Theory and Applications of Categories, Vol. 44, No. 15, 2025, pp. 420–438.

ATOMIC TOPOSES WITH CO-WELL-FOUNDED CATEGORIES OF ATOMS

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ABSTRACT. The atoms of the Schanuel topos can be described as the formal quotients n/G where n is a finite set and G is a subgroup of $\mathrm{Aut}(n)$. We give a general criterion on an atomic site $(\mathcal{A}, J_{\mathrm{at}})$ ensuring that the atoms of $\mathsf{Sh}(\mathcal{A}, J_{\mathrm{at}})$ can be described in a similar fashion, as the formal quotients n/G where $n \in \mathcal{A}$ and $G \subseteq \mathrm{Aut}(n)$ is a "valid" subgroup. It might happen that every group of automorphisms is valid in this sense, and we show that it is the case if and only if the J_{at}-sheaves coincide with the pullback-preserving presheaves. We show that if the criterion is satisfied and the groups $\mathrm{Aut}(n)$ are Noetherian, then $\mathsf{Sh}(\mathcal{A}, J_{\mathrm{at}})$ is locally finitely presentable. By applying this to the Malitz–Gregory atomic topos, we obtain a negative answer to a question of Di Liberti and Rogers: Does every locally finitely presentable topos have enough points? We also provide an example of an atomic topos which is not locally finitely presentable.

1. Introduction

This article is motivated by the following question posed in [DLR24]:

Does every locally finitely presentable sheaf topos have enough points?

As argued in [DLR24], this is a natural inquiry:

- Locally finitely presentable toposes include coherent toposes and presheaf toposes, which do have enough points.
- The analogous question for frames has a positive answer.
- A locally finitely presentable topos has enough points if the slices of its category of finitely presentable objects are essentially countable [DLR24, Cor. 4.1.12].

The same question was raised by Campion on MathOverflow [Cam21]. We show here that the pointless atomic topos of [Mak82, Sec. 5] is a counter-example. In order to do so, we give a combinatorial criterion on an atomic site ensuring that the atoms of its topos of sheaves are not "too complex," and as a consequence that this topos is locally finitely presentable. A crucial property is that these categories of atoms are *co-well-founded* in the sense that every chain stabilizes.

Received by the editors 2024-06-24 and, in final form, 2025-05-01.
Transmitted by Richard Garner. Published on 2025-05-22.
2020 Mathematics Subject Classification: Primary: 03G30; Secondary: 18B25, 18C35.
Key words and phrases: Atomic toposes, toposes without points, nominal sets, locally finitely presentable toposes.

A motivating example As a first example of an atomic topos, we consider the Schanuel topos of nominal sets. It can be defined in many ways; for a complete introduction, we refer to the book [Pit13]. Informally, an object of this topos, a nominal set, can be thought of as a set of *terms* in which a finite number of *variables* may appear. These variables can be *renamed* to yield other terms.

In order to give an idea of what nominal sets look like, we examine two examples.

- The nominal set of *ordered pairs* has terms (a, b) where a and b are variables.
- The nominal set of *unordered pairs* has terms $\{a, b\}$ where a and b are variables.

The difference between these two nominal sets is that exchanging the variables of an unordered pair does nothing, since $\{a, b\} = \{b, a\}$ but $(a, b) \neq (b, a)$ if $a \neq b$. These two examples actually illustrate everything that can happen in a nominal set. More precisely, each nominal set can be decomposed as a potentially infinite union of basic building blocks, its *atoms*. An atom of a nominal set is a term modulo renaming of its variables. This means that we can think of an atom as a term $t(x_1, \ldots, x_n)$ where the x_i are pairwise distinct variables. The number of variables is the *support* of the atom. Another important information is the group $G \subseteq \mathfrak{S}_n$ of permutations of the variables x_i that leave the term unchanged. The atom is in a sense completely determined by the pair (n, G). In categorical terms, an atom of a nominal set is a minimal non-empty sub-object, and two atoms are isomorphic if and only if they have the same invariant (n, G).

Let us determine for instance the atoms of the nominal set of unordered pairs. Modulo renaming, there are two unordered pairs: $\{a, b\}$ with $a \neq b$ and $\{a\}$. The invariants of these atoms are respectively $(2, \mathfrak{S}_2)$ and $(1, \mathfrak{S}_1)$. Similarly, the nominal set of *ordered* pairs also has two atoms, (a, b) with $a \neq b$ and (a, a), but their invariants are $(2, \{\mathrm{id}\})$ and $(1, \mathfrak{S}_1)$.

The Schanuel topos can also be presented as the category of sheaves for the atomic topology on the opposite of the category $\mathsf{FinSetInj}$ of finite sets and injections. There is a direct connection with the explicit description of the atoms given above: each atom is a formal quotient of an object n of $\mathsf{FinSetInj}^{\mathrm{op}}$ by a group of automorphisms of n. The main objective of this paper is to generalize this phenomenon to other categories than $\mathsf{FinSetInj}$.

Overview of the paper In § 2, we present a condition on an atomic site $(\mathcal{A}, J_{\mathrm{at}})$ under which we obtain a concrete description of the category of atoms of $\mathsf{Sh}(\mathcal{A}, J_{\mathrm{at}})$. This condition essentially says that $\mathcal{A}$ is a full subcategory of atoms which is stable under pushouts and co-well-founded. The atoms can then be described as the formal quotients n/G in $[\mathcal{A}^{\mathrm{op}}, \mathsf{Set}]$ where $n \in \mathcal{A}$, $G \subseteq \mathrm{Aut}(n)$ is a subgroup and the pair (n, G) is *atomic* (Definition 2.2.3). We focus in § 2.3 on the case where all the pairs (n, G) are atomic, and we show that this happens precisely when the J_{at}-sheaves coincide with the pullback-preserving functors $\mathcal{A}^{\mathrm{op}} \to \mathsf{Set}$. In § 3, we show that if the groups $\mathrm{Aut}(n)$ where $n \in \mathcal{A}$ are all Noetherian, then the category of atoms is co-well-founded. This implies that $\mathsf{Sh}(\mathcal{A}, J_{\mathrm{at}})$ is locally finitely presentable. We apply this in § 4 to show that a classical example of pointless atomic topos, the Malitz–Gregory topos, is locally finitely presentable. This gives a counter-example to the conjecture that every locally finitely presentable topos has enough points formulated in [DLR24]. Finally, we provide in § 5 an example of an atomic topos which is not locally finitely presentable.

Acknowledgments I thank Morgan Rogers for the discussions about his recent paper [DLR24] with Ivan Di Liberti, from which this document originated. I am grateful to my former post-doc advisor Sam van Gool for his many comments and suggestions. The discussions I had with Victor Iwaniack and Daniela Petrişan were crucial to my understanding of nominal sets and atomic toposes. I sincerely thank the referee for their suggestions and careful corrections, which greatly improved this work. I thank Richard Garner for having a look at the manuscript (and catching a terminological mistake).

Funding The research reported here has been supported financially by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program, grant agreement #670624.

Notations In this paper, topos will mean sheaf topos. The symbol J_{at} denotes the atomic topology on a category left implicit. We will never consider any other topology, so it will often be left implicit. In particular, a sheaf with respect to the atomic topology will simply be referred to as a sheaf. The sheafification of a presheaf F (with respect to the atomic topology) will be denoted by $\mathrm{sh}(F)$. The composite of two arrows $f : A \to B$ and $g : B \to C$ will be denoted by fg or by $g \circ f$.

2. A well-foundedness criterion to describe atoms

A *chain* in a category is a diagram indexed by a well-ordered poset. We say that a chain $(x_i)_{i\in I}$ *stabilizes* at $i \in I$ if the morphisms $x_i \to x_j$ are isomorphisms for all $j \geq i$. If a chain stabilizes at i, then it stabilizes at every $j \geq i$ and the morphisms $x_j \to \mathrm{colim}_k\, x_k$ are isomorphisms for all $j \geq i$. An *ω-chain* is a chain indexed by the ordinal ω.

2.0.1. Definition. *A category* C *is* co-well-founded *if every chain stabilizes. We say that* C *is* well-founded *if* C^{op} *is co-well-founded.*

Equivalently, C is co-well-founded if every ω-chain stabilizes: If there is a chain that does not stabilize, pick an element a_0 in it. Then there is a morphism $a_0 \to a_1$ in the chain which is not an isomorphism. Repeating this process produces an ω-chain that does not stabilize.

The aim of this section is to give an explicit description of the category of atoms of $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$, provided that C satisfies the conditions (C1), (C2) and (C3) below. We express here (C1) and (C2) in an abstract form, but we will give equivalent combinatorial formulations in § 2.1.

(C1) $\mathsf{C}^{\mathrm{op}} \to \mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ is fully faithful and sends each object of C^{op} to an atom.

(C2) C^{op} has pushouts and $\mathsf{C}^{\mathrm{op}} \to \mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ preserves them.

(C3) C is well-founded.

In view of these conditions, the reader might wonder why we chose to work with C and not directly with C^{op}. The reason is that in the examples, C is the category of "finitary" models of some theory, while C^{op} is harder to work with directly. In terms of the syntax-semantics duality, C^{op} lies on the syntactical side while C lies on the semantical side. Both of these perspectives are important.

Conditions (C1) and (C2) ensure that C^{op} describes a category of "representable" atoms stable under pushouts. Condition (C3) will be used to compute coequalizers of morphisms between these representable atoms by iterating pushouts and then quotienting by an automorphism (Lemma 2.2.1). From this, we deduce the description of the category of atoms of $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ presented in Theorem 2.2.6.

2.1. THE COMBINATORIAL CONTENT OF (C1) AND (C2). We say that a category C has *amalgamation* or satisfies the *left Ore condition* if every span admits a cocone.

As can be found in [BD80, Sec. 7(3)] and [Joh02, C2.1.12(c)], (C1) is equivalent to the conjunction of

- C has amalgamation, and
- every morphism in C is the equalizer of all the pairs of maps that it equalizes.

When C has these properties, $(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ is called a *standard* atomic site in [BD80]. In this case, we call the atoms in the image of $\mathsf{C}^{\mathrm{op}} \to \mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ the *representable atoms*.

In order to give the combinatorial content of (C2) in Lemma 2.1.1, we introduce another condition on C:

(C2') C has pullbacks and for every pullback

$$\begin{tikzcd} X \cap Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd} \tag{1}$$

in C, for every pair of parallel morphisms $u, v : Z \rightrightarrows A$ coinciding on $X \cap Y$, there is a morphism $w : A \to A'$ and a sequence of morphisms $k_0 = uw, k_1, k_2, \ldots, k_n = vw : Z \to A'$ such that any two consecutive morphisms in this sequence coincide either on X or on Y.

If C satisfies (C1), then every arrow is a monomorphism. This justifies the notation $X \cap Y$, but in general it is just a pullback.

2.1.1. LEMMA. *Let C be a small category. Then (C2) and (C2') are equivalent.*

Proof. The canonical functor $y : \mathsf{C}^{\mathrm{op}} \to \mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ preserves pushouts if and only if every sheaf $F : \mathsf{C} \to \mathsf{Set}$ preserves pullbacks, using that $\mathrm{Hom}(y(X), F) \cong F(X)$. Suppose that C satisfies (C2'). Let $F : \mathsf{C} \to \mathsf{Set}$ be a sheaf. We show that it preserves the pullback (1). First, note that the sheaf condition implies that F sends every morphism to an injection. We have $F(X \cap Y) \subseteq F(X) \cap F(Y)$ and we must show the reverse inclusion. Let $z \in F(X) \cap F(Y)$. We show that z satisfies the descent condition for the covering $X \cap Y \to Z$. Let $u, v : Z \rightrightarrows A$ be a parallel pair of arrows coinciding on $X \cap Y$. We must show that $F(u)(z) = F(v)(z)$. Let $w : A \to A'$ and $k_0 = uw, k_1, k_2, \ldots, k_n = vw$ a sequence of morphisms as in (C2'). Then $k_0(z) = k_1(z) = \cdots = k_n(z)$. This shows that z descends to $X \cap Y$, and C satisfies (C2).

Suppose now that C satisfies (C2) and consider the square (1). Let $P : \mathsf{C} \to \mathsf{Set}$ be the pushout of $\mathrm{Hom}(X, -)$ and $\mathrm{Hom}(Y, -)$ along $\mathrm{Hom}(Z, -)$ computed in $[\mathsf{C}, \mathsf{Set}]$. Recall that the sheafification functor is written $\mathrm{sh} : [\mathsf{C}, \mathsf{Set}] \to \mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$. Then (C2) says that $\mathrm{sh}(P) \to \mathrm{sh}(\mathrm{Hom}(X \cap Y, -))$ is an isomorphism. We obtain the diagram (2) in $[\mathsf{C}, \mathsf{Set}]$.

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathrm{Hom}(X \cap Y, -) \arrow[d] \\ \mathrm{sh}(P) \arrow[r, "\sim"] & \mathrm{sh}(\mathrm{Hom}(X \cap Y, -)) \end{tikzcd} \tag{2}$$

Let $u, v : Z \rightrightarrows A$ be two arrows that coincide on $X \cap Y$. They represent two elements $[u], [v]$ of $P(A)$ which are sent to the same element of $\mathrm{Hom}(X \cap Y, A)$. Using the commutativity of (2) and the fact that $\mathrm{sh}(P) \to \mathrm{sh}(\mathrm{Hom}(X \cap Y, -))$ is an isomorphism, we obtain that $[u]$ and $[v]$ are sent to the same element of $\mathrm{sh}(P)(A)$. This means that $[u]$ and $[v]$ are locally equal, i.e., that there is a morphism $w : A \to A'$ such that $[uw] = [vw]$ in $P(A')$. The definition of $P(A')$ as a pushout gives the sequence of morphisms in (C2'). ∎

As we saw in the proof, (C2) also means that every sheaf is a pullback-preserving presheaf. We will come back to this in in § 2.3, where we consider the converse implication.

2.2. The atoms of $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$. Given an object A of a category with small colimits an automorphism σ of A, we denote by A/σ the coequalizer of σ and the identity. If $G \subseteq \mathrm{Aut}(A)$ is a subgroup, A/G denotes the common coequalizer of all the arrows in G.

2.2.1. Lemma. *Suppose that C satisfies (C1), (C2) and (C3). Let $\alpha, \beta : n \rightrightarrows m$ be a pair of morphisms between representable atoms in $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$. Then their coequalizer is of the form $m \to m' \to m'/\sigma$ where m' is a representable atom and σ is an automorphism of m'.*

Proof. We build a sequence of atoms

$$m_0 \underset{\beta_0}{\overset{\alpha_0}{\rightrightarrows}} m_1 \underset{\beta_1}{\overset{\alpha_1}{\rightrightarrows}} m_2 \underset{\beta_2}{\overset{\alpha_2}{\rightrightarrows}} \cdots$$

as follows. We start with $m_0 = n$, $m_1 = m$, $\alpha_0 = \alpha$ and $\beta_0 = \beta$. At each step, m_{i+1} is built as the pushout below.

$$\begin{tikzcd} m_i \arrow[r, "\alpha_i"] \arrow[d, "\beta_i"'] & m_{i+1} \arrow[d, "\alpha_{i+1}"] \\ m_{i+1} \arrow[r, "\beta_{i+1}"] & m_{i+2} \end{tikzcd}$$

The coequalizer of α_i and β_i is canonically isomorphic to the coequalizer of α_{i+1} and β_{i+1}. Since every chain of representable atoms stabilizes by (C3), there is some i such that α_i and β_i are isomorphisms. The coequalizer of α_i and β_i is $m_i/(\alpha_i\beta_i^{-1})$, hence the coequalizer of α and β is $m \cong m_1 \to m_i \to m_i/(\alpha_i\beta_i^{-1})$. ∎

In general, the colimit of a diagram in $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ is obtained by sheafifying the colimit of the same diagram considered in $[\mathsf{C}, \mathsf{Set}]$. We will say that m/G is *computed in* $[\mathsf{C}, \mathsf{Set}]$ if there is no need to sheafify when computing this colimit.

2.2.2. Proposition. *Suppose that* C *satisfies (C1), (C2) and (C3). Let* $n \to a$ *be a morphism between atoms of* $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ *where* n *is representable. Then* $n \to a$ *is equal to some composition*

$$n \to m \to m/G \cong a$$

where m *is another representable atom and* $G \subseteq \mathrm{Aut}(m)$. *Moreover,* m *and* G *can be chosen such that the quotient* m/G *is computed in* $[\mathsf{C}, \mathsf{Set}]$.

Proof. Let $n \to m$ be a maximal representable quotient such that $n \to a$ factorizes through it. Since C^{op} is supposed co-well-founded by (C3), such an m exists. Let $G \subseteq \mathrm{Aut}(m)$ be the subgroup of automorphisms fixing $m \to a$. We claim that $m/G = a$. Let x be a representable atom and let $\alpha, \beta : x \rightrightarrows m$ be two morphisms coequalized by $m \to a$. By Lemma 2.2.1, the coequalizer of α and β is of the form $m \to m' \to m'/\sigma$ with m' a representable atom and $\sigma \in \mathrm{Aut}(m')$. But $m \to m'$ must be an isomorphism since m is maximal, so that we can suppose $m = m'$ and the coequalizer is $m \to m/\sigma$ with $\sigma \in G \subseteq \mathrm{Aut}(m)$. Hence we can factorize $m \to \mathrm{coeq}(\alpha, \beta) \to m/G$. But a is the wide pushout of all these coequalizers $\mathrm{coeq}(\alpha, \beta)$ for α and β coequalized by $m \to a$, so that $a = m/G$.

Now, we show that m/G is computed in $[\mathsf{C}, \mathsf{Set}]$. In other words, we must show that $\mathrm{Hom}(-, m)/G$ is a sheaf on the category of representable atoms. Let $f : n \to k$ be a morphism between representable atoms. Since $\mathrm{Hom}(k, m) \to \mathrm{Hom}(n, m)$ is an injective morphism of G-sets, the map $\mathrm{Hom}(k, m)/G \to \mathrm{Hom}(n, m)/G$ is also injective, so the presheaf is separated. To complete the argument, let $[g] \in \mathrm{Hom}(n, m)/G$ be the equivalence class of some $g \in \mathrm{Hom}(n, m)$. We must show that if $[g]$ satisfies the descent condition with respect to the covering $f : n \to k$, then it descends to k. If this descent condition is satisfied by $[g]$, then it is also satisfied by the composite of g and $m \to m/G$. Since $\mathrm{Hom}(-, m/G)$ is a sheaf on the representable atoms (and even on all the atoms), there is a morphism $k \to m/G$ making the diagram below commute.

$$\begin{tikzcd} n \arrow[r, "g"] \arrow[d, "f"'] & m \arrow[d] \\ k \arrow[r] & m/G \end{tikzcd}$$

Since representable atoms are stable under pushouts, we obtain the following diagram with m' representable.

$$\begin{tikzcd} n \arrow[r, "g"] \arrow[d, "f"'] & m \arrow[d] & \\ k \arrow[r] & m' \arrow[r] & m/G \end{tikzcd}$$

By maximality of m, the morphism $m \to m'$ is invertible. This shows that g factors through f, hence $[g]$ descends to k. ∎

Since every atom of $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ is a quotient of a representable one, we obtained that all the atoms are presheaves of the form $\mathrm{Hom}(X,-)/G$ with $X \in \mathsf{C}$ and $G \subseteq \mathrm{Aut}(X)$. We will now characterize in combinatorial terms when, reciprocally, $\mathrm{Hom}(X,-)/G$ is a sheaf, hence an atom of $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$.

2.2.3. Definition. *Given $X \in \mathsf{C}$, we say that a subgroup $G \subseteq \mathrm{Aut}(X)$* reduces *$X$ to a subobject $u : Y \to X$ if for all arrows $f, g : X \rightrightarrows Z$ such that $uf = ug$, there is $\sigma \in G$ with $\sigma f = g$. A pair (X, G) is* atomic *if G does not reduce X to any strict subobject.*

2.2.4. Remark. If (X, G) is atomic, then so is (X, H) for any subgroup $H \subseteq G$.

2.2.5. Lemma. *Suppose that C satisfies (C1), (C2) and (C3). Let $X \in \mathsf{C}$ and let $G \subseteq \mathrm{Aut}(X)$. The pair (X, G) is atomic if and only if $\mathrm{Hom}(X,-)/G$ is a sheaf.*

Proof. As we saw in the proof of Proposition 2.2.2, $\mathrm{Hom}(X,-)/G$ is a sheaf if and only if $\mathrm{Hom}(X,-) \to \mathrm{sh}(\mathrm{Hom}(X,-)/G)$ cannot be factored through any $\mathrm{Hom}(X,-) \to \mathrm{Hom}(Y,-)$ where $Y \subseteq X$ is a strict subobject. We will see that there is such a factorization if and only if G reduces X to Y. Indeed, this factorization means that we have an arrow $\mathrm{Hom}(Y,-) \to \mathrm{sh}(\mathrm{Hom}(X,-)/G)$ making the diagram below commute.

$$\begin{tikzcd} \mathrm{Hom}(X,-) \arrow[r, "\mathrm{id}"] \arrow[d] & \mathrm{Hom}(X,-) \arrow[d] \\ \mathrm{Hom}(Y,-) \arrow[r] & \mathrm{sh}(\mathrm{Hom}(X,-)/G) \end{tikzcd}$$

Hence it is equivalent to $[\mathrm{id}] \in \mathrm{Hom}(X,X)/G$ satisfying the descent condition for $Y \subseteq X$. This descent condition is exactly the definition that G reduces X to Y. ∎

We obtain the following description of the category of atoms of $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$.

2.2.6. Theorem. *Suppose that* C *satisfies (C1), (C2) and (C3). Then the category of atoms of* $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ *is the full subcategory of* $[\mathsf{C}, \mathsf{Set}]$ *of all the quotients* $\mathrm{Hom}(X, -)/G$ *where* (X, G) *is an atomic pair.*

2.2.7. Remark. Given two atomic pairs (X, G) and (Y, H), we can describe concretely the morphisms between the associated atoms. A morphism $[f] : \mathrm{Hom}(Y, -)/H \to \mathrm{Hom}(X, -)/G$ is an equivalence class of morphisms $f : X \to Y$ in C such that $fH \subseteq Gf$, subject to the relation $[f] = [g]$ if and only if $Gf = Gg$.

2.2.8. Remark. Given two atomic pairs (X, G) and (Y, H), an isomorphism between the corresponding atoms is given by an isomorphism $f : X \to Y$ such that $fHf^{-1} = G$. We deduce that the automorphism group of $\mathrm{Hom}(X, -)/G$ is the *normalizer quotient* $N_{\mathrm{Aut}(X)}(G)/G$. The *normalizer* $N_{\mathrm{Aut}(X)}(G)$ of G is the group of all the $f \in \mathrm{Aut}(X)$ such that $fGf^{-1} = G$, and it is the largest subgroup of $\mathrm{Aut}(X)$ in which G is normal. Note that since C is supposed well-founded, every endomorphism $f : X \to X$ is an automorphism, and we have thus even described the monoid of endomorphisms of $\mathrm{Hom}(X, -)/G$.

In general, when $G \subseteq \mathrm{Aut}(Y)$ reduces X to a subobject $u : Y \to X$, we do not necessarily have $\mathrm{sh}(\mathrm{Hom}(X, -)/G) \cong \mathrm{sh}(\mathrm{Hom}(Y, -)/H)$ for some subgroup $H \subseteq \mathrm{Aut}(Y)$. A counter-example is obtained by taking C the category of finite sets of cardinality at most 2 and injections between them. Let $u : \{1\} \hookrightarrow \{1, 2\}$ be the canonical injection and let G be the group of permutations of $\{1, 2\}$. In this case, $\mathrm{sh}(\mathrm{Hom}(\{1, 2\}, -)/G) \cong \mathrm{Hom}(\emptyset, -)$, but the automorphism group of $\{1\}$ is trivial.

As a consequence of Proposition 2.2.2, if Y is the *smallest* subobject to which G reduces X, then $\mathrm{sh}(\mathrm{Hom}(X, -)/G) \cong \mathrm{sh}(\mathrm{Hom}(Y, -)/H)$ for some $H \subseteq \mathrm{Aut}(Y)$. Proposition 2.2.9 below shows that this holds more generally when G *restricts* to $Y \subseteq X$, in addition to reducing X to Y. We say that an endomorphism $\sigma : X \to X$ *restricts* to $u : Y \hookrightarrow X$ if there is a necessarily unique endomorphism $\sigma|_Y : Y \to Y$ such that $u\sigma = \sigma|_Y u$. Automorphisms do not necessarily restrict to automorphisms, but in our situation every endomorphism is an automorphism, because C is well-founded. In this case, if an automorphism restricts to Y, then its inverse also restricts to Y. We say that G *restricts* to Y when all of its elements restrict to Y, and its *restriction* to Y is $G|_Y = \{\sigma|_Y \mid \sigma \in G\}$. The same notations will be used for the co-restriction in the dual situation.

2.2.9. Proposition. *Suppose that* C *satisfies (C1), (C2) and (C3). Let* $u : Y \to X$ *be a subobject and let* G *be a subgroup of* $\mathrm{Aut}(X)$ *that reduces* X *to* Y *and that restricts to* Y*. Then* $\mathrm{sh}(\mathrm{Hom}(X, -)/G) \cong \mathrm{sh}(\mathrm{Hom}(Y, -)/\, G|_Y)$.

Proof. The argument works at a more general level, where $u : Y \to X$ is replaced by a morphism $q : n \twoheadrightarrow m$ between atoms. Let $G \subseteq \mathrm{Aut}(n)$ be a subgroup that co-restricts to m, and let H be its co-restriction to m. Moreover, we suppose that $n \to n/G$ factors through q, which corresponds to the assumption that G reduces X to Y, as we showed in the proof of Lemma 2.2.5.

$$\begin{tikzcd} n \arrow[r, two heads, "q"] \arrow[dr] \arrow[loop above, "G"] & m \arrow[d] \arrow[loop above, "H"] \\ & n/G \end{tikzcd}$$

In this purely categorical situation, it can be shown that $m/H \cong n/G$. We reduce to a set-theoretic situation: it is enough to show that $\mathrm{Hom}(m/G, A) \cong \mathrm{Hom}(n/H, A)$ naturally in A. The morphism q induces an injection $\mathrm{Hom}(m, A) \subseteq \mathrm{Hom}(n, A)$. The group G acts on $\mathrm{Hom}(n, A)$ and restricts to $\mathrm{Hom}(m, A)$.

$$H \circlearrowright \mathrm{Hom}(m, A) \ \subseteq \ \mathrm{Hom}(n, A) \circlearrowleft G$$

The set of fixpoints of G, identified with $\mathrm{Hom}(n/G, A)$, is included in $\mathrm{Hom}(m, A)$. Hence it is equal to the set of fixpoints of H, and we conclude that $\mathrm{Hom}(n/G, A) \cong \mathrm{Hom}(m/H, A)$. ■

2.2.10. Remark. When we motivated Proposition 2.2.9, we said that it generalizes the situation of Proposition 2.2.2 where Y is the *smallest* subobject to which G reduces X. Let us show that it is indeed a generalization, and that G restricts to Y in this case. We will show that any $\sigma \in G$ restricts to $Y \subseteq X$. Let $\sigma \cdot Y$ be the subobject of X obtained as the composite $Y \hookrightarrow X \xrightarrow{\sigma} X$. Since the quotient $\mathrm{Hom}(X, -) \to \mathrm{sh}(\mathrm{Hom}(X, -)/G)$ is invariant under σ, and since it factors through $\mathrm{Hom}(X, -) \to \mathrm{Hom}(Y, -)$, it also factors through $\mathrm{Hom}(X, -) \to \mathrm{Hom}(\sigma \cdot Y, -)$. Hence G also reduces X to $\sigma \cdot Y$ and by minimality of Y, we obtain $\sigma \cdot Y = Y$, so that σ restricts to Y.

2.3. When the sheaves are the pullback-preserving presheaves. According to the description of the Schanuel topos given in the introduction of this paper, *every* pair (X, G) with $X \in \mathsf{FinSetInj}$ and $G \subseteq \mathrm{Aut}(X)$ is atomic. Another well-known fact is that a functor $\mathsf{FinSetInj} \to \mathsf{Set}$ is a sheaf if and only if it preserves pullbacks [Joh02, A2.1.11(h)]. We show here that it is not a coincidence.

2.3.1. Theorem. *Let C be a small category satisfying (C1), (C2) and (C3). Then the following are equivalent:*

1. *The pair (X, G) is atomic for every $X \in \mathsf{C}$ and every subgroup $G \subseteq \mathrm{Aut}(X)$.*
2. *The sheaves are exactly the pullback-preserving presheaves $\mathsf{C} \to \mathsf{Set}$.*

Proof. **1** $\Longrightarrow$ **2.** We already saw in the proof of Lemma 2.1.1 that (C2) implies that every sheaf preserves pullbacks. We will show that any pullback-preserving presheaf $F : \mathsf{C} \to \mathsf{Set}$ can be written as a sum of presheaves of the form $\mathrm{Hom}(X, -)/G$ with $X \in \mathsf{C}$ and $G \subseteq \mathrm{Aut}(X)$. Since 1 says that all the $\mathrm{Hom}(X, -)/G$ are sheaves, and since these sheaves (with respect to the atomic topology) are stable under sums, this will imply 2.

Note that if F preserves pullbacks, then it sends every morphism to an injection because every morphism of C is monic. Given $p \in F(X)$, the *support* of p is the minimal $Y \subseteq X$ such that p is in $F(Y) \subseteq F(X)$. Since C is well-founded, such a minimal Y exists, and since F preserves pullbacks it is unique. Let $\mathcal{A}_F$ be the set of pairs (p, X) with $p \in F(X)$ such that the support of p is X. Let G_p be the group of automorphisms $\sigma : X \to X$ fixing p. For each $(p, X) \in \mathcal{A}_F$, the element p can be seen as a morphism $p : \mathrm{Hom}(X, -) \to F$ and it factors through $\mathrm{Hom}(X, -)/G_p \to F$. We claim that this gives a decomposition

$$F \cong \sum_{(p,X)\in\mathcal{A}_F} \mathrm{Hom}(X, -)/G_p.$$

Let $Y \in \mathsf{C}$ and let $q : \mathrm{Hom}(Y, -) \to F$. We must show that it factors uniquely through one of the $\mathrm{Hom}(X, -)/G_p \to F$. First, the support of q is unique, hence there is a unique $p : \mathrm{Hom}(X, -) \to F$ in $\mathcal{A}_F$ through which q factors. Since $\mathrm{Hom}(Y, -)$ is projective, q can only factor through one $\mathrm{Hom}(X, -)/G_p \to F$. It remains to show that the factorization of q through p is unique modulo G_p. Let $f, g : X \rightrightarrows Y$ such that $F(f)(p) = F(g)(p)$. Take the following pullback.

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "v"'] & X \arrow[d, "f"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

Then u and v are isomorphisms, otherwise the support of $p \in F(X)$ would be smaller than X. We get that $\sigma = v^{-1}u$ is an automorphism fixing p and such that $\sigma \cdot f = g$. This shows that the factorization of q through p is unique and concludes the proof of the decomposition.

2 $\Longrightarrow$ 1. We will show that given any pullback-preserving presheaf $F : \mathsf{C} \to \mathsf{Set}$ and any automorphism group $G \subseteq \mathrm{Aut}(F)$, the quotient F/G in $[\mathsf{C}, \mathsf{Set}]$ is also a pullback-preserving presheaf. Any $f : A \to B$ in C is a monomorphism, hence $F(f) : F(A) \to F(B)$ is an injective morphism of G-sets and $F(f)/G : F(A)/G \to F(B)/G$ is injective too. Consider a pullback in C.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "u"] \\ C \arrow[r, "v"] & D \end{tikzcd}$$

The morphism $F(A)/G \to F(B)/G \times_{F(D)/G} F(C)/G$ is injective since $F(A)/G \to F(B)/G$ is injective. To show that it is surjective, let $b \in F(B)$ and $c \in F(C)$ such that there is $\sigma \in G$ with $\sigma \cdot F(u)(b) = F(v)(c)$. Then $F(u)(\sigma \cdot b) = F(v)(c)$ so there is $a \in F(A)$ with $F(f)(a) = \sigma \cdot b$ and $F(g)(a) = c$. This shows that $F(A)/G \to F(B)/G \times_{F(D)/G} F(C)/G$ is surjective. As a consequence, for any atom n of $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ and any $G \subseteq \mathrm{Aut}(n)$, the quotient n/G computed in $[\mathsf{C}, \mathsf{Set}]$ preserves pullbacks, hence is a sheaf by the assumption.

∎

2.3.2. Remark. The first condition of Theorem 2.3.1 is met when every morphism $f : X \to Y$ sits in a cartesian square of the form (3) below. Indeed, in this situation, if $G \subseteq \mathrm{Aut}(Y)$ reduces Y to X, then there is in particular $\sigma \in G$ such that $\sigma u = v$. But this implies that id_Y factors through f, which means that $X = Y$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "u"] \\ Y \arrow[r, "v"] & Z \end{tikzcd} \tag{3}$$

3. Co-well-founded categories of atoms and finite presentability

Given that the category of atoms of an atomic topos is always an atomic site of presentation, one could wonder whether the conditions (C1), (C2) and (C3) on a small category C transfer to the opposite of the category of atoms of $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$. If this were the case, we would obtain an intrinsic characterization of the toposes of the form $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ where C satisfies (C1), (C2) and (C3). However, (C3) does not transfer, as shown in Example 3.0.1.

3.0.1. Example. Let G be any group, considered as a category with one object denoted $\star$. Then it satisfies conditions (C1), (C2) and (C3), and every pair $(\star, H)$ with $H \subseteq G$ is atomic. The category of atoms of the corresponding atomic topos is well-founded exactly when G is *Noetherian*, which means that every increasing chain of subgroups stabilizes. For instance, the free group F_2 on two generators x and y is not Noetherian. Indeed, let G be the kernel of the morphism $F_2 \to \mathbb{Z}$ sending x and y to 1. It is freely generated by all the elements of the form $x^n y^{-n}$ for $n \in \mathbb{Z}$. (By the Nielsen–Schreier Theorem, every subgroup of F_2 is free.) Hence G is the increasing union of the subgroups $\langle x^n y^{-n} \mid -N < n < N \rangle$ as N goes to infinity, and this chain does not stabilize.

In order to transfer (C3) to the category of atoms, we need a new condition on C:

(C4) The automorphism groups of the objects of C are Noetherian.

3.0.2. Theorem. *Let C be a small category satisfying (C1), (C2), (C3) and (C4). Let $\mathcal{A}$ be the category of atoms of $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$. Then $\mathcal{A}^{\mathrm{op}}$ also satisfies (C1), (C2), (C3) and (C4).*

Proof. Given that $\mathcal{A} \to \mathscr{E}$ is fully faithful and preserves pushouts, and since the induced functor $\mathsf{Sh}(\mathcal{A}, J_{\mathrm{at}}) \to \mathscr{E}$ is an equivalence, we know that $\mathcal{A}^{\mathrm{op}}$ satisfies (C1) and (C2).

As for (C3), we must show that any ω-chain of atoms stabilizes. Let $n \to a_1 \to a_2 \to \cdots$ be an ω-chain of atoms. Suppose that n is representable without loss of generality. By Proposition 2.2.2, we can write $n \to a_1$ as $n \to m_1 \to m_1/G_1 = a_1$. We iterate this process with $m_1 \to a_2$ to obtain $m_1 \to m_2 \to m_2/G_2 = a_2$. We obtain the following diagram.

$$\begin{tikzcd} n \arrow[r] & m_1 \arrow[r] \arrow[d] & m_2 \arrow[r] \arrow[d] & m_3 \arrow[r] \arrow[d] & \cdots \\ & m_1/G_1 \arrow[r] & m_2/G_2 \arrow[r] & m_3/G_3 \arrow[r] & \cdots \end{tikzcd}$$

Since C is well-founded, the sequence $(m_i)_i$ stabilizes at some N. Suppose $m_N = m_{N+i}$ for all i, without loss of generality. Then $G_N \subseteq G_{N+1} \subseteq G_{N+2} \subseteq \cdots$ and by (C4), this sequence also stabilizes. Hence the category of atoms of $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ is co-well-founded.

We now show that $\mathcal{A}$ satisfies (C4). According to Remark 2.2.8, the group of automorphisms of the atom associated to an atomic pair (X, G) is a quotient of a subgroup of $\mathrm{Aut}(X)$, which is Noetherian. It is thus also Noetherian. ∎

In the next proposition, we see that the atomic toposes appearing in Theorem 3.0.2 are locally finitely presentable. For instance, the classifying toposes of profinite groups are of this form. Given an atomic topos $\mathscr{E}$, we denote by $\pi_0 : \mathscr{E} \to \mathsf{Set}$ the functor taking an object to its set of atoms. We say that X is *finite* if $\pi_0(X)$, or equivalently $\mathrm{Sub}_{\mathscr{E}}(X)$, is finite.

3.0.3. Proposition. *If the category of atoms of an atomic topos is co-well-founded, then this topos is locally finitely presentable. In that case, the finitely presentable objects are exactly the finite objects.*

Proof. In an atomic topos, the class of atoms is essentially small, see [Joh02, p. 690] or [BD80, Prop. 9]. Moreover, any object is the filtered colimit of its finite sub-objects. It remains only to show that, under the hypothesis of the proposition, for any atom a, the functor $\mathrm{Hom}(a, -)$ preserves filtered colimits. By [AR94, Cor. 1.7, p. 15], it suffices to show that $\mathrm{Hom}(a, -)$ preserves colimits of chains. Let $(X_i)_{i\in I}$ be a chain in $\mathscr{E}$ and let $X = \mathrm{colim}_i X_i$. Let $a \to X$ be a morphism. The union of the images of $X_i \to X$ is X, hence there is $i \in I$ and an atom $b \in X_i$ whose image coincides with the image of a in X. The sequence of images of $b \in X_i$ in X_j for $j > i$ stabilizes since the category of atoms is co-well-founded. Hence, $a \to X$ factors through $X_j \to X$ for some $j > i$. Now, consider two different morphisms $a \rightrightarrows X_i$ such that the composites $a \rightrightarrows X_i \to X$ are equal. There is an index $j > i$ such that the two morphisms $a \rightrightarrows X_j$ have the same *image* $b \subseteq X_j$, since π_0 is cocontinuous. Since the sequence of atoms obtained as the images of b in X_k for $k > j$ stabilizes, there is some $k > j$ such that the two morphisms $a \rightrightarrows X_k$ are equal. This shows that $\mathrm{Hom}(a, \mathrm{colim}_i X_i) \cong \mathrm{colim}_i \mathrm{Hom}(a, X_i)$. ∎

3.0.4. Remark. We will sketch a different proof of Proposition 3.0.3. Given a small category C, let $\coprod[\mathsf{C}]$ be its free cocompletion under small coproducts and let $\sqcup[\mathsf{C}]$ be its free cocompletion under finite coproducts. It can be shown that $\coprod[\mathsf{Ind}(\mathsf{C})] \simeq \mathsf{Ind}(\sqcup[\mathsf{C}])$, although the author does not know of a short proof or a reference for this fact. If the category $\mathcal{A}$ of atoms is co-well-founded, then $\mathcal{A} \simeq \mathsf{Ind}(\mathcal{A})$ and thus

$$\mathsf{Sh}(\mathcal{A}, J_{\mathrm{at}}) \simeq \coprod[\mathcal{A}] \simeq \coprod[\mathsf{Ind}(\mathcal{A})] \simeq \mathsf{Ind}(\sqcup[\mathcal{A}]).$$

This could be put in comparison with [BP99], where a criterion involving well-foundedness on a pretopos $\mathscr{E}$ is given to guarantee that $\mathsf{Ind}(\mathscr{E})$ is the topos of sheaves on $\mathscr{E}$ with the coherent topology. In general, $\mathsf{Ind}(\mathscr{E})$ is only included in this topos.

3.0.5. Example. In general, every finitely presentable object of an atomic topos is a finite coproduct of atoms, but the converse is not true. Even in a locally finitely presentable atomic topos, there can be atoms which are not finitely presentable. It happens for instance in any of the toposes of Example 3.0.1 obtained from a non-Noetherian group G. The finitely presentable atoms correspond to the finitely generated subgroups of G.

3.0.6. Corollary. *If* C *satisfies (C1), (C2), (C3) and (C4), then* $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ *is locally finitely presentable.*

We will show in § 5 that (C1), (C2) and (C3) are not enough to ensure that $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ is locally finitely presentable, as is the case when C is a group.

4. The Malitz–Gregory atomic topos

The Malitz–Gregory topos is an example of a non-degenerate atomic topos which does not have any points. It was first given in [Mak82, Sec. 5], based on an earlier example in infinitary logic in [Mal68] and [Gre71]. A description can also be found in [Joh02, D3.4.14].

We define in § 4.1 an atomic site of presentation of this topos adapted to our study. This site was obtained by first considering a simpler atomic site and by sheafifying the representable presheaves. We show in § 4.2 that this topos is locally finitely presentable by applying the criterion of § 3.

4.1. Definition of the Malitz–Gregory topos. A *full binary tree* is a tree in which each node is either a leaf, or has exactly two children. More formally, it can be defined as a pointed oriented graph $(X, r_X \in X, \succ \subseteq X^2)$ such that:

- For each $x \in X$, the set $\{y \in X \mid x \succ y\}$ is either empty or of cardinality 2.
- For each node y, there is exactly one path $r_X = x_0 \succ x_1 \succ \cdots \succ x_n = y$.

When $x \succ y$, we say that y is a *child* of x and that x is the *parent* of y. An *embedding* of trees $f : X \to Y$ is an (injective) graph embedding preserving the root.

$\mathbb{I}$-trees We now enrich our notion of tree with a partial labeling of its infinite branches. Let $\mathbb{I}$ be a set. A *branch* of X is an infinite sequence $r_X = x_0 \succ x_1 \succ x_2 \succ \cdots$ Let B_X be the set of branches of X. An $\mathbb{I}$-*tree* is a full binary tree X equipped with a partial function $c_X : B_X \to \mathbb{I}$. An embedding of $\mathbb{I}$-trees is an embedding of full binary trees that preserves the labeling function. This defines the category of $\mathbb{I}$-trees, with embeddings as morphisms. A sub-tree of an $\mathbb{I}$-tree is a sub-object in this category.

An $\mathbb{I}$-tree is *finitary* if it has a finite number of branches, a finite number of elements whose parent is not on a branch, and a total labeling function. Let $\mathsf{T}_{\mathbb{I}}$ be the category of finitary $\mathbb{I}$-trees and embeddings.

4.1.1. Lemma. *The category of* $\mathbb{I}$*-trees is equivalent to* $\mathsf{Ind}(\mathsf{T}_{\mathbb{I}})$.

Proof. First, the category of $\mathbb{I}$-trees has filtered colimits, by taking the union of the underlying sets and labeling functions. Moreover, each $\mathbb{I}$-tree is the filtered union of its finitary sub-trees. We can also show that each finitary $\mathbb{I}$-tree is finitely presentable: Let $\bigcup_{i\in I} X_i$ be a filtered union of $\mathbb{I}$-trees and let $X \subseteq \bigcup_{i\in I} X_i$ be a finitary sub-tree. Each branch of X is labeled, thus it is labeled as a branch of $\bigcup_{i\in I} X_i$. Since the labeling function of $\bigcup_{i\in I} X_i$ is the union of the labeling functions of the X_i, the branch must be contained in one of the X_i. For each element x of the branch, one of its children is contained in X_i, hence the other one too. Only a finite number of elements of X are not covered in this way, and since the union $\bigcup_{i\in I} X_i$ is filtered, X is a sub-tree of one of the X_i. ■

We impose the additional condition that the cardinality of $\mathbb{I}$ is strictly greater than $2^{\aleph_0}$. The *Malitz–Gregory topos* over $\mathbb{I}$ is $\mathsf{Sh}(\mathsf{T}_{\mathbb{I}}^{\mathrm{op}}, J_{\mathrm{at}})$.

4.1.2. Proposition. $\mathsf{Sh}(\mathsf{T}_{\mathbb{I}}^{\mathrm{op}}, J_{\mathrm{at}})$ *has no points.*

Proof. By Diaconescu's theorem, a model of $\mathsf{Sh}(\mathsf{T}_{\mathbb{I}}^{\mathrm{op}}, J_{\mathrm{at}})$ in Set is a flat functor $\mathsf{T}_{\mathbb{I}}^{\mathrm{op}} \to \mathsf{Set}$ continuous with respect to the atomic topology, i.e., which sends every morphism to a surjection. Since flat functors $\mathsf{T}_{\mathbb{I}}^{\mathrm{op}} \to \mathsf{Set}$ can be identified with ind-objects of $\mathsf{T}_{\mathbb{I}}$, and thanks to Lemma 4.1.1, a model is a special $\mathbb{I}$-tree $X \in \mathsf{Ind}(\mathsf{T}_{\mathbb{I}})$. The condition that the corresponding functor sends every morphism to a surjection translates into the fact that each morphism $T \to X$ with $T \in \mathsf{T}_{\mathbb{I}}$ extends along any morphism $T \to T'$ in $\mathsf{T}_{\mathbb{I}}$.

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \\ T' \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \tag{4}$$

However, no $\mathbb{I}$-tree can satisfy this extension condition. For suppose that X is such a tree. Like any tree as we defined them, X has at most a countable number of nodes. Moreover, for each $i \in \mathbb{I}$, there must be a branch labeled by i, by applying the extension property (4) to the inclusion of the one-point tree into an infinite branch labeled by i, together with the other nodes forced to be present in order to have a full binary tree. In particular, the partial function $B_X \to \mathbb{I}$ must be surjective, but it is not possible because $B_X \subseteq \mathscr{P}(T)$ and $\mathbb{I}$ has cardinality strictly bigger than that of $\mathscr{P}(T)$. In conclusion, $\mathsf{Sh}(\mathsf{T}_{\mathbb{I}}^{\mathrm{op}}, J_{\mathrm{at}})$ has no model in Set. ■

4.1.3. Remark. As we will see below, $\mathsf{T}_{\mathbb{I}}$ has amalgamation. However, we will make some arbitrary choices in order to amalgamate trees, and in particular we will not have an amalgamation *functor* as in [DLR24, Dfn. 4.1.3]. This is a substantial difference from the case of the Schanuel topos $\mathsf{Sh}(\mathsf{FinSetInj}^{\mathrm{op}}, J_{\mathrm{at}})$, where amalgamation in $\mathsf{FinSetInj}$ *is* functorial, as the restriction of the pushout functor in FinSet. A special case of [DLR24, Cor. 4.1.9] shows that functorial amalgamation in C implies that $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$ has enough points. We recall the idea here in our case of interest: If C has functorial amalgamation, then we can extend amalgamation to $\mathsf{Ind}(\mathsf{C})$ and, by a small object argument, build an ind-object X satisfying the extension property depicted in (4) above for all $T, T' \in \mathsf{C}$.

Hence, by Proposition 4.1.2, amalgamation in $\mathsf{T}_{\mathbb{I}}$ *cannot* be made functorial. A direct proof of this fact, valid for instance even when $\mathbb{I}$ is a singleton, does not seem obvious.

4.2. The Malitz–Gregory topos is locally finitely presentable. Given two nodes x and y in a tree T, we say that y is a *descendant* of x or that x is an *ancestor* of y if there is a path $x = x_0 \succ x_1 \succ \cdots \succ x_n = y$. Given a node $x \in T$, we denote by $\downarrow_T x$ the set of all the descendants of x equipped with the induced tree structure. We use simply $\downarrow x$ when T is implicit. (It is not a sub-tree because the injection $\downarrow x \to T$ does not preserve the root.)

4.2.1. Lemma. *The category $\mathsf{T}_{\mathbb{I}}$ satisfies (C1).*

Proof. First, we show that $\mathsf{T}_{\mathbb{I}}$ has amalgamation. This means that given two finitary $\mathbb{I}$-trees A and B, and given a common sub-tree $X \subseteq A$, $X \subseteq B$, we can find an $\mathbb{I}$-tree C containing both A and B as sub-trees, such that the intersection contains X. Let W be the complete binary tree of infinite depth, with no labels on its branches. Choose an arbitrary embedding $X \subseteq W$ of trees (not of $\mathbb{I}$-trees). Extend it arbitrarily to two embeddings $A \subseteq W$ and $B \subseteq W$. We now wish to label some of the branches of W so that $A \subseteq W$ and $B \subseteq W$ become embeddings of $\mathbb{I}$-trees. This might be impossible if a branch of A and a branch of B with different labels are sent to the same branch of W. Nonetheless, we can correct these "conflicts" as follows. A conflict is given by a branch of A and a branch of B with different labels that get identified in W. This branch is not in X, since the labels would coincide otherwise. Pick a node x of this branch which is deep enough to ensure that $x \notin X$ and that both $\downarrow_A x$ and $\downarrow_B x$ are composed of a unique branch containing every node or its parent. Modify the embedding of A and B in W from x onward so that the two branches are not identified anymore. Doing so for each conflict creates two new embeddings $A, B \subseteq W$. The union $A \cup B$ is finitary and can be equipped with two labelings so that the embeddings of A and B preserve the labels. This shows that $\mathsf{T}_{\mathbb{I}}$ has amalgamation.

We now show that every morphism of $\mathsf{T}_{\mathbb{I}}$ is a regular monomorphism. As a first step, we show that for each finitary $\mathbb{I}$-tree A, there are two embeddings $A \rightrightarrows A'$ whose equalizer contains only the root of A. If A contains only the root, we can take $A' = A$. If not, let x_1 and x_2 be the two children of the root of A. Let C be a finitary tree such that $\downarrow x_1 \subseteq C$ and $\downarrow x_2 \subseteq C$ (this is possible because $\mathsf{T}_{\mathbb{I}}$ has amalgamation). Let A' be the tree obtained by joining two copies of C by a root. Then there are two obvious embeddings $A \rightrightarrows A'$ and their equalizer contains only the root of A.

Now, let $X \subseteq Y$ be an arbitrary embedding of finitary $\mathbb{I}$-trees. We show it is a regular monomorphism. Let L_X be the set $\{x \in X \mid \downarrow_X x = \{x\}\}$ of leaves of X. Let Y' be the $\mathbb{I}$-tree obtained by replacing each $\downarrow_Y x \subseteq Y$ for $x \in L_X$ by the tree $[\downarrow_Y x]'$ obtained in the previous paragraph, with two embeddings $\downarrow x \rightrightarrows [\downarrow x]'$ whose equalizer is the identity. This gives two embeddings $Y \rightrightarrows Y'$ whose equalizer is X, since every element of $Y \setminus X$ is a descendant of some element of L_X. Hence $X \hookrightarrow Y$ is a regular monomorphism and the atomic topology on $\mathsf{T}_{\mathbb{I}}^{\mathrm{op}}$ is sub-canonical. ∎

4.2.2. LEMMA. *The category* $\mathsf{T}_{\mathbb{I}}$ *satisfies (C2).*

PROOF. We show that $\mathsf{T}_{\mathbb{I}}$ satisfies the equivalent condition (C2'). Consider a pullback as below in $\mathsf{T}_{\mathbb{I}}$.

$$\begin{tikzcd} X \cap Y \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook] \\ Y \arrow[r, hook] & Z \end{tikzcd}$$

Let $u, v : Z \rightrightarrows A$ be two parallel arrows coinciding on $X \cap Y$. Let $L = \{x \in X \cap Y \mid \downarrow_X x = \{x\}\}$. Note that an element of L cannot be a descendant of another element of L. Define $k : Z \to A$ by

$$k(x) = \begin{cases} v(x) & \text{if } x \text{ has an ancestor in } L, \\ u(x) & \text{if } x \text{ has no ancestor in } L. \end{cases}$$

Then k is a tree embedding because it is obtained by modifying the definition of u on $\downarrow_Z x$ for each $x \in L$ to fit another tree embedding v having the same definition on x. Moreover:

- k and u coincide on X because if $x \in X$ has an ancestor in L, then this ancestor is x itself and $u(x) = v(x)$;
- k and v coincide on Y because for each $y \in Y$, either $y \in X \cap Y$ and $u(y) = v(y)$, or $y \notin X \cap Y$ and the closest ancestor of y in $X \cap Y$ is in L.

This shows that $\mathsf{T}_{\mathbb{I}}$ satisfies (C2'), with the identity $w : A \to A$ and the sequence u, k, v. Hence $\mathsf{T}_{\mathbb{I}}$ also satisfies (C2). ∎

4.2.3. LEMMA. $\mathsf{T}_{\mathbb{I}}$ *satisfies (C3) and (C4).*

PROOF. We must show that for every finitary $\mathbb{I}$-tree X, the poset of sub-trees of X is well-founded. Recall that B_X is the set of branches of X. Let F_X be the set of elements of X whose parent is not in a branch of X. If $Y \subseteq X$ is a proper sub-tree, then $|B_Y| \leq |B_X|$, and if $|B_Y| = |B_X|$, then $|F_X| < |F_X|$. Hence $X \mapsto (|B_X|, |F_X|)$ defines a strictly order-preserving map $\mathsf{T}_{\mathbb{I}} \to \omega^2$ and $\mathsf{T}_{\mathbb{I}}$ is well-founded.

To show that the groups of automorphisms of $\mathsf{T}_{\mathbb{I}}$ are Noetherian, note that any morphism $X \to Y$ in $\mathsf{T}_{\mathbb{I}}$ is uniquely determined by a function $B_X \to B_Y$ and a function $F_X \to F_Y$, thus there are only finitely many of them. The automorphism groups are thus finite, and in particular Noetherian. ∎

Using Corollary 3.0.6, we obtain:

4.2.4. COROLLARY. $\mathsf{Sh}(\mathsf{T}_{\mathbb{I}}^{\mathrm{op}}, J_{\mathrm{at}})$ *is a connected locally finitely presentable atomic topos with no points.*

4.2.5. REMARK. The equivalent properties of Theorem 2.3.1 do not hold for $\mathsf{T}_{\mathbb{I}}$. For instance, let n be the atom represented by the unique $\mathbb{I}$-tree having exactly 3 nodes. Then $n/\operatorname{Aut}(n)$ is isomorphic to the atom represented by the tree consisting only of its root.

5. An atomic topos which is not locally finitely presentable

The investigations presented in this paper were motivated by the goal of showing that the Malitz–Gregory atomic topos is locally finitely presentable. In analogy with the case of complete atomic Boolean algebras, I originally thought that every atomic topos is locally finitely presentable. However, Morgan Rogers pointed out my mistake, and we provide here a counter-example.

Given a group $\mathfrak{K}$, we define a category $\mathsf{C}_{\mathfrak{K}}$ as follows. Its objects are $X_0, X_1, X_2, \ldots$ There is no morphism $X_i \to X_j$ when $i > j$. When $i \leq j$, the morphisms $f : X_i \to X_j$ are the tuples $(f_1, \ldots, f_i) \in \mathfrak{K}^i$. The composite of two morphisms $f : X_i \to X_j$ and $g : X_j \to X_k$ is $(f_1 \cdot g_1, \ldots, f_i \cdot g_i) : X_i \to X_k$. We will show that when $\mathfrak{K}$ is not finitely generated, $\mathsf{Sh}(\mathsf{C}_{\mathfrak{K}}^{\mathrm{op}}, J_{\mathrm{at}})$ is not locally finitely presentable.

5.0.1. Remark. If $\mathfrak{K} = (\mathbb{Z}/2\mathbb{Z})^\omega$, which is indeed not finitely generated, $\mathsf{C}_{\mathfrak{K}}$ can be described in a more concrete and "semantical" fashion. Its objects are the $X_n = \{1, \ldots, n\} \times \mathbb{N} \times \{0, 1\}$ where $n \in \mathbb{N}$. The morphisms $X_i \to X_j$ are the injective functions which preserves the first two coordinates, so that in particular $i \leq j$. For a general group $\mathfrak{K}$, we can think of $\mathsf{C}_{\mathfrak{K}}$ in a similar way by replacing $\mathbb{N} \times \{0, 1\}$ by some object $\mathbb{O}$ whose group of symmetries is $\mathfrak{K}$. We can portray an arrow $(f_1, f_2) : X_2 \to X_3$, for instance, in the following way.

$$\begin{array}{ccc} & & \mathbb{O} \\ \mathbb{O} & \xrightarrow{f_2} & \mathbb{O} \\ \mathbb{O} & \xrightarrow{f_1} & \mathbb{O} \end{array}$$

5.0.2. Remark. Another family of counter-examples can be obtained by replacing $\mathsf{C}_{\mathfrak{K}}$ by the free co-affine symmetric monoidal category over $\mathfrak{K}$ seen as a one-object category. The objects of this category are the n-fold tensor products of a generator $\mathbb{O}$ and the morphisms $\mathbb{O}^{\otimes n} \to \mathbb{O}^{\otimes m}$ are the elements of $\mathrm{Inj}(n, m) \times \mathfrak{K}^n$ where $\mathrm{Inj}(n, m)$ is the set of injections $n \to m$. We obtain the Schanuel topos when $\mathfrak{K}$ is the trivial group, but it is not finitely generated. This idea admits further variations, such as replacing the injections by order-preserving functions. Nonetheless, we will concentrate on the category $\mathsf{C}_{\mathfrak{K}}$ described above because the argument is simpler in this case.

It is straightforward to check that $\mathsf{C}_{\mathfrak{K}}$ satisfies (C1), (C2) and (C3). Notice first that two parallel arrows in $\mathsf{C}_{\mathfrak{K}}$ represent always the same subobject.

- For (C1), the minimal amalgamation of X_i and X_j over any X_k is $X_{\max(i,j)}$. Every subobject $X_i \subseteq X_j$ is the equalizer of the identity of X_j and an automorphism $(1, 1, \ldots, 1, \sigma, \sigma, \ldots, \sigma)$, where $\sigma \in \mathfrak{K} \setminus \{1\}$ appears $j - i$ times.
- In the diagram (1) of (C2'), we have $X \cap Y = X$ or $X \cap Y = Y$ since the subobjects of Z are linearly ordered. This implies (C2').
- The category C is well-founded.

5.0.3. Proposition. *If $\mathfrak{K}$ is not finitely generated,* $\mathsf{Sh}(\mathsf{C}_{\mathfrak{K}}^{\mathrm{op}}, J_{\mathrm{at}})$ *is not locally finitely presentable.*

Proof. We first show that no atom is finitely presentable. By Theorem 2.2.6, any atom is of the form $\mathrm{Hom}(X_i, -)/G$ where (X_i, G) is an atomic pair. We will write $\mathrm{Hom}(X_i, -)/G$ as a filtered colimit of atoms which are not isomorphic to it.

For each subgroup $H \subseteq \mathfrak{K}$, we identify $G \times H$ with the subgroup of $\mathrm{Aut}(X_{i+1})$ consisting of the tuples $(f_1, \ldots, f_{i+1})$ where $(f_1, \ldots, f_i) \in G$ and $f_{i+1} \in H$. If H is a proper subgroup of $\mathfrak{K}$, then $(X_{i+1}, G \times H)$ is atomic: This is seen by taking, in the definition of atomic pairs (Dfn. 2.2.3), $f = \mathrm{id}_{X_{i+1}}$ and $g = (1, 1, \ldots, 1, \lambda)$ with $\lambda \in \mathfrak{K} \setminus H$. In particular, if H is finitely generated, then it is proper by assumption on $\mathfrak{K}$ and $(X_{i+1}, G \times H)$ is atomic.

In $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$, the filtered colimit of $\mathrm{Hom}(X_{i+1}, -)/(G \times H)$ as H ranges over the finitely generated subgroups of $\mathfrak{K}$ is $\mathrm{sh}(\mathrm{Hom}(X_{i+1}, -)/(G \times \mathfrak{K}))$. The group $G \times \mathfrak{K}$ reduces X_{i+1} to X_i, and since any automorphism in C restricts to any subobject, we obtain by Proposition 2.2.9 that $\mathrm{sh}(\mathrm{Hom}(X_{i+1}, -)/(G \times \mathfrak{K})) \cong \mathrm{sh}(\mathrm{Hom}(X_i, -)/G) \cong \mathrm{Hom}(X_i, -)/G$.

$$X_{i+1}/(G \times H) \longrightarrow X_{i+1}/(G \times H') \longrightarrow \cdots \longrightarrow \mathrm{sh}(X_{i+1}/(G \times \mathfrak{K})) \cong X_i/G$$

We have written every atom as a filtered colimit of other atoms which are not isomorphic to it. Thus no atom is finitely presentable. In an atomic topos, the finitely presentable objects are the finite coproducts of finitely presentable atoms, hence 0 is the only finitely presentable object of $\mathsf{Sh}(\mathsf{C}^{\mathrm{op}}, J_{\mathrm{at}})$. Since this topos is not trivial, we deduce that it is not locally finitely presentable. ∎

This example complements the proof that the Malitz–Gregory topos is locally finitely presentable, since it shows that it is not simply a consequence of its atomicity. To summarize:

$$\text{atomic} + \text{lfp} \not\Rightarrow \text{enough points} \qquad (\S\ 4)$$
$$\text{atomic} \not\Rightarrow \text{lfp} \qquad (\S\ 5)$$

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