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THE OPLAX LIMIT OF AN ENRICHED CATEGORY

In memory of our colleague Marta Bunge

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ABSTRACT. We show that 2-categories of the form $\mathscr{B}$-**Cat** are closed under slicing, provided that we allow $\mathscr{B}$ to range over bicategories (rather than, say, monoidal categories). That is, for any $\mathscr{B}$-category $\mathbb{X}$, we define a bicategory $\mathscr{B}/\mathbb{X}$ such that $\mathscr{B}\text{-}\mathbf{Cat}/\mathbb{X} \cong (\mathscr{B}/\mathbb{X})\text{-}\mathbf{Cat}$. The bicategory $\mathscr{B}/\mathbb{X}$ is characterized as the oplax limit of $\mathbb{X}$, regarded as a lax functor from a chaotic category to $\mathscr{B}$, in the 2-category **BICAT** of bicategories, lax functors and icons. We prove this conceptually, through limit-preservation properties of the 2-functor **BICAT** $\to$ 2-**CAT** which maps each bicategory $\mathscr{B}$ to the 2-category $\mathscr{B}$-**Cat**. When $\mathscr{B}$ satisfies a mild local completeness condition, we also show that the isomorphism $\mathscr{B}\text{-}\mathbf{Cat}/\mathbb{X} \cong (\mathscr{B}/\mathbb{X})\text{-}\mathbf{Cat}$ restricts to a correspondence between fibrations in $\mathscr{B}$-**Cat** over $\mathbb{X}$ on the one hand, and $\mathscr{B}/\mathbb{X}$-categories admitting certain powers on the other.

1. Introduction

It is well-known that for any monoidal category $\mathscr{V}$ and monoid $M = (M, e\colon I \to M, m\colon M \otimes M \to M)$ therein, the slice category $\mathscr{V}/M$ has a canonical monoidal structure; the unit is e and the monoidal product of objects $(s\colon S \to M)$ and $(t\colon T \to M)$ is

$$S \otimes T \xrightarrow{s \otimes t} M \otimes M \xrightarrow{m} M.$$

Moreover, there is a canonical isomorphism of categories

$$\mathbf{Mon}(\mathscr{V}/M) \cong \mathbf{Mon}(\mathscr{V})/M.$$

This paper originated from a natural generalization of this, replacing the notion of monoid in $\mathscr{V}$ by that of $\mathscr{V}$-category. That is, for any $\mathscr{V}$-category $\mathbb{X}$, there is an appropriate "base" $\mathscr{V}/\mathbb{X}$ admitting a canonical isomorphism of 2-categories

$$(\mathscr{V}/\mathbb{X})\text{-}\mathbf{Cat} \cong \mathscr{V}\text{-}\mathbf{Cat}/\mathbb{X}. \tag{1}$$

Here, the "base" $\mathscr{V}/\mathbb{X}$ is in general not a monoidal category but a bicategory. Enriched category theory over bicategories is developed in, e.g., [BCSW83, Str83]. We recall that, for a bicategory $\mathscr{B}$, a $\mathscr{B}$-category $\mathbb{X}$ is given by

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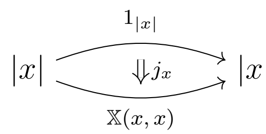
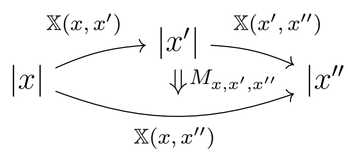
- a set $\mathrm{ob}(\mathbb{X})$;
- a function $|-|\colon \mathrm{ob}(\mathbb{X}) \to \mathrm{ob}(\mathscr{B})$ ($|x|$ is called the *extent* of x);
- for all $x, x' \in \mathrm{ob}(\mathbb{X})$, a 1-cell $\mathbb{X}(x, x')\colon |x| \to |x'|$ in $\mathscr{B}$;
- for all $x \in \mathrm{ob}(\mathbb{X})$, a 2-cell



  in $\mathscr{B}$, where $1_{|x|}$ is the identity 1-cell on $|x|$; and
- for all $x, x', x'' \in \mathrm{ob}(\mathbb{X})$, a 2-cell

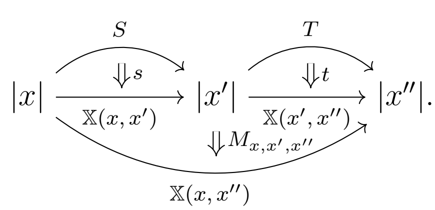


  in $\mathscr{B}$,

subject to the associativity and identity laws, generalizing the usual axioms for a category.

Since the isomorphism (1) already forces us to consider enrichment over bicategories, it is natural to wonder whether there is a generalization of the isomorphism involving a bicategory $\mathscr{B}$ in place of the monoidal category $\mathscr{V}$. Indeed this turns out to be the case: for any bicategory $\mathscr{B}$ and $\mathscr{B}$-category $\mathbb{X}$, there is a bicategory $\mathscr{B}/\mathbb{X}$ with a canonical isomorphism of 2-categories $(\mathscr{B}/\mathbb{X})$-$\mathbf{Cat} \cong \mathscr{B}$-$\mathbf{Cat}/\mathbb{X}$. Thus 2-categories of the form $\mathscr{B}$-$\mathbf{Cat}$ are closed under slicing, provided that we allow $\mathscr{B}$ to range over bicategories.

The construction of $\mathscr{B}/\mathbb{X}$ is simple enough to carry out at this point; see also Remark 4.8 for a more abstract point of view. We set $\mathrm{ob}(\mathscr{B}/\mathbb{X}) = \mathrm{ob}(\mathbb{X})$ and, for all $x, x' \in \mathrm{ob}(\mathscr{B}/\mathbb{X})$, the hom-category $(\mathscr{B}/\mathbb{X})(x, x')$ is the slice category $\mathscr{B}(|x|, |x'|)/\mathbb{X}(x, x')$. The identity 1-cell at x is j_x, and the composite of 1-cells $(s\colon S \to \mathbb{X}(x, x'))\colon x \to x'$ and $(t\colon T \to \mathbb{X}(x', x''))\colon x' \to x''$ is the pasting composite

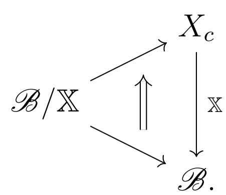


Of course, when both $\mathscr{B}$ and $\mathbb{X}$ have only one object, the construction of $\mathscr{B}/\mathbb{X}$ reduces to that of the slice of a monoidal category over a monoid.

This observation allows one to view (enriched) functors as (enriched) categories, and suggests new perspectives even on notions which are not directly related to enrichment.

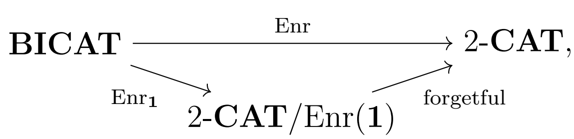
For example, for any (**Set**-)category $\mathbb{X}$, there is a bicategory $\mathbf{Set}/\mathbb{X}$ with an isomorphism $(\mathbf{Set}/\mathbb{X})$-$\mathbf{Cat} \cong \mathbf{Cat}/\mathbb{X}$. Thus we can view functors into $\mathbb{X}$ as enriched categories (see Example 4.6 below and [Gar14] for a related construction), and we may potentially interpret properties of functors via enriched categorical terms. Indeed, we shall show that a functor $\mathbb{Y} \to \mathbb{X}$ is a Grothendieck fibration if and only if the corresponding $\mathbf{Set}/\mathbb{X}$-category $\overline{\mathbb{Y}}$ has powers by a certain class of 1-cells in $\mathbf{Set}/\mathbb{X}$, as well as a $\mathscr{B}$-enriched version of this result.

The notation $\mathscr{B}/\mathbb{X}$ is justified by its characterization as the oplax limit of a 1-cell in a suitable 2-category. To explain this, recall that a $\mathscr{B}$-category $\mathbb{X}$ can be given equivalently as a lax functor $\mathbb{X}\colon X_c \to \mathscr{B}$, where X_c is the chaotic category with the same set of objects as $\mathbb{X}$.[1] Thus we can view the $\mathscr{B}$-category $\mathbb{X}$ as a 1-cell in the 2-category **BICAT** of bicategories, lax functors and icons [Lac10]. The bicategory $\mathscr{B}/\mathbb{X}$ is the oplax limit of this 1-cell in **BICAT**:



(Although **BICAT** is not complete, it does have oplax limits of 1-cells [Lac05, LS12].) This generalizes the characterization of the slice monoidal category $\mathscr{V}/M$ as the oplax limit of the monoid M in $\mathscr{V}$, regarded as a lax monoidal functor from the terminal monoidal category to $\mathscr{V}$, in the 2-category of monoidal categories, lax monoidal functors and monoidal natural transformations.

In this paper, we study properties of the 2-functor $\mathrm{Enr}\colon \mathbf{BICAT} \to 2\text{-}\mathbf{CAT}$ mapping each bicategory $\mathscr{B}$ to the 2-category $\mathscr{B}$-**Cat**, in order to understand the isomorphism $(\mathscr{B}/\mathbb{X})$-$\mathbf{Cat} \cong \mathscr{B}$-$\mathbf{Cat}/\mathbb{X}$ conceptually, as well as to establish further closure properties of 2-categories of the form $\mathscr{B}$-**Cat**. To this end, it is useful to factorize Enr as



where $\mathbf{1}$ is the terminal bicategory. The 2-functor $\mathrm{Enr}_{\mathbf{1}}$ maps each bicategory $\mathscr{B}$ to $\mathscr{B}$-**Cat** equipped with the 2-functor $\mathrm{Enr}(!)\colon \mathscr{B}\text{-}\mathbf{Cat} \to \mathrm{Enr}(\mathbf{1})$ induced from the unique lax functor $!\colon \mathscr{B} \to \mathbf{1}$. The underlying category of $\mathrm{Enr}(\mathbf{1})$ is **Set**, and $\mathrm{Enr}(!)$ can be regarded as $\mathrm{ob}(-)$, mapping each $\mathscr{B}$-category $\mathbb{X}$ to its set of objects $\mathrm{ob}(\mathbb{X})$. (Although Enr is usually denoted simply as $(-)$-**Cat**, we adopted the current notation in order to avoid the potentially misleading expression **1-Cat**.)

In our main theorem (Theorem 2.1), we show that $\mathrm{Enr}_{\mathbf{1}}\colon \mathbf{BICAT} \to 2\text{-}\mathbf{CAT}/\mathrm{Enr}(\mathbf{1})$ preserves *any* limit which happens to exist in **BICAT**. This implies that Enr preserves

[1] Lax functors of this form were studied by Bénabou [Bén67] under the name *polyad*; for the connection with enriched categories see [Str83].

any limit which happens to exist in **BICAT** *and* is preserved by the forgetful 2-functor 2-**CAT**/Enr(**1**) → 2-**CAT**; the latter condition is satisfied whenever the limit in question is small enough to exist in 2-**CAT** and is created by the forgetful 2-functor. In ordinary category theory, the limits created by the forgetful functors from slice categories are precisely the connected limits. In Section 3 we generalize this to 2-categories (or in fact to $\mathscr{V}$-categories where $\mathscr{V}$ is any complete and cocomplete cartesian closed category), introducing the class of **Cat**-connected limits with several characterizations. Thus Enr: **BICAT** → 2-**CAT** preserves any **Cat**-connected limit which happens to exist in **BICAT**. This includes Eilenberg–Moore objects of comonads, for example. Although oplax limits of 1-cells are not **Cat**-connected, the isomorphism $(\mathscr{B}/\mathbb{X})$-**Cat** $\cong$ $\mathscr{B}$-**Cat**$/\mathbb{X}$ is explained via the limit-preservation property of Enr and a 2-categorical argument in Section 4.

Finally, in Section 5, we investigate (internal) fibrations in the 2-category $\mathscr{B}$-**Cat** of $\mathscr{B}$-categories. Specifically, we show that (assuming a mild local completeness condition on $\mathscr{B}$) a $\mathscr{B}$-functor $\mathbb{Y} \to \mathbb{X}$ is a fibration in $\mathscr{B}$-**Cat** if and only if the corresponding $\mathscr{B}/\mathbb{X}$-category $\overline{\mathbb{Y}}$ admits certain powers.

We intend to revisit the results of this paper in the future, in the context of enrichment over pseudo double categories.

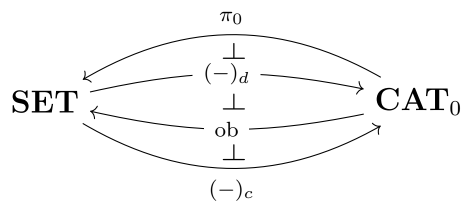
2. The limit-preservation theorem

Size does not play a significant role in this paper; nonetheless we make a few comments here about the issues which arise and our approach to them. The typical *monoidal* categories over which one enriches, such as **Set**, **Cat**, or **Ab**, have small hom-sets but are not themselves small. Thus the corresponding bicategories will not even have small hom-categories. We do still need some control of the size of these bicategories, and therefore fix two Grothendieck universes $\mathcal{U}_0$ and $\mathcal{U}_1$ with $\mathcal{U}_0 \in \mathcal{U}_1$. Sets, categories, etc. in $\mathcal{U}_0$ and $\mathcal{U}_1$ are called *small* and *large* respectively.

Let **BICAT** be the 2-category of large bicategories, lax functors and icons [Lac10, Theorem 3.2], and 2-**CAT** be the 2-category of large 2-categories, 2-functors and 2-natural transformations. We have a 2-functor Enr: **BICAT** → 2-**CAT** sending each bicategory $\mathscr{B}$ to the 2-category $\mathscr{B}$-**Cat** of all small $\mathscr{B}$-categories, $\mathscr{B}$-functors and $\mathscr{B}$-natural transformations. It is the limit-preservation properties of this 2-functor Enr that is our main focus. The limits in question will be 2-limits weighted by 2-functors of the form $F\colon \mathscr{D} \to$ **CAT**, where $\mathscr{D}$ is a large 2-category and **CAT** is the 2-category of large categories.

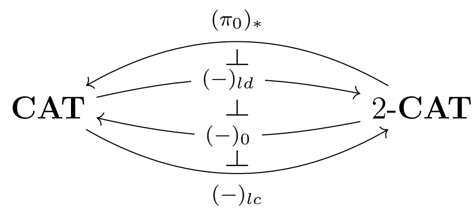
The bicategory **1** with a single 2-cell is the terminal object of **BICAT**, and hence Enr induces the 2-functor $\mathrm{Enr}_{\mathbf{1}}$: **BICAT** → 2-**CAT**/Enr(**1**), where 2-**CAT**/Enr(**1**) denotes the (strict) slice 2-category of 2-**CAT** over Enr(**1**) ∈ 2-**CAT**. The 2-category Enr(**1**) is the locally chaotic 2-category whose underlying category is **Set**. More precisely, the objects of Enr(**1**) can be identified with the small sets, and for each pair of small sets X and Y

we have $\mathrm{Enr}(\mathbf{1})(X,Y) = \mathbf{Set}(X,Y)_c$, where $(-)_c$ appears in the string of adjunctions



(2)

Here, $\mathbf{SET}$ and $\mathbf{CAT}_0$ denote the categories of large sets and of large categories respectively. The (finite-product-preserving) functors in (2) induce 2-adjunctions



Thus we shall write the 2-category $\mathrm{Enr}(\mathbf{1})$ as $\mathbf{Set}_{lc}$.

Explicitly, the 2-functor $\mathrm{Enr}_{\mathbf{1}} \colon \mathbf{BICAT} \to 2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}$ maps each bicategory $\mathscr{B}$ to the 2-category $\mathscr{B}$-$\mathbf{Cat}$ equipped with the 2-functor $\mathrm{ob}(-) \colon \mathscr{B}\text{-}\mathbf{Cat} \to \mathbf{Set}_{lc}$ which extracts the set of objects of a $\mathscr{B}$-category.

2.1. Theorem. *The 2-functor* $\mathrm{Enr}_{\mathbf{1}} \colon \mathbf{BICAT} \to 2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}$ *preserves all weighted limits which happen to exist in* $\mathbf{BICAT}$.

Proof. We shall show the following.

(a) The set $\mathcal{G}$ of all objects of $2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}$ of the form $(\mathbf{2_2} \to \mathbf{Set}_{lc})$, where $\mathbf{2_2}$ denotes the free 2-category on a single 2-cell, is a strong generator of the 2-category $2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}$.

(b) For each object $A \in \mathcal{G}$, the 2-functor $2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}(A, \mathrm{Enr}_{\mathbf{1}}(-)) \colon \mathbf{BICAT} \to \mathbf{CAT}$ is a 2-limit of representable 2-functors, and hence preserves all weighted limits which happen to exist in $\mathbf{BICAT}$.

From these, the main claim follows. Indeed, let $\mathscr{D}$ be a large 2-category, $F \colon \mathscr{D} \to \mathbf{CAT}$ be a 2-functor (the weight) and $S \colon \mathscr{D} \to \mathbf{BICAT}$ be a 2-functor such that the weighted limit $\{F, S\}$ exists in $\mathbf{BICAT}$. Then the weighted limit $\{F, \mathrm{Enr}_{\mathbf{1}} \circ S\}$ exists in $2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}$, because $2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}$ has all (large) weighted limits. We have a comparison 1-cell $M \colon \mathrm{Enr}_{\mathbf{1}}\{F, S\} \to \{F, \mathrm{Enr}_{\mathbf{1}} \circ S\}$ in $2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}$. Now for each $A \in \mathcal{G}$, the functor

$$2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}(A, M) \colon 2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}(A, \mathrm{Enr}_{\mathbf{1}}\{F, S\}) \to 2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}(A, \{F, \mathrm{Enr}_{\mathbf{1}} \circ S\})$$

is an isomorphism by (b), from which we conclude that M is an isomorphism by (a).

$\mathcal{G}$ is a strong generator of 2-$\mathbf{CAT}/\mathbf{Set}_{lc}$ because, given any 1-cell $T\colon (\mathscr{X} \to \mathbf{Set}_{lc}) \to (\mathscr{Y} \to \mathbf{Set}_{lc})$, i.e., a 2-functor $T\colon \mathscr{X} \to \mathscr{Y}$ between 2-categories $\mathscr{X}$ and $\mathscr{Y}$ over $\mathbf{Set}_{lc}$, the condition that 2-$\mathbf{CAT}/\mathbf{Set}_{lc}(A, T)$ be an isomorphism for all $A \in \mathcal{G}$ means that T is bijective on 2-cells.

To show (b), observe that a 2-functor $\mathbf{2_2} \to \mathbf{Set}_{lc}$ corresponds to a parallel pair of functions $f_0, f_1\colon X \to Y$. Such a 2-functor can be seen as an object of 2-$\mathbf{CAT}/\mathbf{Set}_{lc}$. Given $((f_0, f_1)\colon \mathbf{2_2} \to \mathbf{Set}_{lc})$ where $f_0, f_1\colon X \to Y$, first consider the category $\mathbf{2} \times X_c$ where $\mathbf{2} = \{0 < 1\}$ is the two-element chain. We regard $\mathbf{2} \times X_c$ as a bicategory as well. We have the projection functor $\pi\colon \mathbf{2} \times X_c \to X_c$ and the functor $[f_0, f_1]\colon \mathbf{2} \times X_c \to Y_c$ defined by $[f_0, f_1](i, x) = f_i(x)$; these can also be regarded as lax functors, i.e., morphisms in $\mathbf{BICAT}$. The 2-functor

$$2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}((f_0, f_1), \mathrm{Enr}_{\mathbf{1}}(-))\colon \mathbf{BICAT} \to \mathbf{CAT}$$

is the comma object (in $[\mathbf{BICAT}, \mathbf{CAT}]$) as in

$$\begin{array}{ccc} 2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}((f_0, f_1), \mathrm{Enr}_{\mathbf{1}}(-)) & \longrightarrow & \mathbf{BICAT}(X_c, -) \\ \downarrow & \swarrow & \downarrow {\scriptstyle \mathbf{BICAT}(\pi, -)} \\ \mathbf{BICAT}(Y_c, -) & \xrightarrow[\mathbf{BICAT}([f_0, f_1], -)]{} & \mathbf{BICAT}(\mathbf{2} \times X_c, -). \end{array}$$

Indeed, for any bicategory $\mathscr{B} \in \mathbf{BICAT}$, an object of the comma category of the functors $\mathbf{BICAT}([f_0, f_1], \mathscr{B})$ and $\mathbf{BICAT}(\pi, \mathscr{B})$ consists of lax functors $\mathbb{C}\colon X_c \to \mathscr{B}$ and $\mathbb{D}\colon Y_c \to \mathscr{B}$ together with an icon

$$\begin{array}{ccc} \mathbf{2} \times X_c & \xrightarrow{\pi} & X_c \\ {\scriptstyle [f_0, f_1]} \downarrow & \overset{\alpha}{\swarrow} & \downarrow {\scriptstyle \mathbb{C}} \\ Y_c & \xrightarrow[\mathbb{D}]{} & \mathscr{B}. \end{array}$$

This corresponds to $\mathscr{B}$-categories $\mathbb{C}$ and $\mathbb{D}$ with $\mathrm{ob}(\mathbb{C}) = X$ and $\mathrm{ob}(\mathbb{D}) = Y$ such that $|x|_{\mathbb{C}} = |f_i(x)|_{\mathbb{D}}$ for all $x \in X$ and $i \in \{0, 1\}$, together with a 2-cell $\alpha_{(i,x),(i',x')}\colon \mathbb{C}(x, x') \to \mathbb{D}(f_i(x), f_{i'}(x'))$ in $\mathscr{B}$ for all $(i, x), (i', x') \in \mathbf{2} \times X_c$ with $i \leq i'$, satisfying some equations. These latter data in turn correspond to $\mathscr{B}$-functors $F_0\colon \mathbb{C} \to \mathbb{D}$ and $F_1\colon \mathbb{C} \to \mathbb{D}$ (with $\mathrm{ob}(F_i) = f_i$) together with a $\mathscr{B}$-natural transformation $\alpha\colon F_0 \to F_1$. (We record in Lemma 2.2 below an observation which is useful for the verification.)

This gives a bijective correspondence on objects of 2-$\mathbf{CAT}/\mathbf{Set}_{lc}((f_0, f_1), \mathrm{Enr}_{\mathbf{1}}(\mathscr{B}))$ and the comma category of $\mathbf{BICAT}([f_0, f_1], \mathscr{B})$ and $\mathbf{BICAT}(\pi, \mathscr{B})$, which routinely extends to an isomorphism of categories natural in $\mathscr{B}$. ∎

2.2. Lemma. *Let $\mathscr{B}$ be a bicategory, $\mathbb{C}, \mathbb{D}$ be $\mathscr{B}$-categories and $T, S\colon \mathbb{C} \to \mathbb{D}$ be $\mathscr{B}$-functors. To give a $\mathscr{B}$-natural transformation $\alpha\colon T \to S$, i.e., a family of 2-cells*

$$|x| \underset{\mathbb{D}(Tx,Sx)}{\overset{1_{|x|}}{\overset{\Downarrow \alpha_x}{\rightrightarrows}}} |x|$$

in $\mathscr{B}$ for all $x \in \mathbb{C}$, satisfying the naturality axiom saying that for all $x, x' \in \mathbb{C}$,

$$\begin{array}{ccc}
1_{|x'|}.\mathbb{C}(x,x') & \xrightarrow{\alpha_{x'}.T_{x,x'}} & \mathbb{D}(Tx',Sx').\mathbb{D}(Tx,Tx') \\
\cong\big\uparrow & & \big\downarrow M^{\mathbb{D}}_{Tx,Tx',Sx'} \\
\mathbb{C}(x,x') & & \mathbb{D}(Tx,Sx') \\
\cong\big\downarrow & & \big\uparrow M^{\mathbb{D}}_{Tx,Sx,Sx'} \\
\mathbb{C}(x,x').1_{|x|} & \xrightarrow[S_{x,x'}.\alpha_x]{} & \mathbb{D}(Sx,Sx').\mathbb{D}(Tx,Sx)
\end{array} \tag{3}$$

commutes, is equivalent to giving a family of 2-cells

$$|x| \underset{\mathbb{D}(Tx,Sx')}{\overset{\mathbb{C}(x,x')}{\overset{\Downarrow \alpha_{x,x'}}{\rightrightarrows}}} |x'|$$

in $\mathscr{B}$ for all $x, x' \in \mathbb{C}$, such that for all $x, x', x'' \in \mathbb{C}$,

$$\begin{array}{ccc}
\mathbb{C}(x',x'').\mathbb{C}(x,x') & \xrightarrow{M^{\mathbb{C}}_{x,x',x''}} & \mathbb{C}(x,x'') \\
{\scriptstyle S_{x',x''}.\alpha_{x,x'}}\big\downarrow & & \big\downarrow {\scriptstyle \alpha_{x,x''}} \\
\mathbb{D}(Sx',Sx'').\mathbb{D}(Tx,Sx') & \xrightarrow[M^{\mathbb{D}}_{Tx,Sx',Sx''}]{} & \mathbb{D}(Tx,Sx'')
\end{array} \qquad \textit{and}$$

$$\begin{array}{ccc}
\mathbb{C}(x',x'').\mathbb{C}(x,x') & \xrightarrow{M^{\mathbb{C}}_{x,x',x''}} & \mathbb{C}(x,x'') \\
{\scriptstyle \alpha_{x',x''}.T_{x,x'}}\big\downarrow & & \big\downarrow {\scriptstyle \alpha_{x,x''}} \\
\mathbb{D}(Tx',Sx'').\mathbb{D}(Tx,Tx') & \xrightarrow[M^{\mathbb{D}}_{Tx,Tx',Sx''}]{} & \mathbb{D}(Tx,Sx'')
\end{array}$$

commute; the correspondence is given by mapping (α_x) to $(\alpha_{x,x'})$ whose component at (x, x') is the composite (3).

As observed in [Lac10, Section 6.2], the 2-category **BICAT** can be seen as the 2-category of strict algebras, lax morphisms, and algebra 2-cells for a 2-monad T on a certain locally presentable 2-category of **CAT**-enriched graphs, and so by [Lac05] has oplax limits, Eilenberg–Moore objects of comonads, and limits of diagrams containing only strict morphisms; this last class includes in particular products and powers. It also has various other sorts of limits where certain parts of the diagram are required to be pseudofunctors. For a more precise characterization see [LS12].

The case of oplax limits of 1-cells is our motivating example, and is formalized in Section 4, specifically in Theorem 4.5. The case of Eilenberg–Moore objects of comonads is treated in Example 3.9. As a final example, we consider products. In this case, Theorem 2.1 says that, for bicategories $\mathscr{B}$ and $\mathscr{C}$, the diagram

$$\begin{array}{ccc} (\mathscr{B}\times\mathscr{C})\text{-}\mathbf{Cat} & \longrightarrow & \mathscr{C}\text{-}\mathbf{Cat} \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ob}} \\ \mathscr{B}\text{-}\mathbf{Cat} & \xrightarrow[\mathrm{ob}]{} & \mathbf{Set}_{lc} \end{array}$$

is a pullback of 2-categories. In particular, to give a $\mathscr{B}\times\mathscr{C}$-category is equivalent to giving a $\mathscr{B}$-category and a $\mathscr{C}$-category with the same set of objects.

2.3. Remark. It is possible to remove any size-related conditions on the notion of weighted limit in Theorem 2.1. That is, for any (possibly larger than "large") 2-category $\mathscr{D}$ and a weight $F\colon \mathscr{D}\to \mathbf{CAT}'$, where $\mathbf{CAT}'$ is a 2-category of categories in a universe containing $\mathcal{U}_1$, $\mathrm{Enr}_{\mathbf{1}}$ preserves all F-weighted limits which happen to exist in **BICAT**. Indeed, let $S\colon \mathscr{D}\to \mathbf{BICAT}$ be a 2-functor such that $\{F,S\}$ exists in **BICAT**. Then, although a priori we do not know if $\{F,\mathrm{Enr}_{\mathbf{1}}\circ S\}$ exists in $2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}$ or not, we can certainly consider a large enough variant $2\text{-}\mathbf{CAT}'/\mathbf{Set}_{lc}$ in which it does. Then by the above discussion we have $\mathrm{Enr}_{\mathbf{1}}\{F,S\}\cong\{F,\mathrm{Enr}_{\mathbf{1}}\circ S\}$ in $2\text{-}\mathbf{CAT}'/\mathbf{Set}_{lc}$. Since the fully faithful 2-functor $2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}\to 2\text{-}\mathbf{CAT}'/\mathbf{Set}_{lc}$ reflects limits, and $\mathrm{Enr}_{\mathbf{1}}$ does land in $2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}$, we see that the limit $\{F,\mathrm{Enr}_{\mathbf{1}}\circ S\}$ actually exists in $2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}$.

3. Weighted limits created by forgetful 2-functors $\mathscr{K}/A\to\mathscr{K}$

Theorem 2.1 implies that the 2-functor $\mathrm{Enr}\colon \mathbf{BICAT}\to 2\text{-}\mathbf{CAT}$ preserves all weighted limits preserved by the forgetful 2-functor $2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}\to 2\text{-}\mathbf{CAT}$. We now investigate these.

A large part of this section (until the end of Example 3.7) is devoted to the study of this class of limits, which we shall call **Cat**-connected. Since this notion does not require two separate universes, and since it may be of interest in other contexts, we work with a single universe $\mathcal{U}$, whose elements we call *small* sets. (We temporarily ignore $\mathcal{U}_0$ introduced at the beginning of Section 2.) When we later return to the study of **BICAT** and $2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}$, we apply our results in the case $\mathcal{U}=\mathcal{U}_1$, and so speak of **CAT**-connected limits.

In the literature there are (at least) two definitions of creation of limit. Given 2-functors $F\colon \mathscr{D} \to \mathbf{Cat}$, $S\colon \mathscr{D} \to \mathscr{A}$, and $G\colon \mathscr{A} \to \mathscr{B}$, the phrase "$G$ creates the F-weighted limit of S" could mean either of the following.

- For any F-weighted limit $(\{F, GS\}, \mu\colon F \to \mathscr{B}(\{F, GS\}, GS-))$ of GS, there exists a unique F-cylinder $(L, \nu\colon F \to \mathscr{A}(L, S-))$ over S in $\mathscr{A}$ such that $\{F, GS\} = GL$ and $\mu = G_{L,S-} \circ \nu$ hold. Moreover, (L, ν) is an F-weighted limit of S.
- For any F-weighted limit $(\{F, GS\}, \mu)$ of GS, there exists an F-cylinder (L, ν) over S in $\mathscr{A}$ such that the mediating 1-cell $GL \to \{F, GS\}$ is an isomorphism. Moreover, such an F-cylinder (L, ν) is always an F-weighted limit of S.

These two conditions are equivalent when G is the forgetful 2-functor $\mathscr{K}/A \to \mathscr{K}$ from a slice 2-category, since such 2-functors reflect identities and lift invertible 1-cells.

In the following, 1 and $\mathbf{1}$ denote the terminal category and the terminal 2-category, respectively.

3.1. Theorem. *Let $\mathscr{D}$ be a small 2-category and $F\colon \mathscr{D} \to \mathbf{Cat}$ be a 2-functor. Then the following are equivalent.*

(1) *All F-weighted limits are created by the forgetful 2-functor $\mathscr{K}/A \to \mathscr{K}$ for any locally small 2-category $\mathscr{K}$ and $A \in \mathscr{K}$.*

(2) *All F-weighted limits commute with copowers in $\mathbf{Cat}$. In other words, F-weighted limits are preserved by the 2-functor $X \times (-)\colon \mathbf{Cat} \to \mathbf{Cat}$ for any $X \in \mathbf{Cat}$.*

(3) *The F-weighted limit of the unique 2-functor $\mathscr{D} \to \mathbf{1}$ is preserved by any 2-functor $\mathbf{1} \to \mathbf{Cat}$: that is, $X \cong [\mathscr{D}, \mathbf{Cat}](F, \Delta X)$ for any $X \in \mathbf{Cat}$.*

(4) *The F-weighted limit of the unique 2-functor $\mathscr{D} \to \mathbf{1}$ is preserved by any 2-functor $\mathbf{1} \to \mathscr{K}$: that is, $A \cong \{F, \Delta A\}$ for any locally small 2-category $\mathscr{K}$ and $A \in \mathscr{K}$.*

(5) *$F * (-)\colon [\mathscr{D}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{Cat}$ preserves the terminal object. In other words, the F-weighted colimit of $\Delta 1\colon \mathscr{D}^{\mathrm{op}} \to \mathbf{Cat}$ is the terminal category: $F * \Delta 1 \cong 1$.*

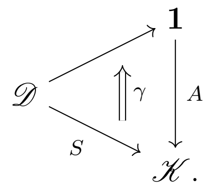
(6) *The (conical) colimit of F is the terminal category: $\Delta 1 * F \cong 1$.*

Proof. [(1) $\Longrightarrow$ (2)] For any $X \in \mathbf{Cat}$, copowers by X are given by $X \times (-)\colon \mathbf{Cat} \to \mathbf{Cat}$, which is the composite of the right adjoint 2-functor $X \times (-)\colon \mathbf{Cat} \to \mathbf{Cat}/X$ and the forgetful 2-functor $\mathbf{Cat}/X \to \mathbf{Cat}$.

[(2) $\Longrightarrow$ (3)] Note that we have $1 \cong \{F, \Delta 1\}$ in $\mathbf{Cat}$. Since $X \times (-)\colon \mathbf{Cat} \to \mathbf{Cat}$ preserves the F-weighted limit $\{F, \Delta 1\}$, we have $X \cong \{F, \Delta X\}$.

[(3) $\Longrightarrow$ (4)] For any $B \in \mathscr{K}$ we have $\mathscr{K}(B, A) \cong [\mathscr{D}, \mathbf{Cat}](F, \Delta\mathscr{K}(B, A))$. This shows that $A \in \mathscr{K}$ is the weighted limit $\{F, \Delta A\}$.

[(4) $\Longrightarrow$ (1)] Let $T\colon \mathscr{D} \to \mathscr{K}/A$ be a 2-functor, with the corresponding oplax cone



In particular, S is the composite of T and the forgetful 2-functor $\mathscr{K}/A \to \mathscr{K}$. Suppose that the weighted limit $\{F, S\}$ exists in $\mathscr{K}$. We have a 1-cell $\{F,\gamma\}\colon \{F,S\} \to \{F,\Delta A\} \cong A$ in $\mathscr{K}$. We claim that the object $(\{F,\gamma\}\colon \{F,S\} \to A) \in \mathscr{K}/A$ is the limit $\{F,T\}$ in $\mathscr{K}/A$. For any $(p\colon B \to A) \in \mathscr{K}/A$, the hom category $(\mathscr{K}/A)((B,p),(\{F,S\},\{F,\gamma\}))$ is given by the equalizer

$$(\mathscr{K}/A)((B,p),(\{F,S\},\{F,\gamma\})) \longrightarrow \mathscr{K}(B,\{F,S\}) \underset{\Delta p}{\overset{\mathscr{K}(B,\{F,\gamma\})}{\rightrightarrows}} \mathscr{K}(B,A),$$

which is easily seen to be canonically isomorphic to $[\mathscr{D},\mathbf{Cat}](F,(\mathscr{K}/A)((B,p),T-))$.

[(4) $\Longrightarrow$ (5)] Applying (4) to $1\colon \mathbf{1} \to \mathbf{Cat}^{\mathrm{op}}$, we obtain $F * \Delta 1 \cong 1$ in $\mathbf{Cat}$.

[(5) $\Longrightarrow$ (3)] For any $X \in \mathbf{Cat}$, we have

$$X \cong [1,X] \cong [F*\Delta 1, X] \cong [\mathscr{D},\mathbf{Cat}](F,[\Delta 1(-),X]) \cong [\mathscr{D},\mathbf{Cat}](F,\Delta X).$$

[(5) $\Longleftrightarrow$ (6)] By $F * \Delta 1 \cong \Delta 1 * F$. ∎

A 2-functor $F\colon \mathscr{D} \to \mathbf{Cat}$ is called $\mathbf{Cat}$-*connected* if F satisfies the equivalent conditions of Theorem 3.1. Similarly, a weighted limit is $\mathbf{Cat}$-*connected* if its weight is so. Note that $F\colon \mathscr{D} \to \mathbf{Cat}$ is connected (in the sense that $[\mathscr{D},\mathbf{Cat}](F,-)\colon [\mathscr{D},\mathbf{Cat}] \to \mathbf{Cat}$ preserves small coproducts) if and only if $[\mathscr{D},\mathbf{Cat}](F,\Delta X) \cong X$ for any small discrete category X, or equivalently just for $X = 1+1$; on the other hand it is $\mathbf{Cat}$-connected if this holds for all small categories X.

3.2. Remark. Theorem 3.1 can be proved more generally for categories enriched over a complete and cocomplete cartesian closed category $\mathscr{V}$ in place of $\mathbf{Cat}$, indeed the proof carries over essentially word-for-word upon replacing each instance of $\mathbf{Cat}$ by $\mathscr{V}$.

We now give a few simple results about $\mathbf{Cat}$-connected weights in order to clarify the scope of the notion.

3.3. Proposition. *If $\mathscr{D}$ has a terminal object, then $F\colon \mathscr{D} \to \mathbf{Cat}$ is $\mathbf{Cat}$-connected if and only if F preserves the terminal object.*

Proof. If $\mathscr{D}$ has a terminal object 1 then the colimit of F is $F(1)$. ∎

3.4. Proposition. *Let $\mathscr{C}$ be a small ordinary category, and $G\colon \mathscr{C} \to \mathbf{Set}$ a functor. This determines a 2-functor $G_d\colon \mathscr{C}_{ld} \to \mathbf{Cat}$, where now $\mathscr{C}_{ld}$ is regarded as a locally discrete 2-category. This G_d sends an object C to the discrete category $G(C)_d$ with object-set $G(C)$. Then G_d is $\mathbf{Cat}$-connected if and only if the corresponding G is connected.*

Proof. Since the functor $(-)_d\colon \mathbf{Set} \to \mathbf{Cat}_0$ preserves colimits, $\mathrm{colim}(G_d) = \mathrm{colim}(G)_d$. ∎

3.5. Proposition. $\Delta 1\colon \mathscr{D} \to \mathbf{Cat}$ *is* $\mathbf{Cat}$*-connected if and only if* $\mathscr{D}_0$ *is connected.*

Proof. The colimit of $\Delta 1\colon \mathscr{D} \to \mathbf{Cat}$ is the discrete category corresponding to the set of connected components of $\mathscr{D}_0$. ∎

3.6. Example. Equifiers are $\mathbf{Cat}$-connected: here it is easiest to verify directly that equifiers in $\mathbf{Cat}$ commute with copowers. Similarly, one verifies that Eilenberg–Moore objects of monads and of comonads are $\mathbf{Cat}$-connected. Equalizers and pullbacks are $\mathbf{Cat}$-connected by Proposition 3.5.

3.7. Example. Non-trivial products are not $\mathbf{Cat}$-connected: they are not even connected. Powers by a category X are limits weighted by $X\colon \mathbf{1} \to \mathbf{Cat}$; since the colimit of such a weight is just X, powers by X are $\mathbf{Cat}$-connected if and only if $X = 1$. Inserters, comma objects and oplax limits of 1-cells are not $\mathbf{Cat}$-connected: in particular they are not preserved by the 2-functor $\mathbb{N}\colon \mathbf{1} \to \mathbf{Cat}$ which picks out the additive monoid $\mathbb{N}$ of natural numbers. More generally, inserters are not preserved by $X\colon \mathbf{1} \to \mathbf{Cat}$ if X has a non-identity endomorphism, while comma objects and oplax limits of 1-cells are not preserved by $X\colon \mathbf{1} \to \mathbf{Cat}$ unless X is discrete.

As anticipated at the beginning of the section, we now take $\mathcal{U}$ to be $\mathcal{U}_1$, and use the resulting notion of $\mathbf{CAT}$-connected limit, involving a large 2-category $\mathscr{D}$ and a 2-functor $F\colon \mathscr{D} \to \mathbf{CAT}$ as weight. Since the inclusion $\mathbf{Cat} \to \mathbf{CAT}$ preserves small limits and small colimits, $\mathbf{Cat}$-connected limits are also $\mathbf{CAT}$-connected. As an immediate consequence of Theorems 2.1 and 3.1, we have:

3.8. Corollary. *The 2-functor* $\mathrm{Enr}\colon \mathbf{BICAT} \to 2\text{-}\mathbf{CAT}$ *preserves all* $\mathbf{CAT}$*-connected limits which happen to exist in* $\mathbf{BICAT}$*.*

3.9. Example. Eilenberg–Moore objects of comonads are $\mathbf{Cat}$-connected (as well as $\mathbf{CAT}$-connected), and exist in $\mathbf{BICAT}$ by the results of [Lac05, LS12], thus they are preserved by Enr. In more detail, a comonad G in $\mathbf{BICAT}$ on a bicategory $\mathscr{B}$ consists of a comonad $G = G_{a,b}$ on each hom-category $\mathscr{B}(a,b)$, together with 2-cells $G_2\colon Gg.Gf \to G(gf)$ for all $f\colon a \to b$ and $g\colon b \to c$, and 2-cells $G_0\colon 1_{Ga} \to G1_a$ for all objects a, subject to various conditions, which say that the $G_{a,b}$, the G_2 and the G_0 can be assembled into an identity-on-objects lax functor $\mathscr{B} \to \mathscr{B}$, in such a way that the counits and comultiplications for the comonads become icons. The Eilenberg–Moore object $\mathscr{B}^G$ is the bicategory with the same objects as $\mathscr{B}$, and with hom-category $\mathscr{B}^G(a,b)$ given by the Eilenberg–Moore category $\mathscr{B}(a,b)^{G_{a,b}}$ of $G_{a,b}$. Corollary 3.8 then says that $\mathscr{B}^G$-$\mathbf{Cat}$ is the Eilenberg–Moore 2-category for the induced 2-comonad on $\mathscr{B}$-$\mathbf{Cat}$.

4. Oplax limits and fibrations

A 1-cell $f\colon A \to B$ in a 2-category $\mathscr{K}$ is called a *fibration*, when $\mathscr{K}(C, f)\colon \mathscr{K}(C, A) \to \mathscr{K}(C, B)$ is a Grothendieck fibration for each $C \in \mathscr{K}$, and

$$\begin{array}{ccc} \mathscr{K}(C,A) & \xrightarrow{\mathscr{K}(c,A)} & \mathscr{K}(D,A) \\ {\scriptstyle \mathscr{K}(C,f)}\downarrow & & \downarrow{\scriptstyle \mathscr{K}(D,f)} \\ \mathscr{K}(C,B) & \xrightarrow[\mathscr{K}(c,B)]{} & \mathscr{K}(D,B) \end{array}$$

is a morphism of fibrations for each $c\colon D \to C$ in $\mathscr{K}$, in the sense that $\mathscr{K}(c, A)$ sends cartesian morphisms (with respect to $\mathscr{K}(C, f)$) to cartesian morphisms (with respect to $\mathscr{K}(D, f)$). If $q\colon F \to B$ and $p\colon E \to B$ are fibrations in $\mathscr{K}$ with the common codomain B, then a 1-cell $r\colon (F, q) \to (E, p)$ in $\mathscr{K}/B$ is a *morphism of fibrations* if for each $C \in \mathscr{K}$, $\mathscr{K}(C, r)$ is a morphism of fibrations, i.e., preserves cartesian morphisms.

As explained by Street [Str74], these notions can be reformulated if the 2-category $\mathscr{K}$ has oplax limits of 1-cells, as we shall henceforth suppose. Recall that the oplax limit of a 1-cell $f\colon A \to B$ in $\mathscr{K}$ is the universal diagram

$$\begin{array}{ccc} & \overset{u_f}{\nearrow} & A \\ B/f & \Uparrow \lambda_f & \downarrow f \\ & \underset{v_f}{\searrow} & B \end{array}$$

wherein we often drop the subscripts f unless multiple oplax limits are being used.

If $\mathscr{K} = \mathbf{Cat}$, then these oplax limits are comma categories, as the notation suggests. On the other hand, we have:

4.1. Example. Let X be a small set, seen as a chaotic bicategory X_c (that is, $(X_c)_{ld}$ or equivalently $(X_c)_{lc}$). To give an X_c-enriched category is just to give a set of objects with a map into X. Similar calculations involving X_c-enriched functors and natural transformations show that the diagram

$$\begin{array}{ccc} & \nearrow & \mathbf{1} \\ X_c\text{-}\mathbf{Cat} & \Uparrow & \downarrow X \\ & \underset{\mathrm{ob}}{\searrow} & \mathbf{Set}_{lc} \end{array}$$

is an oplax limit in 2-$\mathbf{CAT}$; in other words, the 2-category X_c-$\mathbf{Cat}$ is isomorphic to the slice 2-category $\mathbf{Set}_{lc}/X$; this in turn is isomorphic to $(\mathbf{Set}/X)_{lc}$.

The fibrations in $\mathscr{K}$ with codomain B can be understood in terms of a 2-monad T_B on $\mathscr{K}/B$ whose underlying 2-functor maps $f\colon A \to B$ to $v_f\colon B/f \to B$; the component at $f\colon A \to B$ of its unit is the unique map $d = d_f\colon A \to B/f$ with $ud = 1_A$, $vd = f$, and λd equal to the identity 2-cell on f. This 2-monad is *colax idempotent* (has the dual of the "Kock–Zöberlein property"), and so an object $f\colon A \to B$ of $\mathscr{K}/B$ admits the structure of a pseudo T_B-algebra if and only if $d\colon (A,f) \to (B/f, v_f)$ has a right adjoint in $\mathscr{K}/B$; and this in turn is the case if and only if f is a fibration. See for example [Str74, Proposition 3(a)] and [Web07, Theorem 2.7].

Also, if $q\colon F \to B$ and $p\colon E \to B$ are fibrations in $\mathscr{K}$, then a 1-cell $r\colon (F,q) \to (E,p)$ in $\mathscr{K}/B$ admits the structure of a (pseudo) morphism of pseudo T_B-algebras if and only if the mate of the identity 2-cell

$$\begin{array}{ccc} (F,q) & \xrightarrow{\;r\;} & (E,p) \\ {\scriptstyle d_q}\downarrow & & \downarrow{\scriptstyle d_p} \\ (B/q, v_q) & \xrightarrow[T_B r]{} & (B/p, v_p) \end{array}$$

is invertible; and this in turn is the case if and only if r is a morphism of fibrations.

Likewise, the *strict* T_B-algebras are the *split* fibrations in $\mathscr{K}$: those $f\colon A \to B$ for which each $\mathscr{K}(C,f)\colon \mathscr{K}(C,A) \to \mathscr{K}(C,B)$ is a split fibration, and each $\mathscr{K}(c,A)\colon \mathscr{K}(C,A) \to \mathscr{K}(D,A)$ preserves the *chosen* cartesian lifts.

In particular, $v\colon B/f \to B$ is a split fibration for any $f\colon A \to B$, and d exhibits $v\colon B/f \to B$ as the free (split) fibration on f. Thus if $p\colon E \to B$ is a fibration, and $g\colon A \to E$ defines a morphism $(A,f) \to (E,p)$ in $\mathscr{K}/B$, there is an essentially unique morphism of fibrations $r\colon (B/f, v) \to (E,p)$ extending g.

4.2. Proposition. *The 2-functor* $\mathrm{Enr}_{\mathbf{1}}\colon \mathbf{BICAT} \to 2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}$ *factors through the locally full sub-2-category of* $2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}$ *having*

- *the fibrations in* 2-**CAT** *to* $\mathbf{Set}_{lc}$ *as objects, and*
- *the fibration morphisms as 1-cells.*

Proof. First we describe fibrations in 2-**CAT** explicitly. Given a 2-functor $F\colon \mathscr{Y} \to \mathscr{X}$ between 2-categories, a 1-cell $h\colon y' \to y$ in $\mathscr{Y}$ is called *cartesian* (with respect to F) if

$$\begin{array}{ccc} \mathscr{Y}(z,y') & \xrightarrow{\;\mathscr{Y}(z,h)\;} & \mathscr{Y}(z,y) \\ {\scriptstyle F_{z,y'}}\downarrow & & \downarrow{\scriptstyle F_{z,y}} \\ \mathscr{X}(Fz,Fy') & \xrightarrow[\mathscr{X}(Fz,Fh)]{} & \mathscr{X}(Fz,Fy) \end{array}$$

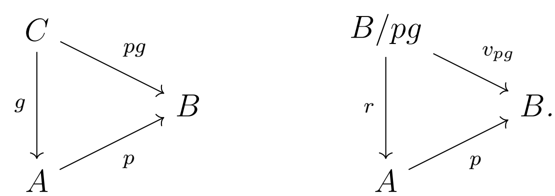
is a pullback in **CAT** for each $z \in \mathscr{Y}$. Then F is a fibration if and only if, for each object $y \in \mathscr{Y}$ and each 1-cell $g\colon x \to Fy$ in $\mathscr{X}$, there is a cartesian morphism $\overline{g}\colon g^*y \to y$ in $\mathscr{Y}$ with $F\overline{g} = g$; such a $\overline{g}$ is called a *cartesian lifting* of g to y. Moreover, given fibrations $F\colon \mathscr{Y} \to \mathscr{X}$ and $G\colon \mathscr{Z} \to \mathscr{X}$ over $\mathscr{X}$, a 2-functor $H\colon \mathscr{Y} \to \mathscr{Z}$ satisfying $F = G \circ H$ is a morphism of fibrations if and only if H preserves cartesian 1-cells. (This is a special case of Proposition 5.3 below, whose proof does not depend on the current proposition.)

For any $\mathscr{B} \in$ **BICAT**, a $\mathscr{B}$-functor $S\colon \mathbb{Y}' \to \mathbb{Y}$ is called *fully faithful* when the 2-cell $S_{y,y'}\colon \mathbb{Y}(y,y') \to \mathbb{Y}'(Sy,Sy')$ in $\mathscr{B}$ is invertible for all $y, y' \in \mathbb{Y}$. It is easy to see that a $\mathscr{B}$-functor is cartesian with respect to $\mathrm{ob}(-)\colon \mathscr{B}\text{-}\mathbf{Cat} \to \mathbf{Set}_{lc}$ if it is fully faithful, and indeed by essential uniqueness of cartesian lifts the reverse implication also holds. The claim follows at once. ∎

4.3. Remark. In the above proposition, we used fibrations in the 2-category 2-**CAT**, called *2-categorical fibrations* in [Gra74, I.2.9]. These were also called *2-fibrations* in [Gra74], but for the purposes of this remark we shall save that name for the more restrictive notion studied by Hermida [Her99]; see also [Bak, Buc14]. In general, $\mathrm{ob}(-)\colon \mathscr{B}\text{-}\mathbf{Cat} \to \mathbf{Set}_{lc}$ is not a 2-fibration in the sense of [Her99]. Indeed, a 2-fibration is a 2-functor which among other things is locally a fibration, but the forgetful functor $\mathscr{B}\text{-}\mathbf{Cat}(\mathbb{X},\mathbb{Y}) \to \mathbf{Set}_{lc}(\mathrm{ob}(\mathbb{X}),\mathrm{ob}(\mathbb{Y}))$ induced by $\mathrm{ob}(-)$ is rarely a fibration of categories.

In general, oplax limits of 1-cells are not preserved by the projection $\mathscr{K}/B \to \mathscr{K}$, but to some extent fibrations can be used to remedy this, as the following result shows.

4.4. Proposition. *Let $p\colon A \to B$ be a fibration in $\mathscr{K}$, and consider a morphism g in $\mathscr{K}/B$ into p, and the (essentially unique) induced morphism r of fibrations, as below*



Then the oplax limit of g in $\mathscr{K}$ is the oplax limit of r in $\mathscr{K}/B$.

Proof. As usual we write A/g for the oplax limit of g in $\mathscr{K}$. We also write $(A,p)/r$ for the oplax limit of r in $\mathscr{K}/B$.

A morphism $D \to A/g$ consists of morphisms $a\colon D \to A$, $c\colon D \to C$, and a 2-cell $\alpha\colon a \to gc$.

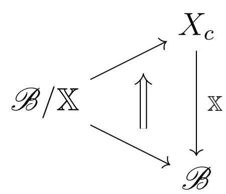
A morphism $D \to B/pg$ consists of morphisms $b\colon D \to B$, $c\colon D \to C$, and a 2-cell $\beta\colon b \to pgc$, and composing with r gives the domain of the cartesian lifting $\overline{\beta}\colon \beta^*gc \to gc$ of β. A morphism $(D,b) \to (A,p)/r$ in $\mathscr{K}/B$ consists of $(b,c,\beta)\colon D \to B/pg$, $a\colon D \to A$, and a 2-cell $\alpha'\colon a \to \beta^*gc$ with $p\alpha'$ equal to the identity on $pa = b$. But by the fibration property of p, to give such an α' is equivalently to give $\alpha\colon a \to gc$ with $p\alpha = \beta$.

This shows that the one-dimensional aspect of the universal properties of A/g and $(A,p)/r$ agree, and similarly the two-dimensional aspects also agree. ∎

We can use this to prove the following key result, already stated in the introduction.

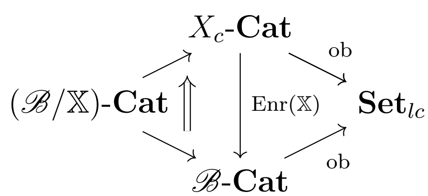
4.5. Theorem. *Let $\mathscr{B}$ be a bicategory and $\mathbb{X}$ a $\mathscr{B}$-category. Then the slice 2-category $\mathscr{B}$-$\mathbf{Cat}/\mathbb{X}$ is isomorphic to $(\mathscr{B}/\mathbb{X})$-$\mathbf{Cat}$ for a bicategory $\mathscr{B}/\mathbb{X}$.*

Proof. If we regard $\mathbb{X}$ as a lax functor $\mathbb{X}\colon X_c \to \mathscr{B}$, where $X = \mathrm{ob}(\mathbb{X})$, we may take its oplax limit



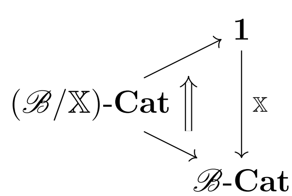
in **BICAT**. Explicitly, $\mathrm{ob}(\mathscr{B}/\mathbb{X}) = \mathrm{ob}(\mathbb{X}) = X$, while the hom $(\mathscr{B}/\mathbb{X})(x, x')$ is given by the slice category $\mathscr{B}(|x|, |x'|)/\mathbb{X}(x, x')$ for all $x, x' \in X$.

It follows by Theorem 2.1 that $(\mathscr{B}/\mathbb{X})$-$\mathbf{Cat}$ is the oplax limit



in 2-$\mathbf{CAT}/\mathbf{Set}_{lc}$.

Now $\mathrm{ob}(-)\colon X_c$-$\mathbf{Cat} \to \mathbf{Set}_{lc}$ is the free fibration on $X\colon \mathbf{1} \to \mathbf{Set}_{lc}$ by Example 4.1, while $\mathrm{Enr}(\mathbb{X})$ is the morphism of fibrations induced by $\mathbb{X}\colon \mathbf{1} \to \mathscr{B}$-$\mathbf{Cat}$ by Proposition 4.2, and so by Proposition 4.4 the diagram



is an oplax limit in 2-$\mathbf{CAT}$. But this says precisely that $(\mathscr{B}/\mathbb{X})$-$\mathbf{Cat} \cong \mathscr{B}$-$\mathbf{Cat}/\mathbb{X}$. ∎

4.6. Example. In particular, when $\mathscr{B}$ is the cartesian monoidal category **Set** regarded as a one-object bicategory, we have for each (**Set**-)category $\mathbb{X}$ the bicategory $\mathbf{Set}/\mathbb{X}$ whose set of objects is $\mathrm{ob}(\mathbb{X})$ and whose hom-category $(\mathbf{Set}/\mathbb{X})(x, x')$ is the slice category $\mathbf{Set}/\mathbb{X}(x, x')$. Each functor $F\colon \mathbb{Y} \to \mathbb{X}$ corresponds to a $\mathbf{Set}/\mathbb{X}$-category $\overline{\mathbb{Y}}$ given as follows: the objects of $\overline{\mathbb{Y}}$ are the same as those of $\mathbb{Y}$, the extent of y in $\overline{\mathbb{Y}}$ is Fy, and the hom $\overline{\mathbb{Y}}(y, y')$ is $F_{y,y'}\colon \mathbb{Y}(y, y') \to \mathbb{X}(Fy, Fy')$. Note that since $\mathbf{Set}/\mathbb{X}(x, x') \simeq \mathbf{Set}^{\mathbb{X}(x,x')}$, $\mathbf{Set}/\mathbb{X}$ is (biequivalent to) the free local cocompletion of $\mathbb{X}$ regarded as a locally discrete bicategory, as pointed out to us by Ross Street.

A variant of $\mathbf{Set}/\mathbb{X}$ is the free quantaloid $\mathcal{P}\mathbb{X}$ over $\mathbb{X}$. Specifically, $\mathcal{P}\mathbb{X}$ is also a bicategory with the same objects as $\mathbb{X}$, but whose hom-category $(\mathcal{P}\mathbb{X})(x, x')$ is the powerset lattice $\mathcal{P}(\mathbb{X}(x, x'))$, which is equivalent to the full subcategory of the slice category $\mathbf{Set}/\mathbb{X}(x, x')$ consisting of the injections to $\mathbb{X}(x, x')$. Accordingly, the $\mathcal{P}\mathbb{X}$-categories correspond to the *faithful* functors $\mathbb{Y} \to \mathbb{X}$ [Gar14, Proposition 3.5].

4.7. Example. Let $\mathscr{B}$ be a bicategory with all right liftings. Then for each $b \in \mathscr{B}$, we have a $\mathscr{B}$-category $\mathbb{B}_b$ whose objects are the 1-cells $f\colon x \to b$ in $\mathscr{B}$ with codomain b, with extent $|(x, f)| = x$, and whose hom $\mathbb{B}_b((x, f), (y, g))\colon x \to y$ is the right lifting of f along g:

$$\begin{array}{ccc} x & \xrightarrow{\mathbb{B}_b((x,f),(y,g))} & y \\ & \Longleftarrow & \\ {\scriptstyle f}\searrow & & \swarrow{\scriptstyle g} \\ & b. & \end{array}$$

(See [GP97, Section 2] for the dual construction.) Given a $\mathscr{B}$-category $\mathbb{X}$, the $\mathscr{B}$-functors $\mathbb{X} \to \mathbb{B}_b$ correspond to the $\mathscr{B}$-presheaves on $\mathbb{X}$ with extent b. Hence if we consider the bicategory $\mathscr{B}/\mathbb{B}_b$, then a $\mathscr{B}/\mathbb{B}_b$-category can be identified with a $\mathscr{B}$-category equipped with a $\mathscr{B}$-presheaf with extent b.

By the universality of right liftings, the bicategory $\mathscr{B}/\mathbb{B}_b$ is canonically isomorphic to the *lax slice* bicategory $\mathscr{B}/\!\!/b$: this has 1-cells with codomain b as objects, and diagrams of the form

$$\begin{array}{ccc} x & \longrightarrow & y \\ & \Longleftarrow & \\ {\scriptstyle f}\searrow & & \swarrow{\scriptstyle g} \\ & b & \end{array}$$

as 1-cells from f to g. Unlike $\mathbb{B}_b$, this lax slice bicategory $\mathscr{B}/\!\!/b$ can be defined even when $\mathscr{B}$ does not have right liftings, and it is true in general that a $\mathscr{B}/\!\!/b$-category corresponds to a $\mathscr{B}$-category equipped with a $\mathscr{B}$-presheaf with extent b. (For a general bicategory $\mathscr{B}$, the notion of $\mathscr{B}$-presheaf can be defined in terms of actions; see [Str83] for a definition of the more general notion of module.)

4.8. Remark. The bicategory $\mathscr{B}/\mathbb{X}$ can be obtained from $\mathbb{X}$ via a change-of-base process for bicategories enriched in a tricategory. Since the theory of tricategory-enriched bicategories, let alone change-of-base for them, has not really been developed in detail, we merely sketch the details. (See [GS16, Section 13] for change-of-base for bicategories enriched over monoidal bicategories.)

We regard $\mathscr{B}$ as a tricategory with no non-identity 3-cells, and we regard the cartesian monoidal 2-category $\mathbf{Cat}$ as a one-object tricategory $\Sigma(\mathbf{Cat})$. There is a lax morphism of tricategories $\Theta\colon \mathscr{B} \to \Sigma(\mathbf{Cat})$ sending each object $b \in \mathscr{B}$ to the unique object of $\Sigma(\mathbf{Cat})$, and sending a 1-cell $f\colon b \to b'$ in $\mathscr{B}$ to the category $\mathscr{B}(b, b')/f$. Composition with Θ

then sends each $\mathscr{B}$-enriched bicategory to a $\Sigma(\mathbf{Cat})$-enriched bicategory. Since $\mathscr{B}$ has no non-identity 3-cells, a $\mathscr{B}$-enriched bicategory is just a $\mathscr{B}$-enriched category; on the other hand, a $\Sigma(\mathbf{Cat})$-enriched bicategory is just a bicategory in the ordinary sense. Applying this to the $\mathscr{B}$-category $\mathbb{X}$ gives the bicategory $\mathscr{B}/\mathbb{X}$.

5. Variation through enrichment

In the paper [BCSW83], the authors showed how fibrations with codomain $\mathbb{X}$ can be seen as certain categories enriched over a bicategory $\mathscr{W}(\mathbb{X})$ depending on the category $\mathbb{X}$. In this section we give a result of the same type, although it differs in several important respects. The bicategory we use is $\mathbf{Set}/\mathbb{X}$ (see Example 4.6), which is like $\mathscr{W}(\mathbb{X})$ in having as objects the objects of $\mathbb{X}$: see Remark 5.1 below for the relationship between the two bicategories. Then we show that fibrations over $\mathbb{X}$ can be identified with $\mathbf{Set}/\mathbb{X}$-categories which have certain powers.

5.1. Remark. Given objects $x, x' \in \mathbb{X}$, a 1-cell in $\mathscr{W}(\mathbb{X})$ from x to x' consists of a presheaf E on $\mathbb{X}$ equipped with maps to $\mathbb{X}(-,x)$ and $\mathbb{X}(-,x')$; in other words, it consists of a *span* of presheaves from $\mathbb{X}(-,x)$ to $\mathbb{X}(-,x')$. Now a 1-cell $S \to \mathbb{X}(x,x')$ in $\mathbf{Set}/\mathbb{X}$ from x to x' determines, via Yoneda, a map $S \cdot \mathbb{X}(-,x) \to \mathbb{X}(-,x')$ of presheaves, where $S \cdot \mathbb{X}(-,x)$ denotes the copower of $\mathbb{X}(-,x)$ by S: the coproduct of S copies of $\mathbb{X}(-,x)$. On the other hand there is the codiagonal $S \cdot \mathbb{X}(-,x) \to \mathbb{X}(-,x)$, and so we obtain a span

$$\mathbb{X}(-,x) \longleftarrow S \cdot \mathbb{X}(-,x) \longrightarrow \mathbb{X}(-,x')$$

of presheaves; that is, a 1-cell in $\mathscr{W}(\mathbb{X})$ from x to x'. This defines the 1-cell part of a homomorphism of bicategories $\mathbf{Set}/\mathbb{X} \to \mathscr{W}(\mathbb{X})$ which is the identity on objects and locally fully faithful. Just as we characterize fibrations over $\mathbb{X}$ as $\mathbf{Set}/\mathbb{X}$-categories with certain limits, so in [BCSW83] these fibrations are seen as $\mathscr{W}(\mathbb{X})$-categories with certain limits; one key difference is that in the case of $\mathscr{W}(\mathbb{X})$ the limits in question are absolute.

In fact we work not just with fibrations of ordinary categories, but rather fibrations in the 2-category $\mathscr{B}$-$\mathbf{Cat}$ of $\mathscr{B}$-enriched categories, as in Section 4. One recovers the case of ordinary categories upon taking $\mathscr{B}$ to be the one-object bicategory $\Sigma(\mathbf{Set})$. We have seen in Theorem 4.5 that, for a $\mathscr{B}$-category $\mathbb{X}$, $\mathscr{B}$-functors with codomain $\mathbb{X}$ correspond to $\mathscr{B}/\mathbb{X}$-enriched categories. We shall see in this section that a $\mathscr{B}$-functor $F\colon \mathbb{Y} \to \mathbb{X}$ is a fibration in $\mathscr{B}$-$\mathbf{Cat}$ if and only if the corresponding $\mathscr{B}/\mathbb{X}$-category $\overline{\mathbb{Y}}$ has certain powers.

First, however, we give an elementary characterization of fibrations in $\mathscr{B}$-$\mathbf{Cat}$. To do this, we start with the fact that every $\mathscr{B}$-category $\mathbb{X}$ has an underlying ordinary category $\mathbb{X}_0$ with the same objects; a morphism $x \to x'$ in $\mathbb{X}_0$ can exist only if x and x' have the same extent ($|x| = |x'|$), in which case it amounts to a 2-cell $1_{|x|} \to \mathbb{X}(x,x')$ in $\mathscr{B}$.[2] We

[2]The assignment $\mathbb{X} \mapsto \mathbb{X}_0$ is the object-part of a 2-functor $\mathscr{B}$-$\mathbf{Cat} \to \mathbf{Cat}$, arising via change-of-base with respect to a lax functor from $\mathscr{B}$ to the cartesian monoidal category $\mathbf{Set}$, seen as a one-object bicategory. The lax functor sends each object b to this unique object; it sends a 1-cell $f\colon b \to c$ to the set $\mathscr{B}(b,c)(1_b, f)$ if $b = c$ and the empty set otherwise; with the evident action on 2-cells.

shall sometimes refer to such morphisms in $\mathbb{X}_0$ simply as morphisms in $\mathbb{X}$. If $f\colon x' \to x''$ is a morphism in $\mathbb{X}$ and x is an object, there is an induced 2-cell $\mathbb{X}(x,f)\colon \mathbb{X}(x,x') \to \mathbb{X}(x,x'')$ defined by pasting $f\colon 1_{|x'|} \to \mathbb{X}(x',x'')$ together with the composition 2-cell $M_{x,x',x''}\colon \mathbb{X}(x',x'').\mathbb{X}(x,x') \to \mathbb{X}(x,x'')$.

5.2. Definition. *Let $F\colon \mathbb{Y} \to \mathbb{X}$ be a $\mathscr{B}$-functor. A morphism $h\colon y' \to y$ in $\mathbb{Y}_0$ is said to be* cartesian *with respect to F if the square*

$$\begin{array}{ccc} \mathbb{Y}(z,y') & \xrightarrow{\mathbb{Y}(z,h)} & \mathbb{Y}(z,y) \\ {\scriptstyle F_{z,y'}}\downarrow & & \downarrow{\scriptstyle F_{z,y}} \\ \mathbb{X}(Fz,Fy') & \xrightarrow[\mathbb{X}(Fz,Fh)]{} & \mathbb{X}(Fz,Fy) \end{array}$$

is a pullback in $\mathscr{B}(|z|,|y|)$ for all objects z in $\mathbb{Y}$.

This implies in particular that h is cartesian with respect to the ordinary functor $F_0\colon \mathbb{Y}_0 \to \mathbb{X}_0$, but in general is stronger than this.

5.3. Proposition. *Suppose that the bicategory $\mathscr{B}$ has pullbacks in each hom-category $\mathscr{B}(a,b)$. A $\mathscr{B}$-functor $F\colon \mathbb{Y} \to \mathbb{X}$ is a fibration in $\mathscr{B}$-**Cat** if and only if, for each object $y \in \mathbb{Y}$ and each morphism $g\colon x \to Fy$ in $\mathbb{X}$ there is a cartesian morphism $\overline{g}\colon g^*y \to y$ in $\mathbb{Y}$ with $F\overline{g} = g$. Given fibrations $F\colon \mathbb{Y} \to \mathbb{X}$ and $G\colon \mathbb{Z} \to \mathbb{X}$, a $\mathscr{B}$-functor $H\colon \mathbb{Y} \to \mathbb{Z}$ with $F = G \circ H$ is a morphism of fibrations if and only if $H\colon \mathbb{Y} \to \mathbb{Z}$ preserves cartesian morphisms.*

Proof. The pullbacks in the hom-categories of $\mathscr{B}$ can be used to construct oplax limits in $\mathscr{B}$-**Cat**, as we shall now show. Let $F\colon \mathbb{Y} \to \mathbb{X}$ be a $\mathscr{B}$-functor; then the oplax limit $\mathbb{L} = \mathbb{X}/F$ has:

- objects given by pairs (g,y), with $y \in \mathbb{Y}$ and $g\colon x \to Fy$ in $\mathbb{X}_0$
- the extent of (g,y) equal to the extent of y (which is also the extent of x)
- homs given by pullbacks as in

$$\begin{array}{ccc} \mathbb{L}((g',y'),(g,y)) & \xrightarrow{U_{(g',y'),(g,y)}} & \mathbb{Y}(y',y) \\ & & \downarrow{\scriptstyle F_{y',y}} \\ {\scriptstyle V_{(g',y'),(g,y)}}\downarrow & & \mathbb{X}(Fy',Fy) \\ & & \downarrow{\scriptstyle \mathbb{X}(g',Fy)} \\ \mathbb{X}(x',x) & \xrightarrow[\mathbb{X}(x',g)]{} & \mathbb{X}(x',Fy) \end{array}$$

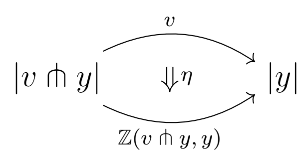
- projections $V\colon \mathbb{L} \to \mathbb{X}$ and $U\colon \mathbb{L} \to \mathbb{Y}$ sending an object (g, y) to x and to y, and defined on homs as in the diagram above.

The diagonal $\mathscr{B}$-functor $D\colon \mathbb{Y} \to \mathbb{L}$ sends an object $z \in \mathbb{Y}$ to $(1_{Fz}, z) \in \mathbb{L}$. Taking $(g', y') = Dz$ in the above diagram gives a pullback

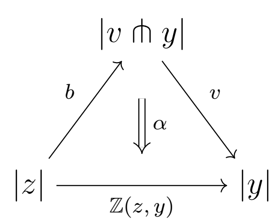
$$\begin{array}{ccc} \mathbb{L}(Dz,(g,y)) & \xrightarrow{U} & \mathbb{Y}(z,y) \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle F_{z,y}} \\ \mathbb{X}(Fz,x) & \xrightarrow[\mathbb{X}(Fz,g)]{} & \mathbb{X}(Fz,Fy). \end{array}$$

Now F is a fibration just when D has a right adjoint in $\mathscr{B}$-**Cat**$/\mathbb{X}$. Such an adjoint is given on objects by a lifting of $g\colon Fx \to y$ to some $\overline{g}\colon g^*y \to y$, and the universal property says that this lifting is cartesian. ∎

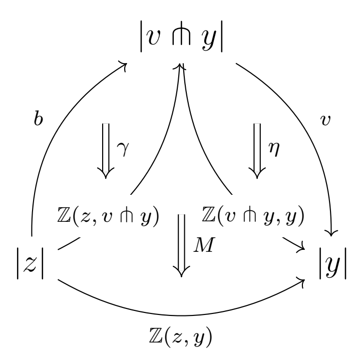
We now turn to the characterization of fibrations of $\mathscr{B}$-categories in terms of $\mathscr{B}/\mathbb{X}$-categories. First recall that if $\mathscr{W}$ is a bicategory and $\mathbb{Z}$ is a $\mathscr{W}$-category then powers in $\mathbb{Z}$ involve an object y of $\mathbb{Z}$ and a 1-cell $v\colon x \to |y|$ in $\mathscr{W}$ with codomain the extent of y. The power of y by v consists of an object $v \pitchfork y$ of $\mathbb{Z}$ with extent $|v \pitchfork y| = x$, together with a 2-cell



such that for all $z \in \mathbb{Z}$ and all

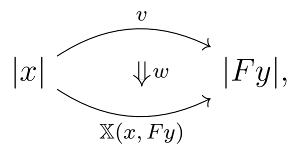


there exists a unique γ making the pasting composite



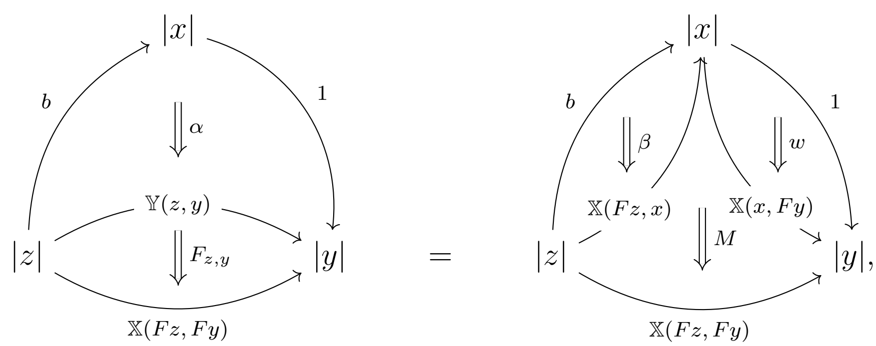
equal to α. (In other words, the pasting of η and M exhibits $\mathbb{Z}(z, v \pitchfork y)$ as the right lifting of $\mathbb{Z}(z,y)$ along v.)

We consider this in the case where $\mathscr{W} = \mathscr{B}/\mathbb{X}$ and $\mathbb{Z}$ is the $\mathscr{B}/\mathbb{X}$-category $\overline{\mathbb{Y}}$ corresponding to a $\mathscr{B}$-functor $F\colon \mathbb{Y} \to \mathbb{X}$. Then an object y of $\overline{\mathbb{Y}}$ is just an object of $\mathbb{Y}$, and the extent of y, as an object of $\mathscr{B}/\mathbb{X}$, is the object Fy of $\mathbb{X}$. A general 1-cell $x \to Fy$ in $\mathscr{B}/\mathbb{X}$ has the form

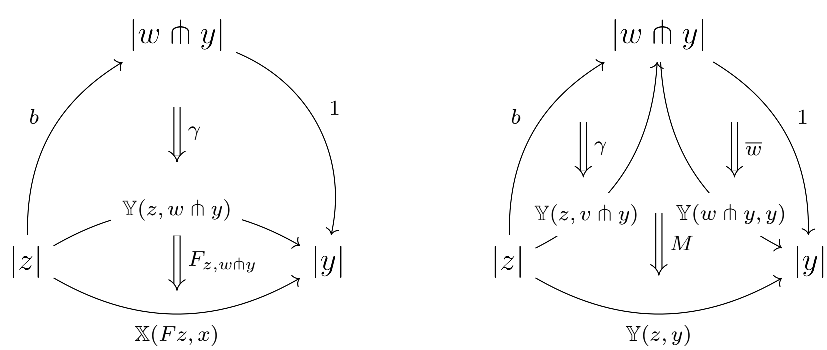


but we shall only consider the special case where $|x| = |Fy|$ and $v = 1_{|x|}$, so that in fact we are dealing with a morphism $w\colon x \to Fy$ in $\mathbb{X}_0$. In general, we call a 1-cell $(w\colon v \to \mathbb{X}(x,x'))\colon x \to x'$ in $\mathscr{B}/\mathbb{X}$ a *singleton* 1-cell if $|x| = |x'|$ and $v = 1_{|x|}$. Note that the category $\mathbb{X}_0$ can be regarded as a sub-bicategory of $\mathscr{B}/\mathbb{X}$ whose 1-cells are the singleton 1-cells. When $\mathscr{B} = \mathbf{Set}$, a 1-cell $x \to x'$ in $\mathbf{Set}/\mathbb{X}$ corresponds to a set v equipped with a function $w\colon v \to \mathbb{X}(x,x')$; in this case, the singleton 1-cells in $\mathbf{Set}/\mathbb{X}$ can be identified with those 1-cells with v a singleton, whence the name singleton.

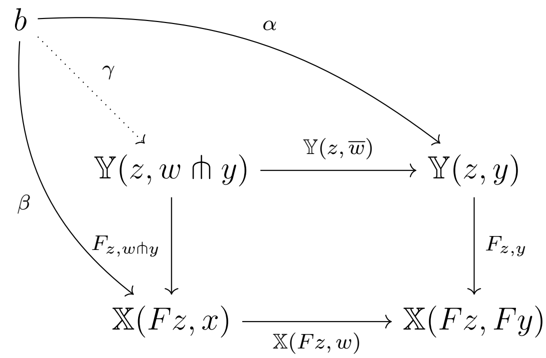
A power of y by $w\colon 1 \to \mathbb{X}(x, Fy)$ then consists of an object $w \pitchfork y$ of $\mathbb{Y}$ with $F(w \pitchfork y) = x$ together with a morphism $\overline{w}\colon w \pitchfork y \to y$ in $\mathbb{Y}_0$ with $F\overline{w} = w$ — that is, a lifting $\overline{w}$ of w — subject to the universal property stating that for all $z \in \mathbb{Y}$, $b\colon |z| \to |y|$, α, and β making



there exists a unique γ making the pasting composites



equal respectively to β and α. But this says exactly that if the exterior of the diagram



in $\mathscr{B}(|z|,|y|)$ commutes, then there is a unique γ making the triangular regions commute; in other words, that the internal square is a pullback. This in turn says that $\overline{w}$ is a cartesian lifting of w. This now proves:

5.4. Proposition. *Let $\mathscr{B}$ be a bicategory in which each hom-category has pullbacks. A $\mathscr{B}$-functor $F\colon \mathbb{Y} \to \mathbb{X}$ is a fibration if and only if the corresponding $\mathscr{B}/\mathbb{X}$-category $\overline{\mathbb{Y}}$ has powers by morphisms in $\mathbb{X}_0$; that is, powers by singleton 1-cells.*

We conclude by strengthening this correspondence to an isomorphism between suitable 2-categories. Let $\mathscr{W}$ be a bicategory and $H\colon \mathbb{Z} \to \mathbb{Z}'$ a $\mathscr{W}$-functor. Suppose that the power $v \pitchfork y$ of $y \in \mathbb{Z}$ by $v\colon x \to |y|$ exists in $\mathbb{Z}$, with the associated 2-cell $\eta\colon v \to \mathbb{Z}(v \pitchfork y, y)$. Then H is said to *preserve* the power $v \pitchfork y$ if the 2-cell $H_{v\pitchfork y,y} \circ \eta\colon v \to \mathbb{Z}'(H(v \pitchfork y), Hy)$ exhibits $H(v \pitchfork y)$ as the power $v \pitchfork Hy$ in $\mathbb{Z}'$.

5.5. Theorem. *Let $\mathscr{B}$ be a bicategory in which each hom-category has pullbacks. The canonical isomorphism of 2-categories $(\mathscr{B}/\mathbb{X})$-**Cat** $\cong \mathscr{B}$-**Cat**$/\mathbb{X}$ restricts to an isomorphism between the locally full sub-2-category of $(\mathscr{B}/\mathbb{X})$-**Cat** having*

- *the $\mathscr{B}/\mathbb{X}$-categories with powers by singleton 1-cells as objects, and*
- *the $\mathscr{B}/\mathbb{X}$-functors preserving these powers as 1-cells,*

*and the locally full sub-2-category of $\mathscr{B}$-**Cat**$/\mathbb{X}$ having*

- *the fibrations to $\mathbb{X}$ as objects, and*
- *the fibration morphisms as 1-cells.*

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