Theory and Applications of Categories, Vol. 41, No. 21, 2024, pp. 686–706.

THE CORE GROUPOID CAN SUFFICE

ROSS STREET

ABSTRACT. This work results from a study of Nicholas Kuhn's paper entitled "Generic representation theory of finite fields in nondescribing characteristic". Our goal is to abstract the categorical structure required to obtain an equivalence between functor categories $[\mathscr{F},\mathscr{V}]$ and $[\mathscr{G},\mathscr{V}]$ where $\mathscr{G}$ is the core groupoid of the category $\mathscr{F}$ and $\mathscr{V}$ is a category of modules over a commutative ring. Examples other than Kuhn's are covered by this general setting.

Introduction

Let $\mathbb{F}$ be a finite field, let $\mathscr{F}$ be the category of finite dimensional vector spaces and linear functions over $\mathbb{F}$, and let $\mathscr{G}$ be the groupoid core of $\mathscr{F}$; that is, the subcategory of $\mathscr{F}$ with all the objects and only the bijective linear functions. Let $\mathscr{V}$ be the category of vector spaces over the complex numbers (say). In [12] André Joyal and I studied a braided monoidal structure on the functor category $[\mathscr{G},\mathscr{V}]$. That work could be regarded as a categorified version of the algebra studied by Sandy Green [8] in the representation theory of the finite general linear groups.

So I was quite interested and surprised when Nicholas Kuhn told Steve Lack and me about his equivalence of categories $[\mathscr{F},\mathscr{V}] \simeq [\mathscr{G},\mathscr{V}]$ (see [15]) when we were working on Dold-Kan-type theorems (see [16] and the related [9]). That proof of equivalence refers to a remarkable piece of linear algebra [14] by our late Australian National University colleague Laci Kovács who built on results in the papers [6, 18].

Given Kovács' result, the present note is a study of the rest of the argument in order to abstract the essential categorical features to cover further examples.

Benjamin Steinberg has recently produced two alternative proofs of the result in [14]; see his excellent book [22] and paper [23].

Now let $\mathscr{V}$ be the monoidal category of modules over a commutative ring R. Generally, we are interested in categories $\mathscr{F}$ for which there is a groupoid $\mathscr{G}$ such that the functor categories $[\mathscr{F},\mathscr{V}]$ and $[\mathscr{G},\mathscr{V}]$ are equivalent. In particular, $\mathscr{G}$ could be the core groupoid of $\mathscr{F}$; that is, the subcategory with the same objects and with only the invertible morphisms. Every category $\mathscr{F}$ gives rise to a $\mathscr{V}$-category (that is, an R-linear category), denoted $R\mathscr{F}$,

The author gratefully acknowledges the support of Australian Research Council Discovery Grant DP190102432.

Received by the editors 2023-12-14 and, in final form, 2024-06-17.

Transmitted by John Bourke and Tom Leinster. Published on 2024-06-19.

2020 Mathematics Subject Classification: 20C33; 16D90; 18M80; 20L05; 18A32; 18B40; 18D60.

Key words and phrases: general linear groupoid; finite field; monoid representation; Joyal species; Morita equivalence; Dold-Kan-type theorems.

with the same objects and with hom R-module $R\mathscr{F}(A,B)$ free on the homset $\mathscr{F}(A,B)$. Indeed, $R\mathscr{F}$ is the free $\mathscr{V}$-category on $\mathscr{F}$ so that the $\mathscr{V}$-functor category $[R\mathscr{F},\mathscr{V}]$ is isomorphic to the ordinary functor category $[\mathscr{F},\mathscr{V}]$ with the pointwise R-linear structure. In these terms, we are interested in when $R\mathscr{F}$ and $R\mathscr{G}$ are Morita equivalent $\mathscr{V}$-categories.

In my joint work [16] with Steve Lack on Dold-Kan-type equivalences, we had many examples of this phenomenon. In Section 5 and Appendix A of that work we showed how to reduce our general setting to the core groupoid case. However, the example of Nick Kuhn, where $\mathscr{F}$ is the category of finite vector spaces over a fixed finite field $\mathbb{F}$ with all $\mathbb{F}$-linear functions and $\mathscr{G}$ is the general linear groupoid over $\mathbb{F}$, does not fit into our theory. Yet the "kernel" of the equivalence is of the same type. The present work shows that the category theory behind the Kuhn result also covers Dold-Kan-type results.

Our main general result expressed as a Morita equivalence appears as Theorem 3.10. The more specific conclusion is Corollary 4.12 which says that, under some finiteness-type conditions on a proper factorization system in a category $\mathscr{F}$, and with the existence of certain idempotents, we have an equivalence of categories

$$[\mathscr{F},\mathscr{X}] \simeq [\mathscr{G},\mathscr{X}]$$

where $\mathscr{G}$ is the core groupoid of $\mathscr{F}$ and $\mathscr{X}$ is any R-linear category admitting finite direct sums and splitting of idempotents.

As mentioned, a monoidal structure on the category of representations of the general linear groupoid over a finite field was defined and studied in [12]. So finally, in Section 5, we examine how such monoidal structures transport across the equivalences of our main theorem.

In some ways the philosophy of this paper works in a direction opposite to the modern trends which accept that we must study representations of categories more general than groups or groupoids. Currently we see work on representations of free categories on graphs (quiver representations), of categories coming from low-dimensional topology, and so on. Our results show that sometimes representations of non-groupoids produce nothing more than representations of their core groupoids.

I am grateful to Alexander Campbell for pointing to two gaps in the version of this note that I presented to the Australian Category Seminar on 13 July 2022. I am grateful to Nicholas Kuhn for his help during the writing of this material. Thanks also to Benjamin Steinberg for his valuable feedback on the first arXiv version of this work. My sincere thanks goes to the editors for their thoughtful consideration and to the anonymous referee who made 42 essential suggestions and corrections.

1. Review of factorization systems

Let $\mathscr{F}$ be a category equipped with a factorization system $(\mathscr{E},\mathscr{M})$; see [7, 13]. That is, $\mathscr{E}$ and $\mathscr{M}$ are sets of morphisms of $\mathscr{F}$ satisfying

FS0. $mw \in \mathscr{M}$ if $m \in \mathscr{M}$ and w is invertible, while $we \in \mathscr{E}$ if $e \in \mathscr{E}$ and w is invertible;

FS1. if $mh = ke$ with $e \in \mathscr{E}$ and $m \in \mathscr{M}$ then there exists a unique ℓ with $\ell e = h$ and $m\ell = k$ (see (1.1));

FS2. every morphism f factors as $f = me$ for some $m \in \mathscr{M}$ and $e \in \mathscr{E}$.

$$\begin{array}{ccc} A & \xrightarrow{\;e\;} & B \\ {\scriptstyle h}\downarrow & \overset{\ell}{\swarrow} & \downarrow{\scriptstyle k} \\ X & \xrightarrow[\;m\;]{} & Y \end{array} \tag{1.1}$$

It follows that $\mathscr{E} \cap \mathscr{M}$ is the set of invertible morphisms and that $\mathscr{E}$ and $\mathscr{M}$ are both closed under composition in $\mathscr{F}$. We identify $\mathscr{E}$, $\mathscr{M}$ and $\mathscr{G} := \mathscr{E} \cap \mathscr{M}$ with the subcategories of $\mathscr{F}$ containing all the objects of $\mathscr{F}$ but only the morphisms in those respective sets. The following result is a well-known way to express usefully the uniqueness of factorization up to isomorphism.

1.1. LEMMA. *Composition in $\mathscr{F}$ induces an isomorphism*

$$\int^{C \in \mathscr{G}} \mathscr{M}(C, B) \times \mathscr{E}(A, C) \cong \mathscr{F}(A, B)$$

which is natural in $A \in \mathscr{E}$ and $B \in \mathscr{M}$.

The factorization system is called *proper* when

FSP. every member of $\mathscr{E}$ is an epimorphism and every member of $\mathscr{M}$ is a monomorphism.

1.2. PROPOSITION. *In a proper factorization system, if $hf \in \mathscr{M}$ then $f \in \mathscr{M}$. Dually, if $fk \in \mathscr{E}$ then $f \in \mathscr{E}$.*

PROOF. Assume $hf \in \mathscr{M}$ and use FS2 to factorize $f = me$ with $e \in \mathscr{E}$ and $m \in \mathscr{M}$. Then $(hf)1 = (hm)e$ so, by FS1, there exists w with $we = 1$ and $hfw = hm$. So e is a split monomorphism. Using properness, we know e is an epimorphism and hence invertible. So $f = me \in \mathscr{M}$ by FS0. ∎

2. Preliminaries in the bicategory of modules

Let $\mathscr{V}$ denote the monoidal category Ab^R of modules over the commutative ring R.

Let $\mathfrak{R}$ denote the bicategory $\mathscr{V}$-Mod. The objects are $\mathscr{V}$-categories (also called R-linear categories and to be thought of as "R-algebras with several objects" in as much as a category is a "monoid with several objects"). The homcategories of $\mathfrak{R}$ are defined to be the $\mathscr{V}$-functor $\mathscr{V}$-categories

$$\mathfrak{R}(\mathscr{A}, \mathscr{B}) = [\mathscr{B}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}] \; ;$$

objects of these homs are called *modules from $\mathscr{A}$ to $\mathscr{B}$*. Composition

$$\mathfrak{R}(\mathscr{B}, \mathscr{C}) \otimes \mathfrak{R}(\mathscr{A}, \mathscr{B}) \xrightarrow{\circ} \mathfrak{R}(\mathscr{A}, \mathscr{C})$$

is defined by

$$(K \circ H)(C, A) = \int^{B} H(B, A) \otimes K(C, B) \ .$$

The identity module $\mathscr{A} \to \mathscr{A}$ is the $\mathscr{V}$-valued hom functor of $\mathscr{A}$.

The homcategories of $\mathfrak{R}$ are all abelian R-linear categories. Consequently, within them, exact sequences have meaning.

For $\mathscr{V}$-functors $\mathscr{A} \xrightarrow{F} \mathscr{C} \xleftarrow{G} \mathscr{B}$, we have the module $\mathscr{C}(G, F) : \mathscr{A} \to \mathscr{B}$ with components $\mathscr{C}(G, F)(B, A) = \mathscr{C}(GB, FA)$. In particular, for each $\mathscr{V}$-functor $F : \mathscr{A} \to \mathscr{B}$ we have the module $F_* = \mathscr{B}(1, F) : \mathscr{A} \to \mathscr{B}$. Also, the module $F^* = \mathscr{B}(F, 1) : \mathscr{B} \to \mathscr{A}$ provides a right adjoint $F_* \dashv F^*$ for F_* in the bicategory $\mathfrak{R}$.

Here is a result on Morita equivalence for R-linear categories; for example, see [25, 26].

2.1. Proposition. *Let $\mathscr{Q}\mathscr{A}$ denote the Cauchy completion of the $\mathscr{V}$-category $\mathscr{A}$; that is, the closure of the representables in $[\mathscr{A}^{\mathrm{op}}, \mathscr{V}]$ under finite direct sums and splittings of idempotents. The following conditions on R-linear categories $\mathscr{A}$ and $\mathscr{B}$ are logically equivalent:*

(i) $\mathscr{A}$ and $\mathscr{B}$ are equivalent in the bicategory $\mathfrak{R}$;

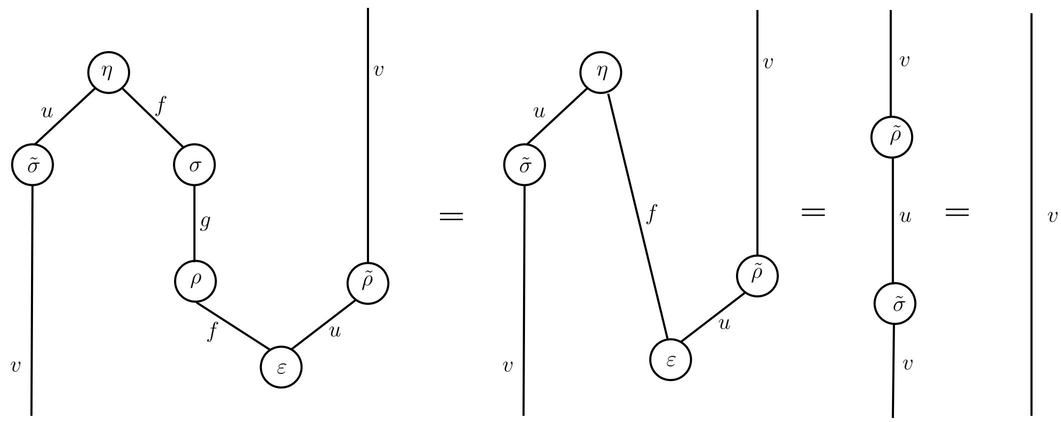
(ii) $[\mathscr{A}^{\mathrm{op}}, \mathscr{V}] \simeq [\mathscr{B}^{\mathrm{op}}, \mathscr{V}]$;

(iii) $[\mathscr{A}, \mathscr{V}] \simeq [\mathscr{B}, \mathscr{V}]$;

(iv) $\mathscr{Q}\mathscr{A} \simeq \mathscr{Q}\mathscr{B}$.

We will need Lemma 2.2 in our bicategory $\mathfrak{R}$. Diagram (2.2) is half of the string proof. The result will be used in the proof of Lemma 3.4.

2.2. Lemma. *In any bicategory $\mathfrak{M}$, suppose $f \dashv u : A \to X$ is an adjunction with counit $\varepsilon : f \circ u \Rightarrow 1_A$ and unit $\eta : 1_X \Rightarrow u \circ f$. Suppose $\omega : f \Rightarrow f$ is an idempotent 2-morphism on f with splitting provided by $\sigma : f \Rightarrow g$ and $\rho : g \Rightarrow f$: that is, $\rho\sigma = \omega$ and $\sigma\rho = 1_g$. The mate $\tilde{\omega} : u \Rightarrow u$ of $\omega : f \Rightarrow f$ under $f \dashv u$ is an idempotent 2-morphism on f satisfying $\varepsilon(\omega u) = \varepsilon(f\tilde{\omega})$. Then, any splitting of $\tilde{\omega}$ delivers a right adjoint v for g. Explicitly, if $\tilde{\rho}\tilde{\sigma} = \tilde{\omega}$ and $\tilde{\sigma}\tilde{\rho} = 1_v$ then $\tilde{\varepsilon} = \varepsilon(\rho \circ \tilde{\rho})$ and $\tilde{\eta} = (\tilde{\sigma} \circ \sigma)\eta$ provide a counit and unit for $g \dashv v$.*



(2.2)

We write RX for the free R-module with basis the set X. This defines the object assignment of the strong-monoidal left adjoint $R : \mathrm{Set} \to \mathscr{V}$ to the forgetful functor. We obtain a 2-functor[1] $R : \mathrm{Cat} \to \mathscr{V}\text{-Cat}$ taking ordinary categories to R-linear categories by applying R on the homsets of the categories. The ordinary functor category

$$[\mathscr{F}, \mathscr{V}]$$

with its pointwise R-linear enrichment is isomorphic to the $\mathscr{V}$-enriched $\mathscr{V}$-functor category

$$[R\mathscr{F}, \mathscr{V}] = \mathfrak{R}(R\mathscr{F}, R1) \ ,$$

where 1 is the terminal object of Cat.

Every module $H : \mathscr{A} \to \mathscr{B}$ is the "kernel" of a functor $\widehat{H} : [\mathscr{B}, \mathscr{V}] \to [\mathscr{A}, \mathscr{V}]$ defined by composition with H in $\mathfrak{R}$ thus: $\widehat{H}(T) = T \circ H : \mathscr{A} \to R1$. That is,

$$\widehat{H}(T)A = \int^{B} H(B, A) \otimes TB \ . \tag{2.3}$$

We obtain a pseudofunctor $\widehat{(-)} : \mathfrak{R}^{\mathrm{op}} \to \mathscr{V}\text{-Cat}$. Moreover, $\widehat{H}$ has a right adjoint $\widetilde{H} : [\mathscr{A}, \mathscr{V}] \to [\mathscr{B}, \mathscr{V}]$ given by right extension along H in $\mathfrak{R}$. Explicitly,

$$\widetilde{H}(F)B = \int_{A} [H(B, A), FA] = [\mathscr{A}, \mathscr{V}](H(B, -), F) \ . \tag{2.4}$$

Return now to $\mathscr{F}$ and its proper factorization system $(\mathscr{E}, \mathscr{M})$ with $\mathscr{G} = \mathscr{E} \cap \mathscr{M}$. We write $\mathscr{M}'$, $\mathscr{E}'$, $\mathscr{G}'$ for the set of morphisms of $\mathscr{F}$ not in $\mathscr{M}$, $\mathscr{E}$, $\mathscr{G}$, respectively.

The $\mathscr{V}$-functor $j : R\mathscr{G} \to R\mathscr{F}$ induced by the inclusion $\mathscr{G} \hookrightarrow \mathscr{F}$ gives the right adjoint $\mathscr{V}$-module $j^* : R\mathscr{F} \to R\mathscr{G}$. So $j^*(A, B) = R\mathscr{F}(jA, B)$ with $A \in R\mathscr{G}$ and $B \in R\mathscr{F}$.

[1] The notation of Eilenberg-Kelly [4] would be R_* for this 2-functor however we would rather overwork the symbol R than the subscript $*$.

2.3. LEMMA. *For $g \in \mathscr{G}(C,A)$ and $h \in \mathscr{F}(B,D)$, there is a commutative square*

$$\begin{array}{ccc} R\mathscr{M}'(A,B) & \xrightarrow{\subseteq} & R\mathscr{F}(A,B) \\ \downarrow & & \downarrow{\scriptstyle R\mathscr{F}(g,h)} \\ R\mathscr{M}'(C,D) & \xrightarrow[\subseteq]{} & R\mathscr{F}(C,D) \end{array} \tag{2.5}$$

so that $M'(A,B) = R\mathscr{M}'(A,B)$ defines a submodule M' of j^*. There is a short exact sequence

$$0 \longrightarrow M' \xrightarrow{\subseteq} j^* \xrightarrow{q} M \longrightarrow 0 \tag{2.6}$$

in $\mathfrak{R}(R\mathscr{F}, R\mathscr{G})$ where $M(A,B) = R\mathscr{M}(jA,B)$. Dually, there is a short exact sequence

$$0 \longrightarrow E' \xrightarrow{\subseteq} j_* \xrightarrow{\tilde{q}} E \longrightarrow 0 \tag{2.7}$$

in $\mathfrak{R}(R\mathscr{G}, R\mathscr{F})$ where $E(B,A) = R\mathscr{E}(B,jA)$.

PROOF. Contrapose the implications $hfg \in \mathscr{M} \;\Rightarrow\; hf = hfgg^{-1} \in \mathscr{M} \;\Rightarrow\; f \in \mathscr{M}$ to prove the first sentence. The R-linear morphism $q_{A,B} : R\mathscr{F}(A,B) \to R\mathscr{M}(A,B)$, defined on generators by $q_{A,B}(f) = f$ for $f \in \mathscr{M}(A,B)$ and $q_{A,B}(f) = 0$ otherwise, has kernel $R\mathscr{M}'(A,B)$. Then, using the universal property oif cokernel, the commutative square (2.5) induces the module structure on M and renders q a module morphism. ∎

2.4. LEMMA. *The family of R-module morphisms*

$$\phi_B : E(B,D) \otimes M(C,B) \to R\mathscr{G}(C,D)\ ,$$

defined by

$$\phi_B(e \otimes m) = \begin{cases} em & \text{for } em \in \mathscr{G} \\ 0 & \text{otherwise} \end{cases}$$

for $e \in \mathscr{E}(B,D)$ and $m \in \mathscr{M}(C,B)$, is dinatural in $B \in \mathscr{F}$, natural in $C, D \in \mathscr{G}$, and induces an invertible morphism $\bar{\phi} : M \circ E \xrightarrow{\cong} 1_{R\mathscr{G}}$ in $\mathfrak{R}$.

PROOF. Dinaturality requires $\phi_B(ef \otimes m) = \phi_{B'}(e \otimes fm)$ for all $e \in \mathscr{E}(B',D)$, $m \in \mathscr{M}(C,B)$ and $f \in \mathscr{F}(B,B')$. This is true since both sides are equal to efm when this is invertible and 0 otherwise. Naturality in $C, D \in \mathscr{G}$ is obvious. So a natural family $\bar{\phi} : \int^B E(B,D) \otimes M(C,B) \to R\mathscr{G}(C,D)$ is induced. An inverse takes $x \in R\mathscr{G}(C,D)$ to the equivalence class of $x \otimes 1_C$; it is obviously a right inverse and also a left inverse since any $e \otimes m$ representing a non-zero equivalence class has $e \otimes m \sim em \otimes 1_C$ with $em \in \mathscr{G}$ by dinaturality of the coend inclusions. ∎

Diagram (2.8) may help in reading Corollaries 2.5 and 2.6.

$$[\mathscr{F},\mathscr{V}] \; \begin{array}{c} \xrightarrow{\;\widehat{E}\;} \\ \perp \\ \xleftarrow[\;\widetilde{E}\;]{} \\ \xleftarrow[\;\widehat{M}\;]{} \end{array} \; [\mathscr{G},\mathscr{V}] \xrightarrow{\;\widehat{E}\circ\widehat{M}\;} [\mathscr{G},\mathscr{V}] \tag{2.8}$$

2.5. Corollary. *The isomorphism $\bar{\phi}$ induces an isomorphism $\widehat{E} \circ \widehat{M} \cong 1_{[\mathscr{G},\mathscr{V}]}$. The mate of this isomorphism, under the adjunction $\widehat{E} \dashv \widetilde{E}$, yields a natural transformation $\Theta : \widehat{M} \to \widetilde{E}$.*

The way we wish to prove $[\mathscr{F},\mathscr{V}] \simeq [\mathscr{G},\mathscr{V}]$ is to show that M and E appear in an adjoint equivalence $R\mathscr{F} \simeq R\mathscr{G}$ in $\mathfrak{R}$. Item (iii) of Corollary 2.6 is a big step towards that in the case where Θ is invertible.

2.6. Corollary. *Suppose Θ too is invertible. Then*

(i) $\widehat{E} \dashv \widehat{M}$*;*

(ii) $\widehat{M}$ *is fully faithful;*

(iii) $\bar{\phi} : M \circ E \xrightarrow{\cong} 1_{R\mathscr{G}}$ *is the counit of an adjunction $M \dashv E$ in $\mathfrak{R}$;*

(iv) $\widetilde{E} \dashv \widetilde{M}$.

Proof. Since $\widehat{E} \dashv \widetilde{E}$ and $\widehat{M} \cong \widetilde{E}$, we obtain (i). Since $\widehat{E} \circ \widehat{M} \cong 1_{[\mathscr{G},\mathscr{V}]}$ is the invertible counit of the adjunction in (i), we obtain (ii). The pseudofunctor $\widehat{(-)} : \mathfrak{R}^{\mathrm{op}} \to \mathscr{V}$-Cat is a biequivalence onto the full subbicategory of $\mathscr{V}$-Cat consisting of the $\mathscr{V}$-categories of the form $[\mathscr{A},\mathscr{V}]$ and left adjoint $\mathscr{V}$-functors between them; so $\widehat{E} \dashv \widehat{M} \dashv \widetilde{M}$ implies (iii). Finally, (iv) is obtained by applying the pseudofunctor $\widetilde{(-)} : \mathfrak{R}^{\mathrm{co}} \to \mathscr{V}$-Cat to $M \dashv E$. ∎

There are alternative formulas for $\widehat{M}$ and $\widetilde{E}$ which can be useful. For $A \in \mathscr{F}$, let Sub$A$ denote a representative set of the isomorphism classes of $\mathscr{M}/A$, and let Quo$A$ denote a representative set of the isomorphism classes of $A/\mathscr{E}$. Then

$$\widehat{M}(T)A \;\cong \sum_{(U \xrightarrow{m} A) \in \mathrm{Sub}A} TU \quad \text{and} \quad \widetilde{E}(T)A \;\cong \prod_{(A \xrightarrow{e} W) \in \mathrm{Quo}A} TW \; . \tag{2.9}$$

In cases where Sub$A$ and Quo$A$ are finite, both the sum and product in (2.9) are direct sums and the components of $\Theta : \widehat{M} \to \widetilde{E}$ transport across the isomorphisms to matrices whose non-zero entries have the form $T(em) : TU \to TW$ for $em \in \mathscr{G}$. Suppose that elements of each Sub$A$ can be listed $m_0, \ldots, m_s$ and those of each Quo$A$ can be listed $e_0, \ldots, e_s$ in such a way that $e_i m_j \in \mathscr{G}$ implies $1 \leqslant i \leqslant j \leqslant s$. Then these matrices are square and triangular with invertible morphisms down the main diagonal, and so (because we have additive inverses in the hom R-modules) are invertible.

One way to obtain listings of the kind just described, and hence the invertibility of Θ, is to produce an order in a manner similar to the one of Proposition 2.9 in [16].

2.7. PROPOSITION. *Suppose $\mathscr{F}$ is the underlying category of a category $\mathscr{P}$ enriched in the cartesian monoidal category of partially ordered sets; so $\mathscr{P}$ is a locally ordered 2-category. Suppose each $m \in \mathscr{M}$ has a right adjoint $m^* \in \mathscr{E}$ in $\mathscr{P}$ with identity unit. Suppose each object of $\mathscr{F}$ has only finitely many automorphisms. Then a reflexive, antisymmetric, transitive relation $m \trianglelefteq n$ is defined on* $\mathrm{Sub}A$ *by: there exists $x \in \mathscr{G}$ with $mx \leqslant n$.*

PROOF. Reflexivity is clear since each $1 \in \mathscr{G}$.

Suppose $m \trianglelefteq n \trianglelefteq \ell$. Then $mx \leqslant n$ and $ny \leqslant \ell$ for some $x, y \in \mathscr{G}$. So $mxy \leqslant ny \leqslant \ell$ yields $m \trianglelefteq \ell$. This proves transitivity.

Suppose $m \trianglelefteq n \trianglelefteq m$. Then $mx \leqslant n$ and $ny \leqslant m$ for some $x, y \in \mathscr{G}$. So $mxy \leqslant ny \leqslant m$. Since m is fully faithful in $\mathscr{P}$, $xy \leqslant 1$ and we have the chain

$$\cdots \leqslant (xy)^r \leqslant (xy)^{r-1} \leqslant \cdots \leqslant (xy) \leqslant 1$$

which cannot be infinite since the domain of m has only finitely many automorphisms. Since xy is invertible, this means $(xy)^r = 1$ for some $r \geqslant 0$. Then $1 = (xy)^r \leqslant \cdots \leqslant (xy) \leqslant 1$ yielding $xy = 1$. So $m = mxy \leqslant ny \leqslant m$ implies $m = ny$. Thus m and n are in the same isomorphism class of $\mathscr{M}/A$; since they are both in $\mathrm{Sub}A$, $m = n$ and we have antisymmetry. ∎

2.8. COROLLARY. *In the situation of Proposition 2.7, suppose* $\mathrm{Sub}A$ *is finite and every $(A \xrightarrow{e} W) \in \mathscr{E}$ is isomorphic to m^* for some $m \in \mathrm{Sub}A$. Then there is a listing of $m_0, \dots, m_s$ of the elements of* $\mathrm{Sub}A$ *such that, together with the listing $m_0^*, \dots, m_s^*$ of* $\mathrm{Quo}A$*, gives an invertible matrix representing $\Theta : \widehat{M} \to \widetilde{E}$.*

PROOF. Take the linear order $m_0, \dots, m_s$ to contain the order $\trianglelefteq$ of the Proposition. Then $x = m_i^* m_j \in \mathscr{G}$ implies $m_i x \leqslant m_j$ and so $m_i \trianglelefteq m_j$ and then $i \leqslant j$. ∎

2.9. REMARK. Under the assumptions in Proposition 2.7 and Corollary 2.8, the functor $\widehat{M} : [\mathscr{G}, \mathscr{V}] \to [\mathscr{F}, \mathscr{V}]$ is actually an equivalence; so $M \dashv E$ is an adjoint equivalence $R\mathscr{F} \simeq R\mathscr{G}$ in $\mathfrak{R}$. A proof using comonadicity is in Section 6 of [16] so we shall not repeat it here. Therefore already we have numerous examples where the core suffices for representations of $\mathscr{F}$ in $\mathscr{V}$. In the next two sections we shall obtain other conditions in order for $\widehat{M}$ to be an equivalence, and we will see that each finite field provides an example satisfying those conditions.

2.10. EXAMPLE. Let $\mathscr{F} = \Delta_\perp$ be the category of non-empty ordinals $\mathbf{a} = \{0, 1, \dots, a-1\}$, $0 < a \in \mathbb{N}$, and order-preserving functions $\xi : \mathbf{a} \to \mathbf{b}$ which also preserve first element. Let $(\mathscr{E}, \mathscr{M})$ be the surjective-injective factorization system. Let $\mathscr{P}$ be the 2-category obtained by ordering the homs of $\mathscr{F}$ with the pointwise order. Each morphism $\xi : \mathbf{a} \to \mathbf{b}$ of $\mathscr{F}$ has a right adjoint ξ^* as since ξ preserves the initial object (and so all joins since $\mathbf{a}$ is linearly ordered). If $\xi \in \mathscr{M}$ then $\xi^* \in \mathscr{E}$ with $1 = \xi^* \xi$, so Corollary 2.8 applies. As the only invertible morphisms in $\mathscr{F}$ are identities, the groupoid $\mathscr{G}$ is discrete with countably many objects; so $[\mathscr{G}, \mathscr{V}]$ is the product $\mathscr{V} \times \mathscr{V} \times \dots$ of countably many copies of $\mathscr{V}$, that is, the category of objects of $\mathscr{V}$ graded by the positive integers.

2.11. Example. Let $\mathscr{A}$ be any category in which every morphism is a monomorphism, pullbacks exist, and each slice category $\mathscr{A}/A$ has finitely many isomorphism classes. Take $\mathscr{F} = \mathscr{A}\sharp$ to be the category of spans in $\mathscr{A}$; that is, the objects are those of $\mathscr{A}$ and the morphisms $f : A \to B$ are isomorphism classes $[f_1, U, f_2]$ of spans $A \xleftarrow{f_1} U \xrightarrow{f_2} B$ in $\mathscr{A}$, where composition uses pullback. Take $\mathscr{M}$ to consist of those spans f with f_1 invertible and $\mathscr{E}$ to consist of those spans f with f_2 invertible. Let $\mathscr{P}$ be the 2-category obtained by declaring $f = [f_1, U, f_2] \leqslant [g_1, V, g_2] = g : A \to B$ when there exists $h : U \to V$ in $\mathscr{A}$ with $g_1 h = f_1$ and $g_2 h = f_2$. Corollary 2.8 applies. The core groupoid $\mathscr{G}$ of $\mathscr{F}$ is isomorphic to the core groupoid of $\mathscr{A}$. An example of this is $\mathscr{A} = \mathrm{FI}$, the category of finite sets and injective functions; so $\mathscr{F} = \mathrm{FI}\sharp$. Then $\mathscr{G} = \mathfrak{S}$, the groupoid of finite sets and bijections, sometimes called *the symmetric groupoid*: so $[\mathscr{G}, \mathscr{V}]$ is the category of R-linear Joyal species [10]; it is equivalent to the product over $n \geqslant 0$ of the categories $[\mathfrak{S}_n, \mathscr{V}]$ of R-linear representations of the groups $\mathfrak{S}_n$. The equivalence $[\mathrm{FI}\sharp, \mathscr{V}] \simeq [\mathfrak{S}, \mathscr{V}]$ appears in the paper [19] by Pirashvili. It was proved independently by Munn [17] and Ponizovskiĭ [20] in the language of inverse semigroups.

3. Main theorem

We continue with a proper factorization system $(\mathscr{E}, \mathscr{M})$ on a category $\mathscr{F}$.

The goal is to find other conditions under which $M : R\mathscr{F} \to R\mathscr{G}$ in $\mathfrak{R}$ is an equivalence. If this is to be the case, M will need to have a left adjoint so we would like to apply Lemma 2.2 to deduce the adjoint from the adjunction $j_* \dashv j^*$. This means we would like M to be a retract of j^*. A natural splitting of the short exact sequence (2.6) would suffice.

A splitting $\rho : M \to j^*$ of the epimorphism $q : j^* \to M$ has components $\rho_{A,B} : M(A, B) \to R\mathscr{F}(A, B)$ which are natural in $A \in \mathscr{G}$ and $B \in \mathscr{F}$. In particular they are natural in $B \in R\mathscr{M}$ and so, by Yoneda, are determined by the value, say $p_A \in R\mathscr{F}(A, A)$, at $1_A \in R\mathscr{M}(A, A)$. Then $\rho_{A,B}(m) = m p_A$. This at least motivates the need for some idempotents p_A if not the need for all the conditions we place on them.

Idempotent Axiom. For each object $A \in \mathscr{F}$, there is a morphism $p_A : A \to A$ in $R\mathscr{F}$ such that

p0. $p_A p_A = p_A$;

p1. if $f \in \mathscr{F}(A, B)$ and $f \in \mathscr{M}'$ then $f p_A = 0$;

p2. if $f \in \mathscr{F}(A, B)$ and $f \in \mathscr{E}'$ then $p_B f = 0$;

p3. if $g \in \mathscr{G}(A, B)$ then $g p_A = p_B g$;

p4. $p'_A = 1_A - p_A \in R(\mathscr{E}'(A, A) \cap \mathscr{M}'(A, A))$.

3.1. Example. Every groupoid $\mathscr{G}$ provides a trivial example with $\mathscr{F} = \mathscr{E} = \mathscr{M} = \mathscr{G}$ and $p_A = 1_A$. In the general case, we can think of $p'_A = 1_A - p_A$ as an obstruction to $\mathscr{F}$'s being a groupoid although we will see in Theorem 3.10 a sense in which $\mathscr{F}$ is not too far from $\mathscr{G}$ at the R-linear level.

Using p1, we have the identity linear function as the left side of a commutative square (3.10). By Proposition 1.2, if $u \in \mathscr{M}'(A, A)$ and $f \in \mathscr{F}(A, B)$ then $fu \in \mathscr{M}'$; so, by p4, $fp'_A \in R\mathscr{M}'(A, B)$ for all $f \in \mathscr{F}(A, B)$. This gives a splitting $\sigma_{A,B}$ of the idempotent $R\mathscr{F}(p'_A, 1_B)$.

$$\begin{array}{ccc} R\mathscr{M}'(A,B) & \xrightarrow{\subseteq} & R\mathscr{F}(A,B) \\ {\scriptstyle 1}\downarrow & \swarrow{\scriptstyle \sigma_{A,B}} & \downarrow{\scriptstyle R\mathscr{F}(p'_A,1_B)} \\ R\mathscr{M}'(A,B) & \xrightarrow[\subseteq]{} & R\mathscr{F}(A,B) \end{array} \tag{3.10}$$

Using p3, we see that the linear functions $R\mathscr{F}(p'_A, 1_B)$ are the components of an idempotent module morphism $\pi' : j^* \to j^*$ which is split by the module morphism $\sigma : j^* \to M'$ and the inclusion $M' \hookrightarrow j^*$. It follows that the short exact sequence (2.6) splits so that, if we put $\pi = 1 - \pi' = R\mathscr{F}(p_A, 1_B)$, we have:

3.2. Lemma. *The idempotent module morphism $\pi = R\mathscr{F}(p_A, 1_B) : j^* \to j^*$ splits as*

$$\begin{array}{ccccc} j^* & & \xrightarrow{\pi} & & j^* \\ & {\scriptstyle q}\searrow & & \nearrow{\scriptstyle \rho} & & \searrow{\scriptstyle q} \\ & & M & \xrightarrow[1_M]{} & & & M \end{array} \tag{3.11}$$

where, for $m \in \mathscr{M}(A, B)$, $\rho_{A,B}(m) = mp_A$ and for $g \in \mathscr{G}(A', A)$ and $f \in \mathscr{F}(B, B')$,

$$M(g, f)m = fmgp_{A'} \ .$$

Dually, we also have:

3.3. Lemma. *The idempotent module morphism $\varpi = R\mathscr{F}(1_A, p_B) : j_* \to j_*$ splits as*

$$\begin{array}{ccccc} j_* & & \xrightarrow{\varpi} & & j_* \\ & {\scriptstyle \tilde{q}}\searrow & & \nearrow{\scriptstyle \tilde{\rho}} & & \searrow{\scriptstyle \tilde{q}} \\ & & E & \xrightarrow[1_E]{} & & & E \end{array} \tag{3.12}$$

where $E(A, B) = R\mathscr{E}(A, B)$, $\tilde{\rho}_{A,B}(e) = p_B e$, and

$$E(f, g)e = p_{B'} gef \ ,$$

for $e \in \mathscr{E}(A, B)$, $f \in \mathscr{F}(A', A)$, $g \in \mathscr{E}((B, B')$.

Recall that, under the conditions of Corollary 2.6, we had the adjunction $M \dashv E$. In the present circumstances, rather, we have the reverse adjunction.

3.4. Lemma. *The idempotent module morphisms $\pi : j^* \to j^*$ and $\varpi : j_* \to j_*$ are mates under the adjunction $j_* \dashv j^*$ in $\mathfrak{R}$. Consequently, $E \dashv M$ in $\mathfrak{R}$.*

Proof. By the Yoneda Lemma or using p3 directly, the square

$$\begin{array}{ccc} R\mathscr{G}(C,D) & \xrightarrow{\;j\;} & R\mathscr{F}(jC,jD) \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle R\mathscr{F}(p_C,1_{jD})} \\ R\mathscr{F}(jC,jD) & \xrightarrow[R\mathscr{F}(1_{jC},p_D)]{} & R\mathscr{F}(jC,jD) \end{array}$$

commutes and the arrows marked j are the components of the unit of $j_* \dashv j^*$. This proves the first sentence. The second sentence uses Lemma 2.2. ∎

Since $\widehat{(-)} : \mathfrak{R}^{\mathrm{op}} \to \mathscr{V}\text{-Cat}$ (see (2.3)) is a pseudofunctor, we have:

3.5. Corollary. *The functor $\widehat{M} : [\mathscr{G},\mathscr{V}] \to [\mathscr{F},\mathscr{V}]$ has right adjoint $\widehat{E}$.*

Since $\widetilde{(-)} : \mathfrak{R}^{\mathrm{co}} \to \mathscr{V}\text{-Cat}$ (see (2.4)), we have:

3.6. Corollary. *The functor $\widetilde{M} : [\mathscr{F},\mathscr{V}] \to [\mathscr{G},\mathscr{V}]$ has right adjoint $\widetilde{E}$.*

3.7. Corollary. *From $\widehat{M} \dashv \widetilde{M}$ and $\widehat{E} \dashv \widetilde{E}$, it follows that $\widehat{E} \cong \widetilde{M}$.*

3.8. Lemma. *The unit $1_{R\mathscr{G}} \Rightarrow M \circ E$ of the adjunction $E \dashv M$ in $\mathfrak{R}$ is invertible. So the unit $1_{[\mathscr{G},\mathscr{V}]} \Rightarrow \widehat{E} \circ \widehat{M}$ of $\widehat{M} \dashv \widehat{E}$ is invertible and $\widehat{M}$ is fully faithful.*

Proof. Note that $(j^* \circ j_*)(C,D) \cong R\mathscr{F}(jC,jD)$ and, from Lemma 2.2, the unit of the adjunction $E \dashv M$ has component at (C,D) equal to the composite

$$R\mathscr{G}(C,D) \xrightarrow{j} R\mathscr{F}(jC,jD) \cong \int^{B\in\mathscr{F}} R\mathscr{F}(B,jD) \otimes R\mathscr{F}(jC,B) \xrightarrow{\int^{B\in\mathscr{F}} q\otimes\tilde{q}} (M\circ E)(C,D)$$

which is inverse to the component of the isomorphism $\bar{\phi}$ of Lemma 2.4.

Alternatively, we take module composition of the splittings of Lemmas 3.2 and 3.3 to obtain the splitting

$$\begin{array}{ccccccc} R\mathscr{F}(jC,jD) & & & \xrightarrow{\pi\circ\varpi = R\mathscr{F}(p_C,p_D)} & & R\mathscr{F}(jC,jD) & \\ & {\scriptstyle q\circ\tilde{q}}\searrow & & \nearrow{\scriptstyle \rho\circ\tilde{\rho}} & & & \searrow{\scriptstyle q\circ\tilde{q}} \\ & & (M\circ E)(C,D) & & \xrightarrow[1]{} & & (M\circ E)(C,D)\ . \end{array}$$

However, we can also compose the splittings as functions to obtain the factorization

$$\begin{array}{ccccccccc} R\mathscr{F}(jC,jD) & & \xrightarrow{R\mathscr{F}(1_C,p_D)} & & R\mathscr{F}(jC,jD) & & \xrightarrow{R\mathscr{F}(p_C,1_D)} & & R\mathscr{F}(jC,jD) \\ & {\scriptstyle \tilde{q}}\searrow & & \nearrow{\scriptstyle \subseteq} & & {\scriptstyle q}\searrow & & \nearrow{\scriptstyle \subseteq} & \\ & & R\mathscr{E}(jC,D) & & & & R\mathscr{M}(C,jD) & & \\ & & & {\scriptstyle q\restriction}\searrow & & \nearrow{\scriptstyle \subseteq} & & & \\ & & & & R\mathscr{G}(C,D) & & & & \end{array}$$

also giving a splitting of the idempotent $\pi \circ \varpi$. The induced isomorphism $R\mathscr{G}(C,D) \cong (M \circ E)(C,D)$ is the component of the unit for $E \dashv M$ at (C,D) as we see by evaluating it and the unit (as given at the beginning of this proof) at a $g \in \mathscr{G}(C,D)$. ∎

3.9. Lemma. *If each object A of $\mathscr{F}$ has only finitely many $\mathscr{M}$-subobjects (that is, each slice category $\mathscr{M}/A$ has only finitely many isomorphism classes) then the counit of the adjunction $E \dashv M$ in $\mathfrak{R}$ is a split epimorphism.*

Proof. First a non-proof! Since $R : \mathrm{Set} \to \mathscr{V}$ preserves colimits and tensor products, Lemma 1.1 yields an isomorphism

$$(E \circ M)(A,B) = \int^{C \in \mathscr{G}} R\mathscr{M}(C,B) \otimes R\mathscr{E}(A,C) \cong R\mathscr{F}(A,B)$$

which in natural in $A \in \mathscr{E}$ and $B \in \mathscr{M}$. However it is not in A and B as objects of $\mathscr{F}$. Moreover, this isomorphism is not the counit $E \circ M \Rightarrow 1_{R\mathscr{F}}$ of $E \dashv M$ as obtained by applying Lemma 2.2.

Now for the proper proof. The counit, which is natural, is the composite

$$(E \circ M)(A,B) = \int^{C \in \mathscr{G}} R\mathscr{M}(C,B) \otimes R\mathscr{E}(A,C) \xrightarrow{\tilde{\rho} \circ \rho} (j_* \circ j^*)(A,B) \xrightarrow{\mathrm{comp}} R\mathscr{F}(A,B)$$

which takes the equivalence class of $m \otimes e \in R\mathscr{M}(C,B) \otimes R\mathscr{E}(A,C)$ to $mp_C e \in R\mathscr{F}(A,B)$. We claim this family has a natural right inverse. In particular, for $A = B$, we must see that the identity 1_A of A is in the image. Take a finite family $(C_i \xrightarrow{m_i} A)_{i=0}^k$ of morphisms in $\mathscr{M}$ representing all isomorphism classes in the ordered set $\mathscr{M}/A$ and with the property that $m_i = m_j n$ for some $C_i \xrightarrow{n} C_j$ implies $i \leqslant j$; we can suppose $C_k = A$ and $m_k = 1_A$. Let

$$\phi_j : \bigoplus_{i \leqslant j} R\mathscr{M}(C_i, A) \otimes R\mathscr{E}(A, C_i) \longrightarrow R\mathscr{F}(A,A)$$

denote the function defined by $\phi_j(m \otimes e) = mp_{C_i} e$ for $m \in R\mathscr{M}(C_i, A)$ and $e \in R\mathscr{E}(A, C_i)$. So the image of ϕ_i is contained in the image of ϕ_j for $i \leqslant j$. We will show that every $f \in \mathscr{F}(A,A)$ is in the image of the function ϕ_k. The proof is by induction on j, where $f = m_j e$ for some $e \in \mathscr{E}$ and unique $j \leqslant k$. Notice that $\mathscr{E}'(C_0, C_0) \cap \mathscr{M}'(C_0, C_0) = \varnothing$: for, if $h \in \mathscr{M}'(C_0, C_0)$ then $h = md$ with $(D \xrightarrow{m} C_0) \in \mathscr{M}$ and $(C_0 \xrightarrow{d} D) \in \mathscr{E}$; so the $\mathscr{M}$-subobject $D \xrightarrow{m_0 m} A$ of A must be isomorphic to m_0 which implies m invertible and hence $h \in \mathscr{E}(C_0, C_0)$. By Idempotent Axiom p4, we have $p'_{C_0} = 0$. So $p_{C_0} = 1_{C_0}$ and $f = m_0 e = m_0 p_{C_0} e = \phi_0(m_0 \otimes e)$ is in the image of ϕ_0. Now take $0 < j \leqslant k$ and assume the result for indices smaller than j. Then $f = m_j e = m_j p_{C_j} e + m_j p'_{C_j} e = \phi_j(m_j \otimes e) + m_j p'_{C_j} e$. By Idempotent Axiom p4, p'_{C_j} is an R-linear combination of terms of the form ne' with $n \in \mathscr{M}$ and $e' \in \mathscr{E}$ non-invertible. So terms of $m_j p'_{C_j} e$ are of the form $m_i e''$ with $i < j$ and $e'' \in \mathscr{E}$. The inductive hypothesis implies all these terms are in the image of ϕ_{j-1}. So f is in the image of ϕ_j.

Hence, $1_A \in \mathscr{F}(A,A)$ is in the image of the A,A component of the counit. Suppose $t \in \int^{C\in\mathscr{G}} R\mathscr{M}(C,A)\otimes R\mathscr{E}(A,C)$ maps to $1_A \in \mathscr{F}(A,A)$. By the Yoneda Lemma, there exists a unique family of morphisms

$$R\mathscr{F}(A,B) \longrightarrow \int^{C\in\mathscr{G}} R\mathscr{M}(C,B)\otimes R\mathscr{E}(A,C)$$

which is natural in $B \in \mathscr{F}$ taking $1_A \in \mathscr{F}(A,A)$ to t. Again by Yoneda, this gives right inverses to the components of the counit. Dually, there is such a family natural in A. By Yoneda yet again, these families agree since they have the same value at the identity. So the counit of $E \dashv M$ has a natural right inverse. ∎

Recall that a $\mathscr{V}$-functor $J : \mathscr{A} \to \mathscr{X}$ is *strongly generating* when the $\mathscr{V}$-functor $\mathscr{X}(-,J) : \mathscr{X} \to [\mathscr{A}^{\mathrm{op}},\mathscr{V}]$ is conservative. When $\mathscr{X}$ is small cocomplete and $\mathscr{A}$ is small, the $\mathscr{V}$-functor $\mathscr{X}(-,J)$ has a left adjoint and J is strongly generating if and only if the counit of the adjunction is a strong epimorphism.

Recall that a $\mathscr{V}$-functor $J : \mathscr{A} \to \mathscr{X}$ is *dense* when the $\mathscr{V}$-functor $\mathscr{X}(-,J) : \mathscr{X} \to [\mathscr{A}^{\mathrm{op}},\mathscr{V}]$ is fully faithful. When $\mathscr{X}$ is small cocomplete and $\mathscr{A}$ is small, the $\mathscr{V}$-functor $\mathscr{X}(-,J)$ has a left adjoint and J is strongly generating if and only if the counit of the adjunction is invertible.

A special case of Theorem 2 of [2] tells says that, if $\mathscr{X}$ is of the form $[\mathscr{B}^{\mathrm{op}},\mathscr{V}]$ with $\mathscr{B}$ small, then $J : \mathscr{A} \to \mathscr{X}$ strongly generating implies J dense.

3.10. THEOREM. *Suppose $(\mathscr{E},\mathscr{M})$ is a proper factorization system on a category $\mathscr{F}$ and assume that the Idempotent Axiom holds. If each object A of $\mathscr{F}$ has only finitely many $\mathscr{M}$-subobjects then the adjunction $E \dashv M$ is an equivalence $R\mathscr{F} \simeq R\mathscr{G}$ in $\mathfrak{R}$.*

PROOF. Using Lemma 3.8 and Corollary 3.7, we know that the unit $1_{R\mathscr{G}} \Rightarrow \widetilde{M}\circ\widehat{M}$ is invertible. So $\widehat{M} : [R\mathscr{G},\mathscr{V}] \to [R\mathscr{F},\mathscr{V}]$ is fully faithful and so is its composite $M' : R\mathscr{G}^{\mathrm{op}} \to [R\mathscr{F},\mathscr{V}]$, $C \mapsto M(C,-)$, with the Yoneda embedding. Moreover, Lemma 3.9 implies M' is strongly generating (the required counit is a split epimorphism and so a strong epimorphism). By Theorem 2 of [2], M' is also dense. So the counit of $E \dashv M$ is invertible. ∎

In the terminology of ring theory in the spirit of [25], Theorem 3.10 implies that $R\mathscr{F}$ and $R\mathscr{G}$ are Morita equivalent several-object R-algebras. In the terminology of enriched category theory, it implies that $R\mathscr{F}$ and $R\mathscr{G}$ are Cauchy equivalent $\mathscr{V}$-categories.

3.11. COROLLARY. *The functor $\widehat{M} : [\mathscr{G},\mathscr{V}] \to [\mathscr{F},\mathscr{V}]$ is an equivalence with inverse equivalence $\widehat{E}$.*

3.12. COROLLARY. *The functor $\widetilde{M} : [\mathscr{F},\mathscr{V}] \to [\mathscr{G},\mathscr{V}]$ is an equivalence with inverse equivalence $\widetilde{E}$.*

3.13. COROLLARY. *The equivalence $\widetilde{M}$ is a retract of the restriction functor*

$$[j,1] : [R\mathscr{F},\mathscr{V}] \to [R\mathscr{G},\mathscr{V}] .$$

Proof. Apply the contravariant functor $[\mathscr{F},\mathscr{V}](-,F)$ to diagram (3.11) of Lemma 3.2 to obtain the idempotent splitting

$$\begin{array}{ccc} [\mathscr{F},\mathscr{V}](R\mathscr{F}(j,-),F) & \xrightarrow{\quad [\mathscr{F},\mathscr{V}](\pi,F) \quad} & [\mathscr{F},\mathscr{V}](R\mathscr{F}(j,-),F) \\ & {\scriptstyle [\mathscr{F},\mathscr{V}](\rho,F)} \searrow \qquad \nearrow {\scriptstyle [\mathscr{F},\mathscr{V}](q,F)} & \\ & [\mathscr{F},\mathscr{V}](M(1_{R\mathscr{G}},-),F) & . \end{array}$$

The top side of the triangle transports across the Yoneda isomorphism to an idempotent on Fj while the bottom vertex is isomorphic to $\widetilde{M}$. ∎

3.14. Corollary. *For any R-linear category $\mathscr{X}$ which has finite direct sums and splitting of idempotents, there is an equivalence*

$$[\mathscr{F},\mathscr{X}] \simeq [\mathscr{G},\mathscr{X}] \ .$$

Proof. See [24] for background. As $R\mathscr{F}$ and $R\mathscr{G}$ are equivalent in $\mathfrak{R}$ (Morita equivalent), their Cauchy completions are equivalent $\mathscr{V}$-categories: $\mathscr{Q}R\mathscr{F} \simeq \mathscr{Q}R\mathscr{G}$. Therefore $[\mathscr{F},\mathscr{X}] \simeq [\mathscr{Q}R\mathscr{F},\mathscr{X}] \simeq [\mathscr{Q}R\mathscr{G},\mathscr{X}] \simeq [\mathscr{G},\mathscr{X}]$ since finite direct sums and splitting of idempotents are preserved by all $\mathscr{V}$-functors (they are absolute colimits). ∎

4. Stiffness and MPK idempotents

This section provides examples of Theorem 3.10 by casting them in a general setting where the Idempotent Axiom holds. The setting assumes a condition we call stiffness which arises in examples from some kind of finiteness. The examples include the Kuhn [15] result and those arising in the Dold-Kan-type work [16].

4.1. Definition. *In any category $\mathscr{F}$, a morphism $f : A \to B$ is called* stiff *when the only endomorphisms of A through which f factors are automorphisms. In other words, $f = (A \xrightarrow{u} A \xrightarrow{v} B)$ implies u invertible. A morphism is* costiff *when it is stiff in the opposite category. A category is called* stiff *when the costiff and stiff morphisms are the $\mathscr{E}$ and $\mathscr{M}$ of a proper factorization system.*

The category of finite sets and the category of finite-dimensional vector spaces over a field are stiff categories: in both costiff means surjective and stiff means injective.

Clearly any stiff endomorphism is an automorphism, so a consequence of stiffness is what might be called "the pigeon-hole principle":

4.2. Proposition. *For all objects A in a stiff category $\mathscr{F}$, the inclusions $\mathscr{G}(A,A) \subseteq \mathscr{M}(A,A)$ and $\mathscr{G}(A,A) \subseteq \mathscr{E}(A,A)$ are equalities. Equivalently, if $\mathscr{G}(A,B) \neq \varnothing$ then $\mathscr{G}(A,B) = \mathscr{E}(A,B) = \mathscr{M}(A,B)$.*

Recall that the set $\mathscr{G}'$ of morphisms is the complement of $\mathscr{G}$ in $\mathscr{F}$. So the identity morphism 1_C of C in the category $\mathscr{F}$ is never in $R\mathscr{G}'(C,C)$. For each C in a stiff $\mathscr{F}$, factorization system properties imply that $R\mathscr{G}'(C,C)$ is a two-sided ideal in $R\mathscr{F}(C,C)$. In particular $R\mathscr{G}'(C,C)$ is an R-algebra, possibly without an identity element. In the case where $\mathscr{F} = \mathrm{vect}_{\mathbb{F}}$ is the category of finite vector spaces over a finite field $\mathbb{F}$ and the factorization is surjective-injective $\mathbb{F}$-linear functions, Laci Kovács [14] showed that there is an identity element making $R\mathscr{G}'(C,C)$ a unital algebra provided the characteristic of $\mathbb{F}$ is invertible in R; also see Subsection 3.1 of Kuhn's paper [15] for helpful material on this. Steinberg has two alternative methods of producing these identity elements; see [22, 23]. A referee for the present paper pointed out that Munn [17] and Ponizovskiĭ [20] independently characterized semisimplicity of semigroup algebras of semigroups satisfying appropriate descending chain conditions. Therefore we make the following definition in the general situation.

4.3. Definition. *An* MPK idempotent *is an element* $\ell_C \in R\mathscr{G}'(C,C)$ *which provides* $R\mathscr{G}'(C,C)$ *with an identity as an* R*-algebra under composition.*

4.4. Example. Return to Example 2.10 where $\mathscr{F} = \Delta_\perp$ and take $C = \mathbf{3} = \{0,1,2\}$. The monoid $\mathscr{F}(C,C) = \Delta_\perp(\mathbf{3},\mathbf{3})$ has six elements, namely, the functions

$$\iota = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \qquad \sigma_0 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \qquad \sigma_1 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\tau = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \qquad \upsilon = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \zeta = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

A presentation of this monoid is provided by the generators σ_0, σ_1 and τ subject to the relations that the generators are idempotents and

$$\sigma_0\sigma_1\sigma_0 = \sigma_0\sigma_1 = \sigma_1\sigma_0\sigma_1$$
$$\tau\sigma_0 = \sigma_0 \ , \ \tau\sigma_1 = \tau = \sigma_0\tau \ , \ \sigma_1\tau = \sigma_1 \ .$$

Note that $\upsilon = \sigma_1\sigma_0$ while $\zeta = \sigma_0\sigma_1$ is a zero in the monoid. The only invertible element is the identity ι. So $\mathscr{G}'(C,C)$ has the five other elements. The MPK idempotent in $R\mathscr{G}'(C,C)$ is $\ell_{\mathbf{3}} = \sigma_0 + \sigma_1 - \upsilon$.

Let us clarify some terminology for use in upcoming Remarks 4.5 and 4.6. For an object C in a category $\mathscr{F}$, we write $\mathscr{F}(C)$ for the full subcategory of $\mathscr{F}$ with the one object C. With the viewpoint that monoids are one object categories, $\mathscr{F}(C)$ is identified with the monoid $\mathscr{F}(C,C)$.

4.5. Remark. Perhaps this is a good point to add a little to Example 4.4 because it does provide an example where a groupoid suffices. The core groupoid of $\Delta_\perp(\mathbf{3})$ is a discrete category with one object. Clearly $R\Delta_\perp(\mathbf{3})$ is not Morita equivalent to $R1$. However, in $R\Delta_\perp(\mathbf{3})$, we have a complete list of orthogonal idempotents (see Appendix B of [16]):

$$e_0 = \sigma_0\sigma_1 \ , \ e_1 = (1-\sigma_0)\sigma_1 \ , \ e_2 = (1-\sigma_1)\sigma_0 \ , \ e_3 = (1-\sigma_1)(1-\sigma_0) \ ,$$

so that, in the Cauchy completion $\mathscr{Q}R\Delta_\perp(\mathbf{3})$ (see Proposition 2.1), we have

$$\mathbf{3} \cong E_0 \oplus E_1 \oplus E_2 \oplus E_3$$

where E_i is the object obtained in splitting the idempotent e_i. If we put $\gamma = \sigma_1\sigma_0 - \sigma_0\sigma_1$ and $\delta = \tau - \sigma_1$, using the relations in the presentation, we obtain the equations

$$e_1\gamma = \gamma = \gamma e_2\ ,\ \delta e_1 = \delta = e_2\delta\ ,\ \gamma\delta = e_1\ ,\ \delta\gamma = e_2\ ,$$

yielding $E_1 \cong E_2$. It follows that we have an isomorphism of the form

$$\mathbf{3} \cong A \oplus B \oplus B \oplus C\ .$$

Transporting the endomorphisms σ_0, σ_1, τ on $\mathbf{3}$ across this isomorphism yields the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\ , \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\ , \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\ .$$

Any endomorphism f of $A \oplus B \oplus B \oplus C$ which commutes with these three matrices must be of the form $f = f_0 \oplus f_1 \oplus f_1 \oplus f_2$. It follows that $R\Delta_\perp(\mathbf{3})$ is $\mathscr{V}$-Morita equivalent to $R3$ where 3 is the discrete category with three objects. In other words, we have an equivalence of R-linear categories

$$[\Delta_\perp(\mathbf{3}), \mathscr{V}] \simeq \mathscr{V} \times \mathscr{V} \times \mathscr{V}\ .$$

Let us clarify some more terminology for use in the upcoming Remark 4.6. A morphism $f : A \to B$ in a category is said to *split* (or to be *von Neumann regular*) when there exists a morphism $g : B \to A$ such that $fgf = f$. A monomorphism splits if and only if it has a left inverse; that is, it is a coretraction. An epimorphism splits if and only if it has a right inverse; that is, it is a retraction. If a morphism f factors as $f = jr$ with r a retraction and j a coretraction then f splits. A category is *von Neumann regular* when every morphism is split; this terminology applies to monoids as one object categories.

Every morphism in the category of sets whose domain is not empty splits. The category of vector spaces over a division ring is von Neumann regular, as are the categories $\mathscr{F}$ of Examples 2.10 and 2.11.

4.6. Remark. It can happen that $R\mathscr{F}(C)$ is semisimple. In particular, this is true when $Y = \mathscr{F}(C)$ is any finite monoid of Lie type and R is a field of characteristic 0; see Corollary 2.9 of [18]. Section 5.4 of Steinberg [22] provides necessary and sufficient conditions on a finite monoid Y and field R in order for RY to be semisimple: it is necessary that Y be regular and the characteristic of R should not divide the order of the group of invertible elements of the monoid pYp for any idempotent $p \in Y$. With semisimplicity of RY, the inclusion of every two-sided ideal J of RY into RY splits as a left module morphism and splits as a right module morphism. The value of the left module splitting at the identity of RY gives a right identity for J and the value of the right module splitting at the identity of RY gives a left identity for J. So J becomes an R-algebra with identity. This is a source of MPK idempotents for categories $\mathscr{F}$ whose endomorphism monoids $\mathscr{F}(C)$ are finite and have $R\mathscr{F}(C)$ semisimple. An anonymous referee has pointed out that the assumption on divisibility of the order of the group of invertible elements can be dropped and you get the condition for a finite regular monoid to be Frobenius, which is the same as saying its algebra is isomorphic to the category algebra of the core groupoid of its Cauchy completion.

4.7. Example. As already mentioned, if $\mathbb{F}$ is a finite field, $\mathscr{F} = \mathrm{vect}_{\mathbb{F}}$, and R is a ring in which the characteristic of $\mathbb{F}$ is invertible then each object $C \in \mathscr{F}$ has a MPK idempotent ℓ_C; see [14].

4.8. Example. Let $\mathbb{F}$ be a finite field. Let $\mathscr{F}$ be the category whose objects (V, α) are finite $\mathbb{F}$-vector spaces V equipped with an $\mathbb{F}$-linear isomorphism $\alpha : V^* \cong V$. A morphism $f : (V, \alpha) \to (W, \beta)$ in $\mathscr{F}$ is an $\mathbb{F}$-linear function $f : V \otimes V \to W \otimes W$ for which there exist $\lambda \in \mathbb{F}$ and $\mathbb{F}$-linear functions $a : V \to W$ and $b : W \to V$ such that $f = a \otimes b^t$ where $ab = \lambda 1_W$, $ba = \lambda 1_V$, and $b^t = \beta b^* \alpha^{-1}$. Composition is that of $\mathbb{F}$-linear functions. Each monoid $\mathscr{F}(V, \alpha)$ is a monoid of Lie type; see Example 2.3 of [18]. By Remark 4.6, every (V, α) has a MPK idempotent.

4.9. Example. Benjamin Steinberg told me that Itamar Stein had proved the semisimplicity of the R-algebra $R\Delta_{\perp}(\mathbf{a})$ for R any field; see Example 2.10 for our notation. The proof has now appeared in [21].

4.10. Lemma. *Assume the pigeon-hole principle and that each object C has a MPK idempotent ℓ_C. Then ℓ_C is a central idempotent in the R-algebra $R\mathscr{F}(C)$ and the morphisms $\ell_C : C \to C$ are the components of a natural endomorphism of the inclusion functor of $j : R\mathscr{G} \to R\mathscr{F}$.*

Proof. Take any $f : C \to C$ in $\mathscr{F}$. Then, by the two-sided ideal property, both $\ell_C f$ and $f\ell_C$ are in $R\mathscr{G}'(C, C)$ so, by the identity element property, $\ell_C f = (\ell_C f)\ell_C = \ell_C(f\ell_C) = f\ell_C$, proving the first clause.

Now take $g \in \mathscr{G}(C, D)$. For any $s \in \mathscr{G}'(D, D)$, we have $g^{-1}sg \in \mathscr{G}'(C, C)$. So $(g^{-1}sg)\ell_C = g^{-1}sg$. This shows that $s(g\ell_C g^{-1}) = s$ yielding that $g\ell_C g^{-1}$ is a right identity in $R\mathscr{G}'(D, D)$. By uniqueness, $g\ell_C g^{-1} = \ell_D$ which proves the claimed naturality. ∎

4.11. Proposition. *Assume that $\mathscr{F}$ is stiff and that each object C has an MPK idempotent ℓ_C. Then the idempotents $p_A = 1_A - \ell_A : A \to A$ satisfy the Idempotent Axiom of Section 3.*

PROOF. Since ℓ_A is an idempotent, so too is its complement p_A yielding condition p0. Lemma 4.10 gives the naturality condition p3. Since $\mathscr{G}'(A,A) = \mathscr{E}'(A,A) \cap \mathscr{M}'(A,A)$ (by Proposition 4.2) and $p'_A = \ell_A \in R\mathscr{G}'(A,A)$, we have p4. Since p1 and p2 are dual, it remains to prove condition p1. Take $f \in \mathscr{F}(A,B)$ with $f \notin \mathscr{M}$. Then f is not stiff so there is a factorization $f = vu$ with $u \in \mathscr{G}'(A,A)$. Consequently, $fp_A = f(1_A - \ell_A) = f - vu\ell_A = f - vu = 0$. ∎

All this leads to the conclusion:

4.12. COROLLARY. *Suppose $(\mathscr{E}, \mathscr{M})$ is a proper factorization system on a stiff category $\mathscr{F}$ where each object C has an MPK idempotent and only finitely many $\mathscr{M}$-subobjects. Let $\mathscr{G}$ denote the core groupoid of $\mathscr{F}$. Let R be a commutative ring and let $\mathscr{X}$ be a category whose homs are enriched in the monoidal category of R-modules. If $\mathscr{X}$ admits finite direct sums and splitting of idempotents then there is an equivalence of enriched categories*

$$[\mathscr{F}, \mathscr{X}] \simeq [\mathscr{G}, \mathscr{X}] .$$

5. Monoidal structure on $[\mathscr{F}, \mathscr{V}]$

In [12] we were interested in a particular monoidal structure on the category $[\mathscr{GL}(q), \mathrm{Vect}_{\mathbb{C}}]$ of complex representations of the general linear groupoid $\mathscr{GL}(q)$ over the field $\mathbb{F}_q$ of cardinality q. That tensor product categorified the multiplication on the class function ring of the finite general linear groups over $\mathbb{F}_q$ due to Green [8]. As $\mathscr{GL}(q)$ is the core groupoid of the category $\mathrm{vect}_{\mathbb{F}_q}$ of finite vector spaces over $\mathbb{F}_q$ with all linear functions, and Corollary 4.12 applies to $\mathscr{F} = \mathrm{vect}_{\mathbb{F}_q}$, it is of interest to see how the monoidal structure on $[\mathscr{GL}(q), \mathrm{Vect}_{\mathbb{C}}]$ manifests itself on $[\mathrm{vect}_{\mathbb{F}_q}, \mathrm{Vect}_{\mathbb{C}}]$ via transport of structure.

The bicategory $\mathfrak{R} = \mathscr{V}$-Mod is autonomous symmetric monoidal (see [3]). The tensor product is the usual tensor product of $\mathscr{V}$-enriched categories as per [4]: take cartesian product of object sets and tensor over R of hom R-modules.

Comonoidales (= pseudocomonoids) in $\mathfrak{R}$ are promonoidal $\mathscr{V}$-categories in the sense of Day [1]. As a consequence of Theorem 3.10, any promonoidal structure on $R\mathscr{G}$ transports across the adjoint equivalence $E \dashv M$ to a promonoidal structure on $R\mathscr{F}$. In the case $\mathscr{F} = \mathrm{vect}_{\mathbb{F}_q}$ and $R = \mathbb{C}$ the complex numbers, a braided promonoidal structure was defined on $R\mathscr{G}$ in Remark 4.3 of [12]. Here we will recall the general setting of Example 4 in Section 2 of [27] and transport the promonoidal structure to $R\mathscr{F}$.

The assumptions of this section are that $\mathscr{F}$ is an abelian category, $\mathscr{E}$ consists of the epimorphisms, $\mathscr{M}$ consists of the monomorphisms, and the Idempotent Axiom of Section 3 holds.

This promonoidal structure (unlike the braiding) actually exists on the ordinary category $\mathscr{G}$ not just the R-linear category $R\mathscr{G}$. It uses the exact sequences of $\mathscr{F}$. We have

$$P_{\mathscr{G}} : \mathscr{G}^{\mathrm{op}} \times \mathscr{G}^{\mathrm{op}} \times \mathscr{G} \longrightarrow \mathrm{Set}$$

defined by

$$P_{\mathscr{G}}(A,B;C) = \left\{(x,y) \;\middle|\; 0 \to A \xrightarrow{x} C \xrightarrow{y} B \to 0 \text{ is a short exact sequence in } \mathscr{F}\right\} ,$$

$P_{\mathscr{G}}(g_1, g_2; g_3)(x, y) = (g_3 x g_1, g_2^{-1} y g_3^{-1})$ for $g_1 \in \mathscr{G}(A', A)$, $g_2 \in \mathscr{G}(B', B)$, $g_3 \in \mathscr{G}(C, C')$, and

$$J_{\mathscr{G}} : \mathscr{G} \longrightarrow \mathrm{Set}$$

defined by

$$J_{\mathscr{G}} A = \begin{cases} 1 & \text{for } A = 0 \\ \varnothing & \text{otherwise.} \end{cases}$$

The associativity isomorphism

$$\alpha : \int^{X} P_{\mathscr{G}}(X, C; D) \times P_{\mathscr{G}}(A, B; X) \cong \int^{Y} P_{\mathscr{G}}(A, Y; D) \times P_{\mathscr{G}}(B, C; Y)$$

is defined by

$$\alpha[(u, v), (x, y)] = [(x', y'), (u', v')]$$

as in the below 3×3 diagram whose rows and columns are short exact sequences in $\mathscr{F}$.

$$\begin{array}{ccccc} A & \xrightarrow{1} & A & \longrightarrow & 0 \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x'} & & \downarrow \\ X & \xrightarrow{u} & D & \xrightarrow{v} & C \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle y'} & & \downarrow{\scriptstyle 1} \\ B & \xrightarrow[u']{} & Y & \xrightarrow[v']{} & C \end{array}$$

Day convolution yields the monoidal structure on $[\mathscr{G}, \mathscr{V}]$ with tensor product

$$(S \otimes_{\mathscr{G}} T)C = \sum_{H \leqslant C} SH \otimes T(C/H)$$

where the sum runs over subobjects H of C in $\mathscr{F}$ and C/H is the quotient object.

The promonoidal structure $RP_{\mathscr{G}}$ on $R\mathscr{G}$ transports across the adjoint equivalence of Theorem 3.10 to a promonoidal structure $RP_{R\mathscr{F}}$ on $R\mathscr{F}$ defined by

$$\begin{aligned} P_{R\mathscr{F}}(A, B; C) &= \int^{A', B', H} M(H, C) \otimes RP_{\mathscr{G}}(A', B'; H) \otimes E(A, A') \otimes E(B, B') \\ &\cong \int^{H} M(H, C) \otimes (E(A, -) \otimes_{\mathscr{G}} E(B, -))H \\ &\cong \sum_{K \leqslant H \leqslant C} R\mathscr{E}(A, jK) \otimes R\mathscr{E}(B, j(H/K)) \ . \end{aligned} \tag{5.13}$$

Day convolution yields the monoidal structure on $[\mathscr{F}, \mathscr{V}]$ with tensor product

$$(F \otimes_{\mathscr{G}} G)C = \sum_{K \leqslant H \leqslant C} \int^{A, B \in \mathscr{F}} R\mathscr{E}(A, jK) \otimes R\mathscr{E}(B, j(H/K)) \otimes FA \otimes GB \ .$$

5.1. Remark. While the promonoidal structure (5.13) on $R\mathscr{F}$ is not the R-linearization of one on $\mathscr{F}$, it does exist at the level of the free category $p\mathscr{F}$ enriched in pointed sets on $\mathscr{F}$. This is clear from the formula for the effect of E on morphisms given in Lemma 3.3.

References

[1] Brian J. Day, On closed categories of functors. *Lecture Notes in Math. 137* (Springer, Berlin 1970) 1–38.

[2] Brian Day and Ross Street, Categories in which all strong generators are dense. *J. Pure Appl. Algebra 43* (1986) 235–242.

[3] Brian Day and Ross Street, Monoidal bicategories and Hopf algebroids. *Advances in Math. 129* (1997) 99–157.

[4] Samuel Eilenberg and G. Max Kelly, Closed categories. *Proceedings of the Conference on Categorical Algebra (La Jolla, 1965)* (Springer-Verlag,1966) 421–562.

[5] Samuel Eilenberg and Saunders Mac Lane, On the groups $H(\Pi, n)$. II. Methods of computation. *Ann. of Math. 60(2)* (1954) 49–139.

[6] D. K. Faddeev, On representations of a full matrix semigroup over a finite field. *Dokl. Akad. Nauk SSSR 230* (1976) 1290–1293.

[7] Peter J. Freyd and G. Max Kelly, Categories of continuous functors I. *J. Pure and Applied Algebra 2* (1972) 169–191; Erratum Ibid. 4 (1974) 121.

[8] J. A. Green, The characters of the finite general linear groups. *Trans. Amer. Math. Soc. 80* (1955), 402-447.

[9] Randall D. Helmstutler, Conjugate pairs of categories and Quillen equivalent stable model categories of functors. *J. Pure and Applied Algebra 218* (2014)1302–1323.

[10] André Joyal, Une theory combinatoire des series formelles. *Advances in Math. 42* (1981) 1–82.

[11] André Joyal, Foncteurs analytiques et espèces de structures. *Lecture Notes in Mathematics 1234* (Springer 1986) 126–159.

[12] André Joyal and Ross Street, The category of representations of the general linear groups over a finite field. *J. Algebra 176(3)* (1995) 908–946.

[13] Mareli Korostenski and Walter Tholen, Factorization systems as Eilenberg-Moore algebras. *J. Pure Appl. Algebra 85* (1993) 57–72.

[14] L.G. (Laci) Kovács, Semigroup algebras of the full matrix semigroup over a finite field. *Proc. Amer. Math. Soc. 116* (1992) 911–919.

[15] Nicholas J. Kuhn, Generic representation theory of finite fields in nondescribing characteristic. *Advances in Mathematics 272* (2015) 598–610.

[16] Stephen Lack and Ross Street, Combinatorial categorical equivalences of Dold-Kan type. *J. Pure Appl. Algebra 219* (2015) 4343–4367; *224* (2020) 1364–1366; also see arxiv.org/abs/1402.7151.

[17] Walter D. Munn, Matrix representations of semigroups. *Proc. Cambridge Philos. Soc. 53* (1957) 5–12.

[18] Jan Okniński and Mohan S. Putcha, Complex representations of matrix semigroups. *Transactions American Math. Society 323(2)* (February 1991) 563–581.

[19] Teimuraz Pirashvili, Dold-Kan type theorem for Γ-groups. *Mathematische Annalen 318* (2000) 277–298.

[20] I.S. Ponizovskiĭ, On matrix representations of associative systems. (Russian) *Mat. Sb. (N.S.) 38(80)* (1956) 241–260.

[21] Itamar Stein, Algebras of reduced E-Fountain semigroups and the generalized right ample identity II. arxiv.org/abs/2209.07112.

[22] Benjamin Steinberg, Representation Theory of Finite Monoids. *Universitext* (Springer International, Switzerland 2016); DOI 10.1007/978-3-319-43932-7.

[23] Benjamin Steinberg, Factoring the Dedekind-Frobenius determinant of a semigroup. *J. Algebra 605* (2022) 1–36.

[24] Ross Street, Absolute colimits in enriched categories. Cahiers de topologie et géométrie différentielle 24 (1983) 377–379

[25] Ross Street, Ideals, radicals, and structure of additive categories. *Applied Categorical Structures 3* (1995) 139–149.

[26] Ross Street, Enriched categories and cohomology with author commentary. *Reprints in Theory and Applications of Categories 14* (2005) 1–18.

[27] Ross Street, Monoidal categories in, and linking, geometry and algebra. *Bulletin of the Belgian Mathematical Society – Simon Stevin 19(5)* (2012) 769–821.

Department of Mathematics, Macquarie University, NSW 2109, Australia
Email: <ross.street@mq.edu.au>

This article may be accessed at http://www.tac.mta.ca/tac/