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INTEGRATION OF 1-FORMS AND CONNECTIONS

To my friend Marta Bunge.

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ABSTRACT. We present a geometric/combinatorial version of the theorem that a flat torsion-free affine connection on a manifold locally may be integrated into an affine structure.

Introduction

We shall present a geometric/combinatorial version of the theorem[1] that *a flat torsion-free affine connection on a manifold locally may be integrated into an affine structure.*

We obtain this integration result via two other integration results: *closed group valued 1-forms locally have primitives*; which in turn implies that *flat connections in groupoids locally have trivializations.*

For each of these integration results, some further conditions have to be imposed. Also, integration here is only formal: it means passing from first order infinitesimal data to higher order infinitesimal data (formal power series). And they do not address convergence questions, and they work over quite general commutative rings, when coordinatized. But largely, our exposition is coordinate free, and is of synthetic/geometric nature. In fact we shall quote notions and arguments from the literature on synthetic differential geometry (SDG), notably [7], [12], and [5].

Via well adapted models of synthetic differential geometry, as constructed by Dubuc, (see [4] or [7]), the results can be interpreted in the category of smooth manifolds in the classical sense. But some of them apply in other categories, e.g. in some categories coming from algebraic geometry. We shall consider the some suitable category of manifolds, e.g. as in [7] I.17. The main thing is that the objects M which we consider come equipped with a reflexive symmetric relation $\sim$ (preserved by the morphisms). For schemes M in algebraic geometry, such $\sim$ was introduced by French algebraic geometry (notably Grothendieck) in the 1960s, via what was called *the first neighbourhood of the diagonal,* $M_{(1)} \subseteq M \times M$. So the notation is that $x \sim y$ iff $(x, y) \in M_{(1)} \subseteq M \times M$.

Part of the notions and proofs we develop in the present paper are phrased entirely in terms of this relation $\sim$ and are purely combinatorial. But to be specific, we consider

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[1] credited by [3] to Auslander and Marcus [1].

manifolds only: the phrase "locally" refers to subsets which are formally open, i.e. closed under the relation $\sim$.

We call $\sim$ the *(first order) neighbour* relation, so $x \sim y$ is read "x and y are neighbours", or even *(first order) infinitesimal* neighbours. The set of neighbours of x, we denote $\mathfrak{M}(x)$, the (first order) *monad*[2] of x. So $x \sim y$ iff $y \in \mathfrak{M}(x)$ iff $(x, y) \in M_{(1)}$.

Note that the relation $\sim$ is not assumed to be transitive. The transitive closure of $\sim$ is an equivalence relation denoted $\sim_\infty$. The equivalence class of x is denoted $\mathfrak{M}_\infty(x)$ (the ∞-*monad* around x).

1. Group valued 1-forms

The following section depends on the axiomatics of synthetic differential geometry; the reader who wants to go straight to the combinatorics, may skip this, and take the conclusion Proposition 1.1, and in more general form, Proposition 1.2, as an axiom.

Let M be a manifold and G a group (not necessarily commutative, multiplication denoted by $*$, unit by 1). Recall (from [12], say) that a G-valued 1-form is a map $\omega : M_{(1)} \to G$ with $\omega(x, x) = 1$ for all $x \in M$. It is *closed* if

$$\omega(x, y) * \omega(y, z) = \omega(x, z), \tag{1}$$

whenever x, y and z are mutual (= pairwise) neighbours (in the sense of $\sim$). In particular, for a closed 1-form ω, we have for mutual neighbours x, y, z that $\omega(x, y) * \omega(y, z)$ is independent of y. We may ask whether this independence of y also applies if we do not assume that $x \sim z$. We shall prove that this is indeed the case, provided that G is a subgroup of the multiplicative monoid of some finite dimensional algebra W. We shall refer to such groups as *matrix groups*; we shall ultimately be interested in the case where W is the algebra of $n \times n$ matrices over R (where R, and hence W (as an R-module), satisfy the basic KL axiom, as in [12] 1.3).

1.1. Proposition. *[Quadrangle Law] Let ω be a closed G-valued form on M, where G is a matrix group. Then for $x \sim y \sim z$, we have that $\omega(x, y) * \omega(y, z)$ is independent of y.*

Proof. The question is local on M, so we may consider it in a formally open chart $U \subseteq V$ (with V a finite dimensional vector space) around x, y, z. This means that ω may be encoded by an (everywhere defined) function $\Omega : U \times V \to W$, with $\omega(x, y) = \omega(x, x) + \Omega(x; y - x)$ for $x \sim y$, and with $\Omega(x; -) : V \to W$ linear (so $\Omega(x; -)$ is the differential of $\omega(x, -)$ at x). Let $d_1 = y - x$ and $d_2 = z - y$, with d_1 and d_2 in $D(V)$. (Recall that the first order monad of $0 \in V$ is denoted $D(V)$, and it is characterized by: $d \in D(V)$ iff any bilinear $V \times V \to R$ vanishes on the (d, d), cf. [12] 1.2.) So the x, y, and z considered are of the form x, $x + d_1$, and $x + d_1 + d_2$, respectively, with d_1 and d_2 in $D(V)$. We calculate for such $(d_1, d_2) \in D(V) \times D(V)$ the expression for $\omega(x, y) * \omega(y, z)$ in terms of Ω, using that $\omega(x, x) = 1$:

[2] the use of word *monad* here is not related to the use of this word in the sense of triples (T, η, μ) in category theory.

$$\begin{aligned}\omega(x,y)*\omega(y,z) &= (1+\Omega(x;d_1))*(1+\Omega(x+d_1;d_2)) \\ &= 1+\Omega(x;d_1)+\Omega(x+d_1;d_2)+\Omega(x;d_1)*\Omega(x+d_1;d_2).\end{aligned}$$

By Taylor expansion, $\Omega(x+d_1;d_2) = \Omega(x;d_2) + d\Omega(x;d_1,d_2)$; substituting this in the two places where $\Omega(x+d_1;d_2)$ occurs, allows us to continue

$$\begin{aligned}= 1+\Omega(x;d_1) \quad &+\Omega(x;d_2)+d\Omega(x;d_1,d_2)+ \\ &+\Omega(x;d_1)*\Omega(x;d_2)+\Omega(x;d_1)*d\Omega(x;d_1,d_2)\end{aligned}$$

The last term here contains d_1 in a bilinear way, so it vanishes. So using that $\Omega(x;-)$ is linear, we conclude

$$\omega(x,y)*\omega(y,z) = 1+\Omega(x;d_1+d_2)+d\Omega(x;d_1,d_2)+\Omega(x;d_1)*\Omega(x;d_2). \tag{2}$$

If $d_1+d_2 \in D(V)$, i.e. if $x \sim z$, we have

$$\omega(x,z) = \Omega(x;d_1+d_2)$$

so if ω is closed, and $d_1+d_2 \in D(V)$, the expression $d\Omega(x;d_1,d_2)+\Omega(x;d_1)*\Omega(x;d_2)$ vanishes. By "2) $\Rightarrow$ 3)" in Proposition 1.3.3 in [12], this implies that the value of the expression in (2), for all $(d_1,d_2) \in D(V)\times D(V)$, only depends on d_1+d_2. ∎

The reason for the name "quadrangle law" is that the conclusion may be expressed by saying that given a $\sim$-quadrangle, meaning four points x, y_1, y_2, z with $x \sim y_1 \sim z$ and $x \sim y_2 \sim z$, we have (for ω closed) that $\omega(x,y_1)*\omega(y_1,z) = \omega(x,y_2)*\omega(y_2,z)$. This equality we shall express as an equality of two "path integrals", or "curve integrals" of the 1-form ω along the periphery of the quadrangle.

We shall, more generally, describe path integrals of a G-valued 1-forms ω along "paths" of arbitrary finite length. We consider the formal (infinitesimal) substitute for the notion of *path* $\underline{x}$, for which the task is to describe the "path integral" $\int_{\underline{x}} \omega \in G$; we define an n-*path* $\underline{x}$ in a manifold M to be an $n+1$-tuple $(x_0, x_1, \ldots, x_n)$ of points in M with $x_i \sim x_{i+1}$ for $i = 0, \ldots, n-1$. The point x_0 is the *domain* of $\underline{x}$, and the point x_n is the *codomain* of $\underline{x}$. If ω is a G-valued 1-form on M, we define the "path integral" $\int_{\underline{x}} \omega$ by

$$\int_{\underline{x}} \omega := \omega(x_0,x_1)*\omega(x_1,x_2)*\ldots*\omega(x_{n-1},x_n). \tag{3}$$

Note that the paths in M form a category, by concatenation of paths; and that $\int\omega$ is takes composition in this category to multiplication $*$ in G.

The following is now a version of the integration result that "closed G-valued 1-forms have primitives". The group G is assumed to be a matrix group.

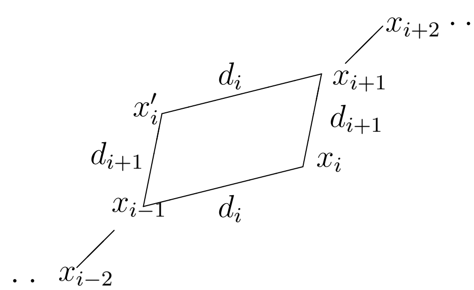
1.2. Proposition. *If ω is a closed G-valued 1-form on a manifold M, then $\int_{\underline{x}} \omega$ only depends on the domain and the codomain of the path $\underline{x}$.*

Note: the reason that we do not have a "simply-connected" assumption on M is that the path notion used here is quite restricted. In fact it implies that $M = \mathfrak{M}_\infty(x)$ for any $x \in M$.

Proof. As in the proof of the Proposition 1.1, we pick an arbitrary chart U contaning all the x_is of the path; so the path with x_0 as domain (say, an n-path) may, in coordinates given by the chart, be presented by a sequence $\underline{d} = d_1, d_2, \ldots, d_n$ (with $d_i \in D(V)$) with $x_i = x_{i-1} + d_i$ for $i = 1, \ldots, n$. From Proposition 1.1 follows that

$$\int_{\underline{x}} \omega = \int_{\underline{x'}} \omega,$$

where $\underline{x'}$ is obtained from $\underline{x}$ by swapping the ith and $(i+1)$st of the d_js ($i = 1, \ldots, n-1$), so as to obtain a new point $x'_i = x_{i-1} + d_{i+1}$ (this x'_i is something that depends on the chart):



We can thus swap any two consecutive entries in the sequence of d_js, without changing the value of the integral; and since neighbour transpositions generate the whole symmetric group S_n of permutations σ of n letters, it follows that (for closed ω)

$$\int_{\underline{x}} \omega = \int_{\sigma(\underline{x})} \omega, \tag{4}$$

where $\sigma(\underline{x})$ replaces the $x_i = x_0 + \sum_{j=1}^{i} d_j$ in the original $\underline{x}$ by $x'_i := x_0 + \sum_{j=1}^{i} d_{\sigma(j)}$. For fixed x_0, we therefore have a map which is invariant under the $n!$ permutations of the n input entries $(d_1, d_2, \ldots, d_n)$, By the "Symmetric Functions Property" in its geometric manifestation, [5] Theorem 2.1, it follows that (4), as a function of the d_is, factors (in fact uniquely) across the addition map $D(V)^n \to D_n(V)$, i.e. it depends only of the sum $\sum d_j$, not on the indivual d_js. Equivalently, $\int_{\underline{x}} \omega$ only depends on x_0 and x_n. This is now a statement which does not mention any particular chart. This proves the Proposition. ∎

There is a similar result for 1-forms with values in (the additive group of) a vector space, - say, in the space of scalars R. The proof is simpler, but similar. It is sketched in [5] Section 3, and it was one of the motivations for that paper.

2. Connections in groupoids

Let M be a set equipped with a reflexive symmetric relation $\sim$. Let $\pi : E \to M$ be a map ("E is a bundle over M"). For $x \in M$, E_x denotes the fibre $\pi^{-1}(x)$ over x. A *(bundle-) connection* on E means (Joyal) that for each $x \sim y$ in M, there is given a map $E_x \to E_y$, typically denoted $\nabla(x, y)$. One requires a unit law: $\nabla(x, x)$ is the identity map of E_x; usually, one also requires that $\nabla(x, y)$ and $\nabla(x, y)$ are inverse to each other, hence both are isomorphisms (bijections). With the inversion law, ∇ may be seen to have its values in the *groupoid* of isomorphisms between the fibres of E. If the fibres of $E \to M$ have some structure, one may consider the subgroupoid of those isomorphisms which preserve the structure.

We are ultimately interested in the bundle $\pi : M_{(1)} \to M$, where π to the pair $x \sim y$ associates x. The fibre over x is thus $\{y \in M \mid x \sim y\}$, i.e. the monad of x; and it has a priori a structure as pointed set, with the chosen point being $x \in \mathfrak{M}(x)$. (More generally, we are also interested in the bundle whose fibre over $x \in M$ is the set of n-paths with domain x. It likewise has the structure of pointed set, with x being the point x. The bundle $M_{(1)} \to M$ is the special case $n = 1$ of such path-bundle.)

We can abstract the bundle-notion of connection, and the resulting family of isomorphisms between the fibres, into the notion of *groupoid valued connection* as follows:

We consider a groupoid $\Phi \rightrightarrows M$ (internal to the category of "spaces" in which we work) where M (= the space of objects of Φ) is equipped with a reflexive symmetric relation $\sim$. Recall from [6], [16] or [12] that a *(groupoid valued) connection* in such groupoid may be defined as a map $\nabla : M_{(1)} \to \Phi$ with $\nabla(x, y)$ an arrow from x to y, with $\nabla(x, x) = 1_x$ and with $\nabla(y, x)$ inverse to $\nabla(x, y)$.

The paths in M form a category $P(\sim) \rightrightarrows M$, with concatenation of paths as composition; in fact, $P(\sim) \rightrightarrows M$ is the free category defined generated by the relation $\sim$ on M. Therefore, the map ∇ from $M_{(1)}$ to Φ extends to a functor[3] $P(\sim) \to \Phi$; explicitly, for a path $\underline{x} = (x_0, x_1, \ldots, x_n)$ in M, one has an arrow from x_0 to x_n in the groupoid Φ, namely the following composite, which we in analogy with (3) denote by $\int_{\underline{x}} \nabla$,

$$\int_{\underline{x}} \nabla := x_0 \xrightarrow{\nabla(x_0,x_1)} x_1 \xrightarrow{\nabla(x_1,x_2)} x_2 \cdot\cdot \xrightarrow{\nabla(x_{n-1},x_n)} x_n.$$

The connection ∇ is called *flat* (or *curvature-free*) if (composing from left to right)

$$\nabla(x, y).\nabla(y, z) = \nabla(x, z), \tag{5}$$

[3]Thinking of the category of paths as a formal version of the category of (Moore-) paths in M, this functor is in terminology from [16] (see also 5.8 in [12]), the *path connection* given by ∇.

whenever $x \sim y$, $y \sim z$ and $x \sim z$, in analogy with (1). In fact (1) may be seen as the special case where the groupoid $\Phi \rightrightarrows M$ is $M \times M \times G$, and the connection is given by $\nabla(x, y) := (x, y, \omega(x, y))$. For a groupoid which is locally of this form, one can locally encode the connection by a G-valued 1-form, which is closed iff the connection is flat.

The following[4] is now an immediate generalization of Proposition 1.2.

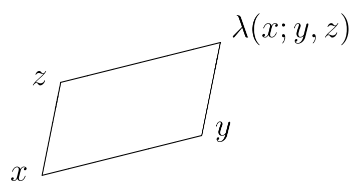
2.1. Proposition. *Assume that, locally, $\Phi \rightrightarrows M$ admits some isomorphisms (over M) with groupoids of the form $M \times M \times G$ for some matrix group G ; then if ∇ is flat, $\int_{\underline{x}} \nabla$ only depends on the endpoints of $\underline{x}$.*

Proof. The auxiliary isomorphism allows us to translate the data of ∇ into a G-valued 1-form ω, which is closed iff ∇ is flat. Then Proposition 1.2 shows the independence. ∎

Note that such an auxiliary isomorphism of Φ with $M \times M \times G$ is not intrinsic to the geometry; but since the conclusion of the Proposition does not mention this auxiliary isomorphism, the conclusion is intrinsic to ∇ and $\Phi \rightrightarrows M$.

3. Affine connections

The formation of parallelograms is a fundamental construction in geometry. It may be formulated in terms of a ternary operation λ: to points x, y, z, the fourth point in the parallelogram spanned by the "vectors" xy and xz is the value $\lambda(x; y, z)$. This may also be seen (in a less symmetric way) as the point obtained by parallel *transport* of z along the "vector" xy,



(Example: in a group, one may put $\lambda(x; y, z) = y \cdot x^{-1} \cdot z$).

The "geometry" of such a ternary operation is the motivation for an ("infinitesimal" combinatorial version of) the notion of *affine connection* in more general manifolds. This was argued in [8] and other places; here, the "vectors" xy and xz have to be sufficiently small, (first order infinitesimals, meaning that $x \sim y$ and $x \sim z$).

To be explicit about the algebra/combinatorics involved: Let M be a set equipped with a symmetric reflexive relation $\sim$. In this context, an *affine connection* λ is a partially defined ternary operation λ, on M, with $\lambda(x; y, z)$ being defined whenever the *book-keeping conditions* $x \sim y$ and $x \sim z$ hold (note that we are not assuming $y \sim z$), in which case one assumes validity of the *book-keeping laws*

$$\lambda(x; y, z) \sim y \text{ and } \lambda(x; y, z) \sim z. \tag{6}$$

[4] I believe that an analogous result was first proved by Virsik, cf. [16], Theorem 7.

The operation λ is required to satisfy two *unit laws*:

$$\lambda(x; x, z) = z \text{ and } \lambda(x; y, x) = y \tag{7}$$

as well as the *inversion law*

$$\lambda(y; x, \lambda(x; y, z)) = z. \tag{8}$$

The book-keeping conditions and book-keeping laws for λ can be visualized by the figure above, with the lines expressing the relation $\sim$; also, the unit laws appear as evident from the geometry of the figure.

A quadrangle which is of the form $x, y, z, \lambda(x; y, z)$, we call a *parallelogram according to* λ.

To emphasize the relation $\sim$ in the book-keeping condition, we also say that λ is *restricted* by $\sim$. The *integration* problem for a $\sim$-restricted λ is essentially to remove the restriction, or at least to replace it by a weaker $\sim$. More precisely, in our case, to replace $\sim$ by its the transitive closure $\sim_\infty$; this amounts in a fully coordinatized situation to *formal* integration, i.e. to the construction of formal power series solutions.

If $\lambda(x; y, z) = \lambda(x; z, y)$ for all $y \sim x \sim z$, the affine connection λ is called *symmetric* or *torsion-free*. The affine connections arising from a Riemannian metric are symmetric. If the connection arises from a group, as described above, it is symmetric iff the group is commutative.

Another property which an affine connection λ may or may not have is *flatness*, which means that parallel translation, using λ, of a point along a path only depends on the endpoints of the path. This is a condition which is best formulated by re-interpreting λ as a groupoid valued connection ∇, in the sense of Section 2, as will be made explicit in Subsection 3.2 below. The notion of flatness of a groupoid valued connection was defined in the previous section.

An unrestricted flat and symmetric λ is essentially the same as a "Schar"-structure, in the sense of Prüfer [15], who derives abelian group structures in terms of such unrestricted operation λ; see also [14]. (A Schar is a set with an ternary operation, satisfying certain equations; [15] denotes the operation $AB^{-1}C$, it corresponds to our $\lambda(B, A, C)$. The flatness is in [15] an associative law of the form $(AB^{-1}C)D^{-1}E = AB^{-1}(CD^{-1}E.)$

3.1. Paths and grids. The following describe some auxiliary concepts, derived from a reflexive relation $\sim$ on a set M. Recall that a *path* $\underline{x}$ of length k, or a k-path, is a sequence $x_0, x_1, \ldots, x_k$ of points in M, with $x_i \sim x_{i+1}$ for $i = 0, \ldots, k-1$. We call x_0 the *domain* of the path, and x_k the *codomain* of the path.

Similarly, a 2-dimensional *grid* Z of size $k \times l$ is a $(k+1) \times (l+1)$ matrix $z_{i,j}$ whose rows and columns are paths.

Let $\lambda(x; y, z)$ be a (partially defined) ternary operation on M, with book-keeping conditions as for an affine connection. Then out of two paths $\underline{x}$ and $\underline{y}$ with common domain $x_0 = y_0$, we can use λ to construct a grid $\underline{z} = \lambda(x_0; \underline{x}, \underline{y})$ of size $k \times l$ by double induction: Initial data $z_{i,0} = x_i$ and $z_{0,j} = y_j$, and

$$z_{i+1,j+1} := \lambda(z_{i,j}; z_{i+1,j}, z_{i,j+1}).$$

The *domain* of the grid is $z_{0,0} = x_0 = y_0$, the *codomain* of this grid is $z_{k,l}$ (A grid of size 2×1 constructed this way is exhibited in (9) below). If $\lambda(x; -, -)$ is symmetric for any x, one has that $\lambda(x_0; \underline{y}, \underline{x})$ is the transpose of $\lambda(x_0; \underline{x}, \underline{y})$ (provided $x_0 = y_0$).

In particular, if λ is symmetric (i.e. $\lambda(x; y, z) = \lambda(x; z, y)$), interchanging the two generating paths gives two grids whose codomains are equal.

3.2. Affine connections as groupoid connections. We shall describe how an affine connection on a manifold M may be seen as a particular case of groupoid valued connection in the sense of Section 2.

The groupoid in question is the groupoid $GL(M) \rightrightarrows M$, where an arrow $x \to y$ is a bijection $\mathfrak{M}(x) \to \mathfrak{M}(y)$ taking x to y. This groupoid is canonically isomorphic to the groupoid consisting of fibrewise linear isomorphims $T_x(M) \to T_y(M)$, see Theorem 4.3.4 in [12]. And this groupoid in turn is, for an n-dimensional manifold, locally isomorphic to the groupoid $M \times M \times GL(n, R)$ (whence the choice of the acronym "GL"). So that Proposition 2.1 applies.

Given an affine connection λ on M. We describe a connection ∇ in the groupoid $GL(M)$:[5] for $x \sim y$ in M, the map $z \mapsto \lambda(x; y, z)$ is a map $\nabla(x, y) : \mathfrak{M}(x) \to \mathfrak{M}(y)$, by the (first) book-keeping law in (6), and it takes $x \in \mathfrak{M}(x)$ to $y \in \mathfrak{M}(y)$ by the second law in (6); so for $z \in \mathfrak{M}(x)$

$$\nabla(x, y)(z) := \lambda(x; y, z) \in \mathfrak{M}(y).$$

Equivalently, this is describing a bundle connection on the bundle $M_{(1)} \to M$. We say that λ is flat if the associated groupoid valued connection ∇ is flat.

Note that y and z play a different role in the interpretation of λ as a bundle connection. We think of x, y as the "active" aspect, and z as the passive: we *transport* z along x, y. In the diagram (9) below, the arrows indicate the active aspect; the lines (as well as the arrows) indicate the relation $\sim$.

Another bundle connection may be constructed, by interchanging the role of the y and z.

Let us describe the composite map $\nabla(x, y_1).\nabla(y_1, y_2)$, i.e. $\lambda(x; y_1, -).\lambda(y_1; y_2, -)$ (composing from left to right); its value in $z \sim x$ can be read off from the diagram (for $x \sim y_1 \sim y_2$)

$$\begin{tikzcd} z \arrow[r, dash] \arrow[d, dash] & \lambda(x; y_1, z) \arrow[r, dash] \arrow[d, dash] & \lambda(y_1; y_2, \lambda(x; y_1, z)) \arrow[d, dash] \\ x \arrow[r] & y_1 \arrow[r] & y_2. \end{tikzcd} \tag{9}$$

Since the local triviality assumptions of Proposition 2.1 are valid for the groupoid $GL(M)$, we have the following special case of Proposition 2.1:

[5]This viewpoint was introduced in [8], see also [12] 2.3.

3.3. PROPOSITION. *For a flat λ, the result of iterated transport of $z \sim x$ along a path $\underline{y}$ with domain x only depends on z and and on the codomain of the path $\underline{y}$.*

By induction in the length l of a path $\underline{z}$, one now may conclude more generally

3.4. PROPOSITION. *For a flat λ, the last column in the grid constructed from two paths $\underline{y}$ and $\underline{z}$ with common domain x only depends on $\underline{z}$ and on the codomain of $\underline{y}$.*

In particular, if λ is also symmetric, the codomain of the grid constructed by λ out of $\underline{y}$ and $\underline{z}$ only depends on the codomains of $\underline{y}$ and $\underline{z}$.

We now state the main integration result. It applies to path connected manifolds M; but recall that the present path notion is of infinitesimal nature, since "paths" are finite chains of first order infinitesimal neighbours $x \sim y$. The combinatorics resides in proving the Cube Lemma, Lemma 3.6 below.

3.5. THEOREM. *If λ is a flat and symmetric affine connection, restricted by $\sim$, then λ extends canonically to an unrestricted flat and symmetric affine connection λ^∞.*

We consider a $\sim$-restricted affine connection λ, assumed to be symmetric and flat.

3.6. LEMMA. *[Cube Lemma] Consider a triple x, y, z of points, each of which is neighbour of a point o. Then there exists a unique point $\Lambda = \Lambda(o; x, y, z)$ which fits into a "cube" shaped diagram*

$$\begin{tikzcd}
z \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & \lambda(o;x,z) \arrow[dr, dash] \arrow[dd, dash] & \\
& \lambda(o;y,z) \arrow[rr, dash] \arrow[dd, dash] & & \Lambda(o;x,y,z) \arrow[dd, dash] \\
o \arrow[rr, dash] \arrow[dr, dash] & & x \arrow[dr, dash] & \\
& y \arrow[rr, dash] & & \lambda(o;x,y)
\end{tikzcd} \tag{10}$$

where the lines indicate the relation $\sim$, and where each of the six faces are parallelograms according to λ.

PROOF. The three faces containing o are by construction λ-parallelograms. The uniqueness of the $\Lambda(o; x, y, z)$ is clear just by (say) the requirement that the right hand face is a λ-parallelogram. So the assertion is that then also the top face and the front face are λ-parallelograms. To say that the right hand face is a λ-parallelogram is to say that

$\Lambda = \lambda(x; \lambda(o; x, y), \lambda(o; x, z))$. There are similar expressions for the top and right hand squares, so the equational content of the Lemma is that these three expressions have equal value. They occur in the three equations (to be proved) in (14), (15), and (16) below.

We have $\lambda(o; x, y) = \lambda(o; y, x)$, and therefore, we have two paths (of length 2) from o to this point, namely $(o, x, \lambda(o; x, y))$ and $(o, y, \lambda(o; y, x))$, and therefore moving z along either of these two paths give same result, by Proposition 2.1.

(If further we happen to have $o \sim \lambda(o; x, y)$, this equals, by flatness, the result $\lambda(o; \lambda(o; x, y), z)$ of moving z directly along the "path" (of length 1) from o to $\lambda(o; x, y)$, and obtain

$$\lambda(o; \lambda(o; x, y), z) = \lambda(x; \lambda(o; x, y), \lambda(o; x, z)).) \tag{11}$$

The formula for the result of moving $z \sim o$ along the path $(o, x, \lambda(o; x, y))$ can be read off from the diagram

$$\begin{tikzcd} z \arrow[r, dash] \arrow[d, dash] & \lambda(o; x, z) \arrow[r, dash] \arrow[d, dash] & \lambda(x; \lambda(o; x, y), \lambda(o; x, z)) \arrow[d, dash] \\ o \arrow[r] & x \arrow[r] & \lambda(o; x, y) \end{tikzcd} \tag{12}$$

and the formula for the result of moving $z \sim o$ along the path $(o, y, \lambda(o; y, x))$ can similarly be read off from the diagram

$$\begin{tikzcd} z \arrow[r, dash] \arrow[d, dash] & \lambda(o; y, z) \arrow[r, dash] \arrow[d, dash] & \lambda(y; \lambda(o; y, x), \lambda(o; y, z)) \arrow[d, dash] \\ o \arrow[r] & y \arrow[r] & \lambda(o; y, x) \end{tikzcd} \tag{13}$$

and using flatness of λ and $\lambda(o; x, y) = \lambda(o; y, x)$, we therefore have the equation (14):

$$\lambda(x; \lambda(o; x, y), \lambda(o; x, z)) = \lambda(y; \lambda(o; y, x), \lambda(o; y, z)) \tag{14}$$

$$\lambda(y; \lambda(o; y, z), \lambda(o; y, x)) = \lambda(z; \lambda(o; z, y), \lambda(o; z, x)) \tag{15}$$

$$\lambda(z; \lambda(o; z, x), \lambda(o; z, y)) = \lambda(x; \lambda(o; x, z), \lambda(o; x, y)) \tag{16}$$

The equations (15) and (16) are permutation instances of (14): the expressions in these two equations come about by cyclically permuting the occurrence of x, y, z in (14), and are therefore valid, since the the book-keeping assumptions on o, x, y, z are symmetric in x, y, z.

Now the by symmetry of λ, the right hand side of (14) equals the left hand side of (15); and the right hand side of (15) equals the left hand side of (16); and the right hand side of (16) equals the left hand side of (14), so we conclude that all the six expressions in these three equations are equal. This proves the Cube Lemma, and hence Theorem 3.5. ∎

3.7. Affine structure. By Theorem 3.5, a flat symmetric $\sim$-restricted affine connection λ extends to a flat symmetric unrestricted connection λ^∞; being flat and symmetric, the Cube Lemma 3.6 therefore applies to λ^∞. But now the book-keeping conditions vacuously hold, and this implies that we can augment the equation triad (14), (15), (16) by some further equations, which are not generally meaningful for the restricted λ. We have for instance validity of (11) for λ^∞.

To exhibit this equation, and two analogous ones, in a more readable way, we found it useful to switch to the following lightweight notation for λ^∞:

$$[xoy] := \lambda^\infty(o; x, y).$$

(Note the change in the ordering of the two first arguments in the notation; we argue for the reasonableness of the notations in the Appendix below.) Then equation (11) reads

$$[[xoy]oz] = [[xoy]x[xoz]],$$

and permutation instances thereof (permuting x, y, z). Combining with the triad of equations (14), (15), (16) (valid for λ^∞), and changing to the lightweight notation, we therefore have equality of all nine expressions in

$$[[xoy]oz] = [[xoy]x[xoz]] = [[yox]y[yoz]]$$

$$[[yoz]ox] = [[yoz]y[yox]] = [[zoy]z[zox]]$$

$$[[zox]oy] = [[zox]z[zoy]] = [[xoz]x[xoy]]$$

In particular, we have $[[xoy]oz] = [[yoz]ox]$, and using symmetry of λ, the right hand side of this may be written $[xo[yoz]]$, so that we have the associative law

$$[[xoy]oz] = [xo[yoz]].$$

Therefore we have, for infinitesimally path connected M,

3.8. Theorem. *For any $o \in M$, the operation $+_o$ given by $x +_o y := [xoy]$ makes M into an abelian group, with o as unit. The inverse of x is $\lambda(x; o, o)$*

Proof. The associative law for $+_o$ was argued above. The inverse of x w.r.to the addition $+_o$ is $\lambda(x; o, o)$; this follows as a substitution instance of the inversion law (8), $\lambda(o; x, \lambda(x; o, o)) = o$. ∎

With this result, the $\Lambda(o; x, y, z)$ in the Cube Lemma is simply $x +_o y +_o z$, and each of the six expressions in represent the various ways this triple sum can be expressed in the terms of x, y, z and the binary $+_o$.

Furthermore, the way is now open to use all of the power of the algebraic theory of abelian groups, e.g. defining n-ary sums relative to $+_o$, or forming affine combinations with integral coefficients; they are independent of the choice of o, since translation by $\lambda(o; o', -)$ is a group homomorphism from $+_o$ to $+_{o'}$.

3.9. SCALARS. Prüfer observed that a "Schar" (a heap) admits affine combinations with coefficients from $\mathbb{Z}$ (an affine combination being a linear combination where the sum of the coefficients is 1). In geometry, one is also interested in affine combinations with more general coefficients, e.g. $\frac{1}{2} \in \mathbb{Q}$, to form the midpoint $\frac{1}{2}x + \frac{1}{2}y$ of x and y. It can be proved that in the context of affine connections λ as considered presently in SDG terms, the data of a *symmetric* affine combination may equivalently be encoded as the data of forming midpoints of any x and y which are second order neighbours, cf. [11] or [12] 8.2. I do not know to what extent flatness of a symmetric affine connection can be formulated in terms of such midpoint formation, or more generally, in terms of binary affine combinations $(1-t)x + ty$; also, I do not know to what extent affine combinations of $n+1$-tuples $x_0, x_1, \ldots, x_n$ with $x_0 \sim x_i$ for $i = 1, \ldots n$ may be formed with more general coefficient than $\mathbb{Z}$. The $\Lambda(o; x, y, z)$ in (10) is an affine combination with four terms, with coefficients from $\mathbb{Z}$.

The question with general coefficients (say, coefficients from $\mathbb{R}$ in well-adapted models for SDG) has been studied in [2].

Appendix on notation

The main structure studied here: affine connections, has been presented with various notations: $\lambda(x; y, z)$, $[yxz]$, $y +_x z$ (and even $\nabla(x, y)(z)$). The first notation reflects the coordinate formulation which affine connections have in terms of Christoffel symbols Γ; thus, in [12] 2.3, $\lambda(x; y, z) = \Gamma(x; y - x, z - x)$, with $\Gamma(x; -, -) : V \times V \to V$ bilinear, where V is the coordinatizing vector space. The notation $[yxz]$ is essentially the one used by Prüfer [15], who writes $(yx^{-1}z)$. The notation $y +_x z$ indicates what [15] and we are aiming for, namely to enter the promised land of abelian groups, with arbitrary x as 0.

Finally, the special role of x in the ternary operation $[yxz] = \lambda(x; y, z)$ is, from the notational viewpoint: that x is the middle entry, respectively that x appears before the semicolon. For the 4-ary operation $\Lambda(o; x, y, z)$, as in the Cube Lemma, the "middle-entry" option is not available, whereas the semicolon option is.

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