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GALOIS DESCENT FOR MOTIVIC THEORIES

BRUNO KAHN

ABSTRACT. We give necessary conditions for a category fibred in pseudo-abelian additive categories over the classifying topos of a profinite group to be a stack; these conditions are sufficient when the coefficients are $\mathbf{Q}$-linear. This applies to pure motives over a field in the sense of Grothendieck, Deligne-Milne and André, to mixed motives in the sense of Nori and to several motivic categories considered in [13]. We also give a simple proof of the exactness of a sequence of motivic Galois groups under a Galois extension of the base field, which applies to all the above (Tannakian) situations. Finally, we clarify the construction of the categories of Chow-Lefschetz motives given in [14] and simplify the computation of their motivic Galois group in the numerical case.

Du blanc! Verse tout, verse de par le diable! Verse deça tout plein, la langue me pelle.
– Lans. tringue.
– A toy, compaing! De hayt! de hayt!
– La! là! là! C'est morfiaillé, cela.
– O lachryma Christi!
– C'est de la Deviniere, c'est vin pineau!
– O le gentil vin blanc!
– Et par mon ame, ce n'est que vin de tafetas.
– Hen, hen, il est à une aureille, bien drappé et de bonne laine.

Rabelais, Gargantua, ch. V.

1. Introduction

The first main result of this article is

1.1. THEOREM. *Let k be a field, $\sim$ an adequate equivalence relation on algebraic cycles with rational coefficients and $\mathbf{Mot}_\sim(k)$ the category of pure motives over k modulo $\sim$, in the sense of Grothendieck. Then the assignment*

$$l \mapsto \mathbf{Mot}_\sim(l)$$

defines a stack of rigid $\otimes$-categories over the small étale site of $\operatorname{Spec} k$*.*

Theorem 1.1 is so easy to prove that it ought to be part of the folklore. Here is a sketch: for l/k a finite Galois extension with group Γ, write $\mathbf{Mot}_\sim(l)[\Gamma]$ for the category of descent

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data on $\mathbf{Mot}_\sim(l)$ relative to Γ. We have to prove that the canonical functor $\mathbf{Mot}_\sim(k) \to \mathbf{Mot}_\sim(l)[\Gamma]$ is an equivalence of categories. Full faithfulness follows from a standard transfer argument, using the facts that the base change functor $f^* : \mathbf{Mot}_\sim(k) \to \mathbf{Mot}_\sim(l)$ has a right adjoint f_* and that the coefficients are $\mathbf{Q}$. For the essential surjectivity, if $(C, (b_g)_{g\in\Gamma}) \in \mathbf{Mot}_\sim(l)[\Gamma]$ is a descent datum, with $C \in \mathbf{Mot}_\sim(l)$, the natural action of Γ on f_*C gives a projector whose image yields the effectivity of the descent datum.

In fact, such a result and sketch of proof hold in much greater generality, which led me to give them an abstract formulation: this is the purpose of Section 2. In Theorem 2.17, we get necessary conditions for a fibered category in additive categories over the classifying topos of a profinite group to be a stack; they are sufficient when the categories are pseudo-abelian and $\mathbf{Q}$-linear. These conditions use a baby "2 functor formalism" (for Galois étale coverings!), see Definition 2.14. In Corollary 3.8, we show how to weaken some hypotheses of this formalism in the presence of a monoidal structure: this allows us to easily prove the stack property for all motivic theories appearing in [13, Th. 4.3 a)], not just for pure motives (Theorem 7.1). It also applies to the related theories of [6] and [1] and to Nori motives [11].

What started me on this work was the desire to clarify and simplify a construction and a reasoning in [14], and descent arguments in its sequel [15]. In [14, §4], I construct a category of Chow-Lefschetz motives over a (possibly non separably closed) field in two steps: first a "crude" category and then a better-behaved one. By hindsight, it became likely that the second step was just the process of creating the associated stack, and this is what is checked in Proposition 8.3. The reasoning I wanted to simplify was the rather ugly recourse to continuous descent data in the proof of [14, Th. 5]: this is done here in Theorem 5.10, which also clarifies the proof of [6, Prop. 6.23 (a)][1] (quoted without comment in [1, 4.6, exemples]). A semisimplicity assumption which appeared in the first version of this paper has now been dropped from this theorem, which makes it also applicable to [12, Th. 4.7] and [11, Th. 9.1.16]. Concerning [15], the reasonings of §§4 and 5 in its first version are greatly clarified by §5 of the present paper.

Note that Theorem 1.1 does not extend to motives over a base S in the sense, say, of Deninger-Murre [7]; indeed, for X, Y smooth projective over S, the presheaf $U \mapsto CH^*(X_U \times_U Y_U)_\mathbf{Q}$ for $U \to S$ étale is already not a sheaf in the Zariski topology! Similarly, Theorem 1.1 is obviously false if the coefficients are not $\mathbf{Q}$-linear (think of the Néron-Severi group of an anisotropic conic). If one wanted to extend it to these two cases, one would probably have to consider (stable?) ∞-categories. Hopefully, the results of this paper will give an insight on what to do in that situation. Indeed, one may wonder if a suitable subset of a six functors formalism can be used to imply descent in the present spirit.

[1] What is not clear in this proof is why the map of motives $\operatorname{Hom}(\bar M, \bar N) \hookrightarrow \underline{\operatorname{Hom}}(M, N)$ on p. 215 exists. For simplicity, take $M = \mathbf{1}$, so that the purported inclusion reads $\operatorname{Hom}(\mathbf{1}, \bar N) \hookrightarrow N$. If we think in terms of representations of the motivic Galois groups, the left hand side is the invariants of the right hand side under the action of the geometric Galois group $G^0(\sigma)$. It is a subrepresentation of N provided $G^0(\sigma)$ is *normal* in the arithmetic Galois group $G(\sigma)$.

Structure of the paper. It is divided in two parts, plus an appendix. Part I contains foundational material: it deals with Galois descent theory from the most general (additive categories) to the most particular (Tannakian categories). Part II concerns applications to various motivic theories: some of those considered in [13], Lefschetz motives à la Milne from [14], 1-motives and Nori motives [11].

Section 2 concerns additive categories without extra structure; actually, the additivity hypothesis only appears from Subsection 2.13 onwards. The main result, Theorem 2.17, says that descent is basically equivalent to a two functors formalism: right adjoints, a base change isomorphism and a trace structure (see Proposition 2.16 for a more precise statement). Two other important ingredients are the construction of a retraction (Lemma 2.11 b)) and a monadic approach (§§2.20 and 2.22): both play a key rôle later. Section 3 adds $\otimes$-structures to the situation; this allows us to simplify the two functors axioms, yielding some conditions which are easy to verify in practice (Corollary 3.8).

In Section 4, we show that a diagram of $\otimes$-categories

$$\begin{tikzcd} \mathcal{A}' \\ \mathcal{A} \arrow[u, "f^*"] \arrow[r, "\gamma"] & \mathcal{B}, \end{tikzcd} \tag{*}$$

where f^* is a descent functor in the sense outlined above, has a categorical push-out in a fashion: see Propositions 4.4 and 4.8. This uses the monadic approach alluded to above.

Section 5 does two things. Suppose given a diagram (*) of Tannakian categories over a field K, with γ faithful and exact, whence a push-out $\mathcal{B}'$ as above. Consider the fibre functor $\omega_{\mathcal{B}} : \mathcal{B} \to \mathbf{Vec}_L$, where L is an extension of K. Replacing γ by $\omega = \omega_{\mathcal{B}} \circ \gamma$ in (*), we get a new diagram whose push-out turns out to be of the form $\mathbf{Vec}_R$ for an étale L-algebra R, and the universal property provides a "fibre functor" $\omega'_{\mathcal{B}} : \mathcal{B}' \to \mathbf{Vec}_R$ and an L-homomorphism $R \to L$ if ω extends to $\mathcal{A}'$. The first main result, Theorem 5.10, is that when $\mathcal{B} = \mathcal{A}$ and $L = K$ the corresponding sequence of Tannakian groups is exact. The second main result, Proposition 5.15, gives a sufficient condition for the composition $\mathcal{B}' \xrightarrow{\omega'_{\mathcal{B}}} \mathbf{Vec}_R \xrightarrow{-\otimes_R L} \mathbf{Vec}_L$ to be faithful; this condition is also necessary when $L = K$. (This will be used in a revised version of [15].)

We reap the fruits of our labour in Part II, showing in §7 that many motivic theories from [13] are stacks, and extending this to 1-motives in §9 and to Nori motives in §10. As indicated above, §8 shows that the construction of [14, §4] is a "stackification". Finally, the appendix contains more foundational material, of a more abstract nature than the one in Part I.

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Terminology. A $\otimes$-category is an additive symmetric monoidal category (the tensor structure being biadditive). A $\otimes$-functor F between $\otimes$-categories $\mathcal{A}, \mathcal{B}$ is an additive symmetric monoidal functor; it is *strong* if the structural morphisms $F(A) \otimes F(B) \to$

$F(A \otimes B)$ are isomorphisms, *lax* in general. If no adjective is used, it means by default that F is strong; we use lax to emphasise the contrary. (See [17, Ch. XI].)

Part I

General theory

2. Stacks over a profinite group

2.1. The set up. Let $\mathcal{E}$ be a category; recall from [10, VI, §§7, 8] that there is a dictionary between fibered categories over $\mathcal{E}$ and pseudo-functors $\mathcal{E} \to \mathbf{Cat}$ whose comparison 2-cocycle consists of natural isomorphisms; we shall adopt here the latter viewpoint, which is also the one of [13].

Take $\mathcal{E} = B\Pi$, where Π is a profinite group and $B\Pi$ is its classifying topos, i.e. the category of finite continuous Π-sets. We want to give conditions for a (contravariant) pseudo-functor $\mathcal{A}$ from $B\Pi$ to the 2-category of categories to be a stack for the natural topology on $B\Pi$; this will be done in Theorem 2.17 when $\mathcal{A}$ takes values in pseudo-abelian $\mathbf{Q}$-linear categories. Recall what being a stack means ([9, Déf. II.1.2.1], [18, Def. 026F]); given $S \in B\Pi$:

1. For any $A, B \in \mathcal{A}(S)$, the presheaf $(U \xrightarrow{f} S) \mapsto \mathcal{A}(U)(f^*A, f^*B)$ on $B\Pi/S$ is a sheaf;
2. Any descent datum relative to a cover $f : T \to S$ is effective.

Here is the special case where f is a Galois covering of connected (= transitive) Π-sets, with $\Gamma = \mathrm{Gal}(f)$; setting $\mathcal{A} = \mathcal{A}(S)$ and $\mathcal{A}' = \mathcal{A}(T)$:

(1G) the map $\mathcal{A}(A, B) \xrightarrow{f^*} \mathcal{A}'(f^*A, f^*B)$ induces an isomorphism $\alpha : \mathcal{A}(A, B) \xrightarrow{\sim} \mathcal{A}'(f^*A, f^*B)^\Gamma$ for any $A, B \in \mathcal{A}$;

(2G) any descent datum relative to f is effective.

In (1G), let us explain the action of Γ on the right hand side: for each $g \in \Gamma$, the equality $fg = f$ and the pseudo-functor structure of $\mathcal{A}$ yield a natural isomorphism $i_g : g^*f^* \overset{\sim}{\Rightarrow} f^*$; these are compatible with the natural isomorphisms $c_{g,h} : h^*g^* \overset{\sim}{\Rightarrow} (gh)^*$. To $\varphi : f^*A \to f^*B$, one associates $\varphi^g = i_g(B)(g^*\varphi)i_g(A)^{-1}$ (right action!). If φ is of the form $f^*\psi$, then $\varphi^g = \varphi$ by the naturality of i_g.

The meaning of (2G) is the following: let $C \in \mathcal{A}'$, provided with isomorphisms $b_g : g^*C \xrightarrow{\sim} C$ verifying the usual 1-coboundary condition

$$b_h \circ h^*b_g = b_{gh} \circ c_{g,h}$$

with respect to the 2-cocycle $c_{g,h}$ (descent datum)[2]. Then there exists $B \in \mathcal{A}$ and an isomorphism $f^*B \xrightarrow{\sim} C$ which induces an isomorphism of descent data, for the canonical descent datum on f^*B implicitly used in the previous paragraph. Moreover, B is unique up to unique isomorphism.

To formalise this, we introduce the category $\mathcal{A}'[\Gamma]$ of descent data: an object is a descent datum as above, and morphisms are the obvious ones.[3] If $(C,(b_g)),(D,(b'_g)) \in \mathcal{A}'[\Gamma]$ and $\varphi \in \mathcal{A}'(C,D)$, one defines φ^g for $g \in \Gamma$ as in the case of effective descent data, generalising the previous construction.

In fact (1G) and (2G) are sufficient to encompass (1) and (2), as shown by the following lemma. Let $B^{\mathrm{gal}}\Pi$ be the full subcategory of $B\Pi$ consisting of those (left) Π-sets Γ where Γ is a finite quotient of Π. We provide it with the topology induced by that of $B\Pi$ (any morphism is a cover). Note that every morphism in $B^{\mathrm{gal}}\Pi$ is a Galois covering. For any site $\mathbf{S}$, let $\mathrm{St}(\mathbf{S})$ be the 2-category of stacks over $\mathbf{S}$. Then

2.2. Lemma. *The restriction 2-functor* $\mathrm{St}(B\Pi) \to \mathrm{St}(B^{\mathrm{gal}}\Pi)$ *is an equivalence of 2-categories.*

Proof sketch. This is a special case of the general fact that stacks over a site only depend on the topos associated to the site [9, Th. II.3.5.1]. For the reader's convenience, let us describe a 2-quasi-inverse:

Start from a stack $\mathcal{A}$ on $B^{\mathrm{gal}}\Pi$. For $S \in B\Pi$ connected, let P be the stabiliser of a point of S: this is an open subgroup of Π. Let $N \subset \Pi$ be an open normal subgroup contained in P: then $\Gamma = \Pi/N$ is finite, and P/N (pseudo-)acts on $\mathcal{A}(\Gamma)$ by restriction of the obvious action of Γ. Define $\mathcal{A}(S)$ to be the category of descent data $\mathcal{A}(\Gamma)(P/N)$. The stack property shows that it does not depend on the choice of N, up to canonical equivalence; choosing another base point in S yields a conjugate of P and also an equivalent category; this equivalence is unique up to a canonical isomorphism because the action of P/N on $\mathcal{A}(S)$ is canonically isomorphic to the identity. In general, we set $\mathcal{A}(S) = \prod_i \mathcal{A}(S_i)$ where the S_i's are the connected components of S. We leave it to the reader to extend this construction to morphisms in order to define a pseudo-functor, and to check the stack property. ■

Lemma 2.2 reduces the study of stacks over $B\Pi$ to that of stacks over $B^{\mathrm{gal}}\Pi$. Moreover, in much of the paper we shall only consider the functor f^* for a fixed Galois $f : T \to S$, so it is convenient to abstract things a little more: thus our setting will be

- two categories $\mathcal{A}$ and $\mathcal{A}'$;
- a pseudo-action of a finite group Γ on $\mathcal{A}'$; we say that $\mathcal{A}'$ is a Γ-*category*;
- a functor $f^* : \mathcal{A} \to \mathcal{A}'$ which pseudo-commutes with the action of Γ (for its trivial action on $\mathcal{A}$).

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[2] In [14, §5], a different convention is used.

[3] Note that $\mathcal{A}'[\Gamma]$ is none else than the Grothendieck construction [10, VI, §8] on the pseudo-functor $\underline{\Gamma} \to \mathcal{A}'$ giving the g^*, where $\underline{\Gamma}$ is the category with one object representing Γ.

As above we have the category of descent data $\mathcal{A}'[\Gamma]$, and there is a functor $\hat{f}^* : \mathcal{A} \to \mathcal{A}'[\Gamma]$ sending A to $(f^*A, (i_g(A)))$; Condition (1G) amounts to say that $\hat{f}^*$ is fully faithful and (2G) amounts to say that it is essentially surjective. When this happens, we say that *f^* has descent.*

2.3. Introducing an adjoint. As a motivation, we start with

2.4. Lemma. *The forgetful functor $\tilde{f}^* : \mathcal{A}'[\Gamma] \to \mathcal{A}'$ sending $(C, (b_g))$ to C is faithful and conservative. If $\mathcal{A}'$ has finite products, $\tilde{f}^*$ has the right adjoint $\tilde{f}_* : D \mapsto \prod_{g\in\Gamma} g^*D$ provided with the descent datum (b_h) given by the isomorphisms $c_{g,h}(D) : h^*g^*D \xrightarrow{\sim} (gh)^*D$ of §2.1; the unit of this adjunction is given by the inverses of the b_g's, and its counit by the projection on the factor $g = 1$.*

Proof. This is readily checked. ∎

Thus, in the presence of finite products in $\mathcal{A}'$, the existence of a right adjoint to f^* is necessary for descent to hold. We shall only assume $\mathcal{A}'$ to have finite products from §2.6 onwards; for now, we just suppose that f^* has a right adjoint f_* with unit η and counit ε, and draw some corresponding identities.

For a descent datum $(C, (b_g))$ and $g \in \Gamma$, we get an endomorphism $[g]$ of f_*C corresponding to ε_C^g by adjunction; in formula:

$$[g] = f_*\varepsilon_C^g \circ \eta_{f_*C}.$$

2.5. Lemma. *a) We have*

$$\varepsilon_C^g = \varepsilon_C \circ f^*[g].$$

b) Let $A \in \mathcal{A}$, $(C, (b_g))$ be a descent datum, and let $g \in \Gamma$. If $\varphi : f^*A \to C$ and $\psi : A \to f_*C$ correspond to each other by adjunction, then φ^g and $[g] \circ \psi$ correspond to each other by adjunction. In particular (taking $A = f_*C$, $\psi = 1_A$), we have $[gh] = [g][h]$ (sic) and $[g]$ is an automorphism.
c) Suppose that C is an effective descent datum f^*A. For any $g \in \Gamma$, we have $\varepsilon_{f^*A}^g \circ f^*\eta_A = 1_{f^*A}$ and $[g] \circ \eta_A = \eta_A$.

Proof. a) This is just the other adjunction identity relating $[g]$ and ε_C^g.

b) We have

$$\varphi = \varepsilon_C \circ f^*\psi, \quad \psi = f_*\varphi \circ \eta_A.$$

The first identity yields

$$\varphi^g = \varepsilon_C^g \circ (f^*\psi)^g = \varepsilon_C^g \circ f^*\psi.$$

By the second identity, the morphism corresponding to φ^g is then

$$f_*\varphi^g \circ \eta_A = f_*\varepsilon_C^g \circ f_*f^*\psi \circ \eta_A = f_*\varepsilon_C^g \circ \eta_{f_*C} \circ \psi = [g] \circ \psi$$

where we used the naturality of η. Hence also the last claim.

c) Indeed, for the first identity,

$$\varepsilon_{f^*A}^g \circ f^*\eta_A = \varepsilon_{f^*A}^g \circ (f^*\eta_A)^g = (\varepsilon_{f^*A} \circ f^*\eta_A)^g = 1_{f^*A}^g = 1_{f^*A}$$

while the second one follows from b) applied to $\varphi = 1_{f^*A}$. ∎

2.6. Cartesianity. Assume now that $\mathcal{A}'$ has finite products. Let $C \in \mathcal{A}'$ and $g \in \Gamma$. We define a morphism $f^* f_* C \to g^* C$ as the composition

$$f^* f_* C \xrightarrow{i_g(f_*C)} g^* f^* f_* C \xrightarrow{g^* \varepsilon_C} g^* C.$$

Collecting over g, we get a morphism

$$f^* f_* C \to \prod_{g \in \Gamma} g^* C. \tag{2.1}$$

2.7. Definition. *The functor f^* is* Cartesian *if (f_* exists and) (2.1) is a natural isomorphism.*

(Suppose that we are in the fibred situation described at the beginning of §2.1. In view of the isomorphism of Π-sets $\coprod_{g \in \Gamma} T \xrightarrow{\sim} T \times_S T$ given by $(g, y) \mapsto (y, gy)$, Definition 2.7 amounts to saying that the "base change morphism" $f^* f_* \Rightarrow (f \times_S 1)_* (1 \times_S f)^*$ in the diagram

$$\begin{array}{ccc}
\mathcal{A}' & \xrightarrow{(1 \times_S f)^*} & \mathcal{A}(T \times_S T) \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (f \times_S 1)_*} \\
\mathcal{A}(S) & \xrightarrow{f^*} & \mathcal{A}(T)
\end{array}$$

is an isomorphism. One should not confuse Definition 2.7 with the notion of a Cartesian morphism in a fibred category.)

Assume f^* Cartesian, and let $(C, (b_g))$ be a descent datum. Composing with the b_g in (2.1), we get an isomorphism

$$f^* f_* C \xrightarrow{u_C} \prod_{g \in \Gamma} C \tag{2.2}$$

whose g-component is given, by definition, by ε_C^g.

2.8. Lemma. *Let $h \in \Gamma$. Then the action of $f^*[h]$ on the left hand side of (2.2) amounts to the action of h by right translation on the indexing set Γ of its right hand side.*

Proof. Let $g \in \Gamma$. Using Lemma 2.5 a) and b), we find

$$\varepsilon_C^g \circ f^*[h] = \varepsilon_C \circ f^*[g] \circ f^*[h] = \varepsilon_C \circ f^*[gh] = \varepsilon_C^{gh}.$$

∎

2.9. Remark. *Assume that $\mathcal{A}$ also has finite products and that f^* commutes with products. In (2.2), take $C = f^* A$ for some $A \in \mathcal{A}$. Then $\varepsilon_{f^*A} = f^* \pi_A$ for*

$$\pi_A : \prod_{g \in \Gamma} A \to A$$

the projection on the factor $g = 1$.

Here are important consequences of cartesianity. First, a definition:

2.10. Definition. *a) (cf. [4, Def. 2.3.7]). A functor $F : \mathcal{C} \to \mathcal{D}$ is* dense *if any object of $\mathcal{D}$ is isomorphic to a retract of $F(C)$ for some $C \in \mathcal{C}$.*
b) A category is pointed *if it has an object which is initial and final.*

Suppose that $\mathcal{A}'$ is pointed. Then any Hom set of $\mathcal{A}'$ has a null element 0; in particular, for $C \in \mathcal{C}$ the projection $\prod_{g\in\Gamma} g^*C \to C$ on the $g = 1$ factor has a section which is the identity on this factor and 0 elsewhere. If f^* is Cartesian, composing with the isomorphism (2.1) we get a section

$$\sigma_C : C \to f^*f_*C \tag{2.3}$$

of ε_C, which is natural in C.

2.11. Lemma. *If $\mathcal{A}'$ is pointed and f^* is Cartesian,*
a) f^* is dense;
b) for any category $\mathcal{B}$ and two functors $a, b : \mathcal{A}' \rightrightarrows \mathcal{B}$, the natural map

$$f_! : \operatorname{Hom}(a, b) \to \operatorname{Hom}(af^*, bf^*)$$

between sets of natural transformations has a canonical retraction ρ; in particular, $f_!$ is injective. Moreover, $v \in \operatorname{Hom}(af^*, bf^*)$ is in the image of $f_!$ if and only if it commutes with the ε_{f^*A} for all $A \in \mathcal{A}$.

Proof. a) Let $C \in \mathcal{A}'$. Then C is a retract of f^*f_*C via the pair $(\varepsilon_C, \sigma_C)$, where σ_C is the section of (2.3).

b) For $v \in \operatorname{Hom}(af^*, bf^*)$, define

$$\rho(v)_C = b(\varepsilon_C)v_{f_*C}a(\sigma_C) : a(C) \to b(C).$$

If $\psi : C \to D$ is a morphism, we have

$$\begin{aligned} b(\psi)\rho(v)_C = b(\psi)b(\varepsilon_C)v_{f_*C}a(\sigma_C) &= b(\varepsilon_D)b(f^*f_*\psi)v_{f_*C}a(\sigma_C) \\ &= b(\varepsilon_D)v_{f_*D}a(f^*f_*\psi)a(\sigma_C) = b(\varepsilon_D)v_{f_*D}a(\sigma_D)a(\psi) = \rho(v)_D a(\psi) \end{aligned}$$

so that $\rho(v)$ is a natural transformation. If $v = f_!(u)$ for some $u \in \operatorname{Hom}(a, b)$, then

$$\rho(v)_C = b(\varepsilon_C)u_{f^*f_*C}a(\sigma_C) = u_C a(\varepsilon_C)a(\sigma_C) = u_C,$$

thus ρ is indeed a retraction of $f_!$.

For the image of $f_!$, the condition is obviously necessary. Suppose that it holds. Then we have, for $A \in \mathcal{A}$,

$$f_!\rho(v)_A = \rho(v)_{f^*A} = b(\varepsilon_{f^*A})v_{f_*f^*A}a(\sigma_{f^*A}) = v_A a(\varepsilon_{f^*A})a(\sigma_{f^*A}) = v_A,$$

so $v = f_!\rho(v)$. ∎

2.12. Remarks. *a) In Lemma 2.11 b), we could have used σ instead of ε for the condition to be in the image of $f_!$.*
b) ρ does not *define a retraction of the functor $f_! : \mathbf{Funct}(\mathcal{A}', \mathcal{B}) \to \mathbf{Funct}(\mathcal{A}, \mathcal{B})$ in general (it need not respect composition). However, it is compatible with composition with a further functor $\mathcal{B} \to \mathcal{C}$.*

2.13. Traces. From now on, we assume $\mathcal{A}$, $\mathcal{A}'$ and f^* (hence also f_*) additive. We write $\bigoplus$ instead of $\prod$. To formulate the result, we need a further definition:

2.14. Definition. *Suppose f^* Cartesian. A* trace structure *on f^* is a natural transformation* tr : $f_* f^* \Rightarrow \mathrm{Id}_{\mathcal{A}}$ *such that, for any $A \in \mathcal{A}$:*

1. *the composition*
$$A \xrightarrow{\eta_A} f_* f^* A \xrightarrow{\mathrm{tr}_A} A \tag{2.4}$$
is multiplication by $|\Gamma|$;
2. *the isomorphism (2.2) for $C = f^* A$ converts $f^* \mathrm{tr}_A$ into the sum map.*

2.15. Main result.

2.16. Proposition. *a) If f^* has descent, then it is Cartesian and has a trace structure.*
b) The converse is true if $\mathcal{A}$ is pseudo-abelian[4] *and $\mathbf{Z}[1/|\Gamma|]$-linear.*

Proof. a) Recall Lemma 2.4. Cartesianity is tautologically true, and the trace morphism is given by $\bigoplus_{g\in\Gamma} g^* C \xrightarrow{(b_g)} C$ for $(C, (b_g)) \in \mathcal{A}'[\Gamma]$. Condition (1) of Definition 2.14 is immediate and Condition (2) is also tautological.

b) We check Conditions (1G) and (2G) of §2.1:

(1G) By adjunction, the map $\mathcal{A}(A, B) \to \mathcal{A}'(f^* A, f^* B)$ may be rewritten as the map

$$\mathcal{A}(A, B) \xrightarrow{a} \mathcal{A}(A, f_* f^* B)$$

induced by the unit morphism η_B. Using (2.4), we get a map b in the opposite direction such that ba is multiplication by $|\Gamma|$; hence a is injective by hypothesis (for this it would suffice that $\mathcal{A}(A, B)$ has no $|\Gamma|$-torsion).

I now claim that $ab = \sum_{g\in\Gamma} g$ for the action of Γ on $\mathcal{A}'(f^* A, f^* B)$ explained in §2.1. By Lemma 2.5 b), it suffices to prove that the composition

$$f_* f^* B \xrightarrow{\mathrm{tr}_B} B \xrightarrow{\eta_B} f_* f^* B$$

is $\sum_{g\in\Gamma} [g]$. By the faithfulness of f^* which has just been established, it suffices to do this after applying f^*. By Condition (2) of the trace structure, this translates as a composition

$$\bigoplus_{g\in\Gamma} f^* B \xrightarrow{\Sigma} f^* B \xrightarrow{\Delta} \bigoplus_{g\in\Gamma} f^* B$$

[4] Also sometimes called Karoubian or idempotent complete.

in which Σ is the sum map and Δ is the diagonal map by Lemma 2.5 c); the claim now follows from Lemma 2.8.

Coming back to the proof of (1G), we find that the composition

$$\mathcal{A}'(f^*A, f^*B)^\Gamma \hookrightarrow \mathcal{A}'(f^*A, f^*B) \xrightarrow{ab} \mathcal{A}'(f^*A, f^*B)^\Gamma$$

is also multiplication by $|\Gamma|$, hence the desired bijectivity of α in (1G).

(2G) Let $(C, (b_g))$ be a descent datum. Consider the idempotent $e_\Gamma = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}[g]$ in $\operatorname{End} f_*C$, and let $A = \operatorname{Im} e_\Gamma$. The adjoint of the inclusion $\iota : A \hookrightarrow f_*C$ yields a morphism $\tilde{\iota} : f^*A \to C$. Let us check that this is a morphism of descent data, and an isomorphism.

The first point amounts to say that $\tilde{\iota}^g = \tilde{\iota}$ for all g which, by Lemma 2.5 b), amounts to $[g] \circ \iota = \iota$ for all g: this is true by definition of ι.

For the second point, we define a morphism $j : C \to f^*A$ as follows. Let $\pi : f_*C \to A$ be the projection associated to the idempotent e_Γ. Then j is the composition

$$C \xrightarrow{\Delta} \bigoplus_{g\in\Gamma} C \xrightarrow{\sim} f^*f_*C \xrightarrow{f^*\pi} f^*A$$

where the first morphism is the diagonal map and the second one is the inverse of the isomorphism (2.2). It remains to show that j is inverse to $\tilde{\iota}$.

By the first point, we have $f^*[g] \circ f^*\iota = f^*\iota$ which means, by Lemma 2.8, that all the components of $f^*\iota$ on (2.2) are equal, i.e. that $\Delta\varepsilon_C f^*\iota = f^*\iota$. Therefore, with an abuse of notation,

$$j\tilde{\iota} = f^*\pi\Delta\varepsilon_C f^*\iota = f^*\pi f^*\iota = 1_{f^*A}.$$

Finally, we have $f^*\iota f^*\pi = f^*e = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} f^*[g]$, hence

$$\tilde{\iota}j = \varepsilon_C f^*\iota f^*\pi\Delta = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} \varepsilon_C f^*[g]\Delta = 1_C$$

as desired. ∎

2.17. Theorem. *a) If $\mathcal{A}$ is a stack over $B\Pi$, then*

(i) *$\mathcal{A}$ commutes with coproducts;*

(ii) *for any Galois covering $f : T \to S$ in $B\Pi$, with S, T connected, f^* is Cartesian and has a trace structure.*

b) Suppose that $\mathcal{A}(S)$ is pseudo-abelian for all S. Then the converse is true if $\mathcal{A}$ is $\mathbf{Z}[1/|\operatorname{Gal}(f)|]$-linear for any f as in (ii) with $S = *$ (the one-point Π-set), e.g. if $\mathcal{A}$ is $\mathbf{Q}$-linear.

Proof. (i) has already been seen. The rest follows from Proposition 2.16 and Lemma 2.2. ∎

2.18. A LEFT ADJOINT STRUCTURE ON f_*. The following is worth noting, but will not be used in the sequel.

Suppose that f^* is Cartesian and has a trace structure. For any $A \in \mathcal{A}$, define $\varepsilon'_A = \mathrm{tr}_A : f_* f^* A \to A$; for any $C \in \mathcal{A}'$, define $\eta'_C : C \to f^* f_* C$ as the inclusion of the $g = 1$ summand in the right hand side of (2.1).

2.19. PROPOSITION. *The natural transformations ε' and η' verify the (left) adjunction identities, provided $\mathcal{A}(A, B)$ has no $|\Gamma|$-torsion for any $A, B \in \mathcal{A}$.*

PROOF. Let $A, C \in \mathcal{A} \times \mathcal{A}'$. We must show that the compositions

$$f^* A \xrightarrow{\eta'_{f^* A}} f^* f_* f^* A \xrightarrow{f^* \varepsilon'_A} f^* A \tag{2.5}$$

and

$$f_* C \xrightarrow{f_* \eta'_C} f_* f^* f_* C \xrightarrow{\varepsilon'_{f_* C}} f_* C \tag{2.6}$$

are equal to the identity. For (2.5), this follows from Property (2) of Definition 2.14. By Proposition 2.16 b) and its proof, f^* is faithful, hence it suffices to prove (2.6) after applying this functor; using (2.2), this reduces to the previous case. ∎

2.20. ALGEBRAS ON A MONAD. Here we study the special case where $\mathcal{A}'$ is the category $\mathcal{A}^M$ of algebras over a additive monad M in $\mathcal{A}$

$$\mathcal{A}^M = \{(A, \varphi) \mid A \in \mathcal{A}, MA \xrightarrow{\varphi} A)\}$$

where φ verifies certain identities [17, VI.2, definition]. The functor f_* sends (A, φ) to A, while f^* sends A to the free algebra with underlying object MA, and the counit ε_A is given by the commutative square expressing the associativity of φ (in particular, $f_* \varepsilon_A = \varphi$). The following lemma is trivial:

2.21. LEMMA. *The forgetful functor f_* is faithful and conservative.* ∎

A homomorphism $[\,] : \Gamma \to \mathrm{End}(M)$ (see Definition A.4) yields a (strict) Γ-action on $\mathcal{A}'$ ($g^*(A, \varphi) = (A, \varphi \circ [g])$, and then we are in a special case of the situation above; in particular, the category $\mathcal{A}'[\Gamma]$ of descent data is defined:

$$\mathcal{A}'[\Gamma] = \mathcal{A}^M[\Gamma] = \{(A, \varphi, b_g) \mid (A, \varphi) \in \mathcal{A}^M, b_g : A \to A\ \}$$

where (b_g) verifies the identities $b_g \circ \varphi \circ [g] = \varphi \circ M(b_g)$ plus the cocycle condition; since the action of Γ is strict, $g \mapsto b_g^{-1}$ is a group homomorphism.

2.22. CODESCENT. The adjunction (f^*, f_*) gives rise to a factorisation of f_* into

$$\mathcal{A}' \xrightarrow{K} \mathcal{A}^M \xrightarrow{U} \mathcal{A} \tag{2.7}$$

for $M = f_* f^*$: the "comparison" functor K is given by $K(C) = (f_* C, f_* \varepsilon_C)$ (*loc. cit.*, VI.3, Th. 1), and U maps (A, φ) to A. It is Γ-equivariant.

2.23. Proposition. *If f^* is Cartesian and $\mathcal{A}'$ is pseudo-abelian, then K is an isomorphism of categories.*

Curiously, this proposition will only be used for a going up result in Theorem 4.6.

Proof. Let $(\partial_0, \partial_1) : C \rightrightarrows D$ be a pair of morphisms in $\mathcal{A}'$. Assume that $(f_*\partial_0, f_*\partial_1)$ has a universal coequaliser in the sense of [17, VI.6]. Let $\widehat{\mathcal{A}'}$ be the additive dual of $\mathcal{A}'$, $y : \mathcal{A}' \to \widehat{\mathcal{A}'}$ the additive Yoneda embedding, and let E be the coequaliser of $(y(\partial_0), y(\partial_1))$. Then $\bigoplus_{g\in\Gamma} g^*E$ is the coequaliser of $(\bigoplus_{g\in\Gamma} g^*y(\partial_0), \bigoplus_{g\in\Gamma} g^*y(\partial_1))$, where g^* is the extension of $g^* : \mathcal{A}' \to \mathcal{A}'$ to $\widehat{\mathcal{A}'}$ via y. By the cartesianity of f^* and by the hypothesis on $(f_*\partial_0, f_*\partial_1)$ applied to f^* and y, $\bigoplus_{g\in\Gamma} g^*E$ is representable, hence so is its direct summand E since $\mathcal{A}'$ is assumed to be pseudo-abelian. The conclusion now follows from Beck's theorem [17, VI.7, Th. 1]. ∎

2.24. A stability property. Suppose that we have naturally commutative diagrams of additive categories and functors

$$\begin{array}{ccc} \mathcal{A}' & \xrightarrow{\gamma'} & \mathcal{B}' \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*_{\mathcal{B}}} \\ \mathcal{A} & \xrightarrow{\gamma} & \mathcal{B} \end{array} \tag{2.8}$$

$$\begin{array}{ccc} \mathcal{A}' & \xrightarrow{\gamma'} & \mathcal{B}' \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*^{\mathcal{B}}} \\ \mathcal{A} & \xrightarrow{\gamma} & \mathcal{B} \end{array} \tag{2.9}$$

where $f_*^{\mathcal{B}}$ is right adjoint to $f^*_{\mathcal{B}}$. The following proposition is trivial but very useful.

2.25. Proposition. *a) Assume that γ' is conservative. If $f^*_{\mathcal{B}}$ is Cartesian, so is f^*.*
b) (see also Theorem 4.6). If moreover γ is fully faithful, a trace structure on $f^*_{\mathcal{B}}$ induces a unique trace structure on f^*. ∎

3. The monoidal case

In this section, we assume that the additive categories $\mathcal{A}$ and $\mathcal{A}'$ are $\otimes$-categories and that the base change functor f^* is a $\otimes$-functor: see *Terminology* at the end of the introduction. We write $\mathbf{1}$ for the unit object of both $\mathcal{A}$ and $\mathcal{A}'$; this will not cause confusion (note that $f^*\mathbf{1} = \mathbf{1}$).

3.1. Weak properties. We have a "projection morphism"

$$A \otimes f_*C \to f_*(f^*A \otimes C)$$

for $(A, C) \in \mathcal{A} \times \mathcal{A}'$, constructed as the adjoint of

$$f^*(A \otimes f_*C) \xrightarrow{\sim} f^*A \otimes f^*f_*C \xrightarrow{1\otimes\varepsilon_C} f^*A \otimes C$$

where the first isomorphism is the inverse of the monoidal structure of f^*. For $C = \mathbf{1}$, we thus get a morphism

$$w_A : A \otimes f_*\mathbf{1} \to f_*f^*A. \tag{3.1}$$

By definition of w_A, we have

3.2. Lemma. *Modulo the monoidal structure of f^*, one has the identity $\varepsilon_{f^*A} \circ f^*w_A = \varepsilon_{f^*\mathbf{1}} \otimes 1_{f^*A}$.* ■

3.3. Definition. *We say that f^* verifies the* weak projection formula *if w_A is an isomorphism for any $A \in \mathcal{A}$, and is* weakly Cartesian *if (2.2) is an isomorphism for $C = \mathbf{1}$.*

3.4. Lemma. *Suppose that f^* verifies the weak projection formula and is weakly Cartesian. Then f^* is Cartesian if and only if, moreover, it is dense (Definition 2.10 a)).*

Proof. "Only if" follows from Lemma 2.11 a). If: for $A \in \mathcal{A}$, consider the diagram

$$\begin{tikzcd}
f^*A \otimes f^*f_*\mathbf{1} \arrow[r, "f^*w_A"] \arrow[d, "1\otimes u_{\mathbf{1}}"'] & f^*f_*f^*A \arrow[d, "u_{f^*A}"'] \\
f^*A \otimes \bigoplus_{g\in\Gamma} \mathbf{1} \arrow[r, "\bigoplus_{g\in\Gamma} e_{f^*A}"] & \bigoplus_{g\in\Gamma} f^*A
\end{tikzcd} \tag{3.2}$$

where e_{f^*A} is the unit constraint: it commutes by Lemma 3.2. The bottom horizontal map is an isomorphism; so are the top and left vertical ones by assumption. Therefore u_{f^*A} is also an isomorphism. By the denseness hypothesis, u_C is then an isomorphism for every $C \in \mathcal{A}'$. ■

Suppose that f^* is Cartesian and admits a trace structure in the sense of Definition 2.14. Then there is a morphism $\mathrm{tr} : f_*\mathbf{1} \to \mathbf{1}$ such that

(1u) the composition

$$\mathbf{1} \xrightarrow{\eta_{\mathbf{1}}} f_*\mathbf{1} \xrightarrow{\mathrm{tr}} \mathbf{1}$$

is multiplication by $|\Gamma|$;

(2u) the isomorphism (2.2) (for $C = \mathbf{1}$) converts $f^*\,\mathrm{tr}$ into the sum map.

3.5. Definition. *We call this a* weak trace structure.

Conversely:

3.6. Proposition. *Suppose that f^* verifies the weak projection formula and is weakly Cartesian. Then a weak trace structure yields a trace structure on f^* by the formula $\mathrm{tr}_A = (1_A \otimes \mathrm{tr}) \circ w_A^{-1}$ for $A \in \mathcal{A}$.*

Proof. The first identity of Definition 2.7 is clear from (1u), and the second one follows from (2u) by using Diagram (3.2) again. ■

3.7. Example. *$\mathcal{A} = \mathbf{Rep}_K(G)$, $\mathcal{A}' = \mathbf{Rep}_K(H)$ for two affine group schemes $G \supseteq H$ over a field K such that $H \lhd G$ and $G/H \simeq \Gamma$. Then f^* identifies with restriction Res^G_H, whose right adjoint is induction Ind^G_H. Cartesianity and weak trace structure follow respectively from the Mackey formula and Frobenius reciprocity.*

3.8. Corollary. *If f^* has descent, it verifies the weak projection formula, is Cartesian and has a weak trace structure; the converse is true if $\mathcal{A}$ is pseudo-abelian and $\mathbf{Z}[1/|\Gamma|]$-linear.*

Proof. Collect Proposition 2.16 and Proposition 3.6. ∎

3.9. A monoidal retraction. We come back to the situation of Lemma 2.11 b), where we now assume that $\mathcal{B}$ is a $\otimes$-category and that a, b are $\otimes$-functors; hence so are also af^* and bf^*. We write $\mathrm{Hom}^\otimes(af^*, bf^*)$ and $\mathrm{Hom}^\otimes(a, b)$ for the sets of *not necessarily unital* $\otimes$-natural transformations, so that $f_!$ carries the latter to the former.

3.10. Proposition. *The retraction ρ of Lemma 2.11 b) carries $\mathrm{Hom}^\otimes(af^*, bf^*)$ to $\mathrm{Hom}^\otimes(a, b)$. If moreover f^* verifies the weak projection formula, then $v \in \mathrm{Hom}(af^*, bf^*)$ is in the image of $f_!$ if and only if it commutes with $\varepsilon_{f^*\mathbf{1}}$.*

Proof. Let $u \in \mathrm{Hom}^\otimes(af^*, bf^*)$, and let $C, D \in \mathcal{A}'$. We have to show that

$$\rho(u)_{C\otimes D} = \rho(u)_C \otimes \rho(u)_D.$$

Using the $\otimes$-structures of f^*, a and b, this amounts to the equality

$$b(\varepsilon_C \otimes \varepsilon_D) \circ u_{f_*C\otimes f_*D} \circ a(\sigma_C \otimes \sigma_D) = b(\varepsilon_{C\otimes D}) \circ u_{f_*(C\otimes D)} \circ a(\sigma_{C\otimes D}).$$

For $C, D \in \mathcal{A}'$, the morphism

$$f^*(f_*C \otimes f_*D) \xrightarrow{\sim} f^*f_*C \otimes f^*f_*D \xrightarrow{\varepsilon_C\otimes\varepsilon_D} C \otimes D,$$

where the first map is the inverse of the (strong) monoidal structure on f^*, yields by adjunction a morphism

$$f_*C \otimes f_*D \to f_*(C \otimes D) \tag{3.3}$$

(lax monoidal structure on f_*). This yields a lax monoidal structure μ rendering the diagram

$$\begin{tikzcd}
f^*f_*C \otimes f^*f_*D \arrow[rr, "\mu"] \arrow[dr, "\varepsilon_C\otimes\varepsilon_D"'] & & f^*f_*(C \otimes D) \arrow[dl, "\varepsilon_{C\otimes D}"] \\
& C \otimes D &
\end{tikzcd}$$

commutative. Using the isomorphisms (2.1), this translates to the following diagram:

$$\begin{tikzcd}
\bigoplus_{g,h} g^*C \otimes h^*D \arrow[rr, "\mu"] \arrow[dr, "\varepsilon_C\otimes\varepsilon_D"'] & & \bigoplus_{g} g^*C \otimes g^*D \arrow[dl, "\varepsilon_{C\otimes D}"] \\
& C \otimes D &
\end{tikzcd}$$

where μ identifies to the obvious projection. We have a dual commutative diagram

$$\begin{tikzcd}
\bigoplus_{g,h} g^*C\otimes h^*D & & \bigoplus_{g} g^*C\otimes g^*D \arrow[ll, "\lambda"'] \\
& C\otimes D \arrow[ul, "\sigma_C\otimes\sigma_D"] \arrow[ur, "\sigma_{C\otimes D}"'] &
\end{tikzcd}$$

where λ is the obvious inclusion. Therefore, it suffices to prove the identity

$$b(\mu)\circ u_{f_*C\otimes f_*D}\circ a(\lambda) = u_{f_*(C\otimes D)}$$

which follows from the naturality of u and the identity $\mu\lambda = 1$.

The last point follows from Lemma 2.11 b), the weak projection formula and Lemma 3.2. ∎

3.11. Remark. *If u is unital, $\rho(u)$ is not necessarily unital. In the situation of Example 3.7, $f_*\mathbf{1} = K[\Gamma]$ with its natural left action by G and u identifies with an element of $G(K)$, while ε sends $\sum_{g\in\Gamma}\lambda_g[g]$ to λ_1 and $\sigma(1) = [1]$. Thus $\rho(u)_\mathbf{1}$ is 1 if $u\in H(K)$ and 0 otherwise. It follows that $\rho(u) = 0$ in the latter case.*

3.12. Monoidal codescent. This is the pendant of Subsection 2.22. Here we simply remark that, if f^* verifies the weak projection formula., the monad M of § 2.22 is

$$MA = f_*\mathbf{1}\otimes A$$

and $\mathcal{A}^M$ is the category of modules in $\mathcal{A}$ over the monoid $f_*\mathbf{1}$ [17, VII.4], see Lemma A.6. This monoid is commutative because the monoidal structures are symmetric.

3.13. Artin objects.

3.14. Definition. *An* Artin object for f^* *is an object $A\in\mathcal{A}$ such that $f^*A\simeq n\mathbf{1}$ for some $n\geq 0$. Artin objects form a (full) rigid $\otimes$-subcategory of $\mathcal{A}$, denoted by $\mathcal{A}^0(f^*)$.*

Let $A\in\mathcal{A}^0(f^*)$ be an Artin object. Then Γ acts on $\operatorname{Hom}(\mathbf{1}, f^*A)\simeq \operatorname{Hom}(f^*\mathbf{1}, f^*A)$ as in §2.1. This defines a Z-linear $\otimes$-functor

$$\begin{aligned}\mathcal{A}^0(f^*) &\to \mathbf{Rep}_Z(\Gamma)\\ A &\mapsto \mathcal{A}'(\mathbf{1}, f^*A)\end{aligned} \tag{3.4}$$

where $Z = \operatorname{End}_{\mathcal{A}(S)}(\mathbf{1}) = \operatorname{End}_{\mathcal{A}^0(f^*)}(\mathbf{1})$, and the right hand side is the category of representations of Γ on free finitely generated Z-modules.

3.15. Lemma. *Under the hypotheses of Corollary 3.8, this functor is an equivalence of categories.*

Proof. Let $\mathcal{A}'[\Gamma]^0$ be the full subcategory of $\mathcal{A}'[\Gamma]$ consisting of those objects $(C,(b_g))$ such that C is isomorphic to $n\mathbf{1}$ for some $n\geq 0$. Then (3.4) factors as a composition

$$\mathcal{A}^0(f^*)\xrightarrow{\hat{f}^*}\mathcal{A}'[\Gamma]^0\xrightarrow{V}\mathbf{Rep}_Z(\Gamma)$$

with $V(C) = \mathcal{A}'(\mathbf{1}, C)$ as before. By definition of Z, V is an equivalence of categories and so is $\hat{f}^*$ by Corollary 3.8. ∎

3.16. An exactness result. We go back to the situation of §3.9. For $u \in \mathrm{Hom}^\otimes(af^*, bf^*)$, we write $u_{|\mathcal{A}^0(f^*)} = 1$ if $u_A : af^*A \to bf^*A$ is the identity for any $A \in \mathcal{A}^0(f^*)$ modulo the isomorphisms

$$af^*A \simeq a(n\mathbf{1}) \simeq n\mathbf{1}_\mathcal{B}, \quad bf^*A \simeq b(n\mathbf{1}) \simeq n\mathbf{1}_\mathcal{B}.$$

3.17. Theorem. *Let $\mathrm{Hom}^{\otimes,\mathbf{1}}(a,b)$ be the subset of $\mathrm{Hom}^\otimes(a,b)$ formed of unital $\otimes$-natural transformations. Suppose that f^* verifies the weak projection formula. Then $u \in \mathrm{Hom}^\otimes(af^*, bf^*)$ is of the form $f_!v$ for a (unique) $v \in \mathrm{Hom}^{\otimes,\mathbf{1}}(a,b)$ if and only $u_{|\mathcal{A}^0(f^*)} = 1$.*

Proof. Uniqueness follows from the existence of the retraction ρ of Lemma 2.11 b). The condition is obviously necessary, and its sufficiency follows from Proposition 3.10 plus the hypothesis on u, since $f_*\mathbf{1} \in \mathcal{A}^0(f^*)$ by the isomorphism (2.1). ∎

4. Morphisms of stacks

4.1. A trivial lemma. The following is obvious:

4.2. Lemma. *Let $F : \mathcal{A} \to \mathcal{B}$ be a morphism of stacks over a site. If F is faithful (resp. fully faithful, an equivalence of categories) locally, it is so globally.* ∎

4.3. Universal extension. Let $\mathcal{A}, \mathcal{A}', f^*$ be as in Section 3. Let $\mathcal{B}$ be a $\otimes$-category and $\gamma : \mathcal{A} \to \mathcal{B}$ be a $\otimes$-functor. We are going to do a reverse construction to that of §2.22.

Recall from §3.12 that (3.3) provides $f_*\mathbf{1}$ with a commutative monoid structure. Then $R = \gamma(f_*\mathbf{1})$ is a commutative monoid of $\mathcal{B}$, and γ induces a functor

$$\mathcal{A}^{f_*\mathbf{1}} \to \mathcal{B}^R =: \mathcal{B}'$$

hence a functor

$$\gamma' : \mathcal{A}' \to \mathcal{B}' \tag{4.1}$$

obtained by composing with the comparison functor K of (2.7): explicitly,

$$\gamma' C = (\gamma f_* C, \gamma f_* \varepsilon_C). \tag{4.2}$$

It comes with a naturally commutative diagram (2.9) in which $f_*^\mathcal{B}$ is the forgetful functor.

Recall that $f_*^\mathcal{B}$ has the left adjoint $f^*_\mathcal{B} : X \mapsto (R \otimes X, \mu \otimes 1_X)$ where μ is the multiplication of R: this is a special case of [17, VI.2, Th. 1]. Therefore we get a base change morphism

$$f^*_\mathcal{B}\gamma \Rightarrow \gamma' f^* \tag{4.3}$$

fitting in Diagram (2.8) (so far it is not necessarily invertible).

Suppose that $\mathcal{B}$ has coequalisers (*e.g* that it is abelian). Since $f_*\mathbf{1}$ is commutative, so is R; by Proposition A.7, $\mathcal{B}'$ acquires a $\otimes$-structure with unit R, and $f^*_\mathcal{B}$ is a $\otimes$-functor.

The action of Γ on $f_*\mathbf{1}$ (§2.3) carries over to R via γ and defines a pseudo-action of Γ on $\mathcal{B}'$ such that γ' is Γ-equivariant. In particular, the category of descent data $\mathcal{B}'[\Gamma]$ is defined (see §2.20).

4.4. Proposition. *Assume that f^* verifies the weak projection formula. Then*
a) The natural transformation (4.3) is invertible.
b) If moreover f^* is Cartesian and $\mathcal{B}$ has cokernels, γ' is a $\otimes$-functor.
c) If moreover $\mathcal{A}$ is $\mathbf{Z}[1/|\Gamma|]$-linear and γ' is dense, $f^*_{\mathcal{B}}$ has descent.

Proof. a) After composition with $f_*^{\mathcal{B}}$, the value of (4.3) on $A \in \mathcal{A}$ becomes

$$R \otimes \gamma(A) \to \gamma(f_* f^* A)$$

which, via the weak projection formula, is the strong monoidality isomorphism of γ; since $f_*^{\mathcal{B}}$ is conservative (Lemma 2.21), we are done.

b) We first provide γ' with an (a priori lax) symmetric monoidal structure. Let $C, D \in \mathcal{A}'$. The lax monoidal structure (3.3) yields a 0-sequence

$$f_*C \otimes f_*\mathbf{1} \otimes f_*D \to f_*C \otimes f_*D \to f_*(C \otimes D)$$

where the first map is the difference of the $f_*\mathbf{1}$ actions on f_*C and f_*D. Applying γ and using its strong monoidality, we get another 0-sequence

$$\gamma f_*C \otimes R \otimes \gamma f_*D \to \gamma f_*C \otimes \gamma f_*D \to \gamma f_*(C \otimes D)$$

which induces the desired natural transformation (compare (4.2) and (A.1)):

$$\gamma' C \otimes \gamma' D \to \gamma'(C \otimes D). \tag{4.4}$$

By a) and the strong monoidality of $f^*_{\mathcal{B}}$, $\gamma' \circ f^*$ is strongly monoidal: in other terms, (4.4) is an isomorphism when C and D are of the form f^*A and f^*B, hence in general by Lemma 2.11 a).

c) If $\mathcal{A}$ is $\mathbf{Z}[1/|\Gamma|]$-linear, so is $\mathcal{B}$; it is also pseudo-abelian since it has cokernels. By Corollary 3.8, it suffices to see that $f^*_{\mathcal{B}}$ verifies the weak projection formula, has a weak trace structure and is Cartesian. The first fact is a tautology, the second follows from the same property for $f_*\mathbf{1}$, as does the weak cartesianity of $f^*_{\mathcal{B}}$. But since γ' and f^* are dense, so is their composition and thus so is $f^*_{\mathcal{B}}$ as well; hence $f^*_{\mathcal{B}}$ is Cartesian by Lemma 3.4. ∎

4.5. Remark. *The density hypothesis on γ' in c) seems artificial, even though it is easy to verify in practice. I don't know how to avoid it.*

We now have a going-down and going-up theorem:

4.6. Theorem. *Under all the hypotheses of Proposition 4.4, (i.e. assuming that $\mathcal{A}$ is $\mathbf{Z}[1/|\Gamma|]$-linear, that f^* verifies the weak projection formula and is Cartesian, and that γ' is dense), γ is fully faithful if and only if γ' is fully faithful.*

Proof. "If" follows from Lemma 4.2 and Proposition 4.4. For "only if", the full faithfulness of γ implies that of $\mathcal{A}^{f_*\mathbf{1}} \to \mathcal{B}^R$. The conclusion then follows from Proposition 2.23. ∎

4.7. "Universal" property of the universal extension.

4.8. Proposition. *Consider Diagram (2.8). Let $\mathcal{C}$ be a $\otimes$-category and let $a : \mathcal{A}' \to \mathcal{C}$, $b : \mathcal{B} \to \mathcal{C}$ be two $\otimes$-functors, provided with a natural $\otimes$-transformation $v : b\gamma \Rightarrow af^*$. Suppose that, as in Theorem 4.6, all the hypotheses of Proposition 4.4 are verified and that, moreover, $\mathcal{C}$ has cokernels. Then there exists a unique $\otimes$-functor $b' : \mathcal{B}' \to \mathcal{C}$ such that $b = b'f_{\mathcal{B}}^*$; it comes with a canonical natural $\otimes$-transformation $u : b'\gamma' \Rightarrow a$.*

Proof. Applying a to the counit of the adjunction (f^*, f_*) yields a morphism

$$a^* f^* f_* \mathbf{1} \to a^* \mathbf{1} = \mathbf{1}.$$

Composing it with v gives another morphism

$$b(R) = b\gamma f_* \mathbf{1} \to \mathbf{1}$$

which is a homomorphism of monoids by construction. By Corollary A.8, this yields the first claim. By Proposition 4.4 a), we then get a natural $\otimes$-transformation $b'\gamma' f^* \Rightarrow af^*$, and Proposition 3.10 provides u. ∎

5. Tannakian categories

5.1. The set-up. Let $\mathcal{A}, \mathcal{A}', f^*$ be again as in Section 3. We add some assumptions: $\mathcal{A}, \mathcal{A}'$ are abelian and rigid, and $Z(\mathcal{A}) \xrightarrow{\sim} Z(\mathcal{A}') = K$, where K is a field of characteristic 0. Throughout, we suppose that f^* satisfies the hypotheses of Theorem 2.17 b), hence satisfies descent.

5.2. Going up. Let $\omega : \mathcal{A} \to \mathbf{Vec}_L$ be a fibre functor, where L is an extension of K (thus $\mathcal{A}$ is a Tannakian category over K). Write $E = \omega(f_* \mathbf{1})$.

5.3. Lemma. *E is an étale L-algebra of dimension $|\Gamma|$.*

Proof. By Cartesianity and the projection formula, we have

$$f_* \mathbf{1} \otimes f_* \mathbf{1} = f_* f^* f_* \mathbf{1} = f_* \prod_\Gamma \mathbf{1} = \prod_\Gamma f_* \mathbf{1}$$

hence

$$E \otimes_L E \xrightarrow{\sim} \prod_\Gamma E$$

where the homomorphism is given by $r \otimes s \mapsto (rg(s))_{g\in\Gamma}$. Here the action of Γ on E is induced by its action on $f_* \mathbf{1}$. The claims follow. ∎

5.4. Remark. *Thus E is a Galois Γ-algebra over L in the sense of [2, 1.3].*

As in (4.1), we get a $\otimes$-functor

$$\tilde{\omega}' : \mathcal{A}' \to (\mathbf{Vec}_L)^E = \mathbf{Vec}_E$$

where the right hand side denotes the $\otimes$-category of E-modules which are finite-dimensional over L (*i.e.* of finite type over E).

5.5. Lemma. *The functor $\tilde{\omega}'$ is exact and faithful.*

Proof. Let $U : \mathbf{Vec}_E \to \mathbf{Vec}_L$ be the forgetful functor. We have $U\tilde{\omega}' = \omega f_*$. The right hand side is exact and faithful as a composition of two such functors. But U is also faithful and exact, hence faithfully exact, hence the conclusion. ∎

5.6. The neutral case. Here we assume $L = K$. Let $G = \mathbf{Aut}^{\otimes}(\omega)$ be the Tannakian group of ω and $H = \mathbf{Aut}^{\otimes}(\omega')$ that of ω' (recall that every $\otimes$-endomorphism of ω or ω' is an automorphism, hence unital, by rigidity [6, Rk. 2.18]). By Tannakian duality, we may then write $\mathcal{A} = \mathbf{Rep}_K(G)$ and $\mathcal{A}' = \mathbf{Rep}_K(H)$.

The $\otimes$-functor f^* induces a homomorphism $i : H \to G$. The equivalence of Lemma 3.15 is induced by ω' since $\omega = \omega' \circ f^*$ (indeed, $\mathcal{A}'(\mathbf{1}, B)$ is functorially isomorphic to $\omega'(B)$ for any split $B \in \mathcal{A}'$). Whence a homomorphism $p : G \to \Gamma$.

5.7. Theorem. *The sequence $1 \to H \xrightarrow{i} G \xrightarrow{p} \Gamma \to 1$ is exact.*

Proof. By Theorem 3.17, it suffices to show that p is epi. By [6, Prop. 2.21 (a)], we must show that every subobject $B \in \mathcal{A}$ of an object $A \in \mathcal{A}^0(f^*)$ belongs to $\mathcal{A}^0(f^*)$; but this is obvious since $\mathbf{1}$ is simple in $\mathcal{A}'$ by [6, Prop. 1.17]. ∎

5.8. Globalisation. Let K be a field of characteristic 0, and let $\mathcal{A}$ be a pseudo-functor from $B^{\mathrm{gal}}\Pi$ to the 2-category $\mathbf{Ex}^{\mathrm{rig}}(K)$ of rigid abelian $\otimes$-categories $\mathcal{C}$ with $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = K$, $\otimes$-functors and $\otimes$-natural transformations. Let $\mathcal{A} = \mathcal{A}(*)$, where $*$ is the terminal object. Define $\mathcal{A}_\infty$ as 2-$\varinjlim_{T \in B^{\mathrm{gal}}(\Pi)} \mathcal{A}(T)$: it belongs to $\mathbf{Ex}^{\mathrm{rig}}(K)$.

Let $\omega_\infty : \mathcal{A}_\infty \to \mathbf{Vec}_K$ be a fibre functor to the category of finite-dimensional K-vector spaces: by restriction, it defines a fibre functor ω_T on $\mathcal{A}(T)$ for every T. For $T = *$, we write $\omega_T = \omega$. Let $G = \mathbf{Aut}^{\otimes}(\omega)$ be the Tannakian group of ω and $H = \mathbf{Aut}^{\otimes}(\omega_\infty)$ the one of ω_∞.

5.9. Lemma. *The natural morphism $H \to \varprojlim_T \mathbf{Aut}^{\otimes}(\omega_T)$ is an isomorphism.*

Proof. It suffices to verify this on R-points for any K-algebra R. Then it follows from the definition of $\mathcal{A}_\infty$. ∎

5.10. Theorem. *Suppose that suppose that f^* satisfies the hypotheses of Theorem 2.17 b) for any Galois $f : T \to *$. Then the sequence*

$$1 \to H \xrightarrow{i} G \xrightarrow{p} \Pi \to 1$$

is exact.

Proof. In view of Lemma 5.9, this follows from Theorem 5.7. ∎

5.11. Remark. *Even if it is not obvious, this proof is inspired by Ayoub's proof of the corresponding theorem in [3, Prop. 5.7], using Hopf algebras. He explained me a version using ind-Tannakian categories, which inspired the retraction of Lemma 2.11 b). Here is this argument, translated from French:*

One has a morphism of ind-Tannakian categories $e^* : T \to T'$, a fibre functor $w'^* : T' \to Vect_K$ [the category of all small K-vector spaces] and one sets $w^* = w'^* \circ e^*$. Assume that for every object $M \in T'$, the morphism $e^* e_* M \otimes_{e^* e_* K} K \to M$ is an isomorphism. Since $w^* w_* K = w'^* e^* e_* w'_* K$ we get that $w^* w_* K \otimes_{w^* e^* e_* K} K \simeq w'^* w'_* K$ as desired.

5.12. Remark. *There is an obvious extension of Theorem 5.10 to the case of $\otimes$-morphisms between two fibre functors, as in Theorem 3.17. Formulating it is left to the reader. Similarly for another extension to Tannakian monoids for fibre functors on not necessarily rigid $\otimes$-categories.*

5.13. Enrichments. Consider now a factorisation of ω

$$\mathcal{A} \xrightarrow{\gamma} \mathcal{B} \xrightarrow{\omega_{\mathcal{B}}} \mathbf{Vec}_L \tag{5.1}$$

where $\mathcal{B}$ is another Tannakian category over K and $\gamma, \omega_{\mathcal{B}}$ are exact and faithful $\otimes$-functors.

Let $\mathcal{B}'$ be the universal extension of §4.3, and take the notation of (2.8) and (2.9). Since $\mathcal{B}$ is rigid, its tensor structure is exact. By Lemmas A.2 d) and A.6, $\mathcal{B}'$ is abelian and the forgetful functor $f_*^{\mathcal{B}}$ is exact.

Suppose that ω is the restriction to $\mathcal{A}$ of a fibre functor $\omega' : \mathcal{A}' \to \mathbf{Vec}_L$. By Proposition 4.8, applied with $(\mathcal{C}, a, b, \beta) \equiv (\mathbf{Vec}_L, \omega, \omega_{\mathcal{B}}, 1)$, there exists a unique $\otimes$-functor $\omega'_{\mathcal{B}} : \mathcal{B}' \to \mathbf{Vec}_L$ such that $\omega' = \omega'_{\mathcal{B}} f_{\mathcal{B}}^*$. It is provided with a natural transformation $v : \omega'_{\mathcal{B}} \gamma' f^* \Rightarrow \omega' f^*$ such that $v_{\mathbf{1}}$ is the morphism $\omega(\varepsilon_{\mathbf{1}}) : E = \omega(f_* \mathbf{1}) \to \omega(\mathbf{1}) = L$.

5.14. Lemma. *The functor $\omega'_{\mathcal{B}}$ is the composition of $\tilde{\omega}'$ and the functor $L \otimes_R - : \mathbf{Vec}_R \to \mathbf{Vec}_L$. It is exact.*

Proof. The first claim follows by the functoriality of the construction of Proposition 4.8. In the composition, the first functor is exact by Lemma 5.5, and the second is exact because the homomorphism $R \to L$ is flat, thanks to Lemma 5.3. ∎

Contrary to Lemma 5.5, $\omega'_{\mathcal{B}}$ is not faithful in general, for example if $\mathcal{B} = \mathbf{Vec}_L$! The following proposition gives a case where it is. Recall that a rigid $\otimes$-category $\mathcal{C}$ is *connected* if $Z(\mathcal{C}) := \mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is a field.

5.15. Proposition. *In the above situation, the functor $\omega'_{\mathcal{B}}$ is faithful if and only if $\mathcal{B}'$ is connected. A sufficient condition is that the restriction of γ to $\mathcal{A}^0(f^*)$ is full.*

Proof. Since $\mathbf{Vec}_L$ is connected, the condition is necessary; the converse follows from [6, Prop. 1.19]. If the restriction of γ to Artin objects is full, then the map

$$K = \mathrm{End}_{\mathcal{A}'}(\mathbf{1}) = \mathcal{A}(\mathbf{1}, f_* f^* \mathbf{1}) \xrightarrow{\gamma} \mathcal{B}(\mathbf{1}, \gamma_s f_* f^* \mathbf{1}) \simeq \mathcal{B}(\mathbf{1}, f_*^{\mathcal{B}} f_{\mathcal{B}}^* \mathbf{1})) = \mathrm{End}_{\mathcal{B}'}(\mathbf{1})$$

is bijective, where we used Proposition 4.4 a) for the isomorphism. ∎

We come back to the neutral case, write $\mathcal{B} = \mathbf{Rep}_K(G')$ and let $\gamma^* : G' \to G$ be the homomorphism dual to γ. Let $H' = \mathrm{Ker}(G' \to G \to \Gamma)$.

5.16. Proposition. *The functor $\omega'_{\mathcal{B}}$ factors as a composition*

$$\mathcal{B}' \xrightarrow{\pi} \mathbf{Rep}_K(H') \xrightarrow{\bar{\omega}'_{\mathcal{B}}} \mathbf{Vec}_K \tag{5.2}$$

where π is a Serre localisation and $\bar{\omega}'_{\mathcal{B}}$ is faithful. Moreover, π is an equivalence of categories if and only if $G' \to \Gamma$ is epi. In particular, the fullness condition is also necessary in Proposition 5.15.

Proof. (5.2) is the canonical factorisation of the exact functor $\omega'_{\mathcal{B}}$ into a Serre localisation followed by a faithful functor. To identify the middle category with $\mathbf{Rep}_K(H')$, we apply Corollary A.8 to the restriction functor $\mathbf{Rep}_K(G') \to \mathbf{Rep}_K(H')$ to factor it through $\mathcal{B}'$. In the last statement, sufficiency follows from Proposition 5.15. For necessity, suppose that π is an equivalence. Then $Z = Z(\mathcal{B}')$ is a field, and we have a factorisation of the identity

$$K = Z(\mathbf{Rep}_K(H)) \xrightarrow{\gamma'} Z \xrightarrow{\omega'_{\mathcal{B}}} Z(\mathbf{Vec}_K) = K$$

hence $Z = K$ and γ' is surjective. As in the proof of Proposition 5.15, this gives that $\mathcal{A}(\mathbf{1}, f_*\mathbf{1}) \xrightarrow{\gamma} \mathcal{B}(\mathbf{1}, R)$ is bijective, from which the fullness of $\gamma_{|\mathcal{A}^0(f^*)}$ easily follows; in turn, this is equivalent to the surjectivity of $G' \to \Pi$. ∎

Part II
Applications

6. The general layout

Let k be a base field. The idea of the applications which follow is to start from the basic functoriality of schemes (or pairs of schemes) over a finite Galois extension l/k, and to transport it to categories of motives through the motive functor. This leads to the following caveat:

In the said categories of schemes, naïve restriction of scalars is left adjoint to restriction of scalars. If the motive functor is contravariant, it will convert this functor into a right adjoint, and we can directly apply the framework of §§2 and 3. This is the case for Chow-Lefschetz motives (§8) and Nori motives (§10), but not for the theories of [13] studied in §7, where the choice was that of a covariant motive functor. This means that in the latter case one must replace these categories by their opposites; of course, this does not affect the stack property. Thus cartesianity will follow from cartesianity for l-schemes X:

$$\coprod_{g\in\Gamma} g_*X \xrightarrow{\sim} X_{(k)} \otimes_k l \tag{6.1}$$

where $\Gamma = \mathrm{Gal}(l/k)$ and g_* is the base change given by $g : \operatorname{Spec} l \to \operatorname{Spec} l$ for $g \in \Gamma$; (6.1) is itself induced by the special case $X = \operatorname{Spec} l$ (Galois theory). Similarly, the weak projection formula will follow from the equality for k-schemes Y:

$$(Y \otimes_k l)_{(k)} = Y \times_{\operatorname{Spec} k} \operatorname{Spec} l. \tag{6.2}$$

Here we write $(-) \otimes_k l$ for extension of scalars from k to l, and $(-)_{(k)}$ for the naïve restriction of scalars from l to k (i.e., composing with the morphism $\operatorname{Spec} l \to \operatorname{Spec} k$).

7. Motivic theories

The following generalises Theorem 1.1 of the introduction:

7.1. Theorem. *All motivic theories $\mathcal{A}$ of [13, Th. 4.3 a)] are stacks for the étale topology on* $\operatorname{Spec} k$ *provided they are* $\mathbf{Q}$*-linear. In particular, this is the case for pure motives à la Grothendieck for any adequate equivalence relation.*

Proof. By Lemma 2.2 and Corollary 3.8, it suffices to check that, for any finite Galois extension $f : T = \operatorname{Spec} l \to S = \operatorname{Spec} k$, f^* verifies the weak projection formula, is Cartesian and has a weak trace structure. Use M generically to denote the "motive" functor $\mathbf{Sm}(-) \to \mathcal{A}(-)$. As explained in §6, we replace $\mathcal{A}(-)$ by $\mathcal{A}^{\mathrm{op}}(-)$ to make M contravariant. By [13, Th. 4.1] and its proof, f_* exists and commutes with naïve restriction of scalars on $\mathbf{Sm}(-)$ via M.

That (2.1) is a natural isomorphism is checked on pseudo-abelian generators of $\mathcal{A}$. Also, f^* commutes with Tate twists when they are present in the theory $\mathcal{A}$. We thus may take $A = M(X)$ for $X \in \mathbf{Sm}(k)$ or $\mathbf{Sm}^{\mathrm{proj}}(k)$, and we are reduced to (6.1). Similarly, (3.1) reduces to (6.2) by the monoidality of M. Finally, we define the weak trace tr by using the (finite) correspondence given by the transpose of graph of the projection $\operatorname{Spec} l \to \operatorname{Spec} k$. The axioms of a weak trace structure follow readily. ∎

7.2. Remark. *The same result holds for the motivic theories of Deligne [6] and André [1], with the same proof.*

8. Chow-Lefschetz motives

8.1. The associated stack. Let $\mathcal{A}_0$ be a fibred category over a site Σ. Recall [9, Th. II.2.1.3] that there is an "associated stack" $\mathcal{A}$ together with a fibred functor $\mathcal{A}_0 \to \mathcal{A}$ which is 2-universal for fibred functors from $\mathcal{A}_0$ to stacks. The stack $\mathcal{A}$ is constructed from $\mathcal{A}_0$ in two steps:

Associated prestack (cf. [9, Lemma II.2.2.2]) $\mathcal{A}_1$: same objects as $\mathcal{A}_0$; for $S \in \Sigma$ and $X, Y \in \mathcal{A}_0(S)$, $\mathcal{A}_1(S)(X, Y)$ is the sheaf associated to the presheaf $(T \to S) \mapsto \mathcal{A}_0(T)(X_T, Y_T)$.

Associated stack (cf. [16, Lemma 3.2]) starting from $\mathcal{A}_1$, for $S \in \Sigma$ an object of $\mathcal{A}(S)$ is a descent datum of $\mathcal{A}_1$ for a suitable cover $(U_i)_{i\in I} \to S$; morphisms are given by refining covers. This operation is fully faithful (loc. cit., Remark 3.2.1).

In the case $\Sigma = B\Pi$, these two constructions translate as follows, with the notation of Section 2: in Step 1, one replaces the groups $\mathcal{A}_0(S)(A,B)$ by $\varinjlim_T \mathcal{A}_0(T)(f^*A, f^*B)^{\mathrm{Gal}(f)}$, where $f : T \to S$ runs through the (finite) Galois coverings of S; for Step 2, we take the 2-colimit of the categories of descent data on $\mathcal{A}_1$. One could do both constructions in one gulp, but this would not be convenient for the next subsection.

8.2. The case of Chow-Lefschetz motives. In [14] we introduced categories of "Chow-Lefschetz motives" $\mathbf{LMot}_\sim(k)$ over a field k (modulo an adequate equivalence relation $\sim$) in two steps: a) by defining "crude" categories $\mathbf{LMot}_\sim(k)_0$ [14, §4.1]; b) by refining this construction [14, §4.2].

8.3. Proposition. $\mathbf{LMot}_\sim$ *is the stack associated to* $(\mathbf{LMot}_\sim)_0$.

Proof. Here we use implicitly Lemma 2.2 to consider only finite Galois extensions l/k. We first prove that $\mathbf{LMot}_\sim$ is a stack. This is essentially done in [14]: the descent property for morphisms is loc. cit., (4.4) and the effectivity of descent data is shown in the proof of Theorem 5 in loc. cit., §5.5 in the same way as here (we were inspired here by this argument). Alternately we may apply Corollary 3.8 of the present paper just as in the proof of Theorem 7.1, using the right adjoint of [14, Lemma 4.5] (note that the isomorphism (2.1) is explicitly proven in this lemma).

In remains to show that the canonical fibred functor $(\mathbf{LMot}_\sim)_0 \to \mathbf{LMot}_\sim$ induces an equivalence on the associated stacks; it suffices to do it for the fibred functor $(\mathbf{LCorr}_\sim)_0 \to \mathbf{LCorr}_\sim$ on categories of correspondences. After forming the associated prestack $(\mathbf{LCorr}_\sim)_1$ as in §8.1, this functor becomes fully faithful. Let l/k be finite Galois, with group Γ; the Γ-equivariant fully faithful functor $\mathbf{LCorr}_\sim(l)_1 \to \mathbf{LCorr}_\sim(l)$ induces a fully faithful functor on the categories of descent data, hence $\mathbf{LCorr}_\sim(k)_1 \to \mathbf{LCorr}_\sim(k)$ factors through a fully faithful functor $\mathbf{LCorr}_\sim(l)_1[\Gamma] \to \mathbf{LCorr}_\sim(k)$, and then through a fully faithful functor 2-$\varinjlim_l \mathbf{LCorr}_\sim(l)_1[\Gamma] \to \mathbf{LCorr}_\sim(k)$.

For its essential surjectivity, let A be an object of $\mathbf{LCorr}_\sim(k)$: by definition, it is an abelian scheme over an étale k-algebra E. Choose l/k and Γ as above such that l splits E. Then the l-scheme $B = \coprod_{\sigma\in \mathrm{Mor}_k(E,l)} \sigma^* A$ is provided with a canonical descent datum $(b_g)_g \in \Gamma$, given by the action of Γ on $\mathrm{Mor}_k(E,l)$, and the object $(B,(b_g)) \in \mathbf{LCorr}_\sim(l)_1[\Gamma]$ maps to A. ∎

9. 1-motives

Let $\mathbf{Mot}_1(k)$ be the category of Deligne 1-motives over a field k. Here there is no need to use the present theory:

9.1. Theorem. *The assignment* $l \mapsto \mathbf{Mot}_1(l)$, *where* l *runs through all finite Galois extensions of* k, *is a stack (compare Lemma 2.2).*

PROOF. This is trivial: we may view $\mathbf{Mot}_1(k)$ as a full subcategory of the category of arrows of the category of locally quasi-projective group schemes. When k varies, the latter is a Galois stack by [10, VIII, Cor. 7.6], hence so is $\mathbf{Mot}_1$ as well. ∎

10. Nori motives

We refer to [11, Ch. 9] for a construction of Nori's category of mixed motives over a subfield k of $\mathbf{C}$. We shall denote it here by $\mathbf{NMot}(k)$ (it is denoted by $\mathcal{MM}(k)$ in *loc. cit.*).

Let l/k be a finite extension, corresponding to $f : \operatorname{Spec} l \to \operatorname{Spec} k$. We write $f^* : \mathbf{NMot}(k) \to \mathbf{NMot}(l)$ for the base change functor denoted by $\operatorname{res}_{l/k}$ in [11, Lemma 9.5.1].

10.1. PROPOSITION. *The functor f^* has a right adjoint f_*. If l/k is Galois, f^* satisfies the weak projection formula, is Cartesian and has a weak trace structure in the sense of Definition 3.5.*

The proof will show that f_* coincides with the functor $\operatorname{cores}_{l/k}$ of [11, Prop. 9.5.3].

PROOF. It is variant of that of Theorem 7.1. The full subcategory $\mathcal{C}$ of $\mathbf{NMot}(l)$ formed of those M's such that f_* is defined at M is closed under kernels; more precisely, if $0 \to M' \to M \to M''$ is an exact sequence in $\mathbf{NMot}(l)$ such that $M, M'' \in \mathcal{C}$, then f_*M is given by $\operatorname{Ker}(f_*M \to f_*M'')$. By a result of Fresán and Jossen [8, Th. 6.3], any object of $\mathbf{NMot}(l)$ is a subobject of an object of the form $H^i_{\mathrm{Nori}}(X,Y)(n)$ for a triple (X,Y,i), hence has a copresentation by objects of this form. Therefore it suffices to check that f_* is defined at such objects.

Recall that $f^*H^i_{\mathrm{Nori}}(X,Y)(n) = H^i_{\mathrm{Nori}}(X_l,Y_l)(n)$, where $X_l = X \otimes_k l$ for a k-scheme X [11, Lemma 9.5.1]. For a l-triple (X,Y,i), write $M = H^i_{\mathrm{Nori}}(X,Y)(n)$ and define $f_*M = H^i_{\mathrm{Nori}}(X_{(k)},Y_{(k)})(n)$ where $(-)_{(k)}$ denotes the (naïve) restriction of scalars. Define a counit morphism $\varepsilon : f^*f_*M \to M$ by the canonical morphism of triples

$$(X,Y,i) \to ((X_{(k)})_l, (Y_{(k)})_l, i).$$

We must show that the composition

$$\mathbf{NMot}(k)(N, f_*M) \xrightarrow{f^*} \mathbf{NMot}(l)(f^*N, f^*f_*M) \xrightarrow{\varepsilon_*} \mathbf{NMot}(l)(f^*N, M) \tag{10.1}$$

is an isomorphism for any $N \in \mathbf{NMot}(k)$. Since f^* is exact [11, Lemma 9.5.1], we reduce to the case where N is of the form $H^j_{\mathrm{Nori}}(X',Y')(m)$ by [8, Th. 6.1] (dual to the previous theorem). Since twisting is invertible and commutes with f^*, playing with powers of $\mathbb{G}_m$ or $\mathbf{P}^1$ (see [11, 9.3.7 and 9.3.8]) we may even assume $n = m = 0$.

For N as above, define a unit morphism $\eta : N \to f_*f^*N$ by the canonical morphism of triples $((X'_l)_{(k)}, (Y'_l)_{(k)}, j) \to (X', Y', j)$. We get another composition

$$\mathbf{NMot}(l)(f^*N, M) \xrightarrow{f_*} \mathbf{NMot}(k)(f_*f^*N, f_*M) \xrightarrow{\eta^*} \mathbf{NMot}(k)(N, f_*M) \tag{10.2}$$

and it suffices to show that it is inverse to (10.1). Since ε and η are the counit and unit of an adjunction between categories of triples, this is true by the functoriality of $H^i_{\mathrm{Nori}}(-)(n)$.

Checking that (2.1) and (3.1) are isomorphisms is done in exactly the same way: note that since f^* is exact, f_* and hence f^*f_*, f_*f^* are left exact. Thus we may reduce to the case of objects of the form $H^i_{\mathrm{Nori}}(X_l, Y_l)(n)$ by diagram chase, using [8, Th. 6.3] again. The isomorphism (2.1) follows from the same in the categories of triples (Galois descent). For (3.1), same reasoning by using the partial monoidality of H^*_{Nori} [11, Prop. 9.3.1], for which we remark that $(\operatorname{Spec} l, \emptyset, i)$ is a good pair.

This proves everything, except the existence of a weak trace structure. For this we need to define a morphism

$$\mathrm{tr}_{l/k} : H^0_{\mathrm{Nori}}(\operatorname{Spec} l, \emptyset) = f_*\mathbf{1}_{\mathbf{NMot}(l)} \to \mathbf{1}_{\mathbf{NMot}(k)} = H^0_{\mathrm{Nori}}(\operatorname{Spec} k, \emptyset)$$

with the properties (1u) and (2u) stated before Definition 3.5. We do as in the proof of Proposition 8.3. The two properties are proven in the same way (or deduced from the existence of a functor from Chow motives to **NMot**). ∎

Proposition 10.1 holds for Nori motives $\mathbf{NMot}(-, A)$ with coefficients in any commutative ring A (same proof). Along with Lemma 2.2 and Corollary 3.8, this yields:

10.2. Theorem. *If A is a $\mathbf{Q}$-algebra, the assignment $l \mapsto \mathbf{NMot}(l, A)$ defines a stack over $(\operatorname{Spec} k)_{\text{ét}}$.* ∎

10.3. Remarks. *Theorem 5.10 provides a proof of [11, Th. 9.1.16].*

A. Monads and monoids

I put here things I didn't find in [17].

A.1. Lemma. *Let (T, η, μ) be a monad in a category $\mathcal{C}$ [17, VI.1]. Then the sequence*

$$T^3 \rightrightarrows T^2 \xrightarrow{\mu} T$$

is a split coequaliser in the sense of [17, VI.6], where the first pair of arrows is $(T\mu, \mu T)$. More precisely, applying this sequence to any object of $\mathcal{C}$ yields a split coequaliser.

Proof. Define $s : T \to T^2$ and $t : T^2 \to T^3$ by $s = \eta T$ and $t = \eta T^2$, and check the identities. ∎

A.2. Lemma. *With the notation of Lemma A.1, let $\mathcal{C}^T$ be the category of T-algebras [17, VI.2]. Then*
a) The forgetful functor $U : \mathcal{C}^T \to \mathcal{C}$ is faithful, conservative and reflects equalisers. In particular, if $\mathcal{C}$ has equalisers then so has $\mathcal{C}^T$.
b) If T preserves coequalisers, then U reflects coequalisers; hence $\mathcal{C}^T$ has coequalisers if $\mathcal{C}$ does.
c) If $\mathcal{C}$ and T are additive, $\mathcal{C}^T$ is additive.
d) If $\mathcal{C}$ is abelian and T is right exact, then $\mathcal{C}^T$ is abelian and U is exact.

Proof. a) The first two properties are obvious. For the third, let $(a;b) : (C_1,\varphi_1) \rightrightarrows (C_2,\varphi_2)$ be two parallel arrows in $\mathcal{C}^T$, and suppose that (Ua,Ub) has an equaliser $c : C \to C_1$. The composition

$$TC \xrightarrow{Tc} TC_1 \xrightarrow{\varphi_1} C_1$$

is equalised by Ua and Ub, hence factors uniquely through C; one checks that the resulting morphism $\varphi : TC \to C$ defines a T-algebra, and then that this T-algebra is an equaliser.

b) For (a,b) as in a), suppose that (Ua,Ub) has a coequaliser $d : C_2 \to D$. By hypothesis, Td is a coequaliser of (TUa,TUb), hence $d\varphi_2$ factors uniquely through a $\psi : TD \to D$. One sees that this is a T-algebra by observing that T^2 also respects coequalisers, and then that it is a coequaliser.

c) is easy and left to the reader. For d), the characterisation of an abelian category by the isomorphism of coimages onto images yields that $\mathcal{C}^T$ is abelian via a), b) and c). Since U is a right adjoint, it is left exact, and it remains to show that it preserves epimorphisms, which follows from b) by viewing 0 as the cokernel of an epimorphism. ■

A.3. Remark. *If the conclusions of d) hold, then conversely T (assumed to be additive) is right exact as the composition of U and its (right exact) left adjoint.*

A.4. Definition. *Let (T',η',μ') be another monad in $\mathcal{C}$.*
a) Let $u,v : T \Rightarrow T'$ be two natural transformations. We write

$$u \bullet v : T^2 \to T'^2$$

for either of the compositions $T^2 \xrightarrow{Tv} TT' \xrightarrow{uT'} T'^2$, $T^2 \xrightarrow{uT} T'T \xrightarrow{T'v} T'^2$.
b) A morphism *from T to T' is a natural transformation $u : T \Rightarrow T'$ such that*

(i) $u\eta = \eta'$;

(ii) $u\mu = \mu'(u \bullet u)$.

A.5. Proposition. *Let $u : T \to T'$ be a morphism of monads as in Definition A.4 b). Then there is a canonical functor $u^* : \mathcal{C}^{T'} \to \mathcal{C}^T$ of "restriction of scalars". If $\mathcal{C}$ has coequalisers and T' is right exact,, u^* has a left adjoint $u_!$ ("extension of scalars").*

Proof. If (C,ψ) is a T'-algebra, then $(C,\psi\circ u_C)$ is a T-algebra. This defines u^*. Suppose now that $\mathcal{C}$ has coequalisers. For $(C,\varphi) \in \mathcal{C}^{T'}$, let D is the coequaliser of $(T'(\varphi),\mu'_C \circ T'(u_C)) : T'TC \rightrightarrows T'C$. If $\pi : T'C \to D$ is the associated morphsm, $\pi \circ \mu'_C$ equalises $({T'}^2(\varphi), T'\mu'_C \circ {T'}^2(u_C))$ because the diagram

$$\begin{array}{ccccc}
{T'}^2TC & \overset{{T'}^2(\varphi)}{\underset{T'(\mu'_C)\circ {T'}^2(u_C)}{\rightrightarrows}} & {T'}^2C & \xrightarrow{T'(\pi)} & T'D \\
\Big\downarrow{\scriptstyle \mu'_{TC}} & & \Big\downarrow{\scriptstyle \mu'_C} & & \\
T'TC & \overset{T'(\varphi)}{\underset{\mu'_C\circ T'(u_C)}{\rightrightarrows}} & T'C & \xrightarrow{\pi} & D
\end{array}$$

commutes thanks to the associativity axiom for μ' and its naturality. The right exactness assumption on T' implies that the top row is a coequaliser, hence the diagram can be completed by a unique map $\psi : T'D \to D$, that one checks to be a T'-algebra morphism. The composition

$$C \xrightarrow{\eta'_C} T'C \xrightarrow{\pi} D$$

defines a morphism of T-algebras $(C, \varphi) \to u^*(D, \psi)$ and one checks that it is universal. ∎

A.6. Lemma. *Let $\mathcal{B}$ be a monoidal category, and let (R, η, μ) be a monoid in $\mathcal{B}$ [17, VII.3]. Then $TX = R \otimes X$ defines a monad in $\mathcal{B}$ provided with a natural isomorphism $TX \otimes Y \xrightarrow{\sim} T(X \otimes Y)$. Conversely, any monad provided with such a natural isomorphism is of this form.*

Proof. The first claim is easy to check by comparing the axioms of a monad and a monoid. For the converse, let $R = T\mathbf{1}$. Then, for any $X \in \mathcal{B}$, one has $TX \xrightarrow{\sim} T(\mathbf{1} \otimes X) \xrightarrow{\sim} R \otimes X$. ∎

A.7. Proposition. *In the situation of Lemma A.6, suppose $\mathcal{B}$ symmetric and R commutative (*i.e.*, $\mu \circ \sigma = \mu$ where σ is the switch of $R \otimes R$).*
a) (cf. [5, Prop. 4.1.10]). Let $\mathcal{B}^R = {}_R\mathbf{Lact}$ be the category of left actions by R [17, VII.4], and let $m_R : \mathcal{B} \to \mathcal{B}^R$, $X \mapsto (R \otimes X, \mu \otimes 1_X)$ be the left adjoint to the forgetful functor U (ibid.). Suppose also that $\mathcal{B}$ has coequalisers and that $- \otimes R$ is right exact (e.g., that $\otimes$ itself is right exact). Then there is a unique symmetric monoidal structure on $\mathcal{B}^R$ such that m_R is a strong $\otimes$-functor.
b) The functor U reflects dualisability. In particular, if $\mathcal{B}$ is rigid, so is $\mathcal{B}^R$.
c) (cf. loc. cit., Rem. 4.1.11). Let S be a second commutative monoid in $\mathcal{B}$, and let $\varphi : R \to S$ be a homomorphism of monoids. Then there is a unique $\otimes$-functor $\varphi_* : \mathcal{B}^R \to \mathcal{B}^S$ such that $m_S = \varphi_* \circ m_R$.

Proof. a) *Existence.* By the hypotheses, to a left action ν of R on $X \in \mathcal{B}$ corresponds a right action given by

$$X \otimes R \xrightarrow{\sigma} R \otimes X \xrightarrow{\nu} X$$

where σ is the symmetry of $\mathcal{B}$, and conversely. We use this remark to switch sides without mention.

For $(X, \nu_X), (Y, \nu_Y) \in \mathcal{B}^R$, define

$$X \otimes_R Y = \mathrm{Coker}(X \otimes R \otimes Y \rightrightarrows X \otimes Y) \tag{A.1}$$

where Coker means coequaliser and the two maps are $\nu_X \otimes 1_Y, 1_X \otimes \nu_Y$. Define $\nu : X \otimes R \otimes Y \to X \otimes_R Y$ via either of these two maps; their associativity shows that ν factors through a morphism $\nu_{X \otimes_R Y} : R \otimes X \otimes_R Y \to X \otimes_R Y$. One checks easily that $(X \otimes_R Y, \nu_{X \otimes_R Y}) \in \mathcal{B}^R$ and that the axioms of a symmetric monoidal structure, with unit R, are satisfied.

Let $X_0, Y_0 \in \mathcal{B}$. We must identify $R \otimes X_0 \otimes Y_0$ with the coequaliser of

$$(R \otimes X_0) \otimes R \otimes (R \otimes Y_0) \rightrightarrows (R \otimes X_0) \otimes (R \otimes Y_0)$$

where, modulo the symmetric constraints, the two morphisms are respectively induced by $\mu \otimes 1_{X_0}$ and $\mu \otimes 1_{Y_0}$. By Lemmas A.6 and A.1, this is true when $X_0 = Y_0 = \mathbf{1}$, and then it is even a split coequaliser. This coequaliser remains a split coequaliser after tensoring it with $X_0 \otimes Y_0$.

Uniqueness. Let $\bullet$ be another solution. Let $(X, \nu_X), (Y, \nu_Y)$ be as above. By adjunction, we have a canonical morphism $X \otimes Y = U(X, \nu_X) \otimes U(Y, \nu_Y) \to U(X \bullet Y, \nu_{X \bullet Y}) =: X \bullet Y$; since R must be the unit of $\bullet$, this morphism must equalise the two morphisms of (A.1), hence induce a morphism $\theta : (X \otimes_R Y, \nu_{X \otimes_R Y}) \to (X, \nu_X) \bullet (Y, \nu_Y)$, which must be an isomorphism when (X, ν_X) and (Y, ν_Y) are in the image of m_R. In general, the counits $m_R U(X, \nu_X) \to (X, \nu_X)$ and $m_R U(Y, \nu_Y) \to (Y, \nu_Y)$ become split epis after applying U; therefore, $U(\theta)$ is an isomorphism and so is θ since U is conservative.

b) Let $(X, \nu_X) \in \mathcal{B}^R$, where X has the dual X^*. Define ν_{X^*} as the composition

$$R \otimes X^* \xrightarrow{1\otimes\eta} R \otimes X^* \otimes X \otimes X^* \xrightarrow{1\otimes\sigma\otimes 1} R \otimes X \otimes X^* \otimes X^*$$
$$\xrightarrow{\nu_X \otimes 1} X \otimes X^* \otimes X^* \xrightarrow{\sigma\otimes 1} X^* \otimes X \otimes X^* \xrightarrow{\varepsilon\otimes 1} X^*$$

where η, ε are the unit and counit of the duality structure for (X, X^*) and σ is the symmetry. One verifies that this makes (X^*, ν_{X^*}) dual to (X, ν_X).

c) By a), we may view S as a commutative monoid in $\mathcal{C} = \mathcal{B}^R$ via φ, and get a strong $\otimes$-structure on $m_S(\mathcal{C}) : \mathcal{C} \to \mathcal{C}^S$. It remains to observe that $\mathcal{C}^S = \mathcal{B}^S$. ∎

This construction has a universal property [5, Prop. 5.3.1][5]:

A.8. Corollary. *In the situation of Proposition A.7, let $\mathcal{C}$ be another $\otimes$-category with coequalisers. Then any strong $\otimes$-functor from $F : \mathcal{B} \to \mathcal{C}$, provided with an algebra homomorphism $\beta : F(R) \to \mathbf{1}_{\mathcal{C}}$, induces a unique strong $\otimes$-functor $\tilde{F} : \mathcal{B}^R \to \mathcal{C}$ provided with a natural $\otimes$-isomorphism $F \xrightarrow{\sim} \tilde{F} \circ m_R$, and conversely.*

Proof. Apply Proposition A.7 c) to $(\mathcal{B}, R, S) \equiv (\mathcal{C}, F(R), \mathbf{1}_{\mathcal{C}})$. In the other direction, the counit of the adjunction (m_R, U) yields a morphism

$$m_R(R) = m_R U(\mathbf{1}_{\mathcal{B}^R}) \to \mathbf{1}_{\mathcal{B}^R}$$

to which we apply $\tilde{F}$. ∎

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CNRS, Sorbonne Université and Université Paris Cité, IMJ-PRG
Case 247
4 place Jussieu
75252 Paris Cedex 05
France
Email: bruno.kahn@imj-prg.fr

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