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REPRESENTING THE LANGUAGE OF A TOPOS AS A QUOTIENT OF THE CATEGORY OF SPANS

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ABSTRACT. We use quotients of span categories to introduce the language of a topos. We also introduce the notion of logical relation and study the quotients of span categories derived from them. As an application we show that the category of Boolean toposes is a reflective subcategory of the category of toposes, when the morphisms are logical functors.

1. Introduction

The Mitchell-Bénabou language [Mac Lane, 1992] is a well-known form of the internal language of an elementary topos. In this approach, types are interpreted as objects of the topos, and variables are interpreted as identity morphisms $1 : A \to A$. More generally, terms of type A in variables x_i of types X_i are interpreted as morphisms from the product $\prod X_i \longrightarrow A$. Formulas of the language are therefore identified with morphisms into the subobject classifier Ω.

A different but related approach is introduced in [Lambek, 1986], where variables are treated as *indeterminate morphisms.* Given an object A in a topos $\mathcal{T}$, a new category $\mathcal{T}[x]$ is constructed by freely adjoining a morphism $x : 1 \longrightarrow A$ to $\mathcal{T}$. This is achieved by forming the free category generated by the graph obtained from the underlying graph of $\mathcal{T}$ by adjoining such a morphism and closing under finite limits. Equivalently, this can be described as the Kleisli category of a cotriple $(S_A, \epsilon_A, \delta_A)$, where $S_A(X) = A \times X$, $\epsilon_A(X) = \pi_X$, and $\delta_A(X) = \langle \pi_A, 1_{A\times X}\rangle$.

However, this construction deals with one indeterminate at a time, and lacks a unified environment for reasoning with multiple variables. In this paper, we extend this framework by constructing a category where *all indeterminate morphisms* are adjoined simultaneously. Our construction uses categories of spans and their quotients to provide such a setting.

- We define, for each object A in a cartesian category $\mathcal{C}$, a stable system $\mathcal{A}$ and form a quotient category $\mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A})$, in which a canonical morphism $x = [!_A, 1_A]_{\mathcal{A}} : 1 \longrightarrow A$ plays the role of the indeterminate morphism.

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- We present a quotient category of spans

$$\mathsf{Span}_{\Pi}(\mathcal{C}, \Pi),$$

  which universally incorporates *all* indeterminate morphisms. Moreover, if $\mathcal{C}$ is cartesian closed, then $\mathsf{Span}_{\Pi}(\mathcal{C}, \Pi)$ is cartesian closed as well.

The category $\mathsf{Span}_{\Pi}(\mathcal{C}, \Pi)$ provides a canonical setting for interpreting terms, formulas, and logical connectives in an internal manner. In this paper, we develop a formulation of the internal language of a topos $\mathcal{T}$ within the structured environment of $\mathsf{Span}_{\Pi}(\mathcal{T}, \Pi)$, where all variables are introduced simultaneously. This unified framework enables a coherent representation of the internal language in which variables and logical constructs coexist as morphisms of a single category.

This paper also investigates conditions under which a quotient category of spans $\mathsf{Span}_{\sim}(\mathcal{T})$ forms a power allegory, ensuring that $\mathsf{Map}(\mathsf{Span}_{\sim}(\mathcal{T}))$ is a topos. Leveraging this framework, we construct, in a universal manner, a Boolean topos associated to each elementary topos. As a consequence, we show that the category of Boolean toposes forms a reflective subcategory of the category of toposes, when morphisms are taken to be logical functors.

2. Preliminaries

We recall some definitions and preliminaries about *span categories*. For more details, see [Hosseini, 2020] and [Hosseini, 2022].

We consider *categories equipped with a stable system of morphisms*; that is, pairs $(\mathcal{C}, \mathcal{S})$ where $\mathcal{C}$ is a category and $\mathcal{S}$ is a collection of morphisms in $\mathcal{C}$ satisfying the following properties:

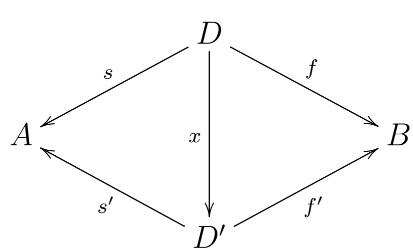
- $\mathcal{S}$ contains all isomorphisms in $\mathcal{C}$ and is closed under composition;
- pullbacks of $\mathcal{S}$-morphisms along arbitrary morphisms exist in $\mathcal{C}$ and belong to $\mathcal{S}$.

For objects A, B in $\mathcal{C}$, a *span* (s, f) with domain A and codomain B consists of a diagram

$$A \xleftarrow{\;s\;} D \xrightarrow{\;f\;} B$$

where $s \in \mathcal{S}$ and f is a morphism in $\mathcal{C}$.

Given another stable system $\mathcal{F}$, we define a morphism $x : (s, f) \longrightarrow (s', f')$ with $x \in \mathcal{F}$ if the following diagram commutes:

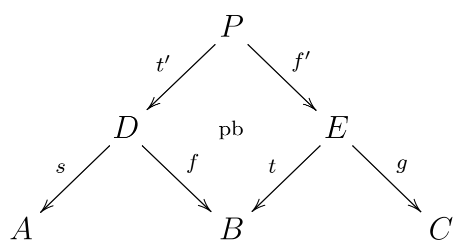


If such a morphism x exists, we write $(s, f) \leq_{\mathcal{F}} (s', f')$. The equivalence relation generated by $\leq_{\mathcal{F}}$ is denoted by $\sim_{\mathcal{F}}$.

We define the quotient category of spans $\mathsf{Span}_{\mathcal{F}}(\mathcal{C}, \mathcal{S})$, where:

- Objects are the same as those of $\mathcal{C}$;
- Morphisms are equivalence classes $[s, f]_{\sim_{\mathcal{F}}}$ of spans under $\sim_{\mathcal{F}}$.

Composition of morphisms $[s, f]_{\sim_{\mathcal{F}}} : A \longrightarrow B$ and $[t, g]_{\sim_{\mathcal{F}}} : B \longrightarrow C$ is defined as $[st', gf']_{\sim_{\mathcal{F}}}$, as in the following diagram:

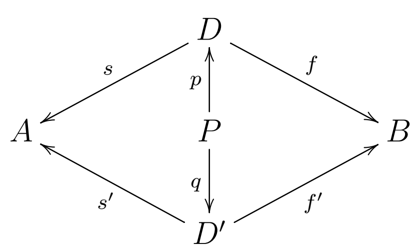


This composition is well-defined. For simplicity, we write $[s, f]_{\mathcal{F}}$ instead of $[s, f]_{\sim_{\mathcal{F}}}$.

In the case where $\mathcal{F} = \mathcal{I}$ is the class of isomorphisms, the category $\mathsf{Span}_{\mathcal{I}}(\mathcal{C}, \mathcal{S})$ is the ordinary category of spans. In this case, we simply write $[s, f]$ for morphisms.

We now state a useful lemma about the equivalence relation $\sim_{\mathcal{F}}$:

2.1. Lemma. [Hosseini, 2022] *Let $\mathcal{F}$ be a stable system. Then $(s, f) \sim_{\mathcal{F}} (s', f')$ if and only if there exist $p, q \in \mathcal{F}$ such that the following diagram commutes:*



To further generalize the relation $\sim_{\mathcal{F}}$, we introduce the notion of a *compatible relation* on $\mathsf{Span}(\mathcal{C}, \mathcal{S})$, which is a relation on spans satisfying:

- only spans with the same domain and codomain may be related;
- vertically isomorphic spans are related;
- the equivalence relation defines a congruence on the category, that is, horizontal composition from either side preserves the relation.

For such a compatible equivalence relation $\sim$, we write the equivalence class of a span (s, f) as $[s, f]_{\sim}$, or simply $[s, f]$ when the context makes it clear which relation is meant. The corresponding quotient category is denoted by

$$\mathsf{Span}_{\sim}(\mathcal{C}, \mathcal{S}).$$

3. Adding indeterminate arrows

Throughout this section, let $\mathcal{C}$ be a cartesian category. As in [Lambek, 1986], for an object $A \in \mathcal{C}$, we aim to add an indeterminate morphism $x : 1 \longrightarrow A$ to $\mathcal{C}$ in a universal way. To achieve this, we define a stable system $\mathcal{A}$ associated with the object A and consider a quotient category of $\mathcal{A}$-spans as a setting where $x : 1 \longrightarrow A$ naturally lives. For an object A in $\mathcal{C}$, define the following class:

$$\mathcal{A} = \{\pi : A^n \times B \longrightarrow B \mid \pi \text{ is a projection}\}.$$

3.1. Lemma. *For every object $A \in \mathcal{C}$, the class*

$$\mathcal{A} = \{\pi : A^n \times B \longrightarrow B \mid \pi \textit{ is a projection}\}$$

is a stable system.

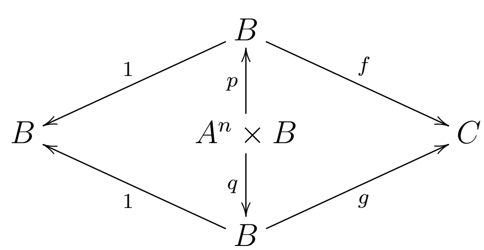
Proof. For $n = 0$, we have $A^n = 1$, so $\mathcal{A}$ contains isomorphisms. Closure under composition and stability under pullbacks are straightforward. ∎

Using $\mathcal{A}$, we define the quotient category of spans:

$$\mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A}).$$

3.2. Proposition. *The map $\mathbf{Q} : \mathcal{C} \longrightarrow \mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A})$ sending a morphism f to $[1, f]_{\mathcal{A}}$ is a functor. Furthermore, if there exists a morphism $1 \longrightarrow A$, then $\mathbf{Q}$ is faithful.*

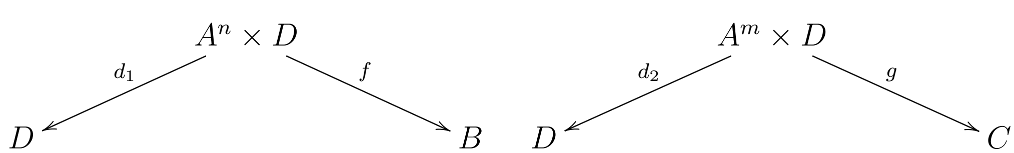
Proof. It is clear that $\mathbf{Q}$ defines a functor. To prove faithfulness, suppose $[1, f]_{\mathcal{A}} = [1, g]_{\mathcal{A}}$ for morphisms $f, g : B \longrightarrow C$ in $\mathcal{C}$. By Lemma 2.1, there exist morphisms $p, q \in \mathcal{A}$ such that the following diagram commutes:



This implies $p = q$. Since there is a morphism $1 \longrightarrow A$, the morphism p is an epimorphism. Therefore, $f = g$, and so $\mathbf{Q}$ is faithful. ∎

3.3. Theorem. *The functor $\mathbf{Q} : \mathcal{C} \longrightarrow \mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A})$ preserves finite products.*

Proof. We show that $B \xleftarrow{[1,\pi_B]_{\mathcal{A}}} B \times C \xrightarrow{[1,\pi_C]_{\mathcal{A}}} C$ is a product in $\mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A})$, where π_B and π_C are the projections in $\mathcal{C}$. Let $B \xleftarrow{[d_1,f]_{\mathcal{A}}} D \xrightarrow{[d_2,g]_{\mathcal{A}}} C$ be a span, where $[d_1, f]_{\mathcal{A}}$ and $[d_2, g]_{\mathcal{A}}$ are represented by the diagrams in $\mathcal{C}$:



Assuming $m \le n$, there exists a projection $\pi : A^n \times D \longrightarrow A^m \times D$. Then,

$$[d_2, g]_{\mathcal{A}} = [d_2\pi, g\pi]_{\mathcal{A}}.$$

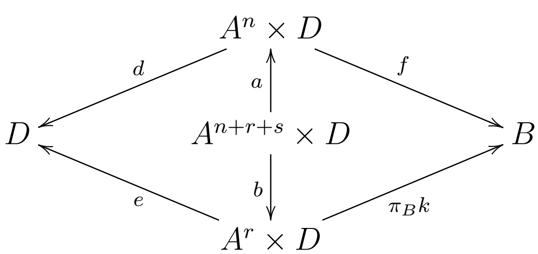
Since both d_1 and $d_2\pi$ are projections from $A^n \times D$ to D, we can assume $n = m$ and $d_1 = d_2$. Let $d = d_1$ and $h = \langle f, g \rangle$. Then,

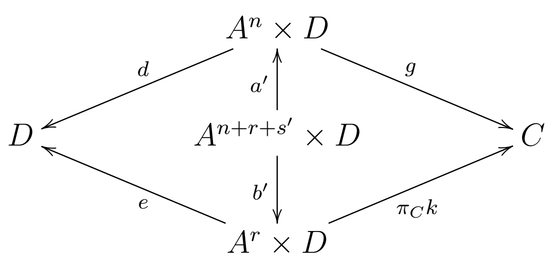
$$[1, \pi_B]_{\mathcal{A}} \circ [d, h]_{\mathcal{A}} = [d, f]_{\mathcal{A}}, \quad [1, \pi_C]_{\mathcal{A}} \circ [d, h]_{\mathcal{A}} = [d, g]_{\mathcal{A}}.$$

To prove uniqueness, suppose $[e, k]_{\mathcal{A}}$ is another morphism such that:

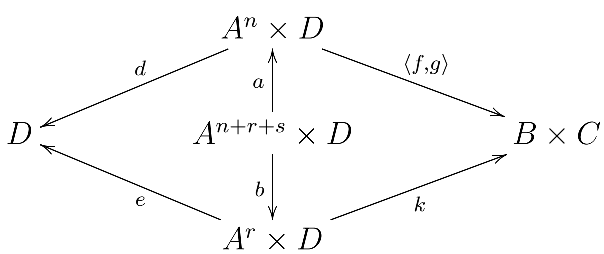
$$[1, \pi_B]_{\mathcal{A}} \circ [e, k]_{\mathcal{A}} = [d, f]_{\mathcal{A}}, \quad [1, \pi_C]_{\mathcal{A}} \circ [e, k]_{\mathcal{A}} = [d, g]_{\mathcal{A}}.$$

By Lemma 2.1, there exist morphisms $a, b, a', b' \in \mathcal{A}$ such that the following diagrams commute:





As before, we may assume $s = s'$, $a = a'$, and $b = b'$. Then the diagram:



commutes, and thus $[e, k]_{\mathcal{A}} = [d, h]_{\mathcal{A}}$. ∎

So far, we have constructed the category $\mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A})$ as a quotient of spans. As mentioned earlier, we will represent the desired indeterminate morphism as a morphism in this category. The morphism $[!_A, 1_A]_{\mathcal{A}} : 1 \to A$ is the indeterminate morphism we are interested in. We denote this morphism by x, and we write the category $\mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A})$ as $\mathcal{C}[x]$.

Morphisms in $\mathcal{C}[x]$ can be interpreted as polynomials in x. The central role of x becomes clearer through a universal property presented in Theorem 3.5. To prove that theorem, we first state the following proposition. Here, x^n denotes the unique morphism

$$x \times \cdots \times x : 1 = 1 \times \cdots \times 1 \longrightarrow A^n = A \times \cdots \times A.$$

3.4. Proposition.

(a) $x^n = [!_{A^n}, 1_{A^n}]$.

(b) $x^n \times 1_B = [\pi, 1_{A^n \times B}]$, *where* $\pi : A^n \times B \longrightarrow B$ *is the projection.*

Proof.

(a) For $n = 2$, the uniqueness of x^2 in the following commutative diagram implies $x^2 = [!_{A^2}, 1_{A^2}]_{\mathcal{A}}$:

$$\begin{array}{ccccc} 1 & \longleftarrow & 1 & \longrightarrow & 1 \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x^2=[!_{A^2},1_{A^2}]_{\mathcal{A}}} & & \downarrow{\scriptstyle x} \\ A & \longleftarrow & A^2 & \longrightarrow & A \end{array}$$

By induction on n, we obtain $x^n = [!_{A^n}, 1_{A^n}]$.

(b) The uniqueness of $x^n \times 1_B$ in the following diagram implies $x^n \times 1_B = [\pi, 1_{A^n \times B}]$:

$$\begin{array}{ccccc} 1 & \longleftarrow & B & \longrightarrow & B \\ {\scriptstyle x^n}\downarrow & & \downarrow{\scriptstyle x^n\times 1_B=[\pi,1_{A^n\times B}]_{\mathcal{A}}} & & \downarrow{\scriptstyle 1} \\ A^n & \longleftarrow & A^n \times B & \longrightarrow & B \end{array}$$

■

The following theorem gives the universal property of $\mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A})$ as a category obtained by freely adding an indeterminate morphism.

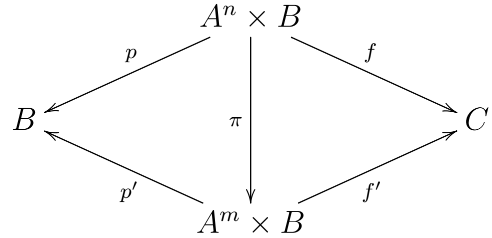
3.5. Theorem. *Let* $\mathbf{F} : \mathcal{C} \longrightarrow \mathcal{C}'$ *be a functor that preserves finite products, and let* $a : 1 \longrightarrow \mathbf{F}(A)$ *be a morphism in* $\mathcal{C}'$*. Then there exists a unique functor* $\mathbf{F}' : \mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A}) \longrightarrow \mathcal{C}'$ *such that* $\mathbf{F}'(x) = a$ *and the following triangle commutes:*

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A}) \\ {\scriptstyle \mathbf{F}}\downarrow & \swarrow{\scriptstyle \mathbf{F}'} & \\ \mathcal{C}' & & \end{array}$$

Proof. Using Proposition 3.4, a morphism $[p, f]$ with $B \xleftarrow{p} A^n \times B \xrightarrow{f} C$ in $\mathcal{C}$ can be written as $[p, f] = [1, f][p, 1] = [1, f](x^n \times 1_B)$. Based on this, we define:

$$\mathbf{F}'[p, f] := \mathbf{F}(f) \circ (a^n \times 1_{\mathbf{F}(B)}).$$

To show that $\mathbf{F}'$ is well defined, suppose $(p, f) \leq_{\mathcal{A}} (p', f')$ via the following diagram:



Then we compute:

$$\mathbf{F}'[p, f] = \mathbf{F}(f)\circ(a^n \times 1_{\mathbf{F}(B)}) = \mathbf{F}(f')\circ\mathbf{F}(\pi)\circ(a^n \times 1_{\mathbf{F}(B)}) = \mathbf{F}(f')\circ(a^m \times 1_{\mathbf{F}(B)}) = \mathbf{F}'[p', f'].$$

By definition of $\mathbf{F}'$, we obtain the commutativity of the triangle, as well as the uniqueness. ∎

The following theorem shows that the construction of indeterminate morphisms is hereditary. This means that for objects A and B in $\mathcal{C}$, one can first add an indeterminate morphism $x : 1 \longrightarrow A$ and then add another indeterminate morphism $y : 1 \longrightarrow B$, or add both of them at once. Before stating the theorem, we define the following classes:

$$\mathcal{B} = \{[1, \pi]_{\mathcal{A}} : B^n \times C \longrightarrow C \mid [1, \pi]_{\mathcal{A}} \text{ is a projection in } \mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A})\}$$
$$\mathcal{A} \circ \mathcal{B} = \{A^n \times B^m \times C \longrightarrow C \mid \pi \text{ is a projection in } \mathcal{C}\}$$

3.6. Theorem. *The category* $\mathsf{Span}_{\mathcal{B}}(\mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A}), \mathcal{B})$ *is isomorphic to* $\mathsf{Span}_{\mathcal{A}\circ\mathcal{B}}(\mathcal{C}, \mathcal{A}\circ\mathcal{B})$.

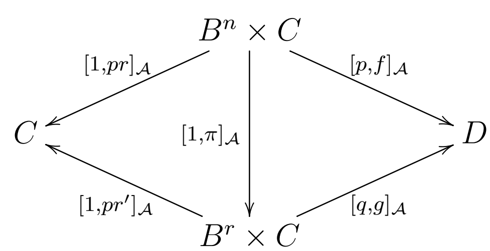
Proof. We define the map

$$[[1, pr]_{\mathcal{A}}, [p, f]_{\mathcal{A}}]_{\mathcal{B}} \longmapsto [pr.p, f]_{\mathcal{A}\circ\mathcal{B}}.$$

To show that this map is well-defined, suppose

$$[[1, pr]_{\mathcal{A}}, [p, f]_{\mathcal{A}}] \leq_{\mathcal{B}} [[1, pr']_{\mathcal{A}}, [q, g]_{\mathcal{A}}]$$

as shown in the diagram, formed in $\mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A})$:



Then we compute:

$$\begin{aligned} [pr'.q, g]_{\mathcal{A}\circ\mathcal{B}} &= [q, g]_{\mathcal{A}\circ\mathcal{B}}[pr', 1]_{\mathcal{A}\circ\mathcal{B}} \\ &= [q, g]_{\mathcal{A}\circ\mathcal{B}}[1, \pi]_{\mathcal{A}\circ\mathcal{B}}[\pi, 1]_{\mathcal{A}\circ\mathcal{B}}[pr', 1]_{\mathcal{A}\circ\mathcal{B}} \\ &= [p, f]_{\mathcal{A}\circ\mathcal{B}}[pr, 1]_{\mathcal{A}\circ\mathcal{B}} \\ &= [pr.p, f]_{\mathcal{A}\circ\mathcal{B}}. \end{aligned}$$

So the map is well defined. It is straightforward to check that this map defines an isomorphism of categories. ∎

3.7. Corollary. *The functor* $\mathcal{C} \longrightarrow \mathsf{Span}_{\mathcal{A}\circ\mathcal{B}}(\mathcal{C}, \mathcal{A}\circ\mathcal{B})$*, defined by* $f \mapsto [1, f]_{\mathcal{A}\circ\mathcal{B}}$*, preserves finite products.*

Proof. This functor is the composition of the following functors, each of which preserves finite products. Therefore, the composition also preserves finite products.

$$\mathcal{C} \longrightarrow \mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A}) \longrightarrow \mathsf{Span}_{\mathcal{B}}(\mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A}), \mathcal{B}) \cong \mathsf{Span}_{\mathcal{A}\circ\mathcal{B}}(\mathcal{C}, \mathcal{A} \circ \mathcal{B})$$

■

As we have seen, adding indeterminate morphisms $x : 1 \longrightarrow A$ and $y : 1 \longrightarrow B$ to the category $\mathcal{C}$ results in the category $\mathsf{Span}_{\mathcal{A}\circ\mathcal{B}}(\mathcal{C}, \mathcal{A} \circ \mathcal{B})$, a quotient of spans. The definition of the compatible system $\mathcal{A} \circ \mathcal{B}$ suggests a natural way to define a quotient category of spans that includes all such indeterminate morphisms by using a more general compatible system. To achieve this, we use the class of all projections, denoted by Π, as a generalization of $\mathcal{A} \circ \mathcal{B}$. It is straightforward to check that Π is a stable system. Hence, we can form the following quotient category:

$$\mathsf{Span}_{\Pi}(\mathcal{C}, \Pi)$$

3.8. Theorem. *The functor* $\mathbf{Q} : \mathcal{C} \longrightarrow \mathsf{Span}_{\Pi}(\mathcal{C}, \Pi)$*, defined by* $f \mapsto [1, f]_{\Pi}$*, preserves finite products.*

Proof. Let $C \xleftarrow{\ \pi_1\ } C \times D \xrightarrow{\ \pi_2\ } D$ be a product diagram in $\mathcal{C}$. We will show that $C \xleftarrow{[1,\pi_1]_{\Pi}} C \times D \xrightarrow{[1,\pi_2]_{\Pi}} D$ is a product diagram in $\mathsf{Span}_{\Pi}(\mathcal{C}, \Pi)$. Suppose we are given a diagram $C \xleftarrow{[p,f]_{\Pi}} E \xrightarrow{[q,g]_{\Pi}} D$ with (p, f) and (q, g) shown as:

$$E \xleftarrow{\ p\ } A \times E \xrightarrow{\ f\ } C \qquad\qquad E \xleftarrow{\ q\ } B \times E \xrightarrow{\ g\ } D$$

in $\mathcal{C}$. By Corollary 3.7, there exists a unique morphism $[r, h]_{\mathcal{A}\circ\mathcal{B}} : E \longrightarrow C \times D$ such that the triangles in the following diagram commute:

$$\begin{array}{ccccc} C & \xleftarrow{[1,\pi_1]_{\mathcal{A}\circ\mathcal{B}}} & C \times D & \xrightarrow{[1,\pi_2]_{\mathcal{A}\circ\mathcal{B}}} & D \\ & {\scriptstyle [p,f]_{\mathcal{A}\circ\mathcal{B}}} \nwarrow & \uparrow {\scriptstyle [r,h]_{\mathcal{A}\circ\mathcal{B}}} & \nearrow {\scriptstyle [q,g]_{\mathcal{A}\circ\mathcal{B}}} & \\ & & E & & \end{array}$$

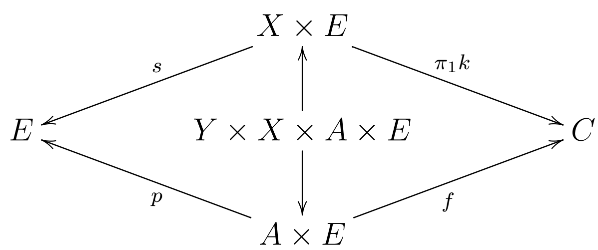
Since $\mathcal{A} \circ \mathcal{B} \subseteq \Pi$, the same morphism $[r, h]_{\Pi}$ also makes the following diagram commute:

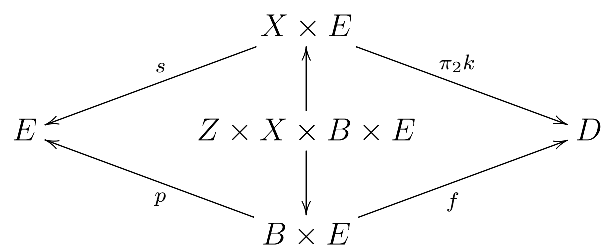
$$\begin{array}{ccccc} C & \xleftarrow{[1,\pi_1]_{\Pi}} & C \times D & \xrightarrow{[1,\pi_2]_{\Pi}} & D \\ & {\scriptstyle [p,f]_{\Pi}} \nwarrow & \uparrow {\scriptstyle [r,h]_{\Pi}} & \nearrow {\scriptstyle [q,g]_{\Pi}} & \\ & & E & & \end{array}$$

To prove uniqueness, suppose another morphism $[s,k]_\Pi$ also satisfies:

$$[1,\pi_1]_\Pi \circ [s,k]_\Pi = [p,f]_\Pi \quad \text{and} \quad [1,\pi_2]_\Pi \circ [s,k]_\Pi = [q,g]_\Pi.$$

By Lemma 2.1, there exist projections such that the following diagrams commute:



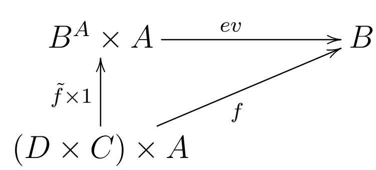


Let $\Pi' = \mathcal{A} \circ \mathcal{B} \circ \mathcal{X} \circ \mathcal{Y} \circ \mathcal{Z}$. An extension of Corollary 3.7 shows that $[s,k]_{\Pi'} = [r,h]_{\Pi'}$, and so $[s,k]_\Pi = [r,h]_\Pi$. Therefore, $[r,h]_\Pi$ is unique, and the diagram is indeed a product in $\mathsf{Span}_\Pi(\mathcal{C},\Pi)$. ∎

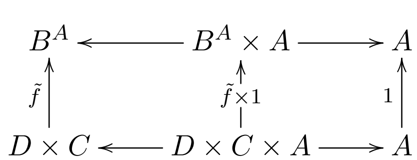
3.9. Theorem. *If $\mathcal{C}$ is a cartesian closed category, then so is $\mathsf{Span}_\Pi(\mathcal{C},\Pi)$.*

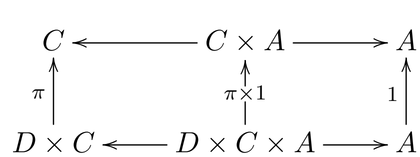
Proof. We want to show that the exponential object B^A in $\mathcal{C}$ is also an exponential object in $\mathsf{Span}_\Pi(\mathcal{C},\Pi)$. We do this by showing that the evaluation map $ev : B^A \times A \longrightarrow B$ in $\mathcal{C}$ induces an evaluation map $[1, ev]_\Pi : B^A \times A \longrightarrow B$ in $\mathsf{Span}_\Pi(\mathcal{C},\Pi)$.

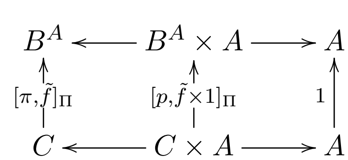
Let $[p,f]_\Pi : C \times A \longrightarrow B$ be a morphism in $\mathsf{Span}_\Pi(\mathcal{C},\Pi)$, where (p,f) is depicted in $\mathcal{C}$ as $C \times A \xleftarrow{p} D \times C \times A \xrightarrow{f} B$. There exists a unique morphism $\tilde{f} : D \times C \longrightarrow B^A$ in $\mathcal{C}$ such that the following diagram commutes:



In the following diagrams, the first two (the left and middle ones) are formed in $\mathcal{C}$ using product diagrams. In each of them, the left square is a pullback. This implies that the right diagram, in $\mathsf{Span}_\Pi(\mathcal{C},\Pi)$, also commutes.



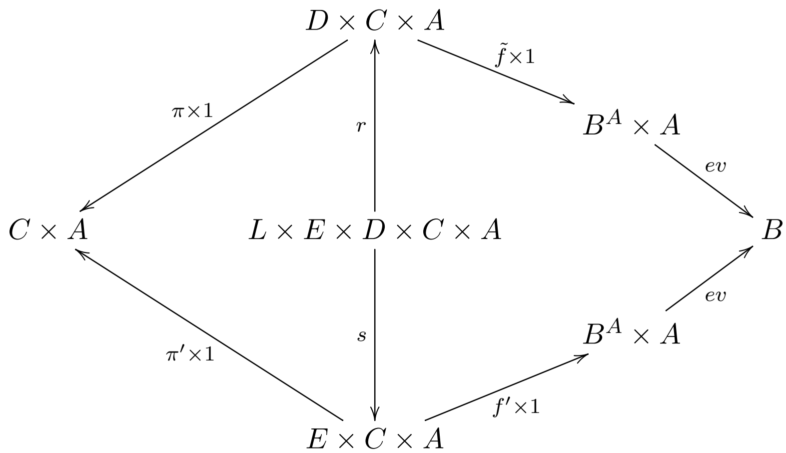




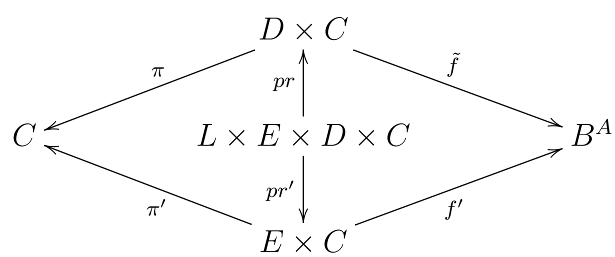
So we have $[p, \tilde{f} \times 1]_\Pi = [\pi, \tilde{f}]_\Pi \times 1$, and therefore

$$[1, ev]_\Pi \circ ([\pi, \tilde{f}]_\Pi \times 1) = [p,f]_\Pi.$$

To prove uniqueness of $[\pi, \tilde{f}]_\Pi$, suppose $[\pi', f']_\Pi$ is another morphism such that $[1, ev]_\Pi \circ ([\pi', f']_\Pi \times 1) = [p, f]_\Pi$. By Lemma 2.1, there exist $r, s \in \Pi$ such that the following diagram commutes:



The projections r and s can be written as $r = pr \times 1 : (L \times E \times D \times C) \times A \longrightarrow D \times C \times A$ and $s = pr' \times 1 : (L \times E \times D \times C) \times A \longrightarrow E \times C \times A$. Then we have $ev(\tilde{f}pr \times 1) = ev(f'pr' \times 1)$, which implies $\tilde{f}pr = f'pr'$. The commutativity of the following diagram shows that $[\pi, \tilde{f}]_\Pi = [\pi', f']_\Pi$, establishing the uniqueness of $[\pi, \tilde{f}]_\Pi$. Therefore, $\mathsf{Span}_\Pi(\mathcal{C}, \Pi)$ is cartesian closed.



■

For each object $A \in \mathcal{C}$, there is a quotient functor $\mathbf{Q} : \mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A}) \longrightarrow \mathsf{Span}_\Pi(\mathcal{C}, \Pi)$ that maps the morphism $x = [!_A, 1_A]_{\mathcal{A}} : 1 \to A \in \mathsf{Span}_{\mathcal{A}}(\mathcal{C}, \mathcal{A})$ to $x = [!_A, 1_A]_\Pi : 1 \to A \in \mathsf{Span}_\Pi(\mathcal{C}, \Pi)$. This means that $\mathsf{Span}_\Pi(\mathcal{C}, \Pi)$ includes all such indeterminate morphisms.

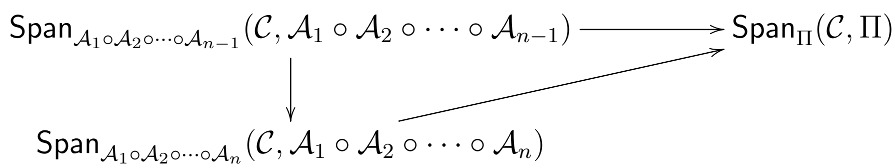
In what follows, we show that $\mathsf{Span}_\Pi(\mathcal{C}, \Pi)$ has this property in a universal way: it is the colimit of a natural diagram in the category Cat. We build this diagram by collecting all quotient functors of the form

$$\mathbf{Q} : \mathsf{Span}_{\mathcal{A}_1 \circ \mathcal{A}_2 \circ \cdots \circ \mathcal{A}_{n-1}}(\mathcal{C}, \mathcal{A}_1 \circ \mathcal{A}_2 \circ \cdots \circ \mathcal{A}_{n-1}) \longrightarrow \mathsf{Span}_{\mathcal{A}_1 \circ \mathcal{A}_2 \circ \cdots \circ \mathcal{A}_n}(\mathcal{C}, \mathcal{A}_1 \circ \mathcal{A}_2 \circ \cdots \circ \mathcal{A}_n),$$

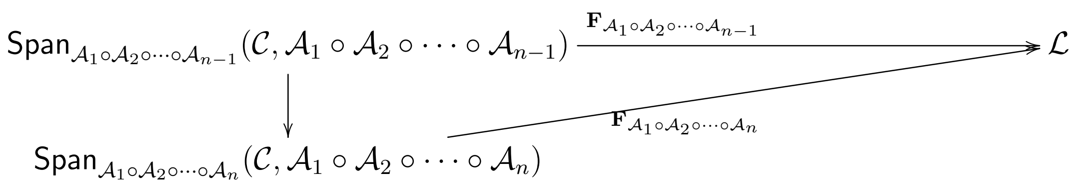
where each $\mathcal{A}_i$ is the compatible system associated to the object A_i, for $1 \le i \le n$.

3.10. Theorem. $\mathsf{Span}_\Pi(\mathcal{C}, \Pi)$ *is the colimit of the above diagram.*

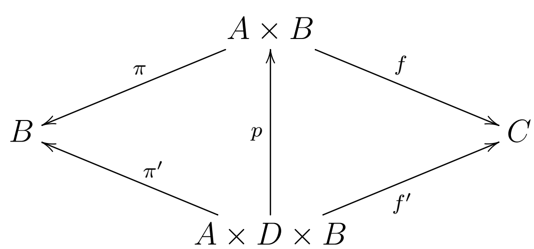
Proof. We first observe that the following diagram forms a natural cocone:



Now suppose we are given another cocone to some category $\mathcal{L}$:



We define a functor $\mathbf{U} : \mathsf{Span}_{\Pi}(\mathcal{C}, \Pi) \longrightarrow \mathcal{L}$ by sending $[\pi, f]_{\Pi} \mapsto \mathbf{F}_{\mathcal{A}}[\pi, f]_{\mathcal{A}}$, where $B \xleftarrow{\pi} A \times B \xrightarrow{f} C$ is a span in $\mathcal{C}$. To check that $\mathbf{U}$ is well defined, suppose we have a commutative diagram where p is a projection:



This implies $[\pi, f]_{\mathcal{A}\circ\mathcal{D}} = [\pi', f']_{\mathcal{A}\circ\mathcal{D}}$, so by naturality:

$$\mathbf{U}[\pi, f]_{\Pi} = \mathbf{F}_{\mathcal{A}}[\pi, f]_{\mathcal{A}} = \mathbf{F}_{\mathcal{A}\circ\mathcal{D}}[\pi, f]_{\mathcal{A}\circ\mathcal{D}} = \mathbf{F}_{\mathcal{A}\circ\mathcal{D}}[\pi', f']_{\mathcal{A}\circ\mathcal{D}} = \mathbf{U}[\pi', f']_{\Pi}.$$

Hence $\mathbf{U}$ is well defined. The uniqueness of $\mathbf{U}$ is straightforward. ∎

4. Language of a topos

So far, we have constructed a quotient of spans that contains all indeterminate morphisms in a universal manner. In this section, we show that for a topos $\mathcal{T}$, the category $\mathsf{Span}_{\Pi}(\mathcal{T}, \Pi)$ can be regarded as a coherent system in which the internal language of the topos $\mathcal{T}$ can be expressed. In our representation of this language, objects of $\mathcal{T}$ are interpreted as types, and morphisms of the form $[!_A, f]_{\Pi} : 1 \to B$ are interpreted as terms of type $B \in \mathcal{T}$.

We denote a term $[!_A, f]_{\Pi} : 1 \to B$ by $\phi(x) : 1 \to B$, where x represents $[!_A, 1_A]_{\Pi} : 1 \to A$. Thus, x is a term of type A, called a variable of type A. Terms of type Ω are referred to as formulas.

4.1. Definition. *Let* $\alpha(x) = [!_A, f]_\Pi : 1 \to D$, $\beta(y) = [!_B, g]_\Pi : 1 \to D$, *and* $\gamma(z) = [!_C, h]_\Pi : 1 \to \mathbf{P}D$. *Then:*

- $\alpha(x) = \beta(y)$ *is the formula* $1 \xrightarrow{\langle\alpha,\beta\rangle} D \times D \xrightarrow{[1,\delta_D]_\Pi} \Omega$
- $\alpha\varepsilon\gamma$ *is the formula* $1 \xrightarrow{\langle\alpha,\gamma\rangle} D \times \mathbf{P}D \xrightarrow{[1,ev]_\Pi} \Omega$

For formulas $\phi(x) = [!_A, f] : 1 \to \Omega$ *and* $\psi(y) = [!_B, g] : 1 \to \Omega$, *define:*

- $\phi \wedge \psi$ *as* $1 \xrightarrow{\langle\phi,\psi\rangle} \Omega \times \Omega \xrightarrow{[1,\wedge]_\Pi} \Omega$
- $\phi \vee \psi$ *as* $1 \xrightarrow{\langle\phi,\psi\rangle} \Omega \times \Omega \xrightarrow{[1,\vee]_\Pi} \Omega$
- $\phi \implies \psi$ *as* $1 \xrightarrow{\langle\phi,\psi\rangle} \Omega \times \Omega \xrightarrow{[1,\implies]_\Pi} \Omega$
- not ϕ *as* $1 \xrightarrow{\phi} \Omega \xrightarrow{[1,\mathrm{not}]_\Pi} \Omega$
- $\forall\phi(x) = [1, \forall_A \tilde{f}]$
- $\exists\phi(x) = [1, \exists_A \tilde{f}]$

  where $\forall_A$ *is the right adjoint and* $\exists_A$ *is the left adjoint to* $\mathbf{P}(!_A) : \Omega \to \mathbf{P}(A)$, *and* $\tilde{f}$ *is obtained from the diagram:*

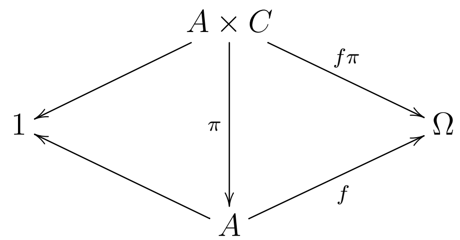
$$\begin{array}{ccc} \mathbf{P}(A) \times A & \xrightarrow{ev} & \Omega \\ {\scriptstyle \tilde{f}\times 1}\uparrow & & \uparrow{\scriptstyle f} \\ 1 \times A & \longrightarrow & A \end{array}$$

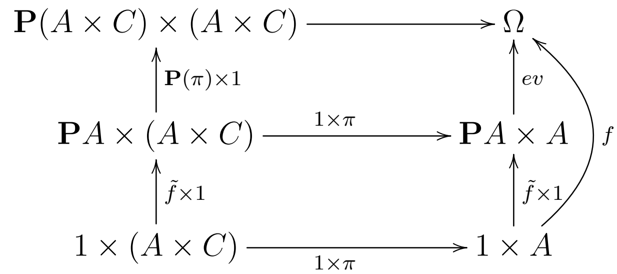
- *For* $\phi(x) = [1, f][!_A, 1] : 1 \to A \to \Omega$, *the expression* $\{x \in A : \phi(x)\}$ *is defined as the unique morphism obtained from the diagram:*

$$\begin{array}{ccc} \mathbf{P}A \times A & \xrightarrow{[1,ev]_\Pi} & \Omega \\ {\scriptstyle \{x\in A:\phi(x)\}\times 1}\uparrow & \nearrow{\scriptstyle [1,f]_\Pi} & \\ 1 \times A & & \end{array}$$

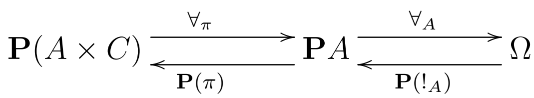
4.2. Proposition. $\forall\phi(x)$ *and* $\exists\phi(x)$ *are well defined.*

Proof. Let the left diagram below be given with $\pi \in \Pi$. We aim to show $\forall_A \tilde{f} = \forall_{A\times C} \widetilde{f\pi}$. The right diagram implies $\widetilde{f\pi} = \mathbf{P}(\pi)\tilde{f}$.

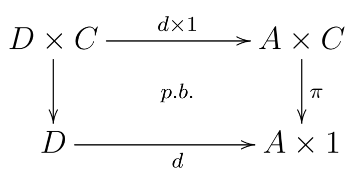


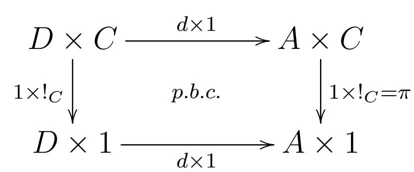


The adjunction diagram below, together with $!_{A\times C} = !_A\pi$, implies $\forall_{A\times C} = \forall_A\forall_\pi$.



The following pullback and pullback-complement squares illustrate the external forms of $\mathbf{P}\pi$ and $\forall_\pi$, respectively. From the right square, we get $\pi^{-1}d = d \times 1$, which implies $\forall_\pi \mathbf{P}\pi = 1$. Therefore, $\forall_{A\times C}\widetilde{f\pi} = \forall_A\forall_\pi\mathbf{P}\pi\tilde{f} = \forall_A\tilde{f}$, so $\forall\phi(x)$ is well defined.





Using [Johnstone, 2002, Lemma 2.3.6], we obtain $\exists_C\mathbf{P}\pi = 1$. Hence, $\exists_{A\times C}\widetilde{f\pi} = \exists_A\exists_C\mathbf{P}\pi\tilde{f} = \exists_A\tilde{f}$, so $\exists\phi(x)$ is also well defined. ■

5. Logical relations on span categories

In [Hosseini, 2022], compatible relations on span categories, in which their quotients are allegories, are studied, and it is shown that for a pullback stable factorization system $(\mathcal{E}, \mathcal{M})$ in a finitely complete category $\mathcal{C}$, $\mathsf{Rel}(\mathcal{C}, \mathcal{E}, \mathcal{M}) \cong \mathsf{Span}_{\mathcal{E}}(\mathcal{C})$ [Hosseini, 2022, Theorem 2.3]. For a regular category $\mathcal{C}$, with $\mathcal{E} = \mathbf{RegEpi}(\mathcal{C})$, $\mathcal{M} = \mathbf{Mono}(\mathcal{C})$, it is well known that $\mathsf{Rel}(\mathcal{C}, \mathcal{E}, \mathcal{M}) \cong \mathsf{Span}_{\mathcal{E}}(\mathcal{C})$ is a tabular allegory and $\mathsf{Map}(\mathsf{Span}_{\mathcal{E}}(\mathcal{C})) \cong \mathcal{C}$. This motivates us to investigate which quotients of $\mathsf{Span}(\mathcal{T})$, for a topos $\mathcal{T}$, are toposes. In [Johnstone, 2002], it is shown that maps of a power allegory form a topos. Inspired by this, we investigate conditions on a compatible relation $\sim$ to make $\mathsf{Span}_\sim(\mathcal{T})$ a power allegory.

Allegories were presented for the first time in [Freyd, 1990] as categories which reflect properties that hold in the category of relations.

5.1. Definition. *An* allegory *is a locally ordered* 2*-category* $\mathcal{A}$ *whose hom-posets have binary intersections, equipped with an anti-involution* $\phi \mapsto \phi^\circ$ *and satisfying the modular law*

$$\psi\phi \cap \chi \leq (\psi \cap \chi\phi^\circ)\phi,$$

whenever this makes sense.

5.2. Definition. *In an allegory, a morphism is called* map *if* $1 \leq r^\circ \cdot r$ *and* $r \cdot r^\circ \leq 1$*. The subcategory of maps of an allegory* $\mathcal{A}$ *is denoted by* $\mathsf{MAP}(\mathcal{A})$*.*

A power allegory is a division allegory with some extra properties. First, we give the definition of a division allegory and then the definition of a power allegory. See [Johnstone, 2002] for more information.

5.3. Definition. [Johnstone, 2002] *An allegory* A *is called a* division allegory *if, for each* $\phi : A \to B$ *and object* C*, the order preserving map* $(-)\phi : \mathcal{A}(B, C) \longrightarrow \mathcal{A}(A, C)$ *has a right adjoint, which we call right division by* ϕ *and denote* $(-)/\phi$*.*

Of course, the anti-involution ensures that if we have right division we also have left division $\phi \setminus (-)$ (right adjoint to $\phi(-)$). We write $(\phi|\psi)$ for

$$(\phi \setminus \psi) \cap (\psi \setminus \phi)^\circ.$$

5.4. Definition. [Johnstone, 2002] *A division allegory* $\mathcal{A}$ *is called a* power allegory *if there is an operation assigning to each object* A *a morphism* $\in_A : PA \to A$ *satisfying* $(\in_A \mid \in_A) = 1_{PA}$ *and*

$$1_B \leq (\phi\backslash \in_A)(\in_A \backslash\phi)$$

for any $\phi : B \to A$*.*

Every topos has an $(\mathbf{Epi}, \mathbf{Mono})$ factorization. In the following, we denote a topos as $\mathcal{T}$ and its epi-mono factorization as $(\mathcal{E}, \mathcal{M})$. Utilizing $(\mathcal{E}, \mathcal{M})$, we define a kind of compatible relation such that the quotient arising from it will be shown to be a power allegory.

5.5. Definition. *For a topos* $\mathcal{T}$*, a compatible relation* $\sim$ *on* $\mathsf{Span}(\mathcal{T})$ *is called* logical *if:*

- $\mathcal{E} \subseteq \sim$
- *for spans* $(f, g), (h, k) : A \to C$ *and a morphism* $a : A \to B$

  $$(f, g) \sim (h, k) \implies (\pi_1\forall_{a\times 1}m, \pi_2\forall_{a\times 1}m) \sim (\pi_1\forall_{a\times 1}n, \pi_2\forall_{a\times 1}n)$$

  where m *and* n *are the* $\mathcal{M}$*-parts of the morphisms* $\langle f, g\rangle : D \longrightarrow A \times C$ *and* $\langle h, k\rangle : D' \longrightarrow A \times C$*, respectively, where* $\langle f, g\rangle$ *and* $\langle h, k\rangle$ *denote the unique morphisms in* $\mathcal{T}$ *induced by the universal property of the product, and* D *and* D' *are the domains of* f *and* h*, respectively*[1]*.*

[1] We use angle brackets to denote the unique morphism resulting from a product diagram. Note that these morphisms are not spans.

First, we show that $\mathsf{Span}_\sim(\mathcal{T})$ is a division allegory, for a logical relation $\sim$. Since $\mathcal{E} \subseteq \sim$, the mapping $Q : \mathsf{Span}_\mathcal{E}(\mathcal{T}) \longrightarrow \mathsf{Span}_\sim(\mathcal{T})$, defined by $Q([f,g]_\mathcal{E}) = [f,g]_\sim$, is a representation of allegories, meaning that Q preserves $^\circ$ and $\cap$.

5.6. Theorem. *For a topos $\mathcal{T}$, $\mathsf{Span}_\mathcal{E}(\mathcal{T})$ is a division allegory.*

Proof. Utilizing [Johnstone, 2002, Theorem 3.4.2] and [Hosseini, 2022, Theorem 4.2], $\mathsf{Span}_\mathcal{E}(\mathcal{T})$ is a division allegory, where $[h,k]_\mathcal{E}/[f,g]_\mathcal{E} := [\pi_1 a, \pi_2 a]_\mathcal{E}$, in which $a = \forall_{g\times 1}(f \times 1)^*(m_{\langle h,k\rangle})$, and $m_{\langle h,k\rangle}$ is the mono part of $\langle h,k\rangle$. ∎

5.7. Theorem. *For a logical relation $\sim$, $\mathsf{Span}_\sim(\mathcal{T})$ is a division allegory and*

$$Q((-)/[f,g]_\mathcal{E}) = (-)/[f,g]_\sim.$$

Proof. Let $(-)/[f,g]_\sim := Q((-)/[f,g]_\mathcal{E})$. It follows from the definition of logical relation that this definition is well-defined. We have

$$[h,k]_\sim = Q[h,k]_\mathcal{E} \le Q(([h,k]_\mathcal{E}[f,g]_\mathcal{E})/[f,g]_\mathcal{E}) = ([h,k]_\sim[f,g]_\sim)/[f,g]_\sim$$

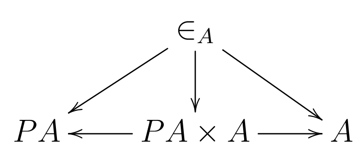
and

$$\begin{aligned} ([r,s]_\sim/[f,g]_\sim)[f,g]_\sim &= Q([r,s]_\mathcal{E}/[f,g]_\mathcal{E})Q[f,g]_\mathcal{E} \\ &= Q(([r,s]_\mathcal{E}/[f,g]_\mathcal{E})[f,g]_\mathcal{E}) \le Q[r,s]_\mathcal{E} = [r,s]_\sim. \end{aligned}$$

Therefore, $(-)/[f,g]_\sim$ is right adjoint to $(-)[f,g]_\sim$. ∎

5.8. Theorem. *$\mathsf{Span}_\mathcal{E}(\mathcal{T})$ is a power allegory.*

Proof. Since $\mathcal{T}$ is a topos, $\mathsf{Rel}(\mathcal{T},\mathcal{E},\mathcal{M})$ is a power allegory with $\in_A : PA \to A$ defined as



By [Hosseini, 2022, Theorem 2.3], we have $\mathsf{Rel}(\mathcal{T},\mathcal{E},\mathcal{M}) \cong \mathsf{Span}_\mathcal{E}(\mathcal{T})$, and $\in_A : PA \to A$ in $\mathsf{Span}_\mathcal{E}(\mathcal{T})$ is defined as in $\mathsf{Rel}(\mathcal{T},\mathcal{E},\mathcal{M})$. ∎

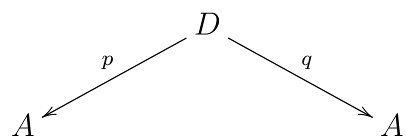
5.9. Theorem. *For a logical relation $\sim$, $\mathsf{Span}_\sim(\mathcal{T})$ is a power allegory and $\mathsf{Map}(\mathsf{Span}_\sim(\mathcal{T}))$ is a topos.*

Proof. Let $\in_A : PA \to A$ in $\mathsf{Span}_\sim(\mathcal{T})$ be $Q(\in_A : PA \to A)$. Since $Q((-)/[f,g]_\mathcal{E}) = (-)/[f,g]_\sim$ and Q is a representation, $\mathsf{Span}_\sim(\mathcal{T})$ is a power allegory. Then by [Johnstone, 2002, Corollary 3.4.7], $\mathsf{Map}(\mathsf{Span}_\sim(\mathcal{T}))$ is a topos. ∎

Denoting $\mathsf{Map}(Q)$ by η:

5.10. Corollary. *$\eta : \mathcal{T} \longrightarrow \mathsf{Map}(\mathsf{Span}_{\sim}(\mathcal{T}))$ is a logical functor.*

In the rest of this section, we present a different kind of compatible relation, generated by a class of *endospans*, that can be considered as an extension of relations generated by classes of morphisms. Utilizing them, we can generate some logical relations. An endospan, as depicted below, is a span in which its domain and codomain are the same.



If in the above endospan $p = q$ and p is an isomorphism, it is called an endospan of an iso.

5.11. Definition.

- *A class of endospans is called* saturated *if it contains all endospans of isos.*
- *Suppose $\mathcal{A}$ is a saturated class of endospans. The compatible relation generated by $\mathcal{A}$, denoted by $\sim_{\mathcal{A}}$, is defined to be the smallest compatible relation $\sim$ on the category $\mathsf{Span}(\mathcal{C})$ such that for all (a, b) in $\mathcal{A}$, $(a, b) \sim (1, 1)$.*

In the next proposition, we explain how this smallest relation is constructed and provide a concrete representation of it.

5.12. Proposition. *For a saturated class of endospans, $\mathcal{A}$, the compatible relation generated by $\mathcal{A}$ is described as follows:*
$(h, k) \sim (r, s) \iff$ *there are decompositions $(h, k) = (h_n, k_n) \cdots (h_1, k_1)$ and $(r, s) = (r_m, s_m) \cdots (r_1, s_1)$, and endospans $(a_1, b_1), \cdots, (a_n, b_n) \in \mathcal{A}$ and $(c_1, d_1), \cdots, (c_m, d_m) \in \mathcal{A}$ such that:*

$$(r_m, s_m)(c_m, d_m) \cdots (r_1, s_1)(c_1, d_1) = (h_n, k_n)(a_n, b_n) \cdots (h_1, k_1)(a_1, b_1)$$

Proof. Obvious. ■

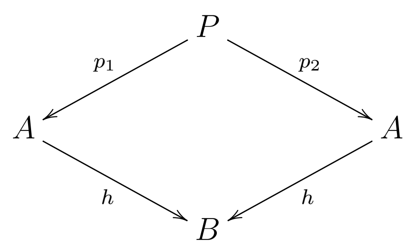
5.13. Example.

- Let $\mathcal{I}$ be the class of all endospans of isos. The compatible relation generated by this class is defined as follows: $(f, g) \sim (h, k)$ if there is an isomorphism ϕ such that $f = h\phi$ and $g = k\phi$. So $\mathsf{Span}_{\sim}(\mathcal{C})$ is the ordinary category of spans.
- For a stable system of morphisms $\mathcal{B}$, we can form a saturated class of endospans containing (b, b) for all $b \in \mathcal{B}$. The compatible relation generated by this class of endospans is equivalent to $\sim_{\mathcal{B}}$.

- For a morphism $f : A \to B$, we can form a saturated endospan class by adding the kernel pair of f to $\mathcal{I}$, the class of all endospans of isos.
- For a morphism $f : A \to B$, a saturated class of endospans can be formed by adding the kernel pair of f to the class of endospans containing (e, e) for epimorphisms e.

5.14. Definition. *For a morphism $f : A \to B$, we define $K(f)$ to be the saturated class of endospans containing the kernel pairs of all morphisms h in which $f = gh$ for some morphism g, and (e, e) for all epimorphisms e.*

The compatible relations generated by $K(f)$ imply $(p_1, p_2) \sim_{K(f)} (1, 1)$, in which p_1, p_2 are obtained by the following pullback diagram, where $f = gh$ for some morphism g:



5.15. Lemma. *Using the above definitions and notations, we have:*

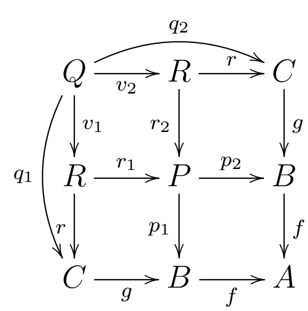
(a) For an epimorphism e, $[1, e]_{K(e)}$ is an isomorphism and its inverse is $[e, 1]_{K(e)}$.

(b) If $f = gh$, then $K(h) \subseteq K(f)$.

Proof. Obvious. ■

The smallest logical relation containing $K(f)$ is denoted by $L(f)$.

5.16. Lemma. *The following diagram is formed by pullbacks, in which g is an epimorphism. Then $\forall_{g\times g}\langle q_1, q_2\rangle = \langle p_1, p_2\rangle$.*



Proof. Let $(g\times g)^{-1}\langle x, y\rangle \leq \langle q_1, q_2\rangle$. Then there is an arrow i such that $(g\times g)^{-1}\langle x, y\rangle = \langle q_1, q_2\rangle i$. Set $(g \times g)^{-1}\langle x, y\rangle = \langle x', y'\rangle$ and $\langle x, y\rangle^{-1}(g \times g) = e$. Since g is an epimorphism, $(g \times g)$ is also an epimorphism, and since epimorphisms are stable under pullbacks in a topos, it follows that e is an epimorphism as well.

We have the following equalizer diagrams:

$$P \xrightarrow{\langle p_1,p_2\rangle} B\times B \overset{f\pi_1}{\underset{f\pi_2}{\rightrightarrows}} A \qquad P \xrightarrow{\langle q_1,q_2\rangle} C\times C \overset{fg\pi_1'}{\underset{fg\pi_2'}{\rightrightarrows}} A$$

We have $fg\pi_1' = f\pi_1(g\times g)$ and $fg\pi_2' = f\pi_2(g\times g)$. So:

$$\begin{aligned} fxe &= f\pi_1\langle x,y\rangle e \\ &= f\pi_1(g\times g)\langle x',y'\rangle \\ &= fg\pi_1'\langle q_1,q_2\rangle i \\ &= fg\pi_2'\langle q_1,q_2\rangle i \\ &= f\pi_2(g\times g)\langle x',y'\rangle \\ &= f\pi_2\langle x,y\rangle e \\ &= fye \end{aligned}$$

Since e is an epimorphism, $fx = fy$. Hence $\langle x,y\rangle \leq \langle p_1,p_2\rangle$. By the following pullback diagrams we get $(g\times g)^{-1}\langle p_1,p_2\rangle = \langle q_1,q_2\rangle$. Then, $\langle x,y\rangle \leq \langle p_1,p_2\rangle$ implies $(g\times g)^{-1}\langle x,y\rangle \leq \langle q_1,q_2\rangle$.

$$\begin{array}{ccccc} Q & \xrightarrow{v_1} & R & \xrightarrow{r_1} & P \\ {\scriptstyle\langle q_1,q_2\rangle}\downarrow & & \downarrow{\scriptstyle\langle r,p_2r_1\rangle} & & \downarrow{\scriptstyle\langle p_1,p_2\rangle} \\ C\times C & \xrightarrow[1\times g]{} & C\times B & \xrightarrow[g\times 1]{} & B\times B \end{array}$$

■

5.17. Corollary. *Suppose $f = gh$ with h an epimorphism. Then $L(g) \subseteq L(f)$.*

Proof. Let $g = uv$ and consequently $f = uvh$. So the kernel pair of v is related to $(1,1)$ by $L(g)$ and the kernel pair of vh is related to $(1,1)$ by $L(f)$. Since the kernel pairs of h and vh are related by $L(f)$, by using $\forall_{h\times h}$ and Lemma 5.16, the kernel pair of v is related to $(1,1)$ by $L(f)$. ■

6. Booleanization of a topos

In this section, we aim to associate a Boolean topos to each topos in a universal way. To achieve this, we introduce a class of morphisms called logical classes, which support certain logical operations. Using this, we construct a quotient of spans, yielding the associated Boolean topos.

6.1. Definition. *Let $\mathcal{T}$ be a topos and let $\mathcal{W}$ be a class of morphisms in $\mathcal{T}$. We call $\mathcal{W}$ a* logical class *if it satisfies the following conditions:*

- *$\mathcal{W}$ is closed under composition, pullbacks, and contains all isomorphisms,*
- *$\mathcal{E} \subseteq \mathcal{W}$,*

- *for each $w \in \mathcal{W}$, its $\mathcal{M}$-part is also in $\mathcal{W}$,*
- *for any monomorphism $m \in \mathcal{W}$, and for any monomorphism f and morphism g in $\mathcal{T}$, the morphism $\forall_g m$ is also in $\mathcal{W}$:*

$$\begin{array}{ccccc} \cdot & \xrightarrow{m} & \cdot & \xrightarrow{f} & \cdot \\ & & & & \downarrow{\scriptstyle g} \\ \cdot & \xrightarrow[\forall_g m]{} & \cdot & \xrightarrow[\forall_g f]{} & \cdot \end{array}$$

6.2. Theorem. *If $\mathcal{W}$ is a logical class, then $\sim_{\mathcal{W}}$ is a logical relation.*

Proof. It can be easily verified. ■

In any topos, the morphism $b : 1{+}1 \to \Omega$ is a monomorphism. Our goal is to make this morphism an isomorphism. Let $\mathcal{B}(\mathcal{T})$ be the smallest logical class containing $b : 1{+}1 \to \Omega$.

6.3. Theorem. *$\mathsf{Map}(\mathsf{Span}_{\mathcal{B}(\mathcal{T})}(\mathcal{T}))$ is a Boolean topos.*

Proof. Since $(b, b) \sim_{\mathcal{B}(\mathcal{T})} (1, 1)$, we have $[b, b]_{\mathcal{B}(\mathcal{T})} = [1, 1]_{\mathcal{B}(\mathcal{T})}$. Thus, $[1, b]_{\mathcal{B}(\mathcal{T})}$ is a retraction. Because b is mono, we get $[b, 1]_{\mathcal{B}(\mathcal{T})}[1, b]_{\mathcal{B}(\mathcal{T})} = 1$. Hence, $[1, b]$ is an isomorphism in $\mathsf{Span}_{\mathcal{B}(\mathcal{T})}(\mathcal{T})$, and therefore also in $\mathsf{Map}(\mathsf{Span}_{\mathcal{B}(\mathcal{T})}(\mathcal{T}))$.

By Theorem 5.10, the functor $\eta : \mathcal{T} \longrightarrow \mathsf{Map}(\mathsf{Span}_{\mathcal{B}(\mathcal{T})}(\mathcal{T}))$ is logical. By [Johnstone, 2002, Corollary 2.2.10], η is cocartesian. Thus,

$$\eta(b : 1 + 1 \to \Omega) = [1, b] : 1 + 1 \to \Omega.$$

So $\mathsf{Map}(\mathsf{Span}_{\mathcal{B}(\mathcal{T})}(\mathcal{T}))$ is a Boolean topos, as required. ■

We now show that this construction is universal.

6.4. Lemma. *For any logical functor $F : \mathcal{T} \longrightarrow \mathcal{T}'$, we have $F(\mathcal{B}(\mathcal{T})) \subseteq \mathcal{B}(\mathcal{T}')$.*

Proof. Since F preserves epis, monos, and $\forall$, one can easily check that $F^{-1}(\mathcal{B}(\mathcal{T}'))$ is a logical class. Because $F(b) = b'$, we have $b \in F^{-1}(\mathcal{B}(\mathcal{T}'))$, and thus $\mathcal{B}(\mathcal{T}) \subseteq F^{-1}(\mathcal{B}(\mathcal{T}'))$. Therefore, $F(\mathcal{B}(\mathcal{T})) \subseteq \mathcal{B}(\mathcal{T}')$. ■

6.5. Theorem. *Let $F : \mathcal{T} \longrightarrow \mathcal{T}'$ be a logical functor.*

(a) The map $PF : \mathsf{Span}_{\mathcal{B}(\mathcal{T})}(\mathcal{T}) \longrightarrow \mathsf{Span}_{\mathcal{B}(\mathcal{T}')}(\mathcal{T}')$ defined by $[f, g]_{\mathcal{B}(\mathcal{T})} \mapsto [Ff, Fg]_{\mathcal{B}(\mathcal{T}')}$ is a representation of allegories.

(b) $\mathsf{Map}(PF) : \mathsf{Map}(\mathsf{Span}_{\mathcal{B}(\mathcal{T})}(\mathcal{T})) \longrightarrow \mathsf{Map}(\mathsf{Span}_{\mathcal{B}(\mathcal{T}')}(\mathcal{T}'))$ is a logical functor.

Proof. For (a), Lemma 6.4 ensures the map is well-defined. The rest follows from the fact that F preserves pullbacks. For (b), the result follows from the definition of $\in_A$ in both allegories. ■

Let $\mathsf{BoolTop}$ denote the category whose objects are Boolean toposes and whose morphisms are logical functors. This forms a subcategory of the category Top of toposes and logical functors. Using Theorem 6.5, we define the functor

$$\mathbf{Bool} : \mathsf{Top} \longrightarrow \mathsf{BoolTop}$$

where $\mathbf{Bool}(F)$ and $\mathbf{Bool}(\mathcal{T})$ denote $\mathsf{Map}(PF)$ and $\mathsf{Map}(\mathsf{Span}_{\mathcal{B}(\mathcal{T})}(\mathcal{T}))$, respectively.

6.6. THEOREM. $\mathsf{BoolTop}$ *is a reflective subcategory of* Top.

PROOF. We show that the functor $\mathbf{Bool}$ is left adjoint to the inclusion functor $\iota : \mathsf{BoolTop} \longrightarrow \mathsf{Top}$. It suffices to show that

$$\eta : \mathcal{T} \longrightarrow \mathsf{Map}(\mathsf{Span}_{\mathcal{B}(\mathcal{T})}(\mathcal{T})) = \iota \cdot \mathbf{Bool}(\mathcal{T})$$

is universal.

Let $F : \mathcal{T} \longrightarrow \mathcal{T}' = \iota(\mathcal{T}')$ be a logical functor. Since F is cocartesian and $\mathcal{T}'$ is a Boolean topos, $F(b)$ is an isomorphism. Theorem 6.5 yields the functor $\mathsf{Bool}(F) : \mathsf{Bool}(\mathcal{T}) \longrightarrow \mathsf{Bool}(\mathcal{T}')$. It is easy to check that $\mathcal{B}(\mathcal{T}') = \mathcal{E}$, hence $\mathsf{Bool}(\mathcal{T}') = \mathcal{T}'$. So we have the commutative triangle:

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\quad \eta \quad} & \mathsf{Map}(\mathsf{Span}_{\mathcal{B}(\mathcal{T})}(\mathcal{T})) = \iota \cdot \mathsf{Bool}(\mathcal{T}) \\ {\scriptstyle F}\downarrow & \swarrow {\scriptstyle \mathsf{Bool}(F)} & \\ \mathcal{T}' & & \end{array}$$

For uniqueness, let $[f, g]_{\mathcal{B}(\mathcal{T})}$ be a map in $\mathsf{Span}_{\mathcal{B}(\mathcal{T})}(\mathcal{T})$. Then $[f, 1]_{\mathcal{B}(\mathcal{T})}$ is an isomorphism with inverse $[1, f]_{\mathcal{B}(\mathcal{T})}$. For any functor G such that $G \circ \eta = F$, we compute:

$$\begin{aligned} G[f, g]_{\mathcal{B}(\mathcal{T})} &= G[1, g]_{\mathcal{B}(\mathcal{T})} \cdot G[f, 1]_{\mathcal{B}(\mathcal{T})} \\ &= G[1, g]_{\mathcal{B}(\mathcal{T})} \cdot (G[1, f]_{\mathcal{B}(\mathcal{T})})^{-1} \\ &= G\eta(g) \cdot (G\eta(f))^{-1} \\ &= F(g) \cdot F(f)^{-1} \\ &= \mathsf{Bool}(F)[f, g]_{\mathcal{B}(\mathcal{T})}. \end{aligned}$$

This completes the proof. ∎

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