta^{[s_0^{R-1}, s_0^R]} \vee \Delta^{[s_0^R, s_0^\ell]} \longrightarrow \Delta^{[s_0^1, s_0^2]} \vee \cdots \Delta^{[s_0^{R-1}, s_0^R]} \vee \Delta^0$$

which commute with the corresponding maps to $\mathcal{J}^R$ (though *not* with the maps to $\mathcal{J}$). Pushing the flag $\underline{S}$ forward along these two maps yields precisely the flag $\underline{S}^R$. This shows us that any two right-equivalent simplices in $\mathcal{P}^j_{\mathcal{J}}$ are sent the same simplex in $\mathfrak{C}[\mathcal{J}^R](j, *_1)$, so we get a well-defined map of simplicial sets

$$\phi : (\mathcal{P}^j_{\mathcal{J}})_{/\sim_R} \to \mathfrak{C}[\mathcal{J}^R](j, *_1).$$

However, the flanked flagged necklace in $\mathfrak{J}^R$ corresponding to $\underline{S}^R$ is totally non-degenerate in the sense of [DS11, Proposition 4.7]. Moreover, it is easy to see that every totally non-degenerate flanked flagged necklace in $\mathfrak{J}^R$ from j to $*_1$ arises in this way. By [DS11, Corollary 4.8], every simplex in $\mathfrak{C}[\mathfrak{J}^R](j,*_1)$ has a unique representative which is a flanked, flagged, totally non-degenerate necklace. Thus, ϕ is an isomorphism, and the proof is complete. ∎

1.25. Remark. It is worth noting that this approach to mapping spaces in quotients of posets is not novel. A particular special case, for instance, is stated in the first bullet point of [Lur09, Example 2.2.2.5]. The necklace characterization is not essential to this statement, but does make the isomorphism of the lemma easier to see.

2. The enhanced twisted arrow category

Our construction of the enhanced twisted arrow category will depend on an upgrade of the cosimplicial object

$$\begin{aligned} \Delta &\longrightarrow \mathrm{Set}_\Delta \\ [n] &\longmapsto \Delta^n \star (\Delta^n)^{\mathrm{op}} \end{aligned}$$

to a cosimplicial object in scaled simplicial sets. For a discussion of the intuition behind this choice of scaling, see the introduction. To simplify some of the discussion to come, we introduce some notational conventions surrounding $\Delta^n \star (\Delta^n)^{\mathrm{op}}$. Note, before we begin, that there is a canonical identification $\Delta^n \star (\Delta^n)^{\mathrm{op}} \cong \Delta^{2n+1}$, which we will often use without comment.

2.1. Notation. In general, we will denote elements of $\Delta^n \star (\Delta^n)^{\mathrm{op}}$ by $i \in \Delta^n$ or $\bar{i} \in (\Delta^n)^{\mathrm{op}}$. Note that under the identification $\Delta^n \star (\Delta^n)^{\mathrm{op}} \cong \Delta^{2n+1}$, $\bar{i}$ is identified with $2n+1-i$. We denote the unique duality on $\Delta^n \star (\Delta^n)^{\mathrm{op}}$ by

$$\begin{aligned} \tau_n : \Delta^n \star (\Delta^n)^{\mathrm{op}} &\longrightarrow (\Delta^n)^{\mathrm{op}} \star \Delta^n \\ i &\longmapsto \bar{i}. \end{aligned}$$

When n is clear from context, we will simply denote τ_n by τ.

2.2. Definition. *We define a cosimplicial object*

$$\begin{aligned} Q : \Delta^{\mathrm{op}} &\longrightarrow \mathrm{Set}_\Delta^{\mathrm{sc}} \\ [n] &\longmapsto \Delta^n \star (\Delta^n)^{\mathrm{op}} \end{aligned}$$

by declaring a non-degenerate 2-simplex $\sigma : \Delta^2 \to \Delta^n \star (\Delta^n)^{\mathrm{op}}$ to be thin if:

- *σ factors through $\Delta^n \subset Q(n)$;*
- *σ factors through $(\Delta^n)^{\mathrm{op}} \subset Q(n)$;*

- *$\sigma = \Delta^{\{i,j,\bar{k}\}}$, where $i < j \leqslant k$; or*
- *$\sigma = \Delta^{\{k,\bar{j},\bar{i}\}}$, where $i < j \leqslant k$.*

Note that the scaling is symmetric under τ_n by definition, i.e. the maps τ_n define dualities on the scaled simplicial sets $Q([n])$.

The 'nerve' operation associated to Q is a functor

$$Q^* : \mathrm{Set}_\Delta^{\mathrm{sc}} \to \mathrm{Set}_\Delta$$

defined by setting $(Q^*X)_n := \mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathrm{sc}}}(Q([n]), X)$.

2.3. Remark. We will often abuse notation and denote $Q([n])$ by $Q(n)$. We will adopt a similar convention for other cosimplicial objects without comment.

2.4. Definition. *Let $\mathbb{C}$ be an ∞-bicategory with underlying ∞-category $\mathcal{C}$. The* enhanced twisted arrow category *of $\mathbb{C}$ is the marked simplicial set*

$$\mathrm{Tw}(\mathbb{C}) := (Q^*\mathbb{C}, E)$$

where the edges of E are precisely those corresponding to maps $\Delta^3_\sharp \to \mathbb{C}$. Note that the inclusions $\Delta^n_\sharp \subset Q(n)$ and $(\Delta^n)^{\mathrm{op}}_\sharp \subset Q(n)$ induce a canonical map

$$\mathrm{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$$

of simplicial sets.

2.5. Remark. It is immediate from the definitions that $\mathrm{Tw}(\mathcal{C})$ is the ∞-categorical twisted arrow category of [Lur11, §4.2]. With some work it can be shown that this is precisely the simplicial subset of $\mathrm{Tw}(\mathbb{C})$ spanned by the marked morphisms.

The immediate aim of this section is to prove the following theorem, which can be seen as an $(\infty, 2)$-categorical analogue of [Lur11, Prop. 4.2.3].

2.6. Theorem. *For any ∞-bicategory $\mathbb{C}$ with underlying ∞-category $\mathcal{C}$, the canonical map*

$$\mathrm{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$$

defines a fibrant marked simplicial set over $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$. In other words, this map is a Cartesian fibration and the Cartesian edges are precisely the marked edges.

The proof of 2.6, while it involves some combinatorial yoga, begins with the usual, straightforward approach: for each $0 < i \leqslant n$, we consider the lifting problems

$$\begin{array}{ccc} (\Lambda^n_i)^\flat & \longrightarrow & \mathrm{Tw}(\mathbb{C}) \\ \downarrow & \nearrow & \downarrow \\ (\Delta^n)^\flat & \longrightarrow & \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \end{array} \tag{1}$$

and pass to adjoint lifting problems. It is worth noting that, in the case $i = n$, we will in fact consider the edge $\Delta^{\{n-1,n\}} \subset \Lambda^n_n$ to be marked.

2.7. Construction. The adjoint lifting problem to (1) will be the extension problem

$$\begin{array}{ccc} (K_i^n)_\dagger & \longrightarrow & \mathbb{C} \\ \big\downarrow & \nearrow & \\ Q(n) & & \end{array} \tag{2}$$

where $(K_i^n)_\dagger \subset Q(n)$ is the scaled simplicial subset consisting of those simplices $\sigma : \Delta^m \to Q(n)$ which fulfill one of the following three conditions.

- σ factors through $\Delta^n \subset Q(n)$.
- σ factors through $(\Delta^n)^{\mathrm{op}} \subset Q(n)$.
- There exists an integer $j \neq i$ such that neither j nor $\overline{j}$ is a vertex of σ.

2.8. Construction. We denote by $Q(n)_\diamond$ the scaled simplicial set defined by adding to the scaling of $Q(n)$ the triangles of the form $\{n-1, n, \overline{j}\}, \{n-1, \overline{n}, \overline{j}\}$ as well as their duals induced by τ. It is immediate to observe that solutions to the lifting problem

$$\begin{array}{ccc} (K_n^n)_\diamond & \longrightarrow & \mathbb{C} \\ \big\downarrow & \nearrow & \\ Q(n)_\diamond & & \end{array}$$

correspond to solutions to (1) with $i = n$, mapping the last edge in Λ_n^n to a marked edge in $\mathrm{Tw}(\mathbb{C})$.

2.9. Construction. Let $0 < i \leqslant n$ and define $\mathcal{K}_i^n$ to be the simplicial set obtained by adding to K_i^n the faces d^0 and d^{2n+1}. Denote by $(\mathcal{K}_i^n)_\dagger$, $(\mathcal{K}_n^n)_\diamond$ the maximal scalings such that the inclusions $(\mathcal{K}_i^n)_\dagger \to Q(n)$ and $(\mathcal{K}_n^n)_\diamond \to Q(n)_\diamond$ are maps of scaled simplicial sets.

Our proof will proceed by showing that both morphisms in each factorization

$$(K_i^n)_\dagger \to (\mathcal{K}_i^n)_\dagger \to Q(n)$$

and

$$(K_n^n)_\diamond \to (\mathcal{K}_n^n)_\diamond \to Q(n)_\diamond$$

are scaled anodyne.

2.10. Lemma.

1. *For $0 < i < n$ the morphism $(\mathcal{K}_i^n)_\dagger \to Q(n)$ is scaled anodyne.*
2. *The morphism $(\mathcal{K}_n^n)_\diamond \to Q(n)_\diamond$ is scaled anodyne.*

Proof. For $0 < i \leqslant n$, we note that unwinding the definition shows that $\mathcal{K}_i^n = \mathcal{S}^{\mathcal{A}_i}$, where $\mathcal{A}_i \subset \mathbf{P}(2n+1)$ is the dull subset containing $\{0\}$, $\{2n+1\}$, and $\{j, \overline{j}\}$ for $0 < j \leqslant n$ such that $j \neq i$. The lemma follows immediately from 1.18. ∎

2.11. Notation. In the coming proofs, given elements $a < b \in [n]$, we will denote by $[a, b] \subset [n]$ the set $\{a, a+1, \ldots, b\}$. We will denote by $\Delta^{[a,b]} \subset \Delta^n$ the corresponding simplex.

2.12. Lemma.

1. *For $0 < i < n$ the morphism $(K^n_i)_\dagger \to (\mathcal{K}^n_i)_\dagger$ is scaled anodyne.*
2. *For $i = n$ the morphism $(K^n_n)_\diamond \to (\mathcal{K}^n_i)_\diamond$ is scaled anodyne.*

Proof. Let $0 < i \leqslant n$ and note that since $d^0 \cap d^{2n+1} \in K^n_i$ it will suffice to show that the top horizontal morphism,

$$\begin{array}{ccc} Y^\varepsilon_i & \longrightarrow & \Delta^{2n} \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle d^\varepsilon} \\ (K^n_i)_* & \longrightarrow & (\Delta^{2n+1})_* \end{array}$$

where $\varepsilon \in \{0, 2n+1\}$ and $* \in \{\dagger, \diamond\}$, is scaled anodyne.

We will first deal with the case $\varepsilon = 0$. Let $1 \leqslant r \leqslant n$ and define

$$\sigma_r : \Delta^{[r,2n+1]} \to \Delta^{2n+1}$$

to be the obvious inclusion. Let us remark that $\sigma_1 = d^0$ and that σ_r factors through d^0 for every possible r. We produce a filtration

$$Y^0_i = X_{n+1} \to X_n \to \cdots \to X_2 \to \Delta^{[1,2n+1]} = X_1$$

where X_r is obtained by adding the simplex σ_r to X_r.

It will thus suffice for us to check that the upper horizontal morphism in the pullback diagram (i.e. the restriction of σ_r to X_{r+1})

$$\begin{array}{ccc} Z_r & \longrightarrow & \Delta^{[r,2n+1]} \\ \big\downarrow & \lrcorner & \big\downarrow \\ X_{r+1} & \longrightarrow & \Delta^{[1,2n+1]} \end{array}$$

is scaled anodyne. However, we can observe that Z_r consists of a union in $\Delta^{[r,2n+1]}$

- The $(2n-r)$-dimensional face d^r.
- The $(2n-r)$-dimensional faces d^{2n+1-j} where $0 \leqslant j < r$ and $j \neq i$.
- The $(2n-r-1)$-dimensional faces given given by those simplices missing a pair of vertices $\{j, 2n+1-j\}$ with $r \leqslant j \leqslant n$ and $j \neq i$.

That is, $Z_r = \mathcal{S}^{\mathcal{A}_r}$, where $\mathcal{A}_r \subsetneq \mathbf{P}(2n+1-r)$ is the dull subset containing

- $\{0\}$.
- The singletons $\{j\}$ for $2n+1-2r < j \leqslant 2n+1-r$ with $j \neq 2n-r-i+1$.
- The sets $\{k, 2n+1-2r-k\}$ for $0 \leqslant k \leqslant n-r$ with $r+k \neq i$.

One can easily verify that the scaling satisfies the conditions of 1.18, and thus $Z_r \to \Delta^{[r,2n+1]}$ is scaled anodyne. Consequently, each step of the filtration

$$Y_i^0 = X_{n+1} \to X_n \to \cdots \to X_2 \to \Delta^{[1,2n+1]} = X_1$$

is scaled anodyne, completing the proof that $Y_i^0 \to \Delta^{[1,2n+1]}$ is scaled anodyne.

We conclude the proof by noting that the case $Y_i^{2n+1} \to \Delta^{[0,2n]}$ is formally dual, so by 1.19, the proof is complete. ∎

2.13. Lemma. *The map* $\mathrm{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ *has the right lifting property with respect to the morphism*

$$(\Lambda^1_1)^\sharp \to (\Delta^1)^\sharp.$$

Proof. Passing to the adjoint lifting problem we find that this will be true so long as the lifting problems

$$\begin{array}{ccc} \mathrm{Sp}^3 & \longrightarrow & \mathbb{C} \\ \big\downarrow & \nearrow & \\ \Delta^3_\sharp & & \end{array}$$

admit solutions, where $\mathrm{Sp}_3 := \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \coprod_{\Delta^{\{2\}}} \Delta^{\{2,3\}}$ is the spine of the 3-simplex. However, the left-most map is clearly scaled anodyne so a solution to this problem is guaranteed by fibrancy. The result now follows. ∎

We have now shown that the map $\mathrm{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ has the right lifting property with respect to all marked anodyne morphisms of types (1) and (2) from [Lur09, Definition 3.1.1.1]. We will complete the proof by showing the lifting properties with respect to morphisms of types (3) and (4) from *loc. cit.*

2.14. Lemma. *The map* $\mathrm{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ *has the right lifting property with respect to the morphism*

$$(\Lambda^2_1)^\sharp \coprod_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat \to (\Delta^2)^\sharp.$$

Proof. Let $(\Delta^2 \star (\Delta^2)^{\mathrm{op}})_\dagger$ denote the scaled simplicial set where the scaling $\dagger$ is obtained from the scaling on $Q(2)$ by additionally scaling the 2-simplices $\Delta^{\{0,\bar{1},\bar{0}\}}$, $\Delta^{\{0,1,\bar{0}\}}$, $\Delta^{\{1,\bar{2},\bar{1}\}}$, and $\Delta^{\{1,2,\bar{1}\}}$.

Considering adjoint lifting problems, it will suffice to show that there is a scaled anodyne extension

$$(\Delta^2 \star (\Delta^2)^{\mathrm{op}})_\dagger \hookrightarrow (\Delta^2 \star (\Delta^2)^{\mathrm{op}})_\ddagger$$

where the scaling ‡ includes $\Delta^{\{0,\overline{2},\overline{0}\}}$ and $\Delta^{\{0,2,\overline{0}\}}$.

To this end, we first consider the 3-simplex $\Delta^{\{0,1,\overline{2},\overline{1}\}}$. Notice that in the †-scaling, the 2-simplices $\Delta^{\{0,1,\overline{2}\}}$, $\Delta^{\{0,1,\overline{1}\}}$, and $\Delta^{\{1,\overline{2},\overline{1}\}}$ are scaled. Thus a pushout along a scaled anodyne morphism of the type described in [Lur09a, Remark 3.1.4] suffices to additionally scale the 2-simplex $\Delta^{\{0,\overline{2},\overline{1}\}}$.

We then turn our attention to the 3-simplex $\Delta^{\{0,\overline{2},\overline{1},\overline{0}\}}$. The 2-simplices $\Delta^{\{0,\overline{1},\overline{0}\}}$ and $\Delta^{\{\overline{2},\overline{1},\overline{0}\}}$ are scaled in the †-scaling, and $\Delta^{\{0,\overline{2},\overline{1}\}}$ is scaled by the previous step. Thus, another pushout of the type described in [Lur09a, Remark 3.1.4] allows us to scale the 2-simplex $\Delta^{\{0,\overline{2},\overline{0}\}}$.

Since the opposites of the scaled anodyne morphisms described in [Lur09a, Remark 3.1.4] are still scaled anodyne morphisms (indeed, the two morphisms described in that remark are each others' opposites), the dual of the above argument shows that we can also scale $\Delta^{\{0,2,\overline{0}\}}$ using a sequence of scaled anodyne pushouts. ■

2.15. Definition. *We will denote by* $\mathbb{1}$ *the nerve of the* walking isomorphism*: the 1-category with two objects* 0 *and* 1 *and a unique morphism between any two objects.*

2.16. Remark. Notice that, since $\mathbb{1}$ is contractible, the canonical inclusion $\Delta^1 \to \mathbb{1}$ is a trivial cofibration in the Kan-Quillen model structure.

2.17. Lemma. *The marked simplicial set* $\mathrm{Tw}(\mathbb{C})$ *has the right extension property with respect to the morphism*

$$\mathbb{1}^\flat \to \mathbb{1}^\sharp.$$

Proof. We consider the adjoint extension problem:

$$\begin{array}{ccc} (\mathbb{1} \star (\mathbb{1})^{\mathrm{op}})_\dagger & \longrightarrow & \mathbb{C} \\ \big\downarrow & & \\ (\mathbb{1} \star (\mathbb{1})^{\mathrm{op}})_\sharp & & \end{array}$$

and note that † is the smallest scaling in which:

- $\mathbb{1}$ and $\mathbb{1}^{\mathrm{op}}$ are each maximally scaled.
- For every k-simplex $\Delta^k \to \mathbb{1}$, the induced inclusion $Q(k) \to \mathbb{2} \star \mathbb{2}^{\mathrm{op}}$ preserves scalings.

We fix the notation that the objects of $\mathbb{1} \star \mathbb{1}^{\mathrm{op}}$ will be denoted by 0, 1, $\overline{0}$, $\overline{1}$, following our convention for $Q(1)$. Notice that every 2-simplex in $\mathbb{1} \star \mathbb{1}^{\mathrm{op}}$ is uniquely specified by an ordered sequence of objects.

To show that we can scale every 2-simplex of $\mathbb{2} \star \mathbb{2}^{\mathrm{op}}$ with a scaled anodyne pushout there are two cases to consider

1. Suppose that we consider a 2-simplex of the form $\Delta^{\{i,\overline{j},\overline{k}\}}$, and consider three sub-cases:

- Suppose that $i = j$. Then we can consider the 1-simplex $\Delta^1 \to \mathbb{1}$ which sends 0 and 1 to k and i, respectively. The corresponding map

$$Q(1) \to \mathbb{1} \star \mathbb{1}^{\mathrm{op}}$$

then sends the scaled 2-simplex $\Delta^{1,\bar{1},\bar{0}}$ to to $\Delta^{\{i,\bar{j},\bar{k}\}}$, and thus that simplex is scaled in $\dagger$.

- Suppose $i = k$. Then we can consider the 3-simplex $\Delta^3 \to \mathbb{1} \star \mathbb{1}^{\mathrm{op}}$ which sends 0, 1, 2 and 3 to i, k, j, and k, respectively.

  We then note that by the previous case, the 2-simplex $\Delta^{\{0,1,2\}}$ is scaled; the 2-simplex $\Delta^{\{0,1,3\}}$ is degenerate and thus scaled; and the 2-simplex $\Delta^{\{1,2,3\}}$ is contained in 2^{op} and thus is scaled. Consequently, we can scale the remaining 2-simplex using an anodyne pushout of the type described in [Lur09a, Remark 3.1.4]

2. Suppose that we consider a 2-simplex of the form $\Delta^{\{i,j,\bar{k}\}}$. Since the cases above only used the scaled anodyne morphisms from of the type described in [Lur09a, Remark 3.1.4], we are free to pass to opposite scaled simplicial sets. The dual of the above cases then shows that $\Delta^{\{i,j,\bar{k}\}}$ can be scaled via a scaled anodyne pushout.

Thus, the morphism $(\mathbb{1} \star \mathbb{1}^{\mathrm{op}})_\dagger \to (\mathbb{1} \star \mathbb{1}^{\mathrm{op}})_\sharp$ is scaled anodyne, and so the desired lift exists. ■

Proof Of Theorem 2.6. We have shown that $\mathrm{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ has the right lifting property with respect to the sets (1), (2), and (3) of marked anodyne morphisms from [Lur09, Definition 3.1.1.1] in Lemmas 2.10, 2.12, 2.13 and 2.14. All that remains is to show that, given a Kan complex K, $\mathrm{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ has the right lifting property with respect to

$$K^\flat \to K^\sharp.$$

Indeed, it suffices to show that $\mathrm{Tw}(\mathbb{C})$ has the extension property with respect to this map.

Moreover, since K is a Kan complex and $\Delta^1 \to \mathbb{1}$ is a trivial cofibration in the Kan-Quillen model structure, every inclusion of an edge $\Delta^1 \to K$ can be extended to a map $\mathbb{1} \to K$. It thus suffices to show that every extension problem

$$\begin{array}{ccc} \mathbb{1}^\flat & \longrightarrow & \mathrm{Tw}(\mathbb{C}) \\ \big\downarrow & & \\ \mathbb{1}^\sharp & & \end{array}$$

has a solution. This is the content of Lemma 2.17, and thus the theorem is proved. ■

3. The functor classified by $\mathrm{Tw}(\mathbb{C})$

Having now established the Cartesian fibrancy of $\mathbb{T}\mathrm{w}(\mathbb{C}) \to \mathcal{C}$, we aim to determine the functor which it classifies. It will come as no surprise to those familiar with other twisted-arrow category constructions that the functor in question will be the *enhanced mapping functor* of [GHN15], i.e., the mapping category functor of $\mathbb{C}$ restricted to $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. The solution to this classification problem will be quite involved and technical, involving a number of intermediate ∞-categories. Where possible, we will attempt to elucidate the meaning and function of these constructions in the text.

The comparison map. We now turn our attention to the first step in our proof: constructing the comparison map. This part of the proof will be quite straightforward and in total analogy with its ∞-categorical counterpart in [Lur11]. To construct the desired map, we fix, once and for all, the following data:

- An ∞-bicategory $\mathbb{C}$ together with its underlying ∞-category $\mathcal{C}$.
- A fibrant Set_Δ^+-enriched category $\mathbb{D}$ and an equivalence $\mathfrak{C}^{\mathrm{sc}}[\mathbb{C}] \to \mathbb{D}$ of Set_Δ^+-enriched categories.
- The maximally-marked (Kan-complex enriched) subcategory $\mathcal{D} \subset \mathbb{D}$.

We note that a simplex σ of $\mathrm{N}^{\mathrm{sc}}(\mathbb{D})$ lies in $\mathrm{N}^{\mathrm{sc}}(\mathcal{D})$ if and only if every 2-simplex of σ is scaled. That is, $\mathrm{N}^{\mathrm{sc}}(\mathcal{D})$ is the underlying ∞-category of the ∞-bicategory $\mathrm{N}^{\mathrm{sc}}(\mathbb{D})$. Since an equivalence of ∞-bicategories induces an equivalence on underlying ∞-categories, our data thus yields a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\simeq} & \mathrm{N}^{\mathrm{sc}}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow[\simeq]{} & \mathrm{N}^{\mathrm{sc}}(\mathbb{D}). \end{array}$$

Since $\mathfrak{C}$ respects monomorphisms by [Lur22, 01GD], the data we have fixed can be summarized in the commutative diagram

$$\begin{array}{ccc} \mathfrak{C}^{\mathrm{sc}}[\mathcal{C}] & \xrightarrow{\simeq} & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathfrak{C}^{\mathrm{sc}}[\mathbb{C}] & \xrightarrow[\simeq]{} & \mathbb{D} \end{array}$$

such that the horizontal arrows are weak equivalences of Set_Δ^+-enriched categories and the vertical arrows are monomorphisms.

With these data in hand, the enhanced mapping functor is the composite

$$F : \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \hookrightarrow \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \xrightarrow{\mathrm{Map}} \mathbb{C}\mathrm{at}_\infty$$

To retain concision, we use the pedestrian notation F for the enhanced mapping functor, rather than the more suggestive $\mathrm{Map}_{\mathbb{D}}$.

3.1. Notation. In the coming proofs, we will make use of the standard left and right cone notations from [Lur09, Ch. 1]. Given a simplicial set X, we will write

$$X^{\triangleright} := X \star \Delta^0$$

and

$$X^{\triangleleft} := \Delta^0 \star X.$$

3.2. Proposition. *There is an map*

$$\beta : \mathrm{Tw}(\mathbb{C}) \to \mathrm{Un}^{+}_{\mathcal{C}\times\mathcal{C}^{\mathrm{op}}}(F)$$

of Cartesian fibrations over $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$.

Proof. The proof proceeds along the same lines as the analogous argument of [Lur11, Prop. 4.2.5]. We define an ancillary simplicial category $\mathcal{E}$ with objects either the objects of $\mathcal{D}^{\mathrm{op}} \times \mathcal{D}$, or a "cone point" v. The mapping spaces will be those of $\mathcal{D}^{\mathrm{op}} \times \mathcal{D}$ if they don't involve v, and will be defined by

$$\begin{aligned} \mathrm{Map}_{\mathcal{E}}(v, (D, D')) &:= \varnothing \\ \mathrm{Map}_{\mathcal{E}}((D, D'), v) &:= \mathrm{Map}_{\mathbb{D}}(D, D') \end{aligned}$$

otherwise.

As in the proof of [Lur11, Prop. 4.2.5], a map over $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ preserving markings — $\beta : \mathrm{Tw}(\mathbb{C}) \to \mathrm{Un}^{+}_{\mathcal{C}\times\mathcal{C}^{\mathrm{op}}}(F)$ — will be equivalent to giving a map

$$\gamma : \mathrm{Tw}(\mathbb{C})^{\triangleright} \to \mathrm{N}(\mathcal{E})$$

such that the diagram

$$\begin{array}{ccccccc} \mathrm{Tw}(\mathbb{C})^{\triangleright} & & & & & \dashrightarrow & \\ \uparrow & & & & & & \\ \mathrm{Tw}(\mathbb{C}) & \longrightarrow & \mathcal{C} \times \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathrm{N}(\mathcal{D}) \times \mathrm{N}(\mathcal{D})^{\mathrm{op}} & \longrightarrow & \mathrm{N}(\mathcal{E}) \end{array}$$

commutes, and such that, for every $f : \Delta^1 \to \mathrm{Tw}(\mathbb{C})$ which is marked, the two-simplex $f * \mathrm{id}_{\Delta^0} : \Delta^1 \star \Delta^0 \to \mathrm{Tw}(\mathbb{C})^{\triangleright} \to \mathrm{N}(\mathcal{E})$ is sent to a scaled 2-simplex in $\mathrm{N}^{\mathrm{sc}}(\mathcal{E})$.

We now define the map γ. On $\mathrm{Tw}(\mathbb{C}) \subset \mathrm{Tw}(\mathbb{C})^{\triangleright}$, the map γ is uniquely determined by the requirement that the diagram above commutes. We send the cone point of $\mathrm{Tw}(\mathbb{C})^{\triangleright}$ to the 'cone point' $v \in \mathrm{N}(\mathcal{E})$. The remainder of the definition amounts to writing down, for every $\sigma : \Delta^n \to \mathrm{Tw}(\mathbb{C})$, the image under γ of the simplex

$$\sigma \star \mathrm{id}_{\Delta^0} : \Delta^{n+1} \to \mathrm{Tw}(\mathbb{C}).$$

We will denote the image of this simplex — which we construct below — by γ_σ.

Given an n-simplex $\sigma : \Delta^n \to \mathrm{Tw}(\mathbb{C})$, we obtain by definition and adjunction a map

$$\nu_\sigma : \mathfrak{C}[\Delta^{2n+1}] \to \mathfrak{C}[\mathbb{C}] \to \mathbb{D}.$$

We now define a map

$$\gamma_\sigma : \mathfrak{C}[\Delta^{n+1}] \to \mathcal{E}$$

On $\mathfrak{C}[\Delta^n] \subset \mathfrak{C}[\Delta^{n+1}]$, this is completely determined by the commutativity condition above. For mapping spaces involving the $(n+1)^{\mathrm{st}}$-vertex, we define the maps

$$\begin{aligned} \zeta_i : \mathfrak{C}[\Delta^{n+1}](i, n+1) &\longrightarrow \mathfrak{C}[\Delta^{2n+1}](i, 2n+1-i) \\ S \cup \{n+1\} &\longmapsto S \cup \tau(S) \end{aligned}$$

where S is considered as a subset of $[n]$, and τ is the involution on vertices of Δ^{2n+1}. We then define

$$\gamma_\sigma : \mathfrak{C}[\Delta^{n+1}](i, n+1) \xrightarrow{\zeta_i} \mathfrak{C}[\Delta^{2n+1}](i, 2n+1-i) \xrightarrow{\nu_\sigma} \mathrm{Map}_{\mathbb{D}}(\nu_\sigma(i), \nu_\sigma(2n+1-i))$$

Completing our definition of γ_σ, and thus of γ.

To see that the map γ respects markings, we need to check that the above condition, namely that, for every marked edge $f : \Delta^1 \to \mathrm{Tw}(\mathbb{C})$, the corresponding 2-simplex

$$f \star \mathrm{id}_{\Delta^0} : \Delta^1 \star \Delta^0 \to \mathrm{Tw}(\mathbb{C})^{\triangleright} \to \mathrm{N}(\mathcal{E})$$

is scaled. Since f is marked, the adjoint map is a map

$$\Delta^3_\sharp \to \mathbb{C}.$$

Thus the composite map

$$v_f : \mathfrak{C}[\Delta^3] \to \mathfrak{C}[\mathbb{C}] \to \mathbb{D}$$

must send every 1-simplex in each mapping space to a marked 1-simplex in the corresponding mapping space of $\mathbb{D}$. Since γ_f is defined on each mapping space $\mathfrak{C}[\Delta^2](i, 2)$ as the composite $\nu_f \circ \zeta_i$, this immediately implies that the simplex of $\mathrm{N}(\mathcal{E})$ represented by γ_f is scaled. ∎

3.3. Remark. The definition of the maps ζ_i which allow us to define the map above are quite ad-hoc in appearance, as indeed are their analogues in the proof of [Lur11, Proposition 4.2.5]. Once we pass to fibers, the map can be much more elegantly defined: in terms of a composite with a map of posets (see 3.23).

The goal of the remainder of this section will be the proof of the following.

3.4. Theorem. *The map β is an equivalence of Cartesian fibrations over $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$.*

Interlude: a compendium of cosimplicial objects. In the sections which follow, there will be a variety of cosimplicial objects in play, each relating to a specific construction necessary for the proof. For ease of reference, we list these here, and describe additional structures (in particular ordered partitions) which will come into play in their study.

3.5. Definition. [The compendium] *We fix, for the rest of the section, the following cosimplicial objects, along with ordered partitions of their* n^{th} *levels.*

1. *A cosimplicial object*
$$\begin{aligned} \bigstar : \Delta &\longrightarrow \mathrm{Set}_\Delta^{\mathrm{sc}} \\ [n] &\longmapsto \Delta^n \star \Delta^0 \end{aligned}$$
where the scaling on $\bigstar(n) = \Delta^n \star \Delta^0$ *is given by declaring every 2-simplex in* $\Delta^n \subset \bigstar(n)$ *to be thin.*
   - *We define an ordered partition* $(J_0^{\bigstar}, J_1^{\bigstar})$ *of* $\bigstar(n)$ *for each* n *by setting* $J_0^{\bigstar} = [n]$ *and* $J_1^{\bigstar} = \{n+1\}$*, where we have identified* $\bigstar(n)$ *with* Δ^{n+1}*.*
2. *A cosimplicial object*
$$\begin{aligned} \boxtimes : \Delta &\longrightarrow \mathrm{Set}_\Delta^{\mathrm{sc}} \\ [n] &\longmapsto \Delta^n \star (\Delta^n)^{\mathrm{op}} \star \Delta^0 \end{aligned}$$
We use our existing notational conventions for objects of $Q(n) \subset \boxtimes(n)$*, and denote the final vertex by* v*. We equip* $\boxtimes(n)$ *with the minimal scaling such that (1) requiring the inclusions* $Q(n) \subset \boxtimes(n)$ *and* $((\Delta^n)^{\mathrm{op}} \star \Delta^0)_\sharp \subset \boxtimes(n)$ *to be maps of scaled simplicial sets; and (2) declaring any simplex of the form* $\Delta^{\{j,\bar{i},v\}}$*, where* $0 \leqslant i \leqslant j \leqslant n$*, to be thin.*
   - *We define an ordered partition* $(J_0^{\boxtimes}, J_1^{\boxtimes})$ *of* $\boxtimes(n)$ *for each* n *by setting* $J_0^{\boxtimes} = [n]$ *and* $J_1^{\boxtimes} = \{n+1, n+2, \ldots 2n+2\}$ *under the identification of* $\boxtimes(n)$ *with* Δ^{2n+2}*.*
3. *A cosimplicial object*
$$\begin{aligned} \square : \Delta &\longrightarrow \mathrm{Set}_\Delta^{\mathrm{sc}} \\ [n] &\longmapsto \Delta^n \times \Delta^1 \end{aligned}$$
where the scaling consists of those triangles factoring through $\Delta^n \times \Delta^{\{1\}}$ *and those described in [Lur09a, 4.1.5], i.e., those triangles of the form* $\Delta^{\{(i,0),(i,1),(j,1)\}}$ *for* $i < j$ *in* $[n]$*.*
   - *We define an ordered partition* $(J_0^{\square}, J_1^{\square})$ *of* $\square(n)$ *by setting* $J_0^{\square} := [n] \times \{0\}$ *and* $J_1 = [n] \times \{1\}$ *under the identification of* $\square$ *with* $\Delta^n \times \Delta^1$*.*

Comparison with outer Cartesian slices. Having established the existence of a comparison map $\beta : \mathrm{Tw}(\mathbb{C}) \to \mathrm{Un}^+_{\mathcal{C}\times\mathcal{C}^{\mathrm{op}}}(F)$ of Cartesian fibrations, we now must pause and circumnavigate our way to a proof that it is an equivalence. The winding route we take will make use of a Cartesian fibration $\mathbb{C}_{/y}$ defined in [GHL19, §2.2]. The utility of $\mathbb{C}_{/y}$ for us lies in the fact that, as established in [GHL19, §2.3], $\mathbb{C}_{/y}$ is classified by the contravariant Yoneda embedding $\mathcal{Y}_y$ on $\mathbb{C}$. In spite of the fact that $\mathbb{C}_{/y} \to \mathcal{C}$ is a Cartesian fibration, we will refer to $\mathbb{C}_{/y}$ as the *outer Cartesian slice category*, in recognition of the fact that our $\mathbb{C}_{/y} \to \mathcal{C}$ is a pullback along the inclusion $\mathcal{C} \to \mathbb{C}$ of an outer Cartesian fibration as defined in [GHL19, §2.1]. We begin by recalling the definition of $\mathbb{C}_{/y}$.

3.6. Definition. *For an object $y \in \mathbb{C}$, we define the* outer Cartesian slice category $\mathbb{C}_{/y}$ *whose n-simplices are given by maps*

$$\sigma : \bigstar(n) \to \mathbb{C}, \text{ such that } \sigma|_{n+1} = y.$$

We equip $\mathbb{C}_{/y}$ with a marking by declaring an edge to be marked precisely when it can be represented by a map $\Delta^2_\sharp \to \mathbb{C}$. Note that the canonical inclusion $\Delta^n_\sharp \subset \bigstar(n)$ induces a map $\mathbb{C}_{/y} \to \mathcal{C}$.

3.7. Proposition. [GHL19, Cor. 2.27] *The functor $\mathbb{C}_{/y} \to \mathcal{C}$ is a Cartesian fibration, and an edge of $\mathbb{C}_{/y}$ is Cartesian if and only if it is marked.*

Since our mode of proof is so circuitous, let us take a moment to sketch the path we will take. We begin by showing that there is a span

$$(\mathbb{C}_{/y})_x \xleftarrow{\sim} \mathcal{M}_{x,y} \xrightarrow{\sim} \mathrm{Tw}(\mathbb{C})_{(x,y)}$$

displaying a weak equivalence of the fibers of $\mathbb{C}_{/y}$ and $\mathrm{Tw}(\mathbb{C})$.

We then show that there is a weak equivalence

$$\mathrm{Un}^+_*(\mathfrak{C}^{\mathrm{sc}}[\mathbb{D}](x,y)) \to \mathrm{Un}^{\mathrm{sc}}_*(\mathfrak{C}^{\mathrm{sc}}[\mathbb{D}](x,y)).$$

Combining [GHL19, Proposition 2.33] and [Lur09a, Theorem 4.2.2], there is an equivalence

$$(\mathbb{C}_{/y})_x \to \mathfrak{C}^{\mathrm{sc}}[\mathbb{C}](x,y)$$

and so we obtain a composite equivalence

$$f : (\mathbb{C}_{/y})_x \to \mathfrak{C}^{\mathrm{sc}}[\mathbb{C}](x,y) \to \mathrm{Un}^{\mathrm{sc}}_*(\mathfrak{C}^{\mathrm{sc}}[\mathbb{D}](x,y)).$$

The final step to showing that β is an equivalence is therefore establishing that the diagram

$$\begin{array}{ccccc}
(\mathbb{C}_{/y})_x & \xleftarrow{\sim} & \mathcal{M}_{x,y} & \xrightarrow{\sim} & \mathrm{Tw}(\mathbb{C})_{(x,y)} \\
\simeq\downarrow f & & & & \downarrow\beta \\
\mathrm{Un}^{\mathrm{sc}}_*(\mathfrak{C}^{\mathrm{sc}}[\mathbb{D}](x,y)) & & \xleftarrow[\simeq]{} & & \mathrm{Un}^+_*(\mathfrak{C}^{\mathrm{sc}}[\mathbb{D}](x,y))
\end{array}$$

commutes up to equivalence.

We begin this journey in the present section by defining the span

$$(\mathbb{C}_{/y})_x \xleftarrow{\sim} \mathcal{M}_{x,y} \xrightarrow{\sim} \mathrm{Tw}(\mathbb{C})_{(x,y)}$$

and showing that its legs are weak equivalences.

3.8. NOTATION. Let $x, y \in \mathbb{C}$. We denote by $\mathrm{Tw}(\mathbb{C})_y$ the pullback

$$\begin{array}{ccc} \mathrm{Tw}(\mathbb{C})_y & \longrightarrow & \mathrm{Tw}(\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow[\mathrm{id}\times\{y\}]{} & \mathcal{C}\times\mathcal{C}^{\mathrm{op}} \end{array}$$

and by $\mathrm{Tw}(\mathbb{C})_{(x,y)}$ the fiber over $(x, y) \in \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$.

3.9. DEFINITION. *For $y \in \mathbb{C}$, we define a simplicial set $\mathcal{M}_y$ whose n-simplices are given by maps*

$$\sigma : \boxtimes(n) \to \mathbb{C} \textit{ such that } \sigma|_{\mathrm{N}\left(J_1^{\boxtimes}\right)} = y.$$

Note that these are, equivalently, maps $\boxtimes(n)^R \to \mathbb{C}$. The inclusion $\Delta^n_\sharp = \mathrm{N}\left(J_0^{\boxtimes}\right)_\sharp \subset \boxtimes(n)$ induces a map $\mathcal{M}_y \to \mathcal{C}$.

3.10. REMARK. We will view $\bigstar(n)$, $\boxtimes(n)$, and $Q(n)$ as equipped with their ordered partitions from 3.5 and 1.22, and consider their right quotients $\bigstar(n)^R$, $\boxtimes(n)^R$, and $Q(n)^R$ as defined in 1.20, each of which piece together to form a cosimplicial object in $\mathrm{Set}_\Delta^{\mathrm{sc}}$. The obvious natural inclusions

$$\bigstar^R \Longrightarrow \boxtimes^R \Longleftarrow Q^R$$

then induce maps

$$\mathbb{C}_{/y} \xleftarrow{\rho} \mathcal{M}_y \xrightarrow{\pi} \mathrm{Tw}(\mathbb{C})_y$$

over $\mathcal{C}$ by restricting simplices (i.e. maps $\boxtimes(n)^R \to \mathbb{C}$) along the inclusions.

3.11. PROPOSITION. *The map $\pi : \mathcal{M}_y \to \mathrm{Tw}(\mathbb{C})_y$ is a trivial Kan fibration.*

PROOF. We first aim to show that the inclusions $i_n : Q(n)^R \to \boxtimes(n)^R$ are scaled trivial cofibrations. To this end, we define a map

$$\begin{aligned} r_n : \boxtimes(n) &\longrightarrow Q(n) \\ i &\longmapsto \begin{cases} i & i < 2n+2 \\ i-1 & i = 2n+2 \end{cases} \end{aligned}$$

We see immediately that r_n descends to a map $r_n : \boxtimes(n)^R \to Q(n)^R$, and that $r_n \circ i_n = \mathrm{id}$. Moreover, one can check that the natural transformation $i_n \circ r_n \Rightarrow \mathrm{id}$ descends to a transformation

$$\Delta^1_\flat \times \boxtimes(n)^R \to \boxtimes(n)^R$$

whose components are degenerate. Consequently, we see that i_n is an equivalence of scaled simplicial sets. We then consider the boundary lifting problem and its associated adjoint problem

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathcal{M}_y \\ \downarrow & \nearrow & \downarrow{\scriptstyle \pi} \\ \Delta^n & \longrightarrow & \mathrm{Tw}(\mathbb{C})_y \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} K^n & \longrightarrow & \mathbb{C} \\ \downarrow & \nearrow & \\ \boxtimes(n)^R & & \end{array} \qquad K^n = \partial(\boxtimes^R)^n \coprod_{\partial(Q^R)^n} Q(n)^R$$

Examining 1.20, we note that we can extend our conventions to $Q(\varnothing)^R = \Delta^0$ and $\boxtimes(\varnothing)^R = \Delta^0$ and rewrite the colimits defined in Notation 1.8 as

$$\partial(\boxtimes^R)^n = \operatorname*{colim}_{I \subsetneq [n]} \boxtimes(I)^R \quad \text{and} \quad \partial(Q^R)^n = \operatorname*{colim}_{I \subsetneq [n]} Q(I)^R,$$

where we now include the empty set in our diagram.

These colimits are indexed over the poset of proper subsets of $[n]$. This poset can be uniquely equipped with the structure of an upwards-directed Reedy structure (in the parlance of [Du08, Definition 13.6]). In particular, this means that, as described in [Lur09, Example A.2.9.22], we can identify the corresponding Reedy model structure with the projective model structure.

Moreover, for every pair of subsets $I, J \subsetneq [n]$, we see that

$$\boxtimes(I)^R \cap \boxtimes(J)^R = \boxtimes(I \cap J)^R$$

when considered as subsets of $\boxtimes(n)$, and the analogous statement holds for $Q(I)^R$. This immediately implies that the diagrams defining $\partial(\boxtimes^R)^n$ and $\partial(Q^R)^n$ are Reedy cofibrant, and thus projectively cofibrant. Consequently, we can write

$$\partial(\boxtimes^R)^n = \operatorname*{hocolim}_{I \subsetneq [n]} \boxtimes(I)^R \quad \text{and} \quad \partial(Q^R)^n = \operatorname*{hocolim}_{I \subsetneq [n]} Q(I)^R.$$

Since the two diagrams are naturally equivalent, this yields an equivalence $\partial(Q^R)^n \xrightarrow{\simeq} \partial(\boxtimes^R)^n$. Since this map is a cofibration, it follows that $Q(n)^R \to K^n$ is an equivalence. Finally we consider the factorization

$$Q(n)^R \to K^n \to \boxtimes(n)^R$$

and we conclude by 2-out-of-3 that the map $K^n \to \boxtimes(n)^R$ is a trivial cofibration. This finishes the proof. ■

3.12. Corollary. *The map $\mathcal{M}_y \to \mathcal{C}$ is a Cartesian fibration, and $\pi : \mathcal{M}_y \to \mathrm{Tw}(\mathbb{C})_y$ is an equivalence of Cartesian fibrations over $\mathcal{C}$.*

3.13. Lemma. *The Cartesian edges of $\mathcal{M}_y$ over $\mathcal{C}$ are precisely those which can be represented by scaled maps $\boxtimes^1_\dagger \to \mathbb{C}$, where $\dagger$ is the extension of the scaling on $\boxtimes^1$ to include*

1. *all 2-simplices in $\Delta^1 \star (\Delta^1)^{\mathrm{op}}$,*
2. *the 2-simplex $\Delta^{\{01,v\}}$.*

PROOF. The projection $\mathcal{M}_y \to \mathcal{C}$ is the composite of the maps

$$\mathcal{M}_y \xrightarrow{\pi} \mathrm{Tw}(\mathbb{C})_y \xrightarrow{\phi} \mathcal{C}$$

where f is a Cartesian fibration, and π is a trivial Kan fibration by Proposition 3.11. As a result, the Cartesian edges of the composite map are precisely those edges f of $\mathcal{M}_y$ such that $\pi(f)$ is Cartesian. This shows that, if we let $\heartsuit$ denote the extension of the scaling on $\boxtimes^1$ defined by condition (1) of the lemma, the Cartesian morphisms of $\mathcal{M}_y$ are the scaling-preserving maps

$$(\boxtimes^1)_{\heartsuit} \to \mathbb{C}.$$

It will thus suffice to show that we can extend the scaling $\heartsuit$ on $\boxtimes^1$ to include the 2-simplex $\Delta^{\{0,1,v\}}$. If we consider the 3-simplex $\Delta^{\{0,1,3,v\}} \subset \boxtimes^1_{\heartsuit}$, we can note that $\Delta^{\{0,1,3\}}$, $\Delta^{\{1,3,v\}}$, and $\Delta^{\{0,3,v\}}$ are scaled. This means that we can scale $\Delta^{\{0,1,v\}}$ using a scaled anodyne pushout along the morphism described in [Lur09a, Remark 3.1.4]. This completes the proof. ∎

3.14. COROLLARY. *The map* $\rho : \mathcal{M}_y \to \mathbb{C}_{/y}$ *is a map of naturally-marked Cartesian fibrations over* $\mathcal{C}$.

3.15. PROPOSITION. *For any* $x \in \mathcal{C}$*, denote the fiber of* $\mathcal{M}_y$ *over* x *by* $\mathcal{M}_{x,y}$*. Then the induced map*

$$\rho : \mathcal{M}_{x,y} \to (\mathbb{C}_{/y})_x$$

is a trivial Kan fibration.

PROOF. We follow effectively the same method as in the proof of Proposition 3.11, now using the two-sided quotients $\widetilde{\bigstar}(n)$ and $\widetilde{\boxtimes}(n)$ of the defining cosimplicial objects. By a nearly identical homotopy colimit argument, it will suffice for us to show that

$$i_n : \widetilde{\bigstar}(n) \to \widetilde{\boxtimes}(n)$$

is an equivalence. However, i_n is already a bijection on objects, so it will suffice for us to show that i_n induces an equivalence on the single non-trivial mapping space. To this end, we make use of the characterization of Lemma 1.24. It will thus suffice to show that the maps of marked simplicial sets

$$\widetilde{s} : \left(\mathcal{P}_{\bigstar(n)}\right)_{/\sim_A} \to \left(\mathcal{P}_{\boxtimes(n)}\right)_{/\sim_A}$$

are equivalences for any n. For the rest of the proof we will abuse notation and denote the nerves of these posets by $\mathcal{P}_{\bigstar}$ and $\mathcal{P}_{\boxtimes}$.

We will work with the unquotiented simplicial sets, and define maps which descend to quotients. Before we can do this, however, we must fix some notation. We denote object $S \in \mathcal{P}_{\boxtimes}$ by triples (S_0, S_1, S_2) of subsets of each of the three joined components in

$\boxtimes(n) = \Delta^n \star (\Delta^n)^{\mathrm{op}} \star \Delta^0$. We will similarly denote objects of $P_\bigstar$ by pairs (S_0, v) of sets. Note that with these new coordinates the unquotient version of $\widetilde{s}$ can be described as

$$\begin{aligned} s:\quad \mathcal{P}_\bigstar &\longrightarrow \mathcal{P}_\boxtimes \\ (S_0, v) &\longmapsto (S_0, \varnothing, v) \end{aligned}$$

We define $\mathcal{P}_G \subset \mathcal{P}_\boxtimes$ as the nerve of the full subposet on those objects of the form

- $(S_0, \varnothing, v)$.
- (S_0, S_1, v) such that
  - $S_1 \neq \varnothing$
  - $S_1 \cup \{v\}$ contains all elements of $\boxtimes(n)$ greater than $\min(S_1)$, and
  - $\tau(S_1) \subset S_0$.

We equip $\mathcal{P}_G$ with the induced marking, producing a factorization $\mathcal{P}_\bigstar \xrightarrow{s_\alpha} \mathcal{P}_G \xrightarrow{s_\beta} \mathcal{P}_\boxtimes$. In light of this fact, we will turn our efforts into showing that s_α, s_β descend to equivalences $\widetilde{s}_\alpha$ and $\widetilde{s}_\beta$. We define a marking-preserving map of posets

$$r_\alpha : \mathcal{P}_G \to \mathcal{P}_\bigstar, (S_0, S_1, v) \mapsto (S_0, v)$$

such that $r_\alpha \circ s_\alpha = \mathrm{id}$. We, moreover, observe that there is a natural transformation $\varepsilon_\alpha : s_\alpha \circ r_\alpha \Rightarrow \mathrm{id}$ whose components are marked in $\mathcal{P}_G$. To check that ε_α (and consequently r_α) factors through the quotient it is enough to note that given k-simplices $\underline{S} \sim_A \underline{T}$ in $\mathcal{P}_G$ then it follows that $\underline{S} \sim_L \underline{T}$. We can now conclude that $\widetilde{s}_\alpha$ is an equivalence.

We define a map of posets

$$\begin{aligned} r_\beta :\qquad P_\boxtimes &\longrightarrow P_G \\ (S_0, S_1, S_2) &\longmapsto \begin{cases} (S_0, \varnothing, v) & \text{if } S_1 = \varnothing \\ (S_0 \cup \tau_n([\min(S_1), v)), [\min(S_1), v), v) & \text{otherwise.} \end{cases} \end{aligned}$$

such that $r_\beta \circ s_\beta = \mathrm{id}$ and note that there is a map $S \to r_\beta(S)$ inducing a natural transformation $\varepsilon_\beta : \mathrm{id} \Rightarrow s_\beta \circ r_\beta$. To show that $\mathcal{P}_G \to \mathcal{P}_\boxtimes$ is an equivalence, it is sufficient to check that r_β preserves markings, that ε_β descends to quotients, and that the components of the natural transformation become equivalences in the fibrant replacement of the localizations. We will prove here that ε_β descends to the quotient leaving the rest of the checks as exercises for the interested reader. Let $\underline{S} \sim_A \underline{T}$ be k-simplices and denote by ${}^{\beta}\underline{S}$ and ${}^{\beta}\underline{T}$ their images under r_β. Let s_0^R, s_0^L be the truncation points for $\underline{S}$ and denote by ${}^{\beta}s_0^R, {}^{\beta}s_0^L$ the truncation points for ${}^{\beta}\underline{S}$. It is immediate that

$${}^{\beta}s_0^R = s_0^R, \qquad {}^{\beta}s_0^L = \begin{cases} \max\left\{s_0^L, \tau({}^{\beta}s_0^R)\right\} & \text{if} \quad (S_0)_1 \neq \varnothing \\ s_0^L \text{ otherwise.} \end{cases}$$

This implies that, in order to show our claim, it suffices to check that for every $\ell \in [k]$ the ambidextrous truncations of ${}^{\beta}S_\ell$, ${}^{\beta}T_\ell$ with respect to s_0^L, s_0^R coincide. For each $\ell \in [k]$, recall that we represent S_ℓ as a tuple $((S_\ell)_0, (S_\ell)_1, (S_\ell)_2)$. If $(S_\ell)_1 \neq \varnothing$ the conclusion follows immediately. We will also assume that $\kappa = \tau(\min((S_l)_1)) < \max((S_l)_0)$, since otherwise we would have ${}^{\beta}S_\ell = {}^{\beta}T_\ell$. Denote by ${}^{\beta}\widehat{S}_\ell^A$ the truncation with respect to our chosen points. Then we observe that

$$ {}^{\beta}\widehat{S}_\ell^A = \begin{cases} [s_0^L, \kappa] \cup (S_\ell)_0^{\geqslant \kappa} \cup [\tau(\kappa), v] & \text{if } s_0^L \leqslant \kappa \\ (S_\ell)_0 \cup [\tau(\kappa), v] \text{ otherwise} \end{cases} $$

where $(S_\ell)_0^{\geqslant \kappa}$ consists of the elements of $(S_\ell)_0$ which are not less than κ. Since this only depends on S_ℓ^A it follows that ${}^{\beta}\widehat{S}_\ell^A = {}^{\beta}\widehat{T}_\ell^A$. ∎

We thus have completed the first step of the proof:

3.16. COROLLARY. *The maps*

$$ \mathbb{C}_{/y} \xleftarrow{\rho} \mathcal{M}_y \xrightarrow{\pi} \mathrm{Tw}(\mathbb{C})_y $$

are equivalences of naturally marked Cartesian fibrations over $\mathcal{C}$.

3.17. REMARK. This would already be sufficient, in light of [GHL19, §2.3], for us to conclude that $\mathrm{Tw}(\mathbb{C})_y$ classifies the restriction to $\mathcal{C}$ of the representable functor defined by y. It is not, however, sufficient to show that $\mathrm{Tw}(\mathbb{C})$ classifies the enhanced mapping functor. We still have work to do.

COMPARING THE COMPARISONS. There is yet another model for the mapping ∞-categories in $\mathbb{C}$, provided by a *coCartesian fibration*, defined in [Lur09a, Notation 4.1.5]. There, for any object x in $\mathbb{C}$, a scaled *coCartesian* fibration $\overline{\mathbb{C}^{x/}} \to \mathbb{C}$ is defined. The pullback of this fibration along the inclusion $\mathcal{C} \to \mathbb{C}$ is a coCartesian fibration, which we will denote by $\mathbb{C}^{x/}$ We recall the definition here.

3.18. DEFINITION. *We define a marked simplicial set* $\mathbb{C}^{x/}$*, the* n*-simplices of which are maps of scaled simplicial sets*

$$ \square^n \to \mathbb{C} $$

which send $\Delta^n \times \{0\}$ *to* x*. A 1-simplex* $\Delta^1 \to \mathcal{C}^{x/}$ *is defined to be marked precisely when it corresponds to a map*

$$ \square^1_\sharp \to \mathbb{C}. $$

The incusion $\Delta^n_\sharp \times \{1\} \to \square^n$ *induces a map* $\mathbb{C}^{x/} \to \mathcal{C}$*, which, by [Lur09a, Proposition 4.1.6] is a naturally marked coCartesian fibration.*

3.19. Remark. Note that in [Lur09a, Notation 4.1.5], Lurie uses the notation $\mathbb{C}^{x/}$ to denote the underlying simplicial set of $\overline{\mathbb{C}^{x/}}$. This is *not* the meaning of $\mathbb{C}^{x/}$ in this paper. To avoid adding additional decorations to the already heavy notation involved in this proof, we use $\mathbb{C}^{x/}$ to mean the pullback

$$\begin{array}{ccc} \mathbb{C}^{x/} & \longrightarrow & \overline{\mathbb{C}^{x/}} \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathbb{C} \end{array}$$

of the scaled coCartesian fibration of [Lur09a, 4.1.5].

By [GHL19, Proposition 2.33] and [Lur09a, Theorem 4.2.2], there are equivalences of marked simplicial sets

$$(\mathbb{C}_{/y})_x \xrightarrow{\sim} (\mathbb{C}^{x/})_y \xrightarrow{\sim} \mathrm{Un}^{\mathrm{sc}}_* (\mathfrak{C}[\mathbb{C}](x,y))$$

where $\mathrm{Un}^{\mathrm{sc}}_*$ is the scaled *coCartesian* unstraightening of [Lur09a, §3.5]. We also have, by 3.2, a comparison map

$$\beta : \mathrm{Tw}(\mathbb{C})_{x,y} \to \mathrm{Un}^{+}_* (\mathfrak{C}[\mathbb{C}](x,y)),$$

where Un^{+}_* is the marked *Cartesian* unstraightening.

We now aim to compare these two comparison maps, using the equivalence between $\mathrm{Tw}(\mathbb{C})_y$ and $\mathbb{C}_{/y}$ of 3.16. The first step is to relate the scaled coCartesian and marked Cartesian straightenings over the point.

3.20. Construction. We denote the former by $\mathrm{St}^{\mathrm{sc}}$ and the latter by St^{+}, leaving the point implicit. These give us two functors

$$\mathrm{St}^{\mathrm{sc}}, \mathrm{St}^{+} : \mathrm{Set}_{\Delta^+} \to \mathrm{Set}^{+}_{\Delta}$$

By [ADS20, Lem. 4.3.3], to display a natural equivalence between them it will suffice to display it on simplices. By definition, we have that $\mathrm{St}^{\mathrm{sc}}(\Delta^n) = \mathfrak{C}^{\mathrm{sc}}[\widetilde{\square}(n)](x,y)$ and $\mathrm{St}^{+}(\Delta^n) = \mathfrak{C}^{\mathrm{sc}}[\widetilde{\bigstar}(n)](x,y)$.

Since the collapse map

$$\begin{array}{ccc} \Delta^n \times \Delta^1 & \longrightarrow & \Delta^n \star \Delta^0 \\ (i,k) & \longmapsto & \begin{cases} i & k=0 \\ n+1 & k=1 \end{cases} \end{array}$$

preserves the scaling and ordered partitions, in this fashion we obtain compatible maps $\theta_n : \mathrm{St}^{\mathrm{sc}}((\Delta^n)^\flat) \to \mathrm{St}^{+}((\Delta^n)^\flat)$ and $\mathrm{St}^{\mathrm{sc}}((\Delta^1)^\sharp) \to \mathrm{St}^{+}((\Delta^n)^\sharp)$ Moreover, the triangles

$$\begin{array}{ccc} \mathrm{St}^{\mathrm{sc}}((\Delta^n)^\flat) & \xrightarrow{\theta_n} & \mathrm{St}^{+}((\Delta^n)^\flat) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & (\Delta^n)^\flat & \end{array}$$

commute, where q is the map π of [Lur09, Prop 3.2.1.14], and p is the map α of [Lur09a, Prop. 3.6.1]. Since both of these are marked equivalences, we have that θ_n is as well. Thus, θ extends to a natural equivalence $\theta : \mathrm{St}^{\mathrm{sc}} \Longrightarrow \mathrm{St}^{+}$.

It immediately follows that

3.21. LEMMA. *The natural transformation* $\mu : \mathrm{Un}^{\mathrm{sc}} \to \mathrm{Un}^{+}$ *adjoint to* θ *is an equivalence.*

It now remains only for us to show

3.22. PROPOSITION. *The diagram*

$$\begin{array}{ccccc} (\mathbb{C}_{/y})_x & \xleftarrow{\sim} & \mathcal{M}_{x,y} & \xrightarrow{\sim} & \mathrm{Tw}(\mathbb{C})_{(x,y)} \\ \simeq \downarrow f & & & & \downarrow \beta \\ \mathrm{Un}^{\mathrm{sc}}_{*}(\mathfrak{C}^{\mathrm{sc}}[\mathbb{D}](x,y)) & & \xleftarrow[\simeq]{} & & \mathrm{Un}^{+}_{*}(\mathfrak{C}^{\mathrm{sc}}[\mathbb{D}](x,y)) \end{array} \tag{3}$$

commutes up to natural equivalence.

To effect a proof, we first note that the maps f and β are both induced by maps of posets. For the reader's convenience, we briefly unwind how in the case of β. For f, we merely state the poset map in question, and leave it to the interested reader to unwind the definitions.

3.23. REMARK. Given an n-simplex σ of $\mathrm{Tw}(\mathbb{C})_{(x,y)}$, the simplex $\beta(\sigma)$ of $\mathrm{Un}^{+}(\mathfrak{C}^{\mathrm{sc}}[\mathbb{D}](x,y))$ is given by pulling back the rigidification $\mathfrak{C}[\tilde{\sigma}]$ of the adjoint map

$$\tilde{\sigma} : \widetilde{Q}(n) \to \mathbb{C}$$

along the maps ζ_i constructed in 3.2. Using the poset-quotient description of the mapping spaces, however, one can easily check that the ζ_i's combine to define a map

$$\begin{aligned} B :\ \mathcal{P}_{\bigstar(n)} &\longrightarrow \mathcal{P}_{Q(n)} \\ (S_0, v) &\longmapsto (S_0, \tau(S_0)) \end{aligned}$$

with associated map on the quotient $\widetilde{B} : \mathfrak{C}^{\mathrm{sc}}[\widetilde{\bigstar}(n)](*_0, *_1) \to \mathfrak{C}^{\mathrm{sc}}[\widetilde{Q}(n)](*_0, *_1)$. That is, $\beta(\sigma)$ is defined by pulling back $\mathfrak{C}[\tilde{\sigma}]$ along $\widetilde{B}$.

More generally, let σ be a simplex of $\mathcal{M}_{x,y}$ and $\tilde{\sigma} : \widetilde{\boxtimes}(n) \to \mathbb{C}$ its adjoint. The right-hand composite $\gamma : \mathcal{M}_{x,y} \to \mathrm{Un}^{\mathrm{sc}}_{*}(\mathfrak{C}^{\mathrm{sc}}[\mathbb{D}](x,y))$ in (3) is given by pulling $\mathfrak{C}[\tilde{\sigma}]$ back along a map $\mathfrak{C}^{\mathrm{sc}}[(\square(n))^{R}](*_0, *_1) \to \mathfrak{C}^{\mathrm{sc}}[\widetilde{\boxtimes}(n)](*_0, *_1)$ induced by[5]

$$\begin{aligned} G :\ P_{\square(n)} &\longrightarrow P_{\boxtimes(n)} \\ (S_0, S_1) &\longmapsto (S_0, \tau(S_0), v). \end{aligned}$$

[5]In point of fact, unraveling the definitions would lead one to believe that map is induced by $(S_0, S_1) \mapsto (S_0, \tau(S_0), \varnothing)$, however, both this map and G lead to the same map on quotients, so the distinction is irrelevant.

The left hand composite $\eta : \mathcal{M}_{x,y} \to \mathrm{Un}^{\mathrm{sc}}_*(\mathfrak{C}^{\mathrm{sc}}[\mathbb{D}](x,y))$ in (3) is given by pulling $\mathfrak{C}[\tilde{\sigma}]$ back along a map $\mathfrak{C}^{\mathrm{sc}}[(\square(n))^R](*_0,*_1) \to \mathfrak{C}^{\mathrm{sc}}[\widetilde{\boxtimes}(n)](*_0,*_1)$ induced by

$$\begin{aligned} H: \quad P_{\square(n)} &\longrightarrow P_{\boxtimes(n)} \\ (S_0,S_1) &\longmapsto (S_0,\varnothing,\{v\}). \end{aligned}$$

With these definitions in place, we can proceed to the final step of our proof.

Proof of 3.22. We will define an explicit homotopy $(\Delta^1)^\sharp \times \mathcal{M}_{x,y} \to \mathrm{Un}^{\mathrm{sc}}_*(\mathfrak{C}^{\mathrm{sc}}[\mathbb{D}](x,y))$ between γ and η.

Given an n-simplex $(\rho,\sigma) : \Delta^n \to (\Delta^1)^\sharp \times \mathcal{M}_{x,y}$, we note that ρ is uniquely specified by $0 \leqslant i \leqslant n+1$:

$$(0,0,\dots,0,\overbrace{1}^{i},1,\dots,1).$$

We define, for $S_0 \subset [n]$, the subset $S_0^{\geqslant i} := \{s \in S_0 \mid s \geqslant i\}$ and then define a map

$$\begin{aligned} h_\rho: \quad P_{\square(n)} &\longrightarrow P_{\boxtimes(n)} \\ (S_0,S_1) &\longrightarrow (S_0,\tau(S_0^{\geqslant i}),v) \end{aligned}$$

Note that, when $i = n+1$ (i.e. ρ is constant on 0) we have that $\tau(S_0^{\geqslant i}) = \varnothing$, so that the map specializes to H. Similarly, when $i = 0$ (i.e. ρ is constant on 1) $S_0^{\geqslant i} = S_0$, so that the map specializes to G.

Let us check that h_ρ descends to quotients. In order to do so, given a k-simplex $\underline{S}$ we compute the ambidextrous truncation $h_\rho(\underline{S})^A_\ell$ in $\mathcal{P}_{\boxtimes(n)}$, as defined in 1.23. Let $l \in [k]$ and denote $(S_l)_0 \cap [s_0^L, n] = \widehat{S}_l$. Then we obtain

$$h_\rho(\underline{S})^A_l = \begin{cases} (\widehat{S}_l, \tau(\widehat{S}_l)) \text{ if } i \leqslant s_0^L \\ \Big(\widehat{S}_l, \tau(\widehat{S}_l) \cap [n+1,\tau(i)]\Big) \text{ if } s_0^L < i < n+1 \\ (\widehat{S}_l, \varnothing, v) \text{ if } i = n+1 \end{cases}$$

since this only depends on the truncation of $\underline{S}$ the claim follows. It is immediate that the maps respect the simplicial identities, so sending a simplex $(\rho,\sigma) \in \Delta^1 \times \mathcal{M}_{x,y}$ to the simplex $h_\rho^*(\mathfrak{C}[\tilde{\sigma}]) \in \mathrm{Un}^{\mathrm{sc}}(\mathfrak{C}[\mathbb{C}](x,y))$ defines a homotopy $\eta \Longrightarrow \gamma$.

To see that it is a marked homotopy, consider the 1-simplex $(0,1)$ in Δ^1, and a degenerate 1-simplex $\sigma : \boxtimes(1)_\sharp \to \mathbb{C}$ in $\mathcal{M}_{x,y}$. This corresponds to a map $(\mathcal{P}_{\boxtimes(1)})^\sharp \to \mathfrak{C}[\mathbb{C}](x,y)$, and so pulling back along $h_{\{0,1\}} : \mathcal{P}_{\square(1)} \to \mathcal{P}_{\boxtimes(1)}$ yields a map

$$h^*_{\{0,1\}}(\gamma) : (\mathcal{P}_{\square(1)})^\sharp \to \mathfrak{C}[\mathbb{C}](x,y),$$

i.e., a marked morphism in $\mathrm{Un}^{\mathrm{sc}}(\mathfrak{C}[\mathbb{C}](x,y))$. We have thus defined a marked homotopy as desired, and the proof is complete. ∎

For completeness, we can now give

Proof of 3.4. Using 3.22, the theorem follows immediately by 2-out-of-3 from 3.16, the equivalence of [GHL19, Prop. 2.33], and 3.21. ■

4. Natural transformations as an end

In this section we will denote by X a maximally scaled simplicial set and by $\mathbb{D}$ an ∞-bicategory that will remain fixed throughout. Given a pair of functors $F, G : X \to \mathbb{D}$ we will denote the associated mapping category in $\mathbb{D}^X$ by $\mathrm{Nat}_X(F,G)$. The aim of this section is to show the next result

4.1. Theorem. *Let X be a maximally scaled simplicial set and let $\mathbb{D}$ be an ∞-bicategory. Given a pair of morphisms $F, G : X \to \mathbb{D}$ of scaled simplicial sets we consider a functor*

$$N_{(F,G)} : \mathrm{Tw}(X)^{\mathrm{op}} \longrightarrow X^{\mathrm{op}} \times X \xrightarrow{F^{\mathrm{op}} \times G} \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \xrightarrow{\mathrm{Map}_{\mathbb{D}}(-,-)} \mathfrak{Cat}_\infty.$$

Then there exists an equivalence of ∞-categories

$$\mathrm{Nat}_X(F,G) \xrightarrow{\simeq} \lim_{\mathrm{Tw}(X)^{\mathrm{op}}} N_{(F,G)}$$

which is natural in both variables.

Following the terminology of [GHN15] or main theorem shows how to express the ∞-category of natural transformations as an end.

4.2. Definition. *Let $\ell : \mathcal{D}^X \times \left(\mathcal{D}^X\right)^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{Tw}(X), \mathcal{D} \times \mathcal{D}^{\mathrm{op}})$ be the functor that maps a simplex of the product $\sigma_1 : X \times \Delta^n \to \mathcal{D}$, $\sigma_2 : X \times (\Delta^n)^{\mathrm{op}} \to \mathcal{D}$ to the composite*

$$\mathrm{Tw}(X) \times \Delta^n \longrightarrow X \times X^{\mathrm{op}} \times \Delta^n \longrightarrow \mathcal{D} \times \mathcal{D}^{\mathrm{op}}.$$

where the second morphism is constructed using the universal property of the product by means of the morphisms

$$X \times X^{\mathrm{op}} \times \Delta^n \longrightarrow X \times \Delta^n \xrightarrow{\sigma_1} \mathcal{D}$$

$$X \times X^{\mathrm{op}} \times \Delta^n \longrightarrow X^{\mathrm{op}} \times \Delta^n \xrightarrow{\sigma_2^{\mathrm{op}}} \mathcal{D}^{\mathrm{op}}$$

In formulas, this means the map

$$X \times X^{\mathrm{op}} \times \Delta^n \longrightarrow \mathcal{D} \times \mathcal{D}^{\mathrm{op}}$$

sends a triple of n-simplices (α, β, γ) to the pair of n-simplices $(\sigma_1(\alpha,\gamma), \sigma_2(\beta,\gamma))$.

We define a marked simplicial set $\mathcal{L}_X$ *equipped with Cartesian fibration to* $\mathcal{D}^X \times \left(\mathcal{D}^X\right)^{\mathrm{op}}$ *via the pullback square*

$$\begin{array}{ccc} \mathcal{L}_X & \longrightarrow & \mathrm{Fun}(\mathrm{Tw}(X)^\sharp, \mathrm{Tw}(\mathbb{D})^\dagger) \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathcal{D}^X \times \left(\mathcal{D}^X\right)^{\mathrm{op}} & \xrightarrow{\ell} & \mathrm{Fun}(\mathrm{Tw}(X), \mathcal{D} \times \mathcal{D}^{\mathrm{op}}). \end{array}$$

4.3. Remark. Given a pair of functors F, G we see that an n-simplex of the fibre $\Delta^n \to (\mathcal{L}_X)_{(F,G)}$ is specified by a map

$$\sigma : \mathrm{Tw}(X)^\natural \times (\Delta^n)^\flat \longrightarrow \mathrm{Tw}(\mathbb{D})^\dagger$$

such that the composite map

$$\mathrm{Tw}(X) \times \Delta^n \xrightarrow{\sigma} \mathrm{Tw}(\mathbb{D}) \longrightarrow \mathcal{D} \times \mathcal{D}^{\mathrm{op}}$$

with the fibration $\mathrm{Tw}(\mathbb{D}) \to \mathcal{D} \times \mathcal{D}^{\mathrm{op}}$ is equal to the totally degenerate n-simplex

$$\mathrm{Tw}(X) \times \Delta^n \longrightarrow X \times X^{\mathrm{op}} \xrightarrow{F \times G^{\mathrm{op}}} \mathcal{D} \times \mathcal{D}^{\mathrm{op}}$$

in $\mathrm{Fun}(\mathrm{Tw}(X), \mathcal{D} \times \mathcal{D}^{\mathrm{op}})$.

That is, each n-simplex $\Delta^n \to (\mathcal{L}_X)_{(F,G)}$ is given by a commutative diagram

$$\begin{array}{ccc} \mathrm{Tw}(X)^\natural \times (\Delta^n)^\flat & \longrightarrow & \mathrm{Tw}(\mathbb{D})^\dagger \\ \big\downarrow & & \big\downarrow \\ X \times X^{\mathrm{op}} & \xrightarrow{F \times G^{\mathrm{op}}} & \mathcal{D} \times \mathcal{D}^{\mathrm{op}} \end{array}$$

where the left-most vertical morphism is given by the composite

$$\mathrm{Tw}(X) \times \Delta^n \to \mathrm{Tw}(X) \to X \times X^{\mathrm{op}}$$

of the projection to $\mathrm{Tw}(X)$ with the fibration $\mathrm{Tw}(X) \to X \times X^{\mathrm{op}}$.

By the universal property of pullback, such simplices are equivalently elements of the ∞-category of Cartesian sections of the Cartesian fibration

$$\mathrm{Tw}(\mathbb{D})^\dagger \times_{\mathcal{D} \times \mathcal{D}^{\mathrm{op}}} \mathrm{Tw}(X)^\natural \longrightarrow \mathrm{Tw}(X)^\natural$$

defined as the pullback

$$\begin{array}{ccc} \mathrm{Tw}(\mathbb{D})^\dagger \times_{\mathcal{D} \times \mathcal{D}^{\mathrm{op}}} \mathrm{Tw}(X)^\natural & \longrightarrow & \mathrm{Tw}(\mathbb{D}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Tw}(X)^\natural & \longrightarrow & \mathcal{D} \times \mathcal{D}^{\mathrm{op}} \end{array}$$

In other words, the fibre $(\mathcal{L}_X)_{(F,G)}$ can be identified with the ∞-category of Cartesian sections of the Cartesian fibration associated to $N_{(F,G)}$. In particular it follows from [Lur09, Cor. 3.3.3.2] that $(\mathcal{L}_X)_{(F,G)}$ is a model for the limit $\lim\limits_{\mathrm{Tw(X)^{op}}} N_{(F,G)}$. We will abuse notation and denote the fiber by $\mathcal{L}_{(F,G)}$ when the indexing ∞-category X is clear from the context.

Recall the canonical map $\mathbb{D}^X \times X \to \mathbb{D}$ and note that since $\mathrm{Tw}(-)$ preserves limits we can use the tensor-hom adjunction to produce

$$u : \mathrm{Tw}\left(\mathbb{D}^X\right) \to \mathrm{Fun}(\mathrm{Tw}(X)^\sharp, \mathrm{Tw}(\mathbb{D})^\dagger)$$

fitting into the commutative diagram

$$\begin{array}{ccc} \mathrm{Tw}\left(\mathbb{D}^X\right) & \xrightarrow{\;u\;} & \mathrm{Fun}(\mathrm{Tw}(X)^\sharp, \mathrm{Tw}(\mathbb{D})^\dagger) \\ \downarrow & & \downarrow \\ \mathcal{D}^X \times \left(\mathcal{D}^X\right)^{\mathrm{op}} & \xrightarrow{\;\ell\;} & \mathrm{Fun}(\mathrm{Tw}(X), \mathcal{D} \times \mathcal{D}^{\mathrm{op}}). \end{array} \tag{4}$$

This in turn yields a map of Cartesian fibrations $\Theta_X : \mathrm{Tw}(\mathbb{D}^X) \to \mathcal{L}_X$ which we call the *canonical comparison map*. The rest of this section is devoted to showing that Θ_X is a fiberwise equivalence for every simplicial set X. Our first observation is that both constructions behave contravariantly in the simplicial set X thus producing functors

$$\mathrm{Tw}\left(\mathbb{D}^{(-)}\right), \mathcal{L}_{(-)} : \mathrm{Set}_\Delta^{\mathrm{op}} \to \mathrm{Set}_\Delta^+$$

equipped with a natural transformation $\Theta : \mathrm{Tw}\left(\mathbb{D}^{(-)}\right) \Longrightarrow \mathcal{L}_{(-)}$. We can now reformulate 4.1 in term of Cartesian fibrations.

4.4. Theorem. *For every simplicial set $X \in \mathrm{Set}_\Delta$, the map $\Theta_X : \mathrm{Tw}\left(\mathbb{D}^X\right) \to \mathcal{L}_X$ is an equivalence of Cartesian fibrations over $\mathcal{D}^X \times \left(\mathcal{D}^X\right)^{\mathrm{op}}$.*

Our proof strategy will consist in reducing the problem to the case $X = \Delta^n$ with $n = 0, 1$. In order to achieve this we will show that the both functors are homotopically well-behaved.

4.5. Proposition. *Let $\alpha : X \to Y$ be a cofibration of simplicial sets. Then for every pair of functors $F, G \in \mathcal{D}^Y$ the induced maps*

$$\mathrm{Tw}\left(\mathbb{D}^Y\right)_{(F,G)} \to \mathrm{Tw}\left(\mathbb{D}^X\right)_{(\alpha^*F, \alpha^*G)}, \; (\mathcal{L}_Y)_{(F,G)} \to (\mathcal{L}_X)_{(\alpha^*F,\alpha^*G)}$$

are fibrations in the Joyal model structure.

Proof. Let us observe that due to Theorem 2.6, the marked simplicial sets $\mathrm{Tw}(\mathbb{D}^Y)_{(F,G)}$ and $\mathrm{Tw}(\mathbb{D}^X)_{(\alpha^*F,\alpha^*G)}$ are fibrant. By [Lur09, Cor. 2.4.6.5], to check that the first map is a Joyal fibration on underlying (Joyal fibrant) simplicial sets, it it suffices to show that the underlying map is an inner fibration and an isofibration. That is, it suffices to solve the lifting problems

$$\begin{array}{ccc} (\Lambda^n_i)^\flat & \longrightarrow & \mathrm{Tw}(\mathbb{D}^Y)_{(F,G)} \\ \downarrow & & \downarrow \\ (\Delta^n)^\flat & \longrightarrow & \mathrm{Tw}(\mathbb{D}^X)_{(\alpha^*F,\alpha^*G)} \end{array} \qquad \begin{array}{ccc} (\Delta^0)^\sharp & \longrightarrow & \mathrm{Tw}(\mathbb{D}^Y)_{(F,G)} \\ \downarrow & & \downarrow \\ (\Delta^1)^\sharp & \longrightarrow & \mathrm{Tw}(\mathbb{D}^X)_{(\alpha^*F,\alpha^*G)} \end{array}$$

with $n \geqslant 2$ and $0 < i < n$. These lifting problems can be easily seen to be equivalent to their adjoint problems (where we are using the notation of the proof of 2.6)

$$\begin{array}{ccc} (K^n_i)_\dagger \times Y \coprod_{(K^n_i)_\dagger \times X} Q(n) \times X & \longrightarrow & \mathbb{D} \\ \downarrow & \nearrow & \\ Q(n) \times Y & & \end{array} \qquad \begin{array}{ccc} \mathrm{Sp}^3 \times Y \coprod_{\mathrm{Sp}^3 \times X} \Delta^3_\sharp \times X & \longrightarrow & \mathbb{D} \\ \downarrow & \nearrow & \\ \Delta^3_\sharp \times Y & & \end{array}$$

which admit solutions by virtue of [Lur09a, Proposition 3.1.8], since the map $(K^n_i)_\dagger \to Q(n)$ is scaled anodyne as shown in Lemma 2.10. The proof for the other functor is almost analogous. First we note that the induced map $\mathrm{Tw}(X) \to \mathrm{Tw}(Y)$ is a cofibration of marked simplicial sets. Let $A^\diamond \to B^\diamond$ be a marked anodyne morphism, then using the pushout-product axiom for marked anodyne maps (see e.g. [Lur09, Prop. 3.1.2.3]) we see that lifting problems of the form

$$\begin{array}{ccc} \mathrm{Tw}(X)^\sharp \times B^\diamond \coprod_{\mathrm{Tw}(X)^\sharp \times A^\diamond} \mathrm{Tw}(Y)^\sharp \times A^\diamond & \longrightarrow & \mathrm{Tw}(\mathbb{D})^\dagger \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Tw}(Y)^\sharp \times B^\diamond & \longrightarrow & \mathcal{D} \times \mathcal{D}^{\mathrm{op}} \end{array}$$

admit a solution. The claim follows immediately from this fact coupled with the above characterization of Joyal fibrations between fibrant objects, [Lur09, Cor. 2.4.6.5]. ∎

4.6. Proposition. *Let $P_i : \mathcal{O}_i \to \mathrm{Set}_\Delta$ with $i = 1, 2$, be two diagrams of simplicial sets such that*

1) *P_1 is a cotower diagram such that for every $\ell \to k$ in $\mathcal{O}_1$ the induced morphism $P_1(\ell) \to P_2(k)$ is a cofibration.*

2) *P_2 is a cospan diagram, such that one leg of the span is mapped to a cofibration.*

Denote by X_i the colimit of P_i and by $\{\beta_j\}_{j \in \mathcal{O}_i}$ the canonical cone of X_i. Given $F, G \in \mathbb{D}^{X_i}$ then it follows that we have equivalences of ∞-categories

$$\mathrm{Tw}(\mathbb{D}^{X_i})_{(F,G)} \simeq \operatorname*{holim}_{j \in \mathcal{O}_i^{\mathrm{op}}} \mathrm{Tw}\left(\mathbb{D}^{P_i(j)}\right)_{(\beta_j^*F, \beta_j^*G)}, \quad \mathcal{L}_{(F,G)} \simeq \operatorname*{holim}_{j \in \mathcal{O}_i^{\mathrm{op}}} \mathcal{L}_{(\beta_j^*F, \beta_j^*G)}$$

Proof. We will show that the functors $\mathrm{Tw}(\mathbb{D}^{-})$ and $\mathcal{L}_{(-)}$ preserves the ordinary limits of shape $\mathcal{O}_i^{\mathrm{op}}$. We consider the composite

$$\mathcal{O}_i^{\mathrm{op}} \xrightarrow{P_i} \mathrm{Set}_\Delta^{\mathrm{op}} \longrightarrow \mathrm{Set}_\Delta^{+}$$

where the last functor is either $\mathrm{Tw}(\mathbb{D}^{-})$ or $\mathcal{L}_{(-)}$. Observe that $\mathrm{Tw}(-)$ preserves limits of scaled simplicial sets since it is constructed as a right adjoint. Moreover, since $\mathbb{D}^{-}$ sends colimits of simplicial sets to limits of scaled simplicial sets we see that

$$\lim_{\mathcal{O}_i^{\mathrm{op}}} \mathrm{Tw}(\mathbb{D}^{P_i(j)}) \cong \mathrm{Tw}(\lim_{\mathcal{O}_i^{\mathrm{op}}} \mathrm{Tw}(\mathbb{D}^{P_i(j)})) \cong \mathrm{Tw}(\mathbb{D}^{X_i}).$$

To prove the claim for $\mathcal{L}_{-}$ first we note that the usual twisted arrow category functor $\mathrm{Tw} : \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ preserves colimits of shape $\mathcal{O}_i$ for $i = 1, 2$. Unraveling the definitions this implies that

$$\lim_{\mathcal{O}_i^{\mathrm{op}}} \mathcal{L}_{P_i(j)} \cong \mathcal{L}_{X_i}.$$

In order to finish the proof we observe that since taking fibers commutes with limits it will be enough to show that both diagrams are injectively fibrant. This follows immediately from our hypothesis and 4.5 together with the duals of [Du08, Prop. 14.5] and [Du08, Prop. 14.10]. ∎

4.7. Lemma. *Let $\iota : \Lambda_i^n \to \Delta^n$ be an inner horn inclusion. Then for every $F, G \in \mathcal{D}^{\Delta^n}$ we have equivalences of ∞-categories*

$$\mathrm{Tw}(\mathbb{D}^{\Delta^n})_{(F,G)} \xrightarrow{\simeq} \mathrm{Tw}(\mathbb{D}^{\Lambda_i^n})_{(\iota^*F,\iota^*G)}, \quad \mathcal{L}_{(F,G)} \xrightarrow{\simeq} \mathcal{L}_{(\iota^*F,\iota^*G)}.$$

Proof. First, let us observe that $\mathbb{D}^{\Delta^n} \to \mathbb{D}^{\Lambda_i^n}$ is a trivial fibration in the scaled model structure. After noticing this, the result follows immediately for Tw. To show the claim for the second functor we just need to show that the inclusion $\iota : \mathrm{Tw}(\Lambda_i^n) \to \mathrm{Tw}(\Delta^n)$ is cofinal. Then the result will follow from the fact that restriction along ι^{op} preserves limits, since $\mathcal{L}_{(F,G)}$ and $\mathcal{L}_{(\iota^*F,\iota^*G))}$ are limits over $\mathrm{Tw}(\Delta^n)^{\mathrm{op}}$ and $\mathrm{Tw}(\Lambda_i^n)^{\mathrm{op}}$, respectively.

To see that ι is cofinal, we check that it satisfies the hypothesis of Quillen's Theorem A. We notice that we can identify $\mathrm{Tw}(\Delta^n)$ with the nerve of the of the poset of subintervals in $[n]$, ordered by reverse inclusion (see, e.g. [DK19, Section 10.1]). Under this identification, $\mathrm{Tw}(\Lambda_i^n)$ is the nerve of the full sub-poset on all objects other than $[n]$. Since this object is inital in $\mathrm{Tw}(\Delta^n)$, we see that $\mathrm{Tw}(\Lambda_i^n)_{[n]/}$ is isomorphic to $\mathrm{Tw}(\Lambda_i^n)$. Since the latter is contractible, the proof is complete. ∎

4.8. Proposition. *Suppose the map Θ_X in 4.4 is an equivalence of Cartesian fibrations for $X = \Delta^n$ with $n = 0, 1$. Then for every $X \in \mathrm{Set}_\Delta$ the map Θ_X is an equivalence of Cartesian fibrations.*

Proof. We will say that a simplicial set X satisfies the property ($\circledast$) if Θ_X is an equivalence of Cartesian fibrations. First we will assume that the simplicial sets Δ^n with $n \geqslant 0$ satisfy ($\circledast$).

As a direct consequence of 4.6 2), we deduce that boundaries $\partial\Delta^n$ fulfill condition ($\circledast$) for $n \geqslant 0$. Let X be an arbitrary simplicial set. We claim that given $n \geqslant 0$ the n-skeleton $\mathrm{sk}_n(X)$ satisfies ($\circledast$). It is clear that the claim holds for $\mathrm{sk}_0(X)$ since it is just a disjoint union of points. Suppose that the claim holds for $\mathrm{sk}_{l-1}(X)$ and let I be the set of non degenerate simplices contained in $\mathrm{sk}_l(X) \setminus \mathrm{sk}_{l-1}(X)$. Given $i \in I$ we can attach that non-degenerate simplex via a pushout square

$$\begin{array}{ccc} \partial\Delta^l & \longrightarrow & \Delta^l \\ \downarrow & & \downarrow \\ \mathrm{sk}_{l-1}(X) & \longrightarrow & N \end{array}$$

4.6 2) implies that Θ_N is an equivalence. Now let us pick a linear order on I and attach one by one all the simplices in I. We can then produce a functor

$$P : I \to \mathrm{Set}_\Delta, \text{ such that } \operatorname*{colim}_I P \cong \mathrm{sk}_l(X).$$

which is an instance of 4.6 1) and therefore the inductive step is proved. The same proposition now applied to $X \cong \mathrm{colim}_{\mathbb{N}}\, \mathrm{sk}_n(X)$ finally shows that Θ_X is an equivalence of Cartesian fibrations provided Θ_{Δ^n} is an equivalence for $n \geqslant 0$.

We will use again induction to show that Θ_{Δ^n} is an equivalence for $n \geqslant 0$. Our ground cases are $n = 0, 1$. Assume the claim holds for $(n-1) \geqslant 1$ and pick an inner horn inclusion $\iota : \Lambda^n_i \to \Delta^n$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Tw}(\mathbb{D}^{\Delta^n})_{(F,G)} & \longrightarrow & (\mathcal{L}_{\Delta^n})_{(F,G)} \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Tw}(\mathbb{D}^{\Lambda^n_i})_{(\iota^*F,\iota^*G)} & \xrightarrow{\simeq} & \left(\mathcal{L}_{\Lambda^n_i}\right)_{(\iota^*F,\iota^*G)} \end{array}$$

where the vertical morphisms are equivalences due to 4.7. It is easy to see that the bottom horizontal morphism is an equivalence by to the inductive hypothesis. The result follows from 2-out-of-3. ■

At this point we have made a drastic reduction in complexity and we are left to show that the object Δ^1 satisfies ($\circledast$), the case of Δ^0 being obvious. We will tackle this last case by a direct computational approach. Before diving into the proof of 4.4 we will take a small detour to analyze the relevant combinatorics. Throughout the rest of this section we will use the coordinates $a \leqslant b$ for Δ^1 instead of the standard $0 \leqslant 1$ notation. In a similar fashion, we denote the coordinates of $\mathrm{Tw}(\Delta^1)$ by $ab \to aa$, $ab \to bb$.

4.9. Definition. *We define a cosimplicial object*

$$\mathcal{R} : \Delta \longrightarrow \mathrm{Set}_\Delta^{\mathrm{sc}}$$
$$[n] \longmapsto (\mathcal{R}(n), T),$$

$$\mathcal{R}(n) = (\Delta^n \times \Delta^1) \star (\Delta^n \times \Delta^1)^{\mathrm{op}} \coprod_{\Delta^n \star (\Delta^n)^{\mathrm{op}}} (\Delta^n \times \Delta^1) \star (\Delta^n \times \Delta^1)^{\mathrm{op}}$$

Where the pushout is taken over the diagram induced by

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\times\{a\}} & \Delta^n \times \Delta^1 \\ {\scriptstyle \times\{a\}}\downarrow & & \\ \Delta^n \times \Delta^1 & & \end{array}$$

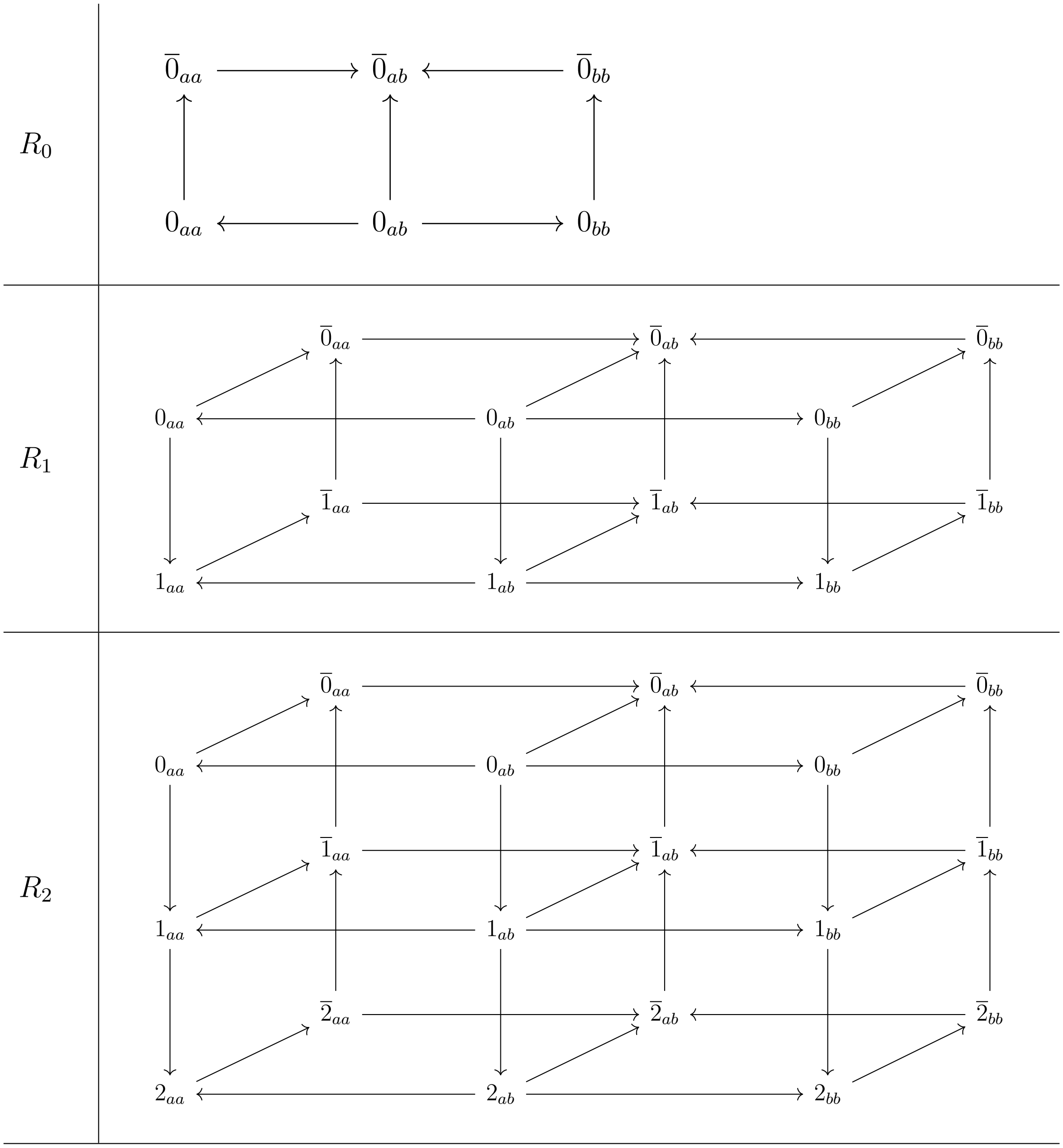
We describe the scaling using the notation of 4.10. T is the scaling which is (1) identical on the two summands and (2) such that the non-degenerate thin 2-simplices of the first summand $(\Delta^n \times \Delta^1) \star (\Delta^n \times \Delta^1)^{\mathrm{op}}$ are those σ such that

- *σ factors through either $(\Delta^n \times \Delta^1)$ or $(\Delta^n \times \Delta^1)^{\mathrm{op}}$.*
- *$i_p < j_q < k_r$ is a simplex in $\Delta^n \times \Delta^1$, and $\sigma = (i_p < j_q < \overline{k_r})$.*
- *$k_r < j_q < i_p$ is a simplex in $\Delta^n \times \Delta^1$ and $\sigma = (i_p < \overline{j_q} < \overline{k_r})$.*
- *$i \leqslant j \leqslant k$ is a simplex of Δ^n and*
  - *$\sigma = i_{ab} < j_{aa} < \overline{k_{aa}}$;*
  - *$\sigma = k_{ab} < \overline{j_{aa}} < \overline{i_{ab}}$;*
  - *$\sigma = i_{aa} < j_{aa} < \overline{k_{ab}}$;*
  - *$\sigma = k_{ab} < \overline{j_{aa}} < \overline{i_{aa}}$;*
  - *$\sigma = i_{ab} < j_{ab} < \overline{k_{aa}}$; or*
  - *$\sigma = k_{aa} < \overline{j_{ab}} < \overline{i_{ab}}$.*

4.10. Remark. We can describe the underlying simplicial set of $\mathcal{R}(n)$ as the nerve of a poset R_n as follows

- The set of objects is given by symbols ℓ_ε where $\ell \in [n]$ and $\varepsilon \in \{ab, aa, bb\}$ together with their formal duals $\overline{\ell}_\varepsilon$.
- We declare $\ell_{ab} \leqslant k_\varepsilon$ where $\varepsilon \in \{ab, aa, bb\}$ if and only if $\ell \leqslant k$. Dually we declare $\overline{\ell}_{ab} \leqslant \overline{k}_\varepsilon$ if and only if $k \leqslant \ell$. Finally we declare $\ell_\varepsilon < \overline{\ell}_\varepsilon$. The ordering on R_n is the minimal one generated by the inequalities above.

We provide graphical representations of the posets for $n \leqslant 2$:



The posets R_n for $n \leq 2$.

4.11. Remark. We observe that the posets above come equipped with an isomorphism $(R_n)^{\mathrm{op}} \cong R_n$ given by applying the "bar operator" $\overline{(-)}$. It is worth pointing out that our scaling is symmetric with respect to this duality.

4.12. Remark. In the following work, we make extensive use of the simple decomposition of $\mathrm{Tw}(\Delta^1)$ into a pushout,

$$\mathrm{Tw}(\Delta^1) \cong \Delta^1 \coprod_{\Delta^{\{0\}}} \Delta^1.$$

taken over the inclusions of the initial vertices. Under this identification, the two induced maps $\Delta^1 \to \mathrm{Tw}(\Delta^1)$ are given by sending $0 \mapsto ab$ and by sending 1 to aa or bb respectively.

4.13. Definition. *We define a cosimplicial object*

$$\begin{aligned} \mathcal{Q} : \Delta &\longrightarrow \mathrm{Set}_\Delta^{\mathrm{sc}} \\ [n] &\longmapsto Q(\Delta^n \times \mathrm{Tw}(\Delta^1)^\sharp) \end{aligned}$$

where Q was already introduced in 2.2 and the functoriality is the obvious one. Since Q preserves colimits we see that $\mathcal{Q}(n)$ splits into

$$Q(\Delta^n \times \mathrm{Tw}(\Delta^1)) \cong Q(\Delta^n \times \Delta^1) \coprod_{Q(\Delta^n)} Q(\Delta^n \times \Delta^1).$$

4.14. Remark. Recall that our definitions imply that a map $\mathcal{Q}(n) \to \mathbb{D}$ corresponds precisely to a functor $\Delta^n \times \mathrm{Tw}(\Delta^1) \to \mathrm{Tw}(\mathbb{D})$. We see that a simplex in $\mathcal{L}_{(F,G)}$ is given by a map $\mathcal{Q}(n) \to \mathbb{D}$ satisfying the obvious conditions after restriction to $\Delta^n \times \mathrm{Tw}(\Delta^1), (\Delta^n \times \mathrm{Tw}(\Delta^1))^{\mathrm{op}} \subset Q(n)$.

4.15. Notation. Let $n \geqslant 0$ and observe that $\mathcal{R}(n)$ fits into a cocone for the colimit defining $\mathcal{Q}(n)$. Then the induced cofibrations $\varepsilon_n : \mathcal{Q}(n) \to \mathcal{R}(n)$, assemble into map of cosimplicial objects $\xi : \mathcal{Q} \Longrightarrow \mathcal{R}$.

4.16. Definition. *We define a cosimplicial object*

$$\begin{aligned} \mathcal{T} : \Delta &\longrightarrow \mathrm{Set}_\Delta^{\mathrm{sc}} \\ [n] &\longmapsto Q(n) \times \Delta^1. \end{aligned}$$

4.17. Remark. Analogously to 4.14, we can identify a simplex $\Delta^n \to \mathrm{Tw}(\mathbb{D}^{\Delta^1})_{(F,G)}$ with a map $\mathcal{T}(n) \to \mathbb{D}$ such that the restrictions to $\Delta^n \times \Delta^1$ and $(\Delta^n)^{\mathrm{op}} \times \Delta^1$ are constant on F and G^{op} respectively.

4.18. Definition. *Define maps of posets*

$$\begin{aligned} \mu_n : \mathcal{T}(n) &\longrightarrow \mathcal{R}(n) \\ (\ell, a) &\longmapsto \ell_{ab} \\ (\bar{\ell}, a) &\longmapsto \bar{\ell}_{aa} \\ (\ell, b) &\longmapsto \ell_{bb} \\ (\bar{\ell}, b) &\longmapsto \bar{\ell}_{ab} \end{aligned}$$

The maps μ_n assemble into a map of cosimplicial objects $\mu : \mathcal{T} \Longrightarrow \mathcal{R}$.

4.19. Remark. At this juncture it is worth noting that the scaling on $\mathcal{R}(n)$ is the minimal scaling such that $\xi : \mathcal{Q} \Longrightarrow \mathcal{R}$ and $\mu : \mathcal{T} \Longrightarrow \mathcal{R}$ respect the scaling, and such that the scaling on $\mathcal{R}(n)$ has the two symmetries previously mentioned.

Let us take a small break to put the previous definitions into perspective. We have defined three cosimplicial objects $\mathcal{R}, \mathcal{Q}$ and $\mathcal{T}$, the last two of which define the simplices of the ∞-categories $\mathrm{Tw}(\mathbb{D}^{\Delta^1})_{(F,G)}$ and $\mathcal{L}_{(F,G)}$ respectively. The proof of 4.4 will rely on identifying $\mathcal{R}$ as an interpolating cosimplicial object between $\mathcal{Q}$ and $\mathcal{T}$. In the next proposition, we will show an equivalence of cosimplicial objects between $\mathcal{Q}$ and $\mathcal{R}$ thus providing a key technical ingredient for the proof of the main theorem. Readers unwilling to join us for this combinatorial ride can safely skip the next proof.

4.20. Proposition. *The map of cosimplicial objects $\xi : \mathcal{Q} \Longrightarrow \mathcal{R}$ is a levelwise trivial cofibration in the scaled model structure.*

Proof. We will prove something stronger, namely, for every $n \geqslant 0$ the map ξ_n is scaled anodyne. Using the description of both $\mathcal{Q}(n)$ and $\mathcal{R}(n)$ as pushouts, we deduce that it will suffice to show that the map

$$Q(\Delta^n \times (\Delta^1)^\sharp) \to \big((\Delta^n \times \Delta^1) \star (\Delta^n \times \Delta^1)^{\mathrm{op}}\big)_\diamond$$

is scaled anodyne, where the subscript $\diamond$ indicates the scaling induced by that of $\mathcal{R}(n)$. Before embarking upon the proof of our claim we will set some notation

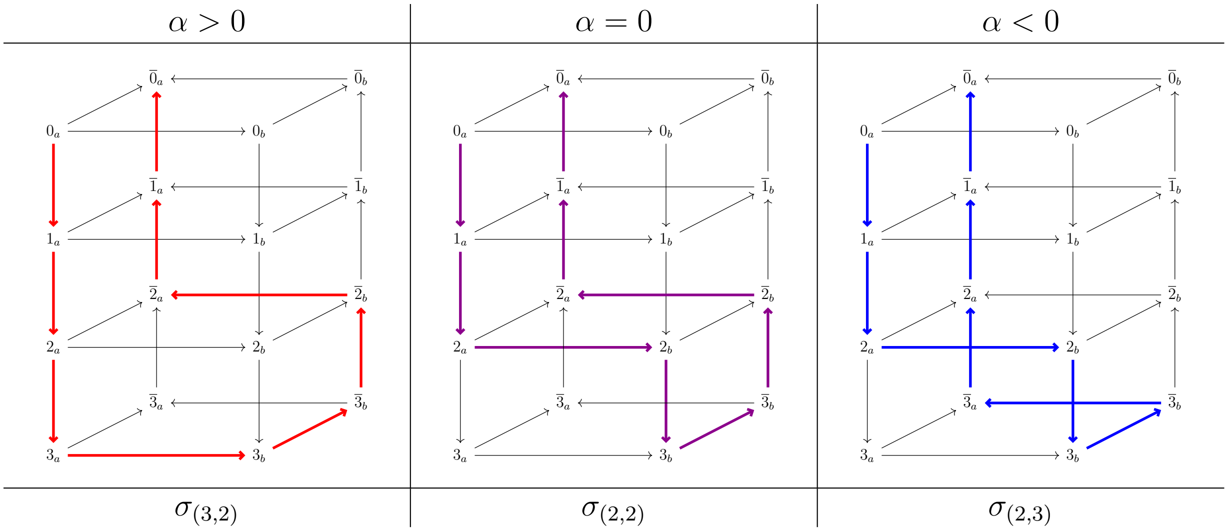
$$Q(\Delta^n \times (\Delta^1)^\sharp) = A^n_\diamond, \quad \big((\Delta^n \times \Delta^1) \star (\Delta^n \times \Delta^1)^{\mathrm{op}}\big)_\diamond = B^n_\diamond.$$

Let (r,s) be a pair of non-negative integers such that $r, s \leqslant n$. We define a simplex

$$\sigma_{(r,s)} : \Delta^{2n+3} \longrightarrow B^n_\diamond$$

$$\ell \longmapsto \begin{cases} \ell_a & \text{if } \ell \leqslant r \\ \ell_b & \text{if } r+1 \leqslant \ell \leqslant n+1 \\ \overline{\ell}_b & \text{if } n+2 \leqslant \ell \leqslant 2n+2-s \\ \overline{\ell}_a & \text{if } 2n+3-s \leqslant \ell \leqslant 2n+3 \end{cases}$$

and note that $B^n_\diamond = \bigcup_{(r,s)} \sigma_{(r,s)}$ as a union of simplicial sets. We further divide the simplices $\sigma_{(r,s)}$ into three families parametrized by $r - s = \alpha$. To illuminate our claims let us include some examples for $n = 3$.



We define $B_\diamond^+$ (resp. $B_\diamond^-$, resp. $B_\diamond^0$) as the union of the simplices $\sigma_{(r,s)}$ such that $\alpha \geqslant 0$ (resp. $\alpha \leqslant 0$, resp. $\alpha = 0$) with the induced scaling. It follows from unwinding the definitions that $A_\diamond^n = B_\diamond^0$ and that $B_\diamond^+ \cap B_\diamond^- = B_\diamond^0$. We have thus produced a pushout square

$$\begin{array}{ccc} A_\diamond^n & \longrightarrow & B_\diamond^+ \\ \downarrow & \ulcorner & \downarrow \\ B_\diamond^- & \longrightarrow & B_\diamond^n. \end{array}$$

We turn now to show that $A_\diamond^n \to B_\diamond^\pm$ is scaled anodyne. First let us tackle the case $\alpha > 0$. To this end we produce a filtration

$$A_\diamond^n = X_0 \to X_1 \to \cdots \to X_{n-1} \to X_n = B_\diamond^+$$

where X_j is the scaled simplicial subset consisting in those simplices contained in some $\sigma_{(r,s)}$ with $\alpha \leqslant j$. We claim that in order to show that $X_{j-1} \to X_j$ is scaled anodyne it suffices to show that top horizontal morphism $f_{(r,s)}$ in the pullback diagram below

$$\begin{array}{ccc} W_{(r,s)} & \xrightarrow{f_{(r,s)}} & \Delta^{2n+3} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \sigma_{(r,s)}} \\ X_{j-1} & \longrightarrow & X_j. \end{array}$$

is scaled anodyne with respect to the induced scaling. We first observe that given pairs (r,s) and (u,v) such that $r-s = u-v = j$, the intersection $\sigma_{(r,s)} \cap \sigma_{(u,v)}$ is already contained in X_{j-1}. To see this, assume without loss of generality that $r > u$, and note that it follows that $\sigma_{(r,s)} \cap \sigma_{(u,v)}$ is contained in $\sigma_{(u,s)}$, and $u - s < j$.

We then choose a linear order on the simplices of the form $\sigma_{(r,s)}$ with $r - s = j$ and write

$$\sigma_0 < \sigma_2 < \cdots < \sigma_\ell$$

to denote this order. Adding these simplices in the chosen order yields a filtration

$$X_{j-1} \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_\ell \longrightarrow X_j.$$

Since the pairwise intersections of the added simplices are already contained in X_{j-1}, for each $0 \leqslant \alpha \leqslant \ell$ we have pushout-pullback diagrams

$$\begin{array}{ccc} W_{(r,s)} & \xrightarrow{f_{(r,s)}} & \Delta^{2n+3} \\ \downarrow & \lrcorner & \downarrow \sigma_{(r,s)} \\ Y_{\alpha-1} & \longrightarrow & Y_\alpha. \end{array}$$

Thus, as expected, it will suffice for us to show that $f_{(r,s)}$ is scaled anodyne.

On inspection, we find that

$$W_{(r,s)} = d_r(\sigma_{(r,s)}) \cup d_{2n+2-s}(\sigma_{(r,s)}).$$

Consequently we can define a dull subset consisting of the sets $\{r\}$, $\{2n+2-s\}$ with pivot point $2n+2-r$. Using 1.18 we conclude that $A^n_\diamond \to B^+_\diamond$ is scaled anodyne.

The case $\alpha < 0$ is a formal dual of the case just proved. To see this we observe that the duality on $\mathcal{R}(n)$ restricts to $(B^+_\diamond)^{\mathrm{op}} \cong B^-_\diamond$ and that our scaling is symmetric. The case $\alpha < 0$ follows, concluding the proof. ■

4.21. Notation. The cosimplicial object $\mathcal{R}$ induces a "nerve" operation

$$\mathrm{Set}^{\mathrm{sc}}_\Delta \longrightarrow \mathrm{Set}_\Delta$$

obtained by taking maps from the cosimplicial object $\mathcal{R}$ into a given scaled simplicial set. For the remaineder of this section, we will use $\mathcal{X}$ to denote the image of the ∞-bicategory $\mathbb{D}$ under this "nerve".

4.22. Corollary. *Let $\mathbb{D}$ be an ∞-bicategory and consider the induced map*

$$\xi^* : \mathcal{X} \to \mathcal{L}_{\Delta^1}.$$

Then the map ξ^* is a trivial Kan fibration. In particular, after passing to fibers we obtain an equivalence of ∞-categories

$$\mathcal{X}_{(F,G)} \xrightarrow{\simeq} \mathcal{L}_{(F,G)}.$$

Proof. Note that as an immediate consequence of 4.20 we obtain a scaled anodyne map $\partial\mathcal{Q}^n \to \partial\mathcal{R}^n$. Consider the morphisms

$$\mathcal{Q}(n) \to \mathcal{Q}(n) \coprod_{\partial\mathcal{Q}^n} \partial\mathcal{R}^n \to \mathcal{R}(n),$$

and note the last map is a trivial cofibration by 2-out-of-3. The reader will observe that the boundary lifting problems

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \xi^* \\ \Delta^n & \longrightarrow & \mathcal{L}_{\Delta^1} \end{array}$$

are in bijection with lifting problems of the form

$$\begin{array}{ccc} \mathcal{Q}(n) \coprod_{\partial\mathcal{Q}^n} \partial\mathcal{R}^n & \longrightarrow & \mathbb{D} \\ \downarrow & \nearrow & \\ \mathcal{R}(n) & & \end{array}$$

and hence the result. ∎

4.23. Construction. We define a map

$$\mathcal{R}(n) \to Q(n)$$

by requiring $\overline{i_{xy}} \mapsto \bar{i}$ and $i_{xy} \mapsto i$. We further define a map

$$\mathcal{R}(n) \to \Delta^1$$

by $i_{xy} \mapsto x; \overline{i_{xy}} \mapsto y$. Note that both of these maps can be easily checked to preserve the scalings. Together, they thus define a map $\psi_n : \mathcal{R}(n) \to \mathcal{T}(n)$ such that $\psi_n \circ \mu_n = \mathrm{id}$ (see 1). Moreover, the ψ_n yield a natural transformation $\psi : \mathcal{R} \to \mathcal{T}$. We denote by $\psi^* : \mathrm{Tw}(\mathbb{D}^{\Delta^1}) \to \mathcal{X}$ the induced map.

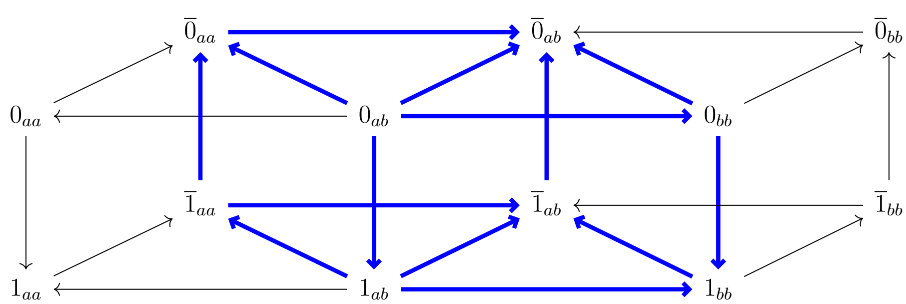


Figure 1: $\mathcal{T}(1)$ pictured in blue as a subset of $\mathcal{R}(1)$ under the inclusion μ_1. The map ψ_1 can be alternately characterized as the unique map such that $\psi_1 \circ \mu_1 = \mathrm{id}$ and ψ preserves $\overline{(-)}$ and its dual.

4.24. Lemma. *The diagram*

$$\begin{tikzcd} \mathrm{Tw}(\mathbb{D}^{\Delta^1}) \arrow[d, "\psi^*"'] \arrow[dr, "\Theta_{\Delta^1}"] & \\ \mathcal{X} \arrow[r, "\xi^*"'] & \mathcal{L}_{\Delta^1} \end{tikzcd}$$

commutes.

Proof. By the universal property of the pullback, the map ξ^* is uniquely determined by a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "g"] \arrow[d, "f"'] & \mathrm{Fun}(\mathrm{Tw}(\Delta^1)^\sharp, \mathrm{Tw}(\mathbb{D})^\natural) \arrow[d] \\ \mathcal{D}^{\Delta^1} \times (\mathcal{D}^{\Delta^1})^{\mathrm{op}} \arrow[r, "\ell"'] & \mathrm{Fun}(\mathrm{Tw}(\Delta^1), \mathcal{D} \times \mathcal{D}^{\mathrm{op}}) \end{tikzcd}$$

induced by composing ξ^* with the defining pullback diagram of $\mathcal{L}_{\Delta^1}$. Thus, since the diagram

$$\begin{tikzcd} \mathrm{Tw}(\mathbb{D}^{\Delta^1}) \arrow[dr, "\psi^*"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right] & & \\ & \mathcal{X} \arrow[r, "g"] \arrow[d, "f"'] & \mathrm{Fun}(\mathrm{Tw}(\Delta^1)^\sharp, \mathrm{Tw}(\mathbb{D})^\natural) \arrow[d] \\ & \mathcal{D}^{\Delta^1} \times (\mathcal{D}^{\Delta^1})^{\mathrm{op}} \arrow[r, "\ell"'] & \mathrm{Fun}(\mathrm{Tw}(\Delta^1), \mathcal{D} \times \mathcal{D}^{\mathrm{op}}) \end{tikzcd}$$

commutes, where the exterior square is the commutative square (4), it follows that the triangle of the lemma also commutes. ■

Proof of 4.4. By virtue of 4.8, it will suffice to show that Θ_X is an equivalence of Cartesian fibrations for $X = \Delta^n$ with $n = 0, 1$. The case $n = 0$ is obvious. To show the case $n = 1$ we observe that due to 4.22 and 4.24 it will suffice to show ψ_* is an equivalence of ∞-categories upon passage to fibers. We further note that since $\mu^* \circ \psi^* = \mathrm{id}$ it will be enough to show that $\varphi^* = \psi^* \circ \mu^*$ is a fiberwise equivalence. Let $\sigma : \Delta^n \to \Delta^1$ and let $j \in [n]$ be the first object such that $\sigma(j) = 1$ if σ is constant on 0 we set the convention $j = n + 1$. Now we can define a map of scaled simplicial sets

$$\varphi^1_\sigma : \mathcal{R}(n) \to \mathcal{R}(n)$$

which leaves every object invariant except those of the form ℓ_{aa} with $\ell < j$ which are sent to ℓ_{ab}. Given $\rho : \Delta^n \to \mathcal{X}_{(F,G)}$ we define a simplex $H(\sigma, \rho) : \Delta^n \to \mathcal{X}_{(F,G)}$ given by the composite

$$\mathcal{R}(n) \xrightarrow{\varphi_\sigma} \mathcal{R}(n) \xrightarrow{\rho} \mathbb{D}$$

This assignment extends to a homotopy $H_1 : \Delta^1 \times \mathcal{X}_{(F,G)} \to \mathcal{X}_{(F,G)}$ which is componentwise an equivalence. This exhibits an equivalence of morphisms $\mathrm{id} \sim (\varphi_0^1)^*_{(F,G)}$ where φ_0^1 denotes the previously defined map with respect to the constant simplex at 0.

Let $\sigma : \Delta^n \to \Delta^1$. Then we define a map of scaled simplicial sets

$$\varphi_\sigma^2 : \mathcal{R}(n) \to \mathcal{R}(n)$$

that leaves every object invariant except those of the form ℓ_{aa} which are sent to ℓ_{ab} and those of the form $\overline{\ell}_{bb}$ with $\ell < j$ with are sent to $\overline{\ell}_{ab}$. We can now define, in perfect analogy to the situation above, a natural equivalence $H_2 : \Delta^1 \times \mathcal{X}_{(F,G)} \to \mathcal{X}_{(F,G)}$ between $\varphi^*_{(F,G)}$ and $(\varphi_0^1)^*_{(F,G)}$, hence the result. ∎

Application: weighted colimits of ∞-categories. We conclude this section (and thereby the paper) with several corollaries of 4.4, and their application to the 2-dimensional universal property of weighted colimits. Because of the technical complexities shunted into the proofs of the properties of $\mathrm{Tw}(\mathbb{D})$, the proof of this 2-universal property is extremely straightforward.

Throughout this section we will fix an ∞-category $\mathcal{C}$ and a pair of functors $F : \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$, $W : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ that we will refer of as the diagram and the weight functors respectively. We will denote by $\mathfrak{Cat}_\infty$ the ∞-bicategory of ∞-categories.

4.25. Definition. *Let $\mathbb{D}$ be an ∞-bicategory. We say that the underlying ∞-category $\mathcal{D}$ is* tensored *over $\mathcal{C}\mathrm{at}_\infty$ with respect to $\mathbb{D}$ if for every $d \in \mathcal{D}$ the mapping functor $\mathrm{Map}_{\mathbb{D}}(d, -)$ has a left adjoint $- \otimes d : \mathcal{C}\mathrm{at}_\infty \to \mathcal{D}$; in this case these adjoints determine an essentially unique functor $\mathcal{C}\mathrm{at}_\infty \times \mathcal{D} \to \mathcal{D}$.*

4.26. Corollary. *Let $\mathbb{D}$ be an ∞-bicategory such that the underlying ∞-category $\mathcal{D}$ is tensored over $\mathcal{C}\mathrm{at}_\infty$ with respect to $\mathbb{D}$. Then for every ∞-category $\mathcal{C}$ the functor category $\mathcal{D}^{\mathcal{C}}$ is tensored over $\mathcal{C}\mathrm{at}_\infty$ with respect to $\mathbb{D}^{\mathcal{C}}$.*

Proof. Combine 4.4 with [GHN15, Lem. 6.7]. ∎

4.27. Corollary. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{E} \to \mathcal{C}$, $\mathcal{E}' \to \mathcal{C}$ be Cartesian fibrations. We denote by $\mathrm{Fun}^{\mathrm{cart}}_{\mathcal{C}}(\mathcal{E}, \mathcal{E}')$ the ∞-category of maps of Cartesian fibrations. Then there is a natural equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{cart}}_{\mathcal{C}}(\mathcal{E}, \mathcal{E}') \xrightarrow{\simeq} \mathrm{Nat}_{\mathcal{C}}(\mathrm{St}(\mathcal{E}), \mathrm{St}(\mathcal{E}))$$

where St denotes the straightening functor over $\mathcal{C}$.

Proof. Combine 4.4 with [GHN15, Prop. 6.9]. ∎

4.28. Remark. It is worth noting that 4.27 can be interpreted as very compelling evidence suggesting that an enhanced version of the straightening functor St, will yield an equivalence of ∞-*bicategories* between the category of Cartesian fibrations over $\mathcal{C}$ and the category of $\mathcal{C}\mathrm{at}_\infty$-valued functors on $\mathcal{C}$.

Recall that in [GHN15, Def. 2.7], the authors define the weighted colimit of F with weight W as the coend

$$\operatorname*{colim}_{\mathrm{Tw}(\mathcal{C})} W \times F.$$

According to this definition the universal property of the weighted colimit is purely 1-dimensional. Our aim in this section is to show that the previous definition is just a shadow of a bicategorical universal property and thus find a bridge between the classical theory of weighted colimits in 2-categories and the realm of ∞-bicategories.

4.29. Definition. *The weighted colimit of F with weight W (if it exists) is defined to be the unique (up to equivalence) ∞-category corepresenting the functor*

$$\mathfrak{C}\mathrm{at}_\infty \xrightarrow{\mathcal{Y}} \mathfrak{C}\mathrm{at}_\infty^{\mathfrak{C}\mathrm{at}_\infty^{\mathrm{op}}} \xrightarrow{F^*} \mathfrak{C}\mathrm{at}_\infty^{\mathcal{C}^{\mathrm{op}}} \xrightarrow{\mathrm{Nat}_{\mathcal{C}^{\mathrm{op}}}(W,-)} \mathfrak{C}\mathrm{at}_\infty$$

where $\mathcal{Y}$ denotes the bicategorical Yoneda embedding.[6] *We will denote weighted colimit by $W \otimes F$. More compactly, this definition means that there is an equivalence $\mathrm{Nat}_{\mathcal{C}^{\mathrm{op}}}(W, \mathrm{Fun}(F(-), \mathcal{X})) \simeq \mathrm{Fun}(W \otimes F, \mathcal{X})$, natural in $\mathcal{X}$.*

4.30. Remark. This definition of weighted colimits was already considered in more generality in [AG20].

4.31. Theorem. *Consider a pair of functors $F : \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$, $W : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$. Then there is an equivalence of ∞-categories*

$$W \otimes F \xrightarrow{\simeq} \operatorname*{colim}_{\mathrm{Tw}(\mathcal{C})} W \times F.$$

Proof. Let $\mathcal{X}$ be an ∞-category. We trace out a chain of equivalences, natural in $\mathcal{X}$. By 4.4, we have

$$\mathrm{Nat}_{\mathcal{C}^{\mathrm{op}}}(W, \mathrm{Fun}(F(-), \mathcal{X})) \simeq \lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Fun}(W(-), \mathrm{Fun}(F(-), \mathcal{X})).$$

A standard chain of manipulations then yields

$$\lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Fun}(W(-), \mathrm{Fun}(F(-), \mathcal{X})) \simeq \lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Fun}(W(-) \times F(-), \mathcal{X})) \simeq \mathrm{Fun}\left(\operatorname*{colim}_{\mathrm{Tw}(\mathcal{C})} W \times F, \mathcal{X}\right)$$

so that $\operatorname*{colim}_{\mathrm{Tw}(\mathcal{C})} W \times F$ satisfies the universal property defining $W \otimes F$, completing the proof. ∎

[6] We are here ignoring some substantial set-theoretic complexities. We should, more properly, fix a nested pair of Grothendieck universes, and consider variants of $\mathfrak{C}\mathrm{at}_\infty$ based on size. In the interest of concision, we will sweep such set-theoretic concerns under the rug, leaving their contemplation to the interested reader.

References

[ADS20] Abellán García, F., Dyckerhoff, T., and Stern, W. H. “A relative 2-nerve”. *Algebr. Geom. Topol.* 20-6 (2020) pp. 3147–3182

[AGS20] Abellán García, F. and Stern, W. H. *Theorem A for marked 2-categories.* J. Pure & Applied Algebra, Vol. 226, Issue 9, 2022

[AGS22] Abellán García, F. and Stern, W. H. “2-Cartesian fibrations II: Higher cofinality”. 2022. arXiv:2201.09589

[AG20] Abellán García, F. *Marked colimits and higher cofinality.* Homotopy Relat. Struct. 17, 1–22 (2022). DOI: s40062-021-00296-2

[BGN18] Barwick, C., Glasman, S. , and Nardin, D. “Dualizing cartesian and cocartesian fibrations”. *Theory and Applications of Categories.* Vol. 33 No. 4 (2018), pp. 67-94.

[DK19] Dyckerhoff, T. and Kapranov, M. “Higher Segal Spaces”. Springer Lecture Notes in Mathematics 2244. *Springer*, 2019.

[Bar17] Barwick, C. “Spectral Mackey Functors and equivariant Algebraic K-Theory (I).” *Advances in Mathematics* Vol. 304 (Jan. 2017), pp. 646-727.

[Cis19] Cisinski, D. “Higher Categories and Homotopical Algebra.” Cambridge University Press, 2019.

[DS11] Dugger, D. and Spivak, David I. “Rigidification of quasi-categories”. *Algebraic & Geometric Topology* 11.1 (jan. 2011), pp. 225–261. DOI: 10.2140/agt.2011.11.225.

[Du08] Dugger, D. “A primer on homotopy colimits”. University of Oregon, 2008.

[GHL19] Gagna, A., Harpaz, Y., Lanari, E. “On the equivalence of all models for $(\infty, 2)$-categories.” *J. London Math. Soc.* 2022.

[GHL20] Gagna, A., Harpaz, Y., Lanari, E. “Fibrations and lax limits of $(\infty, 2)$-categories.” 2020. arXiv:2012.04537

[GHN15] Gepner, D., Haugseng R. and Nikolaus, T. “Lax colimits and free fibrations in ∞-categories”. *Documenta Mathematica* 1.15 (jan. 2015), vol 22. pp. 1225-1266

[Lur11] Lurie, J. “Derived Algebraic Geometry X: Formal Moduli Problems” 2011 available at the author’s webpage: DAG X

[Lur09a] Lurie, J. “$(\infty, 2)$-categories and the Goodwillie Calculus”. 2009. arXiv: 0905.0462

[Lur09] Lurie, J. “Higher Topos Theory”. Princeton University Press, 2009.

[Lur17] Lurie, J. "Higher Algebra". 2017 *Available at the author's webpage.*

[Lur22] Lurie, J. "Kerodon". 2022 https://kerodon.net

THEORY AND APPLICATIONS OF CATEGORIES will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

SUBSCRIPTION INFORMATION Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. Full text of the journal is freely available at http://www.tac.mta.ca/tac/.

INFORMATION FOR AUTHORS LaTeX2e is required. Articles may be submitted in PDF by email directly to a Transmitting Editor following the author instructions at
http://www.tac.mta.ca/tac/authinfo.html.