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A MODEL FOR THE HIGHER CATEGORY OF HIGHER CATEGORIES

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Abstract. We use fibrations of complete Segal spaces as introduced in [Ras22, Ras23a] to construct four complete Segal spaces: *Reedy fibrant simplicial spaces*, *Segal spaces*, *complete Segal spaces*, and *spaces*. Moreover, we show each one comes with a universal fibration that classifies *Reedy left fibrations*, *Segal coCartesian fibrations*, *coCartesian fibrations* and *left fibrations* and prove these are representable fibrations in the sense of [Ras22]. Finally, we use equivalences between quasi-categories and complete Segal spaces constructed in [JT07, Ras21a] to present analogous constructions using fibrations of quasi-categories.

As part of establishing the results, we also develop a theory of *minimal Reedy fibrations* for elegant Reedy categories, which can be of independent interest.

1. Introduction

1.1. The (Higher) Category of (Higher) Categories. *Category theory* has been very effective in the study of a very diverse range of mathematical objects and their relation to each other. We can deduce various formal properties about different mathematical objects (such as the existence of free objects or preservation of universal properties) by using formal categorical results. A key illustration of these powerful methods is the study of sets via the category $\mathcal{S}\mathrm{et}$, which can be realized as the free cocompletion of the category with one object and so interesting properties, such as the fact that algebraic structures in sets are preserved by limits, follows formally [ML98, Rie16].

This powerful perspective has been turned on category theory itself via the study of the (large) category of categories with objects small categories and morphisms functors, and we can similarly now deduce many valuable properties of categories, such as the construction of free categories, by analyzing properties of the category of categories.[1]

While categories are a powerful tool in the study of classical mathematics, they are less suitable for objects that arise in homotopical mathematics. This starts with homotopy types of topological spaces or *Kan complexes* (which can be thought of as homotopical analogues of sets), but also A_∞-*group*s [Sta63] (which, up to homotopy, have a group

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[1] We can in fact deduce all properties of categories by studying the 2-category of categories, which is known as *formal category theory* [Gra74].

structure) and even further *derived schemes* [Toë14]. In order to effectively study such homotopical objects, various notions of weak or homotopical categories have been developed, now known as *higher categories* or $(\infty,1)$-*categories* or simply ∞-*categories* [Ber10]. The most popular model is the model of *quasi-categories* [BV73], and other important models are *Kan enriched categories* [DK80, Ber07a] and *complete Segal spaces* [Rez01], which are related to each other via various equivalences [JT07, Ber07b, Lur09]. These various models of ∞-categories give us the appropriate framework to study concepts such as homotopical algebra and derived geometry.

The analogy with classical category theory would suggest that similar to the category of sets, there is an easily constructed higher category of spaces that can be studied using higher categorical methods. While there is such a higher category, the construction is by no means immediate. The situation gets worse when trying to construct the higher category of higher categories. It is in fact a cruel joke of higher categorical mathematics that the construction of the higher category of higher categories requires us to change models, making the construction quite complicated.

1.2. Strict Categories and Nerves. The easiest way to construct an $(\infty,1)$-category is via Kan enriched categories and so our first way to approach this problem is by constructing a Kan enriched category. Constructing the Kan enriched category of Kan complexes is fairly straightforward and has been known at least since work of Quillen [Qui67]. We can use a similar line of thinking to construct an $(\infty,1)$-category of $(\infty,1)$-categories. Indeed, following work of Rezk, the category of complete Segal spaces is in fact enriched over Kan complexes [Rez01]. On the other hand, the category of quasi-categories is not directly enriched over Kan complexes but rather over quasi-categories themselves, however, we can easily construct a Kan enriched category by taking the underlying Kan complexes of the mapping quasi-categories [RV22]. On the other side, as suggested by the slogan above, while we can construct the category of Kan enriched categories, there is no (known) way to enrich this category over Kan complexes, making it challenging to define the higher category of higher categories with objects Kan enriched categories.

While the construction of the Kan enriched category of spaces and $(\infty,1)$-categories might initially appear to be a satisfactory answer, we are quickly confronted with various challenges. First of all, most of $(\infty,1)$-category theory has not been developed in the context of Kan enriched categories[2]. Historically speaking, most higher categorical concepts have been developed using quasi-categories [Joy08a, Joy08b, Lur09]. Moreover, in an effort to move beyond one specific model Riehl and Verity developed a new method to approach higher category theory model-independently via the notion of an ∞-*cosmos*, which can then, in particular, be applied to quasi-categories, but also complete Segal spaces, Segal categories and even 1-*complicial sets*, but notably not Kan enriched categories [RV22].

This motivates the construction of the higher category of spaces and $(\infty,1)$-categories using more established models, such as quasi-categories and complete Segal spaces. One

[2]In fact even the construction of a simple over-category can be a challenge in this setting, see [Hor19, Appendix A].

first approach might be to simply translate the construction from Kan enriched categories to these models using various *nerve* constructions, which take Kan enriched categories to (bi)simplicial sets. In particular, we can use the *homotopy coherent nerve*, first introduced by Cordier–Porter [CP86] and studied further by Joyal [Joy07] and Lurie [Lur09], to construct quasi-categories out of Kan enriched categories. Similarly we can use the *Rezk nerve*[3] or its variant due to Barwick and Kan to construct complete Segal spaces out of Kan enriched categories [Rez01, Ber09, BK12, Mei16]. These constructions are theoretically very satisfying, however, are computationally very challenging. Indeed, the construction due to Rezk requires a fibrant replacement in the Reedy model structure [Rez01, Section 8], which, while preserving the level-wise homotopy type of the complete Segal space, completely changes the point-set structure. The simplicial nerve [Lur09, Proposition 1.1.5.10] and the nerve by Barwick-Kan [BK12, Theorem 6.1] do not require such additional steps, however, are by definition far more complicated constructions.[4]

1.3. Higher Categories via Fibrations. What makes the nerve constructions so complicated is the fact that maps of spaces (and functors of Kan enriched categories) are by definition strictly functorial, and so we either need to use a very complicated nerve construction (such as the Rezk nerve or simplicial nerve) or use a naive construction and then apply a Reedy fibrant replacement, both with the goal of "destrictifying" the functors in order to allow for the possibility of higher categorical pseudo-functors, whose functoriality only holds up to higher equivalences. Ideally, we could have directly constructed a Kan-enriched category of spaces or ∞-categories where the morphisms directly correspond to some notion of pseudo-functors. However, directly defining pseudo-functors would require specifying an infinite tower of data and so we need to find a way to circumvent this dilemma. We need to choose a notion of functor of spaces (and ∞-categories) that is by definition weaker, yet still manageable. Fortunately, there is already an excellent solution in the category theory literature: *fibrations*.

The idea of using fibrations as a replacement for functors goes back to work of Grothendieck and Bourbaki, who used the (now called) *Grothendieck fibrations* to study set-valued and category-valued functors [Gro03]. In particular, Grothendieck opfibrations over a category $\mathcal{C}$ correspond to a pseudo-functor $\mathcal{C} \to \mathcal{C}\mathrm{at}$. Following the definition of a pseudo-functor [Bén67], a pseudo-functor $[0] \to \mathcal{C}\mathrm{at}$, is the data of a category $\mathcal{C}$ and an automorphism that is naturally isomorphic to the identity. Similarly, a pseudo-functor $[1] \to \mathcal{C}\mathrm{at}$ is the data of a functor $\mathcal{C} \to \mathcal{D}$, and choices of automorphisms of $\mathcal{C}$ and $\mathcal{D}$, that all interact with each other in the appropriate manner. These examples already illustrate that by taking Grothendieck opfibrations over the categories $[n]$, we obtain a much less rigid object than we would if we use the classical nerve, which is defined as $N\mathcal{C}\mathrm{at}_n = \mathrm{Fun}([n], \mathcal{C}\mathrm{at})$ i.e. strict functors out of $[n]$.

This philosophy expands to the ∞-categorical setting. We hence want to construct a quasi-category and complete Segal space of spaces and ∞-categories by choosing an

[3] Called *classifying diagram* in [Rez01].

[4] Indeed, there are several papers dedicated to understanding the left adjoint $\mathfrak{C}$ to the simplicial nerve [DS11, Rie11].

appropriate notion of fibration over the representable diagrams (i.e. the appropriate analogue to $[n]$). This requires us to use the vast literature on fibrations of ∞-categories. Concretely, for many theories of ∞-categories (such as quasi-categories, complete Segal spaces and in fact every other ∞-cosmos) ∞-categorical functors from an ∞-category $\mathcal{C}$ into spaces correspond to *left fibrations*[5] [BdB18, HM15, Ras23b, Cis19]. Similarly, functors from $\mathcal{C}$ valued in ∞-categories are classified by *coCartesian fibrations* over $\mathcal{C}$ [Ras23a, Ras21a, AF20, Lur09, RV22].

1.4. Constructing Complete Segal Spaces via Fibrations. The goal of this paper is to make the intuition outlined in the previous section into precise mathematical statements. Let $\Delta[n,l]$ (Notation 2.3) denote the representable presheaves in the category of bisimplicial sets $\mathrm{Fun}(\mathbb{\Delta}^{op} \times \mathbb{\Delta}^{op}, \mathcal{S}\mathrm{et})$. Then we can construct the bisimplicial set $\mathfrak{S}$ (7) such that $\mathfrak{S}_{n,l}$ is given by the set of left fibrations over $\Delta[n,l]$. Our first major claim is that $\mathfrak{S}$ is an ∞-category of spaces. To this end, let N_Δ denote the *simplicial nerve*, which takes a simplicially enriched category to a strict Segal category (10) and $\underline{\mathcal{K}\mathrm{an}}$ the Kan-enriched category of Kan complexes. We now have the following major result about $\mathfrak{S}$:

1.5. Theorem. [Theorem 3.9] *There is a complete Segal space equivalence* $\mathbb{I} : N_\Delta\underline{\mathcal{K}\mathrm{an}} \to \mathfrak{S}$ *to the complete Segal space* $\mathfrak{S}$*. Moreover, we have a natural bijection* $\mathcal{LF}\mathrm{ib}(-) \cong \mathrm{Hom}(-,\mathfrak{S})$.

The construction of $\mathfrak{S}$ corresponds to a similar result by Kazhdan and Varshavsky [KV14] (Remark 3.3) and generalizes a result by Kapulkin and Lumsdaine [KL21] (Remark 3.11). The bijection between left fibrations over a bisimplicial set X and maps from X into $\mathfrak{S}$ is a manifestation of the *straightening construction*, which at the higher categorical level originated in [Lur09].

Having used left fibrations of simplicial spaces to construct the ∞-category of spaces, we next generalize our result in order to construct the ∞-category of ∞-categories. We in fact obtain a far more general result. Using the observation that every complete Segal space is a Reedy fibrant simplicial space, we first use the theory of *Reedy left fibrations* (Definition 4.3) to construct the ∞-category of Reedy fibrant simplicial spaces.

Concretely, we can construct a bisimplicial set $s\mathfrak{S}$ (16) with $s\mathfrak{S}_{n,l}$ being Reedy left fibrations over $\mathbb{\Delta}[n,l]$. We now claim that $s\mathfrak{S}$ gives us an ∞-category of simplicial spaces. To this end, we have the following result, where $\underline{\mathcal{R}\mathrm{ee}}$ (18) denotes the Kan enriched category of Reedy fibrant simplicial spaces:

1.6. Theorem. [Theorem 4.13] *There is a complete Segal space equivalence* $s\mathbb{I} : N_\Delta\underline{\mathcal{R}\mathrm{ee}} \to s\mathfrak{S}$ *to the complete Segal space* $s\mathfrak{S}$*. Moreover, we have a bijection* $\mathcal{R}\mathrm{ee}\mathcal{LF}\mathrm{ib}(-) \cong \mathrm{Hom}(-, s\mathfrak{S})$.

The construction of the ∞-category of simplicial spaces via fibrations is a new development. Similarly, the bijection between Reedy left fibrations and morphisms into $s\mathfrak{S}$

[5] Also called *discrete coCartesian fibration* over $\mathcal{C}$ in [RV22].

gives us a generalization of the straightening construction to simplicial spaces. While the construction of the complete Segal space of simplicial spaces could have relevance in the study of certain type theories that arise in the work of Riehl and Shulman [RS17] (as further discussed in Remark 4.14) our main focus here is to restrict to the sub-complete Segal space of (complete) Segal spaces. Let $\mathfrak{S}\mathrm{eg}$ be the sub-bisimplicial set of $s\mathfrak{S}$ with $\mathfrak{S}\mathrm{eg}_{n,l}$ consisting of Segal coCartesian fibrations over $\Delta[n,l]$ and similarly $\mathfrak{CSS}$ be the sub-bisimplicial set of $\mathfrak{S}\mathrm{eg}$ with $\mathfrak{CSS}_{n,l}$ consisting of coCartesian fibrations over $\Delta[n,l]$. On the other side, let $\underline{\mathcal{S}\mathrm{eg}}$ ($\underline{\mathcal{CSS}}$) denote the full sub-categories of $\underline{\mathcal{R}\mathrm{ee}}$ with objects (complete) Segal spaces.

1.7. Theorem. [Theorem 4.16] *In the following diagram the top (bottom) horizontal functors are fully faithful functors of strict Segal categories (complete Segal spaces) and the vertical maps are complete Segal space equivalences*

$$\begin{tikzcd} N_\Delta\underline{\mathcal{CSS}} \arrow[r, hook] \arrow[d, "s\mathbb{I}"', "\simeq"] & N_\Delta\underline{\mathcal{S}\mathrm{eg}} \arrow[r, hook] \arrow[d, "s\mathbb{I}"', "\simeq"] & N_\Delta\underline{\mathcal{R}\mathrm{ee}} \arrow[d, "s\mathbb{I}"', "\simeq"] \\ \mathfrak{CSS} \arrow[r, hook] & \mathfrak{S}\mathrm{eg} \arrow[r, hook] & s\mathfrak{S} \end{tikzcd}$$

meaning $\mathfrak{S}\mathrm{eg}$ *is the complete Segal space of Segal spaces and* $\mathfrak{CSS}$ *is the complete Segal space of complete Segal spaces. Moreover, we have bijections*

$$\mathcal{S}\mathrm{egco}\mathcal{C}\mathrm{art}(-) \cong \mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, \mathfrak{S}\mathrm{eg}),$$

$$\mathrm{co}\mathcal{C}\mathrm{art}(-) \cong \mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, \mathfrak{CSS}).$$

The existence of the desired complete Segal spaces of (simplicial) spaces and (complete) Segal spaces with the universal property outlined above implies the existence of universal fibrations, which is the focus of Section 5. We, in particular, establish that the *universal left fibration* is represented by the terminal object (Theorem 5.3) and that the *universal Reedy left fibration*, *Segal coCartesian fibration* and *coCartesian fibration* are represented, in the sense of [Ras22], by the cosimplicial object $\mathbb{\Delta} \to \mathfrak{C}\mathrm{at}_\infty$ taking $[n]$ to n composable morphisms (Theorem 5.8/Corollary 5.9), which has also been discussed in [Ras22, Ste20] (Remark 5.10).

While most of our work focuses on constructing complete Segal spaces, in the last section we use equivalences constructed by Joyal and Tierney [JT07] and its fibrational analogue [Ras23b, Ras21a] to construct various quasi-categories, beginning with the quasi-category of spaces. Let $\mathfrak{S}_{\mathfrak{QC}\mathrm{at}}$ denote the quasi-categorical version of $\mathfrak{S}$, meaning $(\mathfrak{S}_{\mathfrak{QC}\mathrm{at}})_n$ given by left fibrations of simplicial sets over $\Delta[n]$ (which coincides with [Cis19, Definition 5.2.3]). On the other side, let $i_1^*\mathfrak{S}$ denote the underlying quasi-category of $\mathfrak{S}$ [JT07]. Then we have the following:

1.8. Theorem. [Theorem 6.5] *The maps* $\mathfrak{T} : \mathfrak{S}_{\mathfrak{QC}\mathrm{at}} \to i_1^*\mathfrak{S}$ *(22) and* $\mathfrak{I} : i_1^*\mathfrak{S} \to \mathfrak{S}_{\mathfrak{QC}\mathrm{at}}$ *(23) are inverses of quasi-categories.*

This hence recovers the result by Cisinski [Cis19], who proved directly that $\mathfrak{S}_{\mathcal{QC}at}$ is a quasi-category of spaces. As a next step, we can further generalize the result to quasi-category of simplicial spaces (Remark 6.6). Let $s\mathfrak{S}_{\mathcal{QC}at}$ denote the quasi-categorical version of $s\mathfrak{S}$, meaning the simplicial1 set with n-simplices given by Reedy left fibrations of bisimplicial sets over $\Delta[0,n]$. On the other side, let $i_1^* s\mathfrak{S}$ denote the underlying quasi-category of $s\mathfrak{S}$. Then we have the following:

1.9. Theorem. [Theorem 6.10] *The maps* $\mathrm{s}\mathfrak{T} : s\mathfrak{S}_{\mathcal{QC}at} \to i_1^* s\mathfrak{S}$ *and* $\mathrm{s}\mathfrak{I} : i_1^* s\mathfrak{S} \to s\mathfrak{S}_{\mathcal{QC}at}$ *(24) are inverses of quasi-categories.*

We can easily restrict this equivalence to a quasi-category of (complete) Segal spaces to obtain a diagram of equivalences, presented in Corollary 6.11.

1.10. Technical underpinning: Minimal Reedy fibrations. The construction of the complete Segal spaces of interest ($\mathfrak{S}$ and $s\mathfrak{S}$) and their equivalences with the corresponding Segal categories rely on a robust theory of minimal fibrations, both for bisimplicial and trisimplicial sets, which appropriately generalizes the results established for minimal Kan fibrations. Hence, Section A focuses on defining and developing a broad theory of minimal Reedy fibrations, resulting in the following major result:

1.11. Theorem. [Theorem A.11] *Let $\mathcal{R}$ be an elegant Reedy category. Then every Reedy fibration $p : Y \to X$ admits a factorization $p : Y \overset{\simeq}{\twoheadrightarrow} \mathcal{M}\mathrm{in}(Y) \overset{min}{\twoheadrightarrow} X$ into a trivial fibration followed by a minimal fibration, which is unique up to isomorphism.*

This very general result will, in particular, apply to bisimplicial and trisimplicial sets and enable us to prove the results reviewed in Subsection 1.4.

1.12. Background and Notation. We will assume familiarity with standard category theory as can be found in [ML98, Rie16]. Also some familiarity with complete Segal spaces [Rez01] and quasi-categories [Rez17] would be helpful. Moreover, we make extensive use of left and coCartesian fibrations as studied in [Ras23b, Ras22, Ras23a, Ras21a], however, key results have been reviewed when necessary.

Throughout, $\mathbb{\Delta}$ denotes the simplex category. We both use the functor category and the set of functors and hence denote the functor category by $\underline{\mathrm{Fun}}$, whereas the set of functors is Fun. Moreover, for a given functor $F : \mathcal{C} \to \mathcal{D}$, the *set* of objects in the slice category $\underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})_{/F}$ is also denoted by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})_{/F}$.

Finally, we will have a category of small sets, denoted $\mathcal{S}\mathrm{et}$, and for every category $\mathcal{C}$ and functor $X : \mathcal{C} \to \mathcal{S}\mathrm{et}$, we use $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})_{/X}$ to denote the set of objects of the category $\underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{S}\mathrm{et})_{/X}$.

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2. Background & Technicalities

As explained in Subsection 1.3, we want to construct a complete Segal space of spaces $\mathfrak{S}$ (which we will do in Section 3) which is level-wise given by a set of left fibrations over representable objects. However, there are two significant theoretical challenges that we need to overcome. First of all, if we naively define $\mathfrak{S}_{nl}$ as the set of left fibrations over $\Delta[n,l]$ (Notation 2.3), then the functoriality needs to follow from pulling back left fibrations. However, a pullback is only determined up to isomorphisms, hence a functor $\mathbb{\Delta}^{op} \times \mathbb{\Delta}^{op} \to \mathcal{S}\mathrm{et}$ that takes each pair $([n],[l])$ to the set of left fibrations over $\Delta[n,l]$ and each morphisms to the pullback would only be *pseudo-functorial*. In order to avoid this problem, we associate functors to our fibrations, that can then be strictly composed, which is the goal of Subsection 2.11.

Having taking care of the pseudo-functoriality, we can in fact directly define a bisimplicial set $\mathfrak{S}$ with $\mathfrak{S}_{nl}$ given by left fibrations over $\Delta[n,l]$ and we would like to prove that this is in fact a complete Segal space. Here the next problem arises. In order to prove that $\mathfrak{S}$ is a complete Segal space we need to show that for every trivial cofibration in the complete Segal space model structure $i : A \hookrightarrow B$ the map

$$i^* : \mathrm{Hom}(B, \mathfrak{S}) \to \mathrm{Hom}(A, \mathfrak{S})$$

is surjective (Remark 2.6). As we will establish in 7 this is equivalent to $i^* : \mathcal{LF}\mathrm{ib}(B) \to \mathcal{LF}\mathrm{ib}(A)$ being surjective, which means we need to prove that every left fibration over A can be obtained as a pullback of a left fibration over B. By 3 it is immediate that every left fibration is a *homotopy* pullback of a left fibration over B, however we need a strict pullback. In order to guarantee we can obtain this strict pullback, we need to review the theory of minimal fibrations, which is the goal of Subsection 2.23.

2.1. Remark. There are alternative ways to the ones introduced in Subsection 2.11 to avoid the pseudo-functoriality of the pullback of fibrations. For example, we can choose well-orderings on all the fibers of the fibrations as has been done in [KL21, Subsection 2.1] or one can make a choice of a collection of pullbacks as has been done in [Cis19, Subsection 2.1]. On the other hand, all these sources also rely on minimal fibrations (similar to Subsection 2.23) to guarantee the existence of strict lifts.

2.2. Simplicial Objects and Fibrations. In this short section we give a quick review of the necessary simplicial objects, relevant notation and their fibrations.

2.3. Notation. Denote the category of simplicial sets by $s\mathcal{S}\mathrm{et}$ with generators $\Delta[-]$. Moreover, use $\partial\Delta[-]$ to denote its boundary and $\Lambda[-]_i$ its i-th horn [GJ09, Subsection I.3]. Similarly, denote the category of bisimplicial sets by $ss\mathcal{S}\mathrm{et}$ and the generators by

$\Delta[-,-]$. Notice both categories are simplicially enriched and we denote the enrichment by $\mathrm{Map}(-,-)$ [GJ09, Subsection I.V].

For a given bisimplicial set $X_{\bullet\bullet}$ we use the notation X_n to denote the simplicial set $(X_n)_l = X_{nl}$. Our first important fibration of bisimplicial sets are *Reedy fibrations*.

2.4. Definition. *A Reedy fibration is a map of bisimplicial set $p : Y \to X$ that has the right lifting property with respect to the maps*

$$\Delta[n,0] \times \Lambda[0,l]_i \coprod_{\partial\Delta[n,0]\times\Lambda[0,l]_i} \partial\Delta[n,0] \times \Delta[0,l] \hookrightarrow \Delta[n,0] \times \Delta[0,l] \cong \Delta[n,l],$$

for all $n, l \geq 0$ and $0 \leq i \leq l$. A Reedy fibration is moreover trivial if it satisfies the right lifting property with respect to all inclusions of bisimplicial sets.

The Reedy fibrancy condition is equivalent to the map $Y_n \to X_n \times_{M_n X} M_n Y$ being a Kan fibration of simplicial sets, where $M_n Y, M_n X$ are the *matching spaces*, which in the particular case of bisimplicial sets are given by the simplicial set $M_n X = \mathrm{Map}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(\partial\Delta[n,0], X)$. Reedy fibrations are part of a model structure with cofibrations given by inclusions of bisimplicial sets and equivalences given by level-wise Kan equivalences. In particular, all trivial Reedy fibrations are Reedy weak equivalences. See Section A, [Rez01, Subsection 2.4] or [Hov99, Subsection 5.1] for more details. Reedy fibrations can be used to define a prominent model of $(\infty,1)$-categories, complete Segal spaces, which are defined by Rezk [Rez01] and proven to be models of $(\infty,1)$-categories in [Ber07b, JT07, Toë05].

2.5. Definition. *A complete Segal space W is a Reedy fibrant simplicial space that satisfies the following two conditions:*

- **Segal Condition:** *The restriction map*
$$W_n \twoheadrightarrow W_1 \times_{W_0} \ldots \times_{W_0} W_1$$
is a Kan equivalence for all $n \geq 2$.
- **Completeness Conditions:** *The map $W_0 \to (W_0 \times W_0) \times_{(W_1 \times W_1)} W_3$ is a Kan equivalence.*

Complete Segal spaces can be used to do $(\infty,1)$-category theory. Here we think of the objects as the set W_{00} and for two objects x, y in W, the *mapping space* is defined as

$$\mathrm{Map}_W(x,y) = \Delta[0] \times_{(W_0 \times W_0)} W_1. \tag{1}$$

For more details regarding the category theory of complete Segal space see [Rez01, Section 5]. Complete Segal spaces are in fact fibrant objects in a model structure, the *complete Segal space* model structure on $\mathrm{ss}\mathcal{S}\mathrm{et}$, defined originally by Rezk [Rez01, Theorem 7.2].

2.6. Remark. This, in particular, implies that a bisimplicial set W is a complete Segal space if and only if for every trivial cofibration $i : A \to B$ in the complete Segal space model structure, the map $i^* : \mathrm{Hom}(B, W) \to \mathrm{Hom}(A, W)$ is surjective, meaning every map $f : A \to W$ lifts along i.

Next, we have left fibrations.

2.7. Definition. *A left fibration is a Reedy fibration of bisimplicial sets $p : L \to X$ that for all $n, l \geq 0$ satisfies the right lifting property with respect to maps*

$$\Delta[n,0] \times \partial\Delta[0,l] \coprod_{\Delta[0,0]\times\partial\Delta[0,l]} \Delta[0,0] \times \Delta[0,l] \to \Delta[n,0] \times \Delta[0,l] \cong \Delta[n,l],$$

induced by the map $\{0\} : \Delta[0,0] \to \Delta[n,0]$ i.e. the morphism that corresponds to $0 \in \mathrm{Hom}(\Delta[0,0], \Delta[n,0]) = \Delta[n,0]_{00} = \{0, ..., n\}$. By [Ras23b, Proposition 3.7] this is equivalent to being a Reedy fibration and for all n, the map

$$L_n \twoheadrightarrow X_n \times_{X_0} L_0 \tag{2}$$

induced by the inclusion $\{0\} : \Delta[0,0] \to \Delta[n,0]$, being trivial Kan fibrations.

2.8. Remark. If $X = \Delta[0,0]$ then the condition 2 implies L is *homotopically constant*, meaning $L_n \simeq L_0$ for all $n \geq 0$, and so L is (homotopically) uniquely determined by the Kan complex L_0.

While it is generally difficult to construct left fibrations out of general maps of bisimplicial sets, under the right circumstances, we can significantly simplify such computations.

2.9. Lemma. *Let X be a bisimplicial set, such that X_0 is discrete (meaning $X_{0k} \cong X_{00}$), and $p : L \to X$ be a map of bisimplicial sets such that the map $L_n \xrightarrow{\simeq} X_n \times_{X_0} L_0$ is an equivalence. Moreover, let $\hat{p} : \hat{L} \to X$ be a Reedy fibrant replacement of $p : L \to X$ over X via a map $i : L \xrightarrow{\simeq} \hat{L}$. Then $\hat{p} : \hat{L} \to X$ is a left fibration.*

Proof. By assumption, $\hat{p} : \hat{L} \to X$ is a Reedy fibration and hence it suffices to establish the equivalence $\hat{L}_n \xrightarrow{\simeq} X_n \times_{X_0} \hat{L}_0$. For a given element x in X_{00}, let us denote the fiber of $L_0 \to X_0$ over x by $\mathcal{F}\mathrm{ib}_x L_0$ and define $\mathcal{F}\mathrm{ib}_x \hat{L}_0$ analogously. As X_0 is discrete, for every element $x \in X_{00}$, the map $\{x\} : \Delta[0] \to X_0$ is a Kan fibration. So, by pullback stability, the pullback $\mathcal{F}\mathrm{ib}_x \hat{L}_0 \to \hat{L}_0$ is a Kan fibration as well. We now have the following pullback diagram

$$\begin{tikzcd}
\mathcal{F}\mathrm{ib}_x L_0 \arrow[r, two heads] \arrow[d, "\simeq"] \arrow[dr, phantom, "\ulcorner", very near start] & L_0 \arrow[d, "\simeq"] \\
\mathcal{F}\mathrm{ib}_x \hat{L}_0 \arrow[r, two heads] & \hat{L}_0
\end{tikzcd}$$

and so it follow from the right properness of the Kan model structure that the induced map $\mathcal{F}\mathrm{ib}_x L_0 \to \mathcal{F}\mathrm{ib}_x \hat{L}_0$ is also a Kan equivalence. We now have the following commutative diagram

$$\begin{tikzcd} L_n \arrow[r, "\simeq"] \arrow[d, "i_n"', "\simeq"] & L_0 \times_{X_0} X_n \arrow[r, "\cong"] \arrow[d, "(i_0, \mathrm{id}_{X_n})"] & \coprod_{x \in X_{00}} \mathcal{F}\mathrm{ib}_x X_n \times \mathcal{F}\mathrm{ib}_x L_0 \arrow[d, "\simeq"] \\ \hat{L}_n \arrow[r] & \hat{L}_0 \times_{X_0} X_n \arrow[r, "\cong"] & \coprod_{x \in X_{00}} \mathcal{F}\mathrm{ib}_x X_n \times \mathcal{F}\mathrm{ib}_x \hat{L}_0 \end{tikzcd}$$

The top left map is a Kan equivalence by assumption, and the map i_n on the left is a Kan equivalence by definition of Reedy equivalences. Finally, the right hand map is an equivalence by the argument above and the fact that Kan equivalences are stable under products and coproducts. Hence, by 2-out-of-3, the bottom left map is also a Kan equivalence and we are done. ■

Left fibrations and complete Segal space equivalences interact well with each other. Concretely, for every complete Segal space equivalence $i : A \to B$ and left fibration $p : L \to A$, there exists a left fibration $\hat{L} \to B$ obtained as the factorization of $pi : L \to B$ into a trivial cofibration followed by a left fibration $L \overset{\simeq}{\hookrightarrow} \hat{L} \twoheadrightarrow B$ and a homotopy pullback square

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner", very near start] & \hat{L} \arrow[d, two heads] \\ A \arrow[r, "i"] & B \end{tikzcd} \tag{3}$$

formally given as the derived unit of the Quillen equivalence constructed in [Ras23b, Theorem 5.1], meaning L is Reedy equivalent to $i^*\hat{L}$.

2.10. Remark. Analogous to the definition of left fibrations (2) we can also define right fibrations as Reedy fibrations $R \to X$ such that $R_n \to X_n \times_{X_0} R_0$ is an equivalence, this time induced by the map $\{n\} : \Delta[0,0] \to \Delta[n,0]$ [Ras23b, Remark 4.25]. Right fibrations are completely determined by left fibrations. Indeed, let $(-)^{op} : \mathrm{ss}\mathcal{S}\mathrm{et} \to \mathrm{ss}\mathcal{S}\mathrm{et}$ be the automorphism induced by the unique non-trivial automorphism $\sigma \times \sigma$ from $\mathbb{A} \times \mathbb{A}$ to itself. Then a map $R \to X$ is a right fibration if and only if $R^{op} \to X^{op}$ is a left fibration.

2.11. Fibrations vs. Functors. In this subsection we construct a precise way to translate between functors and fibrations to avoid the pseudo-functoriality that arises when using pullback (as discussed in the beginning of Section 2). We will start by reviewing basic facts regarding functors and fibrations as discussed in [MLM94] or [Joh02a, Joh02b].

Recall that a *discrete Grothendieck fibration* is a functor $p : \mathcal{D} \to \mathcal{C}$ such that for every morphism $f : c \to c'$ and chosen lift d' in $\mathcal{D}$ of c' (meaning $p(d') = c'$) there exists a *unique* $\hat{f} : d \to d'$ in $\mathcal{D}$ such that $p(\hat{f}) = f$. For a given functor $F : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$, recall that the *Grothendieck construction* $\int_{\mathcal{C}} F \to \mathcal{C}$ is given as a category over $\mathcal{C}$ with objects pairs $(c, x \in F(c))$ and morphisms $(c, x \in F(c)) \to (d, y \in F(d))$ given by morphisms $f : c \to d$

such that $F(f)(y) = x$. It was shown by Grothendieck [Gro03][6] that this induces a fully faithful functor $\int_{\mathcal{C}} : \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et}) \to \mathcal{C}\mathrm{at}_{/\mathcal{C}}$ with essential image precisely given by discrete Grothendieck fibrations.[7]

Following the convention from Subsection 1.12, we use $\underline{\mathcal{G}\mathrm{roth}}(\mathcal{C})$ to denote the full subcategory of $\mathcal{C}\mathrm{at}_{/\mathcal{C}}$ with objects discrete Grothendieck fibrations of the form $\int_{\mathcal{C}} F$ and $\mathcal{G}\mathrm{roth}(\mathcal{C})$ for the large set of discrete Grothendieck fibrations of the form $\int_{\mathcal{C}} F$, for a given functor $F : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$. From the previous paragraph it follows that $\underline{\mathcal{G}\mathrm{roth}}(\mathcal{C})$ is equivalent to the functor category $\underline{\mathrm{Fun}}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et})$, however, we need an isomorphism and hence state the desired result explicitly.

2.12. Lemma. *Let $\mathcal{C}$ be a small category. There is an isomorphism of categories*

$$\underline{\mathrm{Fun}}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et}) \xrightarrow[\cong]{\int_{\mathcal{C}}} \underline{\mathcal{G}\mathrm{roth}}(\mathcal{C})$$

Proof. We show that the equivalence $\int_{\mathcal{C}}$ is in fact an isomorphism. Fully faithfulness implies that that $\int_{\mathcal{C}}$ is bijective on morphisms, so it suffices to prove that $\int_{\mathcal{C}}$ is bijective on objects. First, $\int_{\mathcal{C}}$ is surjective on objects by definition of the category $\underline{\mathcal{G}\mathrm{roth}}(\mathcal{C})$. Second, for given functors $F, G : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$, $\int_{\mathcal{C}} F = \int_{\mathcal{C}} G$ implies that for all object c in $\mathcal{C}$, we have an equality of fibers

$$\{(c, x) | x \in F(c)\} = \{(c, y) | y \in G(c)\}.$$

This gives us the desired equality $F(c) = G(c)$ for all objects $c \in C$. ∎

This lemma has two important corollaries. Let $\mathcal{G}\mathrm{roth} = \coprod_{\mathcal{C} \in \mathcal{C}\mathrm{at}} \mathcal{G}\mathrm{roth}(\mathcal{C})$ be the large set of all discrete Grothendieck fibrations of the form $\int_{\mathcal{C}} F$. Moreover, recall the notation convention from Subsection 1.12 regarding $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et})_{/F}$.

2.13. Corollary. *Let $F : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$ be a functor. Then the isomorphism in Lemma 2.12 induces a bijection of large sets*

$$\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et})_{/F} \cong \mathcal{G}\mathrm{roth}(\mathcal{C})_{/\int_{\mathcal{C}} F}.$$

2.14. Corollary. *The isomorphism in Lemma 2.12 induces a bijection of sets*

$$\begin{tikzcd} \coprod_{\mathcal{C} \in \mathcal{C}\mathrm{at}} \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et}) \arrow[rr, "\int", "\cong"'] \arrow[dr] & & \mathcal{G}\mathrm{roth} \arrow[dl] \\ & \mathcal{C}\mathrm{at} & \end{tikzcd}.$$

We can now move on to construct a functor with value fibrations, that avoids the pseudo-functoriality of the pullback. Let $\mathcal{A}ll : \mathrm{ss}\mathcal{S}\mathrm{et}^{op} \to \mathcal{S}\mathrm{et}$ be defined as the composition

$$\mathrm{Fun}(\mathbb{\Delta}^{op} \times \mathbb{\Delta}^{op}, \mathcal{S}\mathrm{et})^{op} \xrightarrow{(\int_{\mathbb{\Delta} \times \mathbb{\Delta}})^{op}} \mathcal{G}\mathrm{roth}(\mathbb{\Delta} \times \mathbb{\Delta})^{op} \xrightarrow{\pi^{op}} \mathcal{C}\mathrm{at}^{op} \xrightarrow{\mathrm{Fun}(-, \mathcal{S}\mathrm{et})} \mathcal{S}\mathrm{et}, \qquad (4)$$

[6]See [LR20, Theorem 2.1.2] for a modern treatment of this result.

[7]In fact we have far more general results for Grothendieck fibrations [Joh02a, Theorem B1.3.6].

where $\pi : \mathcal{G}\mathrm{roth} \to \mathcal{C}\mathrm{at}$ is the projection that sends an object $\mathcal{D} \to \mathcal{C}$ to its source $\mathcal{D}$. More concretely, this means $\mathcal{A}\mathrm{ll}(X) = \mathrm{Fun}((\int_{\mathbb{\Delta}\times\mathbb{\Delta}} X)^{op}, \mathcal{S}\mathrm{et})$. Now, we have the following key lemma with regard to $\mathcal{A}\mathrm{ll}(-)$.

2.15. Lemma. *For every bisimplicial set X there is a bijection of sets*

$$\Gamma_X : \mathcal{A}\mathrm{ll}(X) \xrightarrow{\cong} \mathrm{ss}\mathcal{S}\mathrm{et}_{/X}.$$

Proof. First of all, by restricting the bijection in Corollary 2.14 to the fiber over $\int_{\mathbb{\Delta}\times\mathbb{\Delta}} X$, we have the bijection

$$\mathcal{A}\mathrm{ll}(X) \cong \mathcal{G}\mathrm{roth}(\int_{\mathbb{\Delta}\times\mathbb{\Delta}} X).$$

Now, there is an evident discrete Grothendieck fibration $\int_{\mathbb{\Delta}\times\mathbb{\Delta}} X \to \mathbb{\Delta} \times \mathbb{\Delta}$ and so every discrete Grothendieck fibration $\int_{\mathbb{\Delta}\times\mathbb{\Delta}} X$ is simply a discrete Grothendieck fibration into $\mathbb{\Delta} \times \mathbb{\Delta}$ that factors through $\int_{\mathbb{\Delta}\times\mathbb{\Delta}} X$, meaning we have the bijection

$$\mathcal{G}\mathrm{roth}(\int_{\mathbb{\Delta}\times\mathbb{\Delta}} X) \cong \mathcal{G}\mathrm{roth}(\mathbb{\Delta} \times \mathbb{\Delta})_{/\int_{\mathbb{\Delta}\times\mathbb{\Delta}} X}.$$

Finally, by Corollary 2.13, we have the bijection

$$\mathcal{G}\mathrm{roth}(\mathbb{\Delta} \times \mathbb{\Delta})_{/\int_{\mathbb{\Delta}\times\mathbb{\Delta}} X} \cong \mathrm{Fun}(\mathbb{\Delta}^{op} \times \mathbb{\Delta}^{op}, \mathcal{S}\mathrm{et})_{/X} = \mathrm{ss}\mathcal{S}\mathrm{et}_{/X}.$$

Combining these three bijections gives us the desired bijection $\mathcal{A}\mathrm{ll}(X) \cong \mathrm{ss}\mathcal{S}\mathrm{et}_{/X}$. ∎

This bijection is in fact quite well-behaved as we can easily witness by tracing through the definition.

2.16. Lemma. *Let $f : X \to Y$ be a morphism of bisimplicial sets. Then the following square commutes.*

$$\begin{tikzcd} \mathcal{A}\mathrm{ll}(Y) \arrow[r, "\Gamma_Y", "\cong"'] \arrow[d, "\mathcal{A}\mathrm{ll}(f)"'] & \mathrm{ss}\mathcal{S}\mathrm{et}_{/Y} \arrow[d, "f^*"] \\ \mathcal{A}\mathrm{ll}(X) \arrow[r, "\Gamma_X", "\cong"'] & \mathrm{ss}\mathcal{S}\mathrm{et}_{/X} \end{tikzcd}.$$

As a result of this lemma we can think of elements in $\mathcal{A}\mathrm{ll}(X)$ as simplicial spaces over X. Now, we want to prove that $\mathcal{A}\mathrm{ll}(-)$ is representable. Define $\mathrm{ss}\mathfrak{S}\mathrm{et}$ as the bisimplicial set obtained by precomposing $\mathcal{A}\mathrm{ll}(-)$ with the Yoneda embedding $\mathbb{\Delta}^{op} \times \mathbb{\Delta}^{op} \to \mathrm{ss}\mathcal{S}\mathrm{et}^{op}$. Concretely, as we can immediately compute $\int_{\mathbb{\Delta}\times\mathbb{\Delta}} \mathrm{Hom}_{\mathbb{\Delta}\times\mathbb{\Delta}}(-, ([n], [l])) = \mathbb{\Delta}_{/[n]} \times \mathbb{\Delta}_{/[l]}$, we have $\mathrm{ss}\mathfrak{S}\mathrm{et}(n, l) = \mathrm{Fun}((\mathbb{\Delta}_{/[n]})^{op} \times (\mathbb{\Delta}_{/[l]})^{op}, \mathcal{S}\mathrm{et})$. We now have the following key result.

2.17. Corollary. *There is a natural isomorphism $\mathcal{A}\mathrm{ll}(-) \cong \mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, \mathrm{ss}\mathfrak{S}\mathrm{et})$.*

Proof. The functor $\mathrm{Hom}(-, \mathrm{ss}\mathfrak{S}\mathrm{et})$ preserves colimits by definition and $\mathcal{A}\mathrm{ll}(-)$ preserves colimits as the three functors in 4 defining $\mathcal{A}\mathrm{ll}(-)$ preserve colimits. Hence, it suffices to observe a natural isomorphism at the level of representables. However, we have

$$\mathcal{A}\mathrm{ll}(\Delta[n,l]) = \mathrm{Fun}(\mathbb{\Delta}_{/[n]} \times \mathbb{\Delta}_{/[l]}, \mathcal{S}\mathrm{et}) \cong \mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(\Delta[n,l], \mathrm{ss}\mathfrak{S}\mathrm{et})$$

where the last step follows from the Yoneda lemma, giving us the desired natural isomorphism. ∎

The construction of $\mathcal{A}\mathrm{ll}(-)$ and $\mathrm{ss}\mathfrak{S}\mathrm{et}$ is too broad and we often want to restrict it appropriately. We have the following simple observation.

2.18. Lemma. *Let S be a (possibly large) set of morphisms in* $\mathrm{ss}\mathcal{S}\mathrm{et}$*. The following are equivalent.*

1. *The pullback of a morphism in S (along any morphisms) is in S.*
2. *Let $F : (\int_{\mathbb{\Delta}\times\mathbb{\Delta}} X)^{op} \to \mathcal{S}\mathrm{et}$ be a functor such that $\Gamma_X(F)$ is in S, then for any morphism of bisimplicial sets $f : Y \to X$, $\Gamma_Y(\mathcal{A}\mathrm{ll}(f)(F))$ is in S.*

Proof. Condition (1) corresponds to $f^* : \mathrm{ss}\mathcal{S}\mathrm{et}_{/Y} \to \mathrm{ss}\mathcal{S}\mathrm{et}_{/X}$ restricting to the full subset of morphisms in S, whereas condition (2) corresponds to $\mathcal{A}\mathrm{ll}(f) : \mathcal{A}\mathrm{ll}(Y) \to \mathcal{A}\mathrm{ll}(X)$ restricting similarly. By Lemma 2.16 these two conditions are equivalent as the horizontal morphisms Γ_X, Γ_Y are bijections. ∎

We say S is *pullback stable* if it satisfies the equivalent conditions in Lemma 2.18. For a given pullback stable set of morphisms S, let $\mathcal{A}\mathrm{ll}^S(-)$ be the sub-functor of $\mathcal{A}\mathrm{ll}(-)$ with $F \in \mathcal{A}\mathrm{ll}^S(X)$ if and only if $\Gamma_X(F)$ is in S. The functoriality immediately follows from Lemma 2.18. We can similarly define $\mathrm{ss}\mathfrak{S}\mathrm{et}^S$ as the sub-bisimplicial set of $\mathrm{ss}\mathfrak{S}\mathrm{et}$. We now want to deduce a result analogous to Corollary 2.17. For that we need the following additional condition.

2.19. Lemma. *Let S be a pullback stable set of morphisms in* $\mathrm{ss}\mathcal{S}\mathrm{et}$*. The following are equivalent.*

1. *A morphism $Y \to X$ is in S if and only if for all $\Delta[n,l] \to X$, the pullback $Y \times_X \Delta[n,l] \to \Delta[n,l]$ is in S.*
2. *For a functor $F : (\int_{\mathbb{\Delta}\times\mathbb{\Delta}} X)^{op} \to \mathcal{S}\mathrm{et}$ we have $\Gamma_X(F)$ in S if and only if for every functor $G : \mathbb{\Delta}_{/[n]} \times \mathbb{\Delta}_{/[l]} \to \int_{\mathbb{\Delta}\times\mathbb{\Delta}} X$ we have $\Gamma_{\mathbb{\Delta}[n,l]}(F \circ G^{op})$ is in S.*

Proof. First of all, the fully faithfulness of $\int_{\mathbb{\Delta}\times\mathbb{\Delta}}$ gives us a bijection $\mathrm{Hom}(\Delta[n,l], X) \cong \mathrm{Fun}(\mathbb{\Delta}_{/[n]} \times \mathbb{\Delta}_{/[l]}, \int_{\mathbb{\Delta}\times\mathbb{\Delta}} X)$, using the fact that $\int_{\mathbb{\Delta}\times\mathbb{\Delta}} \Delta[n,l] = \mathbb{\Delta}_{/[n]} \times \mathbb{\Delta}_{/[l]}$. This implies that every functor $G : \mathbb{\Delta}_{/[n]} \times \mathbb{\Delta}_{/[l]} \to \int_{\mathbb{\Delta}\times\mathbb{\Delta}} X$ is necessarily of the form $\int_{\mathbb{\Delta}\times\mathbb{\Delta}} f$ for some $f : \Delta[n,l] \to X$.

Now, for a given morphism $f : \Delta[n,l] \to X$, Lemma 2.16 gives us the following diagram

$$\begin{tikzcd} \mathcal{A}\mathrm{ll}(X) \arrow[r, "\Gamma_X", "\cong"'] \arrow[d, "\mathcal{A}\mathrm{ll}(f)"'] & \mathrm{ss}\mathcal{S}\mathrm{et}_{/X} \arrow[d, "f^*"] \\ \mathcal{A}\mathrm{ll}(\Delta[n,l]) \arrow[r, "\Gamma_{\Delta[n,l]}", "\cong"'] & \mathrm{ss}\mathcal{S}\mathrm{et}_{/\Delta[n,l]} \end{tikzcd} .$$

The assumptions are now direct translations along the bijections $\Gamma_{\Delta[n,l]}$ and Γ_X. ∎

A pullback stable set of morphisms that satisfies the equivalent conditions of Lemma 2.19 is called *local*.

2.20. Lemma. *If S is local, then the natural bijection given in Corollary 2.17 restricts to a natural bijection $\mathcal{A}\mathrm{ll}^S(-) \cong \mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, \mathrm{ss}\mathfrak{S}\mathrm{et}^S)$.*

Proof. Let X be a bisimplicial set. As S is local, the bijection

$$\mathcal{A}\mathrm{ll}(X) \cong \lim_{\Delta[n,l]\to X} \mathcal{A}\mathrm{ll}(\Delta[n,l])$$

restricts to a bijection

$$\mathcal{A}\mathrm{ll}^S(X) \cong \lim_{\Delta[n,l]\to X} \mathcal{A}\mathrm{ll}^S(\Delta[n,l]).$$

On the other side, $\mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, \mathrm{ss}\mathfrak{S}\mathrm{et}^S)$ takes colimits to limits, by representability, and so it suffices to establish the result in the particular case of $\Delta[n,l]$, where it follows from the same argument used in Corollary 2.17. ∎

It is useful to have a quick criterion to determine local classes of morphisms with a proof analogous to [Ras23b, Lemma 3.10].

2.21. Corollary. *Let S be a set of morphism of bisimplicial sets determined by a right lifting property with respect to a set of morphisms $A \to \Delta[n,l]$. Then S is pullback stable and in fact local.*

We end this subsection with an elegant example of the previous corollary.

2.22. Example. By Definition 2.4, the large set of Reedy fibrations satisfies the condition of Corollary 2.21. We denote $\mathcal{A}\mathrm{ll}^{\mathcal{R}\mathrm{ee}}(-)$ by $\mathcal{R}\mathrm{ee}(-)$ and $\mathrm{ss}\mathfrak{S}\mathrm{et}^{\mathcal{R}\mathrm{ee}}$ by $\mathfrak{R}\mathrm{ee}$ and notice that by Lemma 2.20 we have a natural bijection

$$\mathcal{R}\mathrm{ee}(-) \cong \mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, \mathfrak{R}\mathrm{ee}).$$

2.23. Minimal Fibrations. In this subsection we introduce minimal Reedy and left fibrations, which play a key role in the construction of strict pullbacks (as discussed in the beginning of Section 2). Recall that a Kan fibration $p : Y \to X$ is a minimal fibration if for any two maps $f, g : \Delta[n] \to Y$, such that f is homotopic to g relative to $\partial\Delta[n]$ and $pf = pg$, then $f = g$. For more details see [GJ09, Diagram I.10.1]. We can now generalize this definition directly.

2.24. Definition. *A Reedy fibration of simplicial spaces $Y \to X$ is minimal if the Kan fibration $Y_n \to X_n \times_{M_n X} M_n Y$ is a minimal Kan fibration for all $n \geq 0$.*

Our aim is to show that a variety of properties about minimal Kan fibrations, as proven in [GJ09, Section I.10] generalize to minimal Reedy fibrations. However, as the proofs are quite technical, the detailed statements and proofs have been relegated to Section A and here we use the implications thereof. First we establish the desired locality property (Lemma 2.19).

2.25. Lemma. *Minimal Reedy fibrations are local.*

Proof. Follows from applying Lemma A.5 to the case of $\mathbb{\Delta}$, by Definition A.2. ∎

One key result regarding minimal Kan fibrations is that every Kan fibration $p : Y \to X$ can be factored into trivial fibration followed by a minimal fibration $Y \to \mathcal{M}\text{in}(Y) \to X$ [GJ09, Proposition 10.3], [Qui68], which generalizes appropriately.

2.26. Proposition. *Every Reedy fibration of simplicial spaces $p : Y \to X$ admits a (up to isomorphism) unique factorization $Y \overset{q}{\twoheadrightarrow} \mathcal{M}\text{in}(Y) \overset{\mathcal{M}\text{in}(p)}{\twoheadrightarrow} X$ into a trivial fibration q followed by a minimal fibration $\mathcal{M}\text{in}(p)$.*

Proof. Follows directly from applying the general theorem for minimal Reedy fibrations Theorem A.11 to the case of $\mathbb{\Delta}$, which indeed applies by Definition A.2 and Example A.9. ∎

Our construction works well for an individual Reedy fibration, however, we would like to have a construction of the minimal Reedy fibration that is consistent with pullback. This requires us to make a globally consistent choice of minimal fibrations for all Reedy fibrations at once, which we achieve in the following way.

Let $\mathcal{R}\text{ee}^{min}(-) \subseteq \mathcal{R}\text{ee}(-)$ be the sub-functor of minimal Reedy fibration and denote the corresponding subobject of the bisimplicial set of $\mathfrak{Ree}$ by $\mathfrak{Ree}^{min}$. By Lemma 2.25 minimal Reedy fibrations are local and so we have a natural bijection

$$\mathcal{R}\text{ee}^{min}(-) \cong \text{Hom}_{ss\mathcal{S}et}(-, \mathfrak{Ree}^{min}) \tag{5}$$

and, in particular, we have a bijection of sets $\mathcal{R}\text{ee}^{min}(\mathfrak{Ree}) \cong \text{Hom}_{ss\mathcal{S}et}(\mathfrak{Ree}, \mathfrak{Ree}^{min})$. Now, applying Proposition 2.26 to the Reedy fibration over $\mathfrak{Ree}$ that corresponds to the identity map in Example 2.22 we obtain a minimal Reedy fibration over $\mathfrak{Ree}$ that by 5 corresponds to a map of bisimplicial sets

$$\mathfrak{Min} : \mathfrak{Ree} \to \mathfrak{Ree}^{min}. \tag{6}$$

Let $\mathcal{M}\text{in} : \mathcal{R}\text{ee}(-) \to \mathcal{R}\text{ee}^{min}(-)$ be the map represented by $\mathfrak{Min}$. This map sends every Reedy fibration to its corresponding unique minimal Reedy fibration constructed in Proposition 2.26. Indeed, by naturality of 5, for every Reedy fibration $Y \to X$, we have the following diagram, where $\mathfrak{Ree}_* \to \mathfrak{Ree}$ is the Reedy fibration corresponding to the identity morphism in Example 2.22

$$\begin{tikzcd} Y \arrow[r] \arrow[d, two heads, "\simeq"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathfrak{Ree}_* \arrow[d, two heads, "\simeq"] \\ \mathcal{M}\mathrm{in}(Y) \arrow[r] \arrow[d, two heads, "min"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{M}\mathrm{in}(\mathfrak{Ree}_*) \arrow[d, two heads, "min"] \\ X \arrow[r] & \mathfrak{Ree} \end{tikzcd},$$

where we are using the fact that both minimal Reedy fibrations and trivial fibrations are stable under pullback and uniqueness of the factorization.

2.27. Remark. The fact that the trivial fibration $\mathfrak{Ree}_* \to \mathcal{M}\mathrm{in}(\mathfrak{Ree}_*)$ comes with a section $\mathcal{M}\mathrm{in}(\mathfrak{Ree}_*) \overset{\simeq}{\hookrightarrow} \mathfrak{Ree}_* \overset{\simeq}{\twoheadrightarrow} \mathcal{M}\mathrm{in}(\mathfrak{Ree}_*)$ implies that for every minimal fibration $Y \to X$ we have $\mathcal{M}\mathrm{in}(Y) = Y$.

How does the minimality construction interact with left fibrations?

2.28. Lemma. *Let $L \to X$ be a left fibration, then $\mathcal{M}\mathrm{in}(L) \to X$ is also a left fibration.*

Proof. We already know that $\mathcal{M}\mathrm{in}(L) \to X$ is a Reedy fibration, hence, by Definition 2.7 it suffices to observe that $\mathcal{M}\mathrm{in}(L)_n \to \mathcal{M}\mathrm{in}(L)_0 \times_{X_0} X_n$ is an equivalence, which follows directly from the fact that L is a left fibration and $L_n \simeq \mathcal{M}\mathrm{in}(L)_n$ for all $n \geq 0$ (Proposition 2.26). ∎

Finally, the key concept that makes minimal Kan fibrations so useful is that every equivalence between two minimal Kan fibrations is in fact an isomorphism [GJ09, Lemma 10.4] and we have the following analogous result.

2.29. Lemma. *Let $p : Y \to X, q : Z \to X$ be two minimal Reedy fibrations and $f : Y \to Z$ a map over X. Then f is a Reedy equivalence if and only if it is an isomorphism.*

Proof. Follows from applying Proposition A.6 to the case $\mathbb{\Delta}$, as the definitions of minimal fibrations coincide by Definition A.2. ∎

The lemma has the following valuable corollary.

2.30. Corollary. *Let $Y \to X, Z \to X$ be two Reedy fibrations over X and let $f : Y \to Z$ be a morphism over X. Then f is an equivalence if and only if $\mathcal{M}\mathrm{in}(Y) \cong \mathcal{M}\mathrm{in}(Z)$.*

Proof. Let us assume f is an equivalence, then the morphism $\mathcal{M}\mathrm{in}(Y) \hookrightarrow Y \to Z \to \mathcal{M}\mathrm{in}(Z)$, where the first map is the section to $Y \to \mathcal{M}\mathrm{in}(Y)$, is an equivalence over X and so the result follows from Lemma 2.29. On the other side, if $\mathcal{M}\mathrm{in}(Y) \cong \mathcal{M}\mathrm{in}(Z)$, then we have the diagram

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, two heads, "\simeq"'] & & Z \arrow[dl, two heads, "\simeq"] \\ & \mathcal{M}\mathrm{in}(Y) & \end{tikzcd}$$

and so the result follows from 2-out-of-3. ∎

3. The Complete Segal Space of Spaces

In this section we finally make the intuition outlined in Subsection 1.3 precise and construct the desired complete Segal space of spaces using left fibrations. Let $\mathcal{LF}\mathrm{ib}$ be the large set of left fibrations. By Definition 2.7 and Corollary 2.21, $\mathcal{LF}\mathrm{ib}$ is local (Lemma 2.19) and so we can take the sub-functor of $\mathcal{A}\mathrm{ll} : \mathrm{ss}\mathcal{S}\mathrm{et}^{op} \to \mathcal{S}\mathrm{et}$ with value left fibrations, that we denote by $\mathcal{LF}\mathrm{ib}(-) : \mathrm{ss}\mathcal{S}\mathrm{et}^{op} \to \mathcal{S}\mathrm{et}$, which, by Lemma 2.20, is represented by a bisimplicial set that we denote by $\mathfrak{S}$, meaning we have a natural bijection

$$\mathcal{LF}\mathrm{ib}(-) \cong \mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, \mathfrak{S}). \tag{7}$$

We now want to prove that $\mathfrak{S}$ is a complete Segal space of spaces. Before we can get to the main result we need the appropriate lemma that helps us understand extension properties of trivial fibrations. For that we can directly generalize [Cis19, Lemma 5.1.20] to bisimplicial sets.

3.1. Lemma. *Let $p : Y \to A$ be a trivial Reedy fibration of bisimplicial sets and $i : A \to B$ an inclusion of bisimplicial sets, inducing an adjunction $i^* \dashv i_*$. Then i_*p is a trivial Reedy fibration and $p \xrightarrow{\cong} i^*i_*p$.*

3.2. Proposition. *The bisimplicial set $\mathfrak{S}$ is a complete Segal space.*

Proof. The argument is analogous to [KL21, Theorem 2.2.1] and [Cis19, Theorem 5.2.10]. By Remark 2.6, we need to prove that for every trivial cofibration $i : A \to B$ in the complete Segal space model structure, the induced map $\mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(B, \mathfrak{S}) \to \mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(A, \mathfrak{S})$ is surjective. By 7, this is equivalent to $\mathcal{LF}\mathrm{ib}(B) \to \mathcal{LF}\mathrm{ib}(A)$ being surjective, which concretely means proving that every left fibration over A is the pullback of a left fibration over B via i.

Fix a left fibration $p : L \to A$. We now have the following diagram

$$\begin{tikzcd}
L \arrow[rr, hook] \arrow[d, two heads, "r"', "\simeq"] \arrow[dd, bend right=60, two heads, "p"'] \arrow[drr, phantom, "\ulcorner" very near start] & & (\hat{r}j)_*L \arrow[d, two heads, "\simeq"', "(\hat{r}j)_*r"] \\
\mathcal{M}\mathrm{in}(L) \arrow[r, hook, "j", "\simeq"'] \arrow[d, two heads, "\mathcal{M}\mathrm{in}(p)"] \arrow[dr, phantom, "\ulcorner" very near start] & \hat{L} \arrow[r, two heads, "\hat{r}", "\simeq"'] \arrow[d, two heads, "\hat{p}"] & \mathcal{M}\mathrm{in}(\hat{L}) \arrow[dl, two heads, "\mathcal{M}\mathrm{in}(\hat{p})"] \\
A \arrow[r, hook, "i"] & B &
\end{tikzcd},$$

which satisfies the following conditions:

- $\mathcal{M}\mathrm{in}(p)$ is a minimal left fibration (Lemma 2.28),
- r a trivial fibration (Proposition 2.26),
- $\hat{p}$ is a left fibration (3),

- j a trivial complete Segal space equivalence (3),
- $\hat{r}$ a trivial fibration (Proposition 2.26),
- $\mathcal{M}\mathrm{in}(\hat{p})$ a minimal left fibration (Lemma 2.28),
- $(\hat{r}j)_*r$ is a trivial fibration (Lemma 3.1).

Now, by the properties of the homotopy pullback square (3), the map $\mathcal{M}\mathrm{in}(L) \to i^*\mathcal{M}\mathrm{in}(\hat{L})$ induced by the pullback is a Reedy equivalence and hence, by Lemma 2.29, a bijection and, by Lemma 3.1, the top rectangle is a pullback. Hence p is the pullback of the left fibration $\mathcal{M}\mathrm{in}(\hat{p}) \circ (\hat{r}j)_*r : (\hat{r}j)_*L \to B$ and we are done. ∎

3.3. Remark. There is a similar result in [KV14, Theorem 2.2.11] without addressing the functoriality of the construction, given that their definition of the simplicial space uses pullbacks [KV14, Main construction 2.2.3].

We now want to understand the mapping spaces of $\mathfrak{S}$. For that we need the following strictification. Combining the locality of left fibrations (7) and minimal Reedy fibrations (5) it follows, by Lemma 2.20, that the set of minimal left fibrations is local and so we have a sub-bisimplicial set of $\mathfrak{S}$, that we denote by $\mathfrak{S}^{min} \hookrightarrow \mathfrak{S}$, and natural bijection

$$\mathcal{LF}\mathrm{ib}^{min}(-) \cong \mathrm{Hom}_{ss\mathcal{S}et}(-, \mathfrak{S}^{min}). \tag{8}$$

On the other hand, the map $\mathfrak{Min} : \mathfrak{Ree} \to \mathfrak{Ree}^{min}$ defined in 6, by Lemma 2.28, restricts to a map $\mathfrak{Min} : \mathfrak{S} \to \mathfrak{S}^{min}$. We now have the following result with regard to these two maps.

3.4. Lemma. *The maps $\mathfrak{S}^{min} \to \mathfrak{S} \to \mathfrak{S}^{min}$ are equivalences of complete Segal spaces.*

Proof. First of all the composition $\mathfrak{S}^{min} \to \mathfrak{S} \to \mathfrak{S}^{min}$ is the identity, as it takes every minimal left fibration to itself (Remark 2.27). Hence, $\mathfrak{S}^{min}$ is a retract of $\mathfrak{S}$ and so a complete Segal space. Next, we prove that $\mathfrak{S} \to \mathfrak{S}^{min}$ is a trivial fibration and this implies that the inclusion is an equivalence as well. Following Definition 2.4, we need to prove that for every inclusion of simplicial spaces $i : A \to B$ the following diagram has a lift

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & \mathfrak{S} \arrow[d, "\mathfrak{Min}"] \\ B \arrow[r] & \mathfrak{S}^{min} \end{tikzcd},$$

which, by 7 and 8, is equivalent to the map

$$\mathcal{LF}\mathrm{ib}(B) \to \mathcal{LF}\mathrm{ib}(A) \times_{\mathcal{LF}\mathrm{ib}^{min}(A)} \mathcal{LF}\mathrm{ib}^{min}(B)$$

being surjective. Unwinding the definitions this means we have the data of the following diagram

$$\begin{tikzcd}
L \arrow[d, two heads, "r", "\simeq"'] & \\
\mathcal{M}\mathrm{in}(L) \arrow[r, hook, "\hat{p}^*i"] \arrow[d, two heads, "p"] \arrow[dr, phantom, "\ulcorner", very near start] & \hat{L} \arrow[d, two heads, "\hat{p}"] \\
A \arrow[r, hook, "i"] & B
\end{tikzcd} ,$$

where r is a trivial fibration and $p, \hat{p}$ are minimal left fibrations and we need to find a left fibration $\tilde{p} : \tilde{L} \to B$, such that $i^*\tilde{p} = pr$ and $\mathcal{M}\mathrm{in}(\tilde{p}) = \hat{p}$. However, by Lemma 3.1, this is given by $(\hat{p}^*i)_*r : (\hat{p}^*i)_*L \to \hat{L}$. Here, the uniqueness of the factorization (Proposition 2.26) and that $(\hat{p}^*i)_*r$ is an equivalence, guarantees that $\mathcal{M}\mathrm{in}((\hat{p}^*i)_*r) \cong \hat{L}$. ■

We now want to use Lemma 3.4 to better understand the mapping spaces (1) of $\mathfrak{S}$. This requires us to better understand minimal left fibrations over $\Delta[1,0]$. First we introduce a notation that will be useful in the next proofs.

3.5. Notation. Let $X \to \Delta[1,l]$ be a map of bisimplicial sets. We use the following three notational conventions

- $X_{/0} = \mathrm{Map}_{/\Delta[1,l]}(d^1 : \Delta[0,l] \to \Delta[1,l], X)$
- $X_{/1} = \mathrm{Map}_{/\Delta[1,l]}(d^0 : \Delta[0,l] \to \Delta[1,l], X)$
- $X_{/01} = \mathrm{Map}_{/\Delta[1,l]}(\mathrm{id}, X)$

Notice, the two maps $d^0, d^1 : \Delta[0,l] \to \Delta[1,l]$ induce maps of simplicial sets $s : X_{/01} \to X_{/0}$ and $t : X_{/01} \to_{/1}$.

3.6. Lemma. *Let L and L' be two left fibrations over $\Delta[1,l]$ such that the two morphisms $t_L : L_{/01} \to L_{/1}$, $t_{L'} : L'_{/01} \to L'_{/1}$ are Kan equivalent morphisms. Then $\mathcal{M}\mathrm{in}(L) = \mathcal{M}\mathrm{in}(L')$.*

Proof. By Corollary 2.30 the result will follow if we can prove that L and L' are equivalent over $\Delta[1,l]$. Now, the projection map $\Delta[1,l] \to \Delta[1,0]$ is a level-wise equivalence and so, by 3, we have level-wise equivalences $L \simeq \hat{L} \times_{\Delta[1,l]} \Delta[1,0]$ and $L' \simeq \hat{L}' \times_{\Delta[1,l]} \Delta[1,0]$ for left fibrations $\hat{L} \to \Delta[1,0]$ and $\hat{L}' \to \Delta[1,0]$. Hence, without loss of generality, we can assume that $l = 0$. Denote the equivalence $t_L \to t_{L'}$ by α.

We now use the adjunction $(s\int_{[1]}, s\mathcal{H}_{[1]})$ as defined in [Ras23b, Lemma 4.9]. Concretely, by definition of $s\mathcal{H}_{[1]}$ we have $s\mathcal{H}_{[1]}(L) = t_L$, $s\mathcal{H}_{[1]}(L') = t_{L'}$ and by [Ras23b, Theorem 4.18] the counit of the adjunction gives us a level-wise equivalence and so we have

$$L \simeq s\int_{[1]} s\mathcal{H}_{[1]}L = s\int_{[1]} t_L \xrightarrow[\simeq]{s\int_{[1]}\alpha} s\int_{[1]} t_{L'} = s\int_{[1]} s\mathcal{H}_{[1]}L' \simeq L'$$

and hence we are done. ■

As explained in Subsection 1.2 one easy way to construct an $(\infty, 1)$-category of spaces is via nerves of Kan enriched categories. We hence want to compare $\mathfrak{S}$ to the nerve of the Kan enriched category of Kan complexes. Following the notational convention of Subsection 1.12, let $\underline{\mathcal{K}\mathrm{an}}$ be the Kan enriched category of Kan complexes, which has objects Kan complexes and for two objects K, L we have

$$\mathrm{Map}_{\underline{\mathcal{K}\mathrm{an}}}(K, L)_n = \mathrm{Hom}_{\underline{\mathcal{K}\mathrm{an}}}(K \times \Delta[n], L) \cong \mathrm{Hom}_{/\Delta[n]}(K \times \Delta[n], L \times \Delta[n]) \quad (9)$$

For further details regarding this Kan enriched category see [GJ09, Subsection I.5].

3.7. Remark. Recall that a Kan complex K is *minimal* if the map $K \twoheadrightarrow \Delta[0]$ is a minimal Kan fibration. Define $\underline{\mathcal{K}\mathrm{an}}^{min}$ as the full Kan enriched subcategory of $\underline{\mathcal{K}\mathrm{an}}$ with objects the minimal Kan complexes and notice the inclusion $\underline{\mathcal{K}\mathrm{an}}^{min} \to \underline{\mathcal{K}\mathrm{an}}$ is an equivalence of Kan enriched categories, in the sense of [Ber07a], as every Kan complex is equivalent to a minimal Kan complex [GJ09, Proposition 10.3].

We now want to apply the nerve to these Kan enriched categories. For a given simplicially enriched category $\mathcal{C}$, let $N_\Delta \mathcal{C}$ be the bisimplicial set with $N_\Delta \mathcal{C}_0 = \mathrm{Obj}_\mathcal{C}$ and

$$N_\Delta \mathcal{C}_n = \coprod_{X_0, ..., X_n} \mathrm{Map}_\mathcal{C}(X_0, X_1) \times ... \times \mathrm{Map}_\mathcal{C}(X_{n-1}, X_n). \quad (10)$$

While $N_\Delta \mathcal{C}$ is not a complete Segal space it is in fact a Segal category [Ber07b, Proposition 8.3] (where the nerve N_Δ is denoted R instead) and we can characterize their equivalences via *Dwyer-Kan* equivalences [Ber07b, Definition 3.9].

3.8. Proposition. *Let $\mathcal{C}$ be a Kan enriched category and W a complete Segal space. A map $F : N_\Delta \mathcal{C} \to W$ is an equivalence in the complete Segal space model structure if* $\mathrm{Obj}_\mathcal{C} \to W_{00}$ *is surjective and for objects x, y in $\mathcal{C}$, the induced map* $\mathrm{Map}_\mathcal{C}(x, y) \to \mathrm{Map}_W(Fx, Fy)$ *is a Kan equivalence.*

Proof. Let I denote the inclusion of Segal precategories into bisimplicial sets, which comes with a right adjoint denoted by R, described explicitly in [Ber07b, Section 6]. The map F factors as $N_\Delta \mathcal{C} \xrightarrow{G} IRW \xrightarrow{C} W$, where C is the counit of the adjunction (I, R), by the universal property of counits [Ber07b, Proposition 6.1]. As W is a complete Segal space, C is a complete Segal space equivalence by [Ber07b, Theorem 6.3], and G is a Dwyer-Kan equivalence of Segal categories by assumption and so also a complete Segal space equivalence, by [Ber07b, Proposition 6.2]. Hence the composition, namely F, is also a complete Segal space equivalence. ∎

We now have the necessary pieces to prove the main result.

3.9. Theorem. *There is a complete Segal space equivalence $\mathbb{I} : N_\Delta \underline{\mathcal{K}\mathrm{an}} \to \mathfrak{S}$ to the complete Segal space $\mathfrak{S}$. Moreover, we have a natural bijection* $\mathcal{LF}\mathrm{ib}(-) \cong \mathrm{Hom}(-, \mathfrak{S})$.

Proof. We have established that $\mathfrak{S}$ is a complete Segal space in Proposition 3.2 and the bijection in 7 and so we only need to prove the equivalence with $N_\Delta\underline{\mathcal{K}\mathrm{an}}$. Let $\mathcal{L}$ be the bisimplicial set given level-wise by

$$\mathcal{L}_n = \coprod_{X_0,\ldots,X_n\in\underline{\mathcal{K}\mathrm{an}}} X_0 \times \mathrm{Map}(X_0,X_1)\times\cdots\times\mathrm{Map}(X_{n-1},X_n). \tag{11}$$

The first two face maps $\mathcal{L}_1 = \coprod_{X_0,X_1} X_0\times\mathrm{Map}(X_0,X_1)\to\coprod_{X_0\in\underline{\mathcal{K}\mathrm{an}}} X_0 = \mathcal{L}_0$ are given by $d_0(x,f)=x$ and $d_1(x,f)=f(x)$ and all higher face maps are given by applying d_0 and d_1 at the appropriate index. Similarly, the degeneracy $s_0:\mathcal{L}_0\to\mathcal{L}_1$ is given by $s_0(x)=(x,\mathrm{id})$ and a general degeneracy inserts an identity at the appropriate index.

There is a projection morphism $\pi_2:\mathcal{L}\to N_\Delta\underline{\mathcal{K}\mathrm{an}}$ that is level-wise given by

$$\begin{aligned}(\pi_2)_n:\mathcal{L}_n &= \coprod_{X_0,\ldots,X_n\in\underline{\mathcal{K}\mathrm{an}}} X_0\times(\mathrm{Map}(X_0,X_1)\times\cdots\times\mathrm{Map}(X_{n-1},X_n))\\ &\to \coprod_{X_0,\ldots,X_n\in\underline{\mathcal{K}\mathrm{an}}}\mathrm{Map}(X_0,X_1)\times\cdots\times\mathrm{Map}(X_{n-1},X_n) \qquad = \underline{\mathcal{K}\mathrm{an}}_n,\end{aligned}$$

meaning it forgets the X_0 component.

As $\underline{\mathcal{K}\mathrm{an}}_0=\mathrm{Obj}_{\underline{\mathcal{K}\mathrm{an}}}$ is a set and the projection map $\pi_2:\mathcal{L}\to N_\Delta\underline{\mathcal{K}\mathrm{an}}$ gives us the strict pullback of simplicial sets, $\mathcal{L}_n\cong\mathcal{L}_0\times_{\mathrm{Obj}_{\underline{\mathcal{K}\mathrm{an}}}} N_\Delta\underline{\mathcal{K}\mathrm{an}}_n$, by Lemma 2.9, the Reedy fibrant replacement $\widehat{\pi_2}:\hat{\mathcal{L}}\to N_\Delta\underline{\mathcal{K}\mathrm{an}}$ over $N_\Delta\underline{\mathcal{K}\mathrm{an}}$ is a left fibration. By 7, this induces a functor $\mathbb{I}:N_\Delta\underline{\mathcal{K}\mathrm{an}}\to\mathfrak{S}$. We want to prove $\mathbb{I}$ is a complete Segal space equivalence.

By Lemma 3.4, it suffices to prove that the composition $N_\Delta\underline{\mathcal{K}\mathrm{an}}\to\mathfrak{S}^{min}$ is an equivalence. Moreover, by Remark 3.7, we can further reduce it to showing that $N_\Delta\underline{\mathcal{K}\mathrm{an}}^{min}\to\mathfrak{S}^{min}$, which we denote by $\mathfrak{Lift}$, is a Dwyer-Kan equivalence. By Remark 2.8 and Lemma 2.29 every minimal left fibration over $L\to\Delta[0,0]$ is uniquely (up to isomorphism) determined by the minimal Kan complex L_0 and so $\mathfrak{Lift}$ is surjective on objects and so, by Proposition 3.8, it suffices to prove $\mathfrak{Lift}$ induces an equivalence of mapping spaces. We will in fact show that the map of simplicial sets $\mathfrak{Lift}_1:N_\Delta\underline{\mathcal{K}\mathrm{an}}_1^{min}\to(\mathfrak{S}^{min})_1$ induces a bijection on mapping spaces.

Before we proceed, we will thoroughly analyze the morphism $\mathfrak{Lift}:N_\Delta\underline{\mathcal{K}\mathrm{an}}^{min}\to\mathfrak{S}^{min}$. By 8, $\mathcal{M}\mathrm{in}(\hat{L})\to N_\Delta\underline{\mathcal{K}\mathrm{an}}$ corresponds to the composition $N_\Delta\underline{\mathcal{K}\mathrm{an}}\to\mathfrak{S}\xrightarrow{\mathfrak{Min}}\mathfrak{S}^{min}$. So, precomposition with the inclusion $N_\Delta\underline{\mathcal{K}\mathrm{an}}^{min}\to N_\Delta\underline{\mathcal{K}\mathrm{an}}$ corresponds to the minimal left fibration $\hat{\mathcal{L}}^{min}=\mathcal{M}\mathrm{in}(\hat{\mathcal{L}})\times_{N_\Delta\underline{\mathcal{K}\mathrm{an}}}N_\Delta\underline{\mathcal{K}\mathrm{an}}^{min}$ over $N_\Delta\underline{\mathcal{K}\mathrm{an}}^{min}$, via pullback. More explicitly for an (n,l)-simplex σ, which corresponds to a morphism $\sigma:\Delta[n,l]\to\mathcal{N}_\Delta\underline{\mathcal{K}\mathrm{an}}^{min}$, $\mathfrak{Lift}(\sigma)$ is the minimal left fibration over $\Delta[n,l]$ obtained by pulling back the minimal left fibration over $\mathfrak{S}^{min}$, which by the pasting lemma for pullbacks is given by $\mathfrak{Lift}(\sigma)=\sigma^*\hat{\mathcal{L}}^{min}\to\Delta[n,l]$.

By definition of the simplicial nerve (10) $(N_\Delta\underline{\mathcal{K}\mathrm{an}}^{min})_{1l}$ is the set of l-morphisms in the simplicially enriched category $\underline{\mathcal{K}\mathrm{an}}^{min}$, which by 9 is explicitly given by a morphism of simplicial sets $X\times\Delta[l]\to Y\times\Delta[l]$ over $\Delta[l]$, where X,Y are minimal Kan complexes. For such a given morphism f, $\mathfrak{Lift}(f)$ is a minimal left fibration over $\Delta[1,l]$ which fits into the following diagram

$$\begin{tikzcd} X \times \Delta[0,l] \coprod Y \times \Delta[0,l] \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & \mathfrak{L}\mathrm{ift}(f) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & \hat{\mathcal{L}}^{min} \arrow[d, two heads] \\ \Delta[0,l] \coprod \Delta[0,l] \arrow[r, hook] \arrow[rr, bend right, "\{X\}\coprod\{Y\}"'] & \Delta[1,l] \arrow[r, "\{f\}"] & N_\Delta \underline{\mathcal{K}\mathrm{an}}^{min} \end{tikzcd},$$

where we are using [GJ09, Corollary 10.8] to deduce that the outer rectangle is also a pullback. This implies the following (using the notation introduced in Notation 3.5):

- $\mathfrak{L}\mathrm{ift}(f)_{/0} = X, \mathfrak{L}\mathrm{ift}(f)_{/1} = Y$,
- $\mathfrak{L}\mathrm{ift}(f)_{/01}$ is the minimal Kan fibration over $X \times Y$ equivalent to the map $(\mathrm{id}, f) : X \to X \times Y$, which determines $\mathfrak{L}\mathrm{ift}(f)_{/01}$ uniquely.

The equivalence $X \simeq \mathfrak{L}\mathrm{ift}(f)_{/01}$ in particular implies that

$$\mathcal{M}\mathrm{in}(\mathfrak{L}\mathrm{ift}(f)_{/01}) = X \tag{12}$$

and $X = \mathcal{M}\mathrm{in}(\mathfrak{L}\mathrm{ift}(f)_{/01}) \to X \times Y \xrightarrow{\pi_2} Y$ is given by f.

We will now prove $\mathfrak{L}\mathrm{ift}_1$ is a bijection by constructing an inverse $\mathcal{S}\mathrm{ec} : (\mathfrak{S}^{min})_1 \to N_\Delta \underline{\mathcal{K}\mathrm{an}}_1^{min}$. For a given minimal left fibration $L \to \Delta[1,l]$, let $\mathcal{S}\mathrm{ec}(L) = \mathcal{M}\mathrm{in}(L_{/01}) \to L_{/1}$ over $\Delta[l]$. We now prove that $\mathcal{S}\mathrm{ec}$ is injective and that $\mathcal{S}\mathrm{ec}$ is a left inverse of $\mathfrak{L}\mathrm{ift}_1$ which will prove they are inverses.

The statement of Lemma 3.6 and the definition of $\mathcal{S}\mathrm{ec}(L)$ as $\mathcal{M}\mathrm{in}(L_{/01}) \to L_{/1}$ immediately implies that $\mathcal{S}\mathrm{ec}$ is injective on minimal left fibrations. Moreover, we observed in 12 that for a given morphism of minimal left fibrations $f : X \times \Delta[l] \to Y \times \Delta[l]$ over $\Delta[l]$, $X = \mathcal{M}\mathrm{in}(\mathfrak{L}\mathrm{ift}(f)_{/01})$ and the composition

$$X = \mathcal{M}\mathrm{in}(\mathfrak{L}\mathrm{ift}(f)_{/01}) \hookrightarrow \mathfrak{L}\mathrm{ift}(f)_{/01} \to \mathfrak{L}\mathrm{ift}(f)_{/1} = Y$$

is given by f. This proves that $\mathcal{S}\mathrm{ec} \circ \mathfrak{L}\mathrm{ift}_1$ is the identity and hence we are done. ■

3.10. Remark. We constructed a complete Segal space of spaces using left fibrations. However, based on Remark 2.10, we could have also used right fibrations to construct a bisimplicial set $\mathfrak{S}^R$ with $\mathfrak{S}^R_{nl}$ given by right fibrations over $\Delta[n,l]$. Now, Remark 2.10 implies that $(-)^{op}$ induces a bijection between $\mathfrak{S}$ and $\mathfrak{S}^R$ that flips the directionality of the morphisms, immediately implying that $\mathfrak{S}^R$ is just the opposite complete Segal space of spaces, $\mathfrak{S}^{op}$.

3.11. Remark. One implication of Theorem 3.9 and the fact that $\mathfrak{S}$ is complete (Proposition 3.2) is that the space of Kan equivalences from X to Y is equivalent to the space of left fibrations over $\Delta[0,\bullet]$ with fiber over the initial vertex in $\Delta[0,\bullet]$ given by X and fiber over terminal vertex in $\Delta[0,\bullet]$ given by Y. As left fibrations $L \to \Delta[0,n]$ are up to homotopy uniquely determined by Kan fibrations over $\Delta[n]$ [Ras23b, Theorem 3.17], this means we are getting an equivalence between the space of equivalences and the space of Kan fibrations over $\Delta[\bullet]$, meaning we get an alternative proof to the simplicial univalence of the universe of Kan fibrations [KL21, Theorem 3.4.1].

4. The Complete Segal Space of (Complete) Segal Spaces

We now want to generalize the results from the previous section from a complete Segal space of spaces to a complete Segal space of complete Segal spaces. Here we rely on the complete Segal object approach to coCartesian fibrations as outlined in the beginning of [Ras21a]. The benefit of this approach is that we can use the results of the previous sections level-wise to immediately deduce the desired results.

Let us start with the appropriate generalizations of Subsection 2.2. The category of *trisimplicial sets* is denoted by $\mathrm{sss}\mathcal{S}\mathrm{et}$ and the generators are denoted by $\Delta[-,-,-]$. We can, analogous to Definition 2.4, characterize Reedy fibrations via right lifting properties against certain morphisms with codomain $\Delta[k,n,l]$. Again, Reedy fibrations are part of a model structure with cofibrations given by inclusions of trisimplicial sets and equivalences given by level-wise Kan equivalences. In particular, all trivial Reedy fibrations are Reedy weak equivalences. See [Ras23a, Subsection 2.8] for more details.

4.1. Notation. For a trisimplicial set X, we use X_k to denote the bisimplicial set with $(X_k)_{nl} = X_{knl}$ and use X_{kn} to denote the simplicial set with $(X_{kn})_l = X_{knl}$.

We have analogous generalizations of left fibrations (Definition 2.7).

4.2. Definition. *Let $\iota_{st} : \mathrm{ss}\mathcal{S}\mathrm{et} \to \mathrm{sss}\mathcal{S}\mathrm{et}$, the standard embedding, take a bisimplicial set X to the trisimplicial set $\iota_{st}(X)$ characterized as $\iota_{st}(X)_{knl} = X_{nl}$. Moreover, for a map of trisimplicial sets $p : Y \to \iota_{st}X$, we denote by $p_k : Y_k \to X$ the map of bisimplicial sets defined as $(p_k)_{nl} = p_{knl}$.*

4.3. Definition. *Let X be a bisimplicial set and $p : L \to X$ be a Reedy fibration of trisimplicial sets.*

- *p is a Reedy left fibration if $p_k : L_k \to X$ (Notation 4.1) is a left fibration for all $k \geq 0$.*
- *p is a Segal coCartesian fibration if it is a Reedy left fibration and it satisfies the Segal condition, meaning the map*

$$L_{kn} \twoheadrightarrow L_{1n} \times_{L_{0n}} \ldots \times_{L_{0n}} L_{1n}$$

is a Kan equivalence for all $k \geq 2$ and $n \geq 0$.

- *p is a coCartesian fibration if it is a Segal coCartesian fibration and it satisfies the completeness condition, meaning the map*

$$L_{0n} \to L_{3n} \times_{(L_{1n} \times L_{1n})} (L_{0n} \times L_{0n})$$

is a Kan equivalence for all $n \geq 0$.

The Segal condition and completeness condition are evident analogues to the ones used to define complete Segal spaces (Definition 2.5) and so we would expect a close connection. Indeed, we have the following remark.

4.4. Remark. By [Ras23a, Theorem 5.7], if $p : L \to X$ is a Reedy left fibration, then p is a Segal coCartesian fibration if and only if for every $x : \Delta[0,0] \to X$, the fiber $x^*L_{\bullet 0}$ is a Segal space and similarly between coCartesian fibrations and complete Segal spaces.

4.5. Remark. Following Remark 2.8, a Reedy left fibration L over $\Delta[0,0]$ is a homotopically constant trisimplicial set, meaning $L_{kn} \simeq L_{k0}$. Similarly, a (Segal) coCartesian fibration over $\Delta[0,0]$ is a homotopically constant trisimplicial set L such that L_0 is a (complete) Segal space.

We have a similar interaction between Reedy left fibrations and complete Segal equivalences as in 3, meaning for every Reedy left fibration $p : L \to A$ and complete Segal space equivalence $A \to B$, we can obtain p as a homotopy pullback square of a Reedy left fibration $\hat{p} : \hat{L} \to B$

$$\begin{tikzcd} L \arrow[r, hook, "j"] \arrow[d, two heads, "p"'] \arrow[dr, phantom, "\ulcorner", very near start] & \hat{L} \arrow[d, two heads, "\hat{p}"] \\ A \arrow[r, "i"] & B \end{tikzcd} \tag{13}$$

where $L \xrightarrow{j} \hat{L} \xrightarrow{\hat{p}} B$ is given via a factorization into a trivial cofibration followed by a Reedy left fibration [Ras23a, Theorem 5.12].

4.6. Remark. Generalizing Remark 2.10, we can analogous to Definition 4.3 define *Reedy right fibrations* as Reedy fibrations $p : R \to X$, such that for all k, $p_k : R_k \to X$ is a right fibration. Moreover, we then define *(Segal) Cartesian fibrations* as Reedy right fibrations, that satisfy the Segal and completeness condition, as described in Definition 4.3. Moreover, by Remark 2.10, a map $R \to X$ is a Reedy right, Segal Cartesian or Cartesian fibration if and only if $R^{op} \to X^{op}$ is a Reedy left, Segal coCartesian or coCartesian fibration, respectively. Here $(-)^{op} : \mathrm{sss}\mathcal{S}\mathrm{et} \to \mathrm{sss}\mathcal{S}\mathrm{et}$ takes $(-)^{op}$ defined in Remark 2.10 level-wise, meaning it is defined as $\mathrm{Fun}(\mathbb{\Delta}^{op}, (-)^{op})$.

We move on to appropriate generalizations of Subsection 2.11. Let $\mathcal{A}\mathrm{ll} : \mathrm{sss}\mathcal{S}\mathrm{et}^{op} \to \mathcal{S}\mathrm{et}$ be the functor $\mathcal{A}\mathrm{ll}(X) = \mathrm{Fun}((\int_{\mathbb{\Delta}\times\mathbb{\Delta}\times\mathbb{\Delta}} X)^{op}, \mathcal{S}\mathrm{et})$. We can restrict this functor by precomposing with the standard embedding $(\iota_{st})^{op} : \mathrm{ss}\mathcal{S}\mathrm{et}^{op} \to \mathrm{sss}\mathcal{S}\mathrm{et}^{op}$ to define $\mathrm{s}\mathcal{A}\mathrm{ll} : \mathrm{ss}\mathcal{S}\mathrm{et}^{op} \to \mathcal{S}\mathrm{et}$. Similar to Lemma 2.15 for every bisimplicial set X we have a bijection

$$\mathrm{s}\mathcal{A}\mathrm{ll}(X) \cong \mathrm{sss}\mathcal{S}\mathrm{et}_{/\iota_{st}X}.$$

Now, define $\mathrm{sss}\mathfrak{S}\mathrm{et} : \mathbb{\Delta}^{op} \times \mathbb{\Delta}^{op} \to \mathcal{S}\mathrm{et}$ as $\mathrm{s}\mathcal{A}\mathrm{ll}(-)$ precomposed with the Yoneda embedding and, analogous to Corollary 2.17 we have a natural isomorphism

$$\mathrm{s}\mathcal{A}\mathrm{ll}(-) \cong \mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, \mathrm{sss}\mathfrak{S}\mathrm{et}). \tag{14}$$

Finally, if S is a local set of morphisms, then, similar to Lemma 2.20, this bijection restricts to a bijection

$$\mathrm{s}\mathcal{A}\mathrm{ll}^S(-) \cong \mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, \mathrm{sss}\mathfrak{S}\mathrm{et}), \tag{15}$$

where $\mathrm{s}\mathcal{A}\mathrm{ll}^S(X) \subseteq \mathrm{s}\mathcal{A}\mathrm{ll}(X)$ is the sub-functor of objects over $\iota_{st}X$ that are in S and $\mathrm{sss}\mathfrak{S}\mathrm{et}^S$ is again the restriction of $\mathrm{s}\mathcal{A}\mathrm{ll}^S(-)$ along the Yoneda embedding. Finally, we have the analogue of Corollary 2.21 for trisimplicial sets.

4.7. Corollary. *Let S be a set of morphisms of trisimplicial sets determined by a right lifting property with respect to a set of morphisms $A \to \Delta[k,n,l]$. Then S is pullback stable and in fact local.*

We can use the corollary to generalize 7. By Definition 4.3, Reedy left fibrations are local and so we get a bisimplicial set $\mathrm{sss}\mathfrak{S}\mathrm{et}^{\mathcal{R}\mathrm{ee}\mathcal{LF}\mathrm{ib}}$ that we denote by $s\mathfrak{S}$ and a natural bijection

$$\mathcal{R}\mathrm{ee}\mathcal{LF}\mathrm{ib}(-) \cong \mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, s\mathfrak{S}). \tag{16}$$

We want to prove that $s\mathfrak{S}$ is a complete Segal space of Reedy fibrant simplicial spaces. This requires us to understand minimal Reedy fibration and minimal Reedy left fibrations of trisimplicial sets, similar to Subsection 2.23. A Reedy fibration of trisimplicial sets $Y \to X$ is *minimal* if for all $k, n \geq 0$ the induced map of simplicial sets (using Notation 4.1)

$$Y_{kn} \twoheadrightarrow M_{kn}Y \times_{M_{kn}X} X_{kn}$$

is a minimal Kan fibration. By Example A.9 all the results in Section A also apply to Reedy fibrations of trisimplicial sets and so can recover the analogous results to Subsection 2.23 with the same proofs. In particular, analogous to Lemma 2.25, Reedy left fibrations are local and, analogous to Proposition 2.26, we have the following factorization.

4.8. Proposition. *Every Reedy fibration of trisimplicial sets $p : Y \to X$ admits a (up to isomorphism) unique factorization $Y \overset{q}{\twoheadrightarrow} \mathcal{M}\mathrm{in}(Y) \overset{\mathcal{M}\mathrm{in}(p)}{\twoheadrightarrow} X$ into a trivial fibration q followed by a minimal fibration $\mathcal{M}\mathrm{in}(p)$.*

Moreover, as being a Reedy left fibration is determined level-wise, by Definition 4.3, it follows directly from Lemma 2.28 that if $L \to X$ is a Reedy left fibration, then $\mathcal{M}\mathrm{in}(L) \to X$ is also a Reedy left fibration. Let $\mathcal{R}\mathrm{ee}\mathcal{LF}\mathrm{ib}^{min}(-) \subseteq \mathcal{R}\mathrm{ee}\mathcal{LF}\mathrm{ib}(-)$ be the sub-functor of minimal Reedy left fibrations and denote the corresponding subobject of the bisimplicial set of $s\mathfrak{S}$ by $s\mathfrak{S}^{min}$. The locality of minimal Reedy left fibrations gives us the natural bijection $\mathcal{R}\mathrm{ee}\mathcal{LF}\mathrm{ib}^{min}(-) \cong \mathrm{Hom}_{\mathrm{sss}\mathcal{S}\mathrm{et}}(-, s\mathfrak{S}^{min})$ and so we can apply Proposition 4.8 to the fibration $s\mathfrak{S}_* \to s\mathfrak{S}$ corresponding to the identity, which by the natural bijection corresponds to a map of bisimplicial sets, $\mathfrak{Min} : s\mathfrak{S} \to s\mathfrak{S}^{min}$, which represents a map

$$\mathrm{s}\mathcal{M}\mathrm{in} : \mathcal{R}\mathrm{ee}\mathcal{LF}\mathrm{ib}(-) \to \mathcal{R}\mathrm{ee}\mathcal{LF}\mathrm{ib}^{min}(-). \tag{17}$$

Naturality again implies that $\mathrm{s}\mathcal{M}\mathrm{in}$ sends a fibration to its corresponding unique minimal Reedy left fibration constructed in Proposition 4.8 and, analogous to Remark 2.27, is the identity when restricted to minimal Reedy left fibrations.

We can now use this result to study the bisimplicial set $s\mathfrak{S}$. Before that we need the following last lemma, which follows analogous to Lemma 3.1.

4.9. Lemma. *Let $p : Y \to A$ be a trivial Reedy fibration of trisimplicial sets and $i : A \to B$ an inclusion of trisimplicial sets, inducing an adjunction $i^* \dashv i_*$. Then i_*p is a trivial Reedy fibration and $p \xrightarrow{\cong} i^*i_*p$.*

4.10. Proposition. *The bisimplicial set $s\mathfrak{S}$ is a complete Segal space.*

Proof. We will follow the steps given in Proposition 3.2. The bijection 16 allows us to again reduce the proof to showing that every Reedy left fibration $L \to A$ can be obtained as the pullback of a Reedy left fibration $R \to B$ along a trivial complete Segal space cofibration $i : A \to B$. We can obtain this lift using the same diagram as in the proof of Proposition 3.2 this time relying on Proposition 4.8 whenever we need a minimal Reedy left fibration, 13 when we need to extend L along $i : A \to B$, and Lemma 4.9 when we need a trivial Reedy fibration. ■

As $s\mathfrak{S}^{min}$ is a retract of $s\mathfrak{S}$, Proposition 4.10 implies that $s\mathfrak{S}^{min}$ is a complete Segal space as well. Using this bijection along with Lemma 4.9 in the proof of Lemma 3.4 we obtain the following result about this retract.

4.11. Lemma. *The maps $s\mathfrak{S}^{min} \to s\mathfrak{S} \to s\mathfrak{S}^{min}$ are equivalences of complete Segal spaces.*

We now want to use these result to finally prove that $s\mathfrak{S}$ is in fact the complete Segal space of Reedy fibrant simplicial spaces. Let $\underline{\mathcal{R}\text{ee}}$ denote the Kan enriched category of Reedy fibrant simplicial spaces, where, by analogy with 9, the mapping spaces for two Reedy fibrant simplicial spaces are defined as follows:

$$\text{Map}_{\underline{\mathcal{R}\text{ee}}}(K, L)_n = \text{Hom}_{\underline{\mathcal{R}\text{ee}}}(K \times \Delta[0, n], L) \cong \text{Hom}_{/\Delta[0,n]}(K \times \Delta[0, n], L \times \Delta[0, n]). \quad (18)$$

4.12. Remark. Similar to Remark 3.7, a Reedy fibrant simplicial space K is *minimal* if the unique map $K \to \Delta[0, 0]$ is a minimal Reedy fibration. Denote by $\underline{\mathcal{R}\text{ee}}^{min}$ the full simplicially enriched subcategory of $\underline{\mathcal{R}\text{ee}}$ consisting of minimal Reedy fibrant simplicial spaces. Notice, the inclusion $\underline{\mathcal{R}\text{ee}}^{min} \hookrightarrow \underline{\mathcal{R}\text{ee}}$ is an equivalent full subcategory, by Proposition 2.26.

Using 10 we obtain a Segal category $N_\Delta\underline{\mathcal{R}\text{ee}}$. We now have the following result.

4.13. Theorem. *There is a complete Segal space equivalence $s\mathbb{I} : N_\Delta\underline{\mathcal{R}\text{ee}} \to s\mathfrak{S}$ to the complete Segal space $s\mathfrak{S}$. Moreover, we have a bijection $\mathcal{R}\text{ee}\mathcal{L}\mathcal{F}\text{ib}(-) \cong \text{Hom}(-, s\mathfrak{S})$.*

Proof. $s\mathfrak{S}$ is a complete Segal space by Proposition 4.10 and the bijection follows from 16. Hence, we only need to prove the equivalence. Let $\text{s}\mathcal{L}_n$ be the trisimplicial set, given level-wise by

$$\text{s}\mathcal{L}_n = \coprod_{X_0,...,X_n \in \underline{\mathcal{R}\text{ee}}} X_0 \times \text{Map}(X_0, X_1) \times ... \times \text{Map}(X_{n-1}, X_n) \quad (19)$$

and with face and degeneracy maps given analogous to the ones for $\mathcal{L}$ (as constructed in Theorem 3.9). We similarly get a projection map $\pi_2 : s\mathcal{L} \to N_\Delta \underline{\mathcal{R}\mathrm{ee}}$ given by forgetting the first coordinate. Again, $N_\Delta \underline{\mathcal{R}\mathrm{ee}}_0 = \mathrm{Obj}_{\underline{\mathcal{R}\mathrm{ee}}}$ is a set and π_2 induces a strict pullback $s\mathcal{L}_n \cong N_\Delta \underline{\mathcal{R}\mathrm{ee}}_n \times_{\mathrm{Obj}_{\underline{\mathcal{R}\mathrm{ee}}}} s\mathcal{L}_0$, and so, by applying Lemma 2.9 level-wise, the Reedy fibrant replacement $\widehat{\pi_2} : \hat{s\mathcal{L}} \to N_\Delta \underline{\mathcal{R}\mathrm{ee}}$ over $N_\Delta \underline{\mathcal{R}\mathrm{ee}}$ is a Reedy left fibration. The bijection 16 gives us a functor $s\mathbb{I} : N_\Delta \underline{\mathcal{R}\mathrm{ee}} \to s\mathfrak{S}$. We want to prove this is a Dwyer-Kan equivalence. Now, combining Lemma 4.11 and Remark 4.12, this proof reduces to showing that $s\mathfrak{L}\mathrm{ift} : N_\Delta \underline{\mathcal{R}\mathrm{ee}}^{min} \to s\mathfrak{S}^{min}$ is a Dwyer-Kan equivalence, which follows from establishing the conditions in Proposition 3.8.

By Remark 4.5, Reedy left fibrations over $\Delta[0,0]$ are determined by a Reedy fibrant simplicial space and so the functor is surjective, meaning we only need to show it is an equivalence of mapping spaces. We can now follow the same steps as in Theorem 3.9 to deduce that $s\mathfrak{L}\mathrm{ift}_1 : N_\Delta \underline{\mathcal{R}\mathrm{ee}}^{min} \to s\mathfrak{S}_1^{min}$ is a bijection by constructing an explicit inverse. ∎

4.14. Remark. General Reedy fibrant simplicial spaces are too broad to give us a model of ∞-categories, however, bisimplicial sets with the Reedy model structure do give us a model topos [Rez10b], which have been established as models for homotopy type theories [Shu19], and so have been studied extensively by Shulman [Shu15a, Shu15b, Shu17]. In particular, bisimplicial sets are the key example of a model of *type theories with shapes* introduced by Riehl and Shulman with the goal of ∞-category theory internal to type theories [RS17]. From this perspective, an explicit construction of a complete Segal space for Reedy fibrant simplicial spaces is an important step towards better understanding its properties as a model of type theories.

We can now restrict this construction in the following two ways to get the results we wanted. Using the fact that (Segal) coCartesian fibration are local (by Definition 4.3) we denote $sss\mathfrak{S}\mathrm{et}^{\mathcal{S}\mathrm{egco}\mathcal{C}\mathrm{art}}$ by $\mathfrak{S}\mathrm{eg}$ and $sss\mathfrak{S}\mathrm{et}^{\mathrm{co}\mathcal{C}\mathrm{art}}$ by $\mathfrak{C}\mathfrak{S}\mathfrak{S}$. Similarly, let $\underline{\mathcal{S}\mathrm{eg}}$ denote the full simplicially enriched category with objects Segal spaces and $\underline{\mathcal{CSS}}$ denote the full simplicially enriched subcategory of complete Segal spaces. We now have the following lemma and theorem, giving us a complete Segal space of (complete) Segal spaces.

4.15. Lemma. *Let W be a complete Segal space. Let $V_{00} \subseteq W_{00}$ be a subset of objects in W closed under equivalences, meaning any object in W equivalent to an element in V_{00} is already in V_{00}. Define $V \subseteq W$ as the sub-bisimplicial set of W with elements in $\sigma \in V_{nl} \subseteq W_{nl}$ if for all $d : W_{nl} \to W_{00}$, $d(\sigma) \in V_{00}$. Then V is a complete Segal space and $V \to W$ is fully faithful.*

Proof. Let $i : A \to B$ be a trivial cofibration in the complete Segal space model structure and let $f : A \to V$ be a map. Then there exists a map $\hat{f} : B \to W$ lifting i. Now, every trivial cofibration in the complete Segal space model structure is surjective on equivalence classes of objects [Ras21b, Lemma 3.54]. Hence, the objects in W that lie in the image of $\hat{f}$ are all equivalent to objects in the image of $f : A \to V \subseteq W$ and hence in V themselves. This proves that the lift factors through V and hence V is a complete Segal space. Finally, for given objects x, y in V, by the definition of V and mapping spaces of complete Segal spaces (1), $\mathrm{Map}_V(x, y) \to \mathrm{Map}_W(x, y)$ is the identity and so we are done. ∎

4.16. THEOREM. *In the following diagram the top (bottom) horizontal functors are fully faithful functors of strict Segal categories (complete Segal spaces) and the vertical maps are complete Segal space equivalences*

$$\begin{tikzcd} N_\Delta \underline{\mathcal{CSS}} \arrow[r, hook] \arrow[d, "s\mathbb{I}"', "\simeq"] & N_\Delta \underline{\mathcal{S}\mathrm{eg}} \arrow[r, hook] \arrow[d, "s\mathbb{I}"', "\simeq"] & N_\Delta \underline{\mathcal{R}\mathrm{ee}} \arrow[d, "s\mathbb{I}"', "\simeq"] \\ \mathfrak{CSS} \arrow[r, hook] & \mathfrak{S}\mathrm{eg} \arrow[r, hook] & s\mathfrak{S} \end{tikzcd}$$

meaning $\mathfrak{S}\mathrm{eg}$ *is the complete Segal space of Segal spaces and* $\mathfrak{CSS}$ *is the complete Segal space of complete Segal spaces. Moreover, we have bijections*

$$\mathcal{S}\mathrm{egco}\mathcal{C}\mathrm{art}(-) \cong \mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, \mathfrak{S}\mathrm{eg}),$$

$$\mathrm{co}\mathcal{C}\mathrm{art}(-) \cong \mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, \mathfrak{CSS}).$$

PROOF. The bijections follow directly from 15 and the fact that (Segal) coCartesian fibrations are local (Definition 4.3). Now, by Theorem 4.13, $s\mathfrak{S}$ is a complete Segal space of bisimplicial sets and, by Lemma 4.15 and Remark 4.4, $\mathfrak{S}\mathrm{eg} \hookrightarrow s\mathfrak{S}$ is a fully faithful inclusion of complete Segal spaces. Here we are using the fact that the Segal condition in Definition 4.3 is by definition up to equivalence and so any Reedy left fibration equivalent to a Segal coCartesian fibration is in fact a Segal coCartesian fibration, proving that the condition in Lemma 4.15 is in fact satisfied. Finally, by Remark 4.5, the objects in $\mathfrak{S}\mathrm{eg}$ are Segal spaces. We can use the same arguments to prove that $\mathfrak{CSS} \to \mathfrak{S}\mathrm{eg}$ is a fully faithful functor of complete Segal spaces with $\mathfrak{CSS}$ having objects complete Segal spaces.

Finally, we have already constructed an equivalence $s\mathbb{I} : N_\Delta \underline{\mathcal{R}\mathrm{ee}} \to s\mathfrak{S}$, so it suffices to show its restriction $N_\Delta \underline{\mathcal{S}\mathrm{eg}} \to s\mathfrak{S}$ has essential image $\mathfrak{S}\mathrm{eg}$. Let X be an object in $\underline{\mathcal{S}\mathrm{eg}}$ i.e. a Segal space. Then $s\mathbb{I}(X)$ is the homotopically constant Reedy left fibration with $s\mathbb{I}(X)_0 = X$, which satisfies the Segal condition by assumption, making it a Segal coCartesian fibration. On the other side, if $L \to \Delta[0, 0, 0]$ is a Segal coCartesian fibration, then L is homotopically constant and L_0 is a Segal space (Remark 4.5), so $s\mathbb{I}(L_0) \simeq L$. This proves $s\mathbb{I} : N_\Delta \mathcal{S}\mathrm{eg} \to \mathfrak{S}\mathrm{eg}$ is an equivalence. We can repeat the same argument to deduce that $s\mathbb{I}$ restricts to an equivalence $s\mathbb{I} : N_\Delta \mathcal{CSS} \to \mathfrak{CSS}$ finishing the proof. ∎

4.17. REMARK. Combining Remark 3.10 and Remark 4.6 directly implies that the bisimplicial set with (k, n)-simplices given by Reedy right fibrations over $\Delta[n, l]$ is precisely the opposite complete Segal space of Reedy fibrant simplicial spaces, $s\mathfrak{S}^{op}$. Moreover, restricting those to (Segal) Cartesian fibrations gives us the opposite complete Segal spaces of (complete) Segal spaces as constructed in Theorem 4.16, $\mathfrak{S}\mathrm{eg}^{op}$ and $\mathfrak{CSS}^{op}$.

5. Universal Fibrations

Up until this point we have constructed various complete Segal spaces that have relevant universal properties in the sense that functors into them correspond to various fibrations over them. This, in particular, implies the existence of a universal fibration corresponding to the identity map. In this section we want to focus on these universal fibrations.

We will, in particular, prove that these universal fibrations are all representable. This gives us a very explicit characterization of the domain of these universal fibrations and helps us establish key properties thereof. In particular, representability of the universal left fibration proves that the complete Segal space of spaces has a "generating object" (which we prove to be the terminal object in 5.3), meaning an object in $\mathfrak{S}$ is uniquely characterized by maps out of the terminal object into it. Similarly, representability of the other universal fibrations proves that the complete Segal space of Reedy fibrant simplicial spaces, and the sub-complete Segal space of (complete) Segal spaces, have a "generating cosimplicial object".

We will start with the case for spaces. Denote by $p_{\mathrm{ss}\mathfrak{S}\mathrm{et}} : \mathrm{ss}\mathfrak{S}\mathrm{et}_* \to \mathrm{ss}\mathfrak{S}\mathrm{et}$ the map that corresponds to the identity map under the bijection in Corollary 2.17. Notice, a map $\Delta[n,l] \to \mathrm{ss}\mathfrak{S}\mathrm{et}_*$ corresponds to a map $\sigma : \Delta[n,l] \to \mathrm{ss}\mathfrak{S}\mathrm{et}$ along with a section of the pullback diagram $\sigma^* p_{\mathrm{ss}\mathfrak{S}\mathrm{et}} : \sigma^*\mathrm{ss}\mathfrak{S}\mathrm{et}_* \to \Delta[n,l]$. This means we have a bijection of sets

$$(\mathrm{ss}\mathfrak{S}\mathrm{et}_*)_{nl} \cong \coprod_{\sigma \in \mathrm{ss}\mathfrak{S}\mathrm{et}_{nl}} \mathrm{Hom}_{/\Delta[n,l]}(\mathrm{id}, \sigma^* p_{\mathrm{ss}\mathfrak{S}\mathrm{et}}). \tag{20}$$

Under this bijection the morphism $p_{\mathrm{ss}\mathfrak{S}\mathrm{et}} : \mathrm{ss}\mathfrak{S}\mathrm{et}_* \to \mathrm{ss}\mathfrak{S}\mathrm{et}$ is the evident projection of a pair $(\sigma, s) \mapsto \sigma$. Moreover, the naturality of the bijection Corollary 2.17 implies the following helpful result.

5.1. Lemma. *The bijection* $\mathcal{A}\mathrm{ll}(-) \cong \mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, \mathrm{ss}\mathfrak{S}\mathrm{et})$ *is induced by pulling back along* $p_{\mathrm{ss}\mathfrak{S}\mathrm{et}}$.

Now, for every local class of morphisms S, we can obtain $p_{\mathrm{ss}\mathfrak{S}\mathrm{et}^S} : \mathrm{ss}\mathfrak{S}\mathrm{et}^S_* \to \mathrm{ss}\mathfrak{S}\mathrm{et}^S$ by using the bijection Lemma 2.20. This map satisfies the following simple, yet useful, lemma that helps us better understand it.

5.2. Lemma. *Let S be a local class of morphisms of bisimplicial sets. Then we have the following pullback square*

$$\begin{tikzcd} \mathrm{ss}\mathfrak{S}\mathrm{et}^S_* \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{ss}\mathfrak{S}\mathrm{et}_* \arrow[d] \\ \mathrm{ss}\mathfrak{S}\mathrm{et}^S \arrow[r, hook] & \mathrm{ss}\mathfrak{S}\mathrm{et} \end{tikzcd}.$$

In particular, elements in $(\mathrm{ss}\mathfrak{S}\mathrm{et}^S_*)_{nl}$ *are morphisms* $p : X \to \Delta[n,l]$ *that are in S along with a choice of section.*

We can now in particular apply this to $S = \mathcal{LF}\mathrm{ib}$ and deduce that the *universal left fibration*, that we denote by $p_{\mathfrak{S}} : \mathfrak{S}_* \to \mathfrak{S}$, has as elements in $(\mathfrak{S}_*)_{nl}$ diagrams of left fibrations $\Delta[n,l] \to L$ over $\Delta[n,l]$. We now want to study the representability of this left fibration. Recall that a left fibration $L \to W$ is *representable* if there exists an object x in W such that

$$L \simeq W_{x/} = W^{\Delta[1,0]} \times_W \Delta[0,0]. \tag{21}$$

See [Ras23b, Definition 3.41, Theorem 3.44] for a more detailed analysis of representable left fibrations of bisimplicial sets.

5.3. Theorem. *The universal left fibration $p_{\mathfrak{S}} : \mathfrak{S}_* \to \mathfrak{S}$ is a representable left fibration, represented by the terminal object. Moreover, the bijection 7 is induced by pulling back the universal left fibration $p_{\mathfrak{S}}$.*

Proof. The fact that the bijection 7 is induced by pulling back $p_{\mathfrak{S}}$ follows directly from Lemma 5.2 and Lemma 5.1. We now want to prove that $p_{\mathfrak{S}} : \mathfrak{S}_* \to \mathfrak{S}$ is representable and concretely represented by the object $\mathrm{id}_{\Delta[0,0]}$ in $\mathfrak{S}$. By [Ras23b, Theorem 3.55] it suffices to prove that $\mathfrak{S}_*$ has an initial object in the fiber of $p_{\mathfrak{S}}$ over $\mathrm{id}_{\Delta[0,0]}$.

By Theorem 3.9, we have an equivalence $\mathbb{I} : N_\Delta\underline{\mathcal{K}\mathrm{an}} \to \mathfrak{S}$, which is induced by a left fibration $\hat{\mathcal{L}}$ over $\mathcal{N}_\Delta\underline{\mathcal{K}\mathrm{an}}$, which implies that $\mathcal{L} \to N_\Delta\underline{\mathcal{K}\mathrm{an}}$, as constructed in 11, is equivalent to the homotopy pullback of $p_{\mathfrak{S}}$ along the complete Segal space equivalence $N_\Delta\underline{\mathcal{K}\mathrm{an}} \to \mathfrak{S}$. Hence, by [Ras23b, Theorem 4.32], we have a complete Segal space equivalence $\mathcal{L} \simeq \mathfrak{S}_*$ that takes $(\Delta[0],0)$ to $\mathrm{id}_{\Delta[0,0]}$, where we used the fact that by 11, $\mathcal{L}_{00} = \coprod_{X\in\mathcal{K}\mathrm{an}} X_0$, meaning objects in $\mathcal{L}$ are of the form $(X, x \in X_0)$, where X is a Kan complex.

This implies that in order to finish the proof we only need to observe that $(\Delta[0],0)$ is initial in $\mathcal{L}$. By definition $\mathcal{L}_1 = \coprod_{X,Y} X \times \mathrm{Map}_{\underline{\mathcal{K}\mathrm{an}}}(X,Y)$ and so for an object (X,x), by 1 the mapping space $\mathrm{Map}_{\mathcal{L}}((\Delta[0],0),(X,x))$ is given via the following pullback

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{L}}((\Delta[0],0),(X,x)) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\ulcorner", very near start] & \Delta[0] \times \mathrm{Map}_{\underline{\mathcal{K}\mathrm{an}}}(\Delta[0],X) \arrow[d, "\cong"', "\mathrm{ev}"] \\
\Delta[0] \arrow[r, "x"] & X
\end{tikzcd}$$

Now the map on the right hand side is a bijection and so $\mathrm{Map}_{\mathcal{L}}((\Delta[0],0),(X,x))$ is bijective to $\Delta[0]$ as well, finishing the proof. ∎

5.4. Remark. One of the main results regarding left fibrations of bisimplicial sets is that they are always fibrations in the complete Segal space model structure [Ras23b, Corollary 5.11]. Using Theorem 5.3 we can deduce the following result more simply. Indeed, Theorem 5.3 and 21 imply that $\mathfrak{S}_* \simeq \mathfrak{S}^{\Delta[1,0]} \times_{\mathfrak{S}} \Delta[0,0]$ is a complete Segal space as the complete Segal model structure is Cartesian (see [Ras23b, Lemma 3.43] for a more detailed argument). As a result, $p_{\mathfrak{S}}$ is a Reedy fibration between complete Segal spaces and so a complete Segal fibration [Rez01, Theorem 7.2]. Now, by Theorem 5.3, every left fibration is a pullback of the complete Segal space fibration $\mathfrak{S}_* \to \mathfrak{S}$ and so a complete Segal space fibration as well.

We now want to generalize Theorem 5.3 to trisimplicial sets. Let $p_{\text{sss}\mathfrak{S}\text{et}} : \text{sss}\mathfrak{S}\text{et}_* \to \text{sss}\mathfrak{S}\text{et}$ be the map that corresponds to the identity map under the bijection 14. First, we want to generalize 20.

5.5. Lemma. *There is a bijection*

$$(\text{sss}\mathfrak{S}\text{et}_*)_{knl} \cong \coprod_{\sigma \in \text{sss}\mathfrak{S}\text{et}_{knl}} \text{Hom}_{/\Delta[k,n,l]}(\Delta[k,n,l], \sigma^* p_{\text{sss}\mathfrak{S}\text{et}}),$$

meaning an element in $(\text{sss}\mathfrak{S}\text{et}_*)_{knl}$ *corresponds to a choice of trisimplicial set* $X \to \Delta[n,l]$ *along with a choice of section for the map of bisimplicial sets* $X_k \to \Delta[n,l]$*.*

Proof. An element in $(\text{sss}\mathcal{S}\text{et}_*)_{knl}$ is given by a map $\Delta[k,n,l] \to \text{sss}\mathcal{S}\text{et}_*$, which is given by a map $\sigma : \Delta[k,n,l] \to \text{sss}\mathfrak{S}\text{et}$ along with a lift. By 14 an element in $\text{sss}\mathfrak{S}\text{et}_{knl}$ is given by a map of trisimplicial sets $\sigma^* p_{\text{sss}\mathfrak{S}\text{et}} \to \Delta[0,n,l]$. Now for such a fixed map, a section precisely corresponds to a choice of element in $(\sigma^* p_{\text{sss}\mathfrak{S}\text{et}})_{knl}$, which is precisely the data of a section $(\sigma^* p_{\text{sss}\mathfrak{S}\text{et}})_k \to \Delta[n,l]$. ■

This also has the following useful implication.

5.6. Lemma. *The bijection* $\text{s}\mathcal{A}\text{ll}(-) \cong \text{Hom}_{\text{ss}\mathcal{S}\text{et}}(-, \text{sss}\mathfrak{S}\text{et})$ *is induced by pulling back along* $p_{\text{sss}\mathfrak{S}\text{et}}$*.*

Now, for every local class of morphisms S, we can obtain $p_{\text{sss}\mathfrak{S}\text{et}^S} : \text{sss}\mathfrak{S}\text{et}_*^S \to \text{sss}\mathfrak{S}\text{et}^S$ by using the bijection 15 and again we have a result analogous to Lemma 5.2.

5.7. Lemma. *Let S be a local class of morphisms of bisimplicial sets. Then we have the following pullback square*

$$\begin{tikzcd} \text{sss}\mathfrak{S}\text{et}_*^S \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \text{sss}\mathfrak{S}\text{et}_* \arrow[d] \\ \text{sss}\mathfrak{S}\text{et}^S \arrow[r, hook] & \text{sss}\mathfrak{S}\text{et} \end{tikzcd}.$$

In particular, elements in $(\text{sss}\mathfrak{S}\text{et}_*^S)_{knl}$ *are morphisms* $p : X \to \Delta[n,l]$ *that are in S along with a choice of section of* $p_k : X_k \to \Delta[n,l]$*.*

We can now apply this lemma to the case $S = \mathcal{R}\text{ee}\mathcal{LF}\text{ib}$ to obtain the universal Reedy left fibration $p_{\mathcal{R}\text{ee}\mathcal{LF}\text{ib}} : s\mathfrak{S}_* \to s\mathfrak{S}$. We want to observe that this Reedy left fibration is representable, in the sense of [Ras22]. Recall from [Ras22, Proposition 2.4] that for a cosimplicial object $X^\bullet : \mathbb{\Delta} \to W$ in a complete Segal space W, we can construct a Reedy left fibration $W_{/X^\bullet} \to W$ uniquely characterized by the fact that $(W_{/X^\bullet})_k \simeq W_{/X_k}$ (where $W_{/X_k}$ was defined in 21). Now, following [Ras22, Definition 2.1], an arbitrary Reedy left fibration $L \to W$ is *representable* if there exists a cosimplicial object $X^\bullet$ in W and an equivalence $W_{/X^\bullet} \simeq L$ over W. We now prove that the universal Reedy left fibration $p_{\mathcal{R}\text{ee}\mathcal{LF}\text{ib}} : s\mathfrak{S}_* \to s\mathfrak{S}$ is representable via a cosimplicial object in $s\mathfrak{S}$.

5.8. Theorem. *The universal Reedy left fibration* $p_{s\mathfrak{S}} : s\mathfrak{S}_* \to s\mathfrak{S}$ *is a representable Reedy left fibration represented by the cosimplicial object* $\mathrm{id}_{\Delta[0,\bullet,0]} : \mathbb{\Delta} \to s\mathfrak{S}$*. Moreover, the bijection 16 is induced by pulling back the universal Reedy left fibration* $p_{s\mathfrak{S}}$*.*

Proof. The fact that the bijection is induced by pullback follows from combining Lemma 5.7 and Lemma 5.6. Now, in order to prove that $p_{s\mathfrak{S}} : s\mathfrak{S}_* \to s\mathfrak{S}$ is representable, by [Ras22, Lemma 4.1], it suffices to prove that the left fibration $(p_{s\mathfrak{S}})_k : (s\mathfrak{S}_*)_k \to s\mathfrak{S}$ is represented by $\mathrm{id}_{\Delta[0,k,0]}$ for all $k \geq 0$. From here on we can follow the steps of the proof of Theorem 5.3.

We need to show that $(s\mathfrak{S}_*)_k$ has an initial object over $\mathrm{id}_{\Delta[0,k,0]}$. By Theorem 4.13, we have an equivalence $s\mathbb{I} : N_\Delta\underline{\mathcal{R}\mathrm{ee}} \to s\mathfrak{S}$, which is induced by the Reedy left fibration $\hat{\mathrm{s}\mathcal{L}}$ over $N_\Delta\underline{\mathcal{R}\mathrm{ee}}$, which implies that the Reedy left fibration $\mathrm{s}\mathcal{L} \to N_\Delta\underline{\mathcal{R}\mathrm{ee}}$, as constructed in 19, is the homotopy pullback of the Reedy left fibration $s\mathfrak{S}_* \to s\mathfrak{S}$, and, in particular, the left fibration $\hat{\mathrm{s}\mathcal{L}}_k \to N_\Delta\underline{\mathcal{R}\mathrm{ee}}$ is the homotopy pullback of the left fibration $(s\mathfrak{S}_*)_k \to s\mathfrak{S}$. Hence, by [Ras23b, Theorem 4.32], we have a complete Segal space equivalence $\mathrm{s}\mathcal{L}_k \simeq (s\mathfrak{S}_*)_k$ that takes $(\Delta[k,0],\mathrm{id}_k)$ to $\mathrm{id}_{\Delta[0,k,0]}$. Here we used the fact that by 19, $\mathrm{s}\mathcal{L}_{k00} = \coprod_{X\in\mathcal{R}\mathrm{ee}} X_{k0}$, meaning objects in $\mathrm{s}\mathcal{L}_k$ are of the form $(X, x \in X_{k0})$, where X is a Reedy fibrant simplicial space.

This implies that in order to finish the proof we only need to observe that $(\Delta[k,0],\mathrm{id}_k)$ is initial in $\mathrm{s}\mathcal{L}_k$. By definition $\mathrm{s}\mathcal{L}_{k1} = \coprod_{X,Y\in\mathcal{R}\mathrm{ee}} X_k \times \mathrm{Map}_{\underline{\mathcal{R}\mathrm{ee}}}(X,Y)$ and so for an object $(X, x \in X_{k0})$, by 1 the mapping space $\mathrm{Map}_{\mathrm{s}\mathcal{L}_k}((\Delta[k,0],\mathrm{id}_k),(X,x))$ is given via the following pullback

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{s}\mathcal{L}_k}((\Delta[k,0],\mathrm{id}_k),(X,x)) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\ulcorner", very near start] & \Delta[k,0]_k \times \mathrm{Map}_{\underline{\mathcal{R}\mathrm{ee}}}(\Delta[k,0],X) \arrow[d, "\mathrm{ev}", "\cong"'] \\
\Delta[0] \arrow[r, "(\mathrm{id}_k,x)"] & \Delta[k,0]_k \times X_k
\end{tikzcd}$$

Now, the map on the right hand side is a bijection, as $\mathrm{Map}_{\underline{\mathcal{R}\mathrm{ee}}}(\Delta[k,0],X) \cong X_k$ by 18. As a result, the mapping space $\mathrm{Map}_{\mathrm{s}\mathcal{L}_k}((\Delta[k,0],\mathrm{id}_k),(X,x))$ is bijective to $\Delta[0]$ as well, finishing the proof. ∎

We now use Theorem 4.16 to get the universal Segal coCartesian fibration, that we denote by $p_{\mathfrak{S}\mathrm{eg}} : \mathfrak{S}\mathrm{eg}_* \to \mathfrak{S}\mathrm{eg}$ and the universal coCartesian fibration, that we denote by $p_{\mathfrak{CSS}} : \mathfrak{CSS}_* \to \mathfrak{CSS}$. Lemma 5.7 and Theorem 5.8 now immediately give the following result.

5.9. Corollary. *The universal (Segal) coCartesian fibration* $p_{\mathfrak{CSS}}$ $(p_{\mathfrak{S}\mathrm{eg}})$ *is represented by* $\Delta[\bullet,0]$*. Moreover, we have pullback squares*

$$\begin{tikzcd}
\mathfrak{CSS}_* \arrow[r, hook] \arrow[d, "p_{\mathfrak{CSS}}"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathfrak{S}\mathrm{eg}_* \arrow[r, hook] \arrow[d, "p_{\mathfrak{S}\mathrm{eg}}"] \arrow[dr, phantom, "\ulcorner", very near start] & s\mathfrak{S}_* \arrow[d, "p_{s\mathfrak{S}}"] \\
\mathfrak{CSS} \arrow[r, hook] & \mathfrak{S}\mathrm{eg} \arrow[r, hook] & s\mathfrak{S}
\end{tikzcd} .$$

Finally, pulling back along the universal fibrations induces bijections given in Theorem 4.16

$$\mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, \mathfrak{S}\mathrm{eg}) \cong \mathcal{S}\mathrm{egco}\mathcal{C}\mathrm{art}(-),$$

$$\mathrm{Hom}_{\mathrm{ss}\mathcal{S}\mathrm{et}}(-, \mathfrak{CSS}) \cong \mathrm{co}\mathcal{C}\mathrm{art}(-).$$

5.10. Remark. The representability of the universal left fibration is well-established (and has for example been studied in [Cis19, Subsection 5.2],) however, the representability of the the universal coCartesian fibration is a more modern phenomena and can be found in [Ras22, Subsection 4.2], [Ste20, Example 3.26], and more recently [CN22]. The representability of the universal Reedy left fibration and the universal Segal coCartesian fibration was not studied before.

6. Comparison with Quasi-Categories

Up until here we constructed the complete Segal spaces of spaces, Reedy fibrant simplicial spaces, and (complete) Segal spaces. In this last section we want to use the fact that we can translate between complete Segal spaces and quasi-categories, another important model of $(\infty, 1)$-categories, to construct quasi-categories of spaces, Reedy fibrant simplicial spaces, and (complete) Segal spaces. This requires us to review left fibrations of simplicial sets [Joy08b, Lur09, Cis19], as well as the translation results between quasi-categories and complete Segal spaces due to Joyal and Tierney [JT07], their generalization to left fibrations in [Ras23b, Appendix B], and their generalization to Reedy left fibrations [Ras21a, Section 1.6].

6.1. Definition. *A left fibration of simplicial sets is map that satisfies the right lifting property with respect to horn inclusions $\Lambda[n]_i \hookrightarrow \Delta[n]$, for $0 \leq i < n$.*

Left fibrations of simplicial sets can be translated to left fibrations of bisimplicial sets (Definition 2.7) and vice versa. Let $i_1^* : \mathrm{ss}\mathcal{S}\mathrm{et} \to \mathrm{s}\mathcal{S}\mathrm{et}$ be the functor that takes a bisimplicial set $X_{\bullet\bullet}$ to the simplicial set $X_{\bullet 0}$ [JT07, Section 4]. Moreover, let $t^! : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathrm{ss}\mathcal{S}\mathrm{et}$ be the functor that takes a simplicial set X to the bisimplicial set $t^! X_{nl} = \mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}(\Delta[n] \times N(I[l]), X)$ [JT07, Theorem 2.12]. Here $I[l]$ is the groupoid with $l + 1$ objects and a unique morphism between any two objects.

These two functors are both right adjoints of Quillen equivalences [JT07, Theorem 4.11, Theorem 4.12], which in particular has the following implications:

6.2. Lemma. *i_1^* takes complete Segal spaces to quasi-categories and $t^!$ takes quasi-categories to complete Segal spaces. Moreover, $i_1^* t^! : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathrm{s}\mathcal{S}\mathrm{et}$ is the identity map and $t^! i_1^* : \mathrm{ss}\mathcal{S}\mathrm{et} \to \mathrm{ss}\mathcal{S}\mathrm{et}$ is equivalent to the identity. Finally, i_1^* preserves and reflects equivalences between complete Segal spaces.*

These results have been generalized in [Ras23b, Theorem B.12, Theorem B.14] to a comparison between left fibrations of simplicial sets and bisimplicial sets, giving us the following valuable result.

6.3. Lemma. *i_1^* takes left fibrations of bisimplicial sets to left fibrations of simplicial sets and $t^!$ takes left fibrations of simplicial sets to left fibrations of bisimplicial sets.*

We now use the ability to translate between quasi-categories and complete Segal spaces to construct additional $(\infty,1)$-categories of spaces. First of all, we can apply i_1^* to the complete Segal space $\mathfrak{S}$ (Theorem 3.9) to obtain the following result.

6.4. Corollary. *$i_1^*\mathfrak{S}$ is a quasi-category of spaces with $i_1^*\mathfrak{S}_n$ given by left fibrations of bisimplicial sets over $\Delta[n,0]$.*

We now want to illustrate how we can use left fibrations of simplicial sets internally to construct a quasi-category of spaces, using analogous steps to Section 3. Let $\mathfrak{S}_{\mathfrak{QCat}}$ be the simplicial set with $(\mathfrak{S}_{\mathfrak{QCat}})_n$ given by left fibrations of simplicial sets over $\Delta[n]$ (where we are using the translation to functors as given in Lemma 2.15 to take care of functoriality).

Now, by Lemma 6.3, $t^!$ preserves left fibrations, and moreover, we have $t^!(\Delta[n]) = \Delta[n,0]$. Hence $t^!$ induces a morphism of quasi-categories

$$\mathfrak{T} : \mathfrak{S}_{\mathfrak{QCat}} \to i_1^*\mathfrak{S}, \tag{22}$$

that takes a left fibration of simplicial sets $L \to \Delta[n]$ to $t^!L \to t^!\Delta[n] = \Delta[n,0]$. Similarly, by Lemma 6.3, i_1^* also preserves left fibrations and $i_1^*(\Delta[n,0]) = \Delta[n]$ and so i_1^* similarly induces a morphism of quasi-categories

$$\mathfrak{I} : i_1^*\mathfrak{S} \to \mathfrak{S}_{\mathfrak{QCat}}, \tag{23}$$

that takes a left fibration of bisimplicial sets $L \to \Delta[n,0]$ to $i_1^*(L) \to i_1^*(\Delta[n,0]) = \Delta[n]$. We now have the following result.

6.5. Theorem. *The maps $\mathfrak{T} : \mathfrak{S}_{\mathfrak{QCat}} \to i_1^*\mathfrak{S}$ and $\mathfrak{I} : i_1^*\mathfrak{S} \to \mathfrak{S}_{\mathfrak{QCat}}$ are inverses of quasi-categories.*

Proof. First, we prove $\mathfrak{S}_{\mathfrak{QCat}}$ is a quasi-category. By Lemma 6.2, $t^! \circ i_1^*$ is the identity and so $\mathfrak{I}\mathfrak{T}$ is the identity as well, meaning $\mathfrak{S}_{\mathfrak{QCat}}$ is a retract of the quasi-category $i_1^*\mathfrak{S}$ (Corollary 6.4) and so a quasi-category as well. We now move on to prove $\mathfrak{T}$ and $\mathfrak{I}$ are inverses of quasi-categories. As $\mathfrak{I}\mathfrak{T}$ is the identity we only need to show $\mathfrak{T}\mathfrak{I} : i_1^*\mathfrak{S} \to i_1^*\mathfrak{S}_{\mathcal{CSS}}$ is equivalent to the identity. By Lemma 6.2 and Lemma 3.4, the two morphisms $i_1^*\mathfrak{S}^{min} \xrightarrow{i_1^*\mathfrak{I}} i_1^*\mathfrak{S} \xrightarrow{i_1^*\mathfrak{Min}} i_1^*\mathfrak{S}^{min}$ are equivalences of quasi-categories, hence it suffices to prove that $i_1^*\mathfrak{I} \circ \mathfrak{T} \circ \mathfrak{I} \circ (i_1^*\mathfrak{Min}) : i_1^*\mathfrak{S}^{min} \to i_1^*\mathfrak{S}^{min}$ is equal to the identity.

Let $L \to \Delta[n,0]$ be a left fibration. By Lemma 6.2, there is an equivalence of complete Segal spaces $t^!i_1^*L \to L$ over $\Delta[n,0]$, which implies they are equivalent left fibrations [Ras23b, Theorem 5.11] and so $\mathcal{M}\mathrm{in}(t^!i_1^*L)$ and $\mathcal{M}\mathrm{in}(L)$ are equal (Lemma 2.29) finishing the proof. ∎

6.6. Remark. The elements in the quasi-category $\mathfrak{S}_{\mathcal{QC}\mathrm{at}}$ are precisely left fibrations over $\Delta[n]$. Hence this construction coincides with the construction of the quasi-category of spaces by Cisinski [Cis19, Theorem 5.2.10, Corollary 5.4.7] and hence gives us an independent proof thereof.

We can take the opposite route to Corollary 6.4 to get the following result.

6.7. Corollary. *$t^!\mathfrak{S}_{\mathcal{QC}\mathrm{at}}$ is a complete Segal space of spaces with $(t^!\mathfrak{S}_{\mathcal{QC}\mathrm{at}})_{nl}$ given by left fibrations of simplicial sets over $\Delta[n] \times N(I[l])$.*

We can now use the results from Section 4, and more precisely apply Lemma 6.2 to Theorem 4.16, to construct quasi-categories of Reedy fibrant simplicial spaces and (complete) Segal spaces.

6.8. Corollary. *$i_1^*s\mathfrak{S}$ is a quasi-category with $i_1^*s\mathfrak{S}_n$ given by Reedy left fibrations of trisimplicial sets over $\Delta[0,n,0]$. Moreover, we have inclusions of quasi-categories*

$$i_1^*\mathfrak{CSS} \hookrightarrow i_1^*\mathfrak{S}\mathrm{eg} \hookrightarrow i_1^*s\mathfrak{S},$$

where $i_1^*\mathfrak{S}\mathrm{eg}$ ($i_1^*\mathfrak{CSS}$) is the quasi-category of (complete) Segal spaces.

We now want to generalize the results in Theorem 6.5 to define the quasi-categorical analogue to $s\mathfrak{S}$, which will then also result in the quasi-categorical analogue to $\mathfrak{S}\mathrm{eg}$ and $\mathfrak{CSS}$. Here we fundamentally rely on work in [Ras21a, Section 1.6]. Recall that a Reedy fibration of bisimplicial sets $L \to \Delta[0,n]$ is a Reedy left fibration of bisimplicial sets if for all $k \geq 0$ the restricted map $L_k \to \Delta[0,n]_k = \Delta[n]$ (Notation 2.3) is a left fibration of simplicial sets. Let $s\mathfrak{S}_{\mathcal{QC}\mathrm{at}}$ be the simplicial set with $(\mathfrak{S}_{\mathcal{QC}\mathrm{at}})_n$ given by Reedy left fibrations of bisimplicial sets over $\Delta[0,n]$ (where we are using the translation to functors as given in Lemma 2.15 to take care of functoriality).

We want to prove that $s\mathfrak{S}_{\mathcal{QC}\mathrm{at}}$ and $i_1^*s\mathfrak{S}$ are equivalent quasi-categories. For that we define analogues to $\mathfrak{T}, \mathfrak{I}$ that we prove are inverses of each other. Let $st^! : \mathrm{ss}\mathcal{S}\mathrm{et} \to \mathrm{sss}\mathcal{S}\mathrm{et}$ be defined as $\mathrm{Fun}(\mathbb{\Delta}^{op}, t^!)$ and similarly, let $si_1^* = \mathrm{Fun}(\mathbb{\Delta}^{op}, i_1^*) : \mathrm{ss}\mathcal{S}\mathrm{et} \to \mathrm{sss}\mathcal{S}\mathrm{et}$. We now have the following facts about si_1^* and $st^!$.

6.9. Lemma. *si_1^* takes Reedy left fibrations of trisimplicial sets to Reedy left fibrations of bisimplicial sets and $st^!$ takes Reedy left fibrations of bisimplicial sets to Reedy left fibrations of trisimplicial sets. Moreover, $si_1^*st^!$ is the identity and $st^!si_1^*$ takes a Reedy left fibration of trisimplicial sets to an equivalent one.*

See [Ras21a, Theorem 1.35] for further details about these functors. Using this result we can generalize the maps of quasi-categories $\mathfrak{T}, \mathfrak{I}$ to maps

$$\begin{aligned} \mathrm{s}\mathfrak{T} &: s\mathfrak{S}_{\mathcal{QC}\mathrm{at}} \to i_1^*s\mathfrak{S}, \\ \mathrm{s}\mathfrak{I} &: i_1^*s\mathfrak{S} \to s\mathfrak{S}_{\mathcal{QC}\mathrm{at}}, \end{aligned} \tag{24}$$

where $\mathrm{s}\mathfrak{T}$ takes a Reedy left fibration of bisimplicial sets $L \to \Delta[0,n]$ to $st^!L \to st^!\Delta[0,n] = \Delta[0,n,0]$ and $\mathrm{s}\mathfrak{I}$ takes a Reedy left fibration of trisimplicial sets $L \to \Delta[0,n,0]$ to

$si_1^*(L) \to si_1^*(\Delta[0,n,0]) = \Delta[0,n]$. Now following the same steps of the proof of Theorem 6.5 (this time with the construction of minimal Reedy left fibrations given in 17) gives us the following result.

6.10. THEOREM. *The maps* $\mathrm{s}\mathfrak{T} : s\mathfrak{S}_{\mathfrak{QCat}} \to i_1^* s\mathfrak{S}$ *and* $\mathrm{s}\mathfrak{I} : i_1^* s\mathfrak{S} \to s\mathfrak{S}_{\mathfrak{QCat}}$ *are inverses of quasi-categories.*

We now proceed to restrict the results to (Segal) coCartesian fibrations. Following [Ras21a, Definition 1.34] a Reedy left fibration of bisimplicial sets $p : L \to \Delta[0,n]$ is a Segal coCartesian fibration if the map $L_k \to L_1 \times_{L_0} ... \times_{L_0} L_1$ is an equivalence of left fibrations for all $k \geq 2$. Moreover, it is a *coCartesian fibration* if it is a Segal coCartesian fibration and the map $L_0 \to L_3 \times_{(L_1 \times L_1)} (L_0 \times L_0)$ is an equivalence of left fibrations. The similarity with the conditions in Definition 4.3 is of course not coincidental and, by [Ras21a, Theorem 1.35], both $st^!$ and si_1^* preserve and reflect (Segal) coCartesian fibrations.

Let $\mathfrak{Seg}_{\mathfrak{QCat}} \hookrightarrow s\mathfrak{S}_{\mathfrak{QCat}}$ denote the full sub-simplicial set consisting of Segal coCartesian fibrations over $\Delta[0,n]$. Similarly, let $\mathfrak{CSS}_{\mathfrak{QCat}} \hookrightarrow \mathfrak{Seg}_{\mathfrak{QCat}}$ denote the full sub-simplicial set consisting of coCartesian fibrations over $\Delta[0,n]$. The fact that $st^!$ and si_1^* preserve and reflect Segal coCartesian fibrations implies that the morphisms of quasi-categories $\mathrm{s}\mathfrak{T}, \mathrm{s}\mathfrak{I}$ restrict to morphisms

$$\mathrm{s}\mathfrak{T} : \mathfrak{Seg}_{\mathfrak{QCat}} \to i_1^*\mathfrak{Seg}, \quad \mathrm{s}\mathfrak{I} : si_1^*\mathfrak{Seg} \to \mathfrak{Seg}_{\mathfrak{QCat}}$$

and remain equivalences after restriction. We have the same equivalences

$$\mathrm{s}\mathfrak{T} : \mathfrak{CSS}_{\mathfrak{QCat}} \to si_1^*\mathfrak{CSS}, \quad \mathrm{s}\mathfrak{I} : i_1^*\mathfrak{CSS} \to \mathfrak{CSS}_{\mathfrak{QCat}},$$

due to the fact that coCartesian fibrations are preserved and reflected by $st^!$ and si_1^*.

Combining these observations gives us the following corollary.

6.11. COROLLARY. *We have the following diagram of quasi-categories, where the horizontal maps are inclusions and the vertical maps equivalences.*

$$\begin{tikzcd}
i_1^*\mathfrak{CSS} \arrow[r, hook] \arrow[d, "\simeq"', "\mathrm{s}\mathfrak{I}"] & i_1^*\mathfrak{Seg} \arrow[r, hook] \arrow[d, "\simeq"', "\mathrm{s}\mathfrak{I}"] & i_1^* s\mathfrak{S} \arrow[d, "\simeq"', "\mathrm{s}\mathfrak{I}"] \\
\mathfrak{CSS}_{\mathfrak{QCat}} \arrow[r, hook] \arrow[d, "\simeq"', "\mathrm{s}\mathfrak{T}"] & \mathfrak{Seg}_{\mathfrak{QCat}} \arrow[r, hook] \arrow[d, "\simeq"', "\mathrm{s}\mathfrak{T}"] & s\mathfrak{S}_{\mathfrak{QCat}} \arrow[d, "\simeq"', "\mathrm{s}\mathfrak{T}"] \\
i_1^*\mathfrak{CSS} \arrow[r, hook] & i_1^*\mathfrak{Seg} \arrow[r, hook] & i_1^* s\mathfrak{S}
\end{tikzcd}$$

Finally, we have the following result analogous to Corollary 6.7.

6.12. COROLLARY. $t^!\mathfrak{CSS}_{\mathfrak{QCat}}$ *is a complete Segal space of complete Segal spaces with* $(t^!\mathfrak{CSS}_{\mathfrak{QCat}})_{nl}$ *given by coCartesian fibrations of bisimplicial sets over* $\Delta[0,n] \times N(I[l])$*.*

A. Minimal Reedy Fibrations

As part of our effort to construct and study the complete Segal spaces of interest, we, in particular, need a theory of minimal fibrations in a variety of settings (examples include Proposition 2.26 and Proposition 4.8). Hence, in this appendix we develop a theory of minimal Reedy fibrations for a wide range of Reedy categories. In particular, we prove that every equivalence between two minimal Reedy fibrations is an isomorphism (Proposition A.6), and every Reedy fibration on an elegant Reedy category (which contains all examples of interest by Example A.9) admits a unique factorization into a trivial fibration followed by a minimal fibration (Theorem A.11). The content of this technical section does not rely on the contents of the remainder of this paper and hence can be read independently.

Recall that a Reedy category is a quadruple $(\mathcal{R}, \deg : \mathrm{Obj}_{\mathcal{R}} \to \mathbb{N}, \mathcal{R}^-, \mathcal{R}^+)$ of a category $\mathcal{R}$, the degree function deg, a wide subcategory $\mathcal{R}^-$ in which all non-identity morphisms decrease the degree, a wide subcategory $\mathcal{R}^+$ in which all non-identity morphisms increase the degree, and in which $(\mathcal{R}^-, \mathcal{R}^+)$ is a factorization system, meaning every map $f : r \to r''$ in $\mathcal{R}$ admits a unique factorization $r \xrightarrow{g} r' \xrightarrow{h} r''$, with g in $\mathcal{R}^-$ and h in $\mathcal{R}^+$. For further details, see the original source due to Reedy [Ree74] or more modern sources, such as [Hov99, Definition 5.2.1] or [Hir03, Definition 15.1.2]. To simplify notation for a given morphism in $\mathcal{R}$ between objects r, r' we adopt the notation $r \to^- r'$ for a non-identity morphism in $\mathcal{R}^-$ and $r \to^+ r'$ for a non-identity morphism in $\mathcal{R}^+$.

For a given object r in the Reedy category $\mathcal{R}$ denote by $\partial(\mathcal{R}^+_{/r})$ $(\partial(\mathcal{R}^-_{r/}))$ the full subcategory of $\mathcal{R}^+_{/r}$ $(\mathcal{R}^-_{r/})$ only lacking the identity. Recall, for a given functor $X : \mathcal{R}^{op} \to \mathcal{S}\mathrm{et}$, we have *latching objects* $L_rX = \mathrm{colim}((\partial(\mathcal{R}^-_{r/}))^{op} \to (\mathcal{R}^-)^{op} \to \mathcal{S}\mathrm{et})$ and *matching objects* $M_rX = \lim((\partial(\mathcal{R}^+_{/r}))^{op} \to (\mathcal{R}^+)^{op} \to \mathcal{S}\mathrm{et})$, which come with canonical natural morphisms $L_rX \to X_r \to M_rX$. See [Hov99, Definition 5.2.2] or [Hir03, Definition 15.2.3] for more details. For a given presheaf $X : \mathcal{R}^{op} \to \mathcal{S}\mathrm{et}$ and element x in X_r, we denote the image in M_rX by ∂x.

For a given object r, denote by $F[r] : \mathcal{R}^{op} \to \mathcal{S}\mathrm{et}$ the presheaf representing r. Recall that for a given object r in $\mathcal{R}$ we define $\partial F[r] = \mathrm{colim}_{r_0 \to^+ r} F[r_0]$ and for every $X : \mathcal{R}^{op} \to \mathcal{C}$, we have the bijections

$$M_rX = \lim_{r_0 \to^+ r} X_{r_0} \cong \lim_{r_0 \to^+ r} \mathrm{Hom}(F[r_0], X) \cong \mathrm{Hom}(\mathrm{colim}_{r_0 \to^+ r} F[r_0], X) = \mathrm{Hom}(\partial F[r], X). \tag{25}$$

We can characterize $\partial F[r]$ more formally. For $n \geq 0$, denote by $\mathcal{R}_{\leq n}$ the full subcategory of $\mathcal{R}$ with objects having degree less than or equal to n . The evident inclusion $\mathcal{R}_{\leq n} \to \mathcal{R}$ induces by restriction and left Kan extension an adjunction

$$\underline{\mathrm{Fun}}(\mathcal{R}^{op}_{\leq n}, \mathcal{S}\mathrm{et}) \underset{tr_n}{\overset{sk_n}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \underline{\mathrm{Fun}}(\mathcal{R}^{op}, \mathcal{S}\mathrm{et})$$

A.1. Remark. We make the following observations about the adjunction (sk_n, tr_n).

- Both sk_n and tr_n preserve colimits, as tr_n also has a right adjoint via right Kan extension.

- The inclusion $\mathcal{R}_{\leq n} \to \mathcal{R}$ is fully faithful, and so the left Kan extension sk_n is also fully faithful [Rie16, Corollary 6.3.9].
- For every object X in $\underline{\mathrm{Fun}}(\mathcal{R}^{op}, \mathcal{S}\mathrm{et})$ we have a colimit diagram $\mathrm{colim}_{n \to +\infty} sk_n tr_n X \cong X$ [Hir03, Proposition 15.1.25].
- For an object r in $\mathcal{R}$ with degree $n+1$, by construction the inclusion $\partial F[r] \to F[r]$ corresponds to the unit map $sk_n tr_n F[r] \to F[r]$.

For a given Reedy category $\mathcal{R}$, the category $\underline{\mathrm{Fun}}(\mathcal{R}^{op}, \mathrm{s}\mathcal{S}\mathrm{et})$ admits a model structure, the *Reedy model structure*, in which a morphism $Y \to X$ is a fibration if for all objects r in $\mathcal{R}$, the map $Y_r \to X_r \times_{M_r X} M_r Y$ is a Kan fibration of simplicial sets and it is a weak equivalence if the map $Y_r \to X_r$ is a Kan equivalence for all objects r in $\mathcal{R}$ [Hir03, Definition 15.3.3, Theorem 15.3.4]. We aim to generalize minimal fibrations from Kan fibrations to Reedy fibrations. We commence with the definition of a minimal Reedy fibration. Recall that minimal Kan fibrations were introduced with the aim of studying key properties of Kan fibrations and their relation to Serre fibrations [Qui68]. We will primarily rely on [GJ09, Section I.10].

A.2. Definition. *A Reedy fibration $Y \to X$ is minimal, if for all objects r in $\mathcal{R}$ the Kan fibration $Y_r \to X_r \times_{M_r X} M_r Y$ is a minimal Kan fibration.*

Before we proceed with a thorough study of minimal Reedy fibrations, let us note that there are several other generalizations of minimal Kan fibrations. First of all, there is a notion of minimal inner fibrations and particularly minimal quasi-categories [Lur09, Section 2.3.3], still working in the context of simplicial sets. There is also a generalization to dendroidal sets [MN16]. Closest to our approach is a generalization to Cisinski model structures on Eilenberg-Zilber Reedy categories [Cis14].

We commence our study by giving an alternative characterization in analogy to [GJ09, Diagram 10.2]. Recall we denote the constant presheaf $\mathcal{R}^{op} \to \mathrm{s}\mathcal{S}\mathrm{et}$ with value $\Delta[n]$ also by $\Delta[n]$. Moreover, for an object r in $\mathcal{R}$ and $n \geq 0$, to simplify notation we use the notation $F[r,n]$ for the presheaf $F[r] \times \Delta[n]$.

A.3. Definition. *Let $\mathcal{R}$ be a Reedy category and $p : Y \to X$ be a morphism in* $\mathrm{Fun}(\mathcal{R}^{op}, \mathrm{s}\mathcal{S}\mathrm{et})$. *Two elements $f, g : F[r,n] \to Y$ are p-equivalent if there exists a commutative diagram of the following form*

$$\begin{tikzcd}
 & \partial F[r,n] \times \Delta[1] \arrow[r] \arrow[d] & \partial F[r,n] \arrow[d] \\
F[r,n] \arrow[r, shift left, "\{0\}"] \arrow[r, shift right, "\{1\}"'] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & F[r,n] \times \Delta[1] \arrow[r] \arrow[d] & Y \arrow[d, two heads] \\
 & F[r,n] \arrow[r] & X
\end{tikzcd} ,$$

A.4. Lemma. *Let $\mathcal{R}$ be a Reedy category. Then a Reedy fibration $p : Y \to X$ is a minimal Reedy fibration if and only if all p-equivalent (r, n)-simplices are equal.*

Proof. By definition $p : Y \to X$ is a minimal fibration if for all objects r in $\mathcal{R}$, $Y_r \to M_rY \times_{M_rX} X_r$ is a minimal fibration. This is equivalent to stating that for every r, every $n \geq 0$ and every $f, g : F[n, r] \to Y$, $f = g$ if there exists a commutative diagram of the following shape

$$\begin{tikzcd}
& \partial\Delta[n] \times \Delta[1] \arrow[r] \arrow[d] & \partial\Delta[n] \arrow[d] \\
\Delta[n] \arrow[r, shift left, "\{0\}"] \arrow[r, shift right, "\{1\}"'] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & \Delta[n] \times \Delta[1] \arrow[r] \arrow[d] & Y_r \arrow[d, two heads] \\
& \Delta[n] \arrow[r] & X_r \times_{M_rX} M_rY
\end{tikzcd} .$$

By 25, we have $M_rY \cong \mathrm{Hom}(\partial F[r], Y)$. So, using the fact that, by [Hir03, Lemma 15.5.1], we have $\partial F[r, n] = F[r] \times \partial\Delta[n] \coprod_{\partial F[r] \times \partial\Delta[n]} \partial F[r] \times \Delta[n]$ and $\mathrm{Hom}(\partial\Delta[n], Y_r) \cong \mathrm{Hom}(F[r] \times \partial\Delta[n], Y)$, it follows from direct inspection that the data of this commutative diagram coincides with the data of the commutative diagram in Definition A.3. ■

Let us establish elementary properties of minimal Reedy fibrations.

A.5. Lemma. *Minimal Reedy fibrations are stable under pullback. Moreover, minimal Reedy fibrations are local in the sense of Lemma 2.19, meaning a Reedy fibration $p : Y \to X$ is minimal if and only if for all morphisms $h : F[r, n] \to X$ the pullback $h^*p : h^*Y \to F[r, n]$ is minimal.*

Proof. Let us assume we have a pullback square

$$\begin{tikzcd}
h^*Y \arrow[r, "j"] \arrow[d, two heads, "h^*p"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, two heads, "p"] \\
Z \arrow[r, "h"] & X
\end{tikzcd} ,$$

where p is minimal. It follows immediately that h^*p is a Reedy fibration and so we only need to establish the minimality property. Let $f, g : F[r, n] \to h^*Y$ be two h^*p-equivalent (r, n)-simplices in h^*Y. Then, jf, jg are p-equivalent (r, n)-simplices in Y and so equal as $p : Y \to X$ is a minimal fibration. On the other hand, $h^*p(f), h^*p(g)$ are equal by assumption of being $h^*(p)$-equivalent. Hence, by the universal property of pullbacks $f = g$ and so $h^*p : h^*Y \to Z$ is minimal.

Now, let us assume $p : Y \to X$ is a Reedy fibration satisfying the locality assumption stated in the lemma. Let $f, g : F[r, n] \to Y$ be two p-equivalent (r, n)-simplices in Y. This means that by assumption there exists a morphism $F[r, n] \times \Delta[1] \to Y$, such that its projection via $p : Y \to X$ factors through a morphism $h : F[r, n] \to X$. This induces the following commutative diagram

$$\begin{tikzcd}
 & \partial F[r,n] \times \Delta[1] \arrow[r] \arrow[d] & \partial F[r,n] \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \partial F[r,n] \arrow[d] \\
F[r,n] \arrow[r, shift left, "\{0\}"] \arrow[r, shift right, "\{1\}"'] \arrow[rr, bend left, "f'"] \arrow[rr, bend right, "g'"'] & F[r,n] \times \Delta[1] \arrow[r] \arrow[d] & h^*Y \arrow[r, "j"] \arrow[d, two heads, "h^*p"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, two heads, "p"] \\
 & F[r,n] \arrow[r, equal] & F[r,n] \arrow[r, "h"] & X
\end{tikzcd} ,$$

where the existence of the morphism $\partial F[r, n] \to h^*Y$ follows from the universal property of pullbacks and the fact that the inclusion $\partial F[r, n] \to X$ factors through $h : F[r, n] \to X$. Moreover, $f', g' : F[r, n] \to h^*Y$ are defined via the universal property of pullbacks, meaning we by definition have $f = jf'$ and $g = jg'$. Now, by assumption h^*p is minimal and so $f' = g'$, which implies that $f = g$ and so, by Lemma A.4, $p : Y \to X$ is minimal as well. ■

We now proceed to generalize the first major property of minimal fibrations, as proven in [GJ09, Lemma 10.4].

A.6. Proposition. *Let $p : Y \to X$ and $q : Z \to X$ be two minimal Reedy fibrations, and $f : Y \to Z$ a level-wise equivalence over X. Then f is an isomorphism.*

Proof. We proceed by Reedy induction. Let r have degree 0. Then $Y_r \to X_r \times_{M_rX} M_rY \cong X_r$ and $Z_r \to X_r \times_{M_rX} M_rZ \cong X_r$ are minimal fibrations and so, by [GJ09, Lemma 10.4], $f_r : Y_r \to Z_r$ is an isomorphism. Moving on to the induction step, let us assume for all objects r in $\mathcal{R}$ with degree less than n, the map f_r is an isomorphism. Now, let r have degree n, then we have the following commutative diagram

$$\begin{tikzcd}
Y_r \arrow[r, "f_r", "\simeq"'] \arrow[d, two heads] & Z_r \arrow[d, two heads] \\
M_rY \times_{M_rX} X_r \arrow[r, "\cong"] & M_rZ \times_{M_rX} X_r
\end{tikzcd} .$$

The two vertical morphisms are minimal Kan fibrations by assumption. Moreover, by definition of the matching object as a limit of lower degree objects and the induction assumption, the bottom map is an isomorphism. Hence, again by [GJ09, Lemma 10.4], f_r is an isomorphism. ■

We now want to generalize the factorization into a trivial fibration followed by a minimal fibration, which necessitates proving the analogue of [GJ09, Lemma 10.2]. However, this actually does not hold in all Reedy categories.

A.7. Example. Let $[1] = \{0 \to 1\}$ be the Reedy category with $\deg(0) = 1$, $\deg(1) = 0$, $[1]^- = [1]$, and $[1]^+$ having only identity maps. Then for a given functor $X : [1]^{op} \to \mathcal{S}et$, $L_1X = X[0]$ and $M_1X = *$, meaning the comparison map $L_1X \to M_1X$ is generally not injective.

We hence restrict our attention to a class of better behaved Reedy categories, known as *elegant Reedy categories* and introduced by Bergner–Rezk [BR13, Definition 3.5]. A Reedy category $\mathcal{R}$ is *elegant* if the category $\mathcal{R}^-$ has pushouts that are preserved by the inclusion into $\mathcal{R}$ and also into $\underline{\mathrm{Fun}}(\mathcal{R}^{op}, \mathcal{S}et)$ via the Yoneda embedding. This condition has the following technical implication.

A.8. Lemma. *Let $\mathcal{R}$ be an elegant Reedy category. Let $r_1 \leftarrow^- r \to^- r_2$ be a cospan in $\mathcal{R}$ with pushout r_3. Then the following is a pushout diagram in $\underline{\mathrm{Fun}}(\mathcal{R}^{op}, \mathcal{S}et)$*

$$\begin{tikzcd} \partial F[r] \arrow[r, "p_1"] \arrow[d, "p_2"] & F[r_1] \arrow[d, "q_1"] \\ F[r_2] \arrow[r, "q_2"] & F[r_3] \end{tikzcd} .$$

Proof. Let $n = \deg(r) - 1$ (note we must have $\deg r > 0$). By Remark A.1, the functor $sk_n tr_n$ preserves colimits. We hence have a pushout diagram

$$\begin{tikzcd} sk_n tr_n F[r] \arrow[r, "sk_n tr_n p_1"] \arrow[d, "sk_n tr_n p_2"] & sk_n tr_n F[r_1] \arrow[d, "sk_n tr_n q_1"] \\ sk_n tr_n F[r_2] \arrow[r, "sk_n tr_n q_2"] & sk_n tr_n F[r_3] \end{tikzcd} .$$

It now follows from Remark A.1 that $sk_n tr_n F[r_i] = F[r_i]$ for $i = 1, 2, 3$ and $sk_n tr_n F[r] = \partial F[r]$, giving us the desired pushout. ∎

While it might appear that the elegance condition restricts the applicability of the results, it does hold in all relevant cases, as the following example demonstrates.

A.9. Example. Let $n, k \geq 0$, then the category $\mathbb{\Delta}^k \times \Theta_n$ is an elegant Reedy category [Hir03, Proposition 15.1.6], [BR13, Corollary 4.5]. Here Θ_n denotes Joyal's Θ_n-category, relevant for the study of (∞, n)-categories [Rez10a]. More generally, elegant Reedy categories include all Eilenberg-Zilber Reedy categories [BR13, Proposition 4.1], and so it generalizes the work in [Cis14].

We can now prove the analogues of [GJ09, Lemma 10.2] for elegant Reedy categories.

A.10. Lemma. *Let $\mathcal{R}$ be an elegant Reedy category and $X : \mathcal{R}^{op} \to \mathcal{S}et$ a functor. Then for all r in $\mathcal{R}$, $L_rX \to M_rX$ is an injection. More explicitly, for two degenerate r-simplices x, y, $\partial x = \partial y$ implies $x = y$.*

Proof. Let us first unwind the statement. By [BR13, Proposition 3.7] an element x in L_rX corresponds to a degenerate element in X_r, which is represented by a morphism $x : F[r] \to^{-} F[r_1] \xrightarrow{x'} X,$. So, we need to prove that for two given morphisms $x : F[r] \to^{-} F[r_1] \xrightarrow{x'} X$, $y : F[r] \to^{-} F[r_2] \xrightarrow{y'} X$, $\partial x = \partial y$ implies $x = y$.

By Lemma A.8, we have $F[r_1 \coprod_r r_2] \cong F[r_1] \coprod_{\partial F[r]} F[r_2]$ and so the equality $\partial x = \partial y$ induces a map $x' + y' : F[r_1] \coprod_{\partial F[r]} F[r_2] \cong F[r_1 \coprod_r r_2] \to X$. Now, the bijection $F[r_1] \coprod_{F[r]} F[r_2] \cong F[r_1 \coprod_r r_2]$ gives us a map $x' + y' : F[r_1] \coprod_{F[r]} F[r_2] \to X$, which by definition of pushouts implies that the two morphisms $x : F[r] \to^{-} F[r_1] \xrightarrow{x'} X$ and $y : F[r] \to^{-} F[r_2] \xrightarrow{y'} X$ are equal. ∎

With this lemma at hand, we can now proceed to the main result (i.e. the analogue to [GJ09, Proposition 10.3]): the construction of the desired unique factorization.

A.11. Theorem. *Let $\mathcal{R}$ be an elegant Reedy category. Then every Reedy fibration $p : Y \to X$ admits a factorization $p : Y \overset{\simeq}{\twoheadrightarrow} \mathcal{M}\mathrm{in}(Y) \overset{min}{\twoheadrightarrow} X$ into a trivial fibration followed by a minimal Reedy fibration, which is unique up to isomorphism.*

Proof. We first show $p : Y \to X$ restricts to a minimal Reedy fibration $\mathcal{M}\mathrm{in}(Y) \to X$, such that the inclusion admits a deformation retract, then show that the retract is a trivial fibration and finally prove uniqueness.

Deformation Retract: First, we construct a retract diagram $\mathcal{M}\mathrm{in}(Y) \overset{j}{\hookrightarrow} Y \overset{q}{\to} \mathcal{M}\mathrm{in}(Y)$ over X, along with a homotopy $h : Y \times \Delta[1] \to Y$ from id_Y to jq, such that $\mathcal{M}\mathrm{in}(p) = pj : \mathcal{M}\mathrm{in}(Y) \to X$ is a minimal Reedy fibration. We will proceed by Reedy induction and construct $\mathcal{M}\mathrm{in}(Y)^{(n)}$ as an object in $\underline{\mathrm{Fun}}((\mathcal{R}^{op} \times \mathbb{\Delta}^{op})_{\leq n}, \mathcal{S}\mathrm{et})$ inductively, adjusting the argument in [GJ09, Proposition 10.3]. More concretely for every $n \in \mathbb{N}$ we show the following:

1. There is a retract diagram $\mathcal{M}\mathrm{in}(Y)^{(n)} \xrightarrow{j^{(n)}} tr_nY \xrightarrow{q^{(n)}} \mathcal{M}\mathrm{in}(Y)^{(n)}$ over tr_nX, appropriately extending the analogous retract diagram for $n-1$.
2. There is a homotopy $h^{(n)} : \Delta[1] \times sk_ntr_nY \to Y$ from the counit $(c_n)_Y : sk_ntr_nY \to Y$ to $(c_n)_Y \circ sk_nj^{(n)} \circ sk_nq^{(n)}$, appropriately extending the homotopy $h^{(n-1)}$, such that $h^{(n)}$ restricted to $\mathcal{M}\mathrm{in}(Y)^{(n)}$ is constant, meaning it is equal to $(c_n)_Y \circ sk_nj^{(n)} \circ \pi_1 : \mathcal{M}\mathrm{in}(Y)^{(n)} \times \Delta[1] \to Y$.
3. Any two $p \circ (c_n)_Y \circ sk_nj^{(n)}$-equivalent (r,k)-simplices (in the sense of Definition A.3) in $sk_n\mathcal{M}\mathrm{in}(Y)^{(n)}$ are equal.

Assuming we have constructed this data for all n, our desired retract follows from taking the following colimit in $\underline{\mathrm{Fun}}(\mathcal{R}^{op}, \mathrm{s}\mathcal{S}\mathrm{et})$

$$\mathcal{M}\mathrm{in}(Y) \overset{j}{\hookrightarrow} Y \overset{q}{\to} \mathcal{M}\mathrm{in}(Y) = \operatorname*{colim}_{n\to\infty}(sk_n\mathcal{M}\mathrm{in}(Y)^{(n)} \xrightarrow{sk_nj^{(n)}} sk_ntr_nY \xrightarrow{sk_nq^{(n)}} sk_n\mathcal{M}\mathrm{in}(Y)^{(n)}),$$

where we are using (Remark A.1). Similarly, we have the homotopy $h = \operatorname*{colim}_{n\to\infty}(h^{(n)} : \Delta[1] \times sk_n tr_n Y \to Y)$, which is a homotopy from $\operatorname*{colim}_{n\to\infty} (c_n)_Y = \mathrm{id}_Y$ to $\operatorname*{colim}_{n\to\infty} (c_n)_Y \circ sk_n j^{(n)} \circ sk_n q^{(n)} = jq$. In addition to that, $\mathcal{M}\mathrm{in}(p) : \mathcal{M}\mathrm{in}(Y) \to X$ is a Reedy fibration, as it is a retract of the Reedy fibration p, and so it is a minimal Reedy fibration, by Lemma A.4, as any two pj-equivalent simplices are equal, by the third condition above.

Let us now move on to constructing $\mathcal{M}\mathrm{in}(Y)^{(n)}$ with the desired properties via Reedy induction. We start with the base case. By definition of Reedy categories, $(\mathcal{R}^{op}\times\mathbb{\Delta}^{op})_{\leq 0}$ is a discrete category. Moreover, by [Hir03, Proposition 15.1.6], $\deg(r,k) = \deg(r)+\deg(k) = 0$ if and only if $\deg(r) = \deg(k) = 0$, which is equivalent to $\deg(r) = 0$ and $k = 0$. Hence $(\mathcal{R}^{op} \times \mathbb{\Delta}^{op})_{\leq 0} \cong \coprod_{r\in\mathcal{R}_{\leq 0}}(r, 0)$. Define $\mathcal{M}\mathrm{in}(Y)^{(0)}_{r0}$ as a choice of representative of each p-equivalence class in Y_{r0}. The inclusion comes with an evident map $q^{(0)} : tr_0 Y_{r0} \to \mathcal{M}\mathrm{in}(Y)^{(0)}_{r0}$ sending each element to the representative of its equivalence class. Now, by assumption every element in Y_{r0} is homotopic to an element in $\mathcal{M}\mathrm{in}(Y)^{(0)}_{r0}$ and so we can readily define a map $h^{(0)} : \Delta[1]\times sk_0 tr_0 Y \to Y$, where we pick identity paths if the element in $sk_0 tr_0 Y_{r0}$ lies in $\mathcal{M}\mathrm{in}(Y)^{(0)}_{r0}$. Finally, if $\deg(r, k) = 0$, then any two $p \circ (c_0)_Y \circ sk_0 j^{(0)}$-equivalent (r, k)-simplices in $sk_0\mathcal{M}\mathrm{in}(Y)^{(0)}_{rk}$ are equal by construction, and for all other objects (r, k) in $\mathcal{R} \times \mathbb{\Delta}$, all (r, k)-simplices in $sk_0\mathcal{M}\mathrm{in}(Y)^{(0)}$ are degenerate and hence any two $p \circ (c_0)_Y \circ sk_0 j^{(0)}$-equivalent (r, k)-simplices in $sk_0\mathcal{M}\mathrm{in}(Y)^{(0)}_{rk}$ are equal by Lemma A.10.

Now, assuming we have defined $\mathcal{M}\mathrm{in}(Y)^{(n)}$ with the desired properties, we now move on to define $\mathcal{M}\mathrm{in}(Y)^{(n+1)}$, proving the induction step. First, for every object (r, k) in $\mathcal{R}\times\mathbb{\Delta}$ with $\deg(r, k) < n+1$ we define $\mathcal{M}\mathrm{in}(Y)^{(n+1)}_{rk} = \mathcal{M}\mathrm{in}(Y)^{(n)}_{rk}$. Next, for every object (r, k) in $\mathcal{R} \times \mathbb{\Delta}$ with $\deg(r, k) = n + 1$ we define $\mathcal{M}\mathrm{in}(Y)^{(n+1)}_{rk} = D_{rk} \cup (R_{rk} \cap M_{rk}) \subseteq Y_{rk}$, where D_{rk}, R_{rk} and M_{rk} are characterized as follows:

- D_{rk} is defined as the set of all degenerate elements in Y_{rk} that lie in the image of the inclusion

$$L_{rk} sk_n \mathcal{M}\mathrm{in}(Y)^{(n)} \xhookrightarrow{L_{rk}((c_n)_Y\circ sk_n j^{(n)})} L_{rk}Y \hookrightarrow Y_{rk} \ .$$

- R_{rk} is a set of one representative from each p-equivalence class in Y_{rk} not equivalent to any degenerate element.

- M_{rk} consists of all elements in Y_{rk} whose image under the projection $Y_{rk} \to M_{rk}Y$ lands inside the image of the inclusion

$$M_{rk} sk_n \mathcal{M}\mathrm{in}(Y)^{(n)} \xhookrightarrow{M_{rk}((c_n)_Y\circ sk_n j^{(n)})} M_{rk}Y \ .$$

The first and last condition guarantee that we have a factorization $L_{rk}\mathcal{M}\mathrm{in}(Y)^{(n+1)} \to \mathcal{M}\mathrm{in}(Y)^{(n+1)}_{rk} \to M_{rk}\mathcal{M}\mathrm{in}(Y)^{(n+1)}$, which, by [Hir03, Theorem 15.2.1], is the necessary and sufficient condition for $tr_{n+1}Y$ to restrict to a sub-functor $\mathcal{M}\text{in}(Y)^{(n+1)} : ((\mathcal{R}\times\mathbb{\Delta})_{\leq n+1})^{op} \to \mathcal{S}\text{et}$ and we denote this natural inclusion by $j^{(n+1)} : \mathcal{M}\text{in}(Y)^{(n+1)} \to tr_{n+1}Y$.

We now proceed to define the retract of $j^{(n+1)}$, denoted $q^{(n+1)} : tr_{n+1}Y \to \mathcal{M}\text{in}(Y)^{(n+1)}$. If x is an (r,k)-simplex with $\deg(r,k) < n+1$, we define $q^{(n+1)}(x) = q^{(n)}(x)$, hence let us assume (r,k) in $\mathcal{R}\times\mathbb{\Delta}$ with $\deg(r,k) = n+1$ and $x \in tr_{n+1}Y_{rk}$. If x is p-equivalent to a degenerate element, then the value of $q^{(n+1)}(x)$ needs to be degenerate and is hence already determined by $q^{(n)}$. So it suffices to consider the case where x is not p-equivalent to a non-degenerate (r,k)-simplex. In order to able to define $q^{(n+1)}(x)$, we first define the two morphisms $h_x, g_x : F[r,k]\times\Delta[1] \to Y$.

If x lies in the image of $(j^{(n+1)})_{rk}$, then we define $h_x, g_x : F[r,k]\times\Delta[1] \to Y$ as the constant map $g_x = h_x = x\circ\pi_1$. If not, then we define $h_x : F[r,k]\times\Delta[1] \to Y$ as a lift to the following commutative diagram

$$\begin{tikzcd}
\partial F[r,k]\times\Delta[1] \coprod_{\partial F[r,k]} F[r,k]\times\{0\} \arrow[r, "h^{(n)}\circ\partial x + x"] \arrow[d, hook, "\simeq"'] & Y \arrow[d, two heads, "p"] \\
F[r,k]\times\Delta[1] \arrow[r, "px\circ\pi_1"'] \arrow[ur, dashed, "h_x"] & X
\end{tikzcd} .$$

By induction assumption $h^{(n)}$ is a homotopy from $(c_n)_Y$ to $(c_n)_Y \circ sk_n(j^{(n)}) \circ sk_n(q^{(n)})$, which means that $h^{(n)}$ restricted to $\partial F[r,k]\times\{1\}$ takes value in the image of $sk_n(j^{(n)}) : sk_n\mathcal{M}\text{in}(Y)^{(n)} \hookrightarrow Y$. Hence, as h_x is defined as a lift of $h^{(n)}$, h_x restricted to $\partial F[r,k]\times\{1\}$ also takes value in $sk_n\mathcal{M}\text{in}(Y)^{(n)} \hookrightarrow Y$. However, $\mathcal{M}\text{in}(Y)^{(n+1)}_{rk}$ consists by definition of a unique representative from each p-equivalence class in Y_{rk} whose boundary lands in $sk_n\mathcal{M}\text{in}(Y)^{(n)}$. Hence, there exists a unique element in $\mathcal{M}\text{in}(Y)^{(n+1)}_{rk}$ that is p-equivalent to $h_x(F[r,k]\times\{1\}) : F[r,k] \to Y$, which we denote by $q^{(n+1)}(x)$, and there exists a morphism $g_x : F[r,k]\times\Delta[1] \to Y$ witnessing the p-equivalence, meaning making the following diagram commute:

$$\begin{tikzcd}
& \partial F[r,k]\times\Delta[1] \arrow[r] \arrow[d] & \partial F[r,k] \arrow[d, "\partial h_x(F[r,k]\times\{1\})"] \\
F[r,k] \arrow[r, shift left, "\{0\}"] \arrow[r, shift right, "\{1\}"'] \arrow[rr, bend left, "h_x(F[r,k]\times\{1\})"] \arrow[rr, bend right, "q^{(n+1)}(x)"'] & F[r,k]\times\Delta[1] \arrow[r, "g_x"] \arrow[d] & Y \arrow[d, two heads, "p"] \\
& F[r,k] \arrow[r] & X
\end{tikzcd} .$$

Our choice of element $q^{(n+1)}(x)$ induces a morphism $q^{(n+1)} : tr_{n+1}Y \to \mathcal{M}\text{in}(Y)^{(n+1)}$, that in fact gives us a retract diagram, as for every element in the image of $j^{(n+1)}$, we defined h_x, g_x to be constant.

Next, we construct the desired homotopy $h^{(n+1)} : \Delta[1]\times sk_{n+1}tr_{n+1}Y \to Y$ from the counit $(c_{n+1})_Y : sk_{n+1}tr_{n+1}Y \to Y$ to $(c_{n+1})_Y \circ sk_{n+1}j^{(n+1)} \circ sk_{n+1}q^{(n+1)}$, appropriately extending the homotopy $h^{(n)}$, such that $h^{(n+1)}$ restricted to $\mathcal{M}\text{in}(Y)^{(n+1)}$ is constant, meaning it is equal to $(c_{n+1})_Y \circ sk_{n+1}j^{(n+1)} \circ \pi_1 : \mathcal{M}\mathrm{in}(Y)^{(n+1)} \times \Delta[1] \to Y$. We consider three separate cases.

For a given object (r,k) in $\mathcal{R} \times \mathbb{\Delta}$ and (r,k)-simplex x, if $\deg(r,k) < n+1$, we define $h_x^{(n+1)} = h_x^{(n)} : F[r,k] \times \Delta[1] \to Y$. Next, if $\deg(r,k) = n+1$ and the (r,k)-simplex x lies in the image of $sk^{(n+1)}j_{rk}^{(n+1)}$, then we define $h_x^{(n+1)} : F[r,k] \times \Delta[1] \to Y$ as the constant path, i.e. the composition $x \circ \pi_1 : F[r,k] \times \Delta[1] \to F[r,k] \to Y$. Finally, let us assume $\deg(r,k) = n+1$ and x is an (r,k)-simplex in $sk_{n+1}tr_{n+1}Y$ that does not lie in the image of $sk^{(n+1)}j_{rk}^{(n+1)}$. Then we construct $h_x^{(n+1)} : F[r,k] \times \Delta[1] \to Y$ in two steps.

Let θ_x be given as a lift to the following commutative diagram

$$\begin{tikzcd} \partial F[r,k] \times \Delta[2] \coprod\limits_{\partial F[r,k] \times \Lambda[2]_1} F[r,k] \times \Lambda[2]_1 \arrow[rr, "(s_1 h_x + (h_x + g_x))"] \arrow[d, hook, "\simeq"] & & Y \arrow[d, two heads, "p"] \\ F[r,k] \times \Delta[2] \arrow[rr, "px \circ \pi_1"] \arrow[urr, dashed, "\theta_x"] & & X \end{tikzcd} .$$

Then, we define $h_x^{(n+1)} : F[r,k] \times \Delta[1] \to Y$ as $\theta_x \circ (\mathrm{id}_{F[r,k]} \times d_1)$, completing our definition of the morphism $h^{(n+1)} : \Delta[1] \times sk_{n+1}tr_{n+1}Y \to Y$. We hence only need to confirm it satisfies the desired conditions.

By construction $h^{(n+1)}$ is compatible with $h^{(n)}$. Moreover, if x is in the image of $sk^{(n+1)}j^{(n+1)}$, then by construction the homotopy is constant, meaning $h^{(n+1)}$ is constant when restricted to the image of $sk^{(n+1)}j^{(n+1)}$. Also, by definition $h_x^{(n+1)}$ restricted to ∂x is given by h_x, which by construction is equal to $h_{\partial x}^{(n)}$. Finally, we have in fact constructed a homotopy from x to $(c_{n+1})_Y \circ sk_{n+1}(j^{(n+1)}) \circ sk_{n+1}(q^{(n+1)})(x)$. Indeed, this is evident for all simplices in the image of $sk^{(n+1)}j^{(n+1)}$ as it is the constant homotopy. For all other cases, for (r,k)-simplices with $\deg(r,k) < n+1$ this follows from the induction assumption and for (r,k)-simplices x with $\deg(r,k) = n+1$, we have a homotopy from $h_x(F[r,k] \times \{0\}) = x$ to $g_x(F[r,k] \times \{1\}) = q^{(n+1)}(x)$.

Finally, if $\deg(r,k) \leq n+1$, then any two $p \circ (c_{n+1})_Y \circ sk_{n+1}j^{(n+1)}$-equivalent (r,k)-simplices in $sk_{n+1}\mathcal{M}\mathrm{in}(Y)_{rk}^{(n+1)}$ are equal by construction, and for all other objects (r,k) in $\mathcal{R} \times \mathbb{\Delta}$, all (r,k)-simplices in $sk_{n+1}\mathcal{M}\mathrm{in}(Y)^{(n+1)}$ are degenerate and hence any two $p \circ (c_{n+1})_Y \circ sk_{n+1}j^{(n+1)}$-equivalent (r,k)-simplices in $sk_{n+1}\mathcal{M}\mathrm{in}(Y)_{rk}^{(n+1)}$ are equal by Lemma A.10.

Trivial Fibration: Next, we show that the map $q : Y \to \mathcal{M}\mathrm{in}(Y)$ is a trivial Reedy fibration. We will follow the argument in the original proof of Quillen [Qui68]. We need to prove that the following commutative diagram admits a lift

$$\begin{tikzcd} \partial F[r,n] \arrow[r, "a"] \arrow[d, hook, "i"'] & Y \arrow[d, "q"] \\ F[r,n] \arrow[r, "b"] & \mathcal{M}\mathrm{in}(Y) \end{tikzcd} .$$

First, we observe that the deformation retract $h : Y \times \Delta[1] \to Y$ restricts along $a : \partial F[r,n] \to Y$ to a map $ha : \partial F[r,n] \times \Delta[1] \to Y$, which is a homotopy from a to $jqa = jbi$. Now, we have the following commutative diagram

$$\begin{tikzcd}
\partial F[r,n] \arrow[r, "\mathrm{id}\times\{0\}"] \arrow[d, hook, "i"'] \arrow[rr, bend left, "a"] & \partial F[r,n] \times \Delta[1] \coprod_{\partial F[r,n]} F[r,n] \arrow[r, "ha+jb"] \arrow[d, hook, "i\times \mathrm{id} \underset{i\times\{1\}}{+} \mathrm{id}\times\{1\}"', "\simeq"] & Y \arrow[r, "q"] \arrow[d, two heads, "p"] & \mathcal{M}\mathrm{in}(Y) \arrow[dl, two heads, "\mathcal{M}\mathrm{in}(p)"] \\
F[r,n] \arrow[r, "\mathrm{id}\times\{0\}"] & F[r,n] \times \Delta[1] \arrow[r, "\mathcal{M}\mathrm{in}(p)b\pi_1"'] \arrow[ur, dashed, "\ell"] & X &
\end{tikzcd}$$

The map $i \times \mathrm{id} +_{i\times\{1\}} \mathrm{id} \times \{1\}$ is a Reedy equivalence and so there exists a map $\ell : F[r,n] \times \Delta[1] \to Y$, such that $\ell k = ha + jb$ and $p\ell = \mathcal{M}\mathrm{in}(p)b\pi_1$. By construction $\ell(\mathrm{id} \times \{0\})i = a$ and so we only need to prove that $q\ell(\mathrm{id} \times \{0\}) = b$.

The map $q\ell : F[r,n] \times \Delta[1] \to \mathcal{M}\mathrm{in}(Y)$ is a homotopy from $q\ell(\mathrm{id} \times \{0\})$ to $qjb = b$ relative to their boundary that is trivial in X, meaning it gives us a $\mathcal{M}\mathrm{in}(p)$-equivalence in the sense of Definition A.3. By minimality of $\mathcal{M}\mathrm{in}(p)$ and Proposition A.6 we get $q\ell(\mathrm{id} \times \{0\}) = b$ and hence q is a trivial fibration.

Uniqueness: Finally, we show the factorization is unique. Let us assume we have a second factorization $Y \xrightarrow{q'} M \xrightarrow{p'} X$. The map $q'j$ is a composition of equivalences and we have

$$p'q'j = pj = \mathcal{M}\mathrm{in}(p)qj = \mathcal{M}\mathrm{in}(p),$$

which implies that $q'j : \mathcal{M}\mathrm{in}(Y) \to M$ is an equivalence between two minimal Reedy fibrations over X. Hence, by Proposition A.6, $q'j$ is an isomorphism and we are done. ■

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