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A FINITE ALGEBRAIC PRESENTATION OF LAWVERE THEORIES IN THE OBJECT-CLASSIFIER TOPOS

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Dedicated to Bill Lawvere with gratitude and admiration

Abstract. Over the topos of sets, the notion of Lawvere theory is infinite countably-sorted algebraic but not one-sorted algebraic. Shifting viewpoint over the object-classifier topos, a finite algebraic presentation of Lawvere theories is considered.

1. Introduction

The notion of Lawvere theory was introduced by Bill Lawvere in his seminal PhD thesis (Lawvere, 1963, Chapter II, Section 1). It initiated the subject of categorical algebra while transforming the subject of universal algebra (Wraith (1969); Adámek, Rosický, and Vitale (2011)).

A pivotal role in the definition of Lawvere theory is played by the category $\mathbb{E} = \mathbb{F}^{\mathrm{op}}$, for $\mathbb{F}$ the category with objects the set of natural numbers $\mathbb{N}$ and morphisms $m \to n$ in $\mathbb{F}$ given by functions $[m] \to [n]$ where $[\ell] = \{ k \in \mathbb{N} \mid k < \ell \}$.

1.1. Definition. *A* Lawvere theory *is a pair* $(\mathcal{L}, L)$ *with* $L : \mathbb{E} \to \mathcal{L}$ *an identity-on-objects finite-product-preserving functor.*

Straightaway, Bill Lawvere introduced and studied the category of Lawvere theories. A Lawvere-theory morphism is a functor between co-slices under $\mathbb{E}$. The most basic example of a Lawvere theory is the initial one $(\mathbb{E}, \mathrm{Id}_{\mathbb{E}})$. The terminal Lawvere theory $(\mathbb{T}, T)$ arises from the identity-on-objects and fully-faithful factorization $\mathbb{E} \to \mathbb{T} \to \mathbf{1}$ of the unique functor from $\mathbb{E}$ to the terminal category $\mathbf{1}$.

The connection with universal algebra was established by Bill Lawvere from the outset. For instance, he showed, both in general and in examples, how Lawvere theories may be described by means of equational presentations (Lawvere, 1963, Chapter II, Section 2).

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In this paper, we will use the notation $x_1, \ldots, x_n \vdash t = u$ for the equation that identifies the terms t and u, both with free variables amongst $x_1, \ldots, x_n$.

The initial Lawvere theory, being presented by no operations and no equations, was defined by Bill Lawvere as the *theory of equality*. On the other hand, the terminal Lawvere theory may be presented by means of any non-empty set of constants O subject to the equations $x \vdash o = x$ for each constant o in O. This example displays an important contrast, often emphasized by Bill Lawvere, between Lawvere theories and equational presentations: the former are representation independent. Indeed, there are in general many equational presentations for the same Lawvere theory and each Lawvere theory determines one variety of algebraic structure or algebraic category (Lawvere, 1963, Chapter III).

Bill Lawvere defined a Lawvere theory $(\mathcal{L}, L)$ to be inconsistent whenever $\mathcal{L}$ is equivalent to either the terminal category or the arrow category. The terminal theory $(\mathbb{T}, T)$ is an example of the first kind. An example of the second kind is the theory presented by no operators subject to the equation $x, y \vdash x = y$. This is a non-initial Lawvere subtheory of the terminal Lawvere theory. Therefore, the category of Lawvere theories has a non-initial sub-terminal object. The fact below then follows from (Adámek, Rosický, and Vitale, 2011, Lemma 11.19).

1.2. Lemma. *The category of Lawvere theories is not one-sorted algebraic.*

In other words, the language of (one-sorted) universal algebra is not expressive enough to describe itself.

It is well-known, however, that the category of Lawvere theories is countably-sorted algebraic; see, for instance, (Adámek, Rosický, and Vitale, 2011, Remark 14.25(1)). One way to see this in direct connection to universal algebra is by means of the notion of *abstract clone*; for which see, for instance, Cohn (1981); Taylor (1993); Grätzer (2008). In the context of many-sorted universal algebra, we will use the notation $x_1 : s_1, \ldots, x_n : s_n \vdash t = u : s$ for the equation that identifies the terms t and u of sort s, both with free variables amongst $x_1, \ldots, x_n$ respectively of sort $s_1, \ldots, s_n$.

The countably-sorted equational presentation of abstract clones has set of sorts $\mathbb{N}$ and operators

$$\mu_{m,n} : m, \underbrace{n, \ldots, n}_{m \text{ times}} \to n \ \ (m, n \in \mathbb{N}) \ \ , \quad \iota^m_i : m \ \ \big(m \in \mathbb{N}, i \in [m]\big)$$

subject to the equations

$$\begin{aligned} & x : \ell, y_0 : m, \ldots, y_{\ell-1} : m, z_0 : n, \ldots, z_{m-1} : n \\ & \quad \vdash \ \mu_{m,n}(\mu_{\ell,m}(x, y_0, \ldots, y_{\ell-1}), z_0, \ldots, z_{m-1}) \\ & \qquad\quad = \ \mu_{\ell,n}(x, \mu_{m,n}(y_0, z_0, \ldots, z_{m-1}), \ldots, \mu_{m,n}(y_{\ell-1}, z_0, \ldots, z_{m-1})) : n \end{aligned}$$

$$x_0 : n, \ldots, x_{m-1} : n \vdash \mu_{m,n}(\iota^m_i, x_0, \ldots, x_{m-1}) = x_i : n$$

$$x : m \vdash \mu_{m,m}(x, \iota^m_0, \ldots, \iota^m_{m-1}) = x : m$$

The idea behind this axiomatization is that the operators $\mu_{m,n}$ model simultaneous substitution (or cartesian multi-composition) while the constants ι_i^n model variables (or projections).

The categories of Lawvere theories and of abstract clones are equivalent[1]; see, for instance, (Taylor, 1973, Appendix). Concisely put, the abstract clone of a Lawvere theory $\mathcal{L}$ consists of the sorted family of sets $\{\, \mathcal{L}(n,1)\,\}_{n\in\mathbb{N}}$ equipped with the operations

$$\mathcal{L}(m,1)\times\mathcal{L}(n,1)^m \xrightarrow{\cong} \mathcal{L}(m,1)\times\mathcal{L}(n,m) \xrightarrow{\circ} \mathcal{L}(n,1) \quad (m,n\in\mathbb{N})$$

$$\pi_{i+1}\in\mathcal{L}(m,1) \quad \big(m\in\mathbb{N}, i\in[m]\big)$$

while the Lawvere theory of an abstract clone

$$\{\, C_n\,\}_{n\in\mathbb{N}} \ , \quad \mu_{m,n} : C_m\times(C_n)^m \to C_n \ \ (m,n\in\mathbb{N}) \ , \quad \iota_i^m\in C_m \ \ \big(m\in\mathbb{N}, i\in[m]\big)$$

has hom-sets $(C_m)^n$, from m to n in $\mathbb{N}$, with composition

$$(C_m)^n\times(C_\ell)^m \xrightarrow{\langle\pi_k\times\mathrm{id}\rangle_{k=1,\ldots,n}} \big(C_m\times(C_\ell)^m\big)^n \xrightarrow{(\mu_{m,\ell})^n} (C_\ell)^n \quad (\ell,m,n\in\mathbb{N})$$

and identities

$$(\iota_0^m,\ldots,\iota_{m-1}^m)\in(C_m)^m \quad (m\in\mathbb{N})$$

We have so far confined our discussion of categorical algebra to universal algebra; that is, on sets. Of course, one of the benefits of the categorical approach is the generalization to other realms. This was recognized early on by Bill Lawvere. In particular, in Lawvere (1969), he put forward the study of more general equational structure emphasizing that this necessitates operations with both arities and co-arities, where the latter embody generalized tupling, parameterization, or indexing. In this paper, we consider such algebraic structure in the object-classifier topos $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{S}et}^{\mathbb{F}}$. Specifically, we present a finite equational presentation of Lawvere theories over $\boldsymbol{\mathcal{F}}$. This result is implicit in Fiore, Plotkin, and Turi (1999) Theorem 3.3 and Proposition 3.4, and we use this occassion to dedicate it to Bill Lawvere. The corresponding *theory of substitution* has two operators, respectively with arity-coarity pairs $(\,V+1\,,\,1\,)$ and $(\,0\,,\,V\,)$, where $V=\mathbb{F}(1,-)$ is the universal object model, subject to four equations. A substitution algebra is then an object A in $\boldsymbol{\mathcal{F}}$ together with operations $s : A^V\times A\to A$ and $v : 1\to A^V$ satisfying natural laws when s is understood as modelling (single variable) substitution and v as (generic) variables. The use of the non-standard arity $V+1$ and of the non-standard co-arity V is fundamental.

The present development streamlines that of Fiore, Plotkin, and Turi (1999) from the viewpoint of a universal characterization of $\mathbb{F}$ as a monoidal theory that emphasizes its structural properties (weakening, contraction, and exchange) and serves as the conceptual foundation (c.f. Fiore (2005, 2006)). In this vein, Sections 2 and 3 introduce the

[1]This was in fact known to Bill Lawvere from the outset. Indeed, he once mentioned to the first author at a category theory conference that he had actually come up with the notion of abstract clone in the process of developing the notion of Lawvere theory.

necessary theory of symmetric (co)monads and distributive laws involving them. Within this framework, Section 4 recasts and generalizes the notion of substitution algebra. Finally, Section 5 outlines, for the first time in print, the isomorphism between abstract clones and substitution algebras. This establishes the main aim of the paper in providing a finite equational presentation of Lawvere theories in the object-classifier topos. From the viewpoint of many-sorted universal algebra, it may be reinterpreted as providing a countably-sorted algebraic presentation of Lawvere theories by an axiomatization of single-variable substitution.

Along the above lines of enquiry, we leave open the conjecture that the notion of Lawvere theory is truly infinite countably-sorted algebraic, in that there are no countably-sorted equational presentations of Lawvere theories with either a finite set of sorts, a finite set of operators, or a finite set of equations.

2. Symmetric monoids and monads

2.1. In the spirit of Lawvere (1963), in Fiore, Plotkin, and Turi (1999), the category $\mathbb{F}$ is viewed as the free cocartesian category on an object 1 with a chosen (strict) coproduct structure

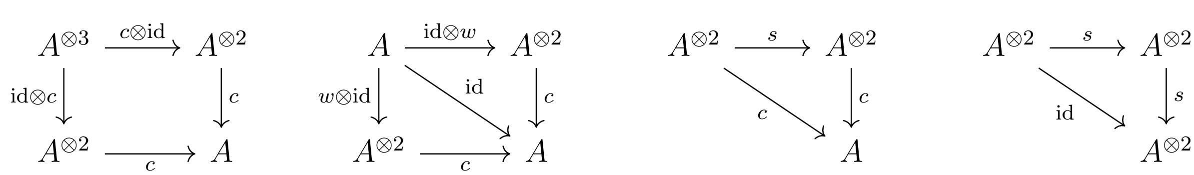
$$n \xrightarrow{\mathrm{old}_n} n+1 \xleftarrow{\mathrm{new}_n} 1$$

Via this coproduct structure, every morphism in $\mathbb{F}$ may be described using the following generating morphisms:

$$\begin{aligned} \mathrm{c} &= [\mathrm{id}_1, \mathrm{id}_1] : 2 \to 1 \\ \mathrm{w} &= \mathrm{old}_0 : 0 \to 1 \\ \mathrm{s} &= [\mathrm{new}_1, \mathrm{old}_1] : 2 \to 2 \end{aligned}$$

As is well-known, the first two maps equip $\mathbb{F}$ with the monad $(\mathrm{Id}+1, \mathrm{id}+\mathrm{w}, \mathrm{id}+\mathrm{c})$ taking a coproduct with 1. This description, however, overlooks the map s. To account for it, we consider the work of Grandis (2001) where $\mathbb{F}$ is instead viewed as the free strict monoidal category on a chosen "symmetric monoid".

2.2. Definition. *Let* $(\mathscr{C}, \otimes, I)$ *be a monoidal category. A* symmetric monoid (A, c, w, s) *in* $\mathscr{C}$ *consists of an object* A *of* $\mathscr{C}$ *and morphisms* $c : A \otimes A \to A$, $w : I \to A$, *and* $s : A \otimes A \to A \otimes A$ *satisfying the following commutative diagrams (where associators and unitors have been omitted):*



$$\begin{array}{ccccc} A^{\otimes 3} & \xrightarrow{s\otimes \mathrm{id}} & A^{\otimes 3} & \xrightarrow{\mathrm{id}\otimes s} & A^{\otimes 3} \\ {\scriptstyle \mathrm{id}\otimes s}\downarrow & & & & \downarrow{\scriptstyle s\otimes \mathrm{id}} \\ A^{\otimes 3} & \xrightarrow[s\otimes \mathrm{id}]{} & A^{\otimes 3} & \xrightarrow[\mathrm{id}\otimes s]{} & A^{\otimes 3} \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{w\otimes \mathrm{id}} & A^{\otimes 2} \\ & {\scriptstyle \mathrm{id}\otimes w}\searrow & \downarrow{\scriptstyle s} \\ & & A^{\otimes 2} \end{array} \qquad \begin{array}{ccccc} A^{\otimes 3} & \xrightarrow{s\otimes \mathrm{id}} & A^{\otimes 3} & \xrightarrow{\mathrm{id}\otimes s} & A^{\otimes 3} \\ {\scriptstyle \mathrm{id}\otimes c}\downarrow & & & & \downarrow{\scriptstyle c\otimes \mathrm{id}} \\ A^{\otimes 2} & & \xrightarrow[s]{\qquad\qquad} & & A^{\otimes 2} \end{array}$$

Note, interestingly, that this definition does not require the tensor product to be braided and, when it is, it may be specialized to the expected definition of a commutative monoid.

2.3. Symmetric monads, comonoids, and comonads. Recalling that the category of endofunctors $\mathrm{Endo}(\mathscr{C})$ on a category $\mathscr{C}$ is strict monoidal, the above allows for the definition of a *symmetric monad* on $\mathscr{C}$ as a symmetric monoid in $\mathrm{Endo}(\mathscr{C})$. There are, of course, appropriate dual definitions of a *symmetric comonoid* and *symmetric comonad*.

2.4. Examples.

1. For a symmetric monoid A, the tensoring with A functor $(-) \otimes A$ is a symmetric monad. In particular, as every object A in a cartesian monoidal category is canonically a symmetric comonoid, the product with A comonad $(-) \times A$ is canonically symmetric.

2. For a symmetric comonoid A in a monoidal closed category, the internal-hom functor $[A, -]$ is a symmetric monad. In particular, for every object A in a cartesian monoidal closed category, the exponentiation by A monad $(-)^A$ is canonically symmetric.

3. Monoidal functors preserve symmetric (co)monoids. In particular, by the convolution monoidal structure (Day (1970); Im and Kelly (1986)), representable presheaves of symmetric (co)monoids are symmetric (co)monoids.

4. A concrete class of examples arises from the general ones above as follows.

   Let (A, c, w, s) be a symmetric monoid in a monoidal small category $\mathscr{C}$. Then, the representable $R_A = \mathscr{C}(A, -)$ in $\boldsymbol{\mathcal{S}et}^{\mathscr{C}}$ is a symmetric comonoid for the convolution monoidal structure and the convolution internal-hom functor $[R_A, -]$ on $\boldsymbol{\mathcal{S}et}^{\mathscr{C}}$ is a symmetric monad. In fact, it is canonically isomorphic to the symmetric monad $(- \otimes A)^\star = \boldsymbol{\mathcal{S}et}^{(-)\otimes A}$ on $\boldsymbol{\mathcal{S}et}^{\mathscr{C}}$ with the simple description below:

$$\begin{aligned} \big(- \otimes A\big)^\star(X) &= X(- \otimes A) \\ \underline{c}_X &= X(- \otimes c) : X(- \otimes A \otimes A) \longrightarrow X(- \otimes A) \\ \underline{w}_X &= X(- \otimes w) : X(-) \longrightarrow X(- \otimes A) \\ \underline{s}_X &= X(- \otimes s) : X(- \otimes A \otimes A) \longrightarrow X(- \otimes A \otimes A) \end{aligned}$$

We will need to consider monoidal notions of symmetric monads.

2.5. Definition. *Let* $(\mathscr{C}, \otimes, I)$ *be a monoidal category. A* lax monoidal *symmetric monad on* $\mathscr{C}$ *is a symmetric monad* $(T, \mu, \eta, \varsigma)$ *on* $\mathscr{C}$ *equipped with a natural transformation* $\ell_{A,B} : T(A) \otimes T(B) \to T(A \otimes B)$ *and a morphism* $e : I \to T(I)$ *satisfying, for every* $A, B \in \mathscr{C}$*, the following commutative diagrams*

$$\begin{array}{ccccc} T^2(A) \otimes T^2(B) & \xrightarrow{\ell_{T(A),T(B)}} & T(T(A) \otimes T(B)) & \xrightarrow{T(\ell_{A,B})} & T^2(A \otimes B) \\ \downarrow{\scriptstyle \mu_A \otimes \mu_B} & & & & \downarrow{\scriptstyle \mu_{A\otimes B}} \\ T(A) \otimes T(B) & & \xrightarrow[\ell_{A,B}]{} & & T(A \otimes B) \end{array}$$

$$\begin{array}{ccc} A \otimes B & & \\ \downarrow{\scriptstyle \eta_A \otimes \eta_B} & \searrow{\scriptstyle \eta_{A\otimes B}} & \\ T(A) \otimes T(B) & \xrightarrow[\ell_{A,B}]{} & T(A \otimes B) \end{array}$$

$$\begin{array}{ccccc} T^2(A) \otimes T^2(B) & \xrightarrow{\ell_{T(A),T(B)}} & T(T(A) \otimes T(B)) & \xrightarrow{T(\ell_{A,B})} & T^2(A \otimes B) \\ \downarrow{\scriptstyle \varsigma_A \otimes \varsigma_B} & & & & \downarrow{\scriptstyle \varsigma_{A\otimes B}} \\ T^2(A) \otimes T^2(B) & \xrightarrow[\ell_{T(A),T(B)}]{} & T(T(A) \otimes T(B)) & \xrightarrow[T(\ell_{A,B})]{} & T^2(A \otimes B) \end{array}$$

as well as expected coherence diagrams involving the unitors and associator of the monoidal structure.

A symmetric monad is called *oplax monoidal* if it satisfies the dual definition, while it is called *monoidal* in the case that the morphisms ℓ and e are isomorphisms. This definition is adapted from Kock (1972), wherein a monad is called lax monoidal if, in the above, such an ℓ and e satisfy the first two diagrams and the coherence conditions. (We caution that our notion differs from Kock's *symmetric monoidal monad*, for which "symmetric" refers to the monoidal tensor, rather than the monad.)

2.6. Example. For every object A in a cartesian monoidal category, the symmetric comonad $(-) \times A$ is oplax monoidal, with structure

$$X \times X' \times Y \xrightarrow{\mathrm{id}\times\mathrm{id}\times\Delta_Y} X \times X' \times Y \times Y \xrightarrow{\cong} X \times Y \times X' \times Y$$

$$1 \times Y \longrightarrow 1$$

2.7. Tensorial strengths. Recall that monoidal monads have an induced (right) *tensorial strength* (Kock (1972)) defined as

$$\mathrm{str}_{A,B} = \; T(A) \otimes B \xrightarrow{\mathrm{id}\otimes\eta_B} T(A) \otimes T(B) \xrightarrow{\ell_{A,B}} T(A \otimes B)$$

that satisfies the following diagrams

$$\begin{array}{ccccc} T^2(A)\otimes B & \xrightarrow{\mathrm{str}_{T(A),B}} & T(T(A)\otimes B) & \xrightarrow{T(\mathrm{str}_{A,B})} & T^2(A\otimes B) \\ {\scriptstyle \mu_A\otimes \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle \mu_{A\otimes B}} \\ T(A)\otimes B & & \xrightarrow[\mathrm{str}_{A,B}]{} & & T(A\otimes B) \end{array}$$

$$\begin{array}{ccc} A\otimes B & & \\ {\scriptstyle \eta_A\otimes \mathrm{id}}\downarrow & \searrow^{\eta_{A\otimes B}} & \\ T(A)\otimes B & \xrightarrow[\mathrm{str}_{A,B}]{} & T(A\otimes B) \end{array}$$

and note that in the case that T is a monoidal symmetric monad it also satisfies

$$\begin{array}{ccccc} T^2(A)\otimes B & \xrightarrow{\mathrm{str}_{T(A),B}} & T(T(A)\otimes B) & \xrightarrow{T(\mathrm{str}_{A,B})} & T^2(A\otimes B) \\ {\scriptstyle \varsigma_A\otimes \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle \varsigma_{A\otimes B}} \\ T^2(A)\otimes B & \xrightarrow[\mathrm{str}_{T(A),B}]{} & T(T(A)\otimes B) & \xrightarrow[T(\mathrm{str}_{A,B})]{} & T^2(A\otimes B) \end{array}$$

Analogously, there is also a (left) tensorial strength defined as

$$\mathrm{str}'_{A,B} = \ A\otimes T(B) \xrightarrow{\eta_A\otimes\mathrm{id}} T(A)\otimes T(B) \xrightarrow{\ell_{A,B}} T(A\otimes B)$$

2.8. Example. For every exponentiable object A in a cartesian category, the symmetric monad $(-)^A$, being a right adjoint, preserves finite products and, in particular, it is a cartesian-monoidal symmetric monad. We therefore have cartesian tensorial strengths:

$$\begin{array}{ccc} X^A\times Y & \xrightarrow{\mathrm{str}_{X,Y}} & (X\times Y)^A \\ \cong\downarrow & & \downarrow\cong \\ Y\times X^A & \xrightarrow[\mathrm{str}'_{Y,X}]{} & (Y\times X)^A \end{array}$$

3. Symmetric distributive laws

3.1. Definition. *Let $\mathscr{C}$ be a category, (T,μ,η,ς) be a symmetric monad on $\mathscr{C}$, and F be an endofunctor on $\mathscr{C}$. A* symmetric distributive law *is a natural transformation $\psi : TF \to FT$ making the following diagrams commute:*

$$\begin{array}{ccccc} T^2F & \xrightarrow{T\psi} & TFT & \xrightarrow{\psi_T} & FT^2 \\ {\scriptstyle \mu_F}\downarrow & & & & \downarrow{\scriptstyle F\mu} \\ TF & & \xrightarrow[\psi]{} & & FT \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\eta_F} & TF \\ & {\scriptstyle F\eta}\searrow & \downarrow{\scriptstyle \psi} \\ & & FT \end{array}$$

$$\begin{array}{ccccc} T^2F & \xrightarrow{T\psi} & TFT & \xrightarrow{\psi_T} & FT^2 \\ {\scriptstyle \varsigma_F}\downarrow & & & & \downarrow{\scriptstyle F\varsigma} \\ T^2F & \xrightarrow[T\psi]{} & TFT & \xrightarrow[\psi_T]{} & FT^2 \end{array}$$

The first two diagrams ask ψ to be a distributive law between the underlying monad T and the endofunctor F, while the third asks ψ to respect ς. One may similarly define a *symmetric codistributive law* between a symmetric comonad T and an endofunctor F.

3.2. Examples.

1. For a symmetric monad $(T, \mu, \eta, \varsigma)$, the natural transformation ς is a symmetric distributive law between T and itself.

2. For an endofunctor F with cartesian strength $\mathrm{str}_{A,B} : F(A) \times B \to F(A \times B)$ each component $\mathrm{str}_{(-),B}$ is a symmetric codistributive law between the symmetric comonad $(-) \times B$ and the endofunctor F.

We highlight a simple lemma to be needed later that illustrates the tensoring of two symmetric (co)distributive laws.

3.3. Lemma. *Let $(T, \ell, e, \mu, \eta, \varsigma)$ be an oplax monoidal symmetric (co)monad on a monoidal category $(\mathscr{C}, \otimes, I)$, and let G_1 and G_2 be two endofunctors on $\mathscr{C}$. If $\psi_1 : TG_1 \to G_1T$ and $\psi_2 : TG_2 \to G_2T$ are symmetric (co)distributive laws, then*

$$\psi_{1,2} \;=\; \Big(\, T(G_1 \otimes G_2) \xrightarrow{\ell_{G_1,G_2}} TG_1 \otimes TG_2 \xrightarrow{\psi_1 \otimes \psi_2} G_1T \otimes G_2T = (G_1 \otimes G_2)T \,\Big)$$

is a symmetric (co)distributive law between the symmetric (co)monad T and the endofunctor $G_1 \otimes G_2$.

3.4. Remark. In the lemma above, note that if ℓ and ψ_1, ψ_2 are isomorphisms then so is $\psi_{1,2}$.

3.5. Definition. *For an endofunctor F on a monoidal category, we let $F^\bullet$ be the endofunctor given by*

$$F^\bullet(X) = F(X) \otimes X$$

3.6. Examples.

1. For an endofunctor F with cartesian strength $\mathrm{str}_{X,Y} : F(X) \times Y \to F(X \times Y)$, using the oplax cartesian-monoidal symmetric comonad $(-) \times Y$ (Example 2.6) and the symmetric codistributive law $\mathrm{str}_{(-),Y}$ between it and F (Example 3.2(2)), we obtain, from Lemma 3.3, a symmetric codistributive law

$$\mathrm{str}^\bullet{}_{(-),Y} : (- \times Y)\, F^\bullet \to F^\bullet\, (- \times Y)$$

It further follows that $\mathrm{str}^\bullet$ is a cartesian strength for $F^\bullet$; explicitly, this is given by

$$F(X) \times X \times Y \xrightarrow{\mathrm{id} \times \Delta} F(X) \times X \times Y \times Y \cong F(X) \times Y \times X \times Y \xrightarrow{\mathrm{str} \times \mathrm{id}} F(X \times Y) \times X \times Y$$

2. For an oplax monoidal symmetric monad T, with symmetry ς, from Example 3.2(1) and Lemma 3.3, we obtain a symmetric distributive law

$$\underline{\varsigma} : T\,T^{\bullet} \to T^{\bullet}\,T$$

explicitly given by

$$T\,T^{\bullet}(X) = T(T(X) \otimes X) \xrightarrow{\ell} TT(X) \otimes T(X) \xrightarrow{\varsigma \otimes \mathrm{id}} TT(X) \otimes T(X) = T^{\bullet}\,T(X)$$

4. Substitution algebras

As recalled in the introduction, in Fiore, Plotkin, and Turi (1999) the algebraic structure of (single variable) substitution and (generic) variables on an object A in the object-classifier topos was axiomatized by means of two operations

$$A^{\mathrm{V}} \times A \to A \quad \text{and} \quad 1 \to A^{\mathrm{V}}$$

subject to four equational laws (see also the last four equations in 4.8). These capture the following properties: (*i*) the substitution on a variable performs the action; (*ii*) the substitution of a variable is a contraction; (*iii*) the substitution for an absent variable has no effect; and (*iv*) the substitution operation is associative. Next, we reconsider and generalize that definition streamlined in the framework of the previous two sections.

4.1. Definition. *A* T-substitution algebra *for a cartesian-monoidal symmetric monad* $(T, \ell, e, \mu, \eta, \varsigma)$ *on a cartesian category* $\mathscr{C}$*, is an object* $A \in \mathscr{C}$ *together with morphisms* $s : T^{\bullet}(A) \to A$ *and* $v : 1 \to T(A)$ *in* $\mathscr{C}$ *such that the following diagrams commute:*

$$\begin{array}{ccc} 1 \times A & & \\ {\scriptstyle v \times \mathrm{id}}\downarrow & \overset{\pi_2}{\underset{\cong}{\searrow}} & \\ T^{\bullet}(A) & \xrightarrow[s]{} & A \end{array} \tag{1}$$

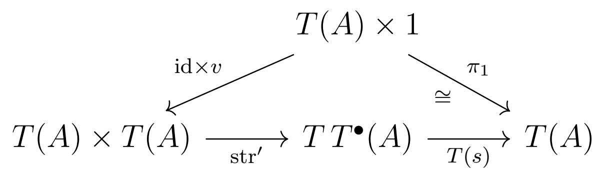
$$\begin{array}{ccccc} T^2(A) \times 1 & & \xrightarrow[\cong]{\pi_1} & & T^2(A) \\ {\scriptstyle \mathrm{id} \times v}\downarrow & & & & \downarrow{\scriptstyle \mu} \\ T^2(A) \times T(A) & \xrightarrow[\ell]{\cong} & T\,T^{\bullet}(A) & \xrightarrow[T(s)]{} & T(A) \end{array} \tag{2}$$

$$\begin{array}{ccc} A \times A & & \\ {\scriptstyle \eta \times \mathrm{id}}\downarrow & \overset{\pi_1}{\searrow} & \\ T^{\bullet}(A) & \xrightarrow[s]{} & A \end{array}$$

$$\begin{array}{ccccccc} T\,T^{\bullet}(A) \times A & \xrightarrow[\cong]{\underline{\varsigma} \times \mathrm{id}} & T^{\bullet}\,T(A) \times A & \xrightarrow{\mathrm{str}^{\bullet}} & T^{\bullet}\,T^{\bullet}(A) & \xrightarrow{T^{\bullet}(s)} & T^{\bullet}(A) \\ {\scriptstyle T(s) \times \mathrm{id}}\downarrow & & & & & & \downarrow{\scriptstyle s} \\ T^{\bullet}(A) & & & \xrightarrow[s]{} & & & A \end{array}$$

Diagram (1) may be naturally considered a left-unit law. As we now show, diagram (2) is equivalent to a right-unit law.

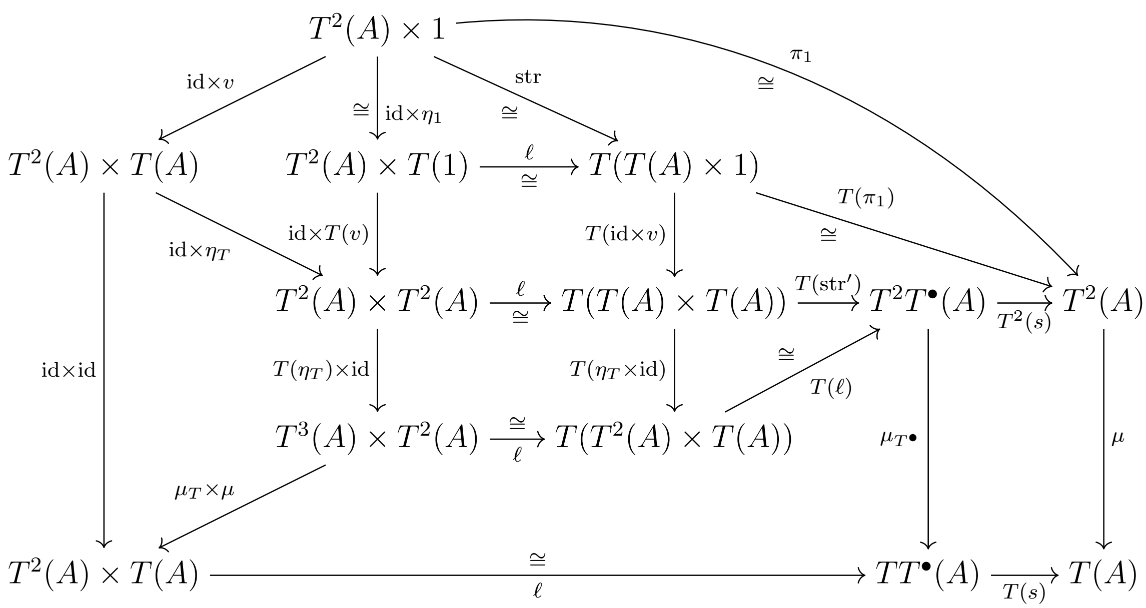
4.2. PROPOSITION. *Diagram (2) commutes if, and only if, so does the following one:*



(3)

PROOF. ($\Rightarrow$) Diagram (3) is readily obtained from diagram (2) by precomposition with the morphism $\eta_T \times \mathrm{id} : T(A) \times 1 \to T^2(A) \times 1$.

($\Leftarrow$) We provide a diagrammatic proof.



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4.3. THEOREM. *T-substitution algebras may be equivalently axiomatized by replacing diagram (2) with diagram (3).*

4.4. DEFINITION. *A homomorphism between T-substitution algebras (A, s, v) and (A', s', v') is a morphism $h : A \to A'$ such that the following diagrams commute:*

We are interested here in the specific category of substitution algebras defined below. In the following, recall Example 2.8.

4.5. Definition. *We let* $\boldsymbol{\mathcal{SA}}$ *be the category of substitution algebras and homomophisms between them for the cartesian-monoidal symmetric monad* $(-)^V$ *on the object-classifier topos* $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{S}et}^{\mathbb{F}}$*, for* $V = \mathbb{F}(1, -)$ *the universal object model.*

4.6. Remark. The equivalence of Theorem 4.3 in the context of $\boldsymbol{\mathcal{SA}}$ accounts for the respective axiomatizations considered in Fiore, Plotkin, and Turi (1999) and in Fiore and Staton (2014).

4.7. Since the universal object model $V \in \boldsymbol{\mathcal{F}}$ is the representable at $1 \in \mathbb{F}$, which with the structure $(\mathrm{c} : 2 \to 1, \mathrm{w} : 0 \to 1, \mathrm{s} : 2 \to 2)$ in $\mathbb{F}$ is the universal symmetric monoid (recall 2.1), the exponentiation by V symmetric monad is equivalently described by $(\delta, \underline{\mathrm{c}}, \underline{\mathrm{w}}, \underline{\mathrm{s}})$ given as follows (see Example 2.4(4)):

$$\begin{aligned}
\delta(A) &= A(- + 1) \\
\underline{\mathrm{c}}_A &= A(- + \mathrm{c}) : A(- + 2) \to A(- + 1) \\
\underline{\mathrm{w}}_A &= A(- + \mathrm{w}) : A(-) \to A(- + 1) \\
\underline{\mathrm{s}}_A &= A(- + \mathrm{s}) : A(- + 2) \to A(- + 2)
\end{aligned}$$

We henceforth adopt this representation. In particular, substitution-algebra structure $s : A^V \times A \to A$ and $v : 1 \to A^V$ on $A \in \boldsymbol{\mathcal{F}}$ is therefore given by natural transformations

$$s_m : A(m+1) \times A(m) \to A(m) \;\; , \quad v_m : 1 \to A(m+1) \qquad\qquad (m \in \mathbb{F})$$

4.8. Equational presentation. It follows from the above that the category of substitution algebras $\boldsymbol{\mathcal{SA}}$ in the object-classifier topos $\boldsymbol{\mathcal{F}}$ has a countably-sorted equational presentation. Indeed, this has set of sorts $\mathbb{N}$ and operators

$$\begin{array}{lll}
\alpha_f : & m \to n & \big(m, n \in \mathbb{N}, f \in \mathbb{F}(m, n)\big) \\
\varsigma_m : & m + 1, m \to m & (m \in \mathbb{N}) \\
\nu : & 1 &
\end{array}$$

subject, for all $\ell, m, n \in \mathbb{N}$, $g \in \mathbb{F}(\ell, m)$, $f \in \mathbb{F}(m, n)$, to the equations:

$$x : \ell \;\vdash \alpha_f\big(\alpha_g(x)\big) = \alpha_{fg}(x) : n$$

$$x : m \;\vdash \alpha_{\mathrm{id}_m}(x) = x : m$$

$$x : m+1, y : m \;\vdash \alpha_f\big(\varsigma_m(x, y)\big) = \varsigma_n\big(\alpha_{f+\mathrm{id}_1}(x), \alpha_f(y)\big) : n$$

$$x : m \;\vdash \varsigma_m\big(\nu_m, x\big) = x : m$$

$$x : m+2 \;\vdash \varsigma_{m+1}\big(x, \nu_m\big) = \alpha_{\mathrm{id}_m + \mathrm{c}}(x) : m+1$$

$$x : m, y : m \;\vdash \varsigma_m\big(\alpha_{\mathrm{id}_m + \mathrm{w}}(x), y\big) = x : m$$

$$\begin{aligned}
&x : m+2, y : m+1, z : m \\
&\quad\vdash \varsigma_m\big(\varsigma_{m+1}(x, y), z\big) = \varsigma_m\Big(\varsigma_{m+1}\big(\alpha_{\mathrm{id}_m + \mathrm{s}}(x), \alpha_{\mathrm{id}_m + \mathrm{w}}(z)\big), \varsigma_m(y, z)\Big) : m
\end{aligned}$$

where $\nu_m = \alpha_{(0 \mapsto m)}(\nu) : m+1$.

The operators α_f together with the first two equations correspond to the presheaf structure of objects in $\boldsymbol{\mathcal{F}}$, the next equation corresponds to the naturality of the substitution operation as embodied by the operators ς_m, and the last four equations correspond to the laws of substitution algebras.

4.9. Corollary. *The category of substitution algebras is countably-sorted algebraic.*

5. Isomorphism theorem

We show that the categories of substitution algebras $\boldsymbol{\mathcal{SA}}$ and of abstract clones $\boldsymbol{\mathcal{AC}}$ are isomorphic. Since, as recalled in the introduction, the category of Lawvere theories is equivalent to that of abstract clones, this establishes the main aim of the paper in exhibiting a finite equational presentation of Lawvere theories in the object-classifier topos.

We will construct inverse functors $\mathrm{S} : \boldsymbol{\mathcal{AC}} \to \boldsymbol{\mathcal{SA}}$ and $\mathrm{C} : \boldsymbol{\mathcal{SA}} \to \boldsymbol{\mathcal{AC}}$. The idea of these constructions is simple: For abstract clones and substitution algebras, respectively understood as modelling simultaneous and single-variable substitution, the first functor expresses single-variable substitution as a special case of simultaneous substitution while the second functor expresses simultaneous substitution as iterated application of single-variable substitution.

5.1. From abstract clones to substitution algebras.

Let (C, μ, ι) be an abstract clone.

5.1.1. For $f \in \mathbb{F}(m, n)$, define

$$C(f) : C_m \to C_n : t \mapsto \mu_{m,n}\big(t, \iota^n_{f(0)}, \dots, \iota^n_{f(m-1)}\big)$$

Then, the families of sets $\{\, C_m \,\}_{m \in \mathbb{N}}$ and of functions $\{\, C(f) : C_m \to C_n \,\}_{m,n \in \mathbb{N}, f \in \mathbb{F}(m,n)}$ determine a presheaf C in $\boldsymbol{\mathcal{F}}$.

5.1.2. For $m \in \mathbb{N}$, define

$$v_m = \iota^{m+1}_m \in C(m+1)$$

Then, the family of elements $\{\, v_m \in C(m+1) \,\}_{m \in \mathbb{N}}$ determines a natural transformation $v : 1 \to \delta(C)$ in $\boldsymbol{\mathcal{F}}$.

5.1.3. For $m \in \mathbb{N}$, define

$$s_m : C(m+1) \times C(m) \to C(m) : (t, u) \mapsto \mu_{m+1,m}\big(t, \iota^m_0, \dots, \iota^m_{m-1}, u\big)$$

Then, the family of functions $\{\, s_{m,n} : C(m+1) \times C(m) \to C(m) \,\}_{m \in \mathbb{N}}$ determines a natural transformation $s : \delta(C) \times C \to C$ in $\boldsymbol{\mathcal{F}}$.

5.1.4. The structure $\mathrm{S}(C, \mu, \iota) = (C, s, v)$ is a substitution algebra and for an abstract-clone homomorphism $h : (C, \mu, \iota) \to (C', \mu', \iota')$, the family of functions $\mathrm{S}(h) = \{\, h_m : C(m) \to C'(m) \,\}_{m \in \mathbb{N}}$ is a substitution-algebra homomorphism $\mathrm{S}(C, \mu, \iota) \to \mathrm{S}(C', \mu', \iota')$. This construction defines a functor $\mathrm{S} : \boldsymbol{\mathcal{AC}} \to \boldsymbol{\mathcal{SA}}$.

5.2. From substitution algebras to abstract clones.

Let (A, s, v) be a substitution algebra.

5.2.1. For $m, n \in \mathbb{N}$, define

$$\varphi_{m,n} : A(n+m) \times (A\,n)^m \to A(n)$$

by induction as

$$\varphi_{0,n} = \quad A(n) \times (A\,n)^0 \xrightarrow[\cong]{\pi_1} A(n)$$

and

$$\begin{array}{ccc}
A(n+m+1) \times (A\,n)^m \times A(n) & \xrightarrow{\varphi_{m+1,n}} & A(n) \\
\cong\downarrow & & \\
A(n+m+1) \times A(n) \times (A\,n)^m & & \uparrow \varphi_{m,n} \\
\mathrm{id}\times A(i\mapsto i)\times \mathrm{id}\downarrow & & \\
A(n+m+1) \times A(n+m) \times (A\,n)^m & \xrightarrow[s_{n+m}\times \mathrm{id}]{} & A(n+m) \times (A\,n)^m
\end{array}$$

5.2.2. For $m, n \in \mathbb{N}$, define

$$\mu_{m,n} : A(m) \times (A\,n)^m \to A(n)$$

as the composite

$$A(m) \times (A\,n)^m \xrightarrow{A(i\mapsto n+i)\times \mathrm{id}} A(n+m) \times (A\,n)^m \xrightarrow{\varphi_{m,n}} A(n)$$

5.2.3. For $m \in \mathbb{N}$ and $i \in [m]$, define

$$\iota_i^m = A\big(0 \mapsto i\big)\big(\, v_0(\,)\,\big) \in A(m)$$

5.2.4. The structure $\mathrm{C}(A, s, v) = (A, \mu, \iota)$ is an abstract clone and for a substitution-algebra homomorphism $h : (A, s, v) \to (A', s', v')$, the family of functions $\mathrm{C}(h) = \{\, h_m : A(m) \to A'(m)\,\}_{m\in\mathbb{N}}$ is an abstract-clone homomorphism $\mathrm{C}(A, s, v) \to \mathrm{C}(A', s', v')$. This construction defines a functor $\mathrm{C} : \boldsymbol{\mathcal{SA}} \to \boldsymbol{\mathcal{AC}}$.

5.3. Theorem. *The functors* $\mathrm{S} : \boldsymbol{\mathcal{AC}} \to \boldsymbol{\mathcal{SA}}$ *and* $\mathrm{C} : \boldsymbol{\mathcal{SA}} \to \boldsymbol{\mathcal{AC}}$ *form an isomorphism of categories.*

5.4. Corollary. *Substitution algebras provide a finite algebraic presentation of Lawvere theories in the object-classifier topos.*

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