# SIMPLICIAL DISTRIBUTIONS, CONVEX CATEGORIES, AND CONTEXTUALITY

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ABSTRACT. The data of a physical experiment can be represented as a presheaf of probability distributions. A striking feature of quantum theory is that those probability distributions obtained in quantum mechanical experiments do not always admit a joint probability distribution, a celebrated observation due to Bell. Such distributions are called contextual. Simplicial distributions are combinatorial models that extend presheaves of probability distributions by elevating sets of measurements and outcomes to spaces. Contextuality can be defined in this generalized setting. This paper introduces the notion of convex categories to study simplicial distributions from a categorical perspective. Simplicial distributions can be given the structure of a convex monoid, a convex category with a single object, when the outcome space has the structure of a group. We describe contextuality as a monoid-theoretic notion by introducing a weak version of invertibility for monoids. Our main result is that a simplicial distribution is noncontextual if and only if it is weakly invertible. Similarly, strong contextuality and contextual fraction can be characterized in terms of invertibility in monoids. Finally, we show that simplicial homotopy can be used to detect extremal simplicial distributions refining the earlier methods based on Čech cohomology and the cohomology of groups.

### 1. Introduction

Physical experiments are probabilistic: When a measurement is performed, the corresponding outcome occurs with a certain probability. Quantum theory comes with an additional constraint prohibiting certain measurements from being performed jointly. Therefore in a quantum mechanical experiment, the data that describes the outcome probabilities consists of a family of probability distributions indexed over subsets of measurements that are allowed to be performed jointly. More precisely, this family is a presheaf of probability distributions since the restriction of probabilities to a smaller subset of measurements, also called marginalization, is compatible. A striking phenomenon in physics, known as Bell's nonlocality and its generalization called contextuality, can be expressed as the nonexistence of a joint probability distribution over the set of all measurements that marginalizes to the distributions of the restricted set of measurements obtained from the experiment. Such a joint distribution always exists in classical theories. In particular, a joint distribution provides a model where all measurement outcomes are assigned before

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the measurement takes place, and the measurement probabilities are obtained by considering all such global assignments with a certain probability. It is a celebrated result of Bell [1] that in quantum theory, the joint distribution does not always exist, i.e., there are contextual families of distributions. Another celebrated result due to Kochen–Specker [2] proves a similar result by showing the impossibility of the global assignments of outcomes in quantum theory. The latter demonstrates a stronger version of contextuality.

There are various approaches to formalizing contextuality. The presheaf approach is introduced in [3]. The ingredients in this approach are (1) a simplicial complex  $\Sigma$  whose vertices represent the set of all measurements, and its simplices are those that can be jointly performed, and (2) a finite set of outcomes for the measurements. For the outcome set, we don't lose any generality by considering the ring  $\mathbb{Z}_d$  of integers modulo d. A distribution on  $(\Sigma, \mathbb{Z}_d)$  is a presheaf of distributions, i.e., a family  $(p_{\sigma})_{\sigma \in \Sigma}$  together with a compatibility condition. Each  $p_{\sigma}$  is a distribution on the set of all functions  $\sigma \longrightarrow \mathbb{Z}_d$ . More formally, let  $D_R$  denote the distribution monad on the category of sets [4], where R is a commutative semiring. Then  $p_{\sigma}$  belongs to  $D_{R}(\mathbb{Z}_{d}^{\sigma})$ . The presheaf approach uses tools from Cech cohomology to study contextuality. Furthermore, this systematic study introduces degrees of contextuality, such as strong contextuality and the more refined measure of contextuality known as the contextual fraction. See also [5-7] that uses presheaf categories to describe contextuality. Another approach to contextuality is the topological approach of [8] based on techniques from the cohomology of groups. This approach introduces cohomology classes that can detect strong contextuality but fail to capture weaker versions, e.g., the famous Clauser–Horne–Shimony–Holt (CHSH) scenario [9]. In [10], co-authored by the current authors, both approaches are unified within the framework of simplicial distributions. This theory goes beyond the usual assumption that measurements and outcomes are represented by finite sets. In this framework, one can study distributions on spaces of measurements and outcomes, where a space is represented by a simplicial set. Simplicial sets are combinatorial objects more expressive than simplicial complexes. They are fundamental objects in modern homotopy theory [11]. A simplicial distribution is defined on a pair (X, Y) of simplicial sets, where X represents the measurements and Y the outcomes. The distribution monad can be elevated to a monad on the category of simplicial sets. Given Y, one can define another simplicial set  $D_R(Y)$ whose simplices are distributions on the set of simplices of Y. A simplicial distribution is a morphism of simplicial sets

$$p: X \longrightarrow D_R(Y).$$

This paper studies simplicial distributions from a categorical perspective by introducing convex categories. In general, we provide an algebraic version of contextuality and contextual fraction as a measure of non-invertibility in convex monoids. Our results on simplicial distributions apply to semirings that behave similarly to canonical examples, such as the semiring of nonnegative reals,  $\mathbb{R}_{\geq 0}$ , and the Boolean semiring,  $\mathbb{B} = \{0, 1\}$ , where  $\vee$ and  $\wedge$  represent the additive and multiplicative structures, respectively. Throughout the paper we will restrict to commutative zero-sum-free integral semirings. Both semirings satisfy these two properties. A semiring R is called

- zero-sum-free if a + b = 0 implies a = b = 0 for all  $a, b \in R$ ,
- *integral* if  $a \cdot b = 0$  implies a = 0 or b = 0 if for all  $a, b \in R$ ,

see [12]. However, for simplicity, readers may assume  $R = \mathbb{R}_{>0}$ .

For a semiring R the algebras of the distribution monad  $D_R : \mathbf{Set} \longrightarrow \mathbf{Set}$  are called R-convex sets. This notion generalizes the usual notion of convexity for  $R = \mathbb{R}_{\geq 0}$ . Let  $\mathbf{Conv}_R$  denote the category of R-convex sets. Every monad has an associated Kleisli category. In the case of  $D_R$ , the morphisms of the Kleisli category  $s\mathbf{Set}_{D_R}$  are the simplicial distributions, i.e., for simplicial sets X, Y the set  $\mathbf{Set}_{D_R}(X, Y)$  of morphisms is given by simplicial set morphisms  $X \longrightarrow D_R(Y)$ . Let  $s\mathbf{Conv}_R$  denote the category of simplicial R-convex sets. The primary examples of such simplicial objects are  $D_R(Y)$  for a given simplicial set Y.

1.1. PROPOSITION. The functor sending a pair (X, Y) of simplicial sets to the set s**Set**(X, Y) of simplicial set morphisms lifts to a functor

$$s\mathbf{Set}(-,-): s\mathbf{Set}^{op} \times s\mathbf{Conv}_R \longrightarrow \mathbf{Conv}_R$$

The main application of this result is to the set of simplicial distributions. By this result the set s**Set** $(X, D_R(Y))$  is an *R*-convex set. Contextuality for simplicial distributions is defined using a comparison map

$$\Theta_{X,Y}: D_R(s\mathbf{Set}(X,Y)) \longrightarrow s\mathbf{Set}(X,D_R(Y)).$$

Under this map, the delta distribution at a simplicial set morphism  $\varphi : X \longrightarrow Y$  is sent to the simplicial distribution given by the composition  $X \xrightarrow{\varphi} Y \xrightarrow{\delta_Y} D_R(Y)$ , called a deterministic distribution. Since the target is *R*-convex,  $\Theta_{X,Y}$  is the unique extension to the free  $D_R$ -algebra, the domain of the map. A simplicial distribution is called noncontextual if it lies in the image of  $\Theta_{X,Y}$ ; otherwise called contextual. We show in Section 2.18 how to realize a presheaf of distributions as a simplicial distribution. In Theorem 2.20 we show that the notion of contextuality for simplicial distributions specializes to the notion for presheaves of distributions originally introduced in [3].

A category-theoretic point of view suggests lifting our analysis from the level of morphism sets to the level of categories. For this, we introduce the notion of convex categories. The first step is to lift  $D_R$  to a monad on the category **Cat** of (locally small) categories. Then we define an *R*-convex category as a  $D_R$ -algebra in **Cat**. Every category enriched over the category of *R*-convex sets is an *R*-convex category. However, the converse does not always hold. The prominent example of a convex category is the Kleisli category **Set**<sub> $D_R$ </sub>, and its simplicial version s**Set**<sub> $D_R$ </sub>, which are not enriched over **Conv**<sub>R</sub>. We can think of  $\Theta_{X,Y}$  assembled into a morphism in **ConvCat**<sub>R</sub> from the free *R*-convex category to the Kleisli category, both of which obtained from the category of simplicial sets:

$$\Theta: D_R(s\mathbf{Set}) \longrightarrow s\mathbf{Set}_{D_R}.$$

For outcome spaces which also have a group structure the convex set of simplicial distributions can be given a monoid structure. Our categorical framework combined with this monoid structure has interesting applications connecting contextuality to a weak notion of invertibility in convex monoids. As we show in Section 2.18 any presheaf of distribution can be realized as a simplicial distribution

$$p: X \longrightarrow D_R(N\mathbb{Z}_d),$$

where  $N\mathbb{Z}_d$  is the nerve of the additive group  $\mathbb{Z}_d$ . The simplicial set  $N\mathbb{Z}_d$  is, in fact, a simplicial group. The group structure on the nerve induces a monoid structure on the convex set  $s\mathbf{Set}(X, D_R(N\mathbb{Z}_d))$  of simplicial distributions. We remark that this monoid structure is new and has not been investigated in the study of contextuality. More precisely,  $s\mathbf{Set}(X, D_R(N\mathbb{Z}_d))$  is a convex monoid, i.e., a convex category with a single object. For a convex monoid M, with the map  $\pi^M : D_R(M) \longrightarrow M$  giving the  $D_R$ -algebra structure, an element  $m \in M$  is called weakly invertible if it lies in the image of the composite  $D_R(M^*) \xrightarrow{D_R(i_M)} D_R(M) \xrightarrow{\pi^M} M$ , where  $i_M : M^* \hookrightarrow M$  is the inclusion of the units. Our main result connects weak invertibility to noncontextuality.

1.2. THEOREM. Given a simplicial set X and a simplicial group Y, a distribution  $p \in s\mathbf{Set}(X, D_R(Y))$  is noncontextual if and only if p is weakly invertible.

Moreover, we introduce the notion of strong invertibility, a monoid-theoretic analogue of strong contextuality. We also introduce a degree of invertibility called invertible fraction which generalizes the notion of noncontextual fraction introduced in [3, 13]. Simplicial distributions are formulated using the theory of simplicial sets. It is a natural question to understand the role of homotopy in the context of simplicial distributions. Embedding presheaves of distributions to our simplicial framework makes homotopical tools available for the study of contextuality. In Corollary 4.25 we show that simplicial homotopy can be used to detect extremal contextual distributions, a question of fundamental importance in the study of polytopes of distributions; see [14–17].

Our paper is organized as follows: In Section 2.1 we recall basics from convex sets and in Section 2.5 we introduce simplicial distributions. Proposition 1.1 (Proposition 2.16) is proved in Section 2.15. In Section 2.18, we show how to describe a presheaf of distributions as a simplicial distribution. In this section, we also provide examples of simplicial distributions, such as the CHSH scenario (Example 2.23). Convex categories are introduced in Section 2.27. Weak invertibility, strong invertibility and invertible fraction are introduced in Sections 3.6, 3.11 and 3.17; respectively. Our main result Theorem 1.2 (Theorem 4.7) is proved in Section 4.1. The relationship between strong invertibility and strong contextuality is studied in Section 4.10. Extremal simplicial distributions and the role of simplicial homotopy are discussed in Section 4.14.

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#### 2. Simplicial distributions

In this section, we introduce simplicial distributions [10] defined over the semiring R. These objects describe distributions on a simplicial set parametrized by another simplicial set. Our main result is that the set of simplicial distributions constitute an R-convex set, in the sense that it is an algebra over the distribution monad. In practice, simplicial distributions come from presheaves of distributions, introduced in [3]. We describe how to embed the theory of presheaves of distributions into the theory of simplicial distributions. Finally, we introduce convex categories and explore their relationship with simplicial distributions.

2.1. ALGEBRAS OVER A MONAD. We recall some basic facts about algebras over a monad from [18, section VI]. A monad on a category **C** is a functor  $T : \mathbf{C} \longrightarrow \mathbf{C}$  together with natural transformations  $\delta : \operatorname{Id}_{\mathbf{C}} \longrightarrow T$  and  $\mu : T^2 \longrightarrow T$  satisfying  $\mu \circ T\mu = \mu \circ \mu T$  and  $\mu \circ \delta T = \mu \circ T\delta = \operatorname{Id}_T$ . A *T*-algebra consists of an object *X* of **C** together with a morphism  $\pi : T(X) \longrightarrow X$  of **C** such that the following diagrams commute

A morphism of *T*-algebras is a morphism  $f: X \longrightarrow Y$  of **C** such that  $\pi^Y \circ T(f) = f \circ \pi^X$ . The category of *T*-algebras will be denoted by  $\mathbf{C}^T$ . The object T(X) together with the structure morphism  $\mu_X$  is called a free *T*-algebra. There is an adjunction  $T: \mathbf{C} \dashv \mathbf{C}^T : U$ where *T* sends an object to the associated free *T*-algebra and *U* is the forgetful functor. The Kleisli category of *T*, denoted by  $\mathbf{C}_T$ , is the category whose objects are the same as the objects of **C** and morphisms  $X \longrightarrow Y$  are given by  $\mathbf{C}(X, T(Y))$ . In fact,  $\mathbf{C}_T$  is equivalent to the full subcategory of free *T*-algebras in  $\mathbf{C}^T$ . See also [19, Chapter 5] and [20, Subsection 5.2].

2.2. REMARK. Under the bijection  $\mathbf{C}^T(T(X), Y) \cong \mathbf{C}(X, Y)$  a morphism  $f: T(X) \longrightarrow Y$  is sent to  $f \circ \delta_X$ . Conversely, under this isomorphism, a morphism  $g: X \longrightarrow Y$  is sent to  $\pi^Y \circ T(g)$ .

2.2.1. CONVEX SETS. The distribution monad  $D_R : \mathbf{Set} \longrightarrow \mathbf{Set}$  is defined as follows:

- For a set X the set  $D_R(X)$  of R-distributions on X consists of functions  $p: X \longrightarrow R$  of finite support, i.e.,  $|\{x \in X : p(x) \neq 0\}| < \infty$ , such that  $\sum_{x \in X} p(x) = 1$ .
- Given a function  $f: X \longrightarrow Y$  the function  $D_R(f): D_R(X) \longrightarrow D_R(Y)$  is defined by

$$p \mapsto \left( y \mapsto \sum_{x \in f^{-1}(y)} p(x) \right).$$

The structure maps of the monad are given as follows:

•  $\delta_X : X \longrightarrow D_R(X)$  sends  $x \in X$  to the delta distribution

$$\delta^x(x') = \begin{cases} 1 & x' = x, \\ 0 & \text{otherwise.} \end{cases}$$

•  $\mu_X: D^2_R(X) \longrightarrow D_R(X)$  sends a distribution P to the distribution

$$\mu_X(P)(x) = \sum_{p \in D_R(X)} P(p)p(x).$$

When  $R = \mathbb{R}_{>0}$  we simply write D for the distribution monad.

2.3. DEFINITION. [12] A  $D_R$ -algebra in the category of sets is called an *R*-convex set. A morphism of *R*-convex sets is given by a morphism of  $D_R$ -algebras. We will denote the category of *R*-convex sets by **Conv**<sub>*R*</sub>.

In the case of  $R = \mathbb{R}_{\geq 0}$  the notion of  $\mathbb{R}_{\geq 0}$ -convexity coincides with the usual notion of convexity, as we recall from [4] in Proposition 2.4 below.

2.3.1. REAL CONVEX SETS. We recall the definition of real convex sets from [4, Definition 3]. A real convex set consists of a set X together with a ternary operation  $\langle -, -, - \rangle$ :  $[0,1] \times X \times X \longrightarrow X$  satisfying the following requirements, for all  $\alpha, \beta \in [0,1]$  and  $x, y, z \in X$ :

- 1.  $\langle \alpha, x, y \rangle = \langle 1 \alpha, y, x \rangle$ .
- 2.  $\langle \alpha, x, x \rangle = x$ .
- 3.  $\langle 0, x, y \rangle = y$ .
- 4. If  $\alpha + (1 \alpha)\beta \neq 0$ , then

$$\langle \alpha, x, \langle \beta, y, z \rangle \rangle = \langle \alpha + (1 - \alpha)\beta, \langle \frac{\alpha}{\alpha + (1 - \alpha)\beta}, x, y \rangle, z \rangle.$$

Given a real convex set X it is sometimes more convenient to use the notation  $\alpha x + (1-\alpha)y$ instead of  $\langle \alpha, x, y \rangle$ . A morphism of convex sets is an function  $f: X \longrightarrow Y$  satisfying

$$f(\alpha x + (1 - \alpha)x') = \alpha f(x) + (1 - \alpha)f(x').$$

for all  $\alpha \in [0, 1]$  and  $x, x' \in X$ . This yields the category of real convex sets denoted by **Conv**.

2.4. PROPOSITION. [4] The category **Conv** is isomorphic to the category **Conv**<sub> $\mathbb{R}_{\geq 0}$ </sub>. Under this isomorphism a real convex set X is sent to the  $\mathbb{R}_{\geq 0}$ -convex set  $(X, \pi^X)$  where

$$\pi^X : D_{\mathbb{R} \ge 0}(X) \longrightarrow X, \quad \pi^X(P) = \sum_{x \in X} P(x)x.$$

Conversely, an  $\mathbb{R}_{>0}$ -convex set  $(X, \pi^X)$  is sent to the real convex set where

$$\langle \alpha, x, y \rangle = \pi^X (\alpha \delta^x + (1 - \alpha) \delta^y).$$

2.5. SIMPLICIAL CONVEX SETS. We begin by introducing simplicial sets. Our main reference is is [11]. The simplex category  $\Delta$  consists of

- the objects  $[n] = \{0, 1, \dots, n\}$  for  $n \ge 0$ , and
- the morphisms  $\theta : [m] \longrightarrow [n]$  given by order preserving functions.

A simplicial set is a functor  $X : \Delta^{\text{op}} \longrightarrow \mathbf{Set}$ . The set of *n*-simplices is usually denoted by  $X_n = X([n])$ . An object [m] in the simplex category gives a simplicial set  $\Delta[m]$  whose set of *n*-simplices is given by  $\Delta([n], [m])$ . It is a well-known fact that every simplicial set can be obtained by "gluing", or more formally as a colimit of, simplicial sets of this form. A morphism of simplicial sets is a natural transformation  $f : X \longrightarrow Y$  between the functors. We will write  $s\mathbf{Set}$  for the category of simplicial sets. This definition can be extended to an arbitrary category  $\mathbf{C}$  and the resulting category is denoted by  $s\mathbf{C}$ .

2.6. PROPOSITION. A monad  $(T, \delta, \mu)$  on **C** extends to a monad  $T : s\mathbf{C} \longrightarrow s\mathbf{C}$  by applying T degree-wise:

- T(X) is the simplicial object with  $(TX)_n = T(X_n)$  and the simplicial structure maps are given by  $d_i^{T(X)} = T(d_i^X)$  and  $s_i^{T(X)} = T(s_i^X)$ .
- $T(f): T(X) \longrightarrow T(Y)$  in degree n is given by  $T(f)_n = T(f_n)$ .
- The monad structure maps  $\delta$  and  $\mu$  are defined by  $(\delta_X)_n = \delta_{X_n}$  and  $(\mu_X)_n = \mu_{X_n}$ .

Moreover, we have  $s(\mathbf{C}^T) \cong (s\mathbf{C})^T$ .

Our main interest is the extension of the distribution monad to the category of simplicial sets, i.e., the functor  $D_R : s\mathbf{Set} \longrightarrow s\mathbf{Set}$ . The associated category of  $D_R$ -algebras will be denoted by  $s\mathbf{Conv}_R$ .

2.6.1. SIMPLICIAL DISTRIBUTIONS. Simplicial distributions are first introduced in [10]. In this section we recall the basic definitions. Let  $s\mathbf{Set}_{D_R}$  denote the Kleisli category of the distributions monad (acting on the category of simplicial sets). Its objects are simplicial sets and morphisms between two simplicial sets X and Y are given by simplicial set morphisms of the form  $p: X \longrightarrow D_R(Y)$ .

2.7. DEFINITION. Morphisms  $s\mathbf{Set}_{D_R}(X, Y)$  of the Kleisli category are called *simplicial* distributions on (X, Y). When the underlying semiring is  $\mathbb{R}_{\geq 0}$  we will call these morphisms simplicial probability distributions. For notational convenience we write  $p_{\sigma}$ , where  $\sigma \in X_n$ , for the distribution  $p_n(\sigma) \in D_R(Y_n)$ .

When X and Y are sets, a map  $X \longrightarrow D(Y)$  is commonly referred to as a kernel (or a parametrized distribution). In this special case, a simplicial distribution corresponds precisely to a kernel. There is a comparison map

$$\Theta_{X,Y}: D_R(s\mathbf{Set}(X,Y)) \longrightarrow s\mathbf{Set}(X,D_R(Y))$$
(2)

defined as follows: Let  $x : \Delta[n] \longrightarrow X$  be an *n*-simplex of X. We can construct a commutative diagram

where the vertical maps are induced by x and the bottom horizontal map is given by the identity map  $D_R(Y_n) \longrightarrow (D_R Y)_n$ . The top horizontal map exists since X is a colimit of its simplices. The  $\Theta$ -map gives rise to an important definition.

2.8. DEFINITION. A simplicial distribution  $p: X \longrightarrow D_R Y$  is called *noncontextual* if p lies in the image of  $\Theta_{X,Y}$ . Otherwise, it is called *contextual*. When we want to refer to the semiring R we say R-contextual, or R-noncontextual.

The map  $\Theta_{X,Y}$  can be given a more explicit description.

2.9. PROPOSITION. For  $d \in D_R(s\mathbf{Set}(X,Y))$  and  $x \in X_n$  we have

$$\Theta(d)_n(x) = \sum_{\varphi \in s \mathbf{Set}(X,Y)} d(\varphi) \delta^{\varphi_n(x)}.$$

That is, for  $y \in Y_n$  we have  $\Theta(d)_n(x) : y \mapsto \sum_{\varphi_n(x)=y} d(\varphi)$ .

In Proposition 2.17 we will show that  $\Theta$  is the unique map in  $\mathbf{Conv}_R$  that makes the following diagram commute

$$s\mathbf{Set}(X,Y) \xrightarrow{(\delta_Y)_*} s\mathbf{Set}(X,D_R(Y))$$

$$\delta_{s\mathbf{Set}(X,Y)} \downarrow \qquad \Theta$$

$$D_R(s\mathbf{Set}(X,Y)) \qquad (3)$$

Alternatively, noncontextual distributions can be described as convex mixtures of distributions in the image of  $(\delta_Y)^*$ .

2.10. DEFINITION. A simplicial distribution on (X, Y) is called *deterministic* if it lies in the image of  $(\delta_Y)_* : s\mathbf{Set}(X, Y) \longrightarrow s\mathbf{Set}(X, D_R(Y))$ . In this case the resulting distribution is denoted by  $\delta^{\varphi} = (\delta_Y)_*(\varphi)$ .

2.11. EXAMPLE. Let  $X = \Delta[n]$  and Y be an arbitrary simplicial set. There is a bijection between the set s**Set** $(\Delta[n], D_R(Y))$  of simplicial distributions and the set  $D_R(Y)_n = D_R(Y_n)$  of *n*-simplices. Similarly, s**Set** $(\Delta[n], Y)$  can be identified with  $Y_n$ . Then the  $\Theta$ -map is the identity map and every simplicial distribution is noncontextual.

Next, we introduce a stronger version of contextuality.

2.12. DEFINITION. The support of a simplicial distribution  $p: X \longrightarrow D_R Y$  is defined by

$$\operatorname{supp}(p) = \{ \varphi \in s \mathbf{Set}(X, Y) : p_n(x)(\varphi_n(x)) \neq 0, \ \forall x \in X_n, \ n \ge 0 \}.$$

We say p is strongly contextual if supp(p) is empty.

2.13. PROPOSITION. If a simplicial distribution p is strongly contextual then it is contextual.

PROOF. Suppose that p is noncontextual, that is there exists  $d \in D_R(s\mathbf{Set}(X,Y))$  such that

$$p_n(x) = \sum_{\varphi \in s \mathbf{Set}(X,Y)} d(\varphi) \delta^{\varphi_n(x)}.$$

Then  $\psi: X \longrightarrow Y$  with  $d(\psi) \neq 0$  will belong to the support of p since

$$p_n(x)(\psi_n(x)) = \sum_{\varphi \in s \mathbf{Set}(X,Y)} d(\varphi) \delta^{\varphi_n(x)}(\psi_n(x)) = d(\psi) + \sum_{\varphi \neq \psi} d(\varphi) \delta^{\varphi_n(x)}(\psi_n(x)) \neq 0.$$

For simplicial probability distributions one can introduce a degree of contextuality generalizing the notion introduced in [3, Subsection 6.1].

2.14. DEFINITION. Let Y be a non-empty simplicial set. The *noncontextual fraction* of a simplicial probability distribution  $p \in s\mathbf{Set}(X, D(Y))$ , which is denoted by NCF(p), is defined to be the supremum of  $\alpha \in [0, 1]$  such that

$$p = \alpha q + (1 - \alpha)s$$

where q and s run over simplicial probability distributions with q noncontextual. The contextual fraction of p is defined to be CF(p) = 1 - NCF(p).

2.15. SIMPLICIAL DISTRIBUTIONS AS A CONVEX SET. In this section we will show that  $s\mathbf{Set}(X,Y)$ , where Y is an object of  $s\mathbf{Conv}_R$ , is an *R*-convex set (Proposition 2.16) and prove a uniqueness result about the comparison map  $\Theta_{X,Y}$  (Proposition 2.17).

In the proof of the next result we will use the fact that there is an equivalence between the category of finitary monads on **Set** and the category of Lawvere theories [21]. Moreover, for a finitary monad  $T : \mathbf{Set} \longrightarrow \mathbf{Set}$  the category of *T*-algebras is equivalent to the category of models of the corresponding Lawvere theory. This equivalence was essentially established by Lintonin [22]; for modern treatments see [23, 24]. Therefore the following result holds more generally for Lawvere theories.

2.16. PROPOSITION. The functor 
$$s\mathbf{Set}(-,-): s\mathbf{Set}^{op} \times s\mathbf{Set} \longrightarrow \mathbf{Set}$$
 lifts to a functor  
 $s\mathbf{Set}(-,-): s\mathbf{Set}^{op} \times s\mathbf{Conv}_R \longrightarrow \mathbf{Conv}_R$ 

PROOF. The finitary monad  $D_R$  defines a Lawvere theory  $T_R$  such that the category of  $D_R$ algebras (*R*-convex sets) is equivalent to the category of models of  $T_R$ , which means that every object in  $\mathbf{Conv}_R$  is a product-preserving functor from  $T_R$  to **Set**. In addition, every object in  $s\mathbf{Conv}_R$  is a product-preserving functor from  $T_R$  to  $s\mathbf{Set}$ . Given a simplicial set X and a simplicial convex set Y, let  $Y': T_R \longrightarrow s\mathbf{Set}$  be the corresponding functor for Y. Then  $s\mathbf{Set}(X, -) \circ Y': T_R \longrightarrow \mathbf{Set}$  is the corresponding functor for  $s\mathbf{Set}(X, Y)$ . This functor preserves products because both Y' and  $s\mathbf{Set}(X, -)$  preserve products.

For a simplicial set Y observe that  $D_R(Y)$  is an object of  $s \mathbf{Conv}_R$ , hence Proposition 2.16 applies. We employ  $s \mathbf{Set}(X, D_R(Y))$  with an *R*-convex set structure by defining

$$\pi^{s\mathbf{Set}(X,D_R(Y))} = (\pi^{D_R(Y)})_* \circ \Theta_{X,D_R(Y)}.$$
(4)

More explicitly, for  $Q \in D_R(s\mathbf{Set}(X, D_R(Y)))$  and  $x \in X_n$  this gives

$$\pi(Q)_n(x) = \sum_{p \in s \operatorname{Set}(X, D_R(Y))} Q(p) \, p_n(x).$$
(5)

2.17. PROPOSITION. The map  $\Theta_{X,Y}$  is the transpose of

$$(\delta_Y)_* : s\mathbf{Set}(X, Y) \longrightarrow s\mathbf{Set}(X, D_R(Y))$$

in  $\mathbf{Conv}_R$ , with respect to the adjunction  $\mathbf{Set} \dashv \mathbf{Conv}_R$ , and it is given by the composite

$$\Theta_{X,Y} = \pi^{s\mathbf{Set}(X,D_R(Y))} \circ D_R((\delta_Y)_*).$$
(6)

In particular, it is the unique map in  $\operatorname{Conv}_R$  that makes Diagram (3) commutes.

**PROOF.** By the naturality of  $\Theta_{X,-}$  we have the following commutative diagram:

that is the following equation holds

$$\Theta_{X,D_R(Y)} \circ D_R((\delta_Y)_*) = (D_R(\delta_Y))_* \circ \Theta_{X,Y}.$$
(8)

We compose with  $(\mu^Y)_*$  on both sides of Equation (8) to obtain

$$(\mu^Y)_* \circ \Theta_{X,D_R(Y)} \circ D_R((\delta_Y)_*) = (\mu^Y)_* \circ (D_R(\delta_Y))_* \circ \Theta_{X,Y}.$$
(9)

Now the last composite on the left-hand side can be rewritten as

$$(\mu^{Y})_{*} \circ \Theta_{X, D_{R}(Y)} = (\pi^{D_{R}(Y)})_{*} \circ \Theta_{X, D_{R}(Y)} = \pi^{s \mathbf{Set}(X, D_{R}(Y))}$$
(10)

and similarly for the right-hand side we have

$$(\mu^{Y})_{*} \circ (D_{R}(\delta_{Y}))_{*} = (\mu^{Y} \circ D_{R}(\delta_{Y}))_{*} = (\mathrm{Id}_{D_{R}(Y)})_{*} = \mathrm{Id}_{s\mathbf{Set}(X, D_{R}(Y))}.$$
 (11)

which proves Equation (6). By Remark 2.2 we see that  $\Theta_{X,Y}$  is the transpose of  $(\delta_Y)_*$  in  $\mathbf{Conv}_R$ .

2.18. SIMPLICIAL DISTRIBUTIONS FROM PRESHEAVES OF DISTRIBUTIONS. Presheaf of distributions can be constructed from the following data:

- A (ordered) simplicial complex  $\Sigma$ .
- The set  $[d] = \{0, 1, \dots, d-1\}$  where  $d \ge 2$ .

The typical case is when d = 2 corresponding to measurements with binary outcomes. It is convenient to identify [d] with the ring  $\mathbb{Z}_d = \{0, 1, \dots, d-1\}$  of integers mod d. The additive group structure becomes useful when we consider various realizations of presheaves of distributions. We will write  $\Sigma_0$  to denote the set of vertices of the simplicial complex. We will think of  $\Sigma$  as a category  $\mathbf{C}_{\Sigma}$  whose objects are the simplices  $\sigma \in \Sigma$ and morphisms are inclusions  $\sigma \hookrightarrow \sigma'$ . An element p of the inverse limit of the composite functor

$$\mathbf{C}_{\Sigma} \xrightarrow{\mathbb{Z}_{d}^{-}} \mathbf{Set} \xrightarrow{D_{R}} \mathbf{Set}$$

is called an *R*-distribution on  $(\Sigma, \mathbb{Z}_d)$  (also known as an *empirical model*). Let  $V \subset U$ be two subsets of  $\Sigma_0$  and  $i: V \hookrightarrow U$  denote the inclusion map. For  $p \in D_R(\mathbb{Z}_d^U)$  we will write  $p|_V$  for the distribution  $D_R(i^*)(p) \in D_R(\mathbb{Z}_d^V)$ , where  $i^*: \mathbb{Z}_d^U \longrightarrow \mathbb{Z}_d^V$  is the restriction map. Then, a presheaf of distributions consists of a tuple  $p = (p_\sigma)_{\sigma \in \Sigma}$  of distributions  $p_\sigma \in D_R(\mathbb{Z}_d^\sigma)$  such that

$$p_{\sigma}|_{\sigma\cap\sigma'} = p_{\sigma'}|_{\sigma\cap\sigma'}, \ \forall \sigma, \sigma' \in \Sigma,$$

We write  $D_R(\Sigma, \mathbb{Z}_d)$  for the set of *R*-distributions on  $(\Sigma, \mathbb{Z}_d)$ . For more details see [3].

For a set U let  $\Delta_U$  denote the simplicial set whose *n*-simplices are given by the set  $U^{n+1}$  and the simplicial structure maps are given by

$$d_i(x_0, x_1, \cdots, x_n) = (x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$
  
$$s_j(x_0, x_1, \cdots, x_n) = (x_0, x_1, \cdots, x_{j-1}, x_j, x_j, x_{j+1}, \cdots, x_n).$$

For the next definition we will use the following notation: For  $p: X \longrightarrow D_R Y$  and  $\sigma \in X_n$ we write  $p_{\sigma}$  for the distribution on  $D_R Y_n$ .

2.19. CONSTRUCTION. [Realization]Given  $(\Sigma, \mathbb{Z}_d)$  we consider the pair of simplicial sets  $(X_{\Sigma}, \Delta_{\mathbb{Z}_d})$  where  $X_{\Sigma}$  is the subsimplicial set of  $\Delta_{\Sigma_0}$  whose *n*-simplices are given by

$$(X_{\Sigma})_n = \{ (x_0, x_1, \cdots, x_n) \in \Sigma_0^{n+1} : x_0 \le x_1 \le \cdots \le x_n, \{ x_0, x_1, \cdots, x_n \} \in \Sigma \}.$$

We define an injective function

Real : 
$$D_R(\Sigma, \mathbb{Z}_d) \longrightarrow s\mathbf{Set}(X_{\Sigma}, D_R(\Delta_{\mathbb{Z}_d}))$$

by sending  $p = (p_{\sigma})_{\sigma \in \Sigma}$  to the simplicial distribution  $\operatorname{Real}(p) : X_{\Sigma} \longrightarrow D_R(\Delta_{\mathbb{Z}_d})$  defined by

$$\operatorname{Real}(p)_{(x_0, x_1, \cdots, x_n)}(a_0, a_1, \cdots, a_n) = p_{\{x_0, x_1, \cdots, x_n\}}(s : x_i \mapsto a_i).$$

2.20. THEOREM. Let  $p = (p_{\sigma})_{\sigma \in \Sigma}$  be an *R*-distribution on  $(\Sigma, \mathbb{Z}_d)$ . Then Real(p) is noncontextual if and only if there exists  $\tilde{p} \in D_R(\mathbb{Z}_d^{\Sigma_0})$  such that  $\tilde{p}|_{\sigma} = p_{\sigma}$  for all  $\sigma \in \Sigma$ .

PROOF. See [10, Theorem B.2].

2.21. REMARK. The condition of being strongly contextual also simplifies in the case of Real(p). The support of the distribution (Definition 2.12) becomes

$$\operatorname{supp}(\operatorname{Real}(p)) = \{ s \in \mathbb{Z}_d^{\Sigma_0} : p_{\sigma}(s|_{\sigma}) \neq 0, \ \forall \sigma \in \Sigma \}.$$

This is precisely the definition of the support of a distribution in  $D_R(\Sigma, \mathbb{Z}_d)$  [3].

Observe that the map  $\eta$  given in the proof of Theorem 2.20 is a morphism of *R*-convex sets when  $R = \mathbb{R}_{\geq 0}$ . This implies that the notion of contextual fraction for simplicial distributions (Definition 2.14) coincides with the corresponding notion for distributions in  $D(\Sigma, \mathbb{Z}_d)$  given in [3].

The realization Real defined in Construction 2.19 has also a version for unordered simplicial complexes.

2.22. EXAMPLE. Let  $\Sigma$  denote the simplicial complex with vertices x, y and a single maximal simplex  $\{x, y\}$ . An element of  $D(\Sigma, \mathbb{Z}_2)$  is simply a distribution  $p \in D(\mathbb{Z}_2^2)$ . Let us write  $p^{ab} = p(a, b)$  where  $a, b \in \mathbb{Z}_2$ . With this notation p can be conveniently represented as a box (table):

$$\begin{array}{c|c} & y \\ \hline x & p^{00} & p^{01} \\ p^{10} & p^{11} \end{array}$$

Let us write  $p_x = p|_{\{x\}}$  and  $p_y = p|_{\{y\}}$ . Then the compatibility relations become

$$p_x^0 = p^{00} + p^{01} p_y^0 = p^{00} + p^{10},$$
(12)

which can be read off from the rows and the columns of the box. Let us describe the corresponding simplicial distribution. The simplicial set  $X_{\Sigma}$  is isomorphic to  $\Delta[1]$ . Therefore

p gives a simplicial distribution of the form  $\Delta[1] \longrightarrow D(\Delta_{\mathbb{Z}_2})$ , again denoted by p for simplicity. Let  $\iota_1$  denote the unique nondegenerate 1-simplex of  $\Delta[1]$ , i.e., the identity map  $[1] \longrightarrow [1]$ . Then the value of the distribution  $p_{\iota_1}$  (see Definition 2.7 for notation) at the 1-simplex (a, b) of  $\Delta_{\mathbb{Z}_2}$  is given by  $p^{ab}$ . The two 0-simplices  $d_1(\iota_1)$  and  $d_0(\iota_1)$  correspond to x and y; respectively. The conditions in Equation (12) are expressed as the simplicial relation  $d_i p_{\iota_1} = p_{d_i \iota_1}$ .

2.23. EXAMPLE. A famous example, known as the CHSH scenario [9], consists of the simplicial complex on the vertex set  $\Sigma_0 = \{x_0, x_1, y_0, y_1\}$  determined by the maximal simplices

 $\{x_0, y_0\}, \{x_0, y_1\}, \{x_1, y_0\}, \{x_1, y_1\}.$ 

The resulting simplicial complex is the boundary of a square:



A distribution  $p = (p_{\sigma})_{\sigma \in \Sigma} \in D(\Sigma, \mathbb{Z}_2)$  consists of distributions  $p_{\{x_i, y_j\}} \in D(\mathbb{Z}_2^{\{x_i, y_j\}})$ together with the compatibility conditions imposed by the inverse limit. Writing  $p_{x_i y_j}^{ab}$  for the probability  $p_{\{x_i, y_j\}}(s)$ , where  $s : \{x_i, y_j\} \longrightarrow \mathbb{Z}_2$  defined by  $s(x_i) = a, s(y_j) = b$ , these conditions can be expressed as

$$\sum_{b \in \mathbb{Z}_2} p_{x_i y_0}^{ab} = \sum_{b \in \mathbb{Z}_2} p_{x_i y_1}^{ab}, \quad \sum_{a \in \mathbb{Z}_2} p_{x_0 y_j}^{ab} = \sum_{a \in \mathbb{Z}_2} p_{x_1 y_j}^{ab}.$$
 (13)

Again a convenient way to represent this data is to use a table of the form

	$y_0$		$y_1$	
$x_0$	$p^{00}_{x_0y_0}\ p^{10}_{x_0y_0}$	$p^{01}_{x_0y_0}\ p^{11}_{x_0y_0}$	$p^{00}_{x_0y_1}\ p^{10}_{x_0y_1}$	$p^{01}_{x_0y_1}\ p^{11}_{x_0y_1}$
$x_1$	$\begin{array}{c} p^{00}_{x_1y_0} \\ p^{10}_{x_1y_0} \end{array}$	$\begin{array}{c} p_{x_1y_0}^{01} \\ p_{x_1y_0}^{11} \\ p_{x_1y_0}^{11} \end{array}$	$\begin{array}{c} p^{00}_{x_1y_1} \\ p^{10}_{x_1y_1} \end{array}$	$\begin{array}{c} p_{x_1y_1}^{01} \\ p_{x_1y_1}^{11} \\ p_{x_1y_1}^{11} \end{array}$

The compatibility conditions in Equation (13) can be read off the table. According to a celebrated theorem due to Fine [25,26], a distribution  $p \in D(\Sigma, \mathbb{Z}_2)$  is noncontextual if and only if the CHSH inequalities are satisfied. Moreover, it is well-known that  $D(\Sigma, \mathbb{Z}_2)$  is a polytope with 16 deterministic vertices and 8 contextual vertices given by the Popescu–Rohrlich (PR) boxes [27].

Next, we describe another way of realizing presheaves of distributions where the target space is distributions on the nerve space  $N\mathbb{Z}_d$ . For a monoid  $(M, \cdot)$  the nerve N(M) is

the simplicial set whose set of *n*-simplices is given by  $M^n$  with the following simplicial structure:

$$d_i(m_1, m_2, \cdots, m_n) = \begin{cases} (m_2, m_3, \cdots, m_n) & i = 0\\ (m_1, \cdots, m_i \cdot m_{i+1}, \cdots, m_n) & 0 < i < n\\ (m_1, m_2, \cdots, m_{n-1}) & i = n \end{cases}$$
$$s_j(m_1, m_2, \cdots, m_n) = (m_1, \cdots, m_j, e_M, m_{j+1}, \cdots, m_n) \quad 0 \le j \le n,$$

where  $e_M$  is the identity element.

We will need the following constructions:

• The join X \* Y of the simplicial sets X, Y has *n*-simplices given by

$$(X * Y)_n = X_n \sqcup \left( \prod_{i+j=n-1} X_i \times Y_j \right) \sqcup Y_n.$$

The simplicial structure maps are induced from that of X and Y; see [28].

• The décalage  $\text{Dec}^0 X$  is the simplicial set obtained from shifting the simplices of X down by one degree [29], i.e.,  $(\text{Dec}^0 X)_n = X_{n+1}$ , and forgetting the first face and degeneracy maps.

2.24. PROPOSITION. Let X be a connected simplicial set. There is a commutative diagram of R-convex sets

**PROOF.** For X connected, there exists a natural bijection

$$s\mathbf{Set}(X, \mathrm{Dec}^{0}Y) \cong s\mathbf{Set}(\Delta[0] * X, Y), \tag{14}$$

see [30, Corollary 2.1], that comes from a more general adjunction. For a simplicial map  $f: \Delta[0] * X \longrightarrow Y$ , the image of the composition  $\Delta[0] * X \xrightarrow{f} Y \xrightarrow{\delta_Y} D_R(Y)$  under this bijection is equal to the composition of the image of f under the same bijection with  $\operatorname{Dec}^0(\delta_Y)$ . Observe also that  $\operatorname{Dec}^0(\delta_Y) = \delta_{\operatorname{Dec}^0 Y}$ , so we have the following diagram in **Set** 

Using Proposition 2.17 we get the result.

There is an isomorphism of simplicial sets

$$\Delta_{\mathbb{Z}_d} \xrightarrow{\cong} \operatorname{Dec}^0(N\mathbb{Z}_d)$$

defined in degree n by sending  $(a_0, a_1, \dots, a_n)$  to the tuple  $(a_0, a_1-a_0, \dots, a_n-a_{n-1})$ . Note that the isomorphism is preserved even after applying the  $D_R$  functor since essentially décalage only shifts the dimension. Then Proposition 2.24 implies that there is a bijection

$$s\mathbf{Set}(X, D_R(\Delta_{\mathbb{Z}_d})) \cong s\mathbf{Set}(\Delta[0] * X, D_R(N\mathbb{Z}_d))$$

and the notion of contextuality for both scenarios  $(\Delta[0] * X, N\mathbb{Z}_d)$  and  $(X, \Delta_{\mathbb{Z}_d})$  coincides.

2.25. EXAMPLE. The transpose of the simplicial set map  $\Delta[1] \longrightarrow D(\Delta_{\mathbb{Z}_2})$  discussed in Example 2.22 under the bijection in (14) is given by

$$p: \Delta[2] \cong \Delta[0] * \Delta[1] \longrightarrow D(N\mathbb{Z}_2).$$

Let  $\iota_2$  denote the unique nondegenerate 2-simplex of  $\Delta[2]$ . Let x, y, z denote the 1simplices of  $\Delta[2]$  given by  $d_2\iota_2$ ,  $d_0\iota_2$  and  $d_1\iota_2$ ; respectively. Then p can be represented as follows:



The distribution  $p_z = d_1 p_{\iota_2}$  is given by

$$p_z^0 = p^{00} + p^{11},$$

and  $p_x^0$ ,  $p_y^0$  are as before; see Equation (12). The formula for  $p_z^0$  implies that z can be interpreted as  $x \oplus y$ , the XOR of x and y. Similarly,  $x = y \oplus z$  and  $z = x \oplus y$ . For more details on this interpretation see [10].

2.26. EXAMPLE. In Example 2.23 the simplicial complex  $\Sigma$  is the boundary of a square. The join  $\Delta[0] * X_{\Sigma}$  is the cone of this one-dimensional space, which is the square consisting of four triangles:



This simplicial set is used in the topological proof of Fine's theorem [10, Section 4.5]. See also [31] for an alternative proof of Fine's theorem that uses a different simplicial realization.

2.27. CONVEX CATEGORIES. Convex categories can be defined in two equivalent ways: (1) through a monadic approach and (2) using enriched categories. In our context, the monadic approach is more suitable, as it aligns more naturally with certain constructions in simplicial distributions. However, the enriched category approach is conceptually simpler.

We begin by extending  $D_R$  to a monad on the category **Cat** of (locally small) categories. Our constructions will depend on the multiplication map: For sets X, Y, we define

$$m_{X,Y}: D_R(X) \times D_R(Y) \longrightarrow D_R(X \times Y),$$
 (15)

by sending (p,q) to the distribution  $p \cdot q$  given by

$$(p \cdot q)(x, y) = p(x)q(y), \ x \in X, \ y \in Y.$$
 (16)

This map is a section of  $D_R(r_1) \times D_R(r_2) : D_R(X \times Y) \longrightarrow D_R(X) \times D_R(Y)$ , where  $r_1 : X \times Y \longrightarrow X$  and  $r_2 : X \times Y \longrightarrow Y$  are the canonical projection maps. The following result is crucial for the upcoming constructions. The proofs are omitted, as they follow from straightforward verifications.

2.28. PROPOSITION. The multiplication map m defined in Equation (15) has the following properties:

1. Naturality: For  $f: X \longrightarrow X'$ ,  $g: Y \longrightarrow Y'$  in **Set**, we have

$$D_R(f \times g) \circ m_{X,Y} = m_{X',Y'} \circ (D_R(f) \times D_R(g)).$$

2. Associativity: For  $X, Y, Z \in \mathbf{Set}$ , we have

$$m_{X \times Y,Z} \circ (m_{X,Y} \times \mathrm{Id}_{D_R(Z)}) = m_{X,Y \times Z} \circ (\mathrm{Id}_{D_R}(X) \times m_{Y,Z}).$$

3. Preserving the unit: For  $X, Y \in \mathbf{Set}$ , the following diagram commutes:



4. Preserving the multiplication: For  $X, Y \in \mathbf{Set}$ , the following diagram commutes:

 $\mathbf{D}$  ( )

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Now, we define a functor  $D_R : \mathbf{Cat} \longrightarrow \mathbf{Cat}$  by specifying

$$\mathbf{C} \mapsto D_R(\mathbf{C})$$
 and  $F \mapsto D_R(F)$ 

for a category **C** and functor  $F : \mathbf{C} \longrightarrow \mathbf{D}$ . The category  $D_R(\mathbf{C})$  consists of the following data:

- the collection of objects of **C**,
- for objects X, Y of **C** the set  $D_R(\mathbf{C}(X, Y))$  of morphisms.

For objects X, Y, Z of **C** we define the composition map

$$*: D_R(\mathbf{C}(X,Y)) \times D_R(\mathbf{C}(Y,Z)) \longrightarrow D_R(\mathbf{C}(X,Z)), \quad (p,q) \mapsto q * p, \tag{17}$$

to be the composite of  $m_{C(X,Y),C(Y,Z)}$  with  $D_R(\circ)$ , where  $\circ$  is the composition in **C**. So we have

$$(q*p)(f) = \sum_{g_2 \circ g_1 = f} q(g_2)p(g_1), \quad \forall f \in \mathbf{C}(X, Z),$$

$$(18)$$

where the sum runs over morphisms  $g_1 : X \longrightarrow Y$  and  $g_2 : Y \longrightarrow Z$  such that  $g_2 \circ g_1 = f$  in **C**.

Next, we describe the functor  $D_R(F)$ : Given a functor  $F : \mathbb{C} \longrightarrow \mathbb{D}$  in  $\mathbb{Cat}$ , specify a functor

$$D_R(F): D_R(\mathbf{C}) \longrightarrow D_R(\mathbf{D}) \tag{19}$$

by defining the following maps:

- $D_R(F)$ :  $Obj(D_R(\mathbf{C})) \longrightarrow Obj(D_R(\mathbf{D}))$  defined as the map F between the objects of the categories.
- $D_R(F)_{X,Y} : D_R(\mathbf{C})(X,Y) \longrightarrow D_R(\mathbf{D})(FX,FY)$  defined by applying  $D_R$  functor to the map  $F_{X,Y}$  between the morphism sets

Finally,  $D_R : \mathbf{Cat} \longrightarrow \mathbf{Cat}$  with the following structure morphisms is a monad:

•  $\delta_{\mathbf{C}}: \mathbf{C} \longrightarrow D_R(\mathbf{C})$  to be the identity map on the objects and

$$\delta_{\mathbf{C}}: \mathbf{C}(X,Y) \longrightarrow D_R(\mathbf{C})(X,Y)$$

to be  $\delta_{\mathbf{C}(X,Y)}$ .

•  $\mu_{\mathbf{C}}: D_R(D_R(\mathbf{C})) \longrightarrow D_R(\mathbf{C})$  to be the identity on the objects and

$$\mu_{\mathbf{C}}: D_R(D_R(\mathbf{C}))(X,Y) \longrightarrow D_R(\mathbf{C})(X,Y)$$

to be  $\mu_{\mathbf{C}(X,Y)}$ .

These maps specify the following natural transformations:

 $\delta : \operatorname{Id}_{\operatorname{Cat}} \longrightarrow D_R \quad \text{and} \quad \mu : D_R D_R \longrightarrow D_R.$  (20)

Therefore we can talk about its algebras.

Note that these constructions work thanks to Proposition 2.28.

2.29. DEFINITION. A  $D_R$ -algebra in **Cat** will be called an *R*-convex category. We will write **ConvCat**<sub>R</sub> for the category of *R*-convex categories.

Convex categories can equivalently be defined as categories enriched over convex sets with the tensor product defined as a coequalizer in  $\mathbf{Conv}_R$ :

$$D_R(D_R(X) \times D_R(Y)) \xrightarrow[D_R(\pi^X \times \pi^Y)]{} D_R(X \times Y) \longrightarrow X \otimes Y$$

A proof of this fact appears in [32, Theorem 2.39].

2.30. REMARK. Here are further remarks on this notion of convex categories:

- This notion of convex category is different than the one introduced in [33, Definition 5.1]. For example, (ℝ, ·) is an ℝ<sub>≥0</sub>-convex category with one object, but not a convex category in the latter sense since coproducts do not exist in this category.
- The isomorphism between the categories of  $\mathbb{R}_{\geq 0}$ -convex sets and real convex sets (Proposition 2.4) extends to the case of  $\mathbb{R}_{\geq 0}$ -convex categories. We will write **ConvCat** for the category **ConvCat**<sub> $\mathbb{R}_{\geq 0}$ </sub> and refer to the objects of this category as real convex categories.
- Categories enriched over *R*-convex sets, where the monoidal structure is the cartesian product, are examples of convex categories. However, the converse is not true. For example, the monoid (*R*, ·) as a category with one object is an *R*<sub>≥0</sub>-convex category, but the product is not a morphism in Conv.
- The Kleisli category  $s\mathbf{Set}_{D_R}$  is an *R*-convex category. Recall that the Kleisli category  $\mathbf{C}_T$  of a monad  $T : \mathbf{C} \longrightarrow \mathbf{C}$  has the same objects as  $\mathbf{C}$  and for objects X, Y its morphisms are given by  $\mathbf{C}(X, TY)$ . For an object X the identity morphism in  $\mathbf{C}_T(X, X)$  is given by  $\delta_X : X \longrightarrow TX$ . The composition of two morphisms  $f: X \longrightarrow TY$  and  $g: Y \longrightarrow TZ$  is defined by

$$g \diamond f : X \xrightarrow{f} TY \xrightarrow{T(g)} T(TZ) \xrightarrow{\mu_Z} TZ.$$

We are interested in the Kleisli category of the distribution monad  $D_R : s\mathbf{Set} \longrightarrow s\mathbf{Set}$ . In this case the composition can be explicitly written as

$$(q \diamond p)_n(x) = \sum_{y \in Y_n} p_n(x)(y)q_n(y),$$

where  $p \in s\mathbf{Set}(X, D_R(Y)), q \in s\mathbf{Set}(Y, D_R(Z))$  and  $x \in X_n$ .

The  $\Theta$ -map defined in Equation (2) can be given a categorical interpretation using the theory of convex categories. Let  $F_T : \mathbf{C} \longrightarrow \mathbf{C}_T$  denote the functor defined as follows:

•  $Obj(\mathbf{C}) \longrightarrow Obj(\mathbf{C}_T)$  is the identity functor.

•  $F_T: \mathbf{C}(X, Y) \longrightarrow \mathbf{C}_T(X, Y)$  is defined to be  $(\delta_Y)_*$ , that is  $f: X \longrightarrow Y$  is sent to the composite  $F_T(f): X \xrightarrow{f} Y \xrightarrow{\delta_Y} TY$ .

The functor  $F_T$  has a right adjoint  $U_T : \mathbf{C}_T \longrightarrow \mathbf{C}$  (see [19, Lemma 5.2.11]). In fact, the monad T arises from this adjunction. In the following we will consider the free convex category  $D_R(s\mathbf{Set})$ , and the Kleisli category  $s\mathbf{Set}_{D_R}$ .

2.31. PROPOSITION. The transpose of the functor  $F_{D_R} : s\mathbf{Set} \longrightarrow s\mathbf{Set}_{D_R}$  with respect to the adjunction  $D_R : \mathbf{Cat} \dashv \mathbf{ConvCat}_R : U$  is the functor  $\Theta : D_R(s\mathbf{Set}) \longrightarrow s\mathbf{Set}_{D_R}$  which is defined as identity on the objects and as the map  $\Theta_{X,Y}$  on morphisms.

PROOF. Using Proposition 2.17 we see that  $\Theta = \pi^{s \operatorname{Set}_{D_R}} \circ D_R(F_{D_R})$ . Then Remark 2.2 implies that  $\Theta$  is the corresponding transpose of  $F_{D_R}$ .

#### 3. Convex monoids

In Section 2.27 we introduced the notion of a convex category. Now, in this section we will specialize to convex monoids, that is convex categories with a single object. We will introduce a weak notion of invertibility for convex monoids. This definition is inspired by the definition of noncontextuality for simplicial distributions. Later in Section 4 we will see that the two notions coincide for cases of interest. For real convex monoids we introduce the notion of invertible fraction to quantify the closeness of an element to being weakly invertible.

The monad  $D_R$  acting on **Cat** lifts to a monad on the category **Mon** of monoids. A  $D_R$ -algebra  $(M, \pi^M)$  over this monad is called an *R*-convex monoid. This is equivalent to saying that  $(M, \pi^M)$  is an *R*-convex set and the map  $\pi^M : D_R(M) \longrightarrow M$  is a homomorphism of monoids. We will write **ConvMon**<sub>R</sub> for the category of *R*-convex monoids. Thinking of convex monoids as monoids enriched in convex sets (Section 2.27), we obtain the following result.

3.1. COROLLARY. Let  $(M, \cdot)$  be a monoid which is also a real convex set. Let  $\pi^M$  denote the structure map of M when regarded as an  $\mathbb{R}_{\geq 0}$ -convex set. Then the following statements are equivalent.

- 1.  $(M, \pi^M)$  is an  $\mathbb{R}_{>0}$ -convex monoid.
- 2. For  $m_i, n_i \in M$  and  $\alpha_i, \beta_i \in [0, 1]$ , where  $1 \leq i \leq k$  and  $\sum_i \alpha_i = \sum_i \beta_i = 1$ , we have

$$\left(\sum_{i} \alpha_{i} m_{i}\right) \cdot \left(\sum_{i} \beta_{i} n_{i}\right) = \sum_{i,j} \alpha_{i} \beta_{j} m_{i} \cdot n_{j}$$

3. For m and n in M the maps  $m \cdot - and - \cdot n$  are in **Conv**.

3.2. REMARK. By part (3) of Corollary 3.1 one can see that the notion of a convex monoid given in [34, Definition 9] is a special case of  $\mathbb{R}_{>0}$ -convex monoid.

As in the case of categories, from now on we will write **ConvMon** for **ConvMon**<sub> $\mathbb{R}_{\geq 0}$ </sub> and refer to the objects of this category as real convex monoids.

3.3. EXAMPLE. A semiring R can be given the structure of an R-convex set by defining  $\pi^R : D_R(R) \longrightarrow R$  as follows:

$$\pi^R(p) = \sum_{x \in R} p(x)x.$$

Since R is commutative  $\pi^R$  is a homomorphism of monoids, hence R becomes an R-convex monoid.

3.4. EXAMPLE. The set  $\mathbb{R}$  of real numbers is a convex set, and with the following product

$$x_1 \bigtriangleup x_2 := x_1 x_2 + (1 - x_1)(1 - x_2), \quad x_1, x_2 \in \mathbb{R},$$

is a monoid with 1 as the identity. In addition, part (3) of Corollary 3.1 holds, hence  $(\mathbb{R}, \pi^{\mathbb{R}}, \Delta)$  is a real convex monoid. Note that  $([0, 1], \pi^{\mathbb{R}}, \Delta)$  is a real subconvex monoid of  $(\mathbb{R}, \pi^{\mathbb{R}}, \Delta)$ .

3.5. EXAMPLE. The set of continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$  is a real convex set. With the composition operation, it is also a monoid but not a real convex monoid.

3.6. WEAK INVERTIBILITY. In this section, we will introduce the notion of weak invertibility for monoids. We begin by relating monoids to groups. For a monoid M let I(M) denote the subset of invertible elements. This construction defines a functor  $I : \mathbf{Mon} \longrightarrow \mathbf{Grp}$ , which turns out to be the right adjoint of the inclusion functor  $j : \mathbf{Grp} \longrightarrow \mathbf{Mon}$ ; see [35, Example 2.1.3 (d)]. The composition of the two adjunctions  $j : \mathbf{Grp} \dashv \mathbf{Mon} : I$  and  $D_R : \mathbf{Mon} \dashv \mathbf{ConvMon}_R : U$  gives us the following adjunction

$$D_R: \mathbf{Grp} \dashv \mathbf{ConvMon}_R: I.$$
(21)

For simplicity, we will use the notation  $M^*$  instead of I(M).

Let  $i_M : M^* \longrightarrow M$  denote the inclusion map. We will consider  $D_R(M^*)$  as a subset of  $D_R(M)$  via the map  $D_R(i_M)$ . The restriction of  $\pi^M : D_R(M) \longrightarrow M$  to  $D_R(M^*)$  will be denoted by  $\tilde{\pi}^M$ .

3.7. DEFINITION. An element  $m \in M$  is called *weakly invertible* if it lies in the image of  $\tilde{\pi}^M : D_R(M^*) \longrightarrow M$ .

In particular, every invertible element is weakly invertible.

3.8. EXAMPLE. In Example 3.3 if we let  $R = \mathbb{R}_{\geq 0}$  then 0 is not weakly invertible since for  $p \in D((\mathbb{R}_{\geq 0})^*)$  we have

$$\tilde{\pi}^{\mathbb{R}_{\geq 0}}(p) = \sum_{x \in \mathbb{R}_{\geq 0}} p(x)x > 0.$$

3.9. EXAMPLE. The invertible elements of the subconvex monoid  $([0,1], \pi^{\mathbb{R}}, \Delta)$  (see Example 3.4) are 0 and 1. For an element  $x \in [0,1]$  we have the distribution  $p = x\delta^1 + (1-x)\delta^0 \in D([0,1]^*)$  that satisfies  $\pi^{\mathbb{R}}(p) = x$ . Therefore every element of [0,1] is weakly invertible.

3.10. PROPOSITION. Let  $n \in M^*$  and  $m \in M$ . Then m is weakly invertible if and only if  $n \cdot m$  is weakly invertible.

**PROOF.** Suppose that  $p \in D_R(M^*)$  with  $\tilde{\pi}^M(p) = m$ . Then we have

$$\tilde{\pi}^M(\delta^n * p) = \tilde{\pi}^M(\delta^n) \cdot \tilde{\pi}^M(p) = n \cdot m$$
(22)

In Equation (22) we used the fact that  $\tilde{\pi}^M(\delta^n) = \pi^M(D_R(i_M)(\delta^n)) = \pi^M(\delta^n) = n$ . This shows that if *m* is weakly invertible then  $n \cdot m$  is weakly invertible. Conversely, if  $n \cdot m$  is weakly invertible then applying this observation to the product  $n \cdot m$  (instead of *m*) we obtain that  $n^{-1} \cdot (n \cdot m) = m$  is weakly invertible.

3.11. STRONG INVERTIBILITY. We introduce a stronger version of invertibility akin to strong contextuality (Definition 2.12) defined in terms of supports.

3.12. DEFINITION. Given an *R*-convex monoid *M*, the *invertible support* of  $m \in M$  is the following subset of  $M^*$ :

$$\operatorname{Isupp}(m) = \{m' \in M^* : \exists P \in D_R(M) \text{ such that } \pi(P) = m \text{ and } P(m') \neq 0\}$$

We say that m is strongly noninvertible if  $\text{Isupp}(m) = \emptyset$ .

3.13. LEMMA. Let M be an R-convex monoid. For  $m_1, m_2 \in M$  we have

 $\mathrm{Isupp}(m_1) \cdot \mathrm{Isupp}(m_2) \subseteq \mathrm{Isupp}(m_1 \cdot m_2).$ 

Furthermore, if  $m_1$  invertible, then  $m_1 \cdot \text{Isupp}(m_2) = \text{Isupp}(m_1 \cdot m_2)$ .

PROOF. Given  $n_1 \in \text{Isupp}(m_1)$ ,  $n_2 \in \text{Isupp}(m_2)$  there exists  $P_1, P_2 \in D_R(M)$  such that  $\pi(P_i) = m_i$  and  $P_i(n_i) \neq 0$ . Therefore  $\pi(P_1 * P_2) = m_1 \cdot m_2$  and

$$P_1 * P_2(n_1 \cdot n_2) = \sum_{x_1 \cdot x_2 = n_1 \cdot n_2} P_1(x_1) P_2(x_2) = P_1(n_1) P_2(n_2) + \dots \neq 0.$$

This means that  $n_1 \cdot n_2 \in \text{Isupp}(m_1 \cdot m_2)$ .

Suppose now that  $m_1 \in M^*$ . Then  $m_1 \cdot \text{Isupp}(m_2) \subseteq \text{Isupp}(m_1) \cdot \text{Isupp}(m_2) \subseteq$ Isupp $(m_1 \cdot m_2)$ . Therefore  $\text{Isupp}(m_1 \cdot m_2) = m_1 \cdot m_1^{-1} \cdot \text{Isupp}(m_1 \cdot m_2) \subseteq m_1 \cdot \text{Isupp}(m_2)$ .

As an immediate consequence of this observation we have the following.

3.14. COROLLARY. Let M be an R-convex monoid. Let  $n \in M^*$  and  $m \in M$ . Then the following are equivalent:

- 1. m is strongly noninvertible.
- 2.  $n \cdot m$  is strongly noninvertible.
- 3.  $m \cdot n$  is strongly noninvertible.

3.15. LEMMA. For a morphism  $f : M_1 \longrightarrow M_2$  in  $\operatorname{ConvMon}_R$  and  $m \in M_1$  we have  $f(\operatorname{Isupp}(m)) \subseteq \operatorname{Isupp}(f(m))$ .

PROOF. Given  $n \in \text{Isupp}(m)$ , there exists  $P \in D_R(M_1)$  such that  $\pi^{M_1}(P) = m$  and  $P(n) \neq 0$ . We have

$$\pi^{M_2}(D_R(f)(P)) = f(\pi^{M_1}(P)) = f(m)$$

and

$$D_R(f)(P)(f(n)) = \sum_{f(n')=f(n)} P(n') = P(n) + \dots \neq 0.$$

Therefore  $f(n) \in \text{Isupp}(f(m))$ .

Next result follows immediately from Lemma 3.15.

3.16. COROLLARY. For a morphism  $f: M_1 \longrightarrow M_2$  in **ConvMon**<sub>R</sub> and  $m \in M_1$  we have the following:

- 1. If  $\text{Isupp}(f(m)) \cap f(M_1^*) = \emptyset$ , then m is strongly noninvertible.
- 2. If f(m) is strongly noninvertible, then m is also strongly noninvertible.

3.17. INVERTIBLE FRACTION. We now give the definition of the invertible fraction for real convex monoids.

3.18. DEFINITION. Let M be a real convex monoid. The *invertible fraction* of  $m \in M$ , denoted by IF(m), is the supremum of

$$\{\sum_{m'\in M^*} P(m'): P \in D(M) \text{ such that } \pi(P) = m\}.$$

The noninvertible fraction of m is defined to be NIF(m) = 1 - IF(m).

We provide an alternative characterization of invertible fraction, which will be useful when comparing this notion to noncontextual fraction.

3.19. PROPOSITION. Let M be a real convex monoid. For  $m \in M$ , the invertible fraction IF(m) is equal to the supremum of

 $\{\alpha \in [0,1] : m_1 \text{ is weakly invertible, } m_2 \in M \text{ such that } m = \alpha m_1 + (1-\alpha)m_2 \}.$ (23)

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**PROOF.** We begin by an observation. For  $P \in D(M)$ , we define

$$I(P)(m) = \begin{cases} \frac{P(m)}{\sum_{n \in M^*} P(n)} & \text{if } m \in M^* \\ 0 & \text{otherwise,} \end{cases}$$

and

$$NI(P)(m) = \begin{cases} 0 & \text{if } m \in M^* \\ \frac{P(m)}{\sum_{n \in M - M^*} P(n)} & \text{otherwise.} \end{cases}$$

Then we have

$$P = \sum_{n \in M^*} P(n)I(P) + \sum_{m \in M - M^*} P(m)NI(P).$$
 (24)

Now, let us denote the set in Equation (23) by A. For  $\alpha \in A$ , there exists  $P_1 \in D(M^*)$ and  $m_2 \in M$  such that

$$m = \alpha \pi^{M}(P_{1}) + (1 - \alpha)m_{2} = \alpha \pi^{M}(P_{1}) + (1 - \alpha)\pi^{M}(\delta^{m_{2}}).$$

By Proposition 2.4 the structure map  $\pi^M$  is a morphism in **Conv**. Therefore we obtain

$$m = \pi^M (\alpha P_1 + (1 - \alpha)\delta^{m_2}).$$

Observe that

$$\sum_{m' \in M^*} (\alpha P_1 + (1 - \alpha)\delta^{m_2})(m') = \alpha \sum_{m' \in M^*} P_1(m') + (1 - \alpha) \sum_{m' \in M^*} \delta^{m_2}(m') \ge \alpha \sum_{m' \in M^*} P_1(m').$$

Since  $P_1 \in D(M^*)$  we have  $\sum_{m' \in M^*} P_1(m') = 1$ , and therefore

$$\sum_{m' \in M^*} (\alpha P_1 + (1 - \alpha)\delta^{m_2})(m') \ge \alpha.$$

This yields that  $IF(m) \ge \sup A$ . Now given  $P \in D(M)$  such that  $\pi(P) = m$ , Equation (24) implies that

$$P = \sum_{n \in M^*} P(n)I(P) + \sum_{m \in M - M^*} P(m)NI(P).$$

Applying  $\pi^M$  we obtain

$$m = \sum_{n \in M^*} P(n) \pi^M(I(P)) + \sum_{m \in M - M^*} P(m) \pi^M(NI(P)).$$

Note that  $\pi^M(I(P))$  is weakly invertible, thus  $\sup A \ge \mathrm{IF}(m)$ .

3.20. PROPOSITION. Let M be a real convex monoid. An element  $m \in M$  is strongly noninvertible if and only if IF(m) = 0.

PROOF. The element m is strongly noninvertible if and only if  $\text{Isupp}(m) = \emptyset$ . The set Isupp(m) is empty if and only if every  $P \in D(M)$  such that  $\pi(P) = m$  satisfies P(m') = 0 for all  $m' \in M^*$ . This is equivalent to the statement that every  $P \in D(M)$  such that  $\pi(P) = m$  satisfies  $\sum_{m' \in M^*} P(m') = 0$ . This means that IF(m) = 0.

3.21. PROPOSITION. Let M be a real convex monoid. For  $m_1, m_2 \in M$  we have

$$\operatorname{IF}(m_1 \cdot m_2) \ge \operatorname{IF}(m_1) \cdot \operatorname{IF}(m_2).$$

PROOF. Let  $P_1, P_2 \in D_R(M)$  such that  $\pi(P_i) = m_i$  for i = 1, 2. Then  $\pi(P_1 * P_2) = m_1 \cdot m_2$ and

$$\sum_{m' \in M^*} (P_1 * P_2)(m') = \sum_{m' \in M^*} \sum_{m'_1 \cdot m'_2 = m'} P_1(m'_1) P(m'_2)$$
$$= \sum_{m'_1 \cdot m'_2 \in M^*} P_1(m'_1) P_2(m'_2)$$
$$\ge \sum_{m'_1, m'_2 \in M^*} P_1(m'_1) P_2(m'_2)$$
$$= \sum_{m'_1 \in M^*} P(m'_1) \sum_{m'_2 \in M^*} P(m'_2).$$

Therefore  $IF(m_1 \cdot m_2) \ge IF(m_1) \cdot IF(m_2)$ .

3.22. COROLLARY. Let M be a real convex monoid. Given  $n \in M^*$  and  $m \in M$  we have  $IF(n \cdot m) = IF(m)$ .

PROOF. By Proposition 3.21 we have  $IF(n \cdot m) \ge IF(n) IF(m) = IF(m)$ . On the other hand,  $m = n^{-1} \cdot (n \cdot m)$ . Therefore  $IF(m) \ge IF(n \cdot m)$ .

3.23. PROPOSITION. Given a morphism  $f: M_1 \longrightarrow M_2$  in **ConvMon** and  $m \in M$  we have  $IF(f(m)) \ge IF(m)$ .

PROOF. For  $P \in D(M_1)$  such that  $\pi^{M_1}(P) = m$  we have  $\pi^{M_2}(D(f)(P)) = f(\pi^{M_1}(P)) = f(m)$  and

$$\sum_{n' \in M_2^*} D(f)(P)(n') = \sum_{n' \in M_2^*} \sum_{f(m)=n'} P(m) \ge \sum_{m' \in M_1^*} P(m').$$

3.24. PROPOSITION. Let M be a real convex monoid. For  $m \in M$  and  $\pi(P) = m$  we have

$$\operatorname{IF}(m) \ge \sum_{x \in M} P(x) \operatorname{IF}(x).$$

PROOF. Let us write  $m_1, m_2, \dots, m_k$  for the distinct elements in  $\{x \in M \mid P(x) \neq 0\}$ . Given  $\epsilon > 0$ , for every  $1 \leq i \leq k$  we choose  $P_i \in D(M)$  such that  $\pi(P_i) = m_i$  and  $\operatorname{IF}(m_i) - \epsilon < \sum_{m' \in M^*} P_i(m')$ . We have  $\sum_{i=1}^k P(m_i) = 1$ . Thus we can define  $Q = \sum_{i=1}^k P(m_i)P_i \in D(M)$ . In fact,  $Q = \mu_M(\tilde{P})$ , where  $\tilde{P} \in D(D(M))$  is defined as the follows:

$$\tilde{P}(S) = \begin{cases} P(m_i) & S = P_i \\ 0 & \text{otherwise} \end{cases}$$

On the other hand,  $D(\pi)(\tilde{P}) \in D(M)$  and for  $x \in M$  we have

$$D(\pi)(\tilde{P})(x) = \sum_{\pi(q)=x} \tilde{P}(q) = \sum_{i:\pi(P_i)=x} \tilde{P}(P_i) = \begin{cases} P(m_i) & x = m_i \\ 0 & \text{otherwise} \end{cases}$$

We obtain that  $D(\pi)(\tilde{P}) = P$ . Thus by the right-hand Diagram in (1), we see that  $\pi(Q) = \pi(P)$ . Therefore  $\pi(Q) = m$ . Using this and by our choice of  $P_1, \dots, P_k$ , we obtain

$$\sum_{m' \in M^*} Q(m') = \sum_{m' \in M^*} \sum_{i=1}^{k} P(m_i) P_i(m')$$
  
=  $\sum_{i=1}^{k} \sum_{m' \in M^*} P(m_i) P_i(m')$   
=  $\sum_{i=1}^{k} P(m_i) (\sum_{m' \in M^*} P_i(m'))$   
>  $\sum_{i=1}^{k} P(m_i) (\text{IF}(m_i) - \epsilon)$   
=  $\sum_{i=1}^{k} P(m_i) \text{IF}(m_i) - \epsilon.$ 

Hence we proved that for every  $\epsilon > 0$  there exists  $Q \in D(M)$  such that  $\pi(Q) = m$  and

$$\sum_{m' \in M^*} Q(m') > \sum_{m \in M} P(m) \operatorname{IF}(m) - \epsilon.$$

This gives the desired result.

3.25. REMARK. Remark 2.21 implies that the notion of contextuality for simplicial distributions and the corresponding notion for presheaves of distributions provided in [3] coincide. Later in Corollary 4.9 we will show that noncontextual fraction of a simplicial distribution  $p: X \longrightarrow D(Y)$  is equal to its invertible fraction (here we assume Y is a simplicial group). With these observations Proposition 3.24 generalizes the first inequality of [13, Theorem 2] satisfies by the contextual fraction of a presheaf of distributions.

### 4. Contextuality and invertibility

Recall that a simplicial distribution  $p: X \longrightarrow D_R(Y)$  is called noncontextual if it lies in the image of  $\Theta_{X,Y}: D_R(s\mathbf{Set}(X,Y)) \longrightarrow s\mathbf{Set}(X,D_R(Y))$  (Definition 2.8). When Y is a simplicial group the set of simplicial distributions  $s\mathbf{Set}(X,D_R(Y))$  is a convex monoid (Corollary 4.3). In this monoid we can consider those elements that are weakly invertible (Definition 3.7). In this section we prove our main result, which is the equivalence of the

two notions for simplicial distributions. We also prove a similar equivalence between their strong versions.

4.1. CONTEXTUALITY AND WEAK INVERTIBILITY. In this section we establish the equivalence of the notions of noncontextuality and weak invertibility.

4.2. LEMMA. Let X be a simplicial set and Y be a simplicial monoid. Then the map  $\Theta_{X,Y} : D_R(s\mathbf{Set}(X,Y)) \longrightarrow s\mathbf{Set}(X,D_R(Y))$  is a homomorphism of monoids.

**PROOF.** Given  $p, q \in D_R(s\mathbf{Set}(X, Y))$ , and  $x \in X_n, y \in Y_n$ , we have

$$\begin{split} \Theta(p*q)_n(x)(y) &= \sum_{\varphi \in s \mathbf{Set}(X,Y): \varphi_n(x)=y} (p*q)(\varphi) \\ &= \sum_{\varphi_n(x)=y} \sum_{\psi \cdot \psi'=\varphi} p(\psi)q(\psi') \\ &= \sum_{\psi_n(x) \cdot \psi'_n(x)=y} p(\psi)q(\psi') \\ &= \sum_{y_1 \cdot y_2=y} \sum_{\psi_n(x)=y_1, \psi'_n(x)=y_2} p(\psi)q(\psi') \\ &= \sum_{y_1 \cdot y_2=y} \left(\sum_{\psi_n(x)=y_1} p(\psi)\right) \left(\sum_{\psi'_n(x)=y_2} q(\psi')\right) \\ &= \sum_{y_1 \cdot y_2=y} \Theta(p)_n(x)(y_1) \cdot \Theta(q)_n(x)(y_2) \\ &= (\Theta(p)_n(x) * \Theta(q)_n(x))(y) \\ &= (\Theta(p) \cdot \Theta(q))_n(x)(y). \end{split}$$

4.3. COROLLARY. The functor  $s\mathbf{Set}(-,-): s\mathbf{Set}^{op} \times s\mathbf{Set} \longrightarrow \mathbf{Set}$  lifts to a functor

$$s\mathbf{Set}(-,-): s\mathbf{Set}^{op} \times s\mathbf{ConvMon}_R \longrightarrow \mathbf{ConvMon}_R$$

In particular, for a simplicial set X and a simplicial monoid Y the set s**Set** $(X, D_R(Y))$  of morphisms is an R-convex monoid.

**PROOF.** Proof follows from Proposition 2.16 and Lemma 4.2.

Next we describe the monoid structure of  $s\mathbf{Set}(X, D(N\mathbb{Z}_2))$  when X is one of the spaces in Example 2.25 and 2.26.

4.4. EXAMPLE. First, let us describe the product for the triangle  $X = \Delta[2]$ . Given simplicial distributions p, q represented by the boxes as in Example 2.22

	y			y	
x	$p_1  p_2$		r	$q_1  q_2$	
	$p_3  p_4$		$q_3  q_4$		

the product  $p \cdot q$  is represented by the box

	y	
r	$p_1 \cdot q_1 + p_2 \cdot q_2 + p_3 \cdot q_3 + p_4 \cdot q_4 \qquad p_1 \cdot q_2 + p_2 \cdot q_1 + p_3 \cdot q_4 + p_4 \cdot q_3$	(25)
	$p_1 \cdot q_3 + p_3 \cdot q_1 + p_2 \cdot q_4 + p_4 \cdot q_2 \qquad p_1 \cdot q_4 + p_2 \cdot q_3 + p_3 \cdot q_2 + p_4 \cdot q_1$	

Similarly, we can describe the product for the square  $X = \Delta[0] * X_{\Sigma}$ . The identity element in s**Set** $(\Delta[0] * X_{\Sigma}, DN\mathbb{Z}_2)$  is given by

	$y_0$	$y_1$	
~	1 0	1 0	
$x_0$	0 0	0 0	
~	1 0	1 0	
$x_1$	0 0	0 0	

The product can be computed considering one box at a time, labeled by  $(x_i, y_j)$ , and using the formula in Equation (25).

Our main result in this section connects noncontextuality and weak invertibility.

4.5. LEMMA. For a group G,  $(D_R(G))^*$  is equal to the image of  $\delta_G : G \longrightarrow D_R(G)$ .

PROOF. The image of  $\delta_G$  is contained in  $(D_R(G))^*$ , since this map is a homomorphism of monoids. For the converse consider  $p = \sum_{g \in G} \alpha_g \delta^g$  with inverse  $q = \sum_{g \in G} \beta_g \delta^g$ , that is

$$p * q(h) = \sum_{g_1g_2 = h} p(g_1)q(g_2) = \begin{cases} 1 & h = e_G \\ 0 & h \neq e_G. \end{cases}$$

The case  $h = e_G$  implies that there exists  $g \in G$  for which  $q(g) \neq 0$ . On the other hand, the second case gives us that  $p(hg^{-1}) = 0$  for all  $h \neq e_G$ , since R is zero-sum-free and integral. Therefore p is the delta distribution  $\delta^{g^{-1}}$ .

4.6. LEMMA. Given a simplicial set X and a simplicial group Y, the set  $(sSet(X, D_R(Y)))^*$ of units is the image of the following map:

$$(\delta_Y)_* : s\mathbf{Set}(X, Y) \hookrightarrow s\mathbf{Set}(X, D_R(Y)).$$

**PROOF.** Since  $(\delta_Y)_*$  is a homomorphism of monoids we have that

$$\operatorname{Im}((\delta_Y)_*) \subseteq (s\mathbf{Set}(X, D_R(Y))^*.$$

For the other direction, consider  $p, q \in s\mathbf{Set}(X, D_R(Y))$  such that  $p \cdot q = e_{s\mathbf{Set}(X, D_R(Y))}$ . This means that we have  $p_n(x) * q_n(x) = e_{D_R(Y_n)}$  for  $x \in X_n$ . By Lemma 4.5 there exists  $y(x) \in Y_n$  such that  $p_n(x) = \delta^{y(x)}$ . Then we define  $\varphi_n : X_n \longrightarrow Y_n$  by  $\varphi_n(x) = y(x)$ . Then compatibility of p and  $\delta_Y$  with the simplicial structure maps implies that  $\varphi : X \longrightarrow Y$ , which is defined in degree n by  $\varphi_n$ , is a simplicial set map such that  $(\delta_Y)_*(\varphi) = p$ . 4.7. THEOREM. Given a simplicial set X and a simplicial group Y, a distribution  $p \in s\mathbf{Set}(X, D_R(Y))$  is noncontextual if and only if p is weakly invertible.

**PROOF.** We have the following commutative diagram:

$$(s\mathbf{Set}(X, D_R(Y)))^* \longleftrightarrow s\mathbf{Set}(X, D_R(Y))$$

$$(\delta_Y)_* \uparrow (\delta_Y)_* (\delta_Y)_* (\delta_Y)_* (26)$$

By Proposition 2.17  $\Theta_{X,Y}$  is the transpose of  $(\delta_Y)_*$  with respect to the adjunction  $D_R$ : **Mon**  $\dashv$  **ConvMon**<sub>R</sub> : U. Similarly, the map  $\tilde{\pi}^{s\mathbf{Set}(X,D_R(Y))}$  is the transpose of the inclusion homomorphism  $(s\mathbf{Set}(X,D_R(Y)))^* \hookrightarrow s\mathbf{Set}(X,D_R(Y))$  with respect to the same adjunction. Therefore we obtain the following commutative diagram:

By Lemma 4.6 the vertical map  $D_R((\delta_Y)_*)$  in Diagram (27) is an isomorphism. Therefore this diagram gives the desired result.

The following example shows that one direction of Theorem 4.7 does not hold when the semiring is not zero-sum-free.

4.8. EXAMPLE. Let X be the simplicial circle  $S^1 = \Delta[1]/\partial \Delta[1]$ . It has one vertex x, and one nondegenerate 1-simplex  $\sigma$ . A distribution  $p \in s\mathbf{Set}(X, D_{\mathbb{R}}(\Delta_{\mathbb{Z}_2}))$  is given by a tuple  $(p^{00}, p^{01}, p^{10}, p^{11})$  where  $p^{00} + p^{10} = d_0(p_{\sigma}) = d_1(p_{\sigma}) = p^{00} + p^{01}$ . Therefore  $p = (p^{00}, p, p, p^{11})$ , which as a box has the following representation

$$\begin{array}{c|c} & x \\ \hline x & p^{00} & p \\ p & p^{11} \end{array}$$

Deterministic distributions on  $(S^1, \Delta_{\mathbb{Z}_2})$  are given by (1, 0, 0, 0) and (0, 0, 0, 1). Therefore the distribution (1, 2, 2, -4) is contextual. But this distribution is weakly invertible, in fact even invertible, with inverse given by  $(\frac{11}{35}, \frac{2}{7}, \frac{2}{7}, \frac{4}{35})$ .

4.9. COROLLARY. Let X be a simplicial set and Y be a simplicial group. For  $p \in s\mathbf{Set}(X, D(Y))$  we have

$$NCF(p) = IF(p).$$

**PROOF.** Follows from Definition 2.14, Proposition 3.19, and Theorem 4.7.

4.10. STRONG CONTEXTUALITY AND STRONG INVERTIBILITY. In this section we make the connection between strong contextuality and strong noninvertibility when  $R = \mathbb{R}_{>0}$ .

Let X be a simplicial set. A simplex  $x \in X_n$  is called degenerate if x belongs to  $\bigcup_{j=0}^{n-1} s_j(X_{n-1})$ ; otherwise it is called nondegenerate.

4.11. LEMMA. Let X be a simplicial set with finitely many nondegenerate simplices and Y be a simplicial group. For  $p \in s\mathbf{Set}(X, D(Y))$  we have

 $\mathrm{Isupp}(p) = (\delta_Y)_*(\mathrm{supp}(p)).$ 

PROOF. A distribution  $q \in \text{Isupp}(p)$  is invertible, and by Lemma 4.6 there exists  $\varphi \in s\mathbf{Set}(X,Y)$  such that  $q = \delta_Y \circ \varphi$ . Also there exists  $Q \in D(s\mathbf{Set}(X,D(Y)))$  such that  $\pi^{s\mathbf{Set}(X,D(Y))}(Q) = p$  and  $Q(\delta_Y \circ \varphi) = Q(q) > 0$ . Using Equation (5), for  $x \in X_n$  we obtain

$$p_n(x)(\varphi_n(x)) = \sum_{\substack{p' \in s \mathbf{Set}(X, D_R(Y))}} Q(p') p'_n(x)(\varphi_n(x))$$
$$\geq Q(\delta_Y \circ \varphi)(\delta_{Y_n} \circ \varphi_n(x))(\varphi_n(x))$$
$$= Q(\delta_Y \circ \varphi)\delta^{\varphi_n(x)}(\varphi_n(x))$$
$$= Q(\delta_Y \circ \varphi) > 0.$$

Therefore  $\varphi \in \operatorname{supp}(p)$  and  $q \in (\delta_Y)_*(\operatorname{supp}(p))$ .

For the converse inclusion, we will show that  $\delta_Y \circ \psi \in \text{Isupp}(p)$  for  $\psi \in \text{supp}(p)$ . We define

$$\alpha = \min\{p_n(x)(\psi_n(x)) : x \in X_n, n \ge 0\}.$$

Observe that  $\alpha > 0$  since X has finitely many simplices and for  $x \in X_n$  we have both

$$p_{n+1}(s_i(x))(\psi_{n+1}(s_i(x)))$$
 and  $p_{n-1}(d_j(x))(\psi_{n-1}(d_j(x)))$ 

are greater than or equal to  $p_n(x)(\psi_n(x))$ . If  $\alpha = 1$ , which implies that  $p = \delta_Y \circ \psi$ , then  $\delta_Y \circ \psi \in \text{Isupp}(p)$ . Then let us suppose that  $\alpha < 1$ . For  $n \ge 0$ , we define  $q_n : X_n \longrightarrow D(Y_n)$  by

$$q_n(x) = \frac{p_n(x) - \alpha \delta^{\psi_n(x)}}{1 - \alpha}, \quad x \in X_n$$

We need to verify that indeed  $q_n(x) \in D(Y_n)$ : Observe that for  $y \in Y_n$ , we have

$$q_n(x)(y) = \begin{cases} \frac{p_n(x)(\psi_n(x)) - \alpha}{1 - \alpha} & \text{if } y = \psi_n(x), \\ \frac{p_n(x)(y)}{1 - \alpha} & \text{otherwise.} \end{cases}$$

By definition of  $\alpha$ , we have  $q_n(x)(y) \ge 0$  for all  $x \in X_n, y \in Y_n$ . In addition, we have

$$\sum_{y \in Y_n} q_n(x)(y) = \sum_{y \in Y_n} \frac{p_n(x)(y) - \alpha \delta^{\psi_n(x)}(y)}{1 - \alpha}$$
$$= \frac{1}{1 - \alpha} \left( \sum_{y \in Y_n} p_n(x)(y) - \alpha \sum_{y \in Y_n} \delta^{\psi_n(x)}(y) \right)$$
$$= \frac{1}{1 - \alpha} (1 - \alpha) = 1.$$

Next we prove that the collection of maps  $\{q_n\}_{n\geq 0}$  form a simplicial set map  $q: X \longrightarrow D(Y)$ . Given  $x \in X_n$  and  $y \in Y_{n-1}$ , we will show that  $(D(d_j) \circ q_n(x))(y) = (q_{n-1} \circ d_j(x))(y)$ . We begin with the case that  $y \neq \psi_{n-1}(d_j(x))$ : We have

$$q_{n-1}(d_j(x))(y) = \frac{p_{n-1}(d_j(x))(y)}{1 - \alpha}$$
  
=  $\frac{1}{1 - \alpha} D(d_j)(p_n(x))(y)$   
=  $\sum_{d_j(y')=y} \frac{p_n(x)(y')}{1 - \alpha}$   
=  $\sum_{d_j(y')=y} q_n(x)(y')$   
=  $D(d_j)(q_n(x))(y).$ 

In the fourth line we used the observation that y' with  $d_j(y') = y$  satisfies  $y' \neq \psi_n(x)$ , otherwise we would have  $y = d_j(y') = d_j(\psi_n(x)) = \psi_{n-1}(d_j(x))$ . Next assume that  $y = \psi_{n-1}(d_j(x)) = d_j(\psi_n(x))$ . Then we have

$$q_{n-1}(d_j(x))(y) = \frac{p_{n-1}(d_j(x))(y) - \alpha}{1 - \alpha}$$
  
=  $\frac{D(d_j)(p_n(x))(y) - \alpha}{1 - \alpha}$   
=  $\frac{\sum_{d_j(y')=y} p_n(x)(y') - \alpha}{1 - \alpha}$   
=  $\frac{p_n(x)(\psi_n(x)) - \alpha}{1 - \alpha} + \sum_{d_j(y')=y, y' \neq \psi_n(x)} \frac{p_n(x)(y')}{1 - \alpha}$   
=  $q_n(x)(\psi_n(x)) + \sum_{d_j(y')=y, y' \neq \psi_n(x)} q_n(x)(y')$   
=  $\sum_{d_j(y')=y} q_n(x)(y')$   
=  $D(d_j)(q_n(x))(y).$ 

Similarly, one can show that  $D(s_j) \circ q_n = q_{n+1} \circ s_j$ . Now we define  $Q \in D(s\mathbf{Set}(X, D(Y)))$  by

$$Q(p') = \begin{cases} 1 - \alpha & \text{if } p' = q \\ \alpha & \text{if } p' = \delta_Y \circ \psi \\ 0 & \text{otherwise.} \end{cases}$$

For  $x \in X_n$ , we have

$$\pi^{s\mathbf{Set}(X,D(Y))}(Q)_n(x) = (1-\alpha)q_n(x) + \alpha(\delta_{Y_n} \circ \psi_n)(x)$$
$$= (1-\alpha)\frac{p_n(x) - \alpha\delta^{\psi_n(x)}}{1-\alpha} + \alpha\delta^{\psi_n(x)}$$
$$= p_n(x).$$

We showed that  $\pi^{s\mathbf{Set}(X,D(Y))}(Q) = p$ , while  $Q(\delta_Y \circ \psi) = \alpha \neq 0$ , which implies that  $\delta_Y \circ \psi \in \mathrm{Isupp}(p)$ .

4.12. COROLLARY. Let X be a simplicial set with finitely many nondegenerate simplices, Y be a simplicial group and  $p \in s\mathbf{Set}(X, D(Y))$ .

1. p is strongly contextual if and only if p is strongly noninvertible.

2. p is strongly contextual if and only if CF(p) = 1.

**PROOF.** The first part follows directly from Lemma 4.11. The second part follows from the first part together with Proposition 3.20, and Corollary 4.9.

4.13. REMARK. Theorem 2.20 together with Remark 2.21 implies that Corollary 4.12 part (2) generalizes Proposition 6.3 in [3].

4.14. Extremal simplicial distributions.

4.15. DEFINITION. Let  $(X, \pi^X)$  be an *R*-convex set. An element  $x \in X$  is called a *vertex*, or an *extreme point*, if x has a unique preimage under  $\pi^X$ .

Recall the definition of deterministic simplicial distribution from Definition 2.10.

4.16. PROPOSITION. For simplicial sets X, Y every deterministic distribution in sSet $(X, D_R(Y))$  is a vertex.

PROOF. Let  $\varphi \in s\mathbf{Set}(X, Y)$ . Suppose that we have  $Q \in D_R(s\mathbf{Set}(X, D_R(Y)))$  such that  $\pi^{s\mathbf{Set}(X, D_R(Y))}(Q) = \delta_Y \circ \varphi$ . This means that

$$\sum_{p \in sSet(X, D_R(Y))} Q(p) p_n(x) = \delta^{\varphi_n(x)}, \ \forall x \in X_n.$$

Given  $q \in s\mathbf{Set}(X, D_R(Y))$ , not equal to  $\delta_Y \circ \varphi$ , there exists  $x \in X_n$  and  $y \in Y_n$  such that  $y \neq \varphi_n(x)$  and  $q_n(x)(y) \neq 0$ . In this case we have

$$\sum_{p \in s \mathbf{Set}(X, D_R(Y))} Q(p) p_n(x)(y) = \delta^{\varphi_n(x)}(y) = 0.$$

Since R is a zero-sum-free semiring, we obtain that  $Q(q)q_n(x)(y) = 0$ . Since R is integral, we have Q(q) = 0. This implies  $Q = \delta^{\delta_Y \circ \varphi}$ .

4.17. PROPOSITION. Let  $f : X \longrightarrow Y$  be a morphism in  $\mathbf{Conv}_R$ . If  $y \in Y$  is a vertex, then every vertex of the R-subconvex set  $f^{-1}(y)$  is a vertex of X.

**PROOF.** Let x be a vertex of  $f^{-1}(y)$  and  $p \in D_R(X)$  such that  $\pi^X(p) = x$ . Then

$$\pi^{Y}(D_{R}(f)(p)) = f(\pi^{X}(p)) = f(x) = y$$

Since y is a vertex of Y, we obtain that  $D_R(f)(p) = \delta^y$ . This means that if  $y' \neq y$  then  $D_R(f)(p)(y') = \sum_{f(x')=y'} p(x') = 0$ . Since R is zero-sum-free we obtain that p(x') = 0 for all  $x' \notin f^{-1}(y)$ . In other words,  $p \in D_R(f^{-1}(y))$ . Finally since y is a vertex of  $f^{-1}(y)$ , we obtain that  $p = \delta^y$ .

Main application of this observation to simplicial distributions is as follows.

4.18. COROLLARY. Let  $f: Z \longrightarrow X$  be a map of simplicial sets. Consider the morphism  $f^*: s\mathbf{Set}(X, D_R(Y)) \longrightarrow s\mathbf{Set}(Z, D_R(Y))$  in  $\mathbf{Conv}_R$ . If  $p \in s\mathbf{Set}(Z, D_R(Y))$  is a vertex then every vertex of  $(f^*)^{-1}(p)$  is a vertex of  $s\mathbf{Set}(X, D_R(Y))$ .

4.19. EXAMPLE. Let  $\overline{\Delta[2]}$  denote the simplicial set obtained by gluing the  $d_1$  and  $d_2$  faces of  $\Delta[2]$ . A simplicial distribution  $p: \overline{\Delta[2]} \longrightarrow D(N\mathbb{Z}_2)$  can be represented by a tuple  $(p^{ab})_{a,b\in\mathbb{Z}_2}$ , as in Example 2.25, where  $p^{01} = p^{11}$ . We will consider the subspace given by the simplicial circle  $S^1 = \Delta[1]/\partial\Delta[1]$ , the  $d_0$ -face of  $\overline{\Delta[2]}$ . A simplicial distribution  $q: S^1 \longrightarrow D(N\mathbb{Z}_2)$  is given by a distribution  $q_{\iota_1} \in D(\mathbb{Z}_2)$ , where  $\iota_1$  is the unique nondegenerate 1-simplex of  $S^1$ . (Note that the set of simplicial distributions on  $(S^1, N\mathbb{Z}_2)$  is in bijective correspondence with the set of simplicial distributions on the pair  $(\Delta[1], N\mathbb{Z}_2)$ ; see Example 2.11. Also observe that  $\overline{\Delta[2]} \cong \Delta[0] * S^1$  and  $s\mathbf{Set}(\overline{\Delta[2]}, D(N\mathbb{Z}_2)) \cong s\mathbf{Set}(S^1, D(\Delta_{\mathbb{Z}_2}))$  by (14). The inclusion  $f: S^1 \hookrightarrow \overline{\Delta[2]}$  as the  $d_0$ -th face induces

$$f^*: s\mathbf{Set}(\overline{\Delta[2]}, D(N\mathbb{Z}_2)) \longrightarrow s\mathbf{Set}(S^1, D(N\mathbb{Z}_2)) \cong [0, 1], \ (p^{ab}) \mapsto 1 - 2p^{01}.$$

The preimage of the vertices  $\{0,1\}$  of [0,1] are given by (0,1/2,0,1/2) and  $(p^{00},0,1-p^{00},0)$ ; respectively. The latter component has the deterministic vertices given by (1,0,0,0) and (0,0,1,0). Therefore this component coincides with the subset of noncontextual distributions. The former component consisting of a single point is a contextual vertex.

4.20. EXAMPLE. Let X denote the square space in Example 2.26 and  $\partial X$  denote its boundary consisting of the nondegenerate 1-simplices given by  $x_i \oplus y_j$  where  $i, j \in \mathbb{Z}_2$ . The inclusion  $f : \partial X \hookrightarrow X$  induces

$$f^*: s\mathbf{Set}(X, D(N\mathbb{Z}_2)) \longrightarrow s\mathbf{Set}(\partial X, D(N\mathbb{Z}_2)) \cong [0, 1]^4$$
(28)

defined by

$$f^*(p_{x_iy_j}^{ab}) = (p_{x_iy_j}^{00} + p_{x_iy_j}^{11})_{i,j \in \mathbb{Z}_2}.$$

There are 16 vertices of the hypercube  $[0, 1]^4$  given by a tuple  $(\delta^{a_{ij}})_{i,j\in\mathbb{Z}_2}$  of delta distributions. The preimage is given by (1) a singleton if  $\sum_{i,j} a_{ij} = 1 \mod 2$ , and otherwise it

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can be identified with (2) the interval [0, 1]. The 8 vertices in case (1) are the PR boxes (hence contextual; see Example 2.23). For example, the preimage of  $(\delta^0, \delta^0, \delta^0, \delta^1)$  is given by

	$y_0$	$y_1$
	1/2 0	1/2 0
$ x_0 $	0 1/2	0 1/2
~	1/2 0	0 1/2
$x_1$	0 1/2	1/2 0

On the other hand, the vertices of case (2) are a pair of deterministic vertices.

4.20.1. HOMOTOPIC SIMPLICIAL DISTRIBUTIONS. Simplicial homotopy can be used to capture strongly contextual simplicial distributions and vertices. We begin with a preliminary observation.

4.21. LEMMA. Let  $f: Z \longrightarrow X$  be a simplicial set map, and  $p \in s\mathbf{Set}(X, D_R(Y))$ .

- 1.  $f^*(\operatorname{supp}(p)) \subseteq \operatorname{supp}(f^*(p)).$
- 2. If  $\operatorname{supp}(f^*(p)) \cap f^*(s\mathbf{Set}(X,Y)) = \emptyset$  then p is strongly contextual.

**PROOF.** Given  $\varphi \in \operatorname{supp}(p)$  and  $x \in Z_n$ , we have

$$f^{*}(p)_{n}(x)(f^{*}(\varphi)_{n}(x)) = (p_{n} \circ f_{n})(x)(\varphi_{n} \circ f_{n}(x)) = p_{n}(f_{n}(x))(\varphi_{n}(f_{n}(x))) \neq 0$$

This proves part (1). If  $\varphi \in \operatorname{supp}(p)$  then by part (1) we have  $f^*(\varphi) \in \operatorname{supp}(f^*(p))$ . This means that  $f^*(\varphi) \in \operatorname{supp}(f^*(p)) \cap f^*(s\mathbf{Set}(X,Y))$ , proving part (2).

Part (2) of Lemma 4.21 is a key observation also used in arguments involving simplicial cohomology [10, Section 5.3] and Čech cohomology [36].

4.22. EXAMPLE. Consider the inclusion  $f: \partial X \longrightarrow X$  in Example 4.20. This map induces

$$f^*: s\mathbf{Set}(X, N\mathbb{Z}_2) \longrightarrow s\mathbf{Set}(\partial X, N\mathbb{Z}_2) \cong \mathbb{Z}_2^4$$

which sends  $\varphi : X \longrightarrow N\mathbb{Z}_2$ , determined by the tuple  $(\varphi_{\sigma_{ij}})_{i,j\in\mathbb{Z}_2}$ , to the tuple  $(\varphi_{\sigma_{ij}}^{00} + \varphi_{\sigma_{ij}}^{11})_{i,j\in\mathbb{Z}_2}$ . In particular, the image of  $f^*$  consists of  $(a_{ij})_{i,j\in\mathbb{Z}_2}$  such that

$$\sum_{i,j} a_{ij} = 0 \mod 2. \tag{29}$$

According to part (2) of Lemma 4.21 a distribution  $p \in s\mathbf{Set}(X, D(N\mathbb{Z}_2))$  is strongly contextual if  $\operatorname{supp}(f^*(p))$  does not contain a tuple  $(a_{ij})_{i,j}$  satisfying Equation (29). Note that PR boxes have this property, thus they are strongly contextual.

Let  $f_0, f_1 : X \longrightarrow Y$  be two simplicial set maps. A simplicial homotopy from  $f_0$  to  $f_1$  is a simplicial set map  $F : X \times \Delta[1] \longrightarrow Y$  such that  $f_0 = F \circ i_1$  and  $f_1 = F \circ i_0$ , where  $i_k : X \cong X \times \Delta[0] \xrightarrow{\operatorname{Id} \times d^k} X \times \Delta[1]$  for k = 0, 1; see [11]. We will write  $f_0 \sim f_1$  if there exists a simplicial homotopy from  $f_0$  to  $f_1$ . If no such homotopy exists we will write  $f_0 \not\sim f_1$ .

4.23. PROPOSITION. Let  $\varphi_1, \varphi_2 \in s\mathbf{Set}(X, Y)$  be such that  $\varphi_1 \not\sim \varphi_2$ . Then every homotopy  $F \in s\mathbf{Set}(X \times \Delta[1], D_R(Y))$  from  $\delta_Y \circ \varphi_1$  to  $\delta_Y \circ \varphi_2$  is strongly contextual.

PROOF. Let  $F \in s\mathbf{Set}(X \times \Delta[1], D_R(Y))$  be a homotopy from  $\delta_Y \circ \varphi_1$  to  $\delta_Y \circ \varphi_2$ . Let  $i = i_0 \sqcup i_1 : X \sqcup X \longrightarrow X \times \Delta[1]$ . Then

$$supp(i^{*}(F)) \cap i^{*}(s\mathbf{Set}(X \times \Delta[1], Y))$$

$$= supp((\delta_{Y} \circ \varphi_{1}) \sqcup (\delta_{Y} \circ \varphi_{2})) \cap i^{*}(s\mathbf{Set}(X \times \Delta[1], Y))$$

$$= supp(\delta_{Y} \circ (\varphi_{1} \sqcup \varphi_{2})) \cap i^{*}(s\mathbf{Set}(X \times \Delta[1], Y))$$

$$= \{\varphi_{1} \sqcup \varphi_{2}\} \cap i^{*}(s\mathbf{Set}(X \times \Delta[1], Y)) = \emptyset.$$

Here we used the fact that  $\operatorname{supp}(\delta_Y \circ (\varphi_1 \sqcup \varphi_2)) = \{\varphi_1 \sqcup \varphi_2\}$ . In addition, the last equation follows from the fact that there exists no homotopy from  $\varphi_1$  to  $\varphi_2$ . By Lemma 4.21 part (2) the distribution F is strongly contextual.

4.24. REMARK. Let X be a simplicial set with finitely many nondegenerate simplices. In the case of  $R = \mathbb{R}_{\geq 0}$  and  $Y \in s\mathbf{Grp}$  there is an alternative proof of Proposition 4.23 that relies on the equivalence of strong contextuality and strong noninvertibility. By Corollary 4.3 the inclusion  $i = i_0 \sqcup i_1 : X \sqcup X \longrightarrow X \times \Delta[1]$  induces a map of real convex monoids:

$$i^*: s\mathbf{Set}(X \times \Delta[1], D(Y)) \longrightarrow s\mathbf{Set}(X \sqcup X, D(Y)).$$

According to Proposition 4.16 the deterministic distribution  $\delta_Y \circ (\varphi_1 \sqcup \varphi_2)$  is a vertex. In particular, its invertible support consists only of  $\delta_Y \circ (\varphi_1 \sqcup \varphi_2)$ . By Proposition 4.6, the horizontal maps of the following diagram are isomorphisms:

$$s\mathbf{Set}(X \times \Delta[1], Y) \xrightarrow{(o_Y)_*} (s\mathbf{Set}(X \times \Delta[1], DY))^*$$
$$\downarrow^{i^*} \qquad \qquad \downarrow^{i^*}$$
$$s\mathbf{Set}(X \sqcup X, Y) \xrightarrow{(\delta_Y)_*} (s\mathbf{Set}(X \sqcup X, DY))^*$$

Since there is no homotopy from  $\varphi_1$  to  $\varphi_2$ , which means that  $\varphi_1 \sqcup \varphi_2 \notin i^*(s\mathbf{Set}(X \times \Delta[1], Y))$ , it turns out that  $\delta_Y \circ (\varphi_1 \sqcup \varphi_2) \notin i^*((s\mathbf{Set}(X \times \Delta[1], DY))^*)$ . We conclude that the intersection of  $\mathrm{Isupp}(i^*(F))$  and  $i^*((s\mathbf{Set}(X \times \Delta[1], DY))^*)$  is empty. By Corollary 3.16 part (1) we obtain that F is strongly noninvertible, and then by Corollary 4.12 part (1) F is strongly contextual.

4.25. COROLLARY. Let  $\varphi_1, \varphi_2 \in s\mathbf{Set}(X, Y)$  be such that  $\varphi_1 \not\sim \varphi_2$ . If  $F \in s\mathbf{Set}(X \times \Delta[1], D_R(Y))$  is the unique homotopy from  $\delta_Y \circ \varphi_1$  to  $\delta_Y \circ \varphi_2$ , then F is a strongly contextual vertex in  $s\mathbf{Set}(X \times \Delta[1], D_R(Y))$ .

PROOF. According to Proposition 4.16, the deterministic distribution  $(\delta_Y \circ \varphi_1) \sqcup (\delta_Y \circ \varphi_2) = \delta_Y \circ (\varphi_1 \sqcup \varphi_2)$  is a vertex in s**Set** $(X \sqcup X, D_R(Y))$ . By Proposition 4.17 the unique preimage of  $(\delta_Y \circ \varphi_1) \sqcup (\delta_Y \circ \varphi_2)$  under  $i^*$  is a vertex in s**Set** $(X \times \Delta[1], D_R(Y))$ . This vertex is strongly contextual by Proposition 4.23.

4.26. EXAMPLE. Let X denote the simplicial set obtained by gluing two copies of  $\Delta[1]$  along their boundaries:



Consider two simplicial set maps  $\varphi, \psi \in s\mathbf{Set}(X, N\mathbb{Z}_2)$ . These maps are determined by the images of the 1-simplices x, y. Let  $\varphi$  be given by  $(x, y) \mapsto (0, 0)$ , and  $\psi$  by  $(x, y) \mapsto (1, 0)$ . Note that  $\varphi \not\sim \psi$ . But the deterministic distributions  $\delta_{N\mathbb{Z}_2} \circ \varphi$  and  $\delta_{N\mathbb{Z}_2} \circ \psi$  are homotopic via the following unique homotopy:



Note that  $(X \times \Delta[1], N\mathbb{Z}_2)$  is another way to describe the CHSH scenario of Example 2.23. The distribution above is a PR box. Each of the 8 distinct PR boxes can be captured in this way as a homotopy.

In general, given the four triangles corresponding to the four boxes in the CHSH scenario, any way of assembling them into a simplicial set provides a description as a simplicial distribution. In addition to the realizations given in this example and Example 2.26 there is another realization where X is a punctured torus; see [10] and [31].

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