

# MULTICATEGORIES FROM SYMMETRIC MONOIDAL CATEGORIES

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**ABSTRACT.** This paper considers the possible underlying multicategories for a symmetric monoidal category, and shows that, up to canonical and coherent isomorphism, there really is only one. As a result, there is a well-defined forgetful functor from symmetric monoidal categories to multicategories, as long as all morphisms of symmetric monoidal categories are at least lax symmetric monoidal.

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## 1. Introduction

The main aim of this paper is to give a rigorous treatment of the folk theorem that a symmetric monoidal category has a canonical underlying multicategory. The problem with this statement is that it is false: a general symmetric monoidal category has many different underlying multicategories, and there is no truly natural choice of which one to use. However, even if we use an absurdly large collection of underlying multicategories, as we will do in this paper, they are all canonically and coherently isomorphic. As a result, it doesn’t matter which ones we use, or even if we use different ones for different symmetric monoidal categories: we still obtain a functor from symmetric monoidal categories (with lax monoidal functors between them) to multicategories, given by whatever underlying multicategory we choose for each symmetric monoidal category.

This underlying multicategory functor has a “weak” left adjoint, whose construction is the same as the actual left adjoint to the underlying multicategory functor from permutative categories and strict maps, as constructed in [3]. However, the “counit” of the

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adjunction is only natural with respect to strict maps, which is far too restrictive in general. The adjunction triangles do however commute, so we do get an adjunction comonad on symmetric monoidal categories, which provides us with a strictification construction different from that given in [5].

Previous work on this issue has concentrated on monoidal categories that need not be symmetric: see, for example, Hermida's work in [4], especially section 7, as well as section 3.3 of Leinster's [6]. In particular, Leinster discusses symmetric multicategories in Section A.2, but doesn't relate them to symmetric monoidal categories, which he doesn't appear to discuss at all.

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## 2. Definitions and Results

We begin by describing the enormous array of possible underlying multicategories for a symmetric monoidal category, and this relies on the categorical operad  $Y$  described in [1] whose algebras are precisely symmetric monoidal categories. The objects of the component category  $Y(n)$  can be thought of as a complete parenthesization of a list of length at least  $n$ , together with a way of permuting and inserting  $n$  variables into slots in the list. The rest of the slots are to be filled with identity elements. There is a unique morphism from any object of  $Y(n)$  to any other, so all diagrams in  $Y(n)$  commute. Then we will describe one underlying multicategory for a symmetric monoidal category  $\mathcal{C}$  for each sequence of functions

$$\kappa_n : \text{Ob}\mathcal{C}^n \rightarrow \text{Ob}Y(n)$$

for  $n \geq 0$ , with no further structure. We give an example of such a sequence at the beginning of Section 7. We will write this underlying multicategory as  $U_\kappa\mathcal{C}$ . The idea is that for each  $n$ -tuple  $\mathbf{x} = (x_1, \dots, x_n)$  of objects of  $\mathcal{C}$ , the image  $\kappa_n\mathbf{x}$  tells us how to combine the objects into a single object using the symmetric monoidal structure of  $\mathcal{C}$ . Formally, we obtain an object  $\kappa_n(\mathbf{x})(\mathbf{x})$  for each  $\mathbf{x} \in \mathcal{C}^n$ ; we will abbreviate this, and write

$$\kappa_n(\mathbf{x})(\mathbf{x}) = \bar{\kappa}\mathbf{x}.$$

For notation, we will always write a morphism set in a multicategory with a semicolon separating the source list from the target object. Then the formal definition of the morphism sets in  $U_\kappa\mathcal{C}$  is almost as follows:

2.1. DEFINITION. Given an  $n$ -tuple  $\mathbf{x} = (x_1, \dots, x_n)$  of objects of  $\mathcal{C}$  and a single object  $y$  of  $\mathcal{C}$ , we define

$$U_\kappa \mathcal{C}(\mathbf{x}; y) = \mathcal{C}(\bar{\kappa}\mathbf{x}, y).$$

The reason this is almost, but not quite, the correct definition is described at the beginning of Section 3.

Our first major theorem is then

2.2. THEOREM. The morphism sets  $U_\kappa \mathcal{C}(\mathbf{x}; y)$  form a multicategory, with objects the objects of  $\mathcal{C}$ .

The structure is given in Section 3; the verification that the properties of a multicategory are satisfied is somewhat lengthy, and deferred until Section 7.

Our next major theorem shows that these underlying multicategories are all essentially the same.

2.3. THEOREM. Let  $\lambda_n : \text{Ob}\mathcal{C}^n \rightarrow \text{Ob}Y(n)$  be another sequence of functions defining another underlying multicategory  $U_\lambda \mathcal{C}$ . Then there is a canonical isomorphism of multicategories

$$U_\kappa \mathcal{C} \rightarrow U_\lambda \mathcal{C}.$$

Further, given a third sequence  $\rho_n : \text{Ob}\mathcal{C}^n \rightarrow \text{Ob}Y(n)$ , the diagram of canonical isomorphisms

$$\begin{array}{ccc} U_\kappa \mathcal{C} & \xrightarrow{\quad} & U_\lambda \mathcal{C} \\ & \searrow \quad \swarrow & \\ & U_\rho \mathcal{C} & \end{array}$$

commutes, so the isomorphisms are both canonical and coherent.

The proof is in Section 4.

In order for theorem 2.3 to give us an actual underlying multicategory functor, we need the morphisms between symmetric monoidal categories to induce morphisms between the underlying multicategories. To this end, we have:

2.4. THEOREM. Let  $\mathcal{C}$  and  $\mathcal{D}$  be symmetric monoidal categories, with arbitrary choices  $U\mathcal{C}$  and  $U\mathcal{D}$  of underlying multicategories. Then a map of multicategories (a multifunctor)

$$U\mathcal{C} \rightarrow U\mathcal{D}$$

determines and is determined by a lax symmetric monoidal functor  $\mathcal{C} \rightarrow \mathcal{D}$ . This assignment produces a fully faithful functor from symmetric monoidal categories with lax symmetric monoidal functors to multicategories with multifunctors.

The proof is in Section 5. As a result, Theorem 2.3 shows that any choices of underlying multicategories produce canonically isomorphic underlying multicategory functors.

2.5. COROLLARY. *Any category of symmetric monoidal categories, all of whose morphism functors are at least lax symmetric monoidal, supports an underlying multicategory functor from symmetric monoidal categories to multicategories.*

Our last major theorem gives this underlying multicategory functor a weak left adjoint, where “weak” means that the counit map is only lax natural.

2.6. THEOREM. *The forgetful functor from symmetric monoidal categories to multicategories has a weak left adjoint, as long as we include strong symmetric monoidal functors as maps of symmetric monoidal categories. The adjunction is weak in the sense that the counit is only natural up to a natural transformation.*

The construction is exactly the same as the left adjoint constructed in [3], Theorem 4.2, in the context of permutative categories and strict maps. The proof is in Section 6. Since the unit of the weak adjunction is actually natural, we do get a comonad, although only a weak monad. We also observe the following in Section 6:

2.7. THEOREM. *The weak left adjoint from Theorem 2.6 converts multicategories and multifunctors into permutative categories and strict maps, so the comonad of the weak adjunction converts symmetric monoidal categories and lax symmetric monoidal functors into permutative categories and strict maps. The (weak) counit of the adjunction is a homotopy equivalence of categories.*

This is not quite the same as the strictification construction due originally to Isbell [5], and made explicit by May in [7]. We discuss the comparison and differences in Section 6.

### 3. Underlying Multicategories: Structure

In this section we give the structure of  $U_\kappa\mathcal{C}$  as a multicategory, following the definition of a multicategory given in [2], Section 2. (There are other sources for the definition, but we will use this one for reference.) The objects of  $U_\kappa\mathcal{C}$  are just the objects of  $\mathcal{C}$ , and we have already almost defined the morphism sets, namely, given a list  $\mathbf{x} = (x_1, \dots, x_n)$  of objects of  $\mathcal{C}$  and a target object  $y$ , we would like to define

$$U_\kappa\mathcal{C}(\mathbf{x}; y) = \mathcal{C}(\bar{\kappa}\mathbf{x}, y) = \mathcal{C}(\kappa_n(\mathbf{x})(\mathbf{x}), y).$$

However, this definition obscures a subtle technical point that will be important later: if  $\mathbf{x} \neq \mathbf{x}'$ , then  $U_\kappa\mathcal{C}(\mathbf{x}; y)$  must be disjoint from  $U_\kappa\mathcal{C}(\mathbf{x}'; y)$ . For example, if  $a, b \in \mathcal{C}$ , and we choose  $\bar{\kappa}(a, b) = a \oplus b = \bar{\kappa}(a \oplus b)$ , the proposed definition would say that

$$U_\kappa\mathcal{C}(a, b; a \oplus b) = \mathcal{C}(a \oplus b, a \oplus b) = U_\kappa\mathcal{C}(a \oplus b; a \oplus b),$$

which makes  $\text{id}_{a \oplus b}$  play the role of both a 2-morphism and a 1-morphism. To avoid this problem rigorously, we would need to define instead

$$U_\kappa\mathcal{C}(\mathbf{x}; y) = (\mathbf{x}, \mathcal{C}(\bar{\kappa}\mathbf{x}, y))$$

in order to explicitly keep track of the source string in  $U_\kappa \mathcal{C}$ . However, to avoid making notation more cumbersome than strictly necessary we will continue to drop the explicit source string, and just use  $\mathcal{C}(\bar{\kappa}\mathbf{x}, y)$ . When necessary, we will indicate whether an element is to be considered an element of a particular arity by using a superscript: so in the example,  $\text{id}_{a \oplus b}^2$  will indicate  $\text{id}_{a \oplus b}$  as an element of  $U_\kappa \mathcal{C}(a, b; a \oplus b)$ , while  $\text{id}_{a \oplus b}^1$  will indicate  $\text{id}_{a \oplus b}$  as an element of  $U_\kappa \mathcal{C}(a \oplus b; a \oplus b)$ .

We must specify identity elements in  $U_\kappa \mathcal{C}(a; a) = \mathcal{C}(\bar{\kappa}(a), a)$  for each object  $a$  of  $\mathcal{C}$ . Now  $\bar{\kappa}(a) = \kappa_1(a)(a)$ , and  $\kappa_1(a)$  is an object of  $Y(1)$ , which also has the canonical identity object  $1 \in Y(1)$ . We therefore have a unique isomorphism  $\omega(a) : \kappa_1(a) \rightarrow 1$  in  $Y(1)$ , and applying this to  $a$  itself, we get a canonical isomorphism in  $\mathcal{C}$ :

$$\omega(a)(a) : \bar{\kappa}(a) = \kappa_1(a)(a) \rightarrow 1 \cdot a = a,$$

which we take as our identity element in  $U_\kappa \mathcal{C}(a; a) = \mathcal{C}(\bar{\kappa}(a), a)$ .

We must give a right action of  $\Sigma_n$  on the morphism sets with source  $n$ -tuples of objects. For  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\sigma \in \Sigma_n$ , let's write

$$\sigma^{-1}\mathbf{x} = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

in accordance with the standard left action of  $\Sigma_n$ . Then we need to define a map

$$\sigma^* : U_\kappa \mathcal{C}(\mathbf{x}; y) \rightarrow U_\kappa \mathcal{C}(\sigma^{-1}\mathbf{x}; y)$$

which we will verify does give a right action in Section 7. We define  $\sigma^*$  by appealing to the definition of an operad action to see that

$$\bar{\kappa}\mathbf{x} = \kappa_n(\mathbf{x})(\mathbf{x}) = (\kappa_n(\mathbf{x}) \cdot \sigma)(\sigma^{-1}\mathbf{x}).$$

Since  $\kappa_n(\sigma^{-1}\mathbf{x})$  and  $\kappa_n(\mathbf{x}) \cdot \sigma$  are both objects of  $Y(n)$  there is a unique isomorphism

$$\theta(\mathbf{x}, \sigma) : \kappa_n(\sigma^{-1}\mathbf{x}) \rightarrow \kappa_n(\mathbf{x}) \cdot \sigma.$$

As a result, we have a canonical map, which is an isomorphism,

$$\bar{\kappa}(\sigma^{-1}\mathbf{x}) = \kappa_n(\sigma^{-1}\mathbf{x})(\sigma^{-1}\mathbf{x}) \xrightarrow{\theta(\mathbf{x}, \sigma)} \kappa_n(\mathbf{x}) \cdot \sigma \sigma^{-1}\mathbf{x} = \kappa_n(\mathbf{x})(\mathbf{x}) = \bar{\kappa}\mathbf{x}$$

which induces our desired map

$$\sigma^* : U_\kappa \mathcal{C}(\mathbf{x}; y) = \mathcal{C}(\bar{\kappa}\mathbf{x}, y) \rightarrow \mathcal{C}(\bar{\kappa}(\sigma^{-1}\mathbf{x}), y) = U_\kappa \mathcal{C}(\sigma^{-1}\mathbf{x}; y).$$

Explicitly, we can say  $\sigma^* = \mathcal{C}(\theta(\mathbf{x}, \sigma), 1)$ .

Finally, we must specify a multiproduct that gives the composition in  $U_\kappa \mathcal{C}$ , and in order to do so efficiently, we need some notation. Suppose we are given a final target object  $z$ , a tuple  $\mathbf{y} = (y_1, \dots, y_n)$  that we will map to  $z$ , and for each  $s$  with  $1 \leq s \leq n$ , a tuple  $\mathbf{x}_s = (x_{s1}, \dots, x_{sj_s})$  that we will map to  $y_s$ . Let  $j = j_1 + \dots + j_n$ , and write  $\odot_s \mathbf{x}_s$  for

the  $j$ -tuple that is the concatenation of all the  $\mathbf{x}_s$ 's. Then we must specify a composition multiproduct

$$\Gamma : U_\kappa \mathcal{C}(\mathbf{y}; z) \times \prod_{s=1}^n U_\kappa \mathcal{C}(\mathbf{x}_s; y_s) \rightarrow U_\kappa \mathcal{C}(\odot_s \mathbf{x}_s; z).$$

But using the definition of  $U_\kappa \mathcal{C}$ , this means giving a map

$$\Gamma : \mathcal{C}(\bar{\kappa} \mathbf{y}, z) \times \prod_{s=1}^n \mathcal{C}(\bar{\kappa} \mathbf{x}_s, y_s) \rightarrow \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{x}_s), z).$$

Now we need just a bit more notation. First, given a list of items such as  $\bar{\kappa} \mathbf{x}_1, \dots, \bar{\kappa} \mathbf{x}_n$ , we write the entire list as  $\langle \bar{\kappa} \mathbf{x}_s \rangle$ , with the subscript presumed to run over appropriate limits. Next, since we are using  $\Gamma$  for our composition in  $U_\kappa \mathcal{C}$ , we will use  $\gamma$  for the operad operation in  $Y$  to avoid confusion. (We used  $\Gamma$  in [1] for the operation in  $Y$ .)

We can now rewrite

$$\prod_{s=1}^n \mathcal{C}(\bar{\kappa} \mathbf{x}_s, y_s) = \mathcal{C}^n(\langle \bar{\kappa} \mathbf{x}_s \rangle, \mathbf{y}),$$

and noting that since  $\kappa_n(\mathbf{y})$  is an object of  $Y(n)$ , it induces a functor  $\mathcal{C}^n \rightarrow \mathcal{C}$ . Further, both  $\kappa_j(\odot_s \mathbf{x}_s)$  and  $\gamma(\kappa_n \mathbf{y}; \langle \kappa_{j_s} \mathbf{x}_s \rangle)$  are objects of  $Y(j)$ , and so there is a unique isomorphism

$$\phi(\mathbf{y}, \langle \mathbf{x}_s \rangle) : \kappa_j(\odot_s \mathbf{x}_s) \rightarrow \gamma(\kappa_n \mathbf{y}; \langle \kappa_{j_s} \mathbf{x}_s \rangle),$$

which we can apply to the object  $\odot_s \mathbf{x}_s$  to obtain an isomorphism which we abusively denote with the same notation:

$$\phi(\mathbf{y}, \langle \mathbf{x}_s \rangle) : \bar{\kappa}(\odot_s \mathbf{x}_s) \rightarrow \gamma(\kappa_n \mathbf{y}; \langle \kappa_{j_s} \mathbf{x}_s \rangle)(\odot_s \mathbf{x}_s) = \kappa_n \mathbf{y} \langle \bar{\kappa} \mathbf{x}_s \rangle,$$

and consequently an induced map

$$\mathcal{C}(\phi(\mathbf{y}, \langle \mathbf{x}_s \rangle), 1) : \mathcal{C}(\kappa_n(\mathbf{y}) \langle \bar{\kappa} \mathbf{x}_s \rangle, z) \rightarrow \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{x}_s), z).$$

We now define our composition multiproduct in  $U_\kappa \mathcal{C}$  as the following composite:

$$\begin{aligned} & \mathcal{C}(\bar{\kappa} \mathbf{y}, z) \times \prod_{s=1}^n \mathcal{C}(\bar{\kappa} \mathbf{x}_s, y_s) \\ & \quad \downarrow = \\ & \mathcal{C}(\bar{\kappa} \mathbf{y}, z) \times \mathcal{C}^n(\langle \bar{\kappa} \mathbf{x}_s \rangle, \mathbf{y}) \\ & \quad \downarrow 1 \times \kappa_n(\mathbf{y}) \\ & \mathcal{C}(\bar{\kappa} \mathbf{y}, z) \times \mathcal{C}(\kappa_n(\mathbf{y}) \langle \bar{\kappa} \mathbf{x}_s \rangle, \bar{\kappa} \mathbf{y}) \\ & \quad \downarrow \circ \\ & \mathcal{C}(\kappa_n(\mathbf{y}) \langle \bar{\kappa} \mathbf{x}_s \rangle, z), \\ & \quad \downarrow \mathcal{C}(\phi(\mathbf{y}, \langle \mathbf{x}_s \rangle), 1) \\ & \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{x}_s), z). \end{aligned}$$

This completes the specification of the structure of  $U_\kappa\mathcal{C}$  as a multicategory. The verification that this structure satisfies the properties of a multicategory is deferred to Section 7.

#### 4. Isomorphisms of Underlying Multicategories

In this section we prove Theorem 2.3, which shows that all the multicategories  $U_\kappa\mathcal{C}$  for all possible values of  $\kappa$  are canonically and coherently isomorphic. This means that we can reasonably speak of “the” underlying multicategory for a symmetric monoidal category, since any one is unique up to a unique canonical isomorphism.

We suppose given two arbitrary sequences

$$\kappa_n : \text{Ob}\mathcal{C}^n \rightarrow \text{Ob}Y(n) \text{ and } \lambda_n : \text{Ob}\mathcal{C} \rightarrow \text{Ob}Y(n),$$

which determine underlying multicategories  $U_\kappa\mathcal{C}$  and  $U_\lambda\mathcal{C}$ . The proof of the theorem consists of the definition of the canonical isomorphism  $U_\kappa\mathcal{C} \rightarrow U_\lambda\mathcal{C}$ , the verification that the coherence diagram commutes, and finally the verification that our map is an isomorphism of multicategories. We begin with the definition of the canonical isomorphism.

**4.1. DEFINITION.** *Let  $\kappa_n : \text{Ob}\mathcal{C}^n \rightarrow \text{Ob}Y(n)$  and  $\lambda_n : \text{Ob}\mathcal{C}^n \rightarrow \text{Ob}Y(n)$  be any two sequences of functions. Then given an object  $\mathbf{x} \in \mathcal{C}^n$ , we have the objects  $\kappa_n\mathbf{x}$  and  $\lambda_n\mathbf{x}$  of  $Y(n)$ , and we define*

$$\alpha(\mathbf{x}) : \lambda_n\mathbf{x} \rightarrow \kappa_n\mathbf{x}$$

*to be the unique isomorphism in  $Y(n)$  with the given source and target.*

*To define an isomorphism of multicategories  $U_\kappa\mathcal{C} \rightarrow U_\lambda\mathcal{C}$ , we must give maps on objects and on sets of morphisms. On objects, we just use  $\text{id}_{\text{Ob}\mathcal{C}}$ , since both underlying multicategories have  $\text{Ob}\mathcal{C}$  as their objects. On morphisms, for each  $\mathbf{x} \in \mathcal{C}^n$ , we have the induced map, which technically should be labeled  $\alpha(\mathbf{x})(\mathbf{x})$ ,*

$$\bar{\lambda}\mathbf{x} = \lambda_n(\mathbf{x})(\mathbf{x}) \xrightarrow{\alpha(\mathbf{x})} \kappa_n(\mathbf{x})(\mathbf{x}) = \bar{\kappa}\mathbf{x},$$

*which in turn induces our desired map*

$$U_\kappa(\mathbf{x}; y) = \mathcal{C}(\bar{\kappa}\mathbf{x}, y) \xrightarrow{\mathcal{C}(\alpha(\mathbf{x}), 1)} \mathcal{C}(\bar{\lambda}\mathbf{x}, y) = U_\lambda(\mathbf{x}; y).$$

It follows immediately from the definition that these maps are coherent, in the sense given in the theorem and in greater detail in the following corollary.

**4.2. COROLLARY.** *Let  $\rho_n : \text{Ob}\mathcal{C}^n \rightarrow \text{Ob}Y(n)$  be a third set of functions defining an underlying multicategory  $U_\rho\mathcal{C}$ , let*

$$\beta(\mathbf{x}) : \rho_n\mathbf{x} \rightarrow \lambda_n\mathbf{x} \text{ and } \delta(\mathbf{x}) : \rho_n\mathbf{x} \rightarrow \kappa_n\mathbf{x}$$

be the unique isomorphisms in  $Y(n)$  inducing the alleged isomorphisms of underlying multicategories  $U_\lambda \mathcal{C} \rightarrow U_\rho \mathcal{C}$  and  $U_\rho \mathcal{C} \rightarrow U_\kappa \mathcal{C}$ . Then the diagram of induced isomorphisms

$$\begin{array}{ccc} U_\kappa \mathcal{C} & \xrightarrow{\cong} & U_\lambda \mathcal{C} \\ & \searrow \cong & \swarrow \cong \\ & U_\rho \mathcal{C} & \end{array}$$

commutes. Further, the induced automorphism on any one underlying multicategory is the identity.

PROOF. The isomorphisms of underlying categories are induced by the diagrams of isomorphisms in  $Y(n)$

$$\begin{array}{ccc} \kappa_n \mathbf{x} & \xleftarrow{\alpha(\mathbf{x})} & \lambda_n \mathbf{x} \\ & \nwarrow \delta(\mathbf{x}) \quad \nearrow \beta(\mathbf{x}) & \\ & \rho_n \mathbf{x} & \end{array}$$

which commute since they are diagrams in  $Y(n)$ , where all diagrams commute. If we set  $\lambda = \kappa$ , then  $\alpha(\mathbf{x}) = \text{id}_{\kappa_n \mathbf{x}}$ , so the automorphism of  $U_\kappa \mathcal{C}$  is the identity. ■

We must show that this definition actually preserves multicategory structure. Since the inducing maps  $\alpha(\mathbf{x})$  are all isomorphisms, the induced maps on morphism sets are all bijections, so this will show that we do have an isomorphism of multicategories. We begin with the identity structure.

4.3. PROPOSITION. *The identity element  $\text{id}_a \in U_\kappa \mathcal{C}(a; a)$  is sent to  $\text{id}_a \in U_\lambda \mathcal{C}(a; a)$ .*

PROOF. We have the unique isomorphisms in  $Y(1)$

$$\omega(a) : \kappa_1(a) \rightarrow 1 \text{ and } \omega'(a) : \lambda_1(a) \rightarrow 1$$

which applied to  $a$  give us the identity elements in  $U_\kappa \mathcal{C}(a; a) = \mathcal{C}(\bar{\kappa}a, a)$  and  $U_\lambda \mathcal{C}(a; a) = \mathcal{C}(\bar{\lambda}a, a)$ . But we also have the diagram

$$\begin{array}{ccc} \lambda_1 a & \xrightarrow{\alpha(a)} & \kappa_1 a \\ & \searrow \omega'(a) \quad \swarrow \omega(a) & \\ & 1 & \end{array}$$

in  $Y(1)$ , where all diagrams commute, and so

$$U_\kappa \mathcal{C}(a; a) = \mathcal{C}(\bar{\kappa}a, a) \xrightarrow{\mathcal{C}(\alpha(a), 1)} \mathcal{C}(\bar{\lambda}a, a) = U_\lambda \mathcal{C}(a; a)$$

sends  $\omega(a)$  to  $\omega'(a)$ , and therefore preserves the identity morphisms. ■

Next, we show that composition is preserved. Following the definition of the composition in  $U_\kappa \mathcal{C}$  given in Section 3, we assume given a final target object  $z$ , an  $n$ -tuple  $\mathbf{y} = (y_1, \dots, y_n)$  mapping to  $z$ , and for each  $1 \leq s \leq n$ , a  $j_s$ -tuple  $\mathbf{x}_s$  that will map to  $y_s$ . Also as before, we say  $j = j_1 + \dots + j_n$ , and the concatenation  $\odot_s \mathbf{x}_s$  of all the  $\mathbf{x}_s$ 's is therefore a  $j$ -tuple.

4.4. PROPOSITION. *The maps of morphism sets defined above preserve composition, meaning that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C}(\bar{\kappa}\mathbf{y}, z) \times \mathcal{C}^n(\langle \bar{\kappa}\mathbf{x}_s \rangle, \mathbf{y}) & \xrightarrow{\Gamma} & \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{x}_s), z) \\ \downarrow \mathcal{C}(\alpha(\mathbf{y}), 1) \times \mathcal{C}^n(\langle \alpha(\mathbf{x}_s) \rangle, 1) & & \downarrow \mathcal{C}(\alpha(\odot_s \mathbf{x}_s), 1) \\ \mathcal{C}(\bar{\lambda}\mathbf{y}, z) \times \mathcal{C}^n(\langle \bar{\lambda}\mathbf{x}_s \rangle, \mathbf{y}) & \xrightarrow{\Gamma} & \mathcal{C}(\bar{\lambda}(\odot_s \mathbf{x}_s), z). \end{array}$$

PROOF. We expand the diagram as follows using the definition of  $\Gamma$ , where  $g$  is induced by the unique isomorphism in  $Y(j)$  given by

$$\gamma(\lambda_n \mathbf{y}; \langle \lambda_{j_s} \mathbf{x}_s \rangle) \rightarrow \gamma(\kappa_n \mathbf{y}; \langle \kappa_{j_s} \mathbf{x}_s \rangle),$$

and  $\phi'(\mathbf{y}, \langle \mathbf{x}_s \rangle) : \bar{\lambda}(\odot_s \mathbf{x}_s) \rightarrow \lambda_n \langle \bar{\lambda} \mathbf{x}_s \rangle$  is the  $\lambda$ -analogue of  $\phi(\mathbf{y}, \langle \mathbf{x}_s \rangle)$ :

$$\begin{array}{ccc} \mathcal{C}(\bar{\kappa}\mathbf{y}, z) \times \mathcal{C}^n(\langle \bar{\kappa}\mathbf{x}_s \rangle, \mathbf{y}) & \xrightarrow{1 \times \kappa_n \mathbf{y}} & \mathcal{C}(\bar{\kappa}\mathbf{y}, z) \times \mathcal{C}(\kappa_n \mathbf{y} \langle \bar{\kappa}\mathbf{x}_s \rangle, \bar{\kappa}\mathbf{y}) \\ \downarrow \mathcal{C}(\alpha(\mathbf{y}), 1) \times \mathcal{C}^n(\langle \alpha(\mathbf{x}_s) \rangle, 1) & & \downarrow \circ \\ \mathcal{C}(\bar{\lambda}\mathbf{y}, z) \times \mathcal{C}^n(\langle \bar{\lambda}\mathbf{x}_s \rangle, \mathbf{y}) & \xrightarrow{\mathcal{C}(\alpha(\mathbf{y}), 1) \times \mathcal{C}(g, \alpha(\mathbf{y})^{-1})} & \mathcal{C}(\kappa_n \mathbf{y} \langle \bar{\kappa}\mathbf{x}_s \rangle, z) \\ \downarrow 1 \times \lambda_n \mathbf{y} & \swarrow \mathcal{C}(g, 1) & \downarrow \mathcal{C}(\phi(\mathbf{y}, \langle \mathbf{x}_s \rangle), 1) \\ \mathcal{C}(\bar{\lambda}\mathbf{y}, z) \times \mathcal{C}(\lambda_n \mathbf{y} \langle \bar{\lambda}\mathbf{x}_s \rangle, \bar{\lambda}\mathbf{y}) & & \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{x}_s), z) \\ \downarrow \circ & \swarrow \mathcal{C}(\phi'(\mathbf{y}, \langle \mathbf{x}_s \rangle), 1) & \downarrow \mathcal{C}(\alpha(\odot_s \mathbf{x}_s), 1) \\ \mathcal{C}(\lambda_n \mathbf{y} \langle \bar{\lambda}\mathbf{x}_s \rangle, z) & \xrightarrow{\mathcal{C}(\phi'(\mathbf{y}, \langle \mathbf{x}_s \rangle), 1)} & \mathcal{C}(\bar{\lambda}(\odot_s \mathbf{x}_s), z). \end{array}$$

The bottom (distorted) square commutes because all the maps are induced by isomorphisms in  $Y(j)$ , where all diagrams commute. The middle (somewhat less distorted) square commutes because we are composing with both  $\alpha(\mathbf{y})$  and its inverse, which then cancel. The top distorted square is the product of two separate squares, and commutes when restricted to the first factor  $\mathcal{C}(\bar{\kappa}\mathbf{y}, z)$  by inspection. This reduces the argument to verifying that the top square commutes when restricted to the second factor, at which point the desired diagram becomes

$$\begin{array}{ccc} \mathcal{C}^n(\langle \bar{\kappa}\mathbf{x}_s \rangle, \mathbf{y}) & \xrightarrow{\kappa_n \mathbf{y}} & \mathcal{C}(\kappa_n \mathbf{y} \langle \bar{\kappa}\mathbf{x}_s \rangle, \bar{\kappa}\mathbf{y}) \\ \downarrow \mathcal{C}^n(\langle \alpha(\mathbf{x}_s) \rangle, 1) & & \downarrow \mathcal{C}(g, \alpha(\mathbf{y})^{-1}) \\ \mathcal{C}^n(\langle \bar{\lambda}\mathbf{x}_s \rangle, \mathbf{y}) & \xrightarrow{\lambda_n \mathbf{y}} & \mathcal{C}(\lambda_n \mathbf{y} \langle \bar{\lambda}\mathbf{x}_s \rangle, \bar{\lambda}\mathbf{y}). \end{array}$$

Tracing a typical element  $\langle f_s \rangle \in \mathcal{C}^n(\langle \bar{\kappa} \mathbf{x}_s \rangle, \mathbf{y})$  through the diagram, we find that commutativity requires us to verify that

$$\lambda_n \mathbf{y} \langle f_s \circ \alpha(\mathbf{x}_s) \rangle = \alpha(\mathbf{y})^{-1} \circ \kappa_n \mathbf{y} \langle f_s \rangle \circ g,$$

or equivalently,

$$\alpha(\mathbf{y}) \circ \lambda_n \mathbf{y} \langle f_s \circ \alpha(\mathbf{x}_s) \rangle = \kappa_n \mathbf{y} \langle f_s \rangle \circ g,$$

where we recall that  $g$  is induced by the unique map in  $Y(j)$

$$\gamma(\lambda_n \mathbf{y}; \langle \lambda_{j_s} \mathbf{x}_s \rangle) \rightarrow \gamma(\kappa_n \mathbf{y}; \langle \kappa_{j_s} \mathbf{x}_s \rangle),$$

therefore giving the canonical isomorphism

$$\lambda_n \mathbf{y} \langle \bar{\lambda} \mathbf{x}_s \rangle \rightarrow \kappa_n \mathbf{y} \langle \bar{\kappa} \mathbf{x}_s \rangle.$$

Our desired equality now becomes the commutativity of the following diagram, in which the top row composes to  $g$ :

$$\begin{array}{ccccc} \lambda_n \mathbf{y} \langle \bar{\lambda} \mathbf{x}_s \rangle & \xrightarrow{\alpha(\mathbf{y})} & \kappa_n \mathbf{y} \langle \bar{\lambda} \mathbf{x}_s \rangle & \xrightarrow{\kappa_n \mathbf{y} \langle \alpha(\mathbf{x}_s) \rangle} & \kappa_n \mathbf{y} \langle \bar{\kappa} \mathbf{x}_s \rangle \\ \lambda_n \mathbf{y} \langle f_s \circ \alpha(\mathbf{x}_s) \rangle \downarrow & & & \searrow \kappa_n \mathbf{y} \langle f_s \circ \alpha(\mathbf{x}_s) \rangle & \downarrow \kappa_n \mathbf{y} \langle f_s \rangle \\ \bar{\lambda} \mathbf{y} & \xrightarrow{\alpha(\mathbf{y})} & & & \bar{\kappa} \mathbf{y}. \end{array}$$

The top row does compose to  $g$ , since it is induced by maps in  $Y(j)$ , where all diagrams commute. The triangle commutes by functoriality of  $\kappa_n \mathbf{y}$ , and the left part of the diagram by naturality of  $\alpha(\mathbf{y})$ . The total diagram therefore commutes, which completes the proof that our map of multicategories preserves composition. ■

To conclude showing that we have a map of multicategories, we must show that the  $\Sigma_n$ -actions on the morphism sets are preserved. This is the content of the following proposition:

**4.5. PROPOSITION.** *Let  $\mathbf{x} \in \mathcal{C}^n$ ,  $y \in \mathcal{C}$ , and  $\sigma \in \Sigma_n$ . Then the following diagram commutes:*

$$\begin{array}{ccc} U_\kappa \mathcal{C}(\mathbf{x}; y) & \xrightarrow{\mathcal{C}(\alpha(\mathbf{x}), 1)} & U_\lambda \mathcal{C}(\mathbf{x}; y) \\ \sigma^* \downarrow & & \downarrow \sigma^* \\ U_\kappa \mathcal{C}(\sigma^{-1} \mathbf{x}; y) & \xrightarrow{\mathcal{C}(\alpha(\sigma^{-1} \mathbf{x}), 1)} & U_\lambda \mathcal{C}(\sigma^{-1} \mathbf{x}; y). \end{array}$$

**PROOF.** We already have the unique isomorphism  $\theta(\mathbf{x}, \sigma) : \kappa_n(\sigma^{-1} \mathbf{x}) \rightarrow \kappa_n \mathbf{x} \cdot \sigma$  in  $Y(n)$  which induces the map  $\sigma^*$  in  $U_\kappa \mathcal{C}$ , and analogously let

$$\pi(\mathbf{x}, \sigma) : \lambda_n(\sigma^{-1} \mathbf{x}) \rightarrow \lambda_n \mathbf{x} \cdot \sigma$$

be the unique isomorphism in  $Y(n)$  inducing the map  $\sigma^*$  in  $U_\lambda \mathcal{C}$ . Then the required square converts and expands as follows:

$$\begin{array}{ccc}
 \mathcal{C}(\bar{\kappa}\mathbf{x}, y) & \xrightarrow{\mathcal{C}(\alpha(\mathbf{x}), 1)} & \mathcal{C}(\bar{\lambda}\mathbf{x}, y) \\
 \downarrow = & & \downarrow = \\
 \mathcal{C}((\kappa_n \mathbf{x} \cdot \sigma)(\sigma^{-1}\mathbf{x}), y) & \xrightarrow{\mathcal{C}(\alpha(\mathbf{x}) \cdot \sigma, 1)} & \mathcal{C}((\lambda_n \mathbf{x} \cdot \sigma)(\sigma^{-1}\mathbf{x}), y) \\
 \downarrow \mathcal{C}(\theta(\mathbf{x}, \sigma), 1) & & \downarrow \mathcal{C}(\pi(\mathbf{x}, \sigma), 1) \\
 \mathcal{C}(\bar{\kappa}(\sigma^{-1}\mathbf{x}), y) & \xrightarrow{\mathcal{C}(\alpha(\sigma^{-1}\mathbf{x}), 1)} & \mathcal{C}(\bar{\lambda}(\sigma^{-1}\mathbf{x}), y).
 \end{array}$$

The top square commutes because both horizontal arrows express the map induced by  $\alpha(\mathbf{x})$ . The bottom square commutes since all the arrows are induced by maps in  $Y(n)$ , where all diagrams commute. The induced map therefore preserves the  $\Sigma_n$ -action. This concludes the proof that we have defined a map of multicategories, which must be an isomorphism since it is a bijection on objects and all morphism sets, and further the inverse is induced by the maps  $\alpha(\mathbf{x})^{-1}$ . ■

## 5. The Relation to Lax Symmetric Monoidal Functors

We have now shown that all of the vast collection of possible underlying multicategories  $U_\kappa \mathcal{C}$  for  $\mathcal{C}$  are canonically and coherently isomorphic, so we obtain an underlying multicategory functor that is unique up to unique isomorphism. This still leaves open the issue of what sort of maps of symmetric monoidal categories can be used to give maps of underlying multicategories. This is answered by Theorem 2.4, whose proof occupies this section.

Before starting the proof of the theorem, it will be convenient to make some assumptions about the sequences  $\{\kappa_n\}$  defining our underlying multicategories. This is justified by the fact that all the underlying multicategories we have defined are canonically and coherently isomorphic, so we can choose any one of them without loss of generality. In particular, throughout this proof, we will assume that both underlying multicategories are defined by sequences of functions that are constant on objects of the same length, so are defined by a sequence of particular objects of  $Y(n)$  for each  $n$ , which we will call just  $\kappa_n$ . We ask that  $\kappa_0 = 0$ , the generator in dimension 0 of the objects of  $Y$ , that  $\kappa_1 = 1$ , the identity element in  $Y(1)$ , and that  $\kappa_2 = m$ , the generator in dimension 2 of the objects of  $Y$ ; note that the action of  $Y$  on any symmetric monoidal category  $\mathcal{C}$  sends 0 to the unit object  $e_{\mathcal{C}}$ , and sends  $m$  to the monoidal product, so  $m \cdot (a, b) = a \oplus b$ . For larger values of  $n$ , we use induction to define

$$\kappa_n = \gamma(m; \kappa_{n-1}, 1).$$

Note however that  $\kappa_1 \neq \gamma(m; \kappa_0, 1)$ , since  $\gamma(m; 0, 1)$  acts on an object  $a$  to produce  $e_{\mathcal{C}} \oplus a$ , while  $\kappa_1 = 1$  acts as the identity.

The effect of our assumed values of the  $\kappa_n$ 's is that we are parenthesizing all products by piling up the parentheses to the left.

We will drop the argument  $\mathbf{x}$  of  $\kappa_n \mathbf{x}$  for an object  $\mathbf{x} \in \mathcal{C}^n$ , since we are assuming  $\kappa_n$  is constant, and instead just write  $\kappa_n$ .

As an additional notational assumption, we suppose give a multifunctor  $\hat{F}$  and wish to produce a lax symmetric monoidal functor  $F$ . Putting the hat on the multifunctor will distinguish it from its induced monoidal functor, while reducing the number of hats in the description.

Now we begin the proof by assuming that we are given a multifunctor  $\hat{F} : U_\kappa \mathcal{C} \rightarrow U_\kappa \mathcal{D}$ ; we must show that this induces a lax symmetric monoidal functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ . In particular, we must produce an induced functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , and show that we have induced natural transformations  $\eta : e_{\mathcal{D}} \rightarrow Fe_{\mathcal{C}}$  and  $\xi : Fa \oplus Fb \rightarrow F(a \oplus b)$ , subject to the following three diagrams, where  $c_a : a \oplus e_{\mathcal{C}} \rightarrow a$  is the unit isomorphism,  $\tau : a \oplus b \rightarrow b \oplus a$  is the commutativity isomorphism, and  $\alpha : (a \oplus b) \oplus c \rightarrow a \oplus (b \oplus c)$  is the associativity isomorphism:

$$\begin{array}{ccc}
 Fa \oplus e_{\mathcal{D}} & \xrightarrow{1 \oplus \eta} & Fa \oplus Fe_{\mathcal{C}} \\
 c_{Fa} \downarrow & & \downarrow \xi \\
 F(a) & \xleftarrow{Fc_a} & F(a \oplus e_{\mathcal{C}}), \\
 \\ 
 (Fa \oplus Fb) \oplus Fc & \xrightarrow{\alpha} & Fa \oplus (Fb \oplus Fc) \\
 \xi \oplus 1 \downarrow & & \downarrow 1 \oplus \xi \\
 F(a \oplus b) \oplus Fc & & Fa \oplus F(b \oplus c) \\
 \xi \downarrow & & \downarrow \xi \\
 F((a \oplus b) \oplus c) & \xrightarrow{F\alpha} & F(a \oplus (b \oplus c)),
 \end{array}$$

and

$$\begin{array}{ccc}
 Fa \oplus Fb & \xrightarrow{\tau} & Fb \oplus Fa \\
 \xi \downarrow & & \downarrow \xi \\
 F(a \oplus b) & \xrightarrow{F\tau} & F(b \oplus a).
 \end{array}$$

We produce our lax monoidal functor as follows.

**5.1. DEFINITION.** *The functor  $F$  is just the underlying functor of the multifunctor. Since we are assuming  $\kappa_1 = 1$ , the identity for  $Y(1)$ , it follows that  $\bar{\kappa}a = a$  for a single object  $a$ , and therefore*

$$U_\kappa \mathcal{C}(a; b) = \mathcal{C}(\bar{\kappa}a, b) = \mathcal{C}(a, b),$$

*and similarly for  $U_\kappa \mathcal{D}$ . We do therefore get a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  by restricting to 1-morphisms.*

*The unit map  $\eta : e_{\mathcal{D}} \rightarrow Fe_{\mathcal{C}}$  arises from the map of 0-morphisms. Since we are assuming  $\kappa_0 = 0 \in Y(0)$ , and since  $\mathcal{C}^0$  is the terminal category with one object which we*

denote  $*$ , and further the action of  $0 \in Y(0)$  by definition selects out the unit element in a symmetric monoidal category, we see that

$$U_\kappa \mathcal{C}(\cdot; y) = \mathcal{C}(\bar{\kappa}(*), y) = \mathcal{C}(0 \cdot *, y) = \mathcal{C}(e_{\mathcal{C}}, y),$$

by the definition of the action of  $Y$  on the symmetric monoidal category  $\mathcal{C}$ , and similarly for  $U_\kappa \mathcal{D}$ . Now given a multifunctor  $\hat{F}$ , we restrict to 0-morphisms and have in particular a map

$$\mathcal{C}(e_{\mathcal{C}}, e_{\mathcal{C}}) = U_\kappa \mathcal{C}(\cdot; e_{\mathcal{C}}) \xrightarrow{\hat{F}_0} U_\kappa \mathcal{D}(\cdot; \hat{F}e_{\mathcal{C}}) = \mathcal{D}(e_{\mathcal{D}}, Fe_{\mathcal{C}}).$$

We define  $\eta : e_{\mathcal{D}} \rightarrow Fe_{\mathcal{C}}$  to be the image of  $\text{id}_{e_{\mathcal{C}}}$  (technically  $\text{id}_{e_{\mathcal{C}}}^0$ ) under this map of 0-morphisms.

We next define the structure map  $\xi : Fa \oplus Fb \rightarrow F(a \oplus b)$ . Since the action of  $Y$  on a symmetric monoidal category sends the generator  $m \in Y(2)$  to the product map, so  $m \cdot (a, b) = a \oplus b$ , and we have assumed  $\kappa_2$  is constant at the element  $m \in Y(2)$ , we can restrict our multifunctor to 2-morphisms, and have

$$\begin{array}{c} \mathcal{C}(a \oplus b, a \oplus b) \\ \downarrow = \\ U_\kappa \mathcal{C}(a, b; a \oplus b) \\ \downarrow \hat{F}_2 \\ U_\kappa \mathcal{D}(\hat{F}a, \hat{F}b; \hat{F}(a \oplus b)) \\ \downarrow = \\ \mathcal{D}(Fa \oplus Fb, F(a \oplus b)), \end{array}$$

and we take  $\xi$  to be the image of  $\text{id}_{a \oplus b}$  (technically  $\text{id}_{a \oplus b}^2$ ) under this map.

We must verify the three coherence diagrams, and begin with the unit coherence diagram.

5.2. LEMMA. *With the above definitions, the diagram*

$$\begin{array}{ccc} Fa \oplus e_{\mathcal{D}} & \xrightarrow{1 \oplus \eta} & Fa \oplus Fe_{\mathcal{C}} \\ c_{Fa} \downarrow & & \downarrow \xi \\ F(a) & \xleftarrow{Fc_a} & F(a \oplus e_{\mathcal{C}}). \end{array}$$

*commutes.*

PROOF. This will follow from the definition of the multicomposition in  $U_\kappa\mathcal{C}$  and  $U_\kappa\mathcal{D}$ , and the multifunctoriality of  $\hat{F}$ . We observe first that we can write  $a \oplus e_C$  as  $\gamma(m; 1, 0) \cdot a$ , and then the unit isomorphism  $c_a : a \oplus e_C \rightarrow a$  is induced by the unique isomorphism  $\gamma(m; 1, 0) \rightarrow 1$  in  $Y(1)$ . Further, this is the inverse of the following isomorphism, which is a special case of the isomorphism  $\phi(\mathbf{y}, \langle \mathbf{x}_s \rangle)$  introduced in the definition of the multiproduct on  $U_\kappa\mathcal{C}$ :

$$\phi((a, e_C), a) : 1 = \kappa_1(a) \rightarrow \gamma(\kappa_2(a, e_C); \kappa_1 a, \kappa_0(*)) = \gamma(m; 1, 0).$$

The key step is to examine the following composition in the multicategory  $U_\kappa\mathcal{C}$ :

$$U_\kappa\mathcal{C}(a, e_C; a \oplus e_C) \times U_\kappa\mathcal{C}(a; a) \times U_\kappa\mathcal{C}(); e_C \rightarrow U_\kappa\mathcal{C}(a; a \oplus e_C).$$

In particular, we look at the triple  $(\text{id}_{a \oplus e_C}^2, \text{id}_a^1, \text{id}_{e_C}^0)$  and find its image in the expanded version of the composition as follows:

$$\begin{array}{c} \mathcal{C}(m \cdot (a, e_C), a \oplus e_C) \times \mathcal{C}^2((a, e_C), (a, e_C)) \\ \downarrow 1 \times m \\ \mathcal{C}(m \cdot (a, e_C), a \oplus e_C) \times \mathcal{C}(m \cdot (a, e_C), m \cdot (a, e_C)) \\ \downarrow \circ \\ \mathcal{C}(m \cdot (a, e_C), a \oplus e_C) \\ \downarrow = \\ \mathcal{C}(\gamma(m; 1, 0) \cdot a, a \oplus e_C) \\ \downarrow \mathcal{C}(\phi((a, e_C), a), 1) \\ \mathcal{C}(a, a \oplus e_C). \end{array}$$

Tracing the triple  $(\text{id}_{a \oplus e_C}, \text{id}_a, \text{id}_{e_C})$  through the composition, we see that at the next-to-the-last step, we have  $\text{id}_{a \oplus e_C}^1$ . This is then composed with  $\phi((a, e_C), a)$ , which is the inverse of the map inducing  $c_a$ , so the composite is  $c_a^{-1}$ .

Now we apply the multifunctor  $\hat{F}$  to the entire composite, so by multifunctoriality, we must end up with  $\hat{F}(c_a^{-1}) = (\hat{F}c_a)^{-1}$ . The starting point is by definition the triple  $(\xi, \text{id}_{Fa}, \eta)$ , and we trace this through the composite

$$U_\kappa\mathcal{D}(Fa, Fe_C; F(a \oplus e_C)) \times U_\kappa\mathcal{D}(Fa; Fa) \times U_\kappa\mathcal{D}(); Fe_C \rightarrow U_\kappa\mathcal{D}(Fa; F(a \oplus e_C)),$$

which expands to

$$\begin{array}{c}
 \mathcal{D}(m \cdot (Fa, Fe_C), F(a \oplus e_C)) \times \mathcal{D}^2((Fa, e_{\mathcal{D}}), (Fa, Fe_C)) \\
 \downarrow 1 \times m \\
 \mathcal{D}(m \cdot (Fa, Fe_C), F(a \oplus e_C)) \times \mathcal{D}(m \cdot (Fa, e_{\mathcal{D}}), m \cdot (Fa, Fe_C)) \\
 \downarrow \circ \\
 \mathcal{D}(m \cdot (Fa, e_{\mathcal{D}}), F(a \oplus e_C)) \\
 \downarrow \mathcal{D}(\phi((Fa, e_{\mathcal{D}}), Fa), 1) \\
 \mathcal{D}(Fa, F(a \oplus e_C)).
 \end{array}$$

Our triple now traces as follows, since the last map is, as before, composition with  $c_{Fa}^{-1}$ :

$$(\xi, \text{id}_{Fa}, \eta) \mapsto (\xi, \text{id}_{Fa \oplus \eta}) \mapsto \xi \circ (\text{id}_{Fa \oplus \eta}) \mapsto \xi \circ (\text{id}_{Fa \oplus \eta}) \circ c_{Fa}^{-1}.$$

Since this must coincide with  $(\hat{F}c_a)^{-1} = (Fc_a)^{-1}$  by multifunctoriality, our desired diagram does commute.  $\blacksquare$

We must verify commutativity of the associativity coherence diagram, and claim:

5.3. LEMMA. *The associativity coherence diagram*

$$\begin{array}{ccc}
 (Fa \oplus Fb) \oplus Fc & \xrightarrow{\alpha} & Fa \oplus (Fb \oplus Fc) \\
 \xi \oplus 1 \downarrow & & \downarrow 1 \oplus \xi \\
 F(a \oplus b) \oplus Fc & & Fa \oplus F(b \oplus c) \\
 \xi \downarrow & & \downarrow \xi \\
 F((a \oplus b) \oplus c) & \xrightarrow{F\alpha} & F(a \oplus (b \oplus c)).
 \end{array}$$

*commutes.*

PROOF. Again we exploit the definition of the multicomposition in the two underlying multicategories, along with the multifunctoriality of  $\hat{F}$ . We note first that by our convention,  $\kappa_3 = \gamma(m; m, 1)$ , and therefore  $\kappa_3(a, b, c) = (a \oplus b) \oplus c$ . Further, the associativity isomorphism  $(a \oplus b) \oplus c \rightarrow a \oplus (b \oplus c)$  is induced by the unique isomorphism in  $Y(3)$

$$\alpha : \kappa_3 = \gamma(m; m, 1) \rightarrow \gamma(m; 1, m).$$

We will need this associativity isomorphism in two different guises: as a 3-morphism in  $U_{\kappa}\mathcal{C}(a, b, c; a \oplus (b \oplus c))$  and as a 1-morphism in  $U_{\kappa}\mathcal{C}((a \oplus b) \oplus c; a \oplus (b \oplus c))$ . Let's call the first of these  $\alpha_3$ , and the second one  $\alpha_1$ .

We begin the argument by showing that the left vertical column in the desired diagram arises from  $\text{id}_{(a \oplus b) \oplus c}^3$  by applying the multifunctor  $\hat{F}$ . We consider the composition in  $U_\kappa \mathcal{C}$  as follows:

$U_\kappa \mathcal{C}(a \oplus b, c; (a \oplus b) \oplus c) \times U_\kappa \mathcal{C}(a, b; a \oplus b) \times U_\kappa \mathcal{C}(c; c) \rightarrow U_\kappa \mathcal{C}(a, b, c; (a \oplus b) \oplus c)$ ,  
and trace the image of the triple  $(\text{id}_{(a \oplus b) \oplus c}^2, \text{id}_{a \oplus b}^2, \text{id}_c^1)$  through its expansion:

$$\begin{array}{c}
 \mathcal{C}(m \cdot (a \oplus b, c), (a \oplus b) \oplus c) \times \mathcal{C}^2((m \cdot (a, b), c), (a \oplus b), c) \\
 \downarrow 1 \times m \\
 \mathcal{C}(m \cdot (a \oplus b, c), (a \oplus b) \oplus c) \times \mathcal{C}(m \cdot (m \cdot (a, b), c), m \cdot (a \oplus b, c)) \\
 \downarrow \circ \\
 \mathcal{C}(m \cdot (m \cdot (a, b), c), (a \oplus b) \oplus c) \\
 \downarrow \mathcal{C}(\phi((a \oplus b, c), ((a, b), c)), 1) \\
 \mathcal{C}((a \oplus b) \oplus c, (a \oplus b) \oplus c).
 \end{array}$$

Now in the last map of this display, we have  $\phi((a \oplus b, c), ((a, b), c)) : \kappa_3 \rightarrow \gamma(\kappa_2; \kappa_2, \kappa_1)$ , but the target here is just

$$\kappa_3 = \gamma(m; m, 1),$$

so the last map is an identity, and our triple ends up at  $\text{id}_{(a \oplus b) \oplus c}^3$ . Applying the multifunctor  $\hat{F}$  throughout will therefore end us up at  $\hat{F}(\text{id}_{(a \oplus b) \oplus c}^3)$ . When we do so, we are looking at the image of the triple

$$(\hat{F}(\text{id}_{(a \oplus b) \oplus c}^2), \hat{F}(\text{id}_{a \oplus b}^2), \hat{F}(\text{id}_c^1)) = (\xi_{a \oplus b, c}, \xi_{a, b}, \text{id}_{\hat{F}c})$$

in the composition in  $U_\kappa \mathcal{D}$  given by

$$\begin{array}{c}
 U_\kappa \mathcal{D}(\hat{F}(a \oplus b), \hat{F}c; \hat{F}((a \oplus b) \oplus c)) \times U_\kappa \mathcal{D}(\hat{F}a, \hat{F}b; \hat{F}(a \oplus b)) \times U_\kappa \mathcal{D}(\hat{F}c, \hat{F}c) \\
 \downarrow \Gamma \\
 U_\kappa \mathcal{D}(\hat{F}a, \hat{F}b, \hat{F}c; \hat{F}((a \oplus b) \oplus c)).
 \end{array}$$

This expands, using the definition of  $\Gamma$ , as follows:

$$\begin{array}{c}
 \mathcal{D}(m \cdot (F(a \oplus b), Fc), F((a \oplus b) \oplus c)) \times \mathcal{D}^2((m \cdot (Fa, Fb), Fc), (F(a \oplus b), Fc)) \\
 \downarrow 1 \times m \\
 \mathcal{D}(m \cdot (F(a \oplus b), Fc), F((a \oplus b) \oplus c)) \times \mathcal{D}(m \cdot (m \cdot (Fa, Fb), Fc), m \cdot (F(a \oplus b), Fc)) \\
 \downarrow \circ \\
 \mathcal{D}(m \cdot (m \cdot (Fa, Fb), Fc), F((a \oplus b) \oplus c)) \\
 \downarrow \mathcal{D}(\phi((F(a \oplus b), Fc), (Fa, Fb, Fc)), 1) \\
 \mathcal{D}((Fa \oplus Fb) \oplus Fc, F((a \oplus b) \oplus c)).
 \end{array}$$

But again, the final map here is just the identity, since  $\kappa_3 = \gamma(m; m, 1)$ . Our source triple therefore traces as

$$(\xi_{a \oplus b, c}, \xi_{a, b}, \text{id}_{Fc}) \mapsto (\xi, \xi \oplus 1) \mapsto \xi \circ (\xi \oplus 1).$$

We may conclude that  $\hat{F}$  sends

$$\text{id}_{(a \oplus b) \oplus c}^3 \in U_\kappa \mathcal{C}(a, b, c; (a \oplus b) \oplus c)$$

to

$$\begin{aligned} \xi \circ (\xi \oplus 1) &\in U_\kappa \mathcal{D}(\hat{F}a, \hat{F}b, \hat{F}c; \hat{F}((a \oplus b) \oplus c)) \\ &= \mathcal{D}((Fa \oplus Fb) \oplus Fc, F((a \oplus b) \oplus c)), \end{aligned}$$

which is the left column of the desired diagram.

Next, we consider the composition in  $U_\kappa \mathcal{C}$

$$U_\kappa \mathcal{C}((a \oplus b) \oplus c; a \oplus (b \oplus c)) \times U_\kappa \mathcal{C}(a, b, c; (a \oplus b) \oplus c) \rightarrow U_\kappa \mathcal{C}(a, b, c; a \oplus (b \oplus c)),$$

and trace the image of the pair  $(\alpha_1, \text{id}_{(a \oplus b) \oplus c}^3)$ , which will turn out to be  $\alpha_3$ . The composition unpacks using the definition just as an ordinary composite, since the usual first step is just the identity, and suppressed, since we are assuming  $\kappa_1 = 1$ :

$$\begin{aligned} &\mathcal{C}((a \oplus b) \oplus c, a \oplus (b \oplus c)) \times \mathcal{C}((a \oplus b) \oplus c, (a \oplus b) \oplus c) \\ &\quad \downarrow \circ \\ &\mathcal{C}((a \oplus b) \oplus c, a \oplus (b \oplus c)). \end{aligned}$$

Tracing the pair  $(\alpha_1, \text{id}_{(a \oplus b) \oplus c}^3)$  through this, we just get  $\alpha$ , but now interpreted as an element of  $U_\kappa \mathcal{C}(a, b, c; (a \oplus b) \oplus c)$ , so we actually end up with  $\alpha_3$ . We can therefore apply  $\hat{F}$  throughout, and find that the pair  $(\hat{F}(\alpha_1), \hat{F}(\text{id}_{(a \oplus b) \oplus c}^3))$  is sent to  $\hat{F}(\alpha_3)$ . But tracing that through the definition, which just unpacks

$$\begin{aligned} &U_\kappa \mathcal{D}(\hat{F}((a \oplus b) \oplus c), \hat{F}(a \oplus (b \oplus c))) \times U_\kappa \mathcal{D}(\hat{F}a, \hat{F}b, \hat{F}c; \hat{F}((a \oplus b) \oplus c)) \\ &\quad \downarrow \Gamma \\ &U_\kappa \mathcal{D}(\hat{F}a, \hat{F}b, \hat{F}c; \hat{F}(a \oplus (b \oplus c))) \end{aligned}$$

as

$$\begin{aligned} &\mathcal{D}(F((a \oplus b) \oplus c), F(a \oplus (b \oplus c))) \times \mathcal{D}((Fa \oplus Fb) \oplus Fc, F((a \oplus b) \oplus c)) \\ &\quad \downarrow \circ \\ &\mathcal{D}((Fa \oplus Fb) \oplus Fc, F(a \oplus (b \oplus c))), \end{aligned}$$

we find that

$$\hat{F}(\alpha_3) = \hat{F}(\alpha_1) \circ \hat{F}(\text{id}_{(a \oplus b) \oplus c}) = F(\alpha) \circ \xi \circ (\xi \oplus 1),$$

where we are justified in removing the hat at the last step since the induced functor  $F$  is just the multifunctor  $\hat{F}$  at the 1-level. This is the counterclockwise direction of our desired diagram.

In the other direction, we begin with the composition in  $U_\kappa \mathcal{C}$  given by

$$U_\kappa \mathcal{C}(a, b \oplus c; a \oplus (b \oplus c)) \times U_\kappa \mathcal{C}(a; a) \times U_\kappa \mathcal{C}(b, c; b \oplus c) \rightarrow U_\kappa \mathcal{C}(a, b, c; a \oplus (b \oplus c)),$$

and trace the triple given by  $(\text{id}_{a \oplus (b \oplus c)}^2, \text{id}_a^1, \text{id}_{b \oplus c}^2)$  through the expanded definition of the composition:

$$\begin{array}{c} \mathcal{C}(m \cdot (a, b \oplus c), a \oplus (b \oplus c)) \times \mathcal{C}^2((a, b \oplus c), (a, b \oplus c)) \\ \downarrow 1 \times m \\ \mathcal{C}(m \cdot (a, b \oplus c), a \oplus (b \oplus c)) \times \mathcal{C}(m \cdot (a, b \oplus c), m \cdot (a, b \oplus c)) \\ \downarrow \circ \\ \mathcal{C}(m \cdot (a, b \oplus c), a \oplus (b \oplus c)) \\ \downarrow \mathcal{C}(\phi((a, b \oplus c), (a, b, c)), 1) \\ \mathcal{C}((a \oplus b) \oplus c, a \oplus (b \oplus c)). \end{array}$$

The triple ends up at  $\text{id}_{a \oplus (b \oplus c)}$  before the final map, but the map  $\phi((a, b \oplus c), (a, b, c))$  is induced by the map in  $Y(3)$

$$\kappa_3 = \gamma(m; m, 1) \rightarrow \gamma(m; 1, m),$$

which is precisely the associativity isomorphism. We can therefore conclude that our triple gets sent to  $\alpha$ , interpreted as  $\alpha_3$ , so applying the multifunctor  $\hat{F}$  throughout, we will end up at  $\hat{F}(\alpha_3)$ .

When we do so, we start with the triple

$$(\hat{F}(\text{id}_{a \oplus (b \oplus c)}^2), \hat{F}(\text{id}_a^1), \hat{F}(\text{id}_{b \oplus c}^2)) = (\xi_{a, b \oplus c}, \text{id}_{\hat{F}a}, \xi_{b, c})$$

in the composition

$$\begin{array}{c} U_\kappa \mathcal{D}(\hat{F}a, \hat{F}(b \oplus c); \hat{F}(a \oplus (b \oplus c))) \times U_\kappa \mathcal{D}(\hat{F}a; \hat{F}a) \times U_\kappa \mathcal{D}(\hat{F}b, \hat{F}c; \hat{F}(b \oplus c)) \\ \downarrow \Gamma \\ U_\kappa \mathcal{D}(\hat{F}a, \hat{F}b, \hat{F}c; \hat{F}(a \oplus (b \oplus c))). \end{array}$$

Expanding using the definition of  $\Gamma$ , we get

$$\begin{array}{c}
\mathcal{D}(m \cdot (Fa, F(b \oplus c)), F(a \oplus (b \oplus c))) \times \mathcal{D}^2((Fa, m \cdot (Fb, Fc)), (Fa, F(b \oplus c))) \\
\downarrow 1 \times m \\
\mathcal{D}(m \cdot (Fa, F(b \oplus c)), F(a \oplus (b \oplus c))) \times \mathcal{D}(m \cdot (Fa, m \cdot (Fb, Fc)), m \cdot (Fa, F(b \oplus c))) \\
\downarrow \circ \\
\mathcal{D}(m \cdot (Fa, m \cdot (Fb, Fc)), F(a \oplus (b \oplus c))) \\
\downarrow \mathcal{D}(\phi((Fa, F(b \oplus c)), (Fa, Fb, Fc)), 1) \\
\mathcal{D}((Fa \oplus Fb) \oplus Fc, F(a \oplus (b \oplus c))).
\end{array}$$

Now  $\phi((Fa, F(b \oplus c)), (Fa, Fb, Fc))$  is again induced by the unique isomorphism in  $Y(3)$

$$\gamma(m; m, 1) \rightarrow \gamma(m; 1, m),$$

which is precisely the isomorphism inducing the associativity isomorphism  $\alpha$ . Tracing our triple  $(\xi_{a, b \oplus c}, \text{id}_{Fa}, \xi_{b, c})$  through the composition, we get

$$(\xi_{a, b \oplus c}, \text{id}_{Fa}, \xi_{b, c}) \mapsto (\xi_{a, b \oplus c}, \text{id}_a \oplus \xi_{b, c}) \mapsto \xi \circ (1 \oplus \xi) \mapsto \xi \circ (1 \oplus \xi) \circ \alpha.$$

We conclude that in this direction, we have

$$\hat{F}(\alpha_3) = \xi \circ (1 \oplus \xi) \circ \alpha,$$

so identifying the two calculations of  $\hat{F}(\alpha_3)$ , we find that

$$F(\alpha) \circ \xi \circ (\xi \oplus 1) = \hat{F}(\alpha_3) = \xi \circ (1 \oplus \xi) \circ \alpha,$$

which says precisely that our desired coherence diagram for associativity does commute. ■

We conclude this direction of the argument by verifying commutativity of the transposition coherence diagram, which we rewrite by switching our two variables.

5.4. LEMMA. *The transposition coherence diagram*

$$\begin{array}{ccc}
Fb \oplus Fa & \xrightarrow{\tau} & Fa \oplus Fb \\
\xi \downarrow & & \downarrow \xi \\
F(b \oplus a) & \xrightarrow{F\tau} & F(a \oplus b)
\end{array}$$

*commutes.*

PROOF. Once again we have a morphism in  $\mathcal{C}$  that represents two different morphisms in  $U_\kappa\mathcal{C}$ : in this case the transposition isomorphism  $\tau \in \mathcal{C}(b \oplus a, a \oplus b)$  represents a 1-morphism in  $U_\kappa\mathcal{C}(b \oplus a; a \oplus b)$  which we will write as  $\tau_1$ , and also a 2-morphism in  $U_\kappa\mathcal{C}(b, a; a \oplus b)$  which we will write as  $\tau_2$ .

Next, since  $\hat{F}$  is a multifunctor, it is in particular equivariant, so the diagram

$$\begin{array}{ccc} U_\kappa\mathcal{C}(a, b; a \oplus b) & \xrightarrow{\hat{F}} & U_\kappa\mathcal{D}(\hat{F}a, \hat{F}b; \hat{F}(a \oplus b)) \\ \tau^* \downarrow & & \downarrow \tau^* \\ U_\kappa\mathcal{C}(b, a; a \oplus b) & \xrightarrow{\hat{F}} & U_\kappa\mathcal{D}(\hat{F}b, \hat{F}a; \hat{F}(a \oplus b)) \end{array}$$

must commute. Tracing the element  $\text{id}_{a \oplus b}^2$  clockwise in the underlying categories from the upper left of the diagram, we have

$$\text{id}_{a \oplus b} \mapsto \xi \mapsto \xi \circ \tau,$$

as an element of  $\mathcal{D}(Fb \oplus Fa, F(a \oplus b))$ , which is the clockwise composite in the diagram we wish to verify.

Now tracing  $\text{id}_{a \oplus b}^2$  counterclockwise, we get

$$\text{id}_{a \oplus b} \mapsto \tau_2 \mapsto \hat{F}(\tau_2).$$

Since the diagram commutes, we conclude that

$$\hat{F}(\tau_2) = \xi \circ \tau.$$

Next, we have the composition in  $U_\kappa\mathcal{C}$

$$U_\kappa\mathcal{C}(b \oplus a, a \oplus b) \times U_\kappa\mathcal{C}(b, a; b \oplus a) \xrightarrow{\Gamma} U_\kappa\mathcal{C}(b, a; a \oplus b),$$

and expanding using the definition of composition in  $U_\kappa\mathcal{C}$ , we see that the pair  $(\tau_1, \text{id}_{b \oplus a}^2)$  gets sent to  $\tau_2$ . Applying the multifunctor  $\hat{F}$ , it follows that the pair  $(\hat{F}(\tau_1), \hat{F}(\text{id}_{b \oplus a}^2))$  gets sent to  $\hat{F}(\tau_2)$ . But this then says that under

$$U_\kappa\mathcal{D}(F(b \oplus a), F(a \oplus b)) \times U_\kappa\mathcal{D}(Fb, Fa; F(b \oplus a)) \xrightarrow{\Gamma} U_\kappa\mathcal{D}(Fb, Fa; F(a \oplus b)),$$

the pair  $(\hat{F}(\tau_1), \xi)$  gets sent to  $\hat{F}(\tau_2)$ . Since  $F$  is just the restriction of  $\hat{F}$  to 1-morphisms, we see that  $\hat{F}(\tau_1) = F(\tau)$ , and further the definition of  $\Gamma$  in the display sends  $(\hat{F}(\tau_1), \xi)$  to  $\hat{F}(\tau_1) \circ \xi$ . Combining the two calculations, we conclude that

$$\xi \circ \tau = F(\tau_2) = F(\tau) \circ \xi,$$

which says that our desired diagram commutes. We therefore do have a lax symmetric monoidal functor. ■

We turn now to the reverse direction of the theorem: given a lax symmetric monoidal functor  $F$ , we must produce a multifunctor  $U_\kappa F : U_\kappa \mathcal{C} \rightarrow U_\kappa \mathcal{D}$  on underlying multicategories. Again, since all underlying multicategories are canonically and coherently isomorphic, it suffices to produce a multifunctor between the underlying multicategories given by the sequence of objects  $\kappa_n \in Y(n)$ , where  $\kappa_0 = 0$ ,  $\kappa_1 = 1$ , and for  $n \geq 2$ ,  $\kappa_n = \gamma(m; \kappa_{n-1}, 1)$ . We begin by generalizing the structure map  $\xi : Fa \oplus Fb \rightarrow F(a \oplus b)$ . Note that we can express  $\xi$  as

$$\xi : \bar{\kappa}(Fa, Fb) \rightarrow F\bar{\kappa}(a, b).$$

5.5. DEFINITION. Let  $\mathbf{x}$  be an object of  $\mathcal{C}^n$ , with  $F\mathbf{x}$  the corresponding object of  $\mathcal{D}^n$ , so

$$F\mathbf{x} = (Fx_1, \dots, Fx_n).$$

We define a map

$$\xi_n : \bar{\kappa}F\mathbf{x} \rightarrow F\bar{\kappa}\mathbf{x}$$

first for  $n = 0$ , in which case  $\mathbf{x} = * \in \mathcal{C}^0$ ,  $\bar{\kappa}\mathbf{x} = e_{\mathcal{C}}$  and  $\bar{\kappa}F\mathbf{x} = e_{\mathcal{D}}$ . Then we define  $\xi_0$  to be

$$\bar{\kappa}F\mathbf{x} = e_{\mathcal{D}} \xrightarrow{\eta} Fe_{\mathcal{C}} = F\bar{\kappa}\mathbf{x}.$$

For  $n \geq 1$  we use induction on  $n$ , with  $\xi_1 = \text{id}_{F\mathbf{x}_1}$ . For  $n \geq 2$ , let  $\hat{\mathbf{x}} = (x_1, \dots, x_{n-1})$ , that is,  $\mathbf{x}$  with the last entry deleted. Note that since  $\kappa_n = \gamma(m; \kappa_{n-1}, 1)$ , we have

$$\bar{\kappa}\mathbf{x} = \bar{\kappa}\hat{\mathbf{x}} \oplus x_n.$$

Then we define  $\xi_n$  as the composite

$$\bar{\kappa}F\mathbf{x} = \bar{\kappa}F\hat{\mathbf{x}} \oplus Fx_n \xrightarrow{\xi_{n-1} \oplus 1} F\bar{\kappa}\hat{\mathbf{x}} \oplus Fx_n \xrightarrow{\xi} F(\bar{\kappa}\hat{\mathbf{x}} \oplus x_n) = F\bar{\kappa}\mathbf{x}.$$

In particular,  $\xi_2 = \xi$ .

5.6. REMARK. We can't actually start our induction at  $n = 0$ , because the definition

$$\kappa_n = \gamma(m; \kappa_{n-1}, 1)$$

does not apply when  $n = 1$ : we have  $\kappa_1 = 1$ , but  $\gamma(m; 0, 1) \neq 1$  in  $Y(1)$ . This is reflected in the fact that in a general symmetric monoidal category, we don't have  $x = e_{\mathcal{C}} \oplus x$ : they're canonically isomorphic, but not equal.

This now allows us to define the structure of our underlying multifunctor  $U_\kappa F$ , which we will often just write as  $F$ .

5.7. DEFINITION. Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a lax symmetric monoidal functor with structure maps  $\eta : e_{\mathcal{D}} \rightarrow Fe_{\mathcal{C}}$  and  $\xi : Fa \oplus Fb \rightarrow F(a \oplus b)$ . We define the underlying multifunctor by giving it on objects and morphism sets. On objects, we just use the map given by the functor  $F$ . For morphism sets, let  $\mathbf{x} \in \mathcal{C}^n$  and  $y \in \mathcal{C}$ ; we must produce a map

$$F_n : U_{\kappa}\mathcal{C}(\mathbf{x}; y) \rightarrow U_{\kappa}\mathcal{D}(F\mathbf{x}; Fy).$$

For all  $n$ , we define  $F_n$  as the composite

$$U_{\kappa}\mathcal{C}(\mathbf{x}; y) = \mathcal{C}(\bar{\kappa}\mathbf{x}, y) \xrightarrow{F} \mathcal{D}(F\bar{\kappa}\mathbf{x}; Fy) \xrightarrow{\mathcal{D}(\xi_n, 1)} \mathcal{D}(\bar{\kappa}F\mathbf{x}; Fy) = U_{\kappa}\mathcal{D}(F\mathbf{x}; Fy).$$

Note that when  $n = 0$ , this becomes

$$U_{\kappa}\mathcal{C}(\cdot; y) = \mathcal{C}(e_{\mathcal{C}}, y) \xrightarrow{F} \mathcal{D}(Fe_{\mathcal{C}}, Fy) \xrightarrow{\mathcal{D}(\eta, 1)} \mathcal{D}(e_{\mathcal{D}}, Fy) = U_{\kappa}\mathcal{D}(\cdot; Fy).$$

This completes the definition of the structure of  $U_{\kappa}F$ . Note that in the special case  $n = 1$ , since  $\bar{\kappa}x = x$  (because  $\kappa_1 = 1$ ),  $F_1$  coincides with the original functor  $F$ .

We must show that  $U_{\kappa}F$  preserves all the multicategory structure. In particular, it must preserve

1. the identity maps,
2. the composition, and
3. the symmetric group actions.

For preservation of the identity maps, we merely note that  $F_1$  coincides with the original functor, as noted above, and therefore the identity maps are preserved.

We turn to verifying the preservation of composition, which requires some preliminary definitions and lemmas.

5.8. DEFINITION. Let  $\mathbf{x} \in \mathcal{C}^j$ , let  $j = q + r$ , and let  $\mathbf{x} = \mathbf{x}' \odot \mathbf{x}''$  with  $\mathbf{x}' \in \mathcal{C}^q$  and  $\mathbf{x}'' \in \mathcal{C}^r$ , so  $\mathbf{x}' = (x_1, \dots, x_q)$  and  $\mathbf{x}'' = (x_{q+1}, \dots, x_j)$ . We allow the possibility that either  $q = 0$  or  $r = 0$ , but do assume that  $j > 0$ . Then we define a map

$$\phi_{qr}\mathbf{x} : \bar{\kappa}\mathbf{x} \rightarrow \bar{\kappa}\mathbf{x}' \oplus \bar{\kappa}\mathbf{x}''$$

by applying the unique isomorphism

$$\phi_{qr} : \kappa_j \rightarrow \gamma(m; \kappa_q, \kappa_r)$$

in  $Y(j)$  to the object  $\mathbf{x}$ .

Our first lemma towards preservation of composition relates these maps  $\phi_{qr}$  to the previously defined  $\xi_j$ 's:

5.9. LEMMA. *The following diagram commutes:*

$$\begin{array}{ccc}
 \bar{\kappa}F\mathbf{x} & \xrightarrow{\phi_{qr}F\mathbf{x}} & \bar{\kappa}F\mathbf{x}' \oplus \bar{\kappa}F\mathbf{x}'' \\
 \downarrow \xi_j & & \downarrow \xi_q \oplus \xi_r \\
 & & F\bar{\kappa}\mathbf{x}' \oplus F\bar{\kappa}\mathbf{x}'' \\
 & & \downarrow \xi \\
 F\bar{\kappa}\mathbf{x} & \xrightarrow{F\phi_{qr}\mathbf{x}} & F(\bar{\kappa}\mathbf{x}' \oplus \bar{\kappa}\mathbf{x}'').
 \end{array}$$

PROOF. We begin with the special case  $r = 0$ , which requires its own argument. Applying the map in  $Y(j)$

$$\phi_{j0} : \kappa_j \rightarrow \gamma(m; \kappa_j, 0)$$

to  $\mathbf{x} \in \mathcal{C}^j$  gives us the map

$$\phi_{j0}\mathbf{x} : \bar{\kappa}\mathbf{x} \rightarrow \bar{\kappa}\mathbf{x} \oplus e_{\mathcal{C}}$$

inverse to the unit map  $c : \bar{\kappa}\mathbf{x} \oplus e_{\mathcal{C}} \rightarrow \bar{\kappa}\mathbf{x}$ . Now our diagram becomes

$$\begin{array}{ccc}
 \bar{\kappa}F\mathbf{x} & \xrightarrow{\phi_{j0}F\mathbf{x}} & \bar{\kappa}F\mathbf{x} \oplus e_{\mathcal{D}} \\
 \downarrow \xi_j & & \downarrow \xi_j \oplus \xi_0 \\
 & & F\bar{\kappa}\mathbf{x} \oplus Fe_{\mathcal{C}} \\
 & & \downarrow \xi \\
 F\bar{\kappa}\mathbf{x} & \xrightarrow{F\phi_{j0}\mathbf{x}} & F(\bar{\kappa}\mathbf{x} \oplus e_{\mathcal{C}}).
 \end{array}$$

But we can expand this as follows:

$$\begin{array}{ccccc}
 \bar{\kappa}F\mathbf{x} & \xrightarrow{\phi_{j0}F\mathbf{x}} & \bar{\kappa}F\mathbf{x} \oplus e_{\mathcal{D}} & & \\
 \downarrow \xi_j & & \downarrow \xi_j \oplus 1 & \searrow \xi_j \oplus \xi_0 & \\
 & & F\bar{\kappa}\mathbf{x} \oplus e_{\mathcal{D}} & \xrightarrow{1 \oplus \xi_0} & F\bar{\kappa}\mathbf{x} \oplus Fe_{\mathcal{C}} \\
 & \nearrow c^{-1} & & & \downarrow \xi \\
 F\bar{\kappa}\mathbf{x} & \xrightarrow{F\phi_{j0}} & F(\bar{\kappa}\mathbf{x} \oplus e_{\mathcal{C}}). & & 
 \end{array}$$

Remembering that  $\phi_{j0}$  coincides with  $c^{-1}$ , the inverse of the unit isomorphism, the left square commutes by naturality of  $c^{-1}$ , and the bottom square by the coherence diagram relating  $\eta$  and  $\xi$ . The triangle commutes by inspection, and now the perimeter gives us our desired diagram.

We also need to consider the special case in which  $q = 0$  and  $r = 1$ . In this case, we have the map

$$\phi_{01} : 1 = \kappa_1 \rightarrow \gamma(m; \kappa_0, \kappa_1) = \gamma(m; 0, 1),$$

which applies to an object  $x \in \mathcal{C}$  to give  $x \rightarrow e_{\mathcal{C}} \oplus x$ , the inverse of the composite

$$e_{\mathcal{C}} \oplus x \xrightarrow{\tau} x \oplus e_{\mathcal{C}} \xrightarrow{c} x,$$

since  $\gamma(m; 0, 1) \cdot \tau = \gamma(m \cdot \tau; 1, 0)$ . Consequently we can write

$$\phi_{01}x = \tau c^{-1} : x \rightarrow e_{\mathcal{C}} \oplus x.$$

Further,  $\tau c^{-1}$  (or equivalently  $c\tau$ ) satisfies the analogous coherence diagram with  $\xi$  that  $c$  itself does, namely

$$\begin{array}{ccc} e_{\mathcal{D}} \oplus Fx & \xrightarrow{\eta \oplus 1} & Fe_{\mathcal{C}} \oplus Fx \\ c\tau \downarrow & & \downarrow \xi \\ Fx & \xleftarrow{F(c\tau)} & F(e_{\mathcal{C}} \oplus x). \end{array}$$

This is because we can expand it as follows:

$$\begin{array}{ccccc} e_{\mathcal{D}} \oplus Fx & \xrightarrow{\eta \oplus 1} & Fe_{\mathcal{C}} \oplus Fx & & \\ \tau \downarrow & & \searrow \tau & & \downarrow \xi \\ Fx \oplus e_{\mathcal{D}} & \xrightarrow{1 \oplus \eta} & Fx \oplus Fe_{\mathcal{C}} & & F(e_{\mathcal{C}} \oplus x) \\ c \downarrow & & \searrow \xi & & \downarrow F\tau \\ Fx & \xleftarrow{Fc} & F(x \oplus e_{\mathcal{C}}). & & \end{array}$$

The top square commutes by naturality of  $\tau$ , the right “square” by the coherence diagram for  $\xi$  and  $\tau$ , and the bottom square by the coherence diagram for  $\eta$  and  $\xi$ . The perimeter traces the claimed diagram.

Now the desired diagram in the case  $q = 0$  and  $r = 1$  becomes

$$\begin{array}{ccc} Fx & \xrightarrow{\tau c^{-1}} & e_{\mathcal{D}} \oplus Fx \\ \downarrow = & & \downarrow \xi_0 \oplus \xi_1 \\ & & Fe_{\mathcal{C}} \oplus Fx \\ & & \downarrow \xi \\ Fx & \xrightarrow{F(\tau c^{-1})} & F(e_{\mathcal{C}} \oplus x). \end{array}$$

But since  $\xi_0 = \eta$  and  $\xi_1 = \text{id}$ , this is just a rearrangement of the previous coherence diagram, so it does commute. This gives us the special case  $q = 0$  and  $r = 1$ .

We now proceed by induction on  $r$  starting at  $r = 1$ . If  $q = 0$ , we have just verified the lemma, and if  $q \geq 1$ , then  $\phi_{qr} = \text{id}$ , and the diagram simply gives the definition of  $\xi_n$ .

Now assume by induction that  $r > 1$  and that the diagram commutes with  $r$  replaced with  $r - 1$ . We examine two diagrams that will need to be pasted together horizontally, and hats always indicate that the last entry is deleted. The first diagram is

$$\begin{array}{ccccc}
\bar{\kappa}F\mathbf{x} & \xrightarrow{=} & \bar{\kappa}F\hat{\mathbf{x}} \oplus Fx_j & \xrightarrow{\phi_{q(r-1)} \oplus 1} & (\bar{\kappa}F\mathbf{x}' \oplus \bar{\kappa}F\hat{\mathbf{x}}'') \oplus Fx_j \\
\downarrow \xi_j & & \downarrow \xi_{j-1} \oplus 1 & & \downarrow (\xi_q \oplus \xi_{r-1}) \oplus 1 \\
& & F\bar{\kappa}\hat{\mathbf{x}} \oplus Fx_j & \xrightarrow{F\phi_{q(r-1)} \oplus 1} & F(\bar{\kappa}\mathbf{x}' \oplus \bar{\kappa}\hat{\mathbf{x}}'') \oplus Fx_j \\
& & \downarrow \xi & & \downarrow \xi \\
F\bar{\kappa}\mathbf{x} & \xrightarrow{=} & F(\bar{\kappa}\hat{\mathbf{x}} \oplus x_j) & \xrightarrow{F(\phi_{q(r-1)} \oplus 1)} & F((\bar{\kappa}\mathbf{x}' \oplus \bar{\kappa}\hat{\mathbf{x}}'') \oplus x_j) \\
\downarrow F\phi_{qr} & & & & \downarrow F\alpha \\
F(\bar{\kappa}\mathbf{x}' \oplus \bar{\kappa}\mathbf{x}'') & \xrightarrow{=} & & & F(\bar{\kappa}\mathbf{x}' \oplus (\bar{\kappa}\hat{\mathbf{x}}'' \oplus x_j)).
\end{array}$$

The top left rectangle commutes by the definition of  $\xi_j$ , the top right rectangle by induction, the middle right rectangle by naturality of  $\xi$ , and the bottom rectangle because it is  $F$  applied to a diagram induced by a diagram in  $Y(j)$ , where all diagrams commute. The whole diagram therefore commutes.

The second diagram is

$$\begin{array}{ccccc}
(\bar{\kappa}F\mathbf{x}' \oplus \bar{\kappa}F\hat{\mathbf{x}}'') \oplus Fx_j & \xrightarrow{\alpha} & \bar{\kappa}F\mathbf{x}' \oplus (\bar{\kappa}F\hat{\mathbf{x}}'' \oplus Fx_j) & \xrightarrow{=} & \bar{\kappa}F\mathbf{x}' \oplus \bar{\kappa}F\mathbf{x}'' \\
\downarrow (\xi_q \oplus \xi_{r-1}) \oplus 1 & & \downarrow \xi_q \oplus (\xi_{r-1} \oplus 1) & & \downarrow \xi_q \oplus \xi_r \\
(F\bar{\kappa}\mathbf{x}' \oplus F\bar{\kappa}\hat{\mathbf{x}}'') \oplus Fx_j & \xrightarrow{\alpha} & F\bar{\kappa}\mathbf{x}' \oplus (F\bar{\kappa}\hat{\mathbf{x}}'' \oplus Fx_j) & & \\
\downarrow \xi \oplus 1 & & \downarrow 1 \oplus \xi & & \downarrow \xi \\
F(\bar{\kappa}\mathbf{x}' \oplus \bar{\kappa}\hat{\mathbf{x}}'') \oplus Fx_j & & F\bar{\kappa}\mathbf{x}' \oplus F(\bar{\kappa}\hat{\mathbf{x}}'' \oplus x_j) & \xrightarrow{=} & F\bar{\kappa}\mathbf{x}' \oplus F\bar{\kappa}\mathbf{x}'' \\
\downarrow \xi & & \downarrow \xi & & \downarrow \xi \\
F((\bar{\kappa}\mathbf{x}' \oplus \bar{\kappa}\hat{\mathbf{x}}'') \oplus x_j) & \xrightarrow{F\alpha} & F(\bar{\kappa}\mathbf{x}' \oplus (\bar{\kappa}\hat{\mathbf{x}}'' \oplus x_j)) & \xrightarrow{=} & F(\bar{\kappa}\mathbf{x}' \oplus \bar{\kappa}\mathbf{x}'') \\
\downarrow F\alpha & & & & \downarrow = \\
F(\bar{\kappa}\mathbf{x}' \oplus (\bar{\kappa}\hat{\mathbf{x}}'' \oplus x_j)) & \xrightarrow{=} & & & F(\bar{\kappa}\mathbf{x}' \oplus \bar{\kappa}\mathbf{x}'').
\end{array}$$

The top left square commutes by naturality of  $\alpha$ , the top right by definition of  $\xi_r$ , the middle left by the coherence diagram for  $\alpha$ , and the middle right and bottom by inspection. The total second diagram therefore also commutes.

The right column of the first diagram coincides with the left column of the second diagram, so we can paste the two diagrams together along their common column. When

we do so, the counterclockwise direction of the total diagram coincides with the counterclockwise direction of our desired diagram, but the clockwise direction requires a bit more work. We need the following diagram to commute:

$$\begin{array}{ccc}
 \bar{\kappa}F\mathbf{x} & \xrightarrow{=} & \bar{\kappa}F\hat{\mathbf{x}} \oplus Fx_j \xrightarrow{\phi_{q(r-1)} \oplus 1} (\bar{\kappa}F\mathbf{x}' \oplus \bar{\kappa}F\hat{\mathbf{x}}'') \oplus Fx_j \\
 \phi_{qr} \downarrow & & \downarrow \alpha \\
 \bar{\kappa}F\mathbf{x}' \oplus \bar{\kappa}F\mathbf{x}'' & \xrightarrow{=} & \bar{\kappa}F\mathbf{x}' \oplus (\bar{\kappa}F\hat{\mathbf{x}}'' \oplus Fx_j).
 \end{array}$$

However, all three maps are induced by maps in  $Y(j)$ , where all diagrams commute. The desired diagram therefore does commute.  $\blacksquare$

We will need a generalization of this lemma, and this requires a bit more notation.

5.10. DEFINITION. Let  $\mathbf{x} \in \mathcal{C}^j$ , and let  $j = j_1 + \cdots + j_n$ , so we can decompose  $\mathbf{x}$  as

$$\mathbf{x} = \odot_s \mathbf{x}_s,$$

where  $\mathbf{x}_s \in \mathcal{C}^{j_s}$  for  $1 \leq s \leq n$ . Regardless of  $\mathbf{x}$ , we have a unique isomorphism

$$\kappa_j \rightarrow \gamma(\kappa_n; \langle \kappa_{j_s} \rangle)$$

in  $Y(j)$ . Note that if we apply the target element  $\gamma(\kappa_n; \langle \kappa_{j_s} \rangle)$  to  $\mathbf{x} = \odot_s \mathbf{x}_s$ , we can write the result as either  $\bar{\kappa}\langle \bar{\kappa}\mathbf{x}_s \rangle$  or  $\kappa_n\langle \bar{\kappa}\mathbf{x}_s \rangle$ , since we have chosen a single object  $\kappa_n \in Y(n)$  as the target of the  $n$ 'th map in our defining sequence  $\{\kappa_n\}$ . We will use  $\kappa_n\langle \bar{\kappa}\mathbf{x}_s \rangle$  since we need the index  $n$  for induction in the next proof. So applying the map in  $Y(n)$  to  $\mathbf{x} = \odot_s \mathbf{x}_s$ , we obtain a map (which is an isomorphism) which we denote

$$\phi\langle j_s \rangle : \bar{\kappa}(\odot_s \mathbf{x}_s) \rightarrow \kappa_n\langle \bar{\kappa}\mathbf{x}_s \rangle.$$

This is a generalization of the previous  $\phi_{qr}$  when  $n = 2$ ,  $j_1 = q$ , and  $j_2 = r$ .

We also generalize lemma 5.9 for these maps, as follows.

5.11. LEMMA. Given  $\mathbf{x} \in \mathcal{C}^j$  and  $j = j_1 + \cdots + j_n$ , note that  $\odot_s F\mathbf{x}_s = F(\odot_s \mathbf{x}_s)$ . Then decomposing  $\mathbf{x}$  as  $\odot_s \mathbf{x}_s$  for  $1 \leq s \leq n$ , the following diagram commutes:

$$\begin{array}{ccc}
 \bar{\kappa}(\odot_s F\mathbf{x}_s) & \xrightarrow{\phi\langle j_s \rangle} & \kappa_n\langle \bar{\kappa}F\mathbf{x}_s \rangle \\
 \downarrow \xi_j & & \downarrow \kappa_n\langle \xi_{j_s} \rangle \\
 & & \kappa_n\langle F\bar{\kappa}\mathbf{x}_s \rangle \\
 & & \downarrow \xi_n \\
 F\bar{\kappa}(\odot_s \mathbf{x}_s) & \xrightarrow{F\phi\langle j_s \rangle} & F\kappa_n\langle \bar{\kappa}\mathbf{x}_s \rangle.
 \end{array}$$

PROOF. We proceed by induction on  $n$ , and the case  $n = 1$  is trivial:  $\phi\langle j \rangle = \text{id}$ ,  $\kappa_1 = 1$ , and  $\xi_1 = \text{id}$  as well.

As with Lemma 5.9, the proof consists of verifying two diagrams that need to be pasted together horizontally. We let  $j' = j - j_n$ , and let hats throughout indicate that the last index has been deleted. The notation  $\phi_{qr}$  is as above from Lemma 5.9. Our first diagram is as follows:

$$\begin{array}{ccccc}
 \bar{\kappa}(\odot_s F \mathbf{x}_s) & \xrightarrow{\phi_{j'j_n}} & \bar{\kappa}(\hat{\odot}_s F \mathbf{x}_s) \oplus \bar{\kappa} F \mathbf{x}_n & \xrightarrow{\phi\langle \hat{j}_s \rangle \oplus 1} & \kappa_{n-1} \langle \widehat{\bar{\kappa} F \mathbf{x}_s} \rangle \oplus \bar{\kappa} F \mathbf{x}_n \\
 \downarrow \xi_j & & \downarrow \xi_{j'} \oplus \xi_{j_n} & & \downarrow \kappa_{n-1} \langle \widehat{\xi_{j_s}} \rangle \oplus 1 \\
 & & F \bar{\kappa}(\hat{\odot}_s \mathbf{x}_s) \oplus F \bar{\kappa} \mathbf{x}_n & \xrightarrow{F \phi\langle \hat{j}_s \rangle \oplus 1} & F \kappa_{n-1} \langle \widehat{\bar{\kappa} \mathbf{x}_s} \rangle \oplus F \bar{\kappa} \mathbf{x}_n \\
 & & \downarrow \xi & & \downarrow \xi_{n-1} \oplus \xi_{j_n} \\
 F \bar{\kappa}(\odot_s \mathbf{b}_s) & \xrightarrow{F \phi_{j'j_n}} & F(\bar{\kappa}(\hat{\odot}_s \mathbf{x}_s) \oplus \bar{\kappa} \mathbf{x}_n) & \xrightarrow{F(\phi\langle \hat{j}_s \rangle \oplus 1)} & F(\kappa_{n-1} \langle \widehat{\bar{\kappa} \mathbf{x}_s} \rangle \oplus \bar{\kappa} \mathbf{x}_n) \\
 \downarrow F \phi\langle j_s \rangle & & & & \downarrow \xi \\
 F \kappa_n \langle \widehat{\bar{\kappa} \mathbf{x}_s} \rangle & \xrightarrow{\quad \quad \quad} & & & F \kappa_n \langle \widehat{\bar{\kappa} \mathbf{x}_s} \rangle
 \end{array}$$

=

The left rectangle commutes since it is an instance of Lemma 5.9. The top right rectangle commutes by induction, and the bottom right rectangle by naturality of  $\xi$ . The bottom triangle commutes since it is  $F$  applied to a diagram induced from maps in  $Y(j)$ , where all diagrams commute. The total diagram therefore commutes.

The second diagram is as follows:

$$\begin{array}{ccc}
 \kappa_{n-1} \langle \widehat{\bar{\kappa} F \mathbf{x}_s} \rangle \oplus \bar{\kappa} F \mathbf{x}_n & \xrightarrow{\quad \quad \quad} & \kappa_n \langle \bar{\kappa} F \mathbf{x}_s \rangle \\
 \downarrow \kappa_{n-1} \langle \widehat{\xi_{j_s}} \rangle \oplus 1 & \searrow \kappa_{n-1} \langle \widehat{\xi_{j_s}} \rangle \oplus \xi_{j_n} & \downarrow \kappa_n \langle \xi_{j_s} \rangle \\
 \kappa_{n-1} \langle \widehat{F \bar{\kappa} \mathbf{x}_s} \rangle \oplus \bar{\kappa} F \mathbf{x}_n & \xrightarrow{1 \oplus \xi_{j_n}} & \kappa_{n-1} \langle \widehat{F \bar{\kappa} \mathbf{x}_s} \rangle \oplus F \bar{\kappa} \mathbf{x}_n \xrightarrow{\quad \quad \quad} \kappa_n \langle F \bar{\kappa} \mathbf{x}_s \rangle \\
 \downarrow \xi_{n-1} \oplus \xi_{j_n} & \swarrow \xi_{n-1} \oplus 1 & \downarrow \xi_n \\
 F \kappa_{n-1} \langle \widehat{\bar{\kappa} \mathbf{x}_s} \rangle \oplus F \bar{\kappa} \mathbf{x}_n & & F \kappa_n \langle \bar{\kappa} \mathbf{x}_s \rangle \\
 \downarrow \xi & & \downarrow \xi_n \\
 F(\kappa_{n-1} \langle \widehat{\bar{\kappa} \mathbf{x}_s} \rangle \oplus \bar{\kappa} \mathbf{x}_n) & \xrightarrow{\quad \quad \quad} & F \kappa_n \langle \bar{\kappa} \mathbf{x}_s \rangle
 \end{array}$$

=

The two triangles commute by inspection. The top (distorted) rectangle commutes since  $\kappa_n \mathbf{y} = \kappa_{n-1} \hat{\mathbf{y}} \oplus y_n$ , where in this case  $\mathbf{y} = \langle \bar{\kappa} F \mathbf{x}_s \rangle$ , and the bottom part commutes by definition of  $\xi_n$ . The total diagram therefore commutes.

Now we paste the two diagrams together along their common column, the right column of the first diagram and the left column of the second diagram, and see that the counterclockwise direction of the total diagram coincides with the counterclockwise direction of our desired diagram. For the clockwise direction, again we need to do a bit more work. We observe that since the top row of the total diagram is induced by maps in  $Y(n)$ , where all diagrams commute, it does coincide with the desired clockwise direction, and the desired diagram therefore does commute. ■

We are now ready to verify that the multifunctor  $F$  preserves composition. Let  $\mathbf{x}_s \in \mathcal{C}^{j_s}$  for  $1 \leq s \leq n$ ,  $\mathbf{y} \in \mathcal{C}^n$ , and  $z \in \mathcal{C}$ . We claim:

5.12. PROPOSITION. *The multifunctor  $F$  preserves composition, meaning the diagram*

$$\begin{array}{ccc} U_\kappa \mathcal{C}(\mathbf{y}; z) \times \prod_{s=1}^n U_\kappa \mathcal{C}(\mathbf{x}_s, y_s) & \xrightarrow{F} & U_\kappa \mathcal{D}(F\mathbf{y}, Fz) \times \prod_{s=1}^n U_\kappa \mathcal{D}(F\mathbf{x}_s, Fy_s) \\ \Gamma \downarrow & & \downarrow \Gamma \\ U_\kappa \mathcal{C}(\odot_s \mathbf{x}_s, z) & \xrightarrow{F} & U_\kappa \mathcal{D}(\odot_s F\mathbf{x}_s, Fz) \end{array}$$

*commutes.*

PROOF. We proceed by a sequence of three diagrams which can be pasted together horizontally, resulting in a total diagram that gives our desired diagram, unpacked using the definitions of our multicategory composition  $\Gamma$  and the multifunctor  $F$ . Note that  $\kappa_n \mathbf{y} = \bar{\kappa} \mathbf{y}$  because of our convention that  $\kappa_n$  is a constant function. The first diagram is as follows:

$$\begin{array}{ccc} & \mathcal{C}(\bar{\kappa} \mathbf{y}, z) \times \mathcal{C}^n(\langle \bar{\kappa} \mathbf{x}_s \rangle, \mathbf{y}) & \\ & \downarrow 1 \times \kappa_n & \\ & \mathcal{C}(\bar{\kappa} \mathbf{y}, z) \times \mathcal{C}(\kappa_n \langle \bar{\kappa} \mathbf{x}_s \rangle, \bar{\kappa} \mathbf{y}) & \\ & \downarrow F & \\ \mathcal{C}(\kappa_n \langle \bar{\kappa} \mathbf{x}_s \rangle, z) & \begin{array}{c} \swarrow \circ \\ \mathcal{D}(F\bar{\kappa} \mathbf{y}, Fz) \times \mathcal{D}(F\kappa_n \langle \bar{\kappa} \mathbf{x}_s \rangle, F\bar{\kappa} \mathbf{y}) \\ \searrow F \end{array} & \\ \downarrow \mathcal{C}(\phi(j_s), 1) & & \downarrow \circ \\ \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{x}_s), z) & \xrightarrow{F} & \mathcal{D}(F\bar{\kappa}(\odot_s \mathbf{x}_s), Fz). \end{array}$$

Both parts of the diagram commute because  $F$  is a functor.

The second diagram, which pastes to the right of the first one, is as follows:

$$\begin{array}{ccc}
\mathcal{C}(\bar{\kappa}\mathbf{y}, z) \times \mathcal{C}^n(\langle \bar{\kappa}\mathbf{x}_s \rangle, \mathbf{y}) & \xrightarrow{F} & \mathcal{D}(F\bar{\kappa}\mathbf{y}, Fz) \times \mathcal{D}^n(\langle F\bar{\kappa}\mathbf{x}_s \rangle, F\mathbf{y}) \\
\downarrow 1 \times \kappa_n & & \downarrow 1 \times \kappa_n \\
\mathcal{C}(\bar{\kappa}\mathbf{y}, z) \times \mathcal{C}(\kappa_n \langle \bar{\kappa}\mathbf{x}_s \rangle, \bar{\kappa}\mathbf{y}) & & \mathcal{D}(F\bar{\kappa}\mathbf{y}, Fz) \times \mathcal{D}(\kappa_n \langle F\bar{\kappa}\mathbf{x}_s \rangle, \bar{\kappa}F\mathbf{y}) \\
\downarrow F & & \downarrow 1 \times \mathcal{D}(1, \xi_n) \\
\mathcal{D}(F\bar{\kappa}\mathbf{y}, Fz) \times \mathcal{D}(F\kappa_n \langle \bar{\kappa}\mathbf{x}_s \rangle, F\bar{\kappa}\mathbf{y}) & \xrightarrow{1 \times \mathcal{D}(\xi_n, 1)} & \mathcal{D}(F\bar{\kappa}\mathbf{y}, Fz) \times \mathcal{D}(\kappa_n \langle F\bar{\kappa}\mathbf{x}_s \rangle, F\bar{\kappa}\mathbf{y}) \\
\downarrow \circ & & \downarrow \circ \\
\mathcal{D}(F\kappa_n \langle \bar{\kappa}\mathbf{x}_s \rangle, Fz) & \xrightarrow{\mathcal{D}(\xi_n, 1)} & \mathcal{D}(\kappa_n \langle F\bar{\kappa}\mathbf{x}_s \rangle, Fz) \\
\downarrow \mathcal{D}(F\phi\langle j_s \rangle, 1) & & \downarrow \mathcal{D}(\kappa_n \langle \xi_{j_s} \rangle, 1) \\
\mathcal{D}(F\bar{\kappa}(\odot_s \mathbf{x}_s), Fz) & \xrightarrow{\mathcal{D}(\xi_j, 1)} & \mathcal{D}(\kappa_n \langle \bar{\kappa}F\mathbf{x}_s \rangle, Fz) \\
& & \downarrow \mathcal{D}(\phi\langle j_s \rangle, 1) \\
& & \mathcal{D}(\bar{\kappa}(\odot_s F\mathbf{x}_s), Fz).
\end{array}$$

The top hexagon is the product of two separate hexagons, and the first factor commutes, since both directions are just  $F$ . For the second factor, we trace a typical element  $\langle f_s \rangle \in \mathcal{C}^n(\langle \bar{\kappa}\mathbf{x}_s \rangle, \mathbf{y})$  through the hexagon, and find that it commutes precisely when

$$F\kappa_n \langle f_s \rangle \circ \xi_n = \xi_n \circ \kappa_n \langle Ff_s \rangle,$$

which is to say that the following diagram commutes:

$$\begin{array}{ccc}
\kappa_n \langle F\bar{\kappa}\mathbf{x}_s \rangle & \xrightarrow{\kappa_n \langle Ff_s \rangle} & \kappa_n F\mathbf{y} \\
\xi_n \downarrow & & \downarrow \xi_n \\
F\kappa_n \langle \bar{\kappa}\mathbf{x}_s \rangle & \xrightarrow{F\kappa_n \langle f_s \rangle} & F\bar{\kappa}\mathbf{y}.
\end{array}$$

But this is just naturality of  $\xi_n$ , so the top hexagon does commute.

The middle rectangle commutes by inspection.

The bottom part of the diagram is  $\mathcal{D}(\_, 1)$  applied to the diagram

$$\begin{array}{ccc}
\bar{\kappa}(\odot_s F\mathbf{x}_s) & \xrightarrow{\xi_j} & F\bar{\kappa}(\odot_s \mathbf{x}_s) \\
\phi\langle j_s \rangle \downarrow & & \downarrow F\phi\langle j_s \rangle \\
\kappa_n \langle \bar{\kappa}F\mathbf{x}_s \rangle & & \\
\kappa_n \langle j_s \rangle \downarrow & & \downarrow \\
\kappa_n \langle F\bar{\kappa}\mathbf{x}_s \rangle & \xrightarrow{\xi_n} & F\kappa_n \langle \bar{\kappa}\mathbf{x}_s \rangle,
\end{array}$$

which is an instance of Lemma 5.11. The second total diagram therefore commutes.

The third diagram we need to paste onto the second one is as follows:

$$\begin{array}{ccccc}
 \mathcal{D}(F\bar{\kappa}\mathbf{y}, Fz) \times \mathcal{D}^n(\langle F\bar{\kappa}\mathbf{x}_s \rangle, F\mathbf{y}) & \xrightarrow{\mathcal{D}(\xi_n, 1) \times \mathcal{D}^n(\langle \xi_{j_s} \rangle, 1)} & \mathcal{D}(\bar{\kappa}F\mathbf{y}, Fz) \times \mathcal{D}^n(\langle \bar{\kappa}F\mathbf{x}_s \rangle, F\mathbf{y}) \\
 \downarrow 1 \times \kappa_n & & \downarrow 1 \times \kappa_n \\
 \mathcal{D}(F\bar{\kappa}\mathbf{y}, Fz) \times \mathcal{D}(\kappa_n \langle F\bar{\kappa}\mathbf{x}_s \rangle, \bar{\kappa}F\mathbf{y}) & & \mathcal{D}(\bar{\kappa}F\mathbf{y}, Fz) \times \mathcal{D}(\kappa_n \langle \bar{\kappa}F\mathbf{x}_s \rangle, \bar{\kappa}F\mathbf{y}) \\
 \downarrow \mathcal{D}(\xi_n, 1) \times 1 & \searrow 1 \times \mathcal{D}(\kappa_n \langle \xi_{j_s} \rangle, 1) & \downarrow \circ \\
 \mathcal{D}(\bar{\kappa}F\mathbf{y}, Fz) \times \mathcal{D}(\kappa_n \langle F\bar{\kappa}\mathbf{x}_s \rangle, \bar{\kappa}F\mathbf{y}) & & \mathcal{D}(\bar{\kappa}F\mathbf{y}, Fz) \times \mathcal{D}(\kappa_n \langle \bar{\kappa}F\mathbf{x}_s \rangle, \bar{\kappa}F\mathbf{y}) \\
 \downarrow 1 \times \mathcal{D}(1, \xi_n) & \searrow \circ & \downarrow \circ \\
 \mathcal{D}(F\bar{\kappa}\mathbf{y}, Fz) \times \mathcal{D}(\kappa_n \langle F\bar{\kappa}\mathbf{x}_s \rangle, F\bar{\kappa}\mathbf{y}) & & \mathcal{D}(\kappa_n \langle \bar{\kappa}F\mathbf{x}_s \rangle, Fz) \\
 \searrow \circ & \downarrow \circ & \downarrow \mathcal{D}(\phi(j_s), 1) \\
 & \mathcal{D}(\kappa_n \langle F\bar{\kappa}\mathbf{x}_s \rangle, Fz) & \mathcal{D}(\bar{\kappa}(\odot_s F\mathbf{x}_s), Fz).
 \end{array}$$

The counterclockwise direction traces the right hand column of the second diagram, so this diagram can be pasted to it. The top pentagon commutes by naturality of  $\kappa_n$ , the left square by naturality of  $\xi_n$ , and the right square by inspection. Now examining the perimeter of the total pasted diagram, we see that the counterclockwise direction gives us the expansion of  $F \circ \Gamma$  in our original claimed diagram, and the clockwise direction gives us the expansion of  $\Gamma \circ F$ . The diagram therefore commutes, and  $F$  preserves composition. ■

Our final step in showing that  $F$  is a multifunctor is preservation of the  $\Sigma_n$ -actions. This is the content of the following proposition:

5.13. PROPOSITION. *Let  $\mathbf{x} \in \mathcal{C}^n$ ,  $y \in \mathcal{C}$ , and  $\sigma \in \Sigma_n$ . Then the following diagram commutes:*

$$\begin{array}{ccc}
 U_\kappa \mathcal{C}(\mathbf{x}; y) & \xrightarrow{F} & U_\kappa \mathcal{D}(F\mathbf{x}; Fy) \\
 \sigma^* \downarrow & & \downarrow \sigma^* \\
 U_\kappa \mathcal{C}(\sigma^{-1}\mathbf{x}; y) & \xrightarrow{F} & U_\kappa \mathcal{D}(\sigma^{-1}F\mathbf{x}; Fy).
 \end{array}$$

PROOF. The map  $\sigma^*$  is induced by the map in  $Y(n)$

$$\theta(\mathbf{x}, \sigma) : \kappa_n(\sigma^{-1}\mathbf{x}) \rightarrow \kappa_n \mathbf{x} \cdot \sigma,$$

but since we are assuming that the map  $\kappa_n : \text{Ob}\mathcal{C}^n \rightarrow \text{Ob}Y(n)$  is constant, we will just abbreviate this to

$$\theta : \kappa_n \rightarrow \kappa_n \cdot \sigma.$$

Now expanding the desired diagram using the definitions, we find that we wish to verify commutativity of the following diagram:

$$\begin{array}{ccccc}
\mathcal{C}(\bar{\kappa}\mathbf{x}, y) & \xrightarrow{F} & \mathcal{D}(F\bar{\kappa}\mathbf{x}, Fy) & \xrightarrow{\mathcal{D}(\xi_n, 1)} & \mathcal{D}(\bar{\kappa}F\mathbf{x}, Fy) \\
= \downarrow & & \downarrow = & & \downarrow = \\
\mathcal{C}((\kappa_n \cdot \sigma)(\sigma^{-1}\mathbf{x}), y) & \xrightarrow{F} & \mathcal{D}(F(\kappa_n \cdot \sigma)(\sigma^{-1}\mathbf{x}), Fy) & \xrightarrow{\mathcal{D}(\xi_n, 1)} & \mathcal{D}((\kappa_n \cdot \sigma)(\sigma^{-1}F\mathbf{x}), Fy) \\
\mathcal{C}(\theta, 1) \downarrow & & \downarrow \mathcal{D}(F\theta, 1) & & \downarrow \mathcal{D}(\theta, 1) \\
\mathcal{C}(\kappa_n \sigma^{-1}\mathbf{x}, y) & \xrightarrow{F} & \mathcal{D}(F\kappa_n \sigma^{-1}\mathbf{x}, Fy) & \xrightarrow{\mathcal{D}(\xi_n, 1)} & \mathcal{D}(\kappa_n \sigma^{-1}F\mathbf{x}, Fy).
\end{array}$$

The top two squares commute by inspection, and the lower left one by functoriality of  $F$ . This reduces the argument to verifying the bottom right square, which is in turn induced from a square in  $\mathcal{D}$  as follows:

$$\begin{array}{ccc}
\kappa_n \sigma^{-1} F\mathbf{x} & \xrightarrow{\xi_n} & F\kappa_n \sigma^{-1} \mathbf{x} \\
\theta \downarrow & & \downarrow F\theta \\
\kappa_n \cdot \sigma \sigma^{-1} F\mathbf{x} & & F\kappa_n \cdot \sigma \sigma^{-1} \mathbf{x} \\
= \downarrow & & \downarrow = \\
\kappa_n F\mathbf{x} & \xrightarrow{\xi_n} & F\kappa_n \mathbf{x}.
\end{array}$$

We need to show that this diagram commutes, and we proceed by induction on  $n$ , starting with  $n = 1$ , in which case there are no non-trivial permutations, so the diagram does commute.

Next, we assume the diagram does commute with  $n$  replaced by  $n-1$ , and first consider the case in which  $\sigma = \hat{\sigma} \oplus 1$  for  $\hat{\sigma} \in \Sigma_{n-1}$ . We write

$$\hat{\theta} : \kappa_{n-1} \rightarrow \kappa_{n-1} \cdot \hat{\sigma}$$

for the unique map in  $Y(n-1)$ . Then the diagram

$$\begin{array}{ccc}
\kappa_n = \gamma(m; \kappa_{n-1}, 1) & \xrightarrow{\theta} & \gamma(m; \kappa_{n-1}, 1) \cdot \sigma \\
& \searrow \gamma(m; \hat{\theta}, 1) & \downarrow = \\
& & \gamma(m; \kappa_{n-1} \cdot \hat{\sigma}, 1)
\end{array}$$

in  $Y(n)$  commutes, since all diagrams commute in  $Y(n)$ . So if we apply this diagram to an object  $\mathbf{x}$ , we have the diagram

$$\begin{array}{ccc}
\kappa_n \mathbf{x} = \kappa_{n-1} \hat{\mathbf{x}} \oplus x_n & \xrightarrow{\theta} & \kappa_n \cdot \sigma \mathbf{x} = \kappa_n (\hat{\sigma} \hat{\mathbf{x}} \oplus x_n) \\
& \searrow \hat{\theta} \oplus 1 & \downarrow = \\
& & \kappa_{n-1} \hat{\sigma} \hat{\mathbf{x}} \oplus x_n,
\end{array}$$

which tells us that in this case, we have  $\theta = \hat{\theta} \oplus 1$ .

Now we can expand our desired diagram as follows:

$$\begin{array}{ccccc}
 \kappa_n \sigma^{-1} F\mathbf{x} = \kappa_{n-1} \hat{\sigma}^{-1} F\hat{\mathbf{x}} \oplus Fx_n & \xrightarrow{\xi_{n-1} \oplus 1} & F\kappa_{n-1} \hat{\sigma}^{-1} \hat{\mathbf{x}} \oplus Fx_n & \xrightarrow{\xi} & F(\kappa_{n-1} \hat{\sigma}^{-1} \hat{\mathbf{x}} \oplus x_n) \\
 \downarrow \theta = \hat{\theta} \oplus 1 & & \downarrow F\hat{\theta} \oplus 1 & & \downarrow F(\hat{\theta} \oplus 1) \\
 \kappa_n \sigma \sigma^{-1} F\mathbf{x} = \kappa_{n-1} \hat{\sigma} \hat{\sigma}^{-1} F\hat{\mathbf{x}} \oplus Fx_n & \xrightarrow{\xi_{n-1} \oplus 1} & F\kappa_{n-1} \hat{\sigma} \hat{\sigma}^{-1} \hat{\mathbf{x}} \oplus Fx_n & \xrightarrow{\xi} & F(\kappa_{n-1} \hat{\sigma} \hat{\sigma}^{-1} \hat{\mathbf{x}} \oplus x_n) \\
 \downarrow = & & \downarrow = & & \downarrow = \\
 \kappa_n F\mathbf{x} = \kappa_{n-1} F\hat{\mathbf{x}} \oplus Fx_n & \xrightarrow{\xi_{n-1} \oplus 1} & F\kappa_{n-1} \hat{\mathbf{x}} \oplus Fx_n & \xrightarrow{\xi} & F\kappa_n \mathbf{x}.
 \end{array}$$

Since  $\xi_n = \xi \circ (\xi_{n-1} \oplus 1)$  and  $\theta = \hat{\theta} \oplus 1$ , the perimeter of the diagram does give our desired diagram. The left part of the diagram commutes by induction, and the right part by naturality of  $\xi$ . This concludes the verification of preservation of the action of  $\sigma$  when  $\sigma = \hat{\sigma} \oplus 1$ .

The remaining case is when  $\sigma$  moves the index  $n$ , but since all elements of  $\Sigma_n$  can be written as a product of transpositions of adjacent indices, we can restrict our attention to such transpositions, and the only one that moves the last index is the transposition of  $n$  and  $n-1$ . So we let  $\sigma$  be this transposition, and the argument will be done when we've verified the square for this particular permutation, which satisfies  $\sigma = \sigma^{-1}$ .

We first look at the inducing map  $\theta : \kappa_n \rightarrow \kappa_n \cdot \sigma$ , and note that applied to an object  $\mathbf{x} \in \mathcal{C}^n$ , we have

$$\sigma \mathbf{x} = (x_1, \dots, x_{n-2}, x_n, x_{n-1}).$$

Therefore

$$\begin{aligned}
 \kappa_n \sigma \mathbf{x} &= \gamma(m; \kappa_{n-1}, 1) \sigma \mathbf{x} = \kappa_{n-1}(x_1, \dots, x_{n-2}, x_n) \oplus x_{n-1} \\
 &= (\kappa_{n-2}(x_1, \dots, x_{n-2}) \oplus x_n) \oplus x_{n-1}.
 \end{aligned}$$

We've already introduced the notation  $\hat{\mathbf{x}}$  for  $\mathbf{x}$  with its last entry removed; now we also need  $\tilde{\mathbf{x}}$  for  $\mathbf{x}$  with its last two entries removed. Then what the last calculation shows is that

$$\kappa_n \sigma \mathbf{x} = (\kappa_{n-2} \tilde{\mathbf{x}} \oplus x_n) \oplus x_{n-1},$$

and we can also expand

$$\kappa_n \mathbf{x} = (\kappa_{n-2} \tilde{\mathbf{x}} \oplus x_{n-1}) \oplus x_n.$$

We can now construct the map  $\theta : \kappa_n \mathbf{x} \rightarrow \kappa_n \sigma \mathbf{x}$  as the composite

$$\begin{array}{c}
 (\kappa_{n-2} \tilde{\mathbf{x}} \oplus x_{n-1}) \oplus x_n \\
 \downarrow \alpha \\
 \kappa_{n-2} \tilde{\mathbf{x}} \oplus (x_{n-1} \oplus x_n) \\
 \downarrow 1 \oplus \tau \\
 \kappa_{n-2} \tilde{\mathbf{x}} \oplus (x_n \oplus x_{n-1}) \\
 \downarrow \alpha^{-1} \\
 (\kappa_{n-2} \tilde{\mathbf{x}} \oplus x_n) \oplus x_{n-1}.
 \end{array}$$

Since both  $\theta$  and this composite are induced from maps in  $Y(n)$ , where all maps are unique, this does coincide with  $\theta$ .

Next, we observe that  $\xi_n : \kappa_n F\mathbf{x} \rightarrow F\kappa_n \mathbf{x}$ , which expands by definition to

$$\kappa_n F\mathbf{x} = (\kappa_{n-1} F\hat{\mathbf{x}}) \oplus Fx_n \xrightarrow{\xi_{n-1} \oplus 1} (F\kappa_{n-1} \hat{\mathbf{x}}) \oplus Fx_n \xrightarrow{\xi} F(\kappa_{n-1} \hat{\mathbf{x}} \oplus x_n) = F\kappa_n \mathbf{x},$$

expands even further as follows, since  $\xi_{n-1} = \xi \circ (\xi_{n-2} \oplus 1)$ :

$$\begin{array}{c}
 \kappa_n F\mathbf{x} \\
 \downarrow = \\
 (\kappa_{n-2} F\tilde{\mathbf{x}} \oplus Fx_{n-1}) \oplus Fx_n \\
 \downarrow (\xi_{n-2} \oplus 1) \oplus 1 \\
 (F\kappa_{n-2} \tilde{\mathbf{x}} \oplus Fx_{n-1}) \oplus Fx_n \\
 \downarrow \xi \oplus 1 \\
 F(\kappa_{n-2} \tilde{\mathbf{x}} \oplus x_{n-1}) \oplus Fx_n \\
 \downarrow \xi \\
 F((\kappa_{n-2} \tilde{\mathbf{x}} \oplus x_{n-1}) \oplus x_n) \\
 \downarrow = \\
 F\kappa_n \mathbf{x}.
 \end{array}$$

We can now fill in the diagram by pasting together two pieces horizontally; the first

piece is as follows, using the  $(\xi_{n-2} \oplus 1) \oplus 1$  part of the above composition:

$$\begin{array}{ccc}
 (\kappa_{n-2}F\tilde{\mathbf{X}} \oplus Fx_n) \oplus Fx_{n-1} & \xrightarrow{(\xi_{n-2} \oplus 1) \oplus 1} & (F\kappa_{n-2}\tilde{\mathbf{X}} \oplus Fx_n) \oplus Fx_{n-1} \\
 \alpha \downarrow & & \downarrow \alpha \\
 \kappa_{n-2}F\tilde{\mathbf{X}} \oplus (Fx_n \oplus Fx_{n-1}) & \xrightarrow{\xi_{n-2} \oplus 1} & F\kappa_{n-2}\tilde{\mathbf{X}} \oplus (Fx_n \oplus Fx_{n-1}) \\
 1 \oplus \tau \downarrow & & \downarrow 1 \oplus \tau \\
 \kappa_{n-2}F\tilde{\mathbf{X}} \oplus (Fx_{n-1} \oplus Fx_n) & \xrightarrow{\xi_{n-2} \oplus 1} & F\kappa_{n-2}\tilde{\mathbf{X}} \oplus (Fx_{n-1} \oplus Fx_n) \\
 \alpha^{-1} \downarrow & & \downarrow \alpha^{-1} \\
 (\kappa_{n-2}F\tilde{\mathbf{X}} \oplus Fx_{n-1}) \oplus Fx_n & \xrightarrow{(\xi_{n-2} \oplus 1) \oplus 1} & (F\kappa_{n-2}\tilde{\mathbf{X}} \oplus Fx_{n-1}) \oplus Fx_n.
 \end{array}$$

The top and bottom rectangles commute by naturality of  $\alpha$ , and the middle one by inspection.

The second piece, to be pasted horizontally to the first one, is

$$\begin{array}{ccc}
 (F\kappa_{n-2}\tilde{\mathbf{X}} \oplus Fx_{n-1}) \oplus Fx_n & \xrightarrow{\xi \circ (\xi \oplus 1)} & F((\kappa_{n-2}\tilde{\mathbf{X}} \oplus x_{n-1}) \oplus x_n) \\
 \alpha \downarrow & & \downarrow F\alpha \\
 F\kappa_{n-2}\tilde{\mathbf{X}} \oplus (Fx_{n-1} \oplus Fx_n) & \xrightarrow{\xi \circ (1 \oplus \xi)} & F(\kappa_{n-2}\tilde{\mathbf{X}} \oplus (x_{n-1} \oplus x_n)) \\
 1 \oplus \tau \downarrow & & \downarrow F(1 \oplus \tau) \\
 F\kappa_{n-2}\tilde{\mathbf{X}} \oplus (Fx_n \oplus Fx_{n-1}) & \xrightarrow{\xi \circ (1 \oplus \xi)} & F(\kappa_{n-2}\tilde{\mathbf{X}} \oplus (x_n \oplus x_{n-1})) \\
 \alpha^{-1} \downarrow & & \downarrow F\alpha^{-1} \\
 (F\kappa_{n-2}\tilde{\mathbf{X}} \oplus Fx_n) \oplus Fx_{n-1} & \xrightarrow{\xi \circ (\xi \oplus 1)} & F((\kappa_{n-2}\tilde{\mathbf{X}} \oplus x_n) \oplus x_{n-1}).
 \end{array}$$

The perimeter of the total pasted diagram gives our desired diagram, so it just remains to verify commutativity of this second piece of the pasting. The top and bottom rectangles both commute by the coherence diagram for  $\alpha$  and  $\xi$ , so this just leaves the middle rectangle. But using generic variables to save space, this can be expanded to

$$\begin{array}{ccccc}
 Fa \oplus (Fb \oplus Fc) & \xrightarrow{1 \oplus \xi} & Fa \oplus F(b \oplus c) & \xrightarrow{\xi} & F(a \oplus (b \oplus c)) \\
 1 \oplus \tau \downarrow & & \downarrow 1 \oplus F\tau & & \downarrow F(1 \oplus \tau) \\
 Fa \oplus (Fc \oplus Fb) & \xrightarrow{1 \oplus \xi} & Fa \oplus F(c \oplus b) & \xrightarrow{\xi} & F(a \oplus (c \oplus b)).
 \end{array}$$

However, the left square commutes by the transposition coherence diagram for  $\xi$ , and the right square by naturality of  $\xi$ . This completes the verification, and so  $F$  does define a multifunctor. ■

## 6. The Weak Left Adjoint

This section is devoted to the proof of Theorem 2.6. As a result of Theorem 2.4, we have a well-defined underlying multicategory functor

$$U : \mathbf{Sym} \rightarrow \mathbf{Mult}$$

with source symmetric monoidal categories and lax monoidal functors, and target multicategories and multifunctors. It is natural to ask for a left adjoint to this construction, and we almost have one: there is a functor

$$L : \mathbf{Mult} \rightarrow \mathbf{Sym}$$

with maps  $\eta : \text{id} \rightarrow UL$  and  $\varepsilon : LU \rightarrow \text{id}$  satisfying the triangle identities for the unit and counit of an adjunction. The only problem is that while  $\eta$  is natural,  $\varepsilon$  is only lax natural: its naturality squares only commute up to a natural map of their own that satisfies a coherence condition. The constructions, which are identical to those in [3], Theorem 4.2, are as follows.

**6.1. CONSTRUCTION.** *Let  $\mathbf{M}$  be a multicategory. We construct a symmetric monoidal category  $L\mathbf{M}$ , which is actually permutative, as follows. The objects of  $L\mathbf{M}$  are the free monoid on the objects of  $\mathbf{M}$ , namely*

$$\coprod_{n \geq 0} \text{Ob}\mathbf{M}^n.$$

*Given objects  $\mathbf{x} = (x_1, \dots, x_j)$  and  $\mathbf{y} = (y_1, \dots, y_n)$  of  $L\mathbf{M}$ , we define the elements of the morphism set  $L\mathbf{M}(\mathbf{x}, \mathbf{y})$  to consist of ordered pairs  $(f, \langle \phi_s \rangle)$ , where  $f : \{1, \dots, j\} \rightarrow \{1, \dots, n\}$  is a function with no further structure, and for  $1 \leq s \leq n$ ,  $\phi_s$  is a morphism in  $\mathbf{M}(\langle x_r \rangle_{f(r)=s}; y_s)$ .*

*Given a third object  $\mathbf{z} = (z_1, \dots, z_p)$  of  $L\mathbf{M}$ , and a morphism  $(g, \langle \psi_t \rangle) \in L\mathbf{M}(\mathbf{y}, \mathbf{z})$ , we define*

$$(g, \langle \psi_t \rangle) \circ (f, \langle \phi_s \rangle) = (g \circ f, \langle \chi_t \rangle),$$

*where the morphisms  $\chi_t : \langle x_r \rangle_{gf(r)=t} \rightarrow z_t$  are given by the composite*

$$\langle x_r \rangle_{gf(r)=t} \cong \bigodot_{g(s)=t} \langle x_r \rangle_{f(r)=s} \xrightarrow{\odot_{g(s)=t} \phi_s} \langle y_s \rangle_{g(s)=t} \xrightarrow{\psi_t} z_t.$$

*Here the first isomorphism simply rearranges the tuple  $\langle x_r \rangle_{gf(r)=t}$  into chunks corresponding to each  $s$  for which  $g(s) = t$ . The symmetric monoidal structure, which is actually permutative, is given by concatenation for the product, and the empty list as the unit.*

*To define the functoriality of this construction, suppose given a multifunctor  $F : \mathbf{M} \rightarrow \mathbf{N}$ . We define  $LF : L\mathbf{M} \rightarrow L\mathbf{N}$ , which will actually be a strict monoidal functor, as follows. On objects, this is just the free monoid functor, so*

$$LF(x_1, \dots, x_j) = (Fx_1, \dots, Fx_j).$$

On morphisms, suppose given  $(f, \langle \phi_s \rangle) : \mathbf{x} \rightarrow \mathbf{y}$ , where  $\mathbf{x}$  and  $\mathbf{y}$  are as above, so we have

$$f : \{1, \dots, j\} \rightarrow \{1, \dots, n\} \text{ and } \phi_s : \langle x_r \rangle_{f(r)=s} \rightarrow y_s \text{ for } 1 \leq s \leq n.$$

Then we just define

$$LF(f, \langle \phi_s \rangle) = (f, \langle F\phi_s \rangle).$$

Functoriality is now straightforward to verify, as is the fact that  $LF$  is strict monoidal, and in fact strict symmetric monoidal.

To show that this construction gives a weak left adjoint to the underlying multicategory construction, we provide a unit and weak counit, and show that the adjunction triangles commute. Since  $LM$  is permutative, we use the usual definition for the underlying multicategory of a permutative category, which in this case becomes

$$ULM(\mathbf{x}_1, \dots, \mathbf{x}_n; \mathbf{y}) = LM(\odot_s \mathbf{x}_s, \mathbf{y}).$$

**6.2. DEFINITION.** Let  $\mathbf{M}$  be a multicategory. We define the unit map  $\eta : \mathbf{M} \rightarrow ULM$  as follows. On objects, this is just the unit map of the free-forgetful adjunction between sets and monoids, so sends an object  $x$  to the list of length 1 with entry  $x$ , i.e., we include  $\text{Ob}\mathbf{M}$  as the objects in level 1 in  $\text{Ob}LM$ . On morphisms, suppose  $\phi \in \mathbf{M}(\mathbf{x}; y)$ , where  $\mathbf{x} = (x_1, \dots, x_j)$ . Then there is exactly one function  $p : \{1, \dots, j\} \rightarrow \{1\}$ , so we send  $\phi$  to the  $j$ -morphism  $(p, \{\phi\})$  of  $ULM$ . We can now check that this defines a map of multicategories, and gives a natural map  $\text{id} \rightarrow UL$ .

For the weak counit, we use the following.

**6.3. DEFINITION.** Let  $\mathcal{C}$  be a symmetric monoidal category. Suppose again that we are using our constant sequence  $\kappa_n : \text{Ob}\mathcal{C}^n \rightarrow \text{Ob}Y(n)$  with  $\kappa_0 = 0$ ,  $\kappa_1 = 1$ , and  $\kappa_n = \gamma(m; \kappa_{n-1}, 1)$  for  $n \geq 2$  to define our underlying multicategory. We define the counit map  $\varepsilon : LUC \rightarrow \mathcal{C}$  as follows. On objects our map sends

$$(y_1, \dots, y_n) = \mathbf{y} \mapsto \kappa_n \mathbf{y} = \bar{\kappa} \mathbf{y}.$$

Now let  $\mathbf{x} = (x_1, \dots, x_j)$ , and suppose given a morphism  $(f, \langle \phi_s \rangle) \in LUC(\mathbf{x}, \mathbf{y})$ , so  $f : \{1, \dots, j\} \rightarrow \{1, \dots, n\}$  and  $\phi_s : \langle x_r \rangle_{f(r)=s} \rightarrow y_s$  in  $UC$ . Let  $j_s = |f^{-1}s|$ , so  $j = j_1 + \dots + j_n$ . Then by definition, we have

$$\phi_s \in \mathcal{C}(\bar{\kappa}(\langle x_r \rangle_{f(r)=s}), y_s),$$

and so

$$\langle \phi_s \rangle \in \mathcal{C}^n(\langle \bar{\kappa}(\langle x_r \rangle_{f(r)=s}) \rangle, \mathbf{y}).$$

Let  $\sigma_f$  be the element of  $\Sigma_j$  for which

$$\sigma_f \cdot \mathbf{x} = \odot_r \langle x_s \rangle_{f(r)=s}.$$

Then we have isomorphisms in  $Y(j)$

$$\kappa_j \xrightarrow{\cong} \kappa_j \cdot \sigma_f \xrightarrow{\cong} \gamma(\kappa_n; \langle \kappa_{j_s} \rangle) \cdot \sigma_f,$$

which induce the isomorphisms in the following composition:

$$\begin{array}{c} \mathcal{C}^n(\langle \bar{\kappa}(\langle x_r \rangle_{f(r)=s}) \rangle, \mathbf{y}) \\ \downarrow \kappa_n \\ \mathcal{C}(\kappa_n \langle \bar{\kappa}(\langle x_r \rangle_{f(r)=s}) \rangle, \bar{\kappa} \mathbf{y}) \\ \downarrow \cong \\ \mathcal{C}(\bar{\kappa}(\odot_s \langle x_r \rangle_{f(r)=s}), \bar{\kappa} \mathbf{y}) \\ \downarrow \cong \\ \mathcal{C}(\bar{\kappa} \mathbf{x}, \bar{\kappa} \mathbf{y}). \end{array}$$

We use the image of  $\langle \phi_s \rangle$  under this composition as our  $\varepsilon(f, \langle \phi_s \rangle)$ . It is an exercise to see that  $\varepsilon$  is a strong symmetric monoidal functor.

Several remarks are in order about this weak counit. First, it is *not* strictly natural, but rather only natural up to a natural transformation, in the sense that if we are given a lax symmetric monoidal functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , we can form the following diagram:

$$\begin{array}{ccc} LUC & \xrightarrow{LUF} & LUD \\ \varepsilon_{\mathcal{C}} \downarrow & & \downarrow \varepsilon_{\mathcal{D}} \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D}. \end{array}$$

For  $\varepsilon$  to be natural, this diagram would have to always commute on the nose, but instead it only commutes up to a natural map, which is an isomorphism if  $F$  is strong monoidal. If  $F$  is strict monoidal, we do get equality. We see this since following an object  $(x_1, x_2)$  of  $LUC$  counterclockwise, it ends up first at  $x_1 \oplus x_2$ , and then  $F(x_1 \oplus x_2)$ . However, clockwise it ends up first at  $(Fx_1, Fx_2)$ , and then at  $Fx_1 \oplus Fx_2$ . We then have to use the lax monoidal structure map

$$\xi_F : Fx_1 \oplus Fx_2 \rightarrow F(x_1 \oplus x_2)$$

in order to give a map between the two ways of traversing the square. For a general object  $\mathbf{x} = (x_1, \dots, x_n)$  of  $LUC$ , we use the map  $\xi_n : \bar{\kappa} F \mathbf{x} \rightarrow F \bar{\kappa} \mathbf{x}$  of Definition 4.5, which is only an equality when  $F$  is a strict monoidal functor. These then combine to give a natural map  $\xi$  from the clockwise direction to the counterclockwise direction, so we have a 2-cell

$$\begin{array}{ccc} LUC & \xrightarrow{LUF} & LUD \\ \varepsilon_{\mathcal{C}} \downarrow & \not\parallel_{\xi_F} & \downarrow \varepsilon_{\mathcal{D}} \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

instead of a commutative diagram. The natural maps  $\xi$  are coherent, in the sense that if we have another lax symmetric monoidal functor  $G : \mathcal{D} \rightarrow \mathcal{E}$ , then we have the following equality of pasting diagrams:

$$\begin{array}{ccccc} LUC & \xrightarrow{LUF} & LUD & \xrightarrow{LUG} & LU\mathcal{E} \\ \varepsilon_C \downarrow & \Downarrow \xi_F & \varepsilon_D \downarrow & \Downarrow \xi_G & \varepsilon_E \downarrow \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} & \xrightarrow{G} & \mathcal{E} \end{array} = \begin{array}{ccc} LUC & \xrightarrow{LU(G \circ F)} & LU\mathcal{E} \\ \varepsilon_C \downarrow & \Downarrow \xi_{GF} & \varepsilon_E \downarrow \\ \mathcal{C} & \xrightarrow{G \circ F} & \mathcal{E}. \end{array}$$

Further,  $\xi_{\text{id}}$  is the identity transformation.

Note that the counit  $\varepsilon$  is strict monoidal if  $\mathcal{C}$  is permutative, but only strong monoidal in general: we have

$$\varepsilon \mathbf{x} \oplus \varepsilon \mathbf{y} = \bar{\kappa} \mathbf{x} \oplus \bar{\kappa} \mathbf{y} = \gamma(m; \kappa_j, \kappa_n) \cdot (\mathbf{x} \odot \mathbf{y}),$$

but

$$\varepsilon(\mathbf{x} \odot \mathbf{y}) = \bar{\kappa}(\mathbf{x} \odot \mathbf{y}) = \kappa_{j+n} \cdot (\mathbf{x} \odot \mathbf{y}),$$

and  $\kappa_{j+n} \neq \gamma(m; \kappa_j, \kappa_n)$  in  $Y(j+n)$ . In all cases, the structure map  $\varepsilon \mathbf{x} \oplus \varepsilon \mathbf{y} \rightarrow \varepsilon(\mathbf{x} \odot \mathbf{y})$  is induced by the unique isomorphism

$$\gamma(m; \kappa_j, \kappa_n) \rightarrow \kappa_{j+n}$$

in  $Y(j+n)$ , which induces an identity if  $\mathcal{C}$  is permutative, but not in general.

Since we have chosen  $\kappa_0 = 0$ , it follows that  $\varepsilon$  is strictly unital:  $\varepsilon() = \kappa_0 * = e_{\mathcal{C}}$ .

We leave to the reader the verification of the coherence diagrams showing that  $\varepsilon$  is a symmetric monoidal functor.

The proof of Theorem 2.6 concludes with the following lemma.

6.4. LEMMA. *The adjunction triangles*

$$\begin{array}{ccc} UC & \xrightarrow{\eta_U} & ULUC \\ & \searrow & \downarrow U\varepsilon \\ & = & UC \end{array} \quad \text{and} \quad \begin{array}{ccc} LM & \xrightarrow{L\eta} & LULM \\ & \searrow & \downarrow \varepsilon_{LM} \\ & = & LM \end{array}$$

*both commute.*

PROOF. For the left triangle, an object in  $UC$  is just an object  $x \in \mathcal{C}$ , which gets sent to a list with one entry in  $ULUC$ , and assuming  $\kappa_1 = 1$  allows us to conclude that this gets sent in turn back to  $\kappa_1 x = x$ . The triangle therefore commutes on objects, and the verification on morphisms is the same.

For the right triangle, an object  $\mathbf{x} \in \text{Ob} \mathbf{M}^j \subset \text{Ob} LM$  gets sent by  $L\eta$  to a single list  $(\mathbf{x})$  in  $LULM$ . This is then concatenated as a list of lists by  $\varepsilon$ , but since there's only one list in the list of lists, it goes back to the original list  $\mathbf{x}$ . The triangle therefore commutes on objects, and a simple check shows that it also commutes on morphisms. We therefore do have a weak adjunction between the underlying multicategory functor and the free symmetric monoidal functor, and the left adjoint actually lands in permutative categories and strict monoidal maps. ■

Because both adjunction triangles commute strictly, and  $\eta$  is strictly natural, we do get a comonad  $LU$  on the category of symmetric monoidal categories and lax symmetric monoidal functors, although the composite  $UL$  is only a lax monad on the category of multicategories.

For the proof of Theorem 2.7, we have already observed that the weak left adjoint  $L$  gives us permutative categories and strict monoidal maps as its output, so the comonad  $LU$  on symmetric monoidal categories does convert lax symmetric monoidal maps of symmetric monoidal categories into strict monoidal maps of permutative categories. The remaining claim of the Theorem is the following proposition:

**6.5. PROPOSITION.** *The weak counit  $\varepsilon : LUC \rightarrow \mathcal{C}$  of the weak adjunction is a homotopy equivalence of categories.*

**PROOF.** The first adjunction triangle shows that the unit  $\eta$  of the adjunction gives a right inverse for  $\varepsilon$ , since a map on underlying multicategories is the same as a lax symmetric monoidal functor on symmetric monoidal categories. We produce a natural map  $\nu : \text{id} \rightarrow \eta \circ \varepsilon$  of functors  $LUC \rightarrow LUC$ , which therefore shows that  $\varepsilon$  and  $\eta$  are inverse homotopy equivalences of categories.

An object  $\mathbf{x} \in LUC$  is sent by  $\eta \circ \varepsilon$  to the list of length 1 consisting of  $\bar{\kappa}\mathbf{x}$ . Let's say that  $\mathbf{x} = (x_1, \dots, x_j)$ . We define  $\nu_{\mathbf{x}} : \mathbf{x} \rightarrow (\bar{\kappa}\mathbf{x})$  as the map in  $LUC$  given by the unique function  $f : \{1, \dots, j\} \rightarrow \{1\}$ , together with the identity  $\text{id}_{\bar{\kappa}\mathbf{x}}$ . It is now easy to check that  $\nu$  is a natural map, which concludes the proof. ■

We end this section by noting that  $\nu$  does *not* have an inverse, since  $f$  is not a bijection. However, there is a comparison map to the strictification construction of Isbell [5], as described explicitly by May in [7], Proposition 4.2. May's construction has as its objects the free monoid on the objects of  $\mathcal{C}$ , but subject to the relation that  $e_{\mathcal{C}} = ()$ : the implication is that  $e_{\mathcal{C}}$  is a strict unit to begin with. However, the construction can be easily modified to use the free monoid with no relations, which are precisely the objects of  $LUC$ . The morphisms are then created by the counit  $\varepsilon$ , rather than being those in Definition 6.1. This still gives us a permutative category and a categorical equivalence with the original symmetric monoidal category. Further, the counit  $\varepsilon$  factors through this construction, and therefore  $LUC$  is also homotopy equivalent to it.

## 7. Underlying Multicategories: Properties

We return finally to verifying that the structure given in Section 3 does satisfy the properties necessary to form a multicategory. We return to our original assumption of an arbitrary sequence of functions

$$\kappa_n : \text{Ob}\mathcal{C}^n \rightarrow \text{Ob}Y(n),$$

which determine the underlying multicategory  $U_{\kappa}\mathcal{C}$ .

To give an example of how involved such a structure can look like, we can consider the free symmetric monoidal category on one object which we describe as follows, using some

notation from [1]. (This example is unnecessary for the subsequent arguments.) Let's call this category  $H$ .

The category  $H$  is a disjoint union of categories  $H(n)$  for  $n \geq 0$ , where the index  $n$  indicates the number of times the freely chosen object is combined with itself. Each category  $H(n)$  has as its objects a set  $Z(n)$ . The sets  $Z(n)$  themselves consist of ordered pairs  $(a, R)$ , where  $a$  is a complete parenthesization of at least  $n$  letters, and  $R$  indicates the slots into which  $n$  copies of the free object are to be inserted, with identities in the rest. To be precise, we define the parenthesization sets  $V(k)$  for  $k \geq 0$  by  $V(0) = \emptyset$ ,  $V(1) = \{1\}$ , and for  $k \geq 2$ ,

$$V(k) = \coprod_{i+j=k} V(i) \times V(j).$$

The idea is that a complete parenthesization of  $k$  letters has a last product, with the left factor and right factors being parenthesizations of smaller numbers of letters. The component  $a$  of  $(a, R)$  is to be an element of  $V(k)$  for  $k \geq n$ .

In order to specify the  $n$  slots into which our free object is to be inserted, we define

$$\mathcal{P}_n(k) = \{R \subset \{1, \dots, k\} : |R| = n\}.$$

Then we define our object set  $Z(n)$  of the component category  $H(n)$  by

$$Z(n) = \coprod_{k \geq n} V(k) \times \mathcal{P}_n(k).$$

For morphisms, if  $(a, R) \in Z(n)$  and  $(b, S) \in Z(m)$ , then there are no morphisms unless  $n = m$ , and if  $n = m$ , the morphism set is a copy of the symmetric group  $\Sigma_n$ . Composition is given by group multiplication.

The symmetric monoidal structure is given as follows. First, the unit object is  $(1, \emptyset) \in Z(0)$ . Next, for the monoidal product, suppose given  $(a, R) \in V(k) \times \mathcal{P}_n(k) \subset Z(n)$  and  $(b, S) \in V(q) \times \mathcal{P}_m(q) \subset Z(m)$ . Then we define

$$(a, R) \oplus (b, S) = ((a, b), R \amalg k + S),$$

Here  $(a, b) \in V(k) \times V(q) \subset V(k+q)$ , and  $R \amalg k + S$  is the subset of  $\{1, \dots, k+q\}$  consisting of  $R$  together with  $k$  added to each element of  $S$ , so this is just the concatenation of  $R$  and  $S$  in  $\{1, \dots, k\} \amalg \{1, \dots, q\} = \{1, \dots, k+q\}$ . We therefore obtain an object of  $Z(n+m)$ . The monoidal product of morphisms is given by the block sum  $\Sigma_n \times \Sigma_m \rightarrow \Sigma_{n+m}$ .

The unit isomorphisms and the associator are all given by identity elements of the appropriate  $\Sigma_n$ . In detail, if given  $(a, R) \in V(k) \times \mathcal{P}_n(k) \subset Z(n)$  then we have

$$(a, R) \oplus (1, \emptyset) = ((a, 1), R) \in V(k+1) \times \mathcal{P}_n(k+1) \subset Z(n)$$

and

$$(1, \emptyset) \oplus (a, R) = ((1, a), 1 + R) \in V(k+1) \times \mathcal{P}_n(k+1) \subset Z(n).$$

In both cases, we use the identity element  $1_n \in \Sigma_n$  as the unit isomorphism to  $(a, R)$ .

For the associator, suppose given

$$\begin{aligned}(a, R) &\in V(k) \times \mathcal{P}_n(k) \subset Z(n), \\ (b, S) &\in V(q) \times \mathcal{P}_m(q) \subset Z(m), \text{ and} \\ (c, T) &\in V(w) \times \mathcal{P}_t(w) \subset Z(t).\end{aligned}$$

Then we have

$$\begin{aligned}((a, R) \oplus (b, S)) \oplus (c, T) &= (((a, b), c), R \amalg k + S \amalg k + q + T), \text{ and} \\ (a, R) \oplus ((b, S) \oplus (c, T)) &= ((a, (b, c)), R \amalg k + S \amalg k + q + T).\end{aligned}$$

Note that the first of these has first component an element of  $V(k+q) \times V(w)$ , while the second has first component an element of  $V(k) \times V(q+w)$ . However, both objects are elements of  $Z(n+m+t)$ , so we can use the identity element  $1_{n+m+t} \in \Sigma_{n+m+t}$  as the associator giving an isomorphism between them.

For the transposition, again suppose given  $(a, R) \in V(k) \times \mathcal{P}_n(k) \subset Z(n)$  and  $(b, S) \in V(q) \times \mathcal{P}_m(q) \subset Z(m)$ . Then

$$\begin{aligned}(a, R) \oplus (b, S) &= ((a, b), R \amalg k + S), \text{ and} \\ (b, S) \oplus (a, R) &= ((b, a), S \amalg q + R).\end{aligned}$$

Both are elements of  $Z(n+m)$ , but we do not use the identity of  $\Sigma_{n+m}$  as the transposition isomorphism, but rather the element  $\tau\langle n, m \rangle$  that transposes a block of length  $n$  and a block of length  $m$ . (We don't want the identity element, since that would actually give the identity as the transposition when  $(b, S) = (a, R)$ .)

All the coherence diagrams that do not involve the transposition commute because all the maps are given by identity elements of a symmetric group. For the two that do involve the transposition, we find that  $\tau^2 = \text{id}$  because  $\tau\langle n, m \rangle \circ \tau\langle m, n \rangle = \text{id}$ . And the hexagon

$$\begin{array}{ccc}((a, R) \oplus (b, S)) \oplus (c, T) & \xrightarrow{\alpha} & (a, R) \oplus ((b, S) \oplus (c, T)) \xrightarrow{\tau} ((b, S) \oplus (c, T)) \oplus (a, R) \\ \tau \oplus 1 \downarrow & & \downarrow \alpha \\ ((b, S) \oplus (a, R)) \oplus (c, T) & \xrightarrow{\alpha} & (b, S) \oplus ((a, R) \oplus (c, T)) \xrightarrow{1 \oplus \tau} (b, S) \oplus ((c, T) \oplus (a, R))\end{array}$$

commutes because  $(1_m \oplus \tau\langle n, t \rangle) \circ (\tau\langle n, m \rangle \oplus 1_t) = \tau\langle n, m+t \rangle$ .

We can now give an example of a sequence of functions  $\{\kappa_n : \text{Ob}H^n \rightarrow \text{Ob}Y(n)\}$  as follows. This example is deliberately complicated in order to show how arbitrary such a choice can be. Suppose given an object  $((a_1, R_1), \dots, (a_n, R_n))$  of  $H^n$ , where  $a_i \in Z(j_i)$  and  $R_i \in \mathcal{P}_{k_i}(j_i)$ , so  $(a_i, R_i) \in \text{Ob}H(k_i)$ , and  $j_i \geq k_i$  for  $1 \leq i \leq n$ . Pick an arbitrary  $\beta \in Z(n)$ . Then using the (non-symmetric) operad structure on  $\{Z(n)\}$  given in [1], we can form

$$\delta = \gamma(\beta; \alpha_1, \dots, \alpha_n) \in Z(j_1 + \dots + j_n).$$

This replaces each parenthesized slot in  $\beta$  with the parenthesization given by the corresponding  $\alpha_i$ . Since  $j_i > 0$  for all  $i$ , we must have  $j_1 + \cdots + j_n \geq n$ , so we can form an object of  $Y(n)$  with  $\delta$  as its first component. Next, let

$$S = \{1, j_1 + 1, j_1 + j_2 + 1, \dots, j_1 + \cdots + j_{n-1} + 1\} \subset \{1, \dots, j_1 + \cdots + j_n\},$$

so  $S \in \mathcal{P}_n(j_1 + \cdots + j_n)$ . Finally, to specify an object of  $Y(n)$ , we need an element of  $\Sigma_n$ , so assuming  $n \geq 2$ , we pick the permutation  $\sigma$  that transposes the first two elements of  $\{1, \dots, n\}$  if  $n$  is odd, and the last two elements if  $n$  is even. Then we assign the object  $(\delta, S, \sigma)$  to the object  $((\alpha_1, R_1), \dots, (\alpha_n, R_n))$  of  $H^n$  in order to define our function  $\kappa_n$ .

We now proceed with the proof that  $U_\kappa \mathcal{C}$  does have the structure of a multicategory in all cases, including the example of  $U_\kappa H$  for the  $H$  and  $\kappa$  given above. We first show that we really do have a right action of  $\Sigma_n$  on the collection of  $n$ -morphisms in  $U_\kappa \mathcal{C}$ , and then proceed to verify the diagrams given in Definition 2.1 of [2].

**7.1. PROPOSITION.** *Let  $\mathbf{x} = (x_1, \dots, x_n) \in \text{Ob} \mathcal{C}^n$  and  $y \in \text{Ob} \mathcal{C}$ . Then the maps*

$$\sigma^* : U_\kappa \mathcal{C}(\mathbf{x}; y) \rightarrow U_\kappa \mathcal{C}(\sigma^{-1} \mathbf{x}; y)$$

*produce a right action of  $\Sigma_n$  on the collection of  $n$ -morphisms of  $U_\kappa \mathcal{C}$ .*

**PROOF.** We must show that  $1 \in \Sigma_n$  produces the identity map, and that given  $\sigma, \tau \in \Sigma_n$ , we have

$$\tau^* \circ \sigma^* = (\sigma \circ \tau)^*.$$

For both of these, recall that  $\sigma^*$  is induced by the unique map in  $Y(n)$

$$\theta(\mathbf{x}, \sigma) : \kappa_n(\sigma^{-1} \mathbf{x}) \rightarrow \kappa_n(\mathbf{x}) \cdot \sigma.$$

But if  $\sigma = 1$ , then source and target are both just  $\kappa_n \mathbf{x} \in Y(n)$ , and objects of  $Y$  have only the identity as automorphisms, so  $\theta(\mathbf{x}, 1) = \text{id}$ , and therefore  $1 \in \Sigma_n$  induces the identity on the collection of  $n$ -morphisms of  $U_\kappa \mathcal{C}$ .

To see that  $\tau^* \circ \sigma^* = (\sigma \circ \tau)^*$ , we examine the following diagram between representing objects:

$$\begin{array}{ccccc}
 \kappa_n(\mathbf{x})(\mathbf{x}) & \xrightarrow{=} & [\kappa_n(\mathbf{x}) \cdot \sigma](\sigma^{-1} \mathbf{x}) & \xrightarrow{\theta(\mathbf{x}, \sigma)} & [\kappa_n(\sigma^{-1} \mathbf{x})](\sigma^{-1} \mathbf{x}) \\
 \downarrow = & & \downarrow = & & \downarrow = \\
 [\kappa_n(\mathbf{x}) \cdot (\sigma \circ \tau)]((\sigma \circ \tau)^{-1} \mathbf{x}) & & & & [\kappa_n(\sigma^{-1} \mathbf{x}) \cdot \tau](\tau^{-1} \sigma^{-1} \mathbf{x}) \\
 \downarrow \theta(\mathbf{x}, \sigma \circ \tau) & \searrow = & \downarrow & \nearrow \theta(\mathbf{x}, \sigma) \cdot \tau & \downarrow \theta(\sigma^{-1} \mathbf{x}, \tau) \\
 & & [(\kappa_n(\mathbf{x}) \cdot \sigma) \cdot \tau](\tau^{-1} \sigma^{-1} \mathbf{x}) & & \\
 \kappa_n((\sigma \circ \tau)^{-1} \mathbf{x})((\sigma \circ \tau)^{-1} \mathbf{x}) & \xrightarrow{=} & & & \kappa_n(\tau^{-1} \sigma^{-1} \mathbf{x})(\tau^{-1} \sigma^{-1} \mathbf{x}).
 \end{array}$$

The perimeter of the diagram gives the desired identity. The top left square commutes, being a diagram of identities on the same object. The lower pentagon commutes since it is induced from a diagram in  $Y(n)$ , where all diagrams commute. This leaves the top right square, which commutes due to the equivariance of the action map of  $Y$  on  $\mathcal{C}$ . In particular, if we explicitly say that  $\xi : Y(n) \rightarrow \mathbf{Cat}(\mathcal{C}^n, \mathcal{C})$  is the action map, then we have

$$\xi(\theta(\mathbf{x}, \sigma) \cdot \tau) = \xi(\theta(\mathbf{x}, \sigma)) \cdot \tau.$$

This is an identity of natural transformations of functors  $\mathcal{C}^n \rightarrow \mathcal{C}$ , which we evaluate at the object  $\tau^{-1}\sigma^{-1}\mathbf{x}$ . Then the left side becomes

$$(\theta(\mathbf{x}, \sigma) \cdot \tau) \cdot (\tau^{-1}\sigma^{-1}\mathbf{x})$$

which is the bottom arrow in the top right square, while the right side becomes

$$\theta(\mathbf{x}, \sigma) \cdot \tau \cdot \tau^{-1}\sigma^{-1}\mathbf{x} = \theta(\mathbf{x}, \sigma) \cdot \sigma^{-1}\mathbf{x},$$

which is the top arrow in the top right square, so that square does commute. The entire diagram therefore commutes.  $\blacksquare$

We must verify the associativity diagram, given as (1) on p. 168 of [2], but we give some notation that will allow us to display it in slightly more compressed form. So suppose given a final target  $d$ . We suppose given a tuple  $\mathbf{c} = (c_1, \dots, c_n)$  that will map to  $d$ , and for each  $s$  with  $1 \leq s \leq n$ , a tuple  $\mathbf{b}_s$  that will map to  $c_s$ . Further, we write the entries in  $\mathbf{b}_s$  as  $b_{st}$  for a second index  $t$ , where we say  $1 \leq t \leq j_s$ , with  $j = j_1 + \dots + j_n$ . We write the concatenation of all the  $\mathbf{b}_s$ 's as  $\odot_s \mathbf{b}_s$ , which is a  $j$ -tuple. For each index pair  $st$ , we assume given a tuple  $\mathbf{a}_{st}$  that will map to  $b_{st}$ . For any fixed  $s$  we write the concatenation of the tuples  $\mathbf{a}_{st}$  as  $\odot_t \mathbf{a}_{st}$ , and the concatenation of all these (so a concatenation of concatenations) as  $\odot_s \odot_t \mathbf{a}_{st}$ . We now claim the following, where  $\Gamma$  is the multiproduct (or composition) in  $U_\kappa \mathcal{C}$ :

7.2. PROPOSITION. *The following associativity diagram in  $U_\kappa \mathcal{C}$  commutes:*

$$\begin{array}{ccc}
 & U_\kappa \mathcal{C}(\mathbf{c}; d) \times \prod_s U_\kappa \mathcal{C}(\odot_t \mathbf{a}_{st}; c_s) & \\
 \text{id} \times \prod_s \Gamma \nearrow & & \searrow \Gamma \\
 U_\kappa \mathcal{C}(\mathbf{c}; d) \times \prod_s \left( U_\kappa \mathcal{C}(\mathbf{b}_s; c_s) \times \prod_t U_\kappa \mathcal{C}(\mathbf{a}_{st}; b_{st}) \right) & & U_\kappa \mathcal{C}(\odot_s \odot_t \mathbf{a}_{st}; d). \\
 \downarrow \cong & & \uparrow \Gamma \\
 U_\kappa \mathcal{C}(\mathbf{c}; d) \times \prod_s U_\kappa \mathcal{C}(\mathbf{b}_s; c_s) \times \prod_s \prod_t U_\kappa \mathcal{C}(\mathbf{a}_{st}; b_{st}) & & \\
 \Gamma \times 1 \searrow & & \uparrow \Gamma \\
 U_\kappa \mathcal{C}(\odot_s \mathbf{b}_s; d) \times \prod_s \prod_t U_\kappa \mathcal{C}(\mathbf{a}_{st}; b_{st}) & & 
 \end{array}$$

PROOF. The basic idea for showing that this diagram commutes is to connect it to the associativity square in  $\mathcal{C}$  displayed below:

$$\begin{array}{ccc}
 \mathcal{C}(\overline{\kappa}(\mathbf{c}), d) \times \mathcal{C}(\overline{\kappa}(\odot_s \mathbf{b}_s), \overline{\kappa}(\mathbf{c})) \times \mathcal{C}(\overline{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), \overline{\kappa}(\odot_s \mathbf{b}_s)) & & \\
 \swarrow \scriptstyle \circ \times 1 & & \searrow \scriptstyle 1 \times \circ \\
 \mathcal{C}(\overline{\kappa}(\odot_s \mathbf{b}_s), d) \times \mathcal{C}(\overline{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), \overline{\kappa}(\odot_s \mathbf{b}_s)) & & \mathcal{C}(\overline{\kappa}(\mathbf{c}), d) \times \mathcal{C}(\overline{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), \overline{\kappa}(\mathbf{c})) \\
 \searrow \scriptstyle \circ & & \swarrow \scriptstyle \circ \\
 & \mathcal{C}(\overline{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), d). &
 \end{array}$$

Unpacking the counterclockwise direction of the desired associativity diagram using the definition of the multiproduct composition  $\Gamma$ , we get the following:

$$\begin{array}{c}
 \mathcal{C}(\overline{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \overline{\kappa}\mathbf{b}_s \rangle, \mathbf{c}) \times \mathcal{C}^j(\odot_s \langle \overline{\kappa}\mathbf{a}_{st} \rangle, \odot_s \mathbf{b}_s) \\
 \downarrow \scriptstyle 1 \times \kappa_n(\mathbf{c}) \times 1 \\
 \mathcal{C}(\overline{\kappa}\mathbf{c}, d) \times \mathcal{C}(\kappa_n(\mathbf{c}) \langle \overline{\kappa}\mathbf{b}_s \rangle, \overline{\kappa}\mathbf{c}) \times \mathcal{C}^j(\odot_s \langle \overline{\kappa}\mathbf{a}_{st} \rangle, \odot_s \mathbf{b}_s) \\
 \downarrow \scriptstyle \circ \times 1 \\
 \mathcal{C}(\kappa_n(\mathbf{c}) \langle \overline{\kappa}\mathbf{b}_s \rangle, d) \times \mathcal{C}^j(\odot_s \langle \overline{\kappa}\mathbf{a}_{st} \rangle, \odot_s \mathbf{b}_s) \\
 \downarrow \scriptstyle \mathcal{C}(\phi(\mathbf{c}, \langle \mathbf{b}_s \rangle), 1) \times 1 \\
 \mathcal{C}(\overline{\kappa}(\odot_s \mathbf{b}_s), d) \times \mathcal{C}^j(\odot_s \langle \overline{\kappa}\mathbf{a}_{st} \rangle, \odot_s \mathbf{b}_s) \\
 \downarrow \scriptstyle 1 \times \kappa_j(\odot_s \mathbf{b}_s) \\
 \mathcal{C}(\overline{\kappa}(\odot_s \mathbf{b}_s), d) \times \mathcal{C}(\kappa_j(\odot_s \mathbf{b}_s)(\odot_s \langle \overline{\kappa}\mathbf{a}_{st} \rangle), \overline{\kappa}(\odot_s \mathbf{b}_s)) \\
 \downarrow \scriptstyle \circ \\
 \mathcal{C}(\kappa_j(\odot_s \mathbf{b}_s)(\odot_s \langle \overline{\kappa}\mathbf{a}_{st} \rangle), d) \\
 \downarrow \scriptstyle \mathcal{C}(\phi(\odot_s \mathbf{b}_s, \langle \odot_s \mathbf{a}_{st} \rangle), 1) \\
 \mathcal{C}(\overline{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), d).
 \end{array}$$

However, the composites and the actions of the morphisms in  $Y$  are essentially indepen-

dent, so we can compress the display to the following:

$$\begin{aligned}
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}\mathbf{b}_s \rangle, \mathbf{c}) \times \mathcal{C}^j(\odot_s \langle \bar{\kappa}\mathbf{a}_{st} \rangle, \odot_s \mathbf{b}_s) \\
& \quad \downarrow 1 \times \kappa_n(\mathbf{c}) \times \kappa_j(\odot_s \mathbf{b}_s) \\
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\kappa_n(\mathbf{c}) \langle \bar{\kappa}\mathbf{b}_s \rangle, \bar{\kappa}\mathbf{c}) \times \mathcal{C}(\kappa_j(\odot_s \mathbf{b}_s)(\odot_s \langle \bar{\kappa}\mathbf{a}_{st} \rangle), \bar{\kappa}(\odot_s \mathbf{b}_s)) \\
& \quad \downarrow 1 \times \mathcal{C}(\phi(\mathbf{c}, \langle \mathbf{b}_s \rangle), 1) \times \mathcal{C}(\phi(\odot_s \mathbf{b}_s, \langle \odot_s \mathbf{a}_{st} \rangle), 1) \\
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{b}_s), \bar{\kappa}\mathbf{c}) \times \mathcal{C}(\bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), \bar{\kappa}(\odot_s \mathbf{b}_s)) \\
& \quad \downarrow \circ \times 1 \\
& \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{b}_s), d) \times \mathcal{C}(\bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), \bar{\kappa}(\odot_s \mathbf{b}_s)) \\
& \quad \downarrow \circ \\
& \mathcal{C}(\bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), d).
\end{aligned}$$

In particular, the last two maps are now the counterclockwise direction in the associativity diagram we know commutes in  $\mathcal{C}$ .

We turn next to the clockwise direction in the desired associativity diagram. Using the definition of the multiproduct in  $U_\kappa \mathcal{C}$ , it unpacks as follows:

$$\begin{aligned}
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}\mathbf{b}_s \rangle, \mathbf{c}) \times \mathcal{C}^j(\odot_s \langle \bar{\kappa}\mathbf{a}_{st} \rangle, \odot_s \mathbf{b}_s) \\
& \quad \downarrow 1 \times 1 \times \odot_s \kappa_{j_s} \mathbf{b}_s \\
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}\mathbf{b}_s \rangle, \mathbf{c}) \times \mathcal{C}^n(\langle \kappa_{j_s} \mathbf{b}_s \langle \bar{\kappa}\mathbf{a}_{st} \rangle \rangle, \langle \bar{\kappa}\mathbf{b}_s \rangle) \\
& \quad \downarrow 1 \times \circ \\
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \kappa_{j_s} \mathbf{b}_s \langle \bar{\kappa}\mathbf{a}_{st} \rangle \rangle, \mathbf{c}) \\
& \quad \downarrow 1 \times \mathcal{C}^n(\langle \phi(\mathbf{b}_s, \langle \mathbf{a}_{st} \rangle) \rangle, 1) \\
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}(\odot_t \mathbf{a}_{st}) \rangle, \mathbf{c}) \\
& \quad \downarrow 1 \times \kappa_n \mathbf{c} \\
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}(\odot_t \mathbf{a}_{st}) \rangle, \bar{\kappa}\mathbf{c}) \\
& \quad \downarrow \circ \\
& \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}(\odot_t \mathbf{a}_{st}) \rangle, d) \\
& \quad \downarrow \mathcal{C}(\phi(\mathbf{c}, \langle \odot_t \mathbf{a}_{st} \rangle), 1) \\
& \mathcal{C}(\bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), d).
\end{aligned}$$

We can rewrite this direction also to delay the compositions to the end, which we omit

and will paste on in the next display. We get the following:

$$\begin{array}{c}
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}\mathbf{b}_s \rangle, \mathbf{c}) \times \mathcal{C}^j(\odot_s \langle \bar{\kappa}\mathbf{a}_{st} \rangle, \odot_s \mathbf{b}_s) \\
\downarrow 1 \times 1 \times \odot_s \kappa_{js} \mathbf{b}_s \\
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}\mathbf{b}_s \rangle, \mathbf{c}) \times \mathcal{C}^n(\langle \kappa_{js} \mathbf{b}_s \langle \bar{\kappa}\mathbf{a}_{st} \rangle \rangle, \langle \bar{\kappa}\mathbf{b}_s \rangle) \\
\downarrow 1 \times 1 \times \mathcal{C}^n(\langle \phi(\mathbf{b}_s, \langle \mathbf{a}_{st} \rangle) \rangle, 1) \\
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}\mathbf{b}_s \rangle, \mathbf{c}) \times \mathcal{C}^n(\langle \bar{\kappa}(\odot_t \mathbf{a}_{st}) \rangle, \langle \bar{\kappa}\mathbf{b}_s \rangle) \\
\downarrow 1 \times \kappa_n \mathbf{c} \times \kappa_n \mathbf{c} \\
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}\mathbf{b}_s \rangle, \bar{\kappa}\mathbf{c}) \times \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}(\odot_t \mathbf{a}_{st}) \rangle, \kappa_n \mathbf{c} \langle \bar{\kappa}\mathbf{b}_s \rangle) \\
\downarrow 1 \times 1 \times \mathcal{C}(\phi(\mathbf{c}, \langle \odot_t \mathbf{a}_{st} \rangle), 1) \\
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}\mathbf{b}_s \rangle, \bar{\kappa}\mathbf{c}) \times \mathcal{C}(\bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), \kappa_n \mathbf{c} \langle \bar{\kappa}\mathbf{b}_s \rangle).
\end{array}$$

We now insert an extra arrow that has no effect on the composite, but allows us to connect the display to the associativity diagram in  $\mathcal{C}$ : we paste the following to the bottom of the previous display, giving the clockwise direction in the desired diagram:

$$\begin{array}{c}
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}\mathbf{b}_s \rangle, \bar{\kappa}\mathbf{c}) \times \mathcal{C}(\bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), \kappa_n \mathbf{c} \langle \bar{\kappa}\mathbf{b}_s \rangle). \\
\downarrow 1 \times \mathcal{C}(\phi(\mathbf{c}, \langle \mathbf{b}_s \rangle), 1) \times \mathcal{C}(1, \phi(\mathbf{c}, \langle \mathbf{b}_s \rangle)^{-1}) \\
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{b}_s), \bar{\kappa}\mathbf{c}) \times \mathcal{C}(\bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), \bar{\kappa}(\odot_s \mathbf{b}_s)) \\
\downarrow 1 \times \circ \\
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), \bar{\kappa}\mathbf{c}) \\
\downarrow \circ \\
\mathcal{C}(\bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), d).
\end{array}$$

Note that the first arrow in the above display composes with  $\phi(\mathbf{c}, \langle \mathbf{b}_s \rangle)$  and its inverse, which then get composed together at the next step. The new arrow therefore has no effect on the composite, so we do actually still have the same clockwise direction in our desired diagram, but the last two arrows now coincide with the clockwise direction in the associativity diagram for  $\mathcal{C}$ . It therefore suffices to show that the two directions coincide at the point that they reach the known associativity diagram.

Another benefit of rewriting the two directions to delay compositions to the end is that we can follow what happens to each of the three factors in the beginning term

$$\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}\mathbf{b}_s \rangle, \mathbf{c}) \times \mathcal{C}^j(\odot_s \langle \bar{\kappa}\mathbf{a}_{st} \rangle, \odot_s \mathbf{b}_s)$$

independently, since the two maps to the known associativity diagram are products of three maps from each of the three factors. Starting with the first term  $\mathcal{C}(\bar{\kappa}\mathbf{c}, d)$ , we see that nothing happens to it in either direction, so the two directions do coincide on that factor.

The maps in either direction on the second factor consist of the same composite, namely

$$\mathcal{C}^n(\langle \bar{\kappa}\mathbf{b}_s \rangle, \mathbf{c}) \xrightarrow{\kappa_n \mathbf{c}} \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}\mathbf{b}_s \rangle, \bar{\kappa}\mathbf{c}) \xrightarrow{\mathcal{C}(\phi(\mathbf{c}, \langle \mathbf{b}_s \rangle), 1)} \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{b}_s), \bar{\kappa}\mathbf{c}).$$

The two directions therefore coincide on that factor as well, reducing the issue to the restrictions to the third factor in either direction.

On the third factor, the counterclockwise direction restricts as follows:

$$\begin{array}{c} \mathcal{C}^j(\odot_s \langle \bar{\kappa}\mathbf{a}_{st} \rangle, \odot_s \mathbf{b}_s) \\ \downarrow \kappa_j(\odot_s \mathbf{b}_s) \\ \mathcal{C}(\kappa_j(\odot_s \mathbf{b}_s)(\odot_s \langle \bar{\kappa}\mathbf{a}_{st} \rangle), \bar{\kappa}(\odot_s \mathbf{b}_s)) \\ \downarrow \mathcal{C}(\phi(\odot_s \mathbf{b}_s, \langle \odot_t \mathbf{a}_{st} \rangle), 1) \\ \mathcal{C}(\bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), \bar{\kappa}(\odot_s \mathbf{b}_s)). \end{array}$$

However, the clockwise direction restricts as

$$\begin{array}{c} \mathcal{C}^j(\odot_s \langle \bar{\kappa}\mathbf{a}_{st} \rangle, \odot_s \mathbf{b}_s) \\ \downarrow \odot_s \kappa_{j_s} \mathbf{b}_s \\ \mathcal{C}^n(\langle \kappa_{j_s} \mathbf{b}_s \langle \bar{\kappa}\mathbf{a}_{st} \rangle \rangle, \langle \bar{\kappa}\mathbf{b}_s \rangle) \\ \downarrow \mathcal{C}^n(\langle \phi(\mathbf{b}_s, \langle \mathbf{a}_{st} \rangle) \rangle, 1) \\ \mathcal{C}^n(\langle \bar{\kappa}(\odot_t \mathbf{a}_{st}) \rangle, \langle \bar{\kappa}\mathbf{b}_s \rangle) \\ \downarrow \kappa_n \mathbf{c} \\ \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}(\odot_t \mathbf{a}_{st}) \rangle, \kappa_n \mathbf{c} \langle \bar{\kappa}\mathbf{b}_s \rangle) \\ \downarrow \mathcal{C}(\phi(\mathbf{c}, \langle \mathbf{a}_{st} \rangle), \phi(\mathbf{c}, \langle \mathbf{b}_s \rangle)^{-1}) \\ \mathcal{C}(\bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), \bar{\kappa}(\odot_s \mathbf{b}_s)). \end{array}$$

Because of the functoriality of  $\kappa_n \mathbf{c}$ , we can rearrange the middle two arrows, so the array

becomes

$$\begin{aligned}
& \mathcal{C}^j(\odot_s \langle \bar{\kappa} \mathbf{a}_{st} \rangle, \odot_s \mathbf{b}_s) \\
& \quad \downarrow \odot_s \kappa_{j_s} \mathbf{b}_s \\
& \mathcal{C}^n(\langle \kappa_{j_s} \mathbf{b}_s \langle \bar{\kappa} \mathbf{a}_{st} \rangle \rangle, \langle \bar{\kappa} \mathbf{b}_s \rangle) \\
& \quad \downarrow \kappa_n \mathbf{c} \\
& \mathcal{C}(\kappa_n \mathbf{c} \langle \kappa_{j_s} \mathbf{b}_s \langle \bar{\kappa} \mathbf{a}_{st} \rangle \rangle, \kappa_n \mathbf{c} \langle \bar{\kappa} \mathbf{b}_s \rangle) \\
& \quad \downarrow \mathcal{C}(\kappa_n \mathbf{c} \langle \phi(\mathbf{b}_s, \langle \mathbf{a}_{st} \rangle) \rangle, 1) \\
& \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}(\odot_t \mathbf{a}_{st}) \rangle, \kappa_n \mathbf{c} \langle \bar{\kappa} \mathbf{b}_s \rangle) \\
& \quad \downarrow \mathcal{C}(\phi(\mathbf{c}, \langle \odot_t \mathbf{a}_{st} \rangle), \phi(\mathbf{c}, \langle \mathbf{b}_s \rangle)^{-1}) \\
& \mathcal{C}(\bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), \bar{\kappa}(\odot_s \mathbf{b}_s)),
\end{aligned}$$

or even more compactly as

$$\begin{aligned}
& \mathcal{C}^j(\odot_s \langle \bar{\kappa} \mathbf{a}_{st} \rangle, \odot_s \mathbf{b}_s) \\
& \quad \downarrow \gamma(\kappa_n \mathbf{c}; \langle \kappa_{j_s} \mathbf{b}_s \rangle) \\
& \mathcal{C}(\kappa_n \mathbf{c} \langle \kappa_{j_s} \mathbf{b}_s \langle \bar{\kappa} \mathbf{a}_{st} \rangle \rangle, \kappa_n \mathbf{c} \langle \bar{\kappa} \mathbf{b}_s \rangle) \\
& \quad \downarrow \mathcal{C}(\kappa_n \mathbf{c} \langle \phi(\mathbf{b}_s, \langle \mathbf{a}_{st} \rangle) \rangle, 1) \\
& \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}(\odot_t \mathbf{a}_{st}) \rangle, \kappa_n \mathbf{c} \langle \bar{\kappa} \mathbf{b}_s \rangle) \\
& \quad \downarrow \mathcal{C}(\phi(\mathbf{c}, \langle \odot_t \mathbf{a}_{st} \rangle), \phi(\mathbf{c}, \langle \mathbf{b}_s \rangle)^{-1}) \\
& \mathcal{C}(\bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), \bar{\kappa}(\odot_s \mathbf{b}_s)).
\end{aligned}$$

The last two arrows are induced by isomorphisms in the operad  $Y$ , which are entirely determined by their sources and targets, so let's just indicate them by isomorphism symbols  $\cong$ . The same is true for the second arrow from the counterclockwise direction. Then asking that the two directions coincide is the same as asking for the commutativity of the following square:

$$\begin{array}{ccc}
\mathcal{C}^j(\odot_s \langle \bar{\kappa} \mathbf{a}_{st} \rangle, \odot_s \mathbf{b}_s) & \xrightarrow{\gamma(\kappa_n \mathbf{c}; \langle \kappa_{j_s} \mathbf{b}_s \rangle)} & \mathcal{C}(\kappa_n \mathbf{c} \langle \kappa_{j_s} \mathbf{b}_s \langle \bar{\kappa} \mathbf{a}_{st} \rangle \rangle, \kappa_n \mathbf{c} \langle \bar{\kappa} \mathbf{b}_s \rangle) \\
\downarrow \kappa_j(\odot_s \mathbf{b}_s) & & \downarrow \cong \\
\mathcal{C}(\kappa_j(\odot_s \mathbf{b}_s)(\odot_s \langle \bar{\kappa} \mathbf{a}_{st} \rangle), \bar{\kappa}(\odot_s \mathbf{b}_s)) & \xrightarrow{\cong} & \mathcal{C}(\bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}), \bar{\kappa}(\odot_s \mathbf{b}_s)).
\end{array}$$

If we trace a typical element  $f \in \mathcal{C}^j(\odot_s \langle \bar{\kappa} \mathbf{a}_{st} \rangle, \odot_s \mathbf{b}_s)$  around the square, we find that

we are asking for commutativity of the following diagram:

$$\begin{array}{ccccc}
 & & \kappa_n \mathbf{c} \langle \kappa_j \mathbf{b}_s (\odot_s \langle \bar{\kappa} \mathbf{a}_{st} \rangle) \rangle & \xrightarrow{\gamma(\kappa_n \mathbf{c} \langle \kappa_{js} \mathbf{b}_s \rangle) f} & \kappa_n \mathbf{c} \langle \bar{\kappa} \mathbf{b}_s \rangle \\
 & \nearrow \cong & \uparrow \cong \phi(\mathbf{c}, \odot_s \mathbf{b}_s) & & \uparrow \cong \phi(\mathbf{c}, \odot_s \mathbf{b}_s) \\
 \bar{\kappa}(\odot_s \odot_t \mathbf{a}_{st}) & & & & \\
 & \searrow \cong & \kappa_j(\odot_s \mathbf{b}_s)(\odot_s \langle \bar{\kappa} \mathbf{a}_{st} \rangle) & \xrightarrow{\kappa_j(\odot_s \mathbf{b}_s) f} & \bar{\kappa}(\odot_s \mathbf{b}_s).
 \end{array}$$

The square commutes by naturality of  $\phi(\mathbf{c}, \odot_s \mathbf{b}_s)$ , and the triangle because it is induced by a diagram in  $Y$ , however not in  $Y(j)$ , but rather in the degree of the total concatenation  $\odot_s \odot_t \mathbf{a}_{st}$ . The diagram therefore does commute, and the composition multiproduct in  $U_\kappa \mathcal{C}$  is associative. ■

Continuing the verification of diagrams from [2], we turn to the unit diagrams, listed as (2) on p. 168 in [2]. We begin with the first of them:

7.3. PROPOSITION. *The following unit diagram in  $U_\kappa \mathcal{C}$  commutes:*

$$\begin{array}{ccc}
 U_\kappa \mathcal{C}(\mathbf{c}; d) \times \{1\}^n & \xrightarrow{\cong} & U_\kappa \mathcal{C}(\mathbf{c}; d). \\
 \text{id} \times 1^n \downarrow & \nearrow \Gamma & \\
 U_\kappa \mathcal{C}(\mathbf{c}; d) \times \prod_{s=1}^n U_\kappa \mathcal{C}(c_s; c_s) & & 
 \end{array}$$

PROOF. Expanding the counterclockwise direction in the diagram using the definitions of the structure maps, we have

$$\begin{aligned}
 & \mathcal{C}(\bar{\kappa} \mathbf{c}, d) \times \{1\}^n \\
 & \downarrow \text{id} \times 1^n \\
 & \mathcal{C}(\bar{\kappa} \mathbf{c}, d) \times \prod_{s=1}^n \mathcal{C}(\bar{\kappa} c_s, c_s) \\
 & \downarrow = \\
 & \mathcal{C}(\bar{\kappa} \mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa} c_s \rangle, \mathbf{c}) \\
 & \downarrow \text{id} \times \kappa_n \mathbf{c} \\
 & \mathcal{C}(\bar{\kappa} \mathbf{c}, d) \times \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa} c_s \rangle, \bar{\kappa} \mathbf{c}) \\
 & \downarrow \text{id} \times \mathcal{C}(\phi(\mathbf{c}, \langle c_s \rangle), 1) \\
 & \mathcal{C}(\bar{\kappa} \mathbf{c}, d) \times \mathcal{C}(\bar{\kappa} \mathbf{c}, \bar{\kappa} \mathbf{c}) \\
 & \downarrow \circ \\
 & \mathcal{C}(\bar{\kappa} \mathbf{c}, d).
 \end{aligned}$$

Since we want this to coincide with the canonical projection, and nothing happens to the first factor until the composition at the end, it suffices to trace the second factor and see that it lands at  $\text{id}_{\bar{\kappa}\mathbf{c}} \in \mathcal{C}(\bar{\kappa}\mathbf{c}, \bar{\kappa}\mathbf{c})$  at the next to the last step.

To see this, note first that the 1's get sent to the maps induced by the unique maps  $\omega(c_s) : \kappa_1(c_s) \rightarrow c_s$  in  $Y(1)$ , by definition of the unit map. Next, observe that

$$\kappa_n \mathbf{c} \langle \bar{\kappa} c_s \rangle = \gamma(\kappa_n \mathbf{c}; \langle \kappa_1 c_s \rangle) \mathbf{c},$$

and the map from there to  $\bar{\kappa}\mathbf{c}$  is the one induced by the maps  $\omega(c_s) : \kappa_1(c_s) \rightarrow 1$  in  $Y(1)$ . But this also gives the inverse to the unique isomorphism

$$\phi(\mathbf{c}, \langle c_s \rangle) : \kappa_n \mathbf{c} \rightarrow \gamma(\kappa_n \mathbf{c}; \langle \kappa_1 c_s \rangle),$$

so we do end up at the identity element at the next to the last step. The claimed diagram therefore does commute.  $\blacksquare$

We next verify the second unit diagram.

7.4. PROPOSITION. *The unit diagram*

$$\begin{array}{ccc} \{1\} \times U_\kappa \mathcal{C}(\mathbf{c}; d) & \xrightarrow{\cong} & U_\kappa \mathcal{C}(\mathbf{c}; d) \\ \downarrow 1 \times \text{id} & \nearrow \Gamma & \\ U_\kappa \mathcal{C}(d; d) \times U_\kappa \mathcal{C}(\mathbf{c}; d) & & \end{array}$$

*commutes.*

PROOF. Again we unpack the counterclockwise direction using the definitions, and we get

$$\begin{array}{c} \{1\} \times \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \\ \downarrow 1 \times \text{id} \\ \mathcal{C}(\bar{\kappa}d, d) \times \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \\ \downarrow \text{id} \times \kappa_1 d \\ \mathcal{C}(\bar{\kappa}d, d) \times \mathcal{C}(\kappa_1 d(\bar{\kappa}\mathbf{c}), \bar{\kappa}d) \\ \downarrow 1 \times \mathcal{C}(\phi(d, \mathbf{c}), 1) \\ \mathcal{C}(\bar{\kappa}d, d) \times \mathcal{C}(\bar{\kappa}\mathbf{c}, \bar{\kappa}d) \\ \downarrow \circ \\ \mathcal{C}(\bar{\kappa}\mathbf{c}, d). \end{array}$$

Looking at what happens before the final composition, the initial  $\{1\}$  gets sent to the canonical map  $\omega(d) : \bar{\kappa}d \rightarrow d$  in  $\mathcal{C}(\bar{\kappa}d, d)$ . We can delay the composition in order to insert

maps induced by  $\omega(d)$  and its inverse as follows, without affecting the outcome:

$$\begin{array}{c}
\{1\} \times \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \\
\downarrow 1 \times \text{id} \\
\mathcal{C}(\bar{\kappa}d, d) \times \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \\
\downarrow \text{id} \times \kappa_1 d \\
\mathcal{C}(\bar{\kappa}d, d) \times \mathcal{C}(\kappa_1 d(\bar{\kappa}\mathbf{c}), \bar{\kappa}d) \\
\downarrow 1 \times \mathcal{C}(\phi(d, \mathbf{c}), 1) \\
\mathcal{C}(\bar{\kappa}d, d) \times \mathcal{C}(\bar{\kappa}\mathbf{c}, \bar{\kappa}d) \\
\downarrow \mathcal{C}(\omega(d)^{-1}, 1) \times \mathcal{C}(1, \omega(d)) \\
\mathcal{C}(d, d) \times \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \\
\downarrow \circ \\
\mathcal{C}(\bar{\kappa}\mathbf{c}, d).
\end{array}$$

But since the first map sends 1 to  $\omega(d)$ , now the initial  $\{1\}$  ends up at  $\text{id}_d \in \mathcal{C}(d, d)$ , so the issue is making sure that the second factor  $\mathcal{C}(\bar{\kappa}\mathbf{c}, d)$  has its identity as the total map. Tracing a typical element  $f \in \mathcal{C}(\bar{\kappa}\mathbf{c}, d)$ , so  $f : \bar{\kappa}\mathbf{c} \rightarrow d$ , we see that the issue is whether the following square commutes:

$$\begin{array}{ccc}
\kappa_1 d(\bar{\kappa}\mathbf{c}) & \xrightarrow{\kappa_1 d(f)} & \bar{\kappa}d \\
\phi(d, \mathbf{c}) \uparrow & & \downarrow \omega(d) \\
\bar{\kappa}\mathbf{c} & \xrightarrow{f} & d.
\end{array}$$

Now  $\phi(d, \mathbf{c})$  arises from the unique isomorphism  $\kappa_n \mathbf{c} \rightarrow \gamma(\kappa_1 d; \kappa_n \mathbf{c})$  in  $Y(n)$ , but this can be expressed as

$$\gamma(\omega(d)^{-1}; \kappa_n \mathbf{c}) : \kappa_n \mathbf{c} = \gamma(1; \kappa_n \mathbf{c}) \rightarrow \gamma(\kappa_1 d; \kappa_n \mathbf{c})$$

by the uniqueness of morphisms in  $Y(n)$ . This means that  $\phi(d, \mathbf{c})$  is really just  $\omega(d)^{-1}$  applied to the object  $\bar{\kappa}\mathbf{c}$ , so we might as well express the diagram as

$$\begin{array}{ccc}
\kappa_1 d(\bar{\kappa}\mathbf{c}) & \xrightarrow{\kappa_1 d(f)} & \bar{\kappa}d \\
\omega(d)^{-1} \uparrow & & \downarrow \omega(d) \\
\bar{\kappa}\mathbf{c} & \xrightarrow{f} & d.
\end{array}$$

Since  $\omega(d)$  is a natural isomorphism, the square commutes by naturality. The second unit diagram therefore commutes. ■

We next verify the commutativity of the equivariance diagram labeled (4) on p. 169 in [2].

7.5. THEOREM. *Let  $\sigma \in \Sigma_n$ . Then the following equivariance diagram in  $U_\kappa \mathcal{C}$  commutes:*

$$\begin{array}{ccc} U_\kappa \mathcal{C}(\mathbf{c}; d) \times \prod_{s=1}^n U_\kappa \mathcal{C}(\mathbf{b}_s; c_s) & \xrightarrow{\Gamma} & U_\kappa \mathcal{C}(\odot_s \mathbf{b}_s; d) \\ \sigma^* \times \sigma^{-1} \downarrow & & \downarrow \sigma \langle j_{\sigma(1)}, \dots, j_{\sigma(n)} \rangle^* \\ U_\kappa \mathcal{C}(\sigma^{-1} \mathbf{c}, d) \times \prod_{s=1}^n U_\kappa \mathcal{C}(\mathbf{b}_{\sigma(s)}, c_{\sigma(s)}) & \xrightarrow{\Gamma} & U_\kappa \mathcal{C}(\odot_s \mathbf{b}_{\sigma(s)}, d). \end{array}$$

PROOF. The counterclockwise direction unpacks as follows:

$$\begin{array}{c} \mathcal{C}(\bar{\kappa} \mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa} \mathbf{b}_s \rangle, \mathbf{c}) \\ \downarrow \sigma^* \times \sigma^{-1} \\ \mathcal{C}(\bar{\kappa}(\sigma^{-1} \mathbf{c}), d) \times \mathcal{C}^n(\langle \bar{\kappa} \mathbf{b}_{\sigma(s)} \rangle, \sigma^{-1} \mathbf{c}) \\ \downarrow 1 \times \kappa_n(\sigma^{-1} \mathbf{c}) \\ \mathcal{C}(\bar{\kappa}(\sigma^{-1} \mathbf{c}), d) \times \mathcal{C}(\kappa_n(\sigma^{-1} \mathbf{c}) \langle \bar{\kappa} \mathbf{b}_{\sigma(s)} \rangle, \bar{\kappa}(\sigma^{-1} \mathbf{c})) \\ \downarrow \circ \\ \mathcal{C}(\kappa_n(\sigma^{-1} \mathbf{c}) \langle \bar{\kappa} \mathbf{b}_{\sigma(s)} \rangle, d) \\ \downarrow \mathcal{C}(\phi(\sigma^{-1} \mathbf{c}, \langle \mathbf{b}_{\sigma(s)} \rangle), 1) \\ \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{b}_{\sigma(s)}), d). \end{array}$$

The clockwise direction unpacks as follows:

$$\begin{array}{c} \mathcal{C}(\bar{\kappa} \mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa} \mathbf{b}_s \rangle, \mathbf{c}) \\ \downarrow 1 \times \kappa_n \mathbf{c} \\ \mathcal{C}(\bar{\kappa} \mathbf{c}, d) \times \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa} \mathbf{b}_s \rangle, \bar{\kappa} \mathbf{c}) \\ \downarrow \circ \\ \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa} \mathbf{b}_s \rangle, d) \\ \downarrow \mathcal{C}(\phi(\mathbf{c}, \langle \mathbf{b}_s \rangle), 1) \\ \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{b}_s), d) \\ \downarrow = \\ \mathcal{C}([\kappa_j(\odot_s \mathbf{b}_s) \cdot \sigma \langle j_{\sigma(1)}, \dots, j_{\sigma(n)} \rangle](\odot_s \mathbf{b}_{\sigma(s)}), d) \\ \downarrow \mathcal{C}(\theta(\odot_s \mathbf{b}_s, \sigma \langle j_{\sigma(1)}, \dots, j_{\sigma(n)} \rangle), 1) \\ \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{b}_{\sigma(s)}), d). \end{array}$$

To begin connecting the two directions, we first decompose the second arrow in the last display, labeled  $\circ$  for composition, as follows:

$$\begin{aligned}
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\kappa_n\mathbf{c}\langle\bar{\kappa}\mathbf{b}_s\rangle, \bar{\kappa}\mathbf{c}) \\
& \quad \downarrow = \\
& \mathcal{C}([\kappa_n\mathbf{c} \cdot \sigma]\sigma^{-1}\mathbf{c}, d) \times \mathcal{C}(\kappa_n\mathbf{c}\langle\bar{\kappa}\mathbf{b}_s\rangle, [\kappa_n\mathbf{c} \cdot \sigma]\sigma^{-1}\mathbf{c}) \\
& \quad \downarrow \mathcal{C}(\theta(\mathbf{c}, \sigma), 1) \times \mathcal{C}(1, \theta(\mathbf{c}, \sigma)^{-1}) \\
& \mathcal{C}(\bar{\kappa}(\sigma^{-1}\mathbf{c}), d) \times \mathcal{C}(\kappa_n\mathbf{c}\langle\bar{\kappa}\mathbf{b}_s\rangle, \bar{\kappa}(\sigma^{-1}\mathbf{c})) \\
& \quad \downarrow \circ \\
& \mathcal{C}(\kappa_n\mathbf{c}\langle\bar{\kappa}\mathbf{b}_s\rangle, d).
\end{aligned}$$

Since we are composing in one factor with  $\theta(\mathbf{c}, \sigma)$  and in the other with its inverse, the composition map remains the same.

Now we connect the two directions by means of the following diagram, which we claim commutes, where the map  $p$  is defined below:

$$\begin{array}{ccc}
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle\bar{\kappa}\mathbf{b}_s\rangle, \mathbf{c}) & \xrightarrow{1 \times \kappa_n\mathbf{c}} & \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\kappa_n\mathbf{c}\langle\bar{\kappa}\mathbf{b}_s\rangle, \bar{\kappa}\mathbf{c}) \\
\downarrow \sigma^* \times \sigma^{-1} & & \downarrow = \\
\mathcal{C}(\bar{\kappa}(\sigma^{-1}\mathbf{c}), d) \times \mathcal{C}^n(\langle\bar{\kappa}\mathbf{b}_{\sigma(s)}\rangle, \sigma^{-1}\mathbf{c}) & & \mathcal{C}([\kappa_n\mathbf{c} \cdot \sigma]\sigma^{-1}\mathbf{c}, d) \times \mathcal{C}(\kappa_n\mathbf{c}\langle\bar{\kappa}\mathbf{b}_s\rangle, [\kappa_n\mathbf{c} \cdot \sigma]\sigma^{-1}\mathbf{c}) \\
\downarrow 1 \times \kappa_n(\sigma^{-1}\mathbf{c}) & & \downarrow \mathcal{C}(\theta(\mathbf{c}, \sigma), 1) \times \mathcal{C}(1, \theta(\mathbf{c}, \sigma)^{-1}) \\
\mathcal{C}(\bar{\kappa}(\sigma^{-1}\mathbf{c}), d) \times \mathcal{C}(\kappa_n(\sigma^{-1}\mathbf{c})\langle\bar{\kappa}\mathbf{b}_{\sigma(s)}\rangle, \bar{\kappa}(\sigma^{-1}\mathbf{c})) & & \mathcal{C}(\bar{\kappa}(\sigma^{-1}\mathbf{c}), d) \times \mathcal{C}(\kappa_n\mathbf{c}\langle\bar{\kappa}\mathbf{b}_s\rangle, \bar{\kappa}(\sigma^{-1}\mathbf{c})) \\
\downarrow \circ & \searrow 1 \times \mathcal{C}(p, 1) & \downarrow \circ \\
\mathcal{C}(\kappa_n(\sigma^{-1}\mathbf{c})\langle\bar{\kappa}\mathbf{b}_{\sigma(s)}\rangle, d) & \xrightarrow{\mathcal{C}(p, 1)} & \mathcal{C}(\kappa_n\mathbf{c}\langle\bar{\kappa}\mathbf{b}_s\rangle, d) \\
\downarrow \mathcal{C}(\phi(\sigma^{-1}\mathbf{c}, \langle\mathbf{b}_{\sigma(s)}\rangle), 1) & & \downarrow \mathcal{C}(\phi(\mathbf{c}, \langle\mathbf{b}_s\rangle), 1) \\
\mathcal{C}(\bar{\kappa}(\odot_s\mathbf{b}_{\sigma(s)}), d) & & \mathcal{C}(\bar{\kappa}(\odot_s\mathbf{b}_s), d) \\
& & \downarrow = \\
& & \mathcal{C}([\kappa_j(\odot_s\mathbf{b}_s) \cdot \sigma\langle j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle](\odot_s\mathbf{b}_{\sigma(s)}), d) \\
& \nwarrow \mathcal{C}(\theta(\odot_s\mathbf{b}_s, \sigma\langle j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle), 1) & \\
& \mathcal{C}(\bar{\kappa}(\odot_s\mathbf{b}_{\sigma(s)}), d) &
\end{array}$$

Here  $p$  is induced by the unique isomorphism

$$\gamma(\kappa_n\mathbf{c}; \langle\kappa_{j_s}\mathbf{b}_s\rangle) \cdot \sigma\langle j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle \rightarrow \gamma(\kappa_n(\sigma^{-1}\mathbf{c}); \langle\kappa_{j_{\sigma(s)}}\mathbf{b}_s\rangle)$$

in  $Y(j)$ , which in turn arises as the composite

$$\begin{aligned}
& \gamma(\kappa_n \mathbf{c}; \langle \kappa_{j_s} \mathbf{b}_s \rangle) \cdot \sigma \langle j_{\sigma(1)}, \dots, j_{\sigma(n)} \rangle \\
& \quad \downarrow = \\
& \gamma(\kappa_n \mathbf{c} \cdot \sigma \cdot \sigma^{-1}; \langle \kappa_{j_s} \mathbf{b}_s \rangle) \cdot \sigma \langle j_{\sigma(1)}, \dots, j_{\sigma(n)} \rangle \\
& \quad \downarrow = \\
& \gamma(\kappa_n \mathbf{c} \cdot \sigma; \langle \kappa_{j_{\sigma(s)}} \mathbf{b}_{\sigma(s)} \rangle) \cdot \sigma^{-1} \langle j_1, \dots, j_n \rangle \cdot \sigma \langle j_{\sigma(1)}, \dots, j_{\sigma(n)} \rangle \\
& \quad \downarrow = \\
& \gamma(\kappa_n \mathbf{c} \cdot \sigma; \langle \kappa_{j_{\sigma(s)}} \mathbf{b}_{\sigma(s)} \rangle) \\
& \quad \downarrow \gamma(\theta(\mathbf{c}, \sigma)^{-1}; 1) \\
& \gamma(\kappa_n(\sigma^{-1} \mathbf{c}); \langle \kappa_{j_{\sigma(s)}} \mathbf{b}_{\sigma(s)} \rangle),
\end{aligned}$$

where the second equality arises from the first equivariance formula for an operad in [8], Definition 1.1(c) on page 2. We are therefore also justified in writing  $p$  as  $\theta(\mathbf{c}, \sigma)^{-1}$ .

The bottom pentagon commutes, since each map is induced by an isomorphism in  $Y(j)$ , where all diagrams commute. The middle (distorted) square commutes since the horizontal arrows are given by the same map  $p$ . This leaves the top hexagon to check, and we can do so on each factor separately, since the hexagon is actually the product of two separate hexagons. On the first factor  $\mathcal{C}(\bar{\kappa} \mathbf{b}_s, d)$ , the counterclockwise direction is given by the map  $\sigma^*$ . But this is defined to be induced by  $\theta(\mathbf{c}, \sigma)$ , which is what induces the clockwise direction. The top portion therefore commutes on the first factor, thus reducing the question to its commutativity on the second factor. This is captured in the perimeter of the following diagram:

$$\begin{array}{ccccc}
& & \mathcal{C}^n(\langle \bar{\kappa} \mathbf{b}_s \rangle, \mathbf{c}) & & \\
& \swarrow \sigma^{-1} & & \searrow \kappa_n \mathbf{c} & \\
\mathcal{C}^n(\langle \bar{\kappa} \mathbf{b}_{\sigma(s)} \rangle, \sigma^{-1} \mathbf{c}) & \xrightarrow{\kappa_n \mathbf{c} \cdot \sigma} & \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa} \mathbf{b}_s \rangle, \bar{\kappa} \mathbf{c}) & & \\
\downarrow \kappa_n(\sigma^{-1} \mathbf{c}) & & \downarrow \mathcal{C}(1, \theta(\mathbf{c}, \sigma)^{-1}) & & \\
\mathcal{C}(\kappa_n(\sigma^{-1} \mathbf{c}) \langle \bar{\kappa} \mathbf{b}_{\sigma(s)} \rangle, \bar{\kappa}(\sigma^{-1} \mathbf{c})) & \xrightarrow{\mathcal{C}(p, 1)} & \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa} \mathbf{b}_s \rangle, \bar{\kappa}(\sigma^{-1} \mathbf{c})). & & 
\end{array}$$

The triangle on top commutes because  $\kappa_n \mathbf{c} \cdot \sigma$  is just the composition of  $\kappa_n \mathbf{c}$  with the permutation given by  $\sigma$ , so composing with  $\sigma^{-1}$  just gives  $\kappa_n \mathbf{c}$ . This reduces us to verifying

commutativity of the bottom square. But we can rewrite the bottom square as follows:

$$\begin{array}{ccc}
 & \mathcal{C}^n(\langle \bar{\kappa} \mathbf{b}_{\sigma(s)} \rangle, \sigma^{-1} \mathbf{c}) & \\
 \swarrow \kappa_n(\sigma^{-1} \mathbf{c}) & & \searrow \kappa_n \mathbf{c} \cdot \sigma \\
 \mathcal{C}(\kappa_n(\sigma^{-1} \mathbf{c}) \langle \bar{\kappa} \mathbf{b}_{\sigma(s)} \rangle, \bar{\kappa}(\sigma^{-1} \mathbf{c})) & \xrightarrow{\mathcal{C}(\theta(\mathbf{c}, \sigma)^{-1}, \theta(\mathbf{c}, \sigma))} & \mathcal{C}(\kappa_n \mathbf{c} \cdot \sigma \langle \bar{\kappa} \mathbf{b}_{\sigma(s)} \rangle, \kappa_n \cdot \sigma(\sigma^{-1} \mathbf{c})) \\
 \searrow \mathcal{C}(\theta(\mathbf{c}, \sigma)^{-1}, 1) & & \swarrow \mathcal{C}(1, \theta(\mathbf{c}, \sigma)^{-1}) \\
 & \mathcal{C}(\kappa_n \mathbf{c} \cdot \sigma \langle \bar{\kappa} \mathbf{b}_{\sigma(s)} \rangle, \bar{\kappa}(\sigma^{-1} \mathbf{c})) & \\
 \downarrow = & & \\
 & \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa} \mathbf{b}_s \rangle, \bar{\kappa}(\sigma^{-1} \mathbf{c})). &
 \end{array}$$

The top triangle commutes by the naturality of  $\theta(\mathbf{c}, \sigma)$ , and the bottom triangle by inspection. This concludes the verification of the first equivariance diagram. ■

The verification that  $U_\kappa \mathcal{C}$  satisfies the requirements for a multicategory concludes with the commutativity of the second equivariance diagram, labeled (4) on p. 169 of [2]. We suppose given permutations  $\tau_s \in \Sigma_{j_s}$  for  $1 \leq s \leq n$ .

7.6. PROPOSITION. *The following equivariance diagram commutes:*

$$\begin{array}{ccc}
 U_\kappa \mathcal{C}(\mathbf{c}; d) \times \prod_{s=1}^n U_\kappa \mathcal{C}(\mathbf{b}_s; c_s) & \xrightarrow{\Gamma} & U_\kappa \mathcal{C}(\odot_s \mathbf{b}_s, d) \\
 \downarrow 1 \times \prod \tau_s^* & & \downarrow (\oplus_s \tau_s)^* \\
 U_\kappa \mathcal{C}(\mathbf{c}; d) \times \prod_{s=1}^n U_\kappa \mathcal{C}(\tau_s^{-1} \mathbf{b}_s, c_s) & \xrightarrow{\Gamma} & U_\kappa \mathcal{C}(\odot_s \tau_s^{-1} \mathbf{b}_s; d).
 \end{array}$$

PROOF. The clockwise direction unpacks as follows:

$$\begin{array}{c}
 \mathcal{C}(\bar{\kappa} \mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa} \mathbf{b}_s \rangle, \mathbf{c}) \\
 \downarrow 1 \times \kappa_n \mathbf{c} \\
 \mathcal{C}(\bar{\kappa} \mathbf{c}, d) \times \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa} \mathbf{b}_s \rangle, \bar{\kappa} \mathbf{c}) \\
 \downarrow \circ \\
 \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa} \mathbf{b}_s \rangle, d) \\
 \downarrow \mathcal{C}(\phi(\mathbf{c}, \langle \mathbf{b}_s \rangle), 1) \\
 \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{b}_s), d) \\
 \downarrow (\oplus_s \tau_s)^* \\
 \mathcal{C}(\bar{\kappa}(\odot_s \tau_s^{-1} \mathbf{b}_s), d).
 \end{array}$$

However, it will be convenient to rearrange this to delay the composition to the end, as follows:

$$\begin{aligned}
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}\mathbf{b}_s \rangle, \mathbf{c}) \\
& \quad \downarrow 1 \times \kappa_n \mathbf{c} \\
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}\mathbf{b}_s \rangle, \bar{\kappa}\mathbf{c}) \\
& \quad \downarrow 1 \times \mathcal{C}(\phi(\mathbf{c}, \langle \mathbf{b}_s \rangle), 1) \\
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{b}_s), \bar{\kappa}\mathbf{c}) \\
& \quad \downarrow 1 \times (\oplus_s \tau_s)^* \\
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\bar{\kappa}(\odot_s \tau_s^{-1} \mathbf{b}_s), \bar{\kappa}\mathbf{c}) \\
& \quad \downarrow \circ \\
& \mathcal{C}(\bar{\kappa}(\odot_s \tau_s^{-1} \mathbf{b}_s), d).
\end{aligned}$$

The counterclockwise direction unpacks as follows:

$$\begin{aligned}
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}\mathbf{b}_s \rangle, \mathbf{c}) \\
& \quad \downarrow 1 \times \prod_s \tau_s^* \\
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}(\tau_s^{-1} \mathbf{b}_s) \rangle, \mathbf{c}) \\
& \quad \downarrow 1 \times \kappa_n \mathbf{c} \\
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}(\tau_s^{-1} \mathbf{b}_s) \rangle, \bar{\kappa}\mathbf{c}) \\
& \quad \downarrow \circ \\
& \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}(\tau_s^{-1} \mathbf{b}_s) \rangle, d) \\
& \quad \downarrow \mathcal{C}(\phi(\mathbf{c}, \langle \tau_s^{-1} \mathbf{b}_s \rangle), 1) \\
& \mathcal{C}(\bar{\kappa}(\odot_s \tau_s^{-1} \mathbf{b}_s), d).
\end{aligned}$$

But again, it will be convenient to rewrite this to delay the composition to the end, as

follows:

$$\begin{array}{c}
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}\mathbf{b}_s \rangle, \mathbf{c}) \\
\downarrow 1 \times \prod_s \tau_s^* \\
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}(\tau_s^{-1}\mathbf{b}_s) \rangle, \mathbf{c}) \\
\downarrow 1 \times \kappa_n \mathbf{c} \\
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}(\tau_s^{-1}\mathbf{b}_s) \rangle, \bar{\kappa}\mathbf{c}) \\
\downarrow 1 \times \mathcal{C}(\phi(\mathbf{c}, \langle \tau_s^{-1}\mathbf{b}_s \rangle), 1) \\
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\bar{\kappa}(\odot_s \tau_s^{-1}\mathbf{b}_s), \bar{\kappa}\mathbf{c}) \\
\downarrow \circ \\
\mathcal{C}(\bar{\kappa}(\odot_s \tau_s^{-1}\mathbf{b}_s), d).
\end{array}$$

We can connect the two directions, which form the perimeter of the following diagram, in which the hexagon is actually the product of two separate hexagons:

$$\begin{array}{ccccc}
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}\mathbf{b}_s \rangle, \mathbf{c}) & & & \\
& \swarrow 1 \times \prod_s \tau_s^* & & \searrow 1 \times \kappa_n \mathbf{c} & \\
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}^n(\langle \bar{\kappa}(\tau_s^{-1}\mathbf{b}_s) \rangle, \mathbf{c}) & & & & \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}\mathbf{b}_s \rangle, \bar{\kappa}\mathbf{c}) \\
\downarrow 1 \times \kappa_n \mathbf{c} & \swarrow 1 \times \mathcal{C}(\kappa_n \mathbf{c} \langle \theta(\mathbf{b}_s, \tau_s) \rangle, 1) & & \searrow 1 \times \mathcal{C}(\phi(\mathbf{c}, \langle \mathbf{b}_s \rangle), 1) & \\
\mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}(\tau_s^{-1}\mathbf{b}_s) \rangle, \bar{\kappa}\mathbf{c}) & & & & \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{b}_s), \bar{\kappa}\mathbf{c}) \\
& \swarrow 1 \times \mathcal{C}(\phi(\mathbf{c}, \langle \tau_s^{-1}\mathbf{b}_s \rangle), 1) & & \nwarrow 1 \times (\oplus_s \tau_s)^* & \\
& \mathcal{C}(\bar{\kappa}\mathbf{c}, d) \times \mathcal{C}(\bar{\kappa}(\odot_s \tau_s^{-1}\mathbf{b}_s), \bar{\kappa}\mathbf{c}) & & & \\
& \downarrow \circ & & & \\
& \mathcal{C}(\bar{\kappa}(\odot_s \tau_s^{-1}\mathbf{b}_s), d). & & & 
\end{array}$$

Since nothing happens to the first factor,  $\mathcal{C}(\bar{\kappa}\mathbf{c}, d)$ , until the end composition, we may ignore it, and concentrate on the second factor,  $\mathcal{C}^n(\langle \bar{\kappa}\mathbf{b}_s \rangle, \mathbf{c})$ . The top square in the diagram becomes

$$\begin{array}{ccc}
\mathcal{C}^n(\langle \bar{\kappa}\mathbf{b}_s \rangle, \mathbf{c}) & \xrightarrow{\kappa_n \mathbf{c}} & \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}\mathbf{b}_s \rangle, \bar{\kappa}\mathbf{c}) \\
\Pi_s \tau_s^* \downarrow & & \downarrow \mathcal{C}(\kappa_n \mathbf{c} \langle \theta(\mathbf{b}_s, \tau_s) \rangle, 1) \\
\mathcal{C}^n(\langle \bar{\kappa}(\tau^{-1}\mathbf{b}_s) \rangle, \mathbf{c}) & \xrightarrow{\kappa_n \mathbf{c}} & \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}(\tau^{-1}\mathbf{b}_s) \rangle, \bar{\kappa}\mathbf{c}).
\end{array}$$

But the maps  $\tau_s^*$  are induced by  $\theta(\mathbf{b}_s, \tau_s)$ , so we can rewrite the left vertical arrow as  $\mathcal{C}^n(\langle \theta(\mathbf{b}_s, \tau_s) \rangle, 1)$ , and the diagram now commutes by functoriality of  $\kappa_n \mathbf{c}$ .

The lower square in our desired diagram becomes

$$\begin{array}{ccc}
 \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa} \mathbf{b}_s \rangle, \bar{\kappa} \mathbf{c}) & \xrightarrow{\mathcal{C}(\phi(\mathbf{c}, \langle \mathbf{b}_s \rangle), 1)} & \mathcal{C}(\bar{\kappa}(\odot_s \mathbf{b}_s), \bar{\kappa} \mathbf{c}) \\
 \downarrow \mathcal{C}(\kappa_n \mathbf{c} \langle \theta(\mathbf{b}_s, \tau_s) \rangle, 1) & & \downarrow (\oplus_s \tau_s)^* \\
 \mathcal{C}(\kappa_n \mathbf{c} \langle \bar{\kappa}(\tau_s^{-1} \mathbf{b}_s) \rangle, \bar{\kappa} \mathbf{c}) & \xrightarrow{\mathcal{C}(\phi(\mathbf{c}, \langle \tau_s^{-1} \mathbf{b}_s \rangle), 1)} & \mathcal{C}(\bar{\kappa}(\tau_s^{-1} \mathbf{b}_s), \bar{\kappa} \mathbf{c}).
 \end{array}$$

Since all the arrows in this diagram are induced from morphisms in  $Y(n)$ , where all diagrams commute, this diagram also commutes. This completes the verification that the second equivariance diagram commutes, and therefore that  $U_\kappa \mathcal{C}$  satisfies the requirements for a multicategory. ■

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