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ON EXTENSIVITY AND COEXTENSIVITY OF MORPHISMS

MICHAEL HOEFNAGEL AND EMMA THEART

Abstract. Extensivity of a category [3] may be described as a property of coproducts—namely, that they are disjoint and universal. An alternative viewpoint is that it is a property of morphisms. This paper explores this point of view through a natural notion of extensive and coextensive morphism. Through these notions, topics in universal algebra, such as the strict refinement and Fraser-Horn properties, take a categorical form and thereby enjoy the benefits of categorical generalisation. On the other hand, the universal algebraic theory surrounding these topics leads to categorical results. One such result we establish in this paper is that a Barr-exact category $\mathbb{C}$ is coextensive if and only if every split monomorphism in $\mathbb{C}$ is coextensive.

1. Introduction

An *extensive category* [3] is, informally speaking, one in which finite coproducts (sums) exist and behave in a manner similar to disjoint unions of sets. Formally, a category $\mathbb{C}$ with finite coproducts is extensive if the canonical functor

$$(\mathbb{C} \downarrow X_1) \times (\mathbb{C} \downarrow X_2) \xrightarrow{+} (\mathbb{C} \downarrow (X_1 + X_2))$$

is an equivalence for any objects X_1 and X_2. Equivalently, extensivity may be formulated internally: a category $\mathbb{C}$ with finite coproducts is extensive if $\mathbb{C}$ admits all pullbacks along coproduct injections, and for any diagram

$$\begin{array}{ccccc} A_1 & \longrightarrow & A & \longleftarrow & A_2 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ X_1 & \longrightarrow & X & \longleftarrow & X_2 \end{array} \tag{1}$$

where the bottom row is a coproduct, the top row is a coproduct if and only if the squares are pullbacks (see Proposition 2.2 in [3]). These two formulations suggest two distinct points of view on the nature of extensivity: the first emphasises it as a property of coproducts, while the second as a property of morphisms. This paper focuses on the second viewpoint and, to that end, defines a morphism $f : A \to X$ in a category $\mathbb{C}$ to be *extensive* if every coproduct injection into X admits a pullback along f, and f satisfies

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the diagrammatic property described above for any diagram such as (1). A category with finite coproducts is then extensive if and only if every morphism is extensive.

The prototypical example of an extensive category is the category Set of sets; yet the category of pointed sets Set_* is not extensive. For instance, if $f\colon A \to X$ is an extensive morphism in a pointed category, the left-hand square in

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longleftarrow & A \\ \downarrow & & \downarrow f & & \downarrow \\ 0 & \longrightarrow & X & \longleftarrow & X \end{array}$$

being a pullback forces $\ker(f) = 0$, since both the top and bottom rows are coproducts. This illustrates that not every morphism of pointed sets is extensive. In fact, in the category of pointed sets, the morphisms f with $\ker(f) = 0$ are precisely the extensive morphisms. To see this, note that the coproduct $X_1 + X_2$ of two pointed sets X_1 and X_2 is their wedge sum, i.e., the usual coproduct followed by the quotient identifying their base points. Then by the dual of Corollary 3.12 it suffices to show that pulling back any morphism $f\colon A \to X_1 + X_2$ in Set_* with trivial kernel along the inclusions of $X_1 \to X_1 + X_2 \leftarrow X_2$ gives a wedge sum (which is (E1) of Definition 2.1), which follows straightforwardly since the preimages of X_1 and X_2 under f only overlap at the basepoint.

Dually, we say that a morphism f in a category $\mathbb{C}$ is *coextensive* if it is extensive in the dual category $\mathbb{C}^{\mathrm{op}}$. The category Grp of groups is not coextensive; however, every product projection $p\colon G \to X$ in Grp where G is a *centerless* (or *perfect*) group is coextensive. As will be shown in Section 3.14, this is a categorical formulation of the fact that centerless (or perfect) groups have the *strict refinement property* in the sense of [4]. In the category Lat of lattices (and lattice homomorphisms), every surjective homomorphism is coextensive, which may be seen as a categorical formulation of the fact that lattices have the *Fraser-Horn* property [6].

There is an interesting interplay between the universal algebraic and the categorical. On the one hand, the theory of coextensive morphisms allows for generalisations of universal algebraic topics to the categorical level. Expressing these algebraic concepts categorically yields natural consequences motivated from the theory of (co)extensive categories. On the other hand, the universal algebraic theory surrounding these algebraic properties suggests corresponding categorical results. For instance, Proposition 4.13 shows that every Barr-exact [1] category $\mathbb{C}$ is coextensive if and only if every split monomorphism is coextensive.

2. Extensive morphisms

For any category $\mathbb{C}$, we will be concerned with two conditions on a morphism $f\colon A \to X$ in $\mathbb{C}$:

(E1) f admits pullbacks along the injections of any coproduct diagram

$$X_1 \to X \leftarrow X_2$$

and the resulting two pullbacks

$$\begin{array}{ccccc} A_1 & \longrightarrow & A & \longleftarrow & A_2 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ X_1 & \longrightarrow & X & \longleftarrow & X_2 \end{array}$$

form a coproduct diagram $A_1 \to A \leftarrow A_2$.

(E2) for any commutative diagram

$$\begin{array}{ccccc} A_1 & \longrightarrow & A & \longleftarrow & A_2 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ X_1 & \longrightarrow & X & \longleftarrow & X_2 \end{array}$$

in $\mathbb{C}$ where the top and bottom rows are coproduct diagrams, both squares are pullbacks.

Then we may restate the main definition of this paper.

2.1. Definition. *A morphism $f\colon A \to X$ in a category $\mathbb{C}$ said to be* extensive *in $\mathbb{C}$ if it satisfies* (E1) *and* (E2)*.*

2.2. Proposition. *In any category $\mathbb{C}$, the composite of two extensive morphisms is extensive.*

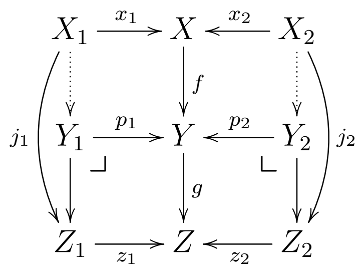
Proof. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be extensive morphisms in $\mathbb{C}$. Suppose we have a coproduct diagram $Z_1 \xrightarrow{z_1} Z \xleftarrow{z_2} Z_2$. By extensivity of g, g admits pullbacks along z_1 and z_2, and these pullback squares together form a coproduct diagram $Y_1 \xrightarrow{y_1} Y \xleftarrow{y_2} Y_2$ in the top row. Since f is extensive it, in turn, admits pullbacks along y_1 and y_2. These pullback diagrams are shown in (2).

$$\begin{array}{ccccc} X_1 & \xrightarrow{x_1} & X & \xleftarrow{x_2} & X_2 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} & \llcorner & \downarrow \\ Y_1 & \xrightarrow{y_1} & Y & \xleftarrow{y_2} & Y_2 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle g} & \llcorner & \downarrow \\ Z_1 & \xrightarrow[z_1]{} & Z & \xleftarrow[z_2]{} & Z_2 \end{array} \tag{2}$$

By pullback pasting, it follows that these composite squares in (2) form pullback diagrams of gf along z_1 and z_2, where the top row is a coproduct. Thus gf satisfies (E1). Next, suppose we have a commutative diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{x_1} & X & \xleftarrow{x_2} & X_2 \\ {\scriptstyle j_1}\downarrow & & {\scriptstyle gf}\downarrow & & \downarrow{\scriptstyle j_2} \\ Z_1 & \xrightarrow[z_1]{} & Z & \xleftarrow[z_2]{} & Z_2 \end{array} \tag{3}$$

in $\mathbb{C}$ where the top and bottom rows are coproduct diagrams. We form the pullbacks of g along z_1 and z_2, and these pullbacks induce the dotted morphisms in (4).



(4)

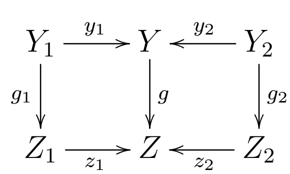
Since the top and middle rows of (4) are coproduct diagrams, it follows by extensivity of f that the top two squares are pullbacks. By pasting the vertical pullbacks together we have that (3) consists of a pair of pullback squares. Hence, gf satisfies (E2). ∎

Given any morphism $\iota\colon A \to X$, we say that ι is a *coproduct inclusion* if there exists a morphism $\iota'\colon B \to X$ such that ι and ι' form a coproduct diagram. The morphism ι' is then called a *complementary coproduct inclusion* of ι. Dually, a morphism $\pi\colon X \to A$ is called a *product projection* if there is a morphism $\pi'\colon X \to B$ such that π and π' form a product diagram, and the morphism π' is called a *complementary product projection* of π.

2.3. Corollary. *If $\mathbb{C}$ is a category with binary coproducts, then $\mathbb{C}$ is extensive if and only if every split epimorphism and every coproduct inclusion is extensive.*

Proof. This follows from the fact that every morphism $f\colon X \to Y$ in such a category $\mathbb{C}$ factors as $X \xrightarrow[\iota_1]{} X+Y \xrightarrow[\langle f,1_Y\rangle]{} Y$ where $\langle f, 1_Y\rangle$ is the morphism induced by 1_Y and f. ∎

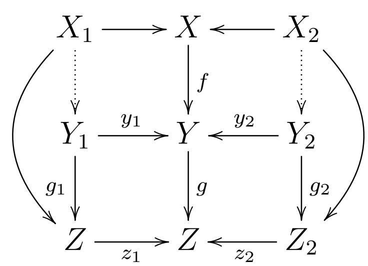
2.4. Lemma. *For any morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in any category $\mathbb{C}$, if gf is extensive and for each coproduct diagram $Y_1 \xrightarrow{y_1} Y \xleftarrow{y_2} Y_2$ there exists a pair of pullback squares*



(5)

where the top and bottom rows are coproduct diagrams, then f is extensive.

Proof. We show that f satisfies (E1). Let $Y_1 \xrightarrow{y_1} Y \xleftarrow{y_2} Y_2$ be any coproduct diagram. Then, the squares in (5) exist by assumption, and therefore, because gf is extensive, we may pull gf back along z_1 and z_2 to form the diagram of solid arrows below



where the dotted arrows are the morphisms induced by these pullbacks, and the top row is a coproduct diagram by extensivity of gf. It then follows that, since the outer and bottom squares are pullbacks, the top two squares are also pullbacks. Hence, f satisfies (E1).

To show that f satisfies (E2), suppose we have a commutative diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{x_1} & X & \xleftarrow{x_2} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow[y_1]{} & Y & \xleftarrow[y_2]{} & Y_2 \end{array}$$

where the top and bottom rows are coproducts. Then, since gf is extensive, it follows that the outer squares in

$$\begin{array}{ccccc} X_1 & \xrightarrow{x_1} & X & \xleftarrow{x_2} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow{y_1} & Y & \xleftarrow{y_2} & Y_2 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_2} \\ Z & \xrightarrow[z_1]{} & Z & \xleftarrow[z_2]{} & Z_2 \end{array}$$

are pullbacks. Hence, so are the top two squares. Therefore, f is extensive. ■

2.5. Lemma. *In any category* $\mathbb{C}$ *with binary coproducts, if* $\iota\colon Y \to Z$ *is a coproduct inclusion which satisfies* (E2)*, then for any coproduct diagram* $Y_1 \xrightarrow{y_1} Y \xleftarrow{y_2} Y_2$*, there exists a pair of pullback squares*

$$\begin{array}{ccccc} Y_1 & \xrightarrow{y_1} & Y & \xleftarrow{y_2} & Y_2 \\ {\scriptstyle \iota_1}\downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota_2} \\ Z_1 & \xrightarrow[z_1]{} & Z & \xleftarrow[z_2]{} & Z_2 \end{array}$$

where the top and bottom rows are coproduct diagrams.

Proof. Since ι is a coproduct inclusion, it has a complementary inclusion $\iota' \colon Y' \to Z$ making $Y \xrightarrow{\iota} Z \xleftarrow{\iota'} Y'$ a coproduct diagram. We may then form the coproduct diagram

$$Y_2 \xrightarrow{j_1} Y_2 + Y' \xleftarrow{j_2} Y' .$$

Consider the commutative diagram:

$$\begin{array}{ccccc} Y_1 & \xrightarrow{y_1} & Y & \xleftarrow{y_2} & Y_2 \\ {\scriptstyle 1_{Y_1}}\downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle j_1} \\ Y_1 & \xrightarrow[\iota y_1]{} & Z & \xleftarrow[\langle \iota y_2, \iota' \rangle]{} & Y_2 + Y' \end{array} \tag{6}$$

It is routine to verify that the bottom row is a coproduct, and hence since ι satisfies (E2), it now follows that both squares in (6) are pullbacks. ■

From the above two lemmas, we get the following proposition.

2.6. Proposition. *Let $\mathbb{C}$ be a category with binary coproducts and $f \colon A \to B$ any morphism in $\mathbb{C}$. Let $\iota \colon B \to X$ be any coproduct inclusion in $\mathbb{C}$. Then, if ιf is extensive and ι satisfies* (E2) *it follows that f is extensive.*

2.7. Corollary. *Let $\mathbb{C}$ be any category with binary coproducts and $\iota \colon A \to X$ any coproduct inclusion in $\mathbb{C}$. If every coproduct inclusion of X is extensive, then every coproduct inclusion of A is extensive.*

2.8. Categories where coproduct inclusions are extensive.

2.9. Definition. *Let $\mathbb{C}$ be a category with finite coproducts which admits all pullbacks along coproduct inclusions. Coproducts are said to be* disjoint *in $\mathbb{C}$ if for any coproduct diagram, $X_1 \xrightarrow{\iota_1} X \xleftarrow{\iota_2} X_2$, the squares*

$$\begin{array}{ccccc} X_1 & \xrightarrow{1_{X_1}} & X_1 & \longleftarrow & 0 \\ {\scriptstyle 1_{X_1}}\downarrow & & \downarrow{\scriptstyle \iota_1} & & \downarrow \\ X_1 & \xrightarrow[\iota_1]{} & X & \xleftarrow[\iota_2]{} & X_2 \end{array}$$

are pullbacks. Note, the left-hand square being a pullback implies that coproduct inclusions are monomorphisms.

2.10. Proposition. *In a category $\mathbb{C}$ with finite coproducts where every coproduct inclusion satisfies* (E2)*, coproducts are disjoint and every coproduct inclusion is a regular monomorphism.*

Proof. It follows immediately that coproducts are disjoint. Consider a coproduct inclusion $X \xrightarrow{\iota} A$, with complementary inclusion $Y \xrightarrow{\iota'} A$. Form also the coproduct diagram $A \xrightarrow{i} A+Y \xleftarrow{i'} Y$. Then, the rows in the following diagram are coproducts

$$\begin{array}{ccccc} X & \xrightarrow{\iota} & A & \xleftarrow{\iota'} & Y \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota+1_Y} & & \downarrow{\scriptstyle 1_Y} \\ A & \xrightarrow[i]{} & A+Y & \xleftarrow[i']{} & Y \end{array}$$

so that both squares are pullbacks. Hence, ι is an equaliser of i and $\iota + 1_Y$. ■

2.11. Proposition. [Coproduct complements are unique] *Let $\mathbb{C}$ be a category with an initial object with disjoint coproducts. Then coproduct complements are unique, i.e., if*

$$X_1 \xrightarrow{\iota_1} X \xleftarrow{\iota_2} X_2, \qquad X_1 \xrightarrow{\iota_1} X \xleftarrow{\iota_2'} X_2' \tag{7}$$

are both coproduct diagrams in $\mathbb{C}$, then there is an isomorphism $\sigma\colon X_2 \to X_2'$ such that $\iota_2'\sigma = \iota_2$.

Proof. Suppose we have two coproduct diagrams as in (7). Then, each right-hand square in

$$\begin{array}{ccccc} 0 & \longrightarrow & X_2 & \xleftarrow{1_{X_2}} & X_2 \\ \downarrow & & \downarrow{\scriptstyle\iota_2} & & \downarrow{\scriptstyle\sigma} \\ X_1 & \xrightarrow[\iota_1]{} & X & \xleftarrow[\iota_2']{} & X_2' \end{array} \qquad \begin{array}{ccccc} 0 & \longrightarrow & X_2' & \longleftarrow & X_2' \\ \downarrow & & \downarrow{\scriptstyle\iota_2'} & & \downarrow \\ X_1 & \xrightarrow[\iota_1]{} & X & \xleftarrow[\iota_2]{} & X_2 \end{array}$$

is a pullback. Thus, $\sigma\colon X_2 \to X_2'$ is an isomorphism such that $\iota_2 = \iota_2'\sigma$. ■

2.12. Proposition. *In a category $\mathbb{C}$ with an initial object and disjoint coproducts, if all coproduct inclusions satisfy* (E1) *then any morphism satisfying* (E1) *is extensive.*

Proof. It suffices to show that every identity morphism in $\mathbb{C}$ satisfies (E2) by Corollary 3.5. Suppose that we are given any diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{a_1} & A & \xleftarrow{a_2} & A_2 \\ {\scriptstyle i_1}\downarrow & & {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle i_2} \\ B_1 & \xrightarrow[b_1]{} & A & \xleftarrow[b_2]{} & B_2 \end{array} \tag{8}$$

where the top and bottom rows are coproduct diagrams. Note that since a_1 and a_2 are monomorphisms, it follows that i_1 and i_2 are monomorphisms. Consider the commutative diagram below.

$$\begin{array}{ccccccc} A_1 & \xrightarrow{i_1} & B_1 & \longleftarrow & 0 & \longleftarrow & 0 \\ {\scriptstyle 1_{A_1}}\downarrow & & \downarrow{\scriptstyle b_1} & & \downarrow & & \downarrow \\ A_1 & \xrightarrow[a_1]{} & A & \xleftarrow[b_2]{} & B_2 & \xleftarrow[i_2]{} & A_2 \end{array}$$

The left square is a pullback (since b_1 is a monomorphism). Further, the right square is a pullback since i_2 is a monomorphism, and the middle square is a pullback since coproducts are disjoint. Therefore, by Proposition 2.11, i_1 is an isomorphism. We can likewise show that i_2 is an isomorphism, so that the two squares in (8) are pullbacks. ∎

2.13. Corollary. *The following are equivalent for a category $\mathbb{C}$ with an initial object.*

1. *Every coproduct inclusion in $\mathbb{C}$ is extensive.*
2. *Coproducts are disjoint, and every coproduct inclusion satisfies* (E1)*.*

2.14. Proposition. *Let $\mathbb{C}$ be a category with binary coproducts where every coproduct inclusion is extensive. Then the pullback of an extensive morphism along a coproduct inclusion exists and is extensive.*

Proof. Let $f\colon A \to B$ be an extensive morphism in $\mathbb{C}$, then f admits pullbacks along coproduct inclusions by virtue of (E1). Now consider the diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{a_1} & A & \xleftarrow{a_2} & A_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow[b_1]{} & B & \xleftarrow[b_2]{} & B_2 \end{array}$$

where the squares are pullbacks. Since f is extensive, the top row is a coproduct and hence a_1 is extensive. Then by Proposition 2.2 it follows that fa_1 is extensive, and hence $b_1 f_1$ is extensive. Since b_1 is a coproduct inclusion, by Proposition 2.6 it follows that f_1 is extensive. ∎

3. Results on coextensive morphisms

From this point forward, we turn our attention exclusively to *coextensive morphisms.* A morphism f in a category $\mathbb{C}$ is called *coextensive* if it is extensive in the dual category $\mathbb{C}^{\mathrm{op}}$. In what follows we will refer to the category-theoretic duals of the conditions (E1) and (E2), as (C1) and (C2), respectively. Thus, the morphism f is coextensive if and only if f satisfies both of the following:

(C1) f admits pushouts along the projections of any product diagram

$$A_1 \leftarrow A \rightarrow A_2$$

and the resulting two pushouts

$$\begin{array}{ccccc} A_1 & \longleftarrow & A & \longrightarrow & A_2 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ X_1 & \longleftarrow & X & \longrightarrow & X_2 \end{array}$$

form a product diagram $X_1 \leftarrow X \rightarrow X_2$.

(C2) for any commutative diagram

$$\begin{array}{ccccc} A_1 & \longleftarrow & A & \longrightarrow & A_2 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ X_1 & \longleftarrow & X & \longrightarrow & X_2 \end{array}$$

in $\mathbb{C}$ where the top and bottom rows are product diagrams, both squares are pushouts.

3.1. COEXTENSIVITY OF IDENTITY MORPHISMS. If $f\colon A \to X$ is any isomorphism in $\mathbb{C}$ then for any morphism $g\colon A \to B$, the square

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1_B} \\ X & \xrightarrow[gf^{-1}]{} & B \end{array}$$

is a pushout in $\mathbb{C}$, and hence the proposition below is immediate.

3.2. PROPOSITION. *In any category $\mathbb{C}$ every isomorphism satisfies* (C1).

If $f\colon A \to X$ and $g\colon B \to Y$ are isomorphisms in $\mathbb{C}$, then any commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \longrightarrow & Y \end{array}$$

is a pushout, leading to the following proposition.

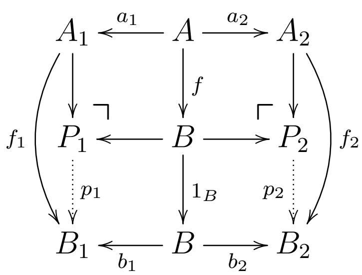
3.3. PROPOSITION. *In a category $\mathbb{C}$ with binary products every isomorphism satisfies* (C2) *if and only if the following condition holds: for any pair of morphisms f_1 and f_2 in $\mathbb{C}$, if their product $f_1 \times f_2$ is an isomorphism, then f_1 and f_2 are isomorphisms.*

3.4. PROPOSITION. *For any morphism $f\colon A \to B$ in $\mathbb{C}$, if f satisfies* (C1) *and 1_B satisfies* (C2) *then f is coextensive.*

PROOF. Consider the following commutative diagram where the rows are products.

$$\begin{array}{ccccc} A_1 & \xleftarrow{a_1} & A & \xrightarrow{a_2} & A_2 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_2} \\ B_1 & \xleftarrow[b_1]{} & B & \xrightarrow[b_2]{} & B_2 \end{array}$$

Since f satisfies (C1), the pushouts of f along a_1 and a_2 exist. Suppose that the top two squares in the following diagram are these pushouts of f



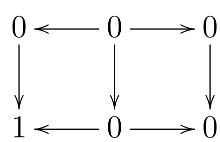
so that the middle row is a product diagram. Since the top two squares are pushouts, there exist morphisms p_1 and p_2 such that the diagram is commutative. Since the middle and bottom rows are products and 1_B satisfies (C2), the bottom two squares are pushouts. Pasting pushouts together, we have that the squares in the initial diagram are pushouts. So f satisfies (C2), and is therefore coextensive. ■

Applying the above results we obtain the following.

3.5. Corollary. *If every identity morphism in* $\mathbb{C}$ *is coextensive, a morphism* $f\colon X \to Y$ *is coextensive if and only if it satisfies* (C1).

3.6. Corollary. *In any category* $\mathbb{C}$*, every isomorphism is coextensive if and only if every identity morphism in* $\mathbb{C}$ *coextensive.*

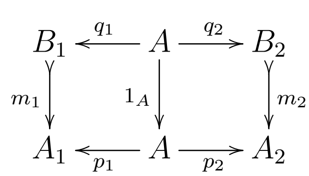
Not every category has (co)extensive identity morphisms. To illustrate this, consider the partially ordered set $\{0, 1\}$ viewed as category. Then the left-hand square in the diagram



is not a pushout. Consequently, the identity morphism on 0 does not satisfy (C2).

Recall that a morphism $e\colon A \to B$ in $\mathbb{C}$ is called an *extremal epimorphism* if for any composable pair of morphisms i and m in $\mathbb{C}$, if $e = mi$ and m is a monomorphism, then m is an isomorphism. Note that if $\mathbb{C}$ has equalisers, then every extremal epimorphism is an epimorphism. The proof of the lemma below is routine and left to the reader.

3.7. Lemma. *Given any commutative diagram*



in $\mathbb{C}$ *where* m_1 *and* m_2 *are monomorphisms, if the bottom row is a product diagram, then so is the top row.*

3.8. Lemma. *Let $\mathbb{C}$ be a finitely complete category where every product projection in $\mathbb{C}$ is an epimorphism. Then, for any two morphisms f_1 and f_2 in $\mathbb{C}$, if their product $f_1 \times f_2$ is a monomorphism then f_1 and f_2 are monomorphisms.*

Proof. Let $f_1 \colon A_1 \to B_1$ and $f_2 \colon A_2 \to B_2$ be any two morphisms in $\mathbb{C}$ such that $f_1 \times f_2$ is a monomorphism. Form the kernel pairs of $f_1 \times f_2$, f_1 and f_2 as below, with α_1 and α_2 induced by K_1 and K_2 respectively.

$$\begin{array}{ccccc}
K_1 & \xleftarrow{\alpha_1} & K & \xrightarrow{\alpha_2} & K_2 \\
{\scriptstyle k_1}\downdownarrows{\scriptstyle k_2} & & {\scriptstyle \ell_1}\downdownarrows{\scriptstyle \ell_2} & & {\scriptstyle m_1}\downdownarrows{\scriptstyle m_2} \\
A_1 & \xleftarrow{\pi_1} & A_1 \times A_2 & \xrightarrow{\pi_2} & A_2 \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_1 \times f_2} & & \downarrow{\scriptstyle f_2} \\
B_1 & \xleftarrow[\mu_1]{} & B_1 \times B_2 & \xrightarrow[\mu_2]{} & B_2
\end{array}$$

Since $f_1 \times f_2$ is monic, $\ell_1 = \ell_2$. Furthermore, the top row is a product since limits commute with limits. Then, from α_1 and α_2 being epimorphisms, we have $k_1 = k_2$ and $m_1 = m_2$ so that f_1 and f_2 are monomorphisms. ∎

3.9. Proposition. *For any object A in $\mathbb{C}$, if the identity morphism 1_A is coextensive, then every product projection of A is an extremal epimorphism. If $\mathbb{C}$ is finitely complete then the converse holds.*

Proof. Suppose that 1_A is coextensive, and that $A_1 \xleftarrow{p_1} A \xrightarrow{p_2} A_2$ is a product diagram. Let $m_1 q_1 = p_1$ be a factorisation of p_1 where m_1 is a monomorphism as in the diagram

$$\begin{array}{ccccc}
I_1 & \xleftarrow{q_1} & A & \xrightarrow{p_2} & A_2 \\
{\scriptstyle m_1}\downarrow & & {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle 1_{A_2}} \\
A_1 & \xleftarrow[p_1]{} & A & \xrightarrow[p_2]{} & A_2
\end{array}$$

By Lemma 3.7, it follows that the top row is a product diagram and hence that the squares are pushouts, so that m_1 is an isomorphism.

Conversely, suppose that $\mathbb{C}$ is finitely complete and that product projections of A are extremal epimorphims. Since all identity morphisms satisfy (C1) by Proposition 3.2, we prove that identity morphisms satisfy (C2). Consider any commutative diagram

$$\begin{array}{ccccc}
A_1 & \xleftarrow{p_1} & A & \xrightarrow{p_2} & A_2 \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle f_2} \\
B_1 & \xleftarrow[q_1]{} & A & \xrightarrow[q_2]{} & B_2
\end{array}$$

where the rows are product diagrams. By Lemma 3.8, it follows that f_1 and f_2 are monomorphisms. Since q_1 is an extremal epimorphism and $f_1 p_1 = q_1$, f_1 is an extremal epimorphism. Similarly, f_2 is an extremal epimorphism. Consequently, f_1 and f_2 are isomorphisms, so that both squares in the original diagram are pushouts. ∎

3.10. Corollary. *If $\mathbb{C}$ is finitely complete, the product projections of $\mathbb{C}$ are extremal epimorphisms if and only if every identity morphism in $\mathbb{C}$ is coextensive.*

3.11. Remark. For a variety of universal algebras $\mathcal{V}$ the condition that identity morphisms in $\mathcal{V}$ be coextensive, i.e., product projections in $\mathcal{V}$ be extremal, is equivalent to the requirement that the algebraic theory of $\mathcal{V}$ admit at least one constant. This is because in any variety the singleton algebra 1 is terminal, and the algebra of constants 0 is initial. Thus, the morphism $0 \to 1$ is extremal (surjective) if and only if $\mathcal{V}$ admits at least one constant.

3.12. Corollary. *If $\mathbb{C}$ is a pointed finitely complete category, then every morphism in $\mathbb{C}$ satisfying* (C1) *is coextensive.*

Proof. Every product projection in a pointed category $\mathbb{C}$ is a split epimorphism and therefore an extremal epimorphism. Thus, by Proposition 3.9, all identity morphisms in $\mathbb{C}$ are coextensive. Therefore, the result follows from Corollary 3.5. ■

Coextensivity of a category with finite products may be expressed in terms of the product functor $\times$. The following proposition provides a similar formulation for the coextensivity of identity morphisms. Recall that a functor $F\colon \mathbb{C} \to \mathbb{D}$ is *conservative* if it reflects isomorphisms, that is, for any morphism f in $\mathbb{C}$ if $F(f)$ is an isomorphism then f is an isomorphism.

3.13. Proposition. *Let $\mathbb{C}$ be a category with binary products. All identity morphisms in $\mathbb{C}$ are coextensive if and only if the functor*

$$\times : (\mathbb{C} \downarrow X_1) \times (\mathbb{C} \downarrow X_2) \longrightarrow (\mathbb{C} \downarrow (X_1 \times X_2))$$

is conservative for all objects X_1 and X_2 in $\mathbb{C}$.

Proof. This result can be derived from the following series of equivalent statements:

(1) The functor $\times$ is conservative for all objects X_1 and X_2;

(2) Each pair of morphisms f_1 and f_2 are isomorphisms whenever $f_1 \times f_2$ is an isomorphism;

(3) All isomorphisms satisfy (C2);

(4) All isomorphisms are coextensive;

(5) All identity morphisms are coextensive.

The equivalence (1) $\Leftrightarrow$ (2) follows readily, (2) $\Leftrightarrow$ (3) is given in Proposition 3.3, (3) $\Leftrightarrow$ (4) follows from Proposition 3.2, while (4) $\Leftrightarrow$ (5) is established in Corollary 3.6. ■

3.14. Coextensivity of product projections. The *strict refinement property* was initially introduced in [4], and has a straightforward generalisation to categories given by the following.

3.15. Definition. *An object X in a category $\mathbb{C}$ is said to have the (finite)* strict refinement property *if for any two (finite) product diagrams $(X \xrightarrow{a_i} A_i)_{i\in I}$ and $(X \xrightarrow{b_j} B_j)_{j\in J}$, there exist families of morphisms $(A_i \xrightarrow{\alpha_{i,j}} C_{i,j})_{i\in I, j\in J}$ and $(B_j \xrightarrow{\beta_{i,j}} C_{i,j})_{i\in I, j\in J}$ such that $\alpha_{i,j}a_i = \beta_{i,j}b_j$ and the diagrams $(A_i \xrightarrow{\alpha_{i,s}} C_{i,s})_{s\in J}$ and $(B_j \xrightarrow{\beta_{t,j}} C_{t,j})_{t\in I}$ are product diagrams for any $i \in I$ and $j \in J$. The category $\mathbb{C}$ satisfies the strict refinement property if every object in $\mathbb{C}$ does.*

The simplest non-trivial strict refinement for an object X is for binary product diagrams.

3.16. Definition. *An object X in a category $\mathbb{C}$ is said to satisfy the* binary strict refinement property *if given any two binary product diagrams*

$$A_1 \xleftarrow{a_1} X \xrightarrow{a_2} A_2, \quad B_1 \xleftarrow{b_1} X \xrightarrow{b_2} B_2,$$

there exists a commutative diagram

$$\begin{array}{ccccc} C_{1,1} & \xleftarrow{\alpha_{1,1}} & A_1 & \xrightarrow{\alpha_{1,2}} & C_{1,2} \\ {\scriptstyle\beta_{1,1}}\uparrow & & \uparrow{\scriptstyle a_1} & & \uparrow{\scriptstyle\beta_{1,2}} \\ B_1 & \xleftarrow{b_1} & X & \xrightarrow{b_2} & B_2 \\ {\scriptstyle\beta_{2,1}}\downarrow & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle\beta_{2,2}} \\ C_{2,1} & \xleftarrow[\alpha_{2,1}]{} & A_2 & \xrightarrow[\alpha_{2,2}]{} & C_{2,2} \end{array}$$

where each edge is a binary product diagram. The category $\mathbb{C}$ then satisfies the binary strict refinement property if every object in $\mathbb{C}$ does.

The proof of the lemma below (which appears as Proposition 2.4 in [9] and also Lemma 1.2 in [2]) is standard and left to the reader.

3.17. Lemma. *Given any reasonably commutative diagram*

$$\begin{array}{ccccc} C_1 & \underset{v_1}{\overset{u_1}{\rightrightarrows}} & X_1 & \xrightarrow{q_1} & Q_1 \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ C_2 & \underset{v_2}{\overset{u_2}{\rightrightarrows}} & X_2 & \xrightarrow[q_2]{} & Q_2 \end{array}$$

in any category, where the top row is a coequaliser diagram and e is an epimorphism, the right-hand square is a pushout if and only if the bottom row is a coequaliser.

Note that in every category with finite products, every product projection admits a kernel pair. Specifically, for every product diagram $X_1 \xleftarrow[p_1]{} X_1 \times X_2 \xrightarrow[p_2]{} X_2$, the morphism p_1 has kernel pair

$$X_1 \times (X_2 \times X_2) \underset{1_{X_1}\times\pi_2}{\overset{1_{X_1}\times\pi_1}{\rightrightarrows}} X_1 \times X_2$$

where π_1 and π_2 are the first and second projection morphisms of the product $X_2 \times X_2$ respectively.

3.18. PROPOSITION. *Let $\mathbb{C}$ be a category with finite products where every product projection in $\mathbb{C}$ is a regular epimorphism. Then binary products in $\mathbb{C}$ are co-disjoint, i.e., coproducts are disjoint in $\mathbb{C}^{\mathrm{op}}$.*

PROOF. For any two objects X, Y in $\mathbb{C}$ we may apply Lemma 3.17 to the diagram

$$\begin{array}{ccccc}
X \times (Y \times Y) & \underset{1_X\times\pi_1}{\overset{1_X\times\pi_2}{\rightrightarrows}} & X \times Y & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
Y \times Y & \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} & Y & \longrightarrow & 1
\end{array}$$

to get that the right square is a pushout. ∎

3.19. LEMMA. *Let $\mathbb{C}$ be a category with binary products, where each projection is a regular epimorphism. For any two morphisms f_1 and f_2, if $f_1 \times f_2$ is a monomorphism, then both f_1 and f_2 are monomorphisms. Consequently, every identity morphism in $\mathbb{C}$ is coextensive.*

PROOF. Let $f_i \colon X_i \to Y_i$ be any morphisms such that $f_1 \times f_2$ is monic and let $a, b : A \to X_1$ with $f_1 a = f_1 b$. Then

$$(f_1 \times f_2)(a \times 1_{X_2}) = (f_1 a) \times f_2 = (f_1 b) \times f_2 = (f_1 \times f_2)(b \times 1_{X_2}),$$

so $a \times 1_{X_2} = b \times 1_{X_2}$. Applying the projection $\pi_1 : X_1 \times X_2 \to X_1$ gives $a\pi = b\pi$ where π is the projection $\pi : A \times X_2 \to A$, and since π is a regular epimorphism, $a = b$. Thus f_1 is monic; similarly for f_2. Then to show that every identity morphism in $\mathbb{C}$ is coextensive, it suffices to show that if $f_1 \times f_2$ is an isomorphism, then both f_1 and f_2 are by Proposition 3.13. This follows easily, since $f_1 \times f_2$ being an isomorphism implies that the f_i is are extremal epimorphisms, which are isomorphisms since they are monomorphisms. ∎

3.20. PROPOSITION. *Let $\mathbb{C}$ be a category with finite products where every product projection is a regular epimorphism. Given any object X in $\mathbb{C}$, then every product projection of X is coextensive if and only if X satisfies the binary strict refinement property.*

PROOF. Suppose first that each product projection of X is coextensive. Let X have the following two product diagrams:

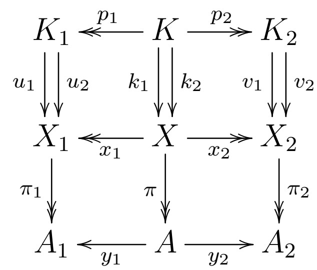
$$A_1 \xleftarrow{a_1} X \xrightarrow{a_2} A_2, \quad B_1 \xleftarrow{b_1} X \xrightarrow{b_2} B_2.$$

By coextensivity of these morphisms, we can form pushouts of each a_i along each b_j, obtaining the binary product diagrams along the edges of the outer square in the refinement diagram.

Conversely, suppose X satisfies the binary strict refinement property. To show that each product projection of X satisfies (C1), let $\pi\colon X \to A$ be any product projection, and consider a product diagram

$$X_1 \xleftarrow{x_1} X \xrightarrow{x_2} X_2.$$

Using the binary strict refinement property, we construct the bottom two squares in the diagram

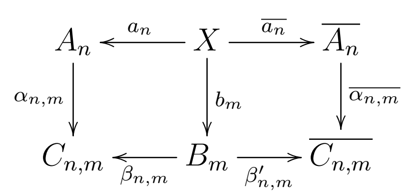


where π_1 and π_2 are product projections and the bottom row is a product diagram. The parallel pairs in the top squares are constructed by taking the kernel pairs of π_1, π, and π_2—which exist since these morphisms are product projections. Hence, each column in this diagram forms a coequaliser diagram. Furthermore, the top row is a product diagram (since the bottom row is), so that p_1 and p_2 are regular epimorphisms. Therefore, by Lemma 3.17, the bottom two squares are pushouts. Thus, π satisfies (C1). Every identity morphism in $\mathbb{C}$ is coextensive by Lemma 3.19, so that the result follows by Proposition 3.4. ∎

The following theorem is an adaptation taken from [11]; we include its proof for the sake of completeness.

3.21. THEOREM. *Let $\mathbb{C}$ be a category with (finite) products and let X be an object with coextensive product projections. Then, X has the (finite) strict refinement property.*

PROOF. Suppose that $(X \xrightarrow{a_i} A_i)_{i\in I}$ and $(X \xrightarrow{b_j} B_j)_{j\in J}$ are any two (finite) product diagrams for X. Let $\overline{A_n}$ be the product of the A_i's where $i \neq n$ and let $\overline{a_n}\colon X \to \overline{A_n}$ be the induced morphism $(a_i)_{i\neq n}$, and similarly let $\overline{B_m}$ be the product of the B_j's where $j \neq m$. For each $n \in I$ and $m \in J$ there is a diagram



where each square is a pushout, and the bottom row is a product diagram, since b_m is a product projection of X. In the diagram

$$\begin{array}{ccccc} A_n & \xleftarrow{\;a_n\;} & X & \xrightarrow{\;\overline{a_n}\;} & \overline{A_n} \\ \Big\downarrow{\scriptstyle (\alpha_{n,j})_{j\in J}} & & \Big\downarrow{\scriptstyle (b_j)_{j\in J}} & & \Big\downarrow{\scriptstyle (\overline{\alpha_{n,j}})_{j\in J}} \\ \prod_{j\in J} C_{n,j} & \xleftarrow[\prod_{j\in J}\beta_{n,j}]{} & \prod_{j\in J} B_j & \xrightarrow[\prod_{j\in J}\beta'_{n,j}]{} & \prod_{j\in J}\overline{C_{n,j}} \end{array}$$

the bottom row is a product diagram, and hence each square is a pushout. Since the central vertical morphism in the diagram is an isomorphism, it follows that the morphism $(\alpha_{n,j})_{j\in J}$ is an isomorphism, and we can similarly obtain $(\beta_{i,m})_{i\in I}$ as an isomorphism. ∎

As a result of Theorem 3.21 and Proposition 3.20, we have the following.

3.22. Corollary. *Let X be an object in a category $\mathbb{C}$ with (finite) products, where every product projection in $\mathbb{C}$ is a regular epimorphism. Then X has the (finite) strict refinement property if and only if every product projection of X is coextensive.*

3.22.1. Examples of coextensive product projections. As a consequence of Hashimoto's theorem [7] for partially ordered sets, every non-empty connected partially ordered set satisfies the strict refinement property. Let Pos denote the category of non-empty partially ordered sets. It is readily seen that every product projection in Pos is a regular epimorphism. Thus, we have the following result.

3.23. Proposition. *Every product projection of a connected partially ordered set is coextensive in* Pos*.*

3.24. Remark. Consider the full subcategory CPos of connected partially ordered sets. Note that CPos has finite products, but does not have equalisers. Moreover CPos has all pushouts, and they are computed as in Pos. Consequently, every product projection in CPos is coextensive.

Since product projections preserve meets and joins, a consequence of Hashimoto's theorem is that every non-empty semi-lattice satisfies the strict refinement property in the category SLat of semi-lattices. This gives rise to the proposition below.

3.25. Proposition. *Every product projection in the category* SLat *of semi-lattices is coextensive.*

Every monoid M admits a *center* $\mathsf{Z}(M)$, which is given by

$$\mathsf{Z}(M) = \{x \in M \mid \forall y \in M[xy = yx]\}.$$

Then M is said to be *centerless* if $\mathsf{Z}(M) = \{0\}$. As a consequence of [14] (see Corollary 2 in Section 5.6), every centerless monoid has the strict refinement property in Mon. For this reason, we have the following result.

3.26. Proposition. *Every product projection of a centerless monoid is coextensive in* Mon.

As shown in [11], in a Barr-exact [1] Mal'tsev category, every centerless object has coextensive product projections.

3.27. $\mathcal{M}$-coextensivity. The main purpose of the paper [11] was to study categorical aspects of the *strict refinement property* [4] for varieties of universal algebras. The approach taken in that paper is object-wise, through a notion introduced there of an $\mathcal{M}$-coextensive object.

3.28. Definition. *Let* $\mathbb{C}$ *be a category and* $\mathcal{M}$ *a class of morphisms from* $\mathbb{C}$*. A commutative square*

$$\begin{array}{ccc} X & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle a} \\ B & \xrightarrow[b]{} & P \end{array}$$

in $\mathbb{C}$ *is called an* $\mathcal{M}$-pushout *if it is a pushout in* $\mathbb{C}$*, and* a, b *are morphisms in* $\mathcal{M}$*.*

3.29. Definition. *Let* $\mathbb{C}$ *be a category and* $\mathcal{M}$ *a class of morphisms in* $\mathbb{C}$*. An object* X *is said to be* $\mathcal{M}$-coextensive *if every morphism in* $\mathcal{M}$ *with domain* X *admits an* $\mathcal{M}$*-pushout along every product projection of* X*, and in each commutative diagram*

$$\begin{array}{ccccc} A_1 & \longleftarrow & X & \longrightarrow & A_2 \\ \downarrow & & \downarrow & & \downarrow \\ B_1 & \longleftarrow & Y & \longrightarrow & B_2 \end{array}$$

where the top row is a product diagram and the vertical morphisms belong to $\mathcal{M}$*, the bottom row is a product diagram if and only if both squares are* $\mathcal{M}$*-pushouts.*

3.30. Proposition. *Let* $\mathcal{M}$ *be a class of morphisms in a category* $\mathbb{C}$*, containing all product projections in* $\mathbb{C}$*, and where* $\mathcal{M}$ *is stable under pushouts along product projections in* $\mathbb{C}$*. Then an object* A *is* $\mathcal{M}$*-coextensive if and only if every morphism in* $\mathcal{M}$ *with domain* A *is coextensive.*

Proof. If A is $\mathcal{M}$-coextensive, then by definition every morphism in $\mathcal{M}$ with domain A is coextensive. Supposing that every morphism in $\mathcal{M}$ with domain A is coextensive, then in any diagram

$$\begin{array}{ccccc} A_1 & \longleftarrow & A & \longrightarrow & A_2 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ X_1 & \longleftarrow & X & \longrightarrow & X_2 \end{array}$$

where the vertical morphisms belong to $\mathcal{M}$, if the top row is a product, then the squares are pushouts, which are in turn $\mathcal{M}$-pushouts (since the product projections of X are in $\mathcal{M}$). Similarly, if the squares are pushouts, then they are $\mathcal{M}$-pushouts (by the assumptions on the class $\mathcal{M}$), and by coextensivity of f the bottom row is a product. ■

3.31. Remark. In any category $\mathbb{C}$, we may take $\mathcal{M}$ to be the class of monomorphisms in $\mathbb{C}$, in which case $\mathcal{M}$ is closed under pullbacks. Then, as defined in [5], an object is called *mono-extensive* if it is $\mathcal{M}$-coextensive in $\mathbb{C}^{\mathrm{op}}$. In what is termed a *unique-decomposition category* (UD-category) in that paper, every object is mono-extensive. Moreover, in a UD-category, every coproduct inclusion is a monomorphism (see axiom UD5 in [5]). Hence, by the dual of Proposition 3.30, every monomorphism in a UD-category is extensive.

3.32. Corollary. *Let $\mathbb{C}$ be a category with finite products in which every product projection is coextensive. Then every object in $\mathbb{C}$ is $\mathcal{M}$-coextensive, where $\mathcal{M}$ is the class of all product projections in $\mathbb{C}$.*

The notion of *Boolean category* introduced in [13] is equivalent to the following.

3.33. Definition. *A category $\mathbb{C}$ with finite coproducts is* Boolean *if it satisfies the following:*

1. *$\mathbb{C}$ admits all pullbacks along coproduct inclusions, and the class of coproduct inclusions is pullback stable;*
2. *Every coproduct inclusion satisfies* (E1)*;*
3. *If $Y \xrightarrow{t} X \xleftarrow{t} Y$ is a coproduct diagram, then X is an initial object.*

The only distinction between a Boolean category and a category with finite coproducts in which every coproduct inclusion is extensive is that coproduct inclusions need not be pullback stable in a category with finite coproducts in which every coproduct inclusion is extensive.

3.34. Commutativity of finite products with coequalisers. Recall that for a category with binary products, we say that binary products commute with coequalisers in $\mathbb{C}$ if for any two coequaliser diagrams

$$C_1 \underset{v_1}{\overset{u_1}{\rightrightarrows}} X_1 \xrightarrow{q_1} Q_1 \qquad C_2 \underset{v_2}{\overset{u_2}{\rightrightarrows}} X_2 \xrightarrow{q_2} Q_2,$$

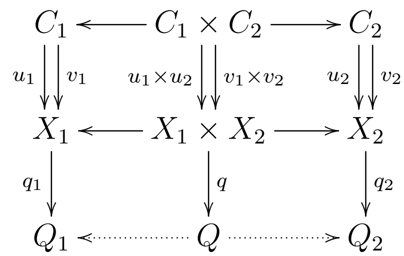
in $\mathbb{C}$, the diagram

$$C_1 \times C_2 \underset{v_1 \times v_2}{\overset{u_1 \times u_2}{\rightrightarrows}} X_1 \times X_2 \xrightarrow{q_1 \times q_2} Q_1 \times Q_2,$$

is a coequaliser diagram. In [8] the commutativity of finite products with coequalisers was considered, and shown to hold in any coextensive category. We remark here that this property is intimately connected to the topic of Huq-centrality of morphisms (see [12]).

3.35. Proposition. *Let $\mathbb{C}$ be a category with binary products and coequalisers, where product projections in $\mathbb{C}$ are epimorphisms, and where every regular epimorphism in $\mathbb{C}$ is coextensive. Then finite products commute with coequalisers in $\mathbb{C}$.*

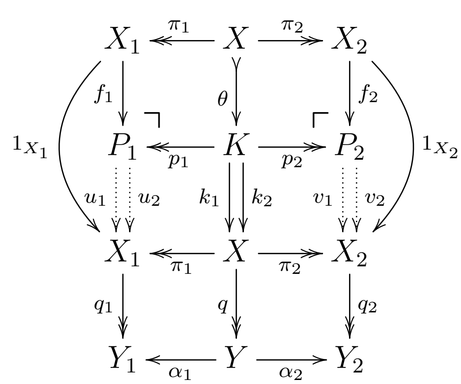
PROOF. Given the two coequaliser diagrams as in the paragraph preceding this proposition, consider the coequaliser $q\colon X_1 \times X_2 \to Q$ of $u_1 \times u_2$ and $v_1 \times v_2$. Then, in the diagram



where the dotted arrows are induced by the coequaliser q (since product projections are epimorphisms), the lower squares are pushouts by Lemma 3.17, and hence the bottom row is a product. ∎

3.36. PROPOSITION. *Let $\mathbb{C}$ be a finitely complete category that admits pushouts of regular epimorphisms along product projections, and suppose that every terminal morphism $X \to 1$ in $\mathbb{C}$ is a regular epimorphism. If finite products commute with coequalisers in $\mathbb{C}$ and every split monomorphism in $\mathbb{C}$ is coextensive, then $\mathbb{C}$ is coextensive.*

PROOF. Suppose that binary products commute with coequalisers in $\mathbb{C}$ and that every split monomorphism is coextensive. Note that regular epimorphisms are stable under binary products in $\mathbb{C}$, so that every product projection $X \times Y \to X$ is a regular epimorphism, since every product projection is a product of a terminal morphism and an identity morphism. It suffices to show that every product projection in $\mathbb{C}$ satisfies (C1) (by Proposition 3.5 and the dual of Corolary 2.3). Let $X_1 \xleftarrow{\pi_1} X \xrightarrow{\pi_2} X_2$ be any product diagram, and suppose that $q\colon X \to Y$ is any product projection of X. Consider the diagram



where (K, k_1, k_2) is the kernel pair of q, θ is the diagonal inclusion and the bottom two squares are the pushouts of q along π_1 and π_2. Since the morphism θ is split monomorphism, we may push out θ along π_1 and π_2, producing a product diagram for K, as well the dotted arrows making the middle squares reasonably commute. Then, since p_1 and p_2 are (regular) epimorphisms, it follows by Lemma 3.17 that q_1 is a coequaliser of u_1 and u_2, and q_2 is a coequaliser of v_1 and v_2. Then, since binary products commute with coequalisers, it follows that $q_1 \times q_2$ is a coequaliser of k_1 and k_2, so that q is isomorphic to $q_1 \times q_2$ and hence the bottom row is a product diagram. ∎

4. Preliminaries on regular categories

Recall that a morphism $f\colon X \to Y$ in a category $\mathbb{C}$ is called a *regular epimorphism* in $\mathbb{C}$ if f is the coequaliser of some parallel pair of morphisms. Recall also that a category $\mathbb{C}$ is *regular* [1] if it has finite limits and coequalisers of kernel pairs, and if the pullback of a regular epimorphism along any morphism is again a regular epimorphism. Listed below are some elementary facts about morphisms in a regular category $\mathbb{C}$:

- Every morphism in $\mathbb{C}$ factors uniquely as a regular epimorphism followed by a monomorphism.
- Regular epimorphisms in $\mathbb{C}$ are stable under finite products.
- Every extremal epimorphism in $\mathbb{C}$ is a regular epimorphism in $\mathbb{C}$.

Given any object X in a regular category $\mathbb{C}$, consider the preorder of all monomorphisms with codomain X. The posetal reflection of this preorder is $\mathsf{Sub}(X)$—the poset of *subobjects* of X. For any morphism $f\colon X \to Y$ in $\mathbb{C}$ there is an induced Galois connection

$$\mathsf{Sub}(X) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{Sub}(Y)$$

given by direct image and inverse image. This is defined as follows: given a subobject A of X represented by a monomorphism $a\colon A_0 \to X$, the direct image $f(A)$ of A along f is defined to be the subobject represented by the monomorphism part of a (regular epimorphism, monomorphism)-factorisation of fa. Given a subobject B of Y represented by a monomorphism $b\colon B_0 \to Y$, we define the inverse image $f^{-1}(B)$ of B to be the subobject of X represented by the monomorphism obtained from pulling back b along f. A *relation* R from X to Y is a subobject of $X \times Y$, i.e., an isomorphism class of monomorphisms with codomain $X \times Y$.

Given such a relation R represented by a monomorphism $(r_1, r_2)\colon R_0 \to X \times Y$, we define the *opposite* relation R^{o} to be the relation represented by the monomorphism $(r_2, r_1)\colon R_0 \to Y \times X$. Given an object X in a category, we write Δ_X for the *diagonal relation*, that is, the relation represented by $(1_X, 1_X)\colon X \to X \times X$, and we write ∇_X for the relation on X represented by the identity morphism on $X \times X$, so that for any relation R on X we have $R \leqslant \nabla_X$.

Regular categories possess a well-behaved composition of relations, which is defined as follows: given a relation R from X to Y and a relation S from Y to Z, and two representatives $(r_1, r_2)\colon R_0 \to X \times Y$ and $(s_1, s_2)\colon S_0 \to Y \times Z$ of R and S respectively, form the pullback of s_1 along r_2 as below

$$\begin{array}{ccc} P & \xrightarrow{p_2} & S_0 \\ {\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle s_1} \\ R_0 & \xrightarrow[r_2]{} & Y \end{array}$$

so that SR is defined to be the relation represented by the monomorphism part of any regular-image factorisation of $(r_1p_1, s_2p_2)\colon P \to X \times Z$. This composition of relations is then associative by virtue of the category $\mathbb{C}$ being regular, and is compatible with composition, i.e., if $R \leqslant S$ and $R' \leqslant S'$, then $RR' \leqslant SS'$ whenever these composites are defined. Any morphism $f\colon X \to Y$ defines a relation, which we shall also denote by$f$, represented by $(1_X, f) : X \to X \times Y$. Given any relations R and S on objects X and Y respectively, note that the image and inverse image under f may be presented as

$$f(R) = fRf^{\circ} \quad \text{and} \quad f^{-1}(S) = f^{\circ}Sf.$$

A relation R on an object X is called:

- *reflexive* whenever $\Delta_X \leqslant R$;
- *symmetric* whenever $R^{\circ} \leqslant R$;
- *transitive* whenever $RR \leqslant R$; an
- *equivalence relation* if R is reflexive, symmetric and transitive.

4.1. Notation. Given any morphism $f\colon X \to Y$, the kernel pair (K, k_1, k_2) of f represents an equivalence relation, which we will denote by $\mathrm{Eq}(f)$ in what follows. Since the pullback of any morphism f along itself gives its kernel pair, note that $\mathrm{Eq}(f) = f^{\circ}f$. Further, any equivalence relation E on an object X is called *effective* if it is the kernel equivalence relation of some morphism, i.e., there is an $f\colon X \to Y$ such that $E = \mathrm{Eq}(f)$.

4.2. Definition. *A regular category $\mathbb{C}$ is said to be* Barr-exact *if every equivalence relation is effective.*

4.3. Calculus of relations in regular categories. The calculus of relations in regular categories is well known, and the formulae presented here are largely folklore, however we were unable to find suitable references for each formula, and include their proofs for this reason. Readers familiar with these formulae may wish to skip to the next section.

Throughout this section, we fix a regular category $\mathbb{C}$.

4.4. Lemma. *Any relation R from X to Y in $\mathbb{C}$ satisfies $\Delta_Y R = R = R\Delta_X$.*

Proof. For any representative $(r_1, r_2)\colon R_0 \to X \times Y$ of R, the following square is a pullback

$$\begin{array}{ccc} R_0 & \xrightarrow{r_2} & Y \\ {\scriptstyle 1_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle 1_Y} \\ R_0 & \xrightarrow[r_2]{} & Y \end{array}$$

which implies that (r_1, r_2) is a representative of $\Delta_Y R$. Hence, $R = \Delta_Y R$. The second equation follows by a similar argument. ■

4.5. Lemma. *For any object X in $\mathbb{C}$ we have $\nabla_X R = \nabla_X = R\nabla_X$ for any reflexive relation R on X.*

Proof. The equation $\nabla_X = \nabla_X R$ holds by the following argument: $\nabla_X = \Delta_X \nabla_X \leqslant R\nabla_X \leqslant \nabla_X$, where the equality follows by Lemma 4.4, the first inequality by the fact that R is reflexive, and the last inequality by the fact that $S \leqslant \nabla_X$ for any relation S on X. The other equation follows similarly. ■

4.6. Lemma. *A morphism $f\colon X \to Y$ in $\mathbb{C}$ is a regular epimorphism if and only if $ff^{\circ} = \Delta_Y$.*

Proof. It is straightforward to see that f is a regular epimorphism if and only if $f(\Delta_X) = \Delta_Y$, and hence

$$f(\Delta_X) = \Delta_Y \Leftrightarrow f\Delta_X f^{\circ} = \Delta_Y \Leftrightarrow ff^{\circ} = \Delta_Y.$$

■

4.7. Lemma. *A reflexive relation R in $\mathbb{C}$ is transitive if and only if $R = RR$.*

Proof. If R is both reflexive and transitive, then $R = \Delta_X R \leqslant RR \leqslant R$, so that $R = RR$. The converse holds trivially. ■

4.8. Lemma. *Let $X_1 \xleftarrow{\pi_1} X \xrightarrow{\pi_2} X_2$ be a product diagram in $\mathbb{C}$. Then the kernel pairs of the projections π_1 and π_2 are given by the relations $\mathrm{Eq}(\pi_1) = \Delta_{X_1} \times \nabla_{X_2}$ and $\mathrm{Eq}(\pi_2) = \nabla_{X_1} \times \Delta_{X_2}$.*

Proof. We label the product projections of the following product diagram as indicated:

$$X_2 \xleftarrow{p_1} X_2^2 \xrightarrow{p_2} X_2.$$

It follows readily that $(1_{X_1} \times p_1, 1_{X_1} \times p_2)$ is the kernel pair of π_1. Hence, $\mathrm{Eq}(\pi_1) = \Delta_{X_1} \times \nabla_{X_2}$. Similarly, $\mathrm{Eq}(\pi_2) = \nabla_{X_1} \times \Delta_{X_2}$. ■

4.9. Lemma. *Let E be any equivalence relation on an object X in $\mathbb{C}$. Then for any product diagram $X_1 \xleftarrow{\pi_1} X \xrightarrow{\pi_2} X_2$ where π_1 and π_2 are regular epimorphisms, $E = \pi_1(E) \times \pi_2(E)$ implies that $\pi_1(E)$ and $\pi_2(E)$ are equivalence relations.*

Proof. It is well-known that reflexivity of relations are preserved by any morphism, and symmetry is preserved by regular epimorphisms. Since π_1 and π_2 are regular epimorphisms, it suffices to show that $\pi_1(E)$ and $\pi_2(E)$ are transitive. Note that

$$\begin{aligned}
\mathrm{Eq}(\pi_1)\,E &= (\Delta_{X_1} \times \nabla_{X_2})(\pi_1(E) \times \pi_2(E)) && \text{(by Lemma 4.8)}\\
&= (\Delta_{X_1}\,\pi_1(E)) \times (\nabla_{X_2}\,\pi_2(E))\\
&= (\pi_1(E)\,\Delta_{X_1}) \times (\pi_2(E)\,\nabla_{X_2}) && \text{(by Lemmas 4.4 and 4.5)}\\
&= (\pi_1(E) \times \pi_2(E))(\Delta_{X_1} \times \nabla_{X_2})\\
&= E\,\mathrm{Eq}(\pi_1).
\end{aligned}$$

Hence,

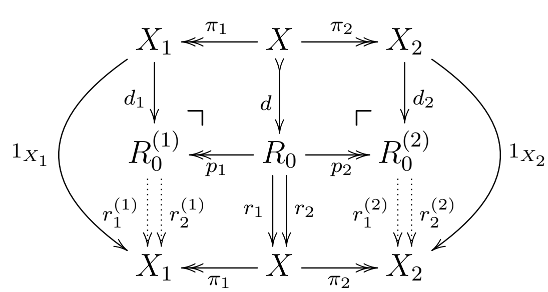
$$\begin{aligned}
(\pi_1(E))(\pi_1(E)) = \pi_1 E\,\pi_1^{\circ}\pi_1 E\,\pi_1^{\circ} &= \pi_1 E\,\mathrm{Eq}(\pi_1)\,E\,\pi_1^{\circ}\\
&= \pi_1 E\,\mathrm{Eq}(\pi_1)\,\pi_1^{\circ} && \text{(by Lemma 4.7)}\\
&= \pi_1 E\,\pi_1^{\circ}\pi_1\pi_1^{\circ} = \pi_1 E\,\pi_1^{\circ} = \pi_1(E). && \text{(by Lemma 4.6)}
\end{aligned}$$

Therefore, $\pi_1(E)$ is transitive by Lemma 4.7, and hence an equivalence relation. It follows similarly that $\pi_2(E)$ is an equivalence relation. ∎

4.10. Coextensivity of split monomorphisms.

4.11. Lemma. *Let $\mathbb{C}$ be a regular category in which every split monomorphism is coextensive. Then, given any product diagram $X_1 \xleftarrow{\pi_1} X \xrightarrow{\pi_2} X_2$ and any reflexive relation R on X, we have $R = \pi_1(R) \times \pi_2(R)$.*

Proof. Suppose that R is a reflexive relation on X represented by (R_0, r_1, r_2), and that $X \xrightarrow{d} R_0$ is the diagonal inclusion. Then we form the diagram

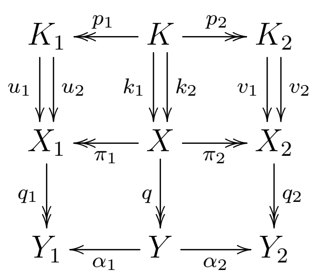


where the top squares are pushouts, and hence the middle row is a product diagram. Moreover, the dotted arrows are induced (reasonably) by the pushouts. By Lemma 3.8 it follows that $r_1^{(1)}, r_2^{(1)}$ and $r_1^{(2)}, r_2^{(2)}$ are jointly monomorphic. Since every identity morphism is coextensive, it follows that all product projections in $\mathbb{C}$ are extremal (and therefore regular) epimorphisms by Proposition 3.9. Hence, p_1 and p_2 are regular epimorphisms. From this it follows that $(R_0^{(1)}, r_1^{(1)}, r_2^{(1)})$ represents $\pi_1(R)$ and $(R_0^{(2)}, r_1^{(2)}, r_2^{(2)})$ represents $\pi_2(R)$ so that $R = \pi_1(R) \times \pi_2(R)$. ∎

4.12. Remark. In the context of a regular majority category [10], given any product diagram $X_1 \xleftarrow{\pi_1} X \xrightarrow{\pi_2} X_2$, and any reflexive relation R on X, we have $R = \pi_1(R) \times \pi_2(R)$. However, it is not true that every split monomorphism in a regular majority category is coextensive. For instance, the variety Lat of lattices is a regular majority category, which does not have coextensive split monomorphisms (as a result of Proposition 4.13).

4.13. Proposition. *Let $\mathbb{C}$ be a Barr-exact category. Then, every split monomorphism in $\mathbb{C}$ is coextensive if and only if $\mathbb{C}$ is coextensive.*

Proof. Suppose that every split monomorphism in $\mathbb{C}$ is coextensive. Note that each product projection is a regular epimorphism by Proposition 3.9. We show that every regular epimorphism is coextensive, and conclude the result by the dual of Corollary 2.3. It suffices to show that every regular epimorphism satisfies (C1) by Proposition 3.5. To this end, let $q : X \to Y$ be any regular epimorphism in $\mathbb{C}$, and let $E = \mathrm{Eq}(q)$ be the equivalence relation represented by the kernel pair (K, k_1, k_2) of q. Given any product diagram $X_1 \xleftarrow[\pi_1]{} X \xrightarrow[\pi_2]{} X_2$, consider the diagram



where (u_1, u_2) and (v_1, v_2) are obtained as representatives of $\pi_1(E)$ and $\pi_2(E)$. By Lemma 4.9, it follows that $\pi_1(E)$ and $\pi_2(E)$ are equivalence relations, so that they are effective. Hence, (K_1, u_1, u_2) and (K_2, v_1, v_2) are kernel pairs and therefore admit coequalisers q_1 and q_2 respectively. Then the morphisms α_1 and α_2 are induced by q being the coequaliser of k_1 and k_2. Since p_1 and p_2 are regular epimorphisms, it follows that the squares are pushouts by Lemma 3.17. Finally, since $E = \pi_1(E) \times \pi_2(E)$ by Lemma 4.11, it follows that (K, k_1, k_2) is a kernel pair of $q_1 \times q_2$. Since $q_1 \times q_2$ is a regular epimorphism, it is the coequaliser of k_1 and k_2. Therefore q and $q_1 \times q_2$ are coequalisers of the same parallel pair, so that the bottom row is a product diagram. ■

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Department of Mathematical Sciences
Stellenbosch University
Private Bag X1
7602, Matieland
South Africa

National Institute for Theoretical and Computational Sciences (NITheCS)
South Africa

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