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THE FUNAYAMA ENVELOPE AS THE TD-HULL OF A FRAME

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Abstract. We introduce proximity morphisms between MT-algebras and show that the resulting category is equivalent to the category of frames. This is done by utilizing the Funayama envelope of a frame, which is viewed as the T_D-hull. Our results have some spatial ramifications, including a generalization of the T_D-duality of Banaschewski and Pultr.

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1. Introduction

Pointfree topology has its origins in the study of topological spaces where the lattice of open sets is taken as the core construct. Although this has been very fruitful (see, e.g., [Joh82, PP12]), it has its own drawbacks because the language is often not expressive enough. This is well manifested when looking at separation axioms. While the language of frames is perfectly adequate to express higher separation axioms, it is less so for the lower ones. For example, being regular means that each open set is the union of open sets whose closure is contained in it. This condition is easy to express in the language of frames using the rather below relation (see, e.g., [PP21, p. 88]). On the other hand, being a T_1-space means that each singleton is closed, which is harder to express since singletons of the space as well as the closure operator cannot be formalized using only the frame of opens. However, it can be done using the powerset algebra equipped with topological closure or interior. This more expressive formalism goes back to Kuratowski [Kur22], and has further been developed by McKinsey and Tarski [MT44] in the form of

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the theory of closure algebras or interior algebras; that is, boolean algebras equipped with an appropriate operator. As is clear from the title of their article, *The Algebra of Topology*, they envisioned an algebraic formalism to reason about topology. This approach turned out to be very beneficial not only for topology, but also for the foundations of mathematics in general, and the connection between intuitionistic and modal logics in particular (see, e.g., [RS63, CZ97, Esa19]).

As was demonstrated in [BR23] (see also [Nöb54, RS63, Nat90]), the standard point-free approach to topology through the frames of opens can be enhanced by considering those interior algebras whose underlying boolean algebra is complete. Indeed, these can naturally be thought of as a pointfree generalization of interior algebras arising as powersets of topological spaces, and give rise to frames by taking the poset of open elements. These algebras were coined MT-algebras, in honor of McKinsey and Tarski. Equipping MT-algebras with an appropriate notion of morphism, we obtain the category **MT**, and the open element functor $\mathcal{O}$ from **MT** to the category **Frm** of frames. Moreover, up to isomorphism, every frame arises as the frame of open elements of some MT-algebra, thanks to the well-known Funayama embedding of each frame into a complete boolean algebra [Fun59]. We call this the Funayama envelope of a frame L and denote it by $\mathcal{F}L$. However, the assignment $L \mapsto \mathcal{F}L$ is not functorial: frame morphisms do not in general lift to complete lattice maps between their Funayama envelopes [BR23, Example 4.4]. To amend this, we introduce a new notion of morphism between MT-algebras, which is based on a proximity-like structure on the MT-algebra, reminiscent of de Vries proximity on a boolean algebra (see [dV62, Bez10]). This modification enables us to obtain $\mathcal{F}$ as a functor from **Frm** to this new category $\mathbf{MT_P}$ of MT-algebras and proximity moprhisms. One of our main results establishes that the functors $\mathcal{O}$ and $\mathcal{F}$ yield an equivalence of these categories.

As we will see, the Funayama envelope of a frame always satisfies T_D-separation. Spatially, this is the separation axiom of Aull and Thron [AT62] stating that each point is locally closed. The MT-version of it states that locally closed elements join-generate the MT-algebra. By contrast, there is no notion of a T_D-frame, only of a T_D-spatial frame (see the next paragraph). This can be explained by the fact that an MT-algebra is T_D iff it is isomorphic to the Funayama envelope of a frame [BR23, Thm. 6.5], yielding a one-to-one corresponence between frames and T_D-algebras. We think of the Funayama envelope as the T_D-hull of a frame, thus providing a useful formalism to capture T_D-separation pointfreely, albeit in the language of MT-algebras rather than frames. As a consequence, we obtain that each MT-algebra has a T_D-reflection, which also happens to be a coreflection. This is explained by the fact that proximity morphisms are weak enough so that not all isomorphisms in this category are structure-preserving bijections. This, in particular, results in an equivalence of $\mathbf{MT_P}$, its full subcategory $\mathbf{TDMT_P}$ of T_D-algebras, and **Frm** (for the reader's convenience, all categories of interest are gathered together in tables at the end of the paper).

Our results have some spatial ramifications, including a further explanation and generalization of the T_D-duality of Banaschewski and Pultr [BP10]. One of their key notions is that of a D-morphism between frames. We generalize this to a D-morphism between MT-algebras. For T_D-algebras, this notion is stronger than that of a proximity morphism. We prove that the category **STDMT** of spatial T_D-algebras is a reflective subcategory of the category $\mathbf{MT_D}$ of MT-algebras and D-morphsims, and is equivalent to the category **TDTop** of T_D-spaces. This yields a pointfree description of the T_D-coreflection of [BP10, 3.7.2], which is not expressible in the language of frames. Another advantage of the MT-approach is that every MT-morphism between T_D-algebras is a D-morphism, which is in contrast with what happens in the setting of frames (where the category $\mathbf{TD\text{-}SFrm_D}$ of T_D-spatial frames with D-morphisms is not a full subcategory of **Frm**).

As we pointed out above, D-morphisms between spatial T_D-algebras correspond to continuous maps between their dual T_D-spaces. We also give a dual characterization of proximity morphisms between spatial T_D-algebras. This is done by introducing the notion of a sober map (that is, a continuous map from one topological space to the soberification of another), thus obtaining a more general duality for spatial T_D-algebras that subsumes the T_D-duality for frames. The latter is the restriction of a new duality between the categories $\mathbf{Top_S}$ of topological spaces and sober maps and $\mathbf{SMT_P}$ of spatial MT-algebras and proximity morphisms. This allows us to not only capture the D-morphisms between T_D-spatial frames as in the T_D-duality of [BP10], but also all frame morphisms.

The ability to describe the T_D-hull of a frame provides further evidence that this enhanced pointfree approach to topology is highly beneficial. For example, it affords sufficiently expressive power to capture lower separation axioms, which have been elusive in locale theory.

2. Preliminaries

In this section we briefly review some well-known facts about frames and MT-algebras that we will use in the rest of the paper.

Frames and co-frames. We recall that a complete lattice L is a *frame* if it satisfies the join-infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\},$$

and a *co-frame* if it satisfies the meet-infinite distributive law

$$a \vee \bigwedge S = \bigwedge \{a \vee s \mid s \in S\}$$

for all $a \in L$ and $S \subseteq L$. A *frame morphism* is a map between frames preserving arbitrary joins and finite meets, and a *co-frame morphism* is defined dually. As usual, we let **Frm** denote the category of frames and frame morphisms.

A typical example of a frame is the complete lattice ΩX of open sets of a topological space X, and of a co-frame the complete lattice ΓX of closed sets of X. The assignment $X \mapsto \Omega X$ is the object part of the functor $\Omega : \mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$, which sends each continuous map $f : X \to Y$ to the preimage map $f^{-1} : \Omega Y \to \Omega X$. The functor Ω has a right adjoint, namely the functor $pt : \mathbf{Frm}^{\mathrm{op}} \to \mathbf{Top}$, which maps each frame L to the space of its *points* (completely prime filters) with the topology given by $\sigma_L[L]$, where $\sigma_L(a) = \{P \in pt\,L \mid a \in P\}$ for each $a \in L$. The functor pt sends each frame morphism $f : L \to M$ to the continuous map $f^{-1} : pt\,M \to pt\,L$.

A frame L is *spatial* provided points of L separate non-comparable elements of L, and a space X is *sober* if each irreducible closed set is the closure of a unique point. The adjunction $\Omega \dashv pt$ restricts to an equivalence between the full subcategories **SFrm** of spatial frames and **Sob** of sober spaces (see [PP12, Ch. II] for details):

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, shift left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathbf{Frm}^{\mathrm{op}} \arrow[l, shift left, "pt"] \\
\mathbf{Sob} \arrow[u, hook, "\text{full}"] \arrow[r, leftrightarrow, "\cong"] & \mathbf{SFrm}^{\mathrm{op}} \arrow[u, hook, "\text{full}"']
\end{tikzcd}$$

More important for our purposes is the T_D-duality of Banaschewski and Pultr [BP10]. We recall [AT62] that a topological space X is a *T_D-space* if each point $x \in X$ is locally closed (closed in some open neighborhood of x). Let **TDTop** be the full subcategory of **Top** consisting of T_D-spaces.[1] If $f : X \to Y$ is a continuous map between T_D-spaces, then $f^{-1} : \Omega Y \to \Omega X$ has an extra property. To describe it, we recall that an element a of a poset P is covered by another element b if $a < b$ and from $a \leq x \leq b$ it follows that $a = x$ or $x = b$. In this case we write $a \lessdot b$. An element a is said to be *covered* if $a \lessdot b$ for some b.

Now, since each $x \in X$ is locally closed, there is $U \in \Omega X$ such that $x \in U$ and $U\backslash\{x\} \in \Omega X$. Therefore, $U\backslash\{x\}$ is covered by U in ΩX. Thus, the filter $F_x := \{U \in \Omega X \mid x \in U\}$ of ΩX is slicing in the following sense:

2.1. Definition. [BP10, Sec. 2.6] A completely prime filter F of a frame L is *slicing* if there exist $b \in F$ and $a \notin F$ with $a \lessdot b$.[2]

The *T_D-spectrum* of a frame L is defined to be the collection $pt_D\,L$ of slicing filters of L, topologized by setting the opens to be the elements of the form $\delta(a) = \{x \in pt_D\,L \mid a \in x\}$. As was shown in [BP10, Lem. 2.6.1], $pt_D\,L$ is a subspace of $pt\,L$. The dual frame morphisms of continuous maps between T_D-spaces have the following extra property:

2.2. Definition. [BP10, Sec. 3.1] A frame morphism $f : L \to M$ is a *D-morphism* provided $f^{-1}(F)$ is a slicing filter of L for each slicing filter F of M.

[1] In [BP10, 3.1] this category is denoted by $\mathbf{Top}_D$.

[2] As follows from [BP10, Lem. 2.6.1], in the definition of a slicing filter it is enough to assume that F is prime as being completely prime is then a consequence. This notion captures locally closed points in that a point (of a T_0-space) is locally closed iff the corresponding completely prime filter F_x is slicing [BP10, Prop. 2.7.1].

Let $\mathbf{Frm_D}$ denote the wide subcategory of $\mathbf{Frm}$ whose morphisms are the D-morphisms. A frame L is said to be *T_D-spatial* if it is isomorphic to ΩX for some T_D-space X. Let $\mathbf{TD\text{-}SFrm_D}$ be the full subcategory of $\mathbf{Frm_D}$ determined by these objects. We then have (see [BP10, Prop. 3.5.1]):

2.3. Theorem. [T_D-duality] *There is an adjunction* (Ω, pt_D) *between* $\mathbf{TDTop}$ *and* $\mathbf{Frm_D}^{\mathrm{op}}$, *which restricts to an equivalence between* $\mathbf{TDTop}$ *and* $\mathbf{TD\text{-}SFrm_D}^{\mathrm{op}}$.

$$\begin{tikzcd}
\mathbf{TDTop} \arrow[r, shift left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathbf{Frm_D}^{\mathrm{op}} \arrow[l, shift left, "pt_D"] \\
\mathbf{TDTop} \arrow[u, leftrightarrow, "="] \arrow[r, leftrightarrow, "\cong"] & \mathbf{TD\text{-}SFrm_D}^{\mathrm{op}} \arrow[u, hook, "\text{full}"']
\end{tikzcd}$$

The pt and pt_D functors are in general different, even for T_D-spatial frames. Hence, the T_D-duality is not a restriction of the $\Omega \dashv pt$ adjunction. But the functor Ω is the same in both cases, thus we do have the following commutative square:

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, "\Omega"] & \mathbf{Frm}^{\mathrm{op}} \\
\mathbf{TDTop} \arrow[u, hook, "\text{full}"] \arrow[r, leftrightarrow, "\cong"] & \mathbf{TD\text{-}SFrm_D}^{\mathrm{op}} \arrow[u, hook, dashed, "\text{nonfull}"']
\end{tikzcd}$$

We emphasize that not all frame morphisms between T_D-spatial frames are D-morphisms. The next example illustrates this.

2.4. Example. Let $X := \{*\}$ be a singleton space and Y the natural numbers equipped with the Alexandroff topology (where opens are precisely the upper sets). It is easy to see that both X and Y are T_D-spaces. Moreover, ΩX is isomorphic to the two-element boolean algebra $2 = \{0, 1\}$ and ΩY is isomorphic to $(\omega + 1)^{\mathrm{op}}$. Define $f : \Omega Y \to \Omega X$ by $f(a) = 1$ iff $a \neq 0$. It is straightforward to check that f is a frame morphism. Furthermore, $F := \{1\}$ is a slicing filter in ΩX, but $f^{-1}(F) = \Omega Y \setminus \{\varnothing\}$ is not a slicing filter in ΩY. Thus, f is not a D-morphism.

2.5. Remark. In the above example, no continuous map between the spaces X and Y can give rise to $f : \Omega Y \to \Omega X$ since otherwise $*$ would have to be mapped to a point whose open neighborhoods are all nonempty opens of Y, and such a point does not exist in Y. In fact, all frame morphisms between T_D-spatial frames that come from continuous maps between T_D-spaces are D-morphisms (as will be evident from Theorem 2.13(2) and Corollary 2.14 below).

Interior algebras and MT-algebras. The following definitions go back to McKinsey and Tarski [MT44] (see also [RS63, Esa19]).

2.6. Definition. An *interior operator* on a boolean algebra A is a unary function $\Box : A \to A$ satisfying Kuratowski's axioms for all $a, b \in A$:

- $\Box 1 = 1$.
- $\Box(a \wedge b) = \Box a \wedge \Box b$.
- $\Box a \leq a$.
- $\Box a \leq \Box\Box a$.

An *interior algebra* is a pair $(A, \Box)$ with A a boolean algebra and $\Box$ an interior operator on A.

Recall (see, e.g., [RS63, Sec. III.3]) that a *morphism of interior algebras* is a boolean homomorphism $f : A \to B$ such that $f(\Box a) = \Box f(a)$ for each $a \in A$. We will be interested in the following weaker condition: $f(\Box a) \leq \Box f(a)$ for each $a \in A$. Such morphisms have been studied in the literature under different names: continuous morphisms [Ghi10], stable morphisms [BBI16], or semi-homomorphisms [BMM08]. For the purposes of this paper, we will follow [Ghi10] in calling them *continuous morphisms*.

2.7. Definition.

(1) Let **Int** be the category of interior algebras and interior algebra morphisms.

(2) Let $\mathbf{Int_C}$ be the category of interior algebras and continuous morphisms.

Clearly **Int** is a wide subcategory of $\mathbf{Int_C}$, and in both categories, compostion is function composition and identity morphisms are identity functions.

2.8. Definition. Let A be an interior algebra.

(1) An element $a \in A$ is *open* if $a = \Box a$.

(2) An element $a \in A$ is *closed* if $a = \Diamond a$ where $\Diamond a = \neg\Box\neg a$.

(3) An element $a \in A$ is *locally closed* if $a = \Box b \wedge \Diamond c$ for some $b, c \in A$.[3]

Let $\mathcal{O}A$, $\mathcal{C}A$, and $\mathcal{LC}A$ be the collections of open, closed, and locally closed elements of A, respectively.

Observe that $\mathcal{O}A$ is a bounded sublattice of A and $\Box : A \to \mathcal{O}A$ is right adjoint to the inclusion $\mathcal{O}A \hookrightarrow A$, yielding that $\mathcal{O}A$ is a Heyting algebra (see, e.g., [Esa19, Sec. 2.5]). Similarly, $\mathcal{C}A$ is a bounded sublattice of A and $\Diamond : A \to \mathcal{C}A$ is left adjoint to the inclusion $\mathcal{C}A \hookrightarrow A$, yielding that $\mathcal{C}A$ is a co-Heyting algebra. We point out that the implication on $\mathcal{O}A$ is given by $u \to v = \Box(\neg u \vee v)$ for all $u, v \in \mathcal{O}A$, and the co-implication on $\mathcal{C}A$ by $c \leftarrow d = \Diamond(d \wedge \neg c)$ for all $c, d \in \mathcal{C}A$. Moreover, $\mathcal{LC}A$ is closed under finite meets, and closing $\mathcal{LC}A$ under finite joins gives the boolean subalgebra of A generated by $\mathcal{O}A$ (or $\mathcal{C}A$).

[3]Equivalently, a is locally closed provided $a = u \wedge \Diamond a$ for some $u \in \mathcal{O}A$.

2.9. DEFINITION.

(1) An interior algebra $(A, \Box)$ is a *McKinsey-Tarski algebra* or an *MT-algebra* if A is a complete boolean algebra.

(2) An *MT-morphism* between MT-algebras M and N is a complete boolean homomorphism $h : M \to N$ such that $h(\Box a) \le \Box h(a)$ for each $a \in M$.

(3) Let $\mathbf{MT}$ be the category of MT-algebras and MT-morphisms.

Since each MT-algebra M is complete and $\Box : M \to \mathcal{O}M$ is right adjoint to the inclusion, $\mathcal{O}M$ is a subframe of M. In fact, we can equivalently think of MT-algebras as pairs (A, L) where A is a complete boolean algebra and L is a subframe of A. Then the interior operator on A is given by $\Box a = \bigvee\{b \in L \mid b \le a\}$. Moreover, if $f : M \to N$ is an MT-morphism, then its restriction $f|_{\mathcal{O}M} : \mathcal{O}M \to \mathcal{O}N$ is a frame morphism that sends identity morphisms to identity morphisms and respects composition. We thus obtain:

2.10. THEOREM. [BR23, Thm. 3.10] *The assignment $M \mapsto \mathcal{O}M$ and $f \mapsto f|_{\mathcal{O}M}$ yields a functor $\mathcal{O} : \mathbf{MT} \to \mathbf{Frm}$.*

A typical example of an MT-algebra is the powerset algebra $(\mathcal{P}X, \Box)$ of a topological space X, where $\Box$ is the interior operator on X. The assignment $X \mapsto \mathcal{P}X$ extends to a functor $\mathcal{P} : \mathbf{Top} \to \mathbf{MT}^{\mathrm{op}}$, where a continuous map $f : X \to Y$ is sent to the MT-morphism $f^{-1} : \mathcal{P}Y \to \mathcal{P}X$. To define a functor in the opposite direction, for an MT-algebra M, let $\mathit{at}\, M$ be the set of atoms of M. For $a \in M$, define

$$\eta_M(a) = \{x \in \mathit{at}\, M \mid x \le a\}.$$

Then $\{\eta_M(a) \mid a \in M\}$ is a topology on $\mathit{at}\, M$, so $\mathcal{P}\,\mathit{at}\, M$ is an MT-algebra and $\eta_M : M \to \mathcal{P}\,\mathit{at}\, M$ is an onto MT-morphism. Moreover, if $f : M \to N$ is an MT-moprhism, then it has a left adjoint (since it is a complete boolean homomorphism). The restriction of the left adjoint is then a well-defined continuous map $f^* : \mathit{at}\, N \to \mathit{at}\, M$. This defines a functor $\mathit{at} : \mathbf{MT}^{\mathrm{op}} \to \mathbf{Top}$, which is right adjoint to $\mathcal{P}$.

We call M *spatial* provided $\eta_M : M \to \mathcal{P}\,\mathit{at}\, M$ is one-to-one (in which case it is an isomorphism of MT-algebras). Let $\mathbf{SMT}$ be the full subcategory of $\mathbf{MT}$ consisting of spatial MT-algebras. For each $X \in \mathbf{Top}$, let $\varepsilon_X : X \to \mathit{at}\,\mathcal{P}X$ be given by $\varepsilon_X(x) = \{x\}$. Then ε_X is a homeomorphism and we have (see [BR23, Thm. 3.22]):

2.11. THEOREM. [MT-duality] *$(\mathcal{P}, \mathit{at})$ is an adjunction between $\mathbf{Top}$ and $\mathbf{MT}^{\mathrm{op}}$ whose unit is given by $\varepsilon : 1_{\mathbf{Top}} \to \mathit{at} \circ \mathcal{P}$ and counit by $\eta : \mathcal{P} \circ \mathit{at} \to 1_{\mathbf{MT}}$. This adjunction restricts to an equivalence between $\mathbf{Top}$ and $\mathbf{SMT}^{\mathrm{op}}$.*

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, shift left, "\mathcal{P}"] \arrow[r, phantom, "\perp"] \arrow[d, leftrightarrow, "="'] & \mathbf{MT}^{\mathrm{op}} \arrow[l, shift left, "\mathit{at}"] \\
\mathbf{Top} \arrow[r, leftrightarrow, "\cong"] & \mathbf{SMT}^{\mathrm{op}} \arrow[u, hook, "\text{full}"']
\end{tikzcd}$$

We conclude this preliminary section by recalling the MT-algebra analogues of T_D and T_0-spaces. An element of an MT-algebra M is *saturated* if it is a meet from $\mathcal{O}M$. Let $\mathcal{S}M$ be the collection of saturated elements of M. We call $a \in M$ *weakly locally closed* if $a = s \wedge c$ where $s \in \mathcal{S}M$ and $c \in \mathcal{C}M$. Let $\mathcal{WLC}M$ be the collection of weakly locally closed elements of M.

2.12. Definition. An MT-algebra M is said to be a *T_D-algebra* if $\mathcal{LC}M$ join-generates M and a *T_0-algebra* if $\mathcal{WLC}M$ join-generates M.

Let **TDMT** be the full subcategory of **MT** consisting of T_D-algebras, **STDMT** the full subcategory of **TDMT** consisting of spatial T_D-algebras, and define **T0MT** and **ST0MT** similarly. Also, let **TDTop** be the full subcategory of **Top** consisting of T_D-spaces, and define **T0Top** similarly. Then MT-duality restricts to yield the following (see [BR23, Thms. 5.7, 6.4]):

2.13. Theorem.

(1) *The adjunction* $(\mathcal{P}, at)$ *restricts to* **T0Top** *and* $\mathbf{T0MT}^{\mathrm{op}}$, *yielding an equivalence between* **T0Top** *and* $\mathbf{ST0MT}^{\mathrm{op}}$.

(2) *The adjunction* $(\mathcal{P}, at)$ *further restricts to* **TDTop** *and* $\mathbf{TDMT}^{\mathrm{op}}$, *yielding an equivalence between* **TDTop** *and* $\mathbf{STDMT}^{\mathrm{op}}$.

Putting Theorems 2.3 and 2.13(2) together, we conclude:

2.14. Corollary. $\mathbf{TD\text{-}SFrm_D}$ *is equivalent to* **STDMT**.

As we pointed out after Theorem 2.3, the inclusion $\mathbf{TD\text{-}SFrm_D} \hookrightarrow \mathbf{Frm}$ is not full. By contrast, the inclusion $\mathbf{STDMT} \hookrightarrow \mathbf{MT}$ is full. Moreover, while the T_D-duality for frames is not a restriction of the $\Omega \dashv pt$ adjuntion (since pt_D is not in general the same as pt), the T_D-duality for MT-algebras is obtained by restricting the adjunction $\mathcal{P} \dashv at$. This is summarized in the two diagrams below:

$$
\begin{tikzcd}
\mathbf{Top} \arrow[r, "\Omega"] & \mathbf{Frm}^{\mathrm{op}} \\
\mathbf{TDTop} \arrow[u, hook, "\text{full}"] \arrow[r, leftrightarrow, "\cong"] & \mathbf{TD\text{-}SFrm_D}^{\mathrm{op}} \arrow[u, hook, dashed, "\text{nonfull}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Top} \arrow[r, shift left, "\mathcal{P}"] \arrow[r, phantom, "\perp"] & \mathbf{MT}^{\mathrm{op}} \arrow[l, shift left, "at"] \\
\mathbf{TDTop} \arrow[u, hook, "\text{full}"] \arrow[r, leftrightarrow, "\cong"] & \mathbf{STDMT}^{\mathrm{op}} \arrow[u, hook, "\text{full}"']
\end{tikzcd}
$$

3. Proximity morphisms between MT-algebras

In this section, we show that each MT-algebra can be equipped with a proximity relation, which is a weakening of a de Vries proximity on a boolean algebra [dV62, Bez10]. This gives rise to a new category $\mathbf{MT_P}$ of MT-algebras and proximity morphisms between them. In Section 4, it will be shown that this category is equivalent to **Frm**.

Given boolean algebras A, B, with B a subalgebra of A, we define a binary relation $\prec_B$ on A by

$$a \prec_B c \Longleftrightarrow \exists b \in B : a \leq b \leq c.$$

It is straightforward to verify that this relation satisfies the following conditions:

(S1) $1 \prec_B 1$;

(S2) $a \prec_B c$ implies $a \leq c$;

(S3) $a \leq a' \prec_B c' \leq c$ implies $a \prec_B c$;

(S4) $a \prec_B c, d$ implies $a \prec_B c \wedge d$;

(S5) $a \prec_B c$ implies $\neg c \prec_B \neg a$;

(S6) $a \prec_B c$ implies that there is $b \in B$ with $a \prec_B b \prec_B c$.

3.1. Remark. The above are standard proximity axioms on a boolean algebra, except (S6) is a strengthening of the usual in-betweenness axiom. However, in general, $\prec_B$ is not a de Vries proximity on A since it is not necessarily the case that $a = \bigvee\{c \in A \mid c \prec_B a\}$. In fact, $\prec_B$ is a de Vries proximity on A if and only if B join-generates A.

In our considerations, A will be an MT-algebra and B will be the boolean subalgebra of A generated by $\mathcal{O}A$. We recall that in the powerset algebra of a topological space, the elements of the boolean subalgebra generated by the frame of opens are exactly the finite unions of locally closed subsets, and are called constructible sets (see, e.g., [Har77, p. 94]). Analogously:

3.2. Definition. An element a of an MT-algebra M is *constructible* provided a is a finite join from $\mathcal{LC}M$. Let $\mathcal{C}ons\, M$ be the set of constructible elements of M.

Note that $\mathcal{C}ons\, M$ is the boolean subalgebra of M generated by $\mathcal{O}M$, and thus one can consider the associated binary relation, $\prec_{\mathcal{C}ons\, M}$. To simplify notation, we omit the subscript.

3.3. Definition. Let M be an MT-algebra. An element a is *constructibly below* b, or "*cons-below*", if $a \prec b$ for the binary relation associated with $\mathcal{C}ons\, M$.

Interestingly, the cons-below relation on an MT-algebra M is a de Vries proximity precisely when M is a T_D-algebra:

3.4. Lemma. *For any MT-algebra M, the cons-below relation is a de Vries proximity on M iff M is a T_D-algebra.*

Proof. By Remark 3.1, the cons-below relation is a de Vries proximity on M iff $\mathcal{C}ons\, M$ join-generates M. Since each element of $\mathcal{C}ons\, M$ is a finite join from $\mathcal{LC}M$, the latter condition is equivalent to M being a T_D-algebra. ■

Next, by analogy with de Vries algebras, we define proximity morphisms between MT-algebras.

3.5. Definition. For $M, N \in \mathbf{MT}$, a map $f : M \to N$ is a *proximity morphism* provided the following conditions are satisfied:

(P1) $f|_{\mathcal{O}(M)} : \mathcal{O}M \to \mathcal{O}N$ is a frame morphism.

(P2) $f(a \wedge b) = f(a) \wedge f(b)$ for each $a, b \in M$.

(P3) $f(\bigvee S) = \bigvee\{f(s) \mid s \in S\}$ for each finite $S \subseteq \mathcal{LC}M$.

(P4) $f(a) = \bigvee\{f(x) \mid x \in \mathcal{LC}M,\, x \leq a\}$ for each $a \in M$.

3.6. Remark. Since each element of $\mathcal{C}ons\, M$ is a finite join from $\mathcal{LC}M$, (P4) is equivalent to

$$f(a) = \bigvee\{f(b) \mid b \in \mathcal{C}ons\, M,\, b \leq a\} \quad \text{for each } a \in M.$$

3.7. Lemma. *Let $f : M \to N$ be a proximity morphism between MT-algebras.*

(1) *$f(\neg x) = \neg f(x)$ for each $x \in \mathcal{O}M \cup \mathcal{C}M$.*

(2) *$f|_{\mathcal{C}M} : \mathcal{C}M \to \mathcal{C}N$ is a co-frame morphism.*

(3) *If $x \in \mathcal{LC}M$ then $f(x) \in \mathcal{LC}N$.*

(4) *$f|_{\mathcal{C}ons\, M} : \mathcal{C}ons\, M \to \mathcal{C}ons\, N$ is a boolean homomorphism.*

Proof. (1) Let $x \in \mathcal{O}M \cup \mathcal{C}M$. Then $\neg x \in \mathcal{C}M \cup \mathcal{O}M$. Thus, since $\mathcal{O}M \cup \mathcal{C}M \subseteq \mathcal{LC}M$, by (P3), $f(x) \vee f(\neg x) = f(x \vee \neg x) = f(1) = 1$. Moreover, by (P2),

$$f(x) \wedge f(\neg x) = f(x \wedge \neg x) = f(0) = 0.$$

Therefore, $f(\neg x) = \neg f(x)$.

(2) Since $\mathcal{C}M \subseteq \mathcal{LC}M$, the restriction $f|_{\mathcal{C}M}$ preserves finite joins by(P3). We show that it preserves arbitrary meets. Let $S \subseteq \mathcal{C}M$. Then $\neg s \in \mathcal{O}M$ for each $s \in S$, so $\bigvee\{\neg s \mid s \in S\} \in \mathcal{O}M$. Therefore, by (P1) and (1),

$$\begin{aligned} f\left(\bigwedge S\right) &= f\left(\neg \bigvee\{\neg s \mid s \in S\}\right) = \neg f\left(\bigvee\{\neg s \mid s \in S\}\right) \\ &= \neg \bigvee\{f(\neg s) \mid s \in S\} = \neg \bigvee\{\neg f(s) \mid s \in S\} \\ &= \bigwedge f[S]. \end{aligned}$$

Thus, $f|_{\mathcal{C}M} : \mathcal{C}M \to \mathcal{C}N$ is a co-frame morphism.

(3) This follows from (P1), (P2), and (2).

(4) Since each element of $\mathcal{C}ons\, M$ is a finite join from $\mathcal{LC}M$, it follows from (P3) and (3) that $f|_{\mathcal{C}ons\, M}$ is well defined. Moreover, by (P1)–(P3), it is a bounded lattice homomorphism, and thus a boolean homomorphism. ∎

3.8. LEMMA. *Let $f : M \to N$ be a map between MT-algebras satisfying (P1), (P2), and (P4). The following are equivalent:*

(1) *f satisfies (P3); that is, f is a proximity morphism.*

(2) *$a_1 \prec b_1$ and $a_2 \prec b_2$ imply $f(a_1 \vee a_2) \prec f(b_1) \vee f(b_2)$ for each $a_i, b_i \in M$.*

(3) *$a \prec b$ implies $\neg f(\neg a) \prec f(b)$ for each $a, b \in M$.*

PROOF. It is sufficient to prove that (1)⇔(2) since (2)⇔(3) follows from [Bez12, Lem. 2.2] and [BH14, Prop. 7.4].

(1)⇒(2): Suppose $a_1 \prec b_1$ and $a_2 \prec b_2$. Then there exist finite $S_1, S_2 \subseteq \mathcal{LC}M$ such that $a_1 \le \bigvee S_1 \le b_1$ and $a_2 \le \bigvee S_2 \le b_2$. Thus, $a_1 \vee a_2 \le \bigvee S_1 \vee \bigvee S_2 \le b_1 \vee b_2$. By (P2), f is order preserving. Therefore, by (1),

$$f(a_1 \vee a_2) \le f\left(\bigvee S_1 \vee \bigvee S_2\right) = \bigvee f[S_1] \vee \bigvee f[S_2] \le f(b_1) \vee f(b_2).$$

Consequently, $f(a_1 \vee a_2) \prec f(b_1) \vee f(b_2)$ since $\bigvee f[S_1] \vee \bigvee f[S_2] \in \mathcal{C}ons\, N$ by Lemma 3.7(3).

(2)⇒(1): Let $S \subseteq \mathcal{LC}M$ be finite. Since f is order preserving, $\bigvee f[S] \le f(\bigvee S)$. For the reverse inequality, since $s \prec s$ for each $s \in S$, (2) implies $f(\bigvee S) \prec \bigvee f[S]$. Thus, $f(\bigvee S) \le \bigvee f[S]$, and hence f satisfies (P3). ■

We next show that the MT-algebras and proximity morphisms between them form a category, however neither the composition is usual function composition nor the identity morphisms are identity functions. The composition of proximity morphisms between MT-algebras is defined as for de Vries algebras:

3.9. DEFINITION. For proximity morphisms $f : M_1 \to M_2$ and $g : M_2 \to M_3$, define

$$(g \star f)(a) = \bigvee \{g(f(x)) \mid x \in \mathcal{LC}M_1,\ x \le a\}.$$

It is immediate that if $x \in \mathcal{LC}M_1$ then $(g \star f)(x) = (g \circ f)(x)$.

3.10. LEMMA. *Let $f : M_1 \to M_2$, $g : M_2 \to M_3$, and $h : M_3 \to M_4$ be proximity morphisms. For each $a \in M_1$, we have*

$$((h \star g) \star f)(a) = \bigvee \{h(g(f(x))) \mid x \in \mathcal{LC}M_1,\ x \le a\} = (h \star (g \star f))(a).$$

PROOF. Let $a \in M_1$. Then

$$\begin{aligned}
((h \star g) \star f)(a) &= \bigvee \{(h \star g)(f(x)) \mid x \in \mathcal{LC}M_1,\ x \le a\} \\
&= \bigvee \{(h \circ g)(f(x)) \mid x \in \mathcal{LC}M_1,\ x \le a\} && \text{since } f(x) \in \mathcal{LC}M_2 \\
&= \bigvee \{h((g \circ f)(x)) \mid x \in \mathcal{LC}M_1,\ x \le a\} \\
&= \bigvee \{h((g \star f)(x)) \mid x \in \mathcal{LC}M_1,\ x \le a\} && \text{since } x \in \mathcal{LC}M_1 \\
&= (h \star (g \star f))(a).
\end{aligned}$$

■

3.11. Definition. For an MT-algebra M, define $1_M : M \to M$ by

$$1_M(a) = \bigvee\{x \in \mathcal{LC}M \mid x \leq a\} \text{ for each } a \in M.$$

3.12. Lemma.

(1) 1_M *is a proximity morphism for each MT-algebra* M.

(2) *For each proximity morphism* $f : M \to N$ *between MT-algebras we have*

$$1_N \star f = f = f \star 1_M.$$

Proof. (1) By the definition of 1_M, $1_M(x) = x$ for each $x \in \mathcal{LC}M$. In particular, 1_M is identity on $\mathcal{O}M$, and hence (P1) holds. In view of Section 3.6, an argument similar to [Bez10, Lem. 4.8] yields that (P2) and Lemma 3.8(3) hold. It is also immediate from the definition that (P4) holds. Thus, 1_M is a proximity morphism by Lemma 3.8.

(2) Let $a \in M$. Then

$$\begin{aligned}
(1_N \star f)(a) &= \bigvee\{1_N(f(x)) \mid x \in \mathcal{LC}M,\, x \leq a\} \\
&= \bigvee\{f(x) \mid x \in \mathcal{LC}M,\, x \leq a\} && \text{since } f(x) \in \mathcal{LC}N \\
&= f(a) \\
&= \bigvee\{f(1_M(x)) \mid x \in \mathcal{LC}M,\, x \leq a\} && \text{since } x \in \mathcal{LC}M \\
&= (f \star 1_M)(a).
\end{aligned}$$

∎

As an immediate consequence of Lemmas 3.10 and 3.12 we obtain:

3.13. Theorem. *The MT-algebras and proximity morphisms between them form a category* $\mathbf{MT_P}$ *where composition is given by* $\star$ *and identity morphisms are* 1_M.

Proof. In view of Lemmas 3.10 and 3.12, we only need to check that if $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are proximity morphisms, then so is $g \star f : M_1 \to M_3$. For this it is sufficient to verify (P1)–(P4).

(P1) For $u \in \mathcal{O}M_1$, we have $(g \star f)(u) = (g \circ f)(u)$. Thus, $(g \star f)|_{\mathcal{O}M_1}$ is a frame morphism.

(P2) For $a, b \in M_1$, since $\mathcal{LC}M_1$ is closed under finite meets, we have

$$\begin{aligned}
&(g \star f)(a) \wedge (g \star f)(b) \\
&\quad= \bigvee\{g(f(x)) \mid x \in \mathcal{LC}M_1,\, x \leq a\} \wedge \bigvee\{g(f(y)) \mid y \in \mathcal{LC}M_1,\, y \leq b\} \\
&\quad= \bigvee\{g(f(x)) \wedge g(f(y)) \mid x, y \in \mathcal{LC}M_1,\, x \leq a,\, y \leq b\} \\
&\quad= \bigvee\{g(f(x \wedge y)) \mid x, y \in \mathcal{LC}M_1,\, x \leq a,\, y \leq b\} \\
&\quad= \bigvee\{g(f(z)) \mid z \in \mathcal{LC}M_1,\, z \leq a \wedge b\} \\
&\quad= (g \star f)(a \wedge b).
\end{aligned}$$

(P3) Let $S \subseteq \mathcal{LC}M$. By (P2), $g \star f$ is order preserving. Thus,

$$\bigvee\{(g\star f)(s) \mid s\in S\} \le (g\star f)\left(\bigvee S\right).$$

For the reverse inequality, since $(g\star f)(a) \le (g\circ f)(a)$ for each $a \in M_1$ and f, g are proximity morphisms, we obtain

$$\begin{aligned}(g\star f)\left(\bigvee S\right) \le (g\circ f)\left(\bigvee S\right) = g\left(\bigvee f[S]\right) &= \bigvee\{g(f(s)) \mid s\in S\}\\ &= \bigvee\{(g\star f)(s) \mid s \in S\}.\end{aligned}$$

(P4) For $a \in M_1$, we have

$$\begin{aligned}(g\star f)(a) &= \bigvee\{g(f(x)) \mid x\in \mathcal{LC}M_1,\ x\le a\}\\ &= \bigvee\{(g\star f)(x) \mid x\in\mathcal{LC}M_1,\ x\le a\} \qquad \text{since } x\in\mathcal{LC}M_1.\end{aligned}$$

■

Not surprisingly, isomorphims in $\mathbf{MT_P}$ are not structure-preserving bijections:

3.14. Example. Let $\Box$ be the identity on the two-element boolean algebra 2. Then $(2, \Box)$ is an MT-algebra. Also, let $M = \{0, a, b, 1\}$ be the four-element boolean algebra. Then $(M, \Box)$ is an MT-algebra, where $\Box : M \to M$ is defined by

$$\Box a = \begin{cases} 1 & \text{if } a = 1\\ 0 & \text{otherwise.}\end{cases}$$

Observe that $1_M = \Box$ and 1_2 is the identity on 2. Since $2 \subseteq M$, we may view 1_M as a proximity morphism $f : M \to 2$ and 1_2 as a proximity morphism $g : 2 \to M$. We then have $g \star f = 1_M$ and $f \star g = 1_2$. Thus, g is the inverse of f in $\mathbf{MT_P}$, and hence f is an isomorphism in $\mathbf{MT_P}$. However, f is clearly not a structure-preserving bijection.

In Proposition 4.22, we will characterize isomorphisms in $\mathbf{MT_P}$, from which we derive that isomorphisms between T_D-algebras are indeed structure-preserving bijections (observe that M in the above example is not a T_D-algebra). This requires more machinery, which we turn to next.

4. Funayama envelope

It is a consequence of a well-known result of Funayama [Fun59] that each frame embeds into a complete boolean algebra. In this section, we use this to define the Funayama envelope of a frame L, denoted by $\mathcal{F}L$, and show that it may be identified with a T_D-algebra whose opens are isomorphic to L. For this reason, we think of $\mathcal{F}L$ as the T_D-hull of L. We prove that this assignment extends to categorical equivalences between $\mathbf{Frm}$, $\mathbf{MT_P}$, and the full subcategory $\mathbf{TDMT_P}$ of $\mathbf{MT_P}$ consisting of T_D-algebras. The equivalence of the last two categories is explained by the fact that isomorphisms in $\mathbf{MT_P}$ are not structure-preserving bijections. We show that this unusual phenomenon disappears in $\mathbf{TDMT_P}$.

The Funayama envelope of a frame can be constructed by taking the MacNeille completion of its boolean envelope (see below).

Boolean envelope of a frame. Recalling the categories **Int** and $\mathbf{Int_C}$ (Definition 2.7), we have:

4.1. Definition.

(1) An interior algebra A is *essential* if the least boolean subalgebra of A generated by $\mathcal{O}A$ coincides with A.[4]

(2) Let $\mathbf{Ess_C}$ be the full subcategory of $\mathbf{Int_C}$ consisting of essential algebras, and define **Ess** similarly (as a full subcategory of **Int**).

Clearly **Ess** is a wide subcategory of $\mathbf{Ess_C}$. These two categories are closely related to the following categories:

4.2. Definition.

(1) Let **Heyt** be the category of Heyting algebras and Heyting homomorphisms, and let **Bool** be the full subcategory of **Heyt** consisting of boolean algebras.

(2) Let **DLat** be the category of bounded distributive lattices and bounded lattice homomorphisms, and let **HLat** be the full subcategory of **DLat** consisting of Heyting algebras.

Clearly, **Heyt** is a wide subcategory of **HLat**. To connect these two categories with **Ess** and $\mathbf{Ess_C}$, we recall the definition of the boolean envelope of a distributive lattice (see, e.g., [BD74, Sec. V.4]), which is the reflector $\mathcal{B} : \mathbf{DLat} \to \mathbf{Bool}$.

The *boolean envelope* or *free boolean extension* of a bounded distributive lattice L is a pair $(\mathcal{B}L, e)$, where $\mathcal{B}L$ is a boolean algebra and $e : L \to \mathcal{B}L$ is a bounded lattice embedding satisfying the following universal mapping property: for any boolean algebra A and a bounded lattice homomorphism $h : L \to A$, there is a unique boolean homomorphism $\mathcal{B}h : \mathcal{B}L \to A$ such that $\mathcal{B}h \circ e = h$; i.e., the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "e"] \arrow[d, "h"'] & \mathcal{B}L \arrow[dl, dashed, "\mathcal{B}h"] \\ A & \end{tikzcd}$$

[4]Esakia introduced essential algebras under the name of skeletal algebras (see [Esa19, Def. 2.5.6]). Since the name "skeletal" is overused in topology, we prefer the name essential. This is justified by the fact that we can think of A as an essential extension of $\mathcal{O}A$ in that for each congruence Θ of the interior algebra A, if Θ is not identity then neither is $\Theta \cap (\mathcal{O}A \times \mathcal{O}A)$.

We identify L with it image $e[L]$ and treat L as a bounded sublattice of $\mathcal{B}L$ which generates $\mathcal{B}L$. If L is a Heyting algebra, then the embedding $L \hookrightarrow \mathcal{B}L$ has a right adjoint $\Box : \mathcal{B}L \to L$ and $(\mathcal{B}L, \Box)$ is an essential interior algebra (see, e.g., [Esa19, Sec. 2.5]). Moreover, each bounded lattice homomorphism $h : L_1 \to L_2$ lifts uniquely to a continuous morphism $\mathcal{B}h : \mathcal{B}L_1 \to \mathcal{B}L_2$. Furthermore, h is a Heyting homomorphism iff $\mathcal{B}h$ is a morphism of interior algebras (see, e.g., [BMM08, Sec. 2.2]). We thus obtain:

4.3. Theorem.

(1) **Ess** *is a coreflective subcategory of* **Int** *that is equivalent to* **Heyt**.

(2) $\mathbf{Ess_C}$ *is a coreflective subcategory of* $\mathbf{Int_C}$ *that is equivalent to* **HLat**.

Proof. For (1) see [Esa19, Thm. 2.5.11], and (2) is proved similarly (see, e.g., [BMM08, Thm. 2.14]). ∎

We next restrict the equivalence in Theorem 4.3(2) to constructible algebras.

4.4. Definition.

(1) We call A *constructible* if it is essential and $\mathcal{O}A$ is a frame.

(2) A continuous morphism $f : A \to B$ between constructible algebras is a *constructible morphism* if $f|_{\mathcal{O}A} : \mathcal{O}A \to \mathcal{O}B$ is a frame morphism.

(3) Let **Cons** be the category of constructible algebras and constructible morphisms.

Note that **Cons** is a non-full subcatgeory of $\mathbf{Ess_C}$ since not every bounded lattice homomorphism between frames is a frame morphism. However, isomorphisms in **Cons** are isomorphisms in $\mathbf{Ess_C}$. We have the following consequence of Theorem 4.3(2):

4.5. Theorem. **Frm** *is equivalent to* **Cons**.

Proof. For a Heyting algebra L, we have that L is a frame iff $(\mathcal{B}L, \Box)$ is a constructible algebra. Indeed, if L is a frame, then $(\mathcal{B}L, \Box)$ is an essential interior algebra for which $\mathcal{O}\mathcal{B}L$ is a frame since $\mathcal{O}\mathcal{B}L = L$ (recall that we identify L with $e[L]$). Thus, $(\mathcal{B}L, \Box)$ is a constructible algebra. For the same reason, if $(\mathcal{B}L, \Box)$ is constructible then L must be a frame. Moreover, for a bounded lattice homomorphism $h : L_1 \to L_2$ between frames, since $\mathcal{O}\mathcal{B}L_i = L_i$ $(i = 1, 2)$ and $\mathcal{B}h|_{L_1} = h$, we have that h is a frame morphism iff $\mathcal{B}h$ is a constructible morphism. Thus, since isomorphisms in **Cons** are isomorphisms in $\mathbf{Ess_C}$, the equivalence of Theorem 4.3(2) restricts to an equivalence between **Frm** and **Cons**. ∎

Funayama envelope of a frame.

4.6. PROPOSITION. $\mathcal{O} : \mathbf{MT_P} \to \mathbf{Frm}$ *is a functor.*

PROOF. By Theorem 2.10, $\mathcal{O}M$ is a frame; by (P1), if $f : M \to N$ is a proximity morphism, $\mathcal{O}f : \mathcal{O}M \to \mathcal{O}N$ is a frame morphism. Since $\mathcal{O}M \subseteq \mathcal{L}\mathcal{C}M$, the restriction of 1_M is the identity on $\mathcal{O}M$, so $\mathcal{O}(1_M) = 1_{\mathcal{O}M}$. For the same reason, if $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are proximity morphisms then

$$\mathcal{O}(g \star f) = (g \star f)|_{\mathcal{O}M_1} = (g \circ f)|_{\mathcal{O}M_1} = g|_{\mathcal{O}M_2} \circ f|_{\mathcal{O}M_1} = \mathcal{O}g \circ \mathcal{O}f.$$

Thus, $\mathcal{O} : \mathbf{MT_P} \to \mathbf{Frm}$ is a functor. ∎

We show that $\mathcal{O}$ is an equivalence by describing its quasi-inverse using Funayama's result [Fun59] that there is a frame embedding of each frame L into a complete boolean algebra B, where B can be constructed as the MacNeille completion[5] of the boolean envelope of L [Grä11].[6]

For a frame L, let $\overline{\mathcal{B}L}$ be the MacNeille completion of its boolean envelope. We lift the interior operator $\Box : \mathcal{B}L \to \mathcal{B}L$ to $\overline{\Box} : \overline{\mathcal{B}L} \to \overline{\mathcal{B}L}$ by

$$\overline{\Box}a = \bigvee\{\Box b \mid b \in \mathcal{B}L \text{ and } b \leq a\}.$$

Then $\left(\overline{\mathcal{B}L}, \overline{\Box}\right)$ is an MT-algebra such that $\mathcal{O}\mathcal{F}L \cong L$ (see e.g., [BR23, p. 8]).

4.7. DEFINITION. For a frame L, we call the MT-algebra $\left(\overline{\mathcal{B}L}, \overline{\Box}\right)$ the *Funayama envelope* of L and denote it by $\mathcal{F}L$.

Note that the assignment $L \mapsto \mathcal{F}L$ cannot be extended to a functor $\mathbf{Frm} \to \mathbf{MT}$ with $f = \mathcal{O}\mathcal{F}f$ for every frame morphism f. Indeed, there are frame morphisms that do not lift to complete boolean homomorphisms between their Funayama envelopes. For the reader's convenience, we illustrate this with an example from [BR23, Example 4.4], detailed below.

4.8. EXAMPLE. Equip $\mathbb{N}$ with the cofinite topology and let $\Omega(\mathbb{N})$ be its frame of opens. The boolean envelope of $\Omega(\mathbb{N})$ is isomorphic to the boolean algebra of finite and cofinite subsets of $\mathbb{N}$, and its MacNeille completion is isomorphic to $\mathscr{P}(\mathbb{N})$. We thus identify $\mathcal{F}\Omega(\mathbb{N})$ with $\mathscr{P}(\mathbb{N})$. Consider the frame morphism $f : \Omega(\mathbb{N}) \to 2$ given by $f(U) = 0$ iff $U = \varnothing$. Then $\mathcal{F}f(F) = 0$ for every finite subset $F \subseteq \mathbb{N}$. Therefore, if $\mathcal{F}f$ is a complete boolean homomorphism, then $\mathcal{F}f(\mathbb{N}) = \varnothing$, a contradiction.

By contrast, we will show that the Funayama envelope does extend to a functor $\mathcal{F} : \mathbf{Frm} \to \mathbf{MT_P}$ that is a quasi-inverse of $\mathcal{O}$. This is, in fact, our main motivation for introducing proximity morphisms.

[5]Recall that the *MacNeille completion* of a partially ordered set P is a complete lattice $\overline{P}$ together with an embedding $e : P \to \overline{P}$ such that $e[P]$ is both join-dense and meet-dense in $\overline{P}$ (see, e.g., [BD74, p. 237].

[6]B can also be realized as the booleanization of the frame of nuclei of L [Joh82]. As was shown in [BGJ13], the two constructions yield the same object up to isomorphism.

4.9. Lemma. *Each frame morphism $h : L_1 \to L_2$ extends to a proximity morphism $\mathcal{F}h : \mathcal{F}L_1 \to \mathcal{F}L_2$ given by*

$$\begin{aligned} \mathcal{F}h(a) &= \bigvee\{\mathcal{B}h(x) \mid x \in \mathcal{LCF}L_1,\, x \le a\} \\ &= \bigvee\{\mathcal{B}h(b) \mid b \in \mathcal{B}L_1,\, b \le a\}. \end{aligned}$$

Proof. By identifying L_1 with its image in $\mathcal{B}L_1$, we have $\mathcal{OF}L_1 = L_1$, and similarly for L_2. Since $\mathcal{LCF}L_1 \subseteq \mathcal{B}L_1$ and each element of $\mathcal{B}L_1$ is a finite join from $\mathcal{LCF}L_1$, we have

$$\bigvee\{\mathcal{B}h(b) \mid b \in \mathcal{B}L_1,\, b \le a\} = \bigvee\{\mathcal{B}h(x) \mid x \in \mathcal{LCF}L_1,\, x \le a\}.$$

We show that $\mathcal{F}h$ satisfies (P1)–(P4) of Definition 3.5. Clearly $\mathcal{F}h|_{\mathcal{B}L_1} = \mathcal{B}h$. In particular, $\mathcal{F}h|_{L_1} = h$, and so (P1) holds. By [Bez10, Lem. 4.8], we have that $\mathcal{F}h(a \wedge b) = \mathcal{F}h(a) \wedge \mathcal{F}h(b)$, and hence (P2) holds. Since $\mathcal{LCF}L_1 \subseteq \mathcal{B}L_1$ and $\mathcal{B}h$ is a boolean homomorphism, for each finite $S \subseteq \mathcal{LCF}L_1$, we get

$$\mathcal{F}h\left(\bigvee S\right) = \mathcal{B}h\left(\bigvee S\right) = \bigvee \mathcal{B}h[S] = \bigvee \mathcal{F}h[S],$$

and thus (P3) holds. Finally,

$$\mathcal{F}h(a) = \bigvee\{\mathcal{B}h(x) \mid x \in \mathcal{LCF}L_1,\, x \le a\} = \bigvee\{\mathcal{F}h(x) \mid x \in \mathcal{LCF}L_1,\, x \le a\},$$

and so (P4) holds, yielding that $\mathcal{F}h$ is a proximity morphism. ∎

4.10. Proposition. *$\mathcal{F} : \mathbf{Frm} \to \mathbf{MT_P}$ is a functor.*

Proof. As we saw above, $\mathcal{F}$ is well defined both on objects and morphisms of **Frm**. We show that $\mathcal{F}$ sends identity morphisms to identity moprhisms and preserves composition. Let $a \in \mathcal{F}L$. Since $\mathcal{B}(1_L) = 1_{\mathcal{B}L}$, we obtain

$$\mathcal{F}(1_L)(a) = \bigvee\{\mathcal{B}(1_L)(x) \mid x \in \mathcal{LC}L_),\, x \le a\} = \bigvee\{x \in \mathcal{LC}L_1 \mid x \le a\} = 1_{\mathcal{F}L}(a).$$

Therefore, $\mathcal{F}(1_L) = 1_{\mathcal{F}L}$. Now, let $f : L_1 \to L_2$ and $g : L_2 \to L_3$ be frame morphisms. Then

$$\begin{aligned} (\mathcal{F}g \star \mathcal{F}f)(a) &= \bigvee\{\mathcal{F}g(\mathcal{F}f(x)) \mid x \in \mathcal{LCF}L_1,\, x \le a\} \\ &= \bigvee\{\mathcal{F}g(\mathcal{F}f(b)) \mid b \in \mathcal{B}L_1,\, b \le a\} \\ &= \bigvee\{\mathcal{F}g(\mathcal{B}f(b)) \mid b \in \mathcal{B}L_1,\, b \le a\} && (\mathcal{F}f|_{\mathcal{B}L_1} = \mathcal{B}f) \\ &= \bigvee\{\mathcal{B}g(\mathcal{B}f(b)) \mid b \in \mathcal{B}L_1,\, b \le a\} && (\mathcal{F}g|_{\mathcal{B}L_2} = \mathcal{B}g; \\ & && \;b \in \mathcal{B}L_1 \Rightarrow \mathcal{B}f(b) \in \mathcal{B}L_2) \\ &= \bigvee\{\mathcal{B}(g \circ f)(b) \mid b \in \mathcal{B}L_1,\, b \le a\} && (\mathcal{B}g \circ \mathcal{B}f = \mathcal{B}(g \circ f)) \\ &= \mathcal{F}(g \circ f)(a). \end{aligned}$$

∎

As we pointed out earlier, each frame L is isomorphic to $\mathcal{O}\mathcal{F}L$. We denote the isomorphism by $\rho_L : L \to \mathcal{O}\mathcal{F}L$. By identifying L with $\mathcal{O}\mathcal{F}L$, we view ρ_L as the identity on L. In addition, if M is an MT-algebra, then the boolean envelope $\mathcal{B}\mathcal{O}M$ of $\mathcal{O}M$ embeds into M by [Esa19, Prop.2.5.9], and hence so does $\mathcal{F}\mathcal{O}M$ by [BD74, Thm. XII.3.4]. We thus identify $\mathcal{F}\mathcal{O}M$ with its image in M.

4.11. Lemma. *For $M \in \mathbf{MT_P}$, define $\zeta_M : \mathcal{F}\mathcal{O}M \to M$ by*

$$\zeta_M(a) = \bigvee_M \{x \in \mathcal{L}\mathcal{C}M \mid x \le a\}$$

and $\varphi_M : M \to \mathcal{F}\mathcal{O}M$ by

$$\varphi_M(b) = \bigvee_{\mathcal{F}\mathcal{O}M} \{x \in \mathcal{L}\mathcal{C}M \mid x \le b\}.$$

Then ζ_M and φ_M are mutually inverse proximity isomorphisms.

Proof. Since each element of $\mathcal{C}ons\, M$ is a finite join from $\mathcal{L}\mathcal{C}M$, we have

$$\zeta_M(a) = \bigvee_M \{x \in \mathcal{L}\mathcal{C}M \mid x \le a\}.$$

Thus, it satisfies (P4). Since ζ_M is identity on both $\mathcal{L}\mathcal{C}M$ and $\mathcal{O}M$, it also satisfies (P3) and (P1). Finally, it satisfies (P2) by [Bez10, Lem. 4.8]. Therefore, ζ_M is a proximity morphism. That φ_M is a proximity morphism is proved similarly. It is left to show that ζ_M and φ_M are mutually inverse in $\mathbf{MT_P}$. Since $\zeta_M(x) = \varphi_M(x) = x$ for each $x \in \mathcal{L}\mathcal{C}M$, for $a \in M$, we have

$$\begin{aligned}(\zeta_M \star \varphi_M)(a) &= \bigvee_M \{\zeta_M(\varphi_M(x)) \mid x \in \mathcal{L}\mathcal{C}M,\, x \le a\} \\ &= \bigvee_M \{x \in \mathcal{L}\mathcal{C}M \mid x \le a\} = 1_M(a);\end{aligned}$$

and for $b \in \mathcal{F}\mathcal{O}M$, we have

$$\begin{aligned}(\varphi_M \star \zeta_M)(b) &= \bigvee_{\mathcal{F}\mathcal{O}M} \{\varphi_M(\zeta_M(x)) \mid x \in \mathcal{L}\mathcal{C}M,\, x \le b\} \\ &= \bigvee_{\mathcal{F}\mathcal{O}M} \{x \in \mathcal{L}\mathcal{C}M \mid x \le b\} = 1_{\mathcal{F}\mathcal{O}M}(b),\end{aligned}$$

concluding the proof. ∎

4.12. Remark. In general, the $\mathbf{MT_P}$-isomorphisms produced in the above result are not bijections. For example, consider the MT-algebra $M = \{0, a, b, 1\}$ of Example 3.14. We have $\mathcal{FO}M = \{0, 1\}$, and so $(\zeta_M \star \varphi_M)(a) = 0 \neq a$. This behavior occurs as composition in $\mathbf{MT_P}$ is given by $\star$ rather than by usual composition of functions, and identities in $\mathbf{MT_P}$ are not identity maps.

4.13. Lemma.

(1) $\rho : 1_{\mathbf{Frm}} \to \mathcal{OF}$ *is a natural transformation.*

(2) $\zeta : \mathcal{FO} \to 1_{\mathbf{MT_P}}$ *is a natural transformation.*

Proof. (1) Let $f : L_1 \to L_2$ be a frame morphism. We must show that the following diagram commutes.

$$\begin{tikzcd} L_1 \arrow[r, "f"] \arrow[d, "\rho_{L_1}"'] & L_2 \arrow[d, "\rho_{L_2}"] \\ \mathcal{OF}L_1 \arrow[r, "\mathcal{OF}f"'] & \mathcal{OF}L_2 \end{tikzcd}$$

As before, we identify L with $\rho_L[L]$ and assume that $L \subseteq \mathcal{F}L$. Since the functor $\mathcal{O}$ sends a proximity morphism to its restriction to the frame of opens, commutativity of the diagram amounts to showing that $\mathcal{F}f(a) = f(a)$ for each $a \in L_1$, which follows from the definition of $\mathcal{F}f$.

(2) Let $g : M_1 \to M_2$ be a proximity morphism between MT-algebras. We must show that the following diagram commutes.

$$\begin{tikzcd} M_1 \arrow[r, "g"] & M_2 \\ \mathcal{FO}M_1 \arrow[r, "\mathcal{FO}g"'] \arrow[u, "\zeta_{M_1}"] & \mathcal{FO}M_2 \arrow[u, "\zeta_{M_2}"'] \end{tikzcd}$$

First let $x \in \mathcal{LC}M_1$. Then $g(x) \in \mathcal{LC}M_2$ by Lemma 3.7(3). Thus, $\mathcal{FO}g(x) = g(x)$, and hence

$$\zeta_{M_2}(\mathcal{FO}g(x)) = \zeta_{M_2}(g(x)) = \bigvee\{y \in \mathcal{LC}M_2 \mid y \leq g(x)\} = g(x) = g(\zeta_{M_1}(x)),$$

where the last equality holds since $\zeta_{M_1}(x) = x$. Now let $a \in \mathcal{FO}M_1$. Then

$$\begin{aligned}(\zeta_{M_2} \star \mathcal{FO}g)(a) &= \bigvee\{\zeta_{M_2}(\mathcal{FO}g(x)) \mid x \in \mathcal{LC}M_1,\ x \leq a\} \\ &= \bigvee\{g(\zeta_{M_1}(x)) \mid x \in \mathcal{LC}M_1,\ x \leq a\} = (g \star \zeta_{M_1})(a).\end{aligned}$$

∎

4.14. Theorem. *The functors $\mathcal{O}$ and $\mathcal{F}$ establish an equivalence of* $\mathbf{MT_P}$ *and* $\mathbf{Frm}$.

Proof. As we observed before Lemma 4.11, the natural transformation ρ is an isomorphism on all components. By Lemma 4.11, so is the natural transformation ζ. Thus, it suffices to show that these are the unit and counit of the adjunction $\mathcal{F} \dashv \mathcal{O}$.

Let $M \in \mathbf{MT_P}$. In view of our identifications, $\mathcal{O}\zeta_M$ and $\rho_{\mathcal{O}M}$ are identities. Hence, for $u \in \mathcal{O}M$, we have

$$\mathcal{O}\zeta_M \circ \rho_{\mathcal{O}M}(u) = \mathcal{O}\zeta_M(u) = u.$$

Let $L \in \mathbf{Frm}$. For similar reasons, ρ_L and $\mathcal{B}\rho_L$ are identities. Therefore, for $b \in \mathcal{B}L$,

$$(\zeta_{\mathcal{F}L} \circ \mathcal{F}\rho_L)(b) = \zeta_{\mathcal{F}L}(\mathcal{B}\rho_L(b)) = \zeta_{\mathcal{F}L}(b) = b.$$

Thus, for $a \in \mathcal{F}L$,

$$(\zeta_{\mathcal{F}L} \circ \mathcal{F}\rho_L)(a) = \bigvee\{\zeta_{\mathcal{F}L}(\mathcal{F}\rho_L(b)) \mid b \in \mathcal{B}L,\, b \le a\} = \bigvee\{b \in \mathcal{B}L \mid b \le a\} = a.$$

■

In Example 3.14 we have seen that $\mathbf{MT_P}$-isomorphisms are not necessarily structure-preserving bijections. The fact that $\mathcal{O} : \mathbf{MT_P} \to \mathbf{Frm}$ establishes an equivalence of categories now gives us a characterization of such morphisms (see, e.g., [AHS06, Prop. 7.47]).

4.15. Proposition. *Let $f : M \to N$ be a proximity morphism of MT-algebras.*

(1) *f is an isomorphism iff $\mathcal{O}f$ is an isomorphism of frames.*

(2) *f is a monomorphism iff $\mathcal{O}f$ is a monomorphism of frames.*

(3) *f is an epimorphism iff $\mathcal{O}f$ is an epimorphism of frames.*

Note that, apart from isomorphisms not being bijections between the underlying sets, in $\mathbf{MT_P}$ we also have monomorphisms that are not injective and epimorphisms that are not surjective:

4.16. Example. In Example 3.14, the maps f and g are both isomorphisms hence both are monic and epic. However, f is not injective and g is not surjective.

This counterintuitive behavior disappears when we restrict our attention to T_D-algebras.

4.17. Proposition. [BR23, Thm. 6.5] *An MT-algebra M is T_D iff $M \cong \mathcal{F}\mathcal{O}M$.*

4.18. Definition. Let $\mathbf{TDMT_P}$ be the full subcategory of $\mathbf{MT_P}$ consisting of T_D-algebras.

We have the following:

4.19. Theorem.

(1) $\mathbf{TDMT_P}$ *is equivalent to* $\mathbf{MT_P}$.

(2) $\mathbf{Frm}$ *is equivalent to* $\mathbf{TDMT_P}$.

PROOF. (1) Let $e : \mathbf{TDMT_P} \to \mathbf{MT_P}$ be the inclusion functor. By Lemma 4.11 and [ML98, Prop. IV.4.2], we obtain an adjoint equivalence between $\mathbf{TDMT_P}$ and $\mathbf{MT_P}$ via the functors e and $\mathcal{F}\mathcal{O}$, making $\mathcal{F}\mathcal{O} : \mathbf{MT_P} \to \mathbf{TDMT_P}$ a quasi-inverse of e.

(2) Apply (1) and Theorem 4.14. ■

The isomorphism $\varphi_M : M \to \mathcal{F}\mathcal{O}M$ may be seen as the T_D-reflection of M. Indeed, $\mathcal{F}\mathcal{O}M$ is always a T_D-algebra, and if N is a T_D-algebra and $f : M \to N$ is a proximity morphism, we may define a proximity morphism $\widehat{f} : \mathcal{F}\mathcal{O}M \to N$ by setting $\widehat{f} = f \star \zeta_M$. We then have a commutative diagram in $\mathbf{MT_P}$:

$$\begin{tikzcd} M \arrow[r, shift left, "\varphi_M"] \arrow[d, "f"'] & \mathcal{F}\mathcal{O}M \arrow[l, shift left, "\zeta_M"] \arrow[dl, "\widehat{f}"] \\ N & \end{tikzcd}$$

4.20. REMARK. By definition, up to isomorphism, $\mathcal{O}\varphi_M = \rho_{\mathcal{O}M}$ is the identity in $\mathbf{Frm}$. Therefore, the T_D-reflection does not do anything in $\mathbf{Frm}$. In fact, for frames there is no concept of the T_D-reflection since the language of frames is less expressive than that of MT-algebras.

Since every MT-algebra is isomorphic to its T_D-reflection, by considering the inverse of the above isomorphism, we see that $\mathbf{TDMT_P}$ is also a coreflective subcategory of $\mathbf{MT_P}$, with the coreflector given by the counit ζ.

We conclude this section by showing that, unlike the situation in $\mathbf{MT_P}$, isomorphisms in $\mathbf{TDMT_P}$ are structure-preserving bijections. For this we use the following lemma, which is a consequence of [Esa19, Thm. 2.5.11].

4.21. LEMMA. *For $f : L \to M$ a frame isomorphism, $\mathcal{B}f : \mathcal{B}L \to \mathcal{B}M$ is a boolean isomorphism.*

PROOF. By Theorem 4.3(1), **Heyt** is equivalent to **Ess**. Therefore, Heyting isomorphims $H_1 \to H_2$ correspond to interior algebra isomorphisms $\mathcal{B}H_1 \to \mathcal{B}H_2$. But frame isomorphisms between frames are Heyting algebra isomorphisms, and interior algebra isomorphisms are boolean isomorphisms, so the result follows. ■

4.22. PROPOSITION. *A proximity map $f : M \to N$ between T_D-algebras is an isomorphism in $\mathbf{MT_P}$ iff it is an order-isomorphism.*

PROOF. First suppose that $f : M \to N$ is a proximity isomorphism between T_D-algebras. By Lemma 4.15(1), $\mathcal{O}f$ is an isomorphism of frames. Therefore, by Lemma 4.21, $\mathcal{B}\mathcal{O}f : \mathcal{B}\mathcal{O}M \to \mathcal{B}\mathcal{O}N$ is a boolean isomorphism. Thus, it can be lifted to an isomorphism between $\mathcal{F}\mathcal{O}M$ and $\mathcal{F}\mathcal{O}N$ (see, e.g., [DP02, Thm. 7.41(ii)]). As M and N are T_D-algebras, they are order-isomorphic to $\mathcal{F}\mathcal{O}M$ and $\mathcal{F}\mathcal{O}N$, and since the isomorphism lifting $\mathcal{B}\mathcal{O}f$ preserves arbitrary joins, it must coincide with $\mathcal{F}\mathcal{O}f = f$.

Conversely, suppose that $f : M \to N$ is an order-isomorphism. Then its inverse $f^{-1} : N \to M$ is an order-isomorphism. Therefore, for $a \in M$, we have

$$\begin{aligned}(f^{-1} \star f)(a) &= \bigvee\{f^{-1}(f(x)) \mid x \in \mathcal{L}\mathcal{C}M,\, x \le a\}\\ &= \bigvee\{x \in \mathcal{L}\mathcal{C}M \mid x \le a\}\\ &= 1_M(a).\end{aligned}$$

A similar argument yields that $f \star f^{-1} = 1_N$. Thus, f is a proximity isomorphism. ∎

4.23. Remark. The category $\mathbf{TDMT_P}$ has the following additional pleasant features:

(1) Identities in $\mathbf{TDMT_P}$ are identity functions. In fact, an MT-algebra M is T_D iff the identity 1_M in $\mathbf{MT_P}$ is the identity function. Indeed,

$$\begin{aligned}M \text{ is } T_D &\iff \forall a \in M,\ a = \bigvee\{x \in \mathcal{L}\mathcal{C}M \mid x \le a\}\\ &\iff \forall a \in M,\ a = 1_M(a).\end{aligned}$$

(2) The category $\mathbf{TDMT}$ is a wide subcategory of $\mathbf{TDMT_P}$. For, if $f : M \to N$ is a $\mathbf{TDMT}$-morphism, it preserves all finite meets and joins by definition. By Theorem 2.10, its restriction $\mathcal{O}f : \mathcal{O}M \to \mathcal{O}N$ is a frame morphism. For $a \in M$, because M is T_D, $a = \bigvee\{x \in \mathcal{L}\mathcal{C}M \mid x \le a\}$. Since f preserves all joins, $f(a) = \bigvee\{f(x) \mid x \in \mathcal{L}\mathcal{C}M,\, x \le a\}$, so it is a $\mathbf{TDMT_P}$-morphism.

Fig. 1 summarizes the relationship between the categories introduced in this section. The connecting "arrows" should be understood as follows:

- red two-sided arrows denote categorical equivalence;
- solid black hooks denote full embeddings, with reflections and coreflections noted;
- dashed black hooks denote non-full embeddings;
- blue hooks denote wide embeddings;
- squiggly lines denote same objects but different morphisms.

(The same color coding will be used in the rest of the paper.)

5. T_D-duality for MT-algebras

In this section, we generalize the T_D-duality of Banaschewski and Pultr [BP10] to the setting of MT-algebras. This is done by generalizing the notion of a D-morphism between frames to that of a D-morphism between MT-algebras. For T_D-algebras, this notion is stronger than that of a proximity morphism. We prove that the category $\mathbf{STDMT}$ of

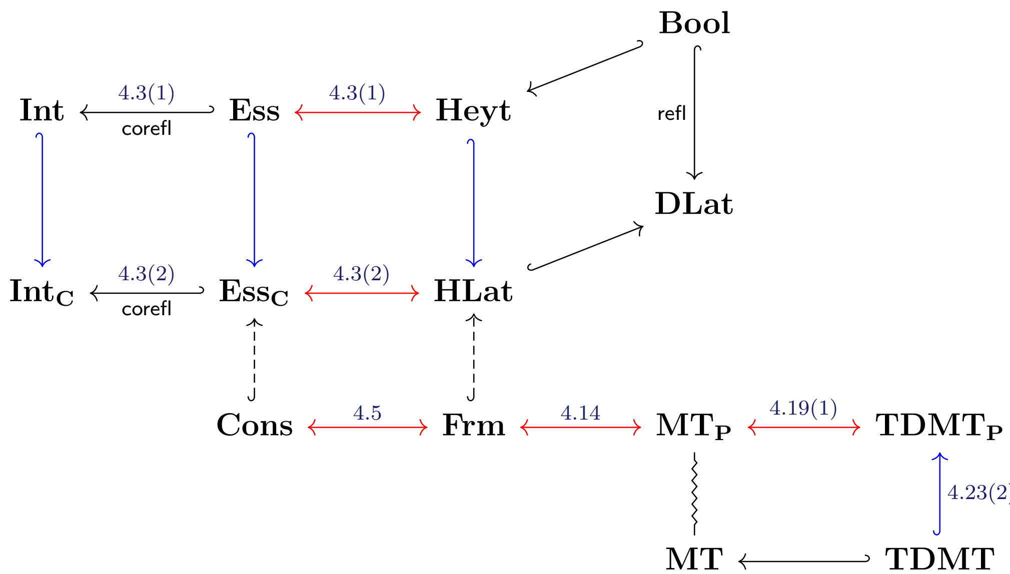

Figure 1: Relationship between categories

spatial T_D-algebras is a reflective subcategory of the category $\mathbf{MT_D}$ of MT-algebras and D-morphsims, and is equivalent to the category **TDTop** of T_D-spaces. This, in particular, yields a generalization of the T_D-coreflection [BP10, 3.7.2] from T_0-spaces to arbitrary ones. We argue that the MT setting is more natural for the T_D-duality than the frame setting by observing that, unlike the case of T_D-spatial frames, the spatial T_D-algebras form a full subcategory of **MT**.

T_D-spectra of MT-algebras. In this subsection, we introduce the T_D-spectrum of an MT-algebra and connect it to the T_D-spectrum of a frame.

5.1. Definition. For an MT-algebra M, let $at_D\, M$ be the collection of its locally closed atoms.

We view $at_D\, M$ as a subspace of the spectrum $at\, M$ of M as defined in Section 2. To connect $at_D\, M$ to $pt_D\, \mathcal{O}M$, we recall:

5.2. Lemma.

(1) [BR23, Prop. 4.8] *For every MT-algebra M, there is a continuous map $\theta : at\, M \to pt\, \mathcal{O}M$ given by $\theta(x) = {\uparrow}x \cap \mathcal{O}M$.*

(2) [BR23, Prop. 4.10] *If M is a T_0-algebra then θ is a subspace embedding.*

We show that for T_0-algebras, the above embedding yields a homeomorphism between $at_D\, M$ and $pt_D\, \mathcal{O}M$.

For this, we use the following:

5.3. Lemma. *For a T_0-algebra M, an element $x \in M$ is an atom iff for each $u \in \mathcal{O}M$ we have $x \le u$ iff $x \not\le \neg u$.*

Proof. Clearly if $x \in M$ is an atom, then the condition in the statement is satisfied. For the converse, if $x \in M$ satisfies the condition, then $x \neq 0$. Let $y \leq x$ with $y \neq 0$. We show that $x \leq y$. Since M is T_0, $y = \bigwedge S \wedge \neg v$ for some $S \subseteq \mathcal{O}M$ and $v \in \mathcal{O}M$. If $x \nleq y$ then either $x \nleq u$ for some $u \in S$ or $x \nleq \neg v$. By assumption, in the former case we get that $x \leq \neg u$, and in the latter case that $x \leq v$. In both cases, $x \leq \bigvee\{\neg u \mid u \in S\} \vee v = \neg y$. Therefore, $y \leq x \leq \neg y$, a contradiction. ■

Recall (see, e.g., [PP12, Sec. II.3.3]) that in a frame L, we have a bijection between completely prime filters and prime elements given by $P \mapsto \bigvee(L\backslash P)$. Moreover, since every slicing filter is completely prime, one is able to identify those prime elements that arise from slicing filters.

5.4. Lemma. [BP10, Prop. 2.6.2] *For a completely prime filter P, the following are equivalent:*

(1) *P is a slicing filter;*

(2) *The corresponding prime is a covered prime;*

(3) *The corresponding prime is completely meet-irreducible.*

We are ready to prove the main result of this subsection.

5.5. Theorem. *For a T_0-algebra M, the embedding $\theta : \mathit{at}\, M \to \mathit{pt}\,\mathcal{O}M$ restricts and co-restricts to a homeomorphism $\theta' : \mathit{at}_D\, M \to \mathit{pt}_D\,\mathcal{O}M$.*

$$
\begin{tikzcd}
\mathit{at}\, M \arrow[r, "\theta"] & \mathit{pt}\,\mathcal{O}M \\
\mathit{at}_D\, M \arrow[u, hook] \arrow[r, "\theta'"] & \mathit{pt}_D\,\mathcal{O}M \arrow[u, hook]
\end{tikzcd}
$$

Proof. To see that θ' is well defined, let $x \in \mathit{at}_D\, M$. Since $x \in \mathcal{L}\mathcal{C}M$, $x = u \wedge \Diamond x$ for some $u \in \mathcal{O}M$ (see footnote 3). Therefore, $u \vee \neg\Diamond x \in {\uparrow}x \cap \mathcal{O}M$ and $\neg\Diamond x \notin {\uparrow}x \cap \mathcal{O}M$. We show that $\neg\Diamond x \lessdot u \vee \neg\Diamond x$. Suppose $\neg\Diamond x \leq v \leq u \vee \neg\Diamond x$ for some $v \in \mathcal{O}M$. Since x is an atom, either $x \leq v$ or $x \leq \neg v$. In the former case, $x \vee \neg\Diamond x \leq v$, so $(u \wedge \Diamond x) \vee \neg\Diamond x \leq v$, and hence $u \vee \neg\Diamond x \leq v$. In the latter case, $\Diamond x \leq \neg v$ since $\neg v$ is closed, so $v \leq \neg\Diamond x$. Thus, ${\uparrow}x \cap \mathcal{O}M$ is a slicing filter.

That θ' is one-to-one follows from Lemma 5.2(2). To show it is onto, we need to show that every slicing filter $F \subseteq \mathcal{O}M$ is of the form ${\uparrow}x \cap \mathcal{O}M$ for some locally closed atom x. Let

$$x = \bigwedge F \wedge \bigwedge\{\neg a \mid a \notin F\}.$$

5.6. Claim. x is an atom.

Proof of Claim. Since for each $u \in \mathcal{O}M$ we have $x \le u$ or $x \le \neg u$, if $x \neq 0$, it is an atom by Lemma 5.3. Thus, it is enough to show that $x \neq 0$. Let p be the covered prime corresponding to F. Since $p = \bigvee (L \setminus F)$, we have $x = 0$ iff $\bigwedge F \le p$. Indeed,

$$\begin{aligned} x = 0 &\implies \bigwedge F \wedge \bigwedge \{\neg a \mid a \notin F\} = 0 \\ &\implies \bigwedge F \le \neg \bigwedge \{\neg a \mid a \notin F\} = \bigvee \{a \mid a \notin F\} = p. \end{aligned}$$

Conversely,

$$\begin{aligned} \bigwedge F \le p &\implies \bigwedge F \le \bigvee \{a \mid a \notin F\} \\ &\implies \bigwedge F \wedge \neg \bigvee \{a \mid a \notin F\} = 0 \\ &\implies \bigwedge F \wedge \bigwedge \{\neg a \mid a \notin F\} = 0 \\ &\implies x = 0. \end{aligned}$$

But $\bigwedge F \le p$ iff $\bigwedge \{u \vee p \mid u \in F\} = p$. Since for $A \subseteq \mathcal{O}M$, $\bigwedge A \in \mathcal{O}M$ implies $\bigwedge A = \bigwedge_{\mathcal{O}M} A$, the last condition implies that $\bigwedge_{\mathcal{O}M} \{u \vee p \mid u \in F\} = p$. However, because p is covered, by Lemma 5.4 this means that $u \le p$ for some $u \in F$. The obtained contradiction proves that x is an atom. ■

5.7. Claim. $F = {\uparrow} x \cap \mathcal{O}M$.

Proof of Claim. Let $u \in \mathcal{O}M$. First suppose that $u \in F$. Then $x \le \bigwedge F \le u$, and so $u \in {\uparrow} x \cap \mathcal{O}M$. Next suppose that $u \notin F$. Then $x \le \neg u$. By Claim 5.6, x is an atom, so $x \not\le u$, and hence $u \notin {\uparrow} x \cap \mathcal{O}M$. ■

5.8. Claim. x is locally closed.

Proof of Claim. By Claim 5.7, $F = {\uparrow} x \cap \mathcal{O}M$. Since F is slicing, there exist $a, b \in \mathcal{O}M$ such that $a \lessdot b$, $a \notin {\uparrow} x \cap \mathcal{O}M$, and $b \in {\uparrow} x \cap \mathcal{O}M$. By Claim 5.6, x is an atom. This together with M being a T_0-algebra yields that $x = \bigwedge S \wedge \neg v$ for some $S \subseteq \mathcal{O}M$ and $v \in \mathcal{O}M$. For each $u \in S$, we have $a \le a \vee (b \wedge u) \le b$. Since $a \lessdot b$, either $a = a \vee (b \wedge u)$ or $b = a \vee (b \wedge u)$. In the former case, $x \le b \wedge u \le a$, and hence $a \in {\uparrow} x \cap \mathcal{O}M$, a contradiction. Therefore, $b = a \vee (b \wedge u)$, and thus

$$b \wedge \neg a = [a \vee (b \wedge u)] \wedge \neg a = b \wedge u \wedge \neg a \le u.$$

Since this is true for each $u \in S$, we obtain that $b \wedge \neg a \le \bigwedge S$. Consequently,

$$x \le b \wedge \neg a \wedge \neg v \le \bigwedge S \wedge \neg v \le x,$$

yielding that $x = b \wedge \neg a \wedge \neg v$. Thus, x is locally closed since b is open and $\neg a \wedge \neg v$ is closed. ■

Consequently, since $\theta : \mathit{at}\, M \to \mathit{pt} \mathcal{O}M$ is a subspace embedding by Lemma 5.2, $\theta' : \mathit{at}_D\, M \to \mathit{pt}_D\, \mathcal{O}M$ is a homeomorphism. ■

5.9. Corollary. *For a T_D-algebra M, $\mathit{at}\, M$ is homeomorphic to $\mathit{pt}_D\, \mathcal{O}M$.*

Proof. By Theorem 5.5, there is a homeomorphism $\theta' : \mathit{at}_D\, M \cong \mathit{pt}_D\, \mathcal{O}M$. Since M is a T_D-algebra, $\mathit{at}_D\, M = \mathit{at}\, M$, yielding the result. ∎

The assumption in Theorem 5.5 that M is a T_0-algebra is necessary. (Note that the assumption is used to show that $\theta' : \mathit{at}_D\, M \to \mathit{pt}_D\, \mathcal{O}M$ is onto.)

5.10. Example. In the MT-algebra M of Example 3.14, ${\uparrow} a \cap \mathcal{O}M = \{1\}$ is a slicing filter. But $a \in \mathit{at}\, M$ is not locally closed because $\mathcal{LC}M = \{0, 1\}$. Thus, θ' is not onto.

T_D-reflection of MT-algebras and T_D-coreflection of topological spaces. We now focus our attention on morphisms and look at the MT-analogues of D-morphisms of Banaschewski and Pultr (see Definition 2.2), which we will also call D-morphisms. Our aim is to show that the spatial T_D-algebras form a full reflective subcategory of the category of MT-algebras and D-morphisms, thus yielding a pointfree version of the T_D-coreflection of T_0-spaces defined in [BP10]. We emphasize that this T_D-reflection is not expressible in the language of frames.

5.11. Definition.

(1) We call a continuous map *locally closed* if it maps locally closed points to locally closed points.

(2) Let $\mathbf{Top_{LC}}$ be the wide subcategory of $\mathbf{Top}$ whose morphisms are locally closed maps.

We point out that identity maps are locally closed and that the composition of two locally closed maps is locally closed, so $\mathbf{Top_{LC}}$ indeed forms a category. We let $\mathbf{T0Top_{LC}}$ be the full subcategory of $\mathbf{Top_{LC}}$ consisting of T_0-spaces[7], and note that $\mathbf{TDTop}$ is a full subcategory of $\mathbf{T0Top_{LC}}$ since every continuous map between T_D-spaces is automatically locally closed.

We show that locally closed maps between T_0-spaces can be seen as topological duals of D-morphisms. For this we recall the following result from [BP10, Prop. 2.7.1]:

5.12. Lemma. *For a T_0-space X, every slicing filter of ΩX is of the form $F_x = \{U \in \Omega X \mid x \in U\}$ for some locally closed $x \in X$.*

5.13. Proposition. *A continuous map $f : X \to Y$ between T_0-spaces is locally closed iff Ωf is a D-morphism.*

[7] As follows from Proposition 5.13, $\mathbf{T0Top_{LC}}$ is precisely the category ${}_D\mathbf{Top}$ defined in [BP10, 3.7.2].

PROOF. Suppose that $f : X \to Y$ is a locally closed map between T_0-spaces. Consider a slicing filter of ΩX, which by Lemma 5.12 is of the form F_x for some locally closed $x \in X$. Then $f(x) \in Y$ is locally closed. We have

$$(\Omega f)^{-1}(F_x) = \{U \in \Omega Y \mid f^{-1}(U) \in F_x\} = \{U \in \Omega Y \mid x \in f^{-1}(U)\} = F_{f(x)}.$$

Since $f(x)$ is locally closed, $F_{f(x)}$ is slicing, and hence Ωf is a D-morphism.

For the converse, suppose that Ωf is a D-morphism. If $x \in X$ is locally closed, the same computation as above shows that $(\Omega f)^{-1}(F_x) = F_{f(x)}$, and because Ωf is a D-morphism, $F_{f(x)}$ is a slicing filter. Thus, $f(x)$ is locally closed by Lemma 5.12, and hence f is a locally closed map. ∎

We next introduce D-morphisms for MT-algebras.

5.14. DEFINITION.

(1) An MT-morphism f is a *D-morphism* if the left adjoint f^* maps locally closed atoms to locally closed atoms.

(2) Let $\mathbf{MT_D}$ be the category of MT-algebras and D-morphisms. We also let $\mathbf{SMT_D}$ be the full subcategory of $\mathbf{MT_D}$ consisting of spatial MT-algebras and $\mathbf{ST0MT_D}$ the full subcategory of $\mathbf{SMT_D}$ consisting of spatial T_0-algebras.

We point out that identity maps are D-morphisms and that the composition of two D-morphisms is a D-morphism, so $\mathbf{MT_D}$ indeed forms a category. Also, note that **STDMT** is a full subcategory of $\mathbf{ST0MT_D}$ since each MT-morphism between T_D-algebras is automatically a D-morphism (because every atom in a T_D-algebra is locally closed). The following result holds for arbitrary (not only T_0) spaces.

5.15. LEMMA. *A continous map $f : X \to Y$ is locally closed iff $f^{-1} : \wp Y \to \wp X$ is a D-morphism.*

PROOF. For each $x \in X$, we have

$$(f^{-1})^*(\{x\}) = \bigcap\{S \in \wp Y \mid \{x\} \subseteq f^{-1}(S)\} = \bigcap\{S \in \wp Y \mid f(x) \in S\} = \{f(x)\}.$$

The result follows since a point in a space is locally closed iff the corresponding singleton is a locally closed element in the MT-algebra of all subsets. ∎

5.16. REMARK. The above result is no longer true if we replace the functor $\wp$ with Ω: consider the inclusion $\{0\} \subseteq \{0, 1\}$ where both sets are given the trivial topology. The dualization of this map is the identity on the two-element frame, which is a D-morphism. But $\{0\}$ is locally closed in $\{0\}$, and not in $\{0, 1\}$. Of course, by Proposition 5.13, it does remain true for Ω if the spaces under consideration are T_0.

As an immediate consequence of Theorems 2.11, 2.13(1), and the above lemma we obtain:

5.17. Theorem. *$\mathbf{Top_{LC}}$ is equivalent to $\mathbf{SMT_D}^{\mathrm{op}}$, and $\mathbf{T0Top_{LC}}$ is equivalent to $\mathbf{ST0MT_D}^{\mathrm{op}}$.*

The above equivalences further restrict to give the equivalence of Theorem 2.13(2). We thus arrive at the following commutative diagram:

$$\begin{tikzcd}
\mathbf{Top_{LC}} \arrow[r, leftrightarrow] & \mathbf{SMT_D}^{\mathrm{op}} \\
\mathbf{T0Top_{LC}} \arrow[u, hook] \arrow[r, leftrightarrow] & \mathbf{ST0MT_D}^{\mathrm{op}} \arrow[u, hook] \\
\mathbf{TDTop} \arrow[u, hook] \arrow[r, leftrightarrow] & \mathbf{STDMT}^{\mathrm{op}} \arrow[u, hook]
\end{tikzcd}$$

We next study the relationship between D-morphisms of MT-algebras and D-morphisms of frames. Recalling Theorem 2.10, we have:

5.18. Lemma. *For a complete boolean homomorphism $f : M \to N$ between MT-algebras, we have*

$$(\mathcal{O}f)^{-1}(\uparrow x \cap \mathcal{O}N) = \uparrow f^*(x) \cap \mathcal{O}M,$$

for all atoms $x \in N$.

Proof. By the adjointness property, we have that $x \leq f(a)$ iff $f^*(x) \leq a$ for each $a \in M$. This, by definition, means that for each $a \in \mathcal{O}M$,

$$a \in (\mathcal{O}f)^{-1}(\uparrow x \cap \mathcal{O}N) \iff x \leq f(a) \iff f^*(x) \leq a \iff a \in \uparrow f^*(x) \cap \mathcal{O}M.$$

■

5.19. Proposition.

(1) *An MT-morphism $f : M \to N$ between T_0-algebras is a D-morphism iff $\mathcal{O}f$ is a D-morphism.*

(2) *Any MT-morphism $f : M \to N$ between T_D-algebras is a D-morphism.*

Proof. (1) Let $f : M \to N$ be an MT-morphism between T_0-algebras. First suppose that $\mathcal{O}f$ is a D-morphism, and that $x \in N$ is a locally closed atom. By Theorem 5.5, $\uparrow x \cap \mathcal{O}M$ is a slicing filter, and hence so is $\uparrow f^*(x) \cap \mathcal{O}M$ by Lemma 5.18. Thus, $f^*(x)$ is locally closed by reapplying Theorem 5.5.

Next suppose that f is a D-morphism. Let $F \subseteq \mathcal{O}N$ be a slicing filter. By Theorem 5.5, $F = \uparrow x \cap \mathcal{O}N$ for some locally closed atom $x \in N$. By assumption, $f^*(x) \in M$ is a locally closed atom. By Theorem 5.5, $\uparrow f^*(x) \cap \mathcal{O}M$ is a slicing filter, and hence so is $(\mathcal{O}f)^{-1}(\uparrow x \cap \mathcal{O}N)$ by Lemma 5.18. Thus, $\mathcal{O}f$ is a D-morphism.

(2) This follows immediately from the fact that in a T_D-algebra all atoms are locally closed. ■

By [BP10, Sec. 3.7.2], T_D-spaces are a coreflective subcategory of the category of T_0-spaces and locally closed maps. The T_D-coreflection of a T_0-space X is $pt_D\,\Omega X \subseteq X$, which up to homeomorphism is the inclusion of locally closed points of X into it. We next define a pointfree analogue of this construction, without restricting to T_0 objects on either side.

5.20. Definition. For an MT-algebra M, define $\chi_M : M \to \mathscr{P}\, at_D\, M$ by

$$\chi_M(a) = \{x \in at_D\, M \mid x \le a\}.$$

It follows directly from the definition that $\chi_M(a \wedge b) = \chi_M(a) \cap \chi_M(b)$, and $\chi_M(\bigvee_i a_i) = \bigcup_i \chi_M(a_i)$ because atoms are completely join-prime. Thus, $\chi_M[\mathcal{O}M]$ is a subframe of $\mathscr{P}\, at_D\, M$. We will regard $\mathscr{P}\, at_D\, M$ as an MT-algebra whose opens are precisely this subframe. Thus, $\chi_M : M \to \mathscr{P}\, at_D\, M$ is an MT-morphism onto a spatial T_D-algebra.

5.21. Lemma. *The map $\chi_M : M \to \mathscr{P}\, at_D\, M$ is a D-morphism.*

Proof. As observed above, the map is an MT-morphism. The atoms of $\mathscr{P}\, at_D\, M$ are the singletons. For $x \in at_D\, M$, we have $\chi_M^*(\{x\}) = x$, which is locally closed. Thus, χ_M is a D-morphism. ∎

5.22. Theorem. *The category* **STDMT** *is a full reflective subcategory of* $\mathbf{MT_D}$.

Proof. The subcategory is full by Lemma 5.19(2). For any MT-algebra M, by Lemma 5.21, the map $\chi_M : M \to \mathscr{P}\, at_D\, M$ is a D-morphism onto a spatial T_D-algebra. Suppose that $f : M \to N$ is a D-morphism with N a spatial T_D-algebra. Define $\widehat{f} : \mathscr{P}\, at_D\, M \to N$ by $\widehat{f}(S) = \bigvee\{f(x) \mid x \in S\}$. We show that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\chi_M"] \arrow[d, "f"'] & \mathscr{P}\, at_D\, M \arrow[dl, "\widehat{f}"] \\ N & \end{tikzcd}$$

For $a \in M$, $\widehat{f}(\chi_M(a)) = \bigvee\{f(x) \mid x \in at_D\, M,\ x \le a\}$. It is clear, then, that $\widehat{f}(\chi_M(a)) \le f(a)$. For the other inequality, since N is spatial it suffices to show that $y \le f(a)$ implies $y \le \widehat{f}(\chi_M(a))$ for all $y \in at\, N$. If $y \le f(a)$, then $f^*(y) \le a$. By assumption on f, $f^*(y) \in at_D\, M$. Therefore,

$$y \le f(f^*(y)) \le \bigvee\{f(x) \mid x \in at_D\, M,\ x \le a\}.$$

Thus, $y \le \widehat{f}(\chi_M(a))$, as desired. Finally, we show that $\widehat{f}$ is an MT-morphism:

$$\begin{aligned} \widehat{f}\left(\bigcup_i S_i\right) &= \bigvee\{f(x) \mid x \in \bigcup_i S_i\} = \bigvee \bigvee_i \{f(x) \mid x \in S_i\} \\ &= \bigvee_i \bigvee\{f(x) \mid x \in S_i\} = \bigvee_i \widehat{f}(S_i); \end{aligned}$$

it is left to see that $\widehat{f}$ maps opens to opens. However, the commutativity of the diagram gives $\widehat{f}(\chi_M(a)) = f(a)$ for $a \in \mathcal{O}M$, and the result follows. ∎

5.23. Remark. It follows from [BP10, Prop. 3.5.1] that the category $\mathbf{TD\text{-}SFrm_D}$ of T_D-spatial frames is a full reflective subcategory of $\mathbf{Frm_D}$, the T_D-spatialization being the reflector. Theorem 5.22 provides a generalization of this, but also an improvement since $\mathbf{TD\text{-}SFrm_D}$ is not a full subcategory of $\mathbf{Frm}$ (see Example 2.4), while $\mathbf{STDMT}$ is a full subcategory of $\mathbf{MT}$ (see the end of Section 2).

We conclude this subsection by showing that the coreflection in [BP10, Sec. 3.7.2] may be obtained as the dualization of the above reflection.

5.24. Definition. For a topological space X, let X_D be the subspace of X consisting of locally closed points.

Since $x \in X$ is locally closed if and only if $\{x\}$ is locally closed in the MT-algebra $\mathscr{P}X$, we have a homeomorphism $h_D : X_D \cong \mathit{at}_D\,\mathscr{P}X$ given by $h_D(x) = \{x\}$. From now on, we will identify these spaces.

5.25. Lemma. *The inclusion $i_D : X_D \subseteq X$ is such that $\mathscr{P}(i_D) = \chi_{\mathscr{P}X}$.*

Proof. Since $\mathscr{P}(i_D) = {i_D}^{-1}$, $\mathscr{P}(i_D)(Y) = Y \cap X_D$ for each $Y \subseteq X$. Therefore, under the identification described above,

$$\mathscr{P}(i_D)(Y) = Y \cap X_D = \{\{x\} \in \mathit{at}_D\,\mathscr{P}\,X \mid \{x\} \subseteq Y\} = \chi_{\mathscr{P}\,X}(Y).$$

■

5.26. Theorem. *The category* $\mathbf{TDTop}$ *is a full coreflective subcategory of* $\mathbf{Top_{LC}}$. *The coreflection is given by the inclusion $X_D \subseteq X$.*

Proof. Let X be a space. By definition, the inclusion $X_D \subseteq X$ is a locally closed map. Suppose that Y is a T_D-space and $f : Y \to X$ is a locally closed map. By Lemma 5.15, $f^{-1} : \mathscr{P}X \to \mathscr{P}Y$ is an $\mathbf{MT_D}$-morphism. By Theorem 5.22, there is an $\mathbf{MT_D}$-morphism $\widehat{f^{-1}} : \mathscr{P}\,\mathit{at}_D\,\mathscr{P}X \to \mathscr{P}Y$ such that the diagram on the right commutes:

$$\begin{tikzcd} X_D \arrow[r, "\subseteq"] & X \\ Y \arrow[u, dashed, "\widehat{f}"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathscr{P}X \arrow[r, "\chi_{\mathscr{P}X}"] \arrow[d, "f^{-1}"'] & \mathscr{P}\,\mathit{at}_D\,\mathscr{P}X \arrow[dl, "\widehat{f^{-1}}"] \\ \mathscr{P}Y & \end{tikzcd}$$

By Lemma 5.25 and Theorem 2.13(2), there must be a locally closed map $\widehat{f} : Y \to X_D$ such that the diagram on the left commutes. ■

5.27. Remark. The above theorem yields the T_D-coreflection of Banaschewski and Pultr. In [BP10, 3.7.2] it was described as the embedding $pt_D\,\Omega X \to X$ for every T_0-space X. One of the advantages of our approach is that we do not have to restrict to T_0-spaces.

5.28. Remark. As we saw above, the T_D-coreflection of a space is neatly captured by dualizing the spatial T_D-reflection of $\mathbf{MT_D}$. By contrast, the frame setting is not expressive enough for this purpose. Indeed, we recall from [BP10, 3.7.2] that the T_D-spatialization of a frame L is given by

$$\begin{aligned} \sigma_L : L &\to \Omega\, pt_D\, L, \\ a &\mapsto \{P \in pt_D\, L \mid a \in P\}. \end{aligned}$$

If we dualize σ_L using pt_D, we obtain a homeomorphism $pt_D\, \Omega\, pt_D\, L \to pt_D\, L$, which is a trivial T_D-coreflection. On the other hand, if we dualize σ_L using pt, we obtain $pt\, \sigma_L : pt\, \Omega\, pt_D\, L \to pt\, L$. Since $pt_D\, L$ is the subspace of locally closed points of $pt\, L$, this gives the inclusion of the soberification of $pt_D\, L$ into $pt\, L$, which is not the T_D-coreflection. In fact, the soberification of a T_D-space is T_D only in the trivial case where the starting space is both sober and T_D. We will explore the interplay between soberification and the T_D axiom in Section 6.

6. Duality for spatial MT-algebras and proximity morphisms

In this final section, we generalize the duality of Theorem 2.11 between **Top** and **SMT** to incorporate proximity morphisms between spatial MT-algebras. This is done by introducing the notion of a sober map, a continuous map from one space to the soberification of another, and by showing that frame morphisms between spatial frames and their corresponding proximity morphisms between spatial MT-algebras are characterized by sober maps. As a corollary, we obtain the topological and MT analogues of the category of T_D-spatial frames and frame morphisms.

We begin by recalling that **Sob** is a reflective subcategory of **Top**, and that the reflector $\mathcal{s} : \mathbf{Top} \to \mathbf{Sob}$ is given by the soberification $pt\,\Omega$ (see, e.g., [Joh82, p. 44]). The unit $\lambda : 1_{\mathbf{Top}} \to \mathcal{s}$ is given by $\lambda_X(x) = F_x$ for each $X \in \mathbf{Top}$ and $x \in X$.

6.1. Definition. For topological spaces X and Y, we call a continuous map $f : X \to \mathcal{s}Y$ a *sober map* from X to Y, and denote it by $f : X \rightsquigarrow Y$.

If $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$ are two sober maps then their composition $g \bullet f : X \rightsquigarrow Z$ is given by $\lambda_{\mathcal{s}Z}^{-1} \circ \mathcal{s}g \circ f : X \to \mathcal{s}Z$, which is well defined since $\lambda_{\mathcal{s}Z} : \mathcal{s}Z \to \mathcal{s}\mathcal{s}Z$ is a homeomorphism. By identifying $\mathcal{s}Z$ with $\mathcal{s}\mathcal{s}Z$, the composition $g \bullet f$ can be described as $\mathcal{s}g \circ f$. By this identification, we have that $\mathcal{s}\lambda_X = \lambda_{\mathcal{s}X}$ is the identity on $\mathcal{s}X$. Consequently, since λ is a natural transformation, for each $f : X \rightsquigarrow Y$, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda_X"'] & \mathcal{s}Y \arrow[d, "\lambda_{\mathcal{s}Y}"] \\ \mathcal{s}X \arrow[r, "\mathcal{s}f"'] & \mathcal{s}Y = \mathcal{s}\mathcal{s}Y \end{tikzcd}$$

Therefore, $f \bullet \lambda_X = \mathpzc{s}f \circ \lambda_X = \lambda_{\mathpzc{s}Y} \circ f = f$. Similarly, for each $g : Y \rightsquigarrow X$, we have that $\lambda_X \bullet g = g$. We thus arrive at the following new category:

6.2. Definition. Let $\mathbf{Top_S}$ be the category of topological spaces and sober maps between them, where composition is given by $\bullet$ and identity morphisms are λ_X.

6.3. Remark. A sober map $f : X \rightsquigarrow Y$ is a $\mathbf{Top_S}$-isomorphism iff $\mathpzc{s}f : \mathpzc{s}X \to \mathpzc{s}Y$ is a homeomorphism. To see this, suppose there is $g : Y \rightsquigarrow X$ such that $g \bullet f = \lambda_X$ and $f \bullet g = \lambda_Y$, so $\mathpzc{s}g \circ f = \lambda_X$ and $\mathpzc{s}f \circ g = \lambda_Y$ (see the left diagram below). By applying $\mathpzc{s}$ to the former, $\mathpzc{s}(\mathpzc{s}g \circ f) = \mathpzc{s}\mathpzc{s}g \circ \mathpzc{s}f = \mathpzc{s}\lambda_X$. Therefore, by identifying $\mathpzc{s} = \mathpzc{s}\mathpzc{s}$, we have that $\mathpzc{s}g \circ \mathpzc{s}f$ is the identity on $\mathpzc{s}X$. Similarly, $\mathpzc{s}f \circ \mathpzc{s}g$ is the identity on $\mathpzc{s}Y$, and hence $\mathpzc{s}f$ is a homeomorphism.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\lambda_X"'] & \mathpzc{s}Y \arrow[dl, shift left, "\mathpzc{s}g"] \\
\mathpzc{s}X \arrow[ur, shift left, "\mathpzc{s}f"] & Y \arrow[u, "\lambda_Y"'] \arrow[l, "g"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\lambda_X"'] & \mathpzc{s}Y \arrow[dl, shift left, "g'"] \\
\mathpzc{s}X \arrow[ur, shift left, "\mathpzc{s}f"] & Y \arrow[u, "\lambda_Y"']
\end{tikzcd}$$

Conversely, suppose $\mathpzc{s}f$ is a homeomorphism (see the right diagram, where we identify $\mathpzc{s} = \mathpzc{s}\mathpzc{s}$). Then there is a continuous map $g' : \mathpzc{s}Y \to \mathpzc{s}X$ which is inverse to $\mathpzc{s}f$. Let $g : Y \rightsquigarrow X$ be given by $g = g' \circ \lambda_Y$. By identifying $\mathpzc{s} = \mathpzc{s}\mathpzc{s}$, we have that $\mathpzc{s}g' = g'$ and $\mathpzc{s}\lambda_Y$ is the identity. Therefore, since g' is the inverse of $\mathpzc{s}f$,

$$g \bullet f = \mathpzc{s}g \circ f = \mathpzc{s}g' \circ \mathpzc{s}\lambda_Y \circ f = g' \circ f = g' \circ \mathpzc{s}f \circ \lambda_X = \lambda_X.$$

Similarly, $f \bullet g = \mathpzc{s}f \circ g = \mathpzc{s}f \circ g' \circ \lambda_Y = \lambda_Y$. Consequently, g is the inverse of f, so f is a $\mathbf{Top_S}$-isomorphism.

Our aim is to show that $\mathbf{Top_S}$ is dually equivalent to the full subcategory $\mathbf{SMT_P}$ of $\mathbf{MT_P}$ consisting of spatial MT-algebras. To define a functor from $\mathbf{SMT_P}^{\mathrm{op}}$ to $\mathbf{Top_S}$, we need the following lemma, where η is the counit of MT-duality (see Theorem 2.11).

6.4. Lemma. *Suppose $f : M \to N$ is a proximity morphism between spatial MT-algebras. Define $at_{\mathrm{S}}\, f : at\, N \rightsquigarrow at\, M$ by*

$$at_{\mathrm{S}}\, f(y) = \{\eta_M(a) \mid a \in \mathcal{O}M,\ y \leq f(a)\}$$

for each $y \in at\, N$. Then at_{S} is a sober map.

Proof. Since f is a proximity morphism, its restriction $f|_{\mathcal{O}M} : \mathcal{O}M \to \mathcal{O}N$ is a frame morphism, so $pt(f|_{\mathcal{O}M}) : pt\,\mathcal{O}N \to pt\,\mathcal{O}M$ is a continuous map, as is $\theta : at\, N \to pt\,\mathcal{O}N$ by Lemma 5.2(1). Because M is spatial, $\eta_M : \mathcal{O}M \to \Omega\, at\, M$ is an isomorphism, so there is a homeomorphism

$$\psi := pt(\eta_M^{-1}) : pt\,\mathcal{O}M \to pt\,\Omega\, at\, M = \mathpzc{s}\, at\, M.$$

The composition

$$\mathit{at}\, N \xrightarrow{\theta} \mathit{pt}\,\mathcal{O}N \xrightarrow{\mathit{pt}(f|_{\mathcal{O}M})} \mathit{pt}\,\mathcal{O}M \xrightarrow{\psi} \mathscr{s}\,\mathit{at}\, M$$

is clearly a sober map. But

$$\begin{aligned}\psi \circ \mathit{pt}(f|_{\mathcal{O}M}) \circ \theta(y) &= \psi \circ \mathit{pt}(f|_{\mathcal{O}M})(\{b \in \mathcal{O}N \mid y \leq b\}) \\ &= \psi(\{a \in \mathcal{O}M \mid y \leq f(a)\}) \\ &= \{\eta_M(a) \mid a \in \mathcal{O}M,\, y \leq f(a)\} \\ &= \mathit{at}_{\mathrm{S}}\, f(y)\end{aligned}$$

for each $y \in \mathit{at}\, N$, completing the proof. ∎

The inverse of $\psi : \mathit{pt}\,\mathcal{O}M \to \mathscr{s}\,\mathit{at}\, M$ is given by $\mathscr{s}\theta$ (where $\theta : \mathit{at}\, M \to \mathit{pt}\,\mathcal{O}M$ is defined in Lemma 5.2). Indeed, recalling the counit $\sigma : 1_{\mathbf{Frm}} \to \Omega \circ \mathit{pt}$ from Section 2, for $x \in \mathit{pt}\,\mathcal{O}M$,

$$\begin{aligned}(\mathscr{s}\theta \circ \psi)(x) &= \mathscr{s}\theta(\eta_M[x]) \\ &= \{U \in \Omega\, \mathit{pt}\,\mathcal{O}M \mid \theta^{-1}(U) \in \eta_M[x]\} \\ &= \{U \in \Omega\, \mathit{pt}\,\mathcal{O}M \mid \exists u \in x : \theta^{-1}(U) = \eta_M(u)\} \\ &= \{U \in \Omega\, \mathit{pt}\,\mathcal{O}M \mid \exists u \in x : U = \sigma_{\mathcal{O}M}(u)\} \\ &= \{U \in \Omega\, \mathit{pt}\,\mathcal{O}M \mid x \in U\} \\ &= \lambda_{\mathit{pt}\,\mathcal{O}M}(x) = x,\end{aligned} \tag{♥}$$

where the last equality is true by identifying $\mathscr{s} = \mathscr{s}\mathscr{s}$ since $\mathit{pt}\,\mathcal{O}M$ is sober. Moreover, for each $x \in \mathit{at}\, M$,

$$\begin{aligned}\psi \circ \theta(x) &= \psi({\uparrow}x \cap \mathcal{O}M) \\ &= \eta_M[{\uparrow}x \cap \mathcal{O}M] \\ &= \{\eta_M(a) \mid a \in \mathcal{O}M,\, x \leq a\} \\ &= \lambda_{\mathit{at}\, M}.\end{aligned}$$

Thus, for each $y \in \mathscr{s}\,\mathit{at}\, M$,

$$y = \mathscr{s}(\lambda_{\mathit{at}\, M})(y) = \mathscr{s}(\psi \circ \theta)(y) = \psi \circ \mathscr{s}\theta(y).$$

6.5. Proposition. $\mathit{at}_{\mathrm{S}} : \mathbf{SMT_P}^{\mathrm{op}} \to \mathbf{Top_S}$ *is a functor.*

Proof. For a spatial MT-algebra M, let $\mathit{at}_{\mathrm{S}}\, M = \mathit{at}\, M$ and for a proximity morphism $f : M \to N$ between spatial MT-algebras, let $\mathit{at}_{\mathrm{S}}\, f : \mathit{at}\, N \rightsquigarrow \mathit{at}\, M$ be defined as in Lemma 6.4. Then at_{S} is well defined on both objects and morphisms. Moreover, for proximity morphisms $f : M_1 \to M_2$ and $g : M_2 \to M_3$, by Lemma 6.4 and (♥),

$$\begin{aligned}\mathit{at}_{\mathrm{S}}\, f \bullet \mathit{at}_{\mathrm{S}}\, g &= \mathscr{s}\,\mathit{at}_{\mathrm{S}}\, f \circ \mathit{at}_{\mathrm{S}}\, g \\ &= \psi_{M_1} \circ \mathit{pt}(f|_{\mathcal{O}M_1}) \circ \mathscr{s}\theta_{M_2} \circ \psi_{M_2} \circ \mathit{pt}(g|_{\mathcal{O}M_2}) \circ \theta_{M_3} \\ &= \psi_{M_1} \circ \mathit{pt}(g|_{\mathcal{O}M_2} \circ f|_{\mathcal{O}M_1}) \circ \theta_{M_3} \\ &= \psi_{M_1} \circ \mathit{pt}((g \star f)|_{\mathcal{O}M_1}) \circ \theta_{M_3} \\ &= \mathit{at}_{\mathrm{S}}(g \star f).\end{aligned}$$

Furthermore, for $y \in at_{\mathrm{S}}\, M$,

$$\begin{aligned} at_{\mathrm{S}}\, 1_M(y) &= \{\eta_M(a) \mid a \in \mathcal{O}M,\, y \leq 1_M(a)\} \\ &= \{\eta_M(a) \mid a \in \mathcal{O}M,\, y \leq a\} \\ &= \{\eta_M(a) \mid a \in \mathcal{O}M,\, y \in \eta_M(a)\} \\ &= F_y = \lambda_{at_{\mathrm{S}}\, M}(y). \end{aligned}$$

Thus, $at_{\mathrm{S}} : \mathbf{SMT_P}^{\mathrm{op}} \to \mathbf{Top_S}$ is a well-defined functor. ∎

To define a functor in the other direction, we require the following:

6.6. Lemma. *If $f : M_1 \to M_2$ is a proximity morphism and $g : M_2 \to M_3$ is an MT-morphism, then $g \circ f : M_1 \to M_3$ is a proximity morphism.*

Proof. Since g is an MT-morphism, it satisfies (P1)–(P3), and hence so does the composition $g \circ f$. For (P4), observe that

$$\begin{aligned} gf(a) &= g\left(\bigvee\{f(x) \mid x \in \mathcal{LC}(M_1),\, x \leq a\}\right) \\ &= \bigvee\{g(f(x)) \mid x \in \mathcal{LC}(M_1),\, x \leq a\} \end{aligned}$$

for each $a \in M_1$. Thus, $g \circ f$ is a proximity morphism. ∎

Let M, N be MT-algebras. If $h : \mathcal{O}M \to \mathcal{O}N$ is a frame morphism, then h lifts to a proximity morphism given by the following composition in $\mathbf{MT_P}$:

$$M \xrightarrow{\varphi_M} \mathcal{F}\mathcal{O}M \xrightarrow{\mathcal{F}h} \mathcal{F}\mathcal{O}N \xrightarrow{\zeta_N} N.$$

For a topological space X,

$$\Omega(\lambda_X) = \mathcal{P}(\lambda_X)|_{\mathcal{O}\,\mathcal{P}\,X} : \mathcal{O}\,\mathcal{P}\,\mathcal{s}X \to \mathcal{O}\,\mathcal{P}\,X$$

is a frame isomorphism. Therefore, the frame isomorphism $\Omega(\lambda_X)^{-1}$ lifts to a proximity morphism $h_X : \mathcal{P}\,X \to \mathcal{P}\,\mathcal{s}X$, which is a proximity isomorphism by Proposition 4.15(1).

6.7. Definition. For a topological space X, let $h_X : \mathcal{P}\,X \to \mathcal{P}\,\mathcal{s}X$ be the lift of $\Omega(\lambda_X)^{-1}$ described above.

Recall from Section 3 that for proximity morphisms f, g and a locally closed element x, we have $(g \star f)(x) = gf(x)$. Therefore, for $D \in \mathcal{LC}\,\mathcal{P}\,X$, by Lemma 4.11 we have

$$\begin{aligned} h_X(D) &= \zeta_{\mathcal{P}\,Y} \circ \mathcal{F}\Omega(\lambda_X)^{-1} \circ \varphi_{\mathcal{P}\,X}(D) \\ &= \zeta_{\mathcal{P}\,Y} \circ \mathcal{F}\Omega(\lambda_X)^{-1}(D) \\ &= \zeta_{\mathcal{P}\,Y}(\mathcal{B}\Omega(\lambda_X)^{-1}(D)) \\ &= \mathcal{B}\Omega(\lambda_X)^{-1}(D). \end{aligned} \tag{♣}$$

6.8. Lemma. *If $f : X \rightsquigarrow Y$ is a sober map then $\mathcal{P}_{\mathrm{P}}\, f := \mathcal{P}\, f \circ h_Y$ is a proximity morphism from $\mathcal{P}\,Y$ to $\mathcal{P}\,X$.*

Proof. By definition, f is a continuous map from X to $\mathcal{s}Y$. This means that the map $\mathcal{P}\, f = f^{-1} : \mathcal{P}\,\mathcal{s}Y \to \mathcal{P}\,X$ is an MT-morphism. Consequently, $\mathcal{P}\, f \circ h_Y : \mathcal{P}\,Y \to \mathcal{P}\,X$ is a proximity morphism by Lemma 6.6. ∎

6.9. Lemma. *Let $f : M \to N$ and $g : M \to N$ be proximity morphisms. If $f(u) = g(u)$ for all $u \in \mathcal{O}M$, then $f = g$.*

Proof. Let $x \in \mathcal{L}\mathcal{C}M$. Then $x = u \wedge \neg v$ for some $u, v \in \mathcal{O}M$. Therefore,

$$f(x) = f(u \wedge \neg v) = f(u) \wedge \neg f(v) = g(u) \wedge \neg g(v) = g(u \wedge \neg v) = g(x).$$

Thus, for $a \in M$,

$$f(a) = \bigvee\{f(x) \mid x \in \mathcal{L}\mathcal{C}M,\, x \leq a\} = \bigvee\{g(x) \mid x \in \mathcal{L}\mathcal{C}M,\, x \leq a\} = g(a).$$

Consequently, $f = g$. ∎

We point out that for $U \in \Omega Y$,

$$\begin{aligned} \mathscr{P}_{\mathrm{P}}\, f(U) &= (\mathscr{P}\, f \circ h_Y)(U) \\ &= f^{-1}(\Omega(\lambda_Y)^{-1}(U)) \\ &= \{x \in X \mid f(x) \in \Omega(\lambda_Y)^{-1}(U)\} \\ &= \{x \in X \mid U \in f(x)\}. \end{aligned} \tag{$\spadesuit$}$$

This will be used in what follows.

6.10. Proposition. *$\mathscr{P}_{\mathrm{P}} : \mathbf{Top_S} \to \mathbf{SMT_P}^{\mathrm{op}}$ is a functor.*

Proof. For $X \in \mathbf{Top_S}$, let $\mathscr{P}_{\mathrm{P}}\, X = \mathscr{P}\, X$ and for a sober map $f : X \rightsquigarrow Y$, let $\mathscr{P}_{\mathrm{P}}\, f : \mathscr{P}_{\mathrm{P}}\, Y \to \mathscr{P}_{\mathrm{P}}\, X$ be defined as in Lemma 6.8. Then $\mathscr{P}_{\mathrm{P}}$ is well defined both on objects and morphisms. For sober maps $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$, we show that $\mathscr{P}_{\mathrm{P}}(g \bullet f) = \mathscr{P}_{\mathrm{P}}\, f \star \mathscr{P}_{\mathrm{P}}\, g$. By Lemma 6.9, it suffices to show that they agree on open elements. Let $U \in \mathcal{O}\,\mathscr{P}\, Z = \Omega Z$. Then

$$\mathscr{P}_{\mathrm{P}}(g \bullet f)(U) = \mathscr{P}_{\mathrm{P}}(\mathscr{s}g \circ f)(U) = \mathscr{P}(\mathscr{s}g \circ f) \circ h_Z(U) = \mathscr{P}\, f \circ \mathscr{P}\, \mathscr{s}g \circ h_Z(U)$$

and since $\star$ is usual composition on open elements,

$$\mathscr{P}_{\mathrm{P}}\, f \star \mathscr{P}_{\mathrm{P}}\, g(U) = \mathscr{P}_{\mathrm{P}}\, f \circ \mathscr{P}_{\mathrm{P}}\, g(U) = \mathscr{P}\, f \circ h_Y \circ \mathscr{P}\, g \circ h_Z(U).$$

Thus, it is enough to show that $h_Y \circ \mathscr{P}\, g(V) = \mathscr{P}(\mathscr{s}g(V))$ for all $V \in \mathcal{O}\,\mathscr{P}\, \mathscr{s}Z = \Omega \mathscr{s}Z$. Using ($\clubsuit$), we have

$$h_Y \circ \mathscr{P}\, g(V) = \Omega\lambda_Y^{-1}(g^{-1}(V)) = (\mathscr{s}g)^{-1}(V) = \mathscr{P}\, \mathscr{s}g(V),$$

where the second equality holds because

$$\begin{aligned} z \in \Omega(\lambda_Y)((\mathscr{s}g)^{-1}(V)) &\iff \lambda_Y(z) \in (\mathscr{s}g)^{-1}(V) \iff \mathscr{s}g(\lambda_Y(z)) \in V \\ &\iff g(z) \in V \iff z \in g^{-1}(V). \end{aligned}$$

Finally, for $W \in \mathcal{O}\,\mathscr{P}\, X$, by ($\spadesuit$),

$$\mathscr{P}_{\mathrm{P}}\, \lambda_X(W) = \{x \in X \mid W \in \lambda_X(x)\} = \{x \in X \mid x \in W\} = W = \Omega(1_{\mathscr{P}\, X})(W).$$

Thus, by Lemma 6.9, $\mathscr{P}_{\mathrm{P}}\, \lambda_X = 1_{\mathscr{P}\, X}$. ∎

We next connect **Top** with $\mathbf{Top_S}$ and **MT** with $\mathbf{MT_P}$.

6.11. PROPOSITION.

(1) $\Lambda : \mathbf{Top} \to \mathbf{Top_S}$ *is a functor given by* $\Lambda X = X$ *for each topological space* X *and* $\Lambda f = \lambda_Y \circ f$ *for each continuous map* $f : X \to Y$.

(2) $\Gamma : \mathbf{MT} \to \mathbf{MT_P}$ *is a functor given by* $\Gamma M = M$ *for each MT-algebra* M *and* $\Gamma g = g \circ 1_N$ *for each MT-morphism* $g : M \to N$.

PROOF. (1) It is sufficient to show that Λ preserves composition and identities. The latter is immediate since $\Lambda 1_X = \lambda_X \circ 1_X = \lambda_X$. For composition, let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Then

$$\Lambda g \bullet \Lambda f = (\lambda_Z \circ g) \bullet (\lambda_Y \circ f) = \mathscr{s}\lambda_Z \circ \mathscr{s}g \circ \lambda_Y \circ f = \mathscr{s}g \circ \lambda_Y \circ f = \lambda_Z \circ g \circ f = \Lambda(g \circ f),$$

where the third equality holds because $\mathscr{s}\lambda_Z$ is the identity and the fourth because λ is a natural transformation.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda_X"] & Y \arrow[r, "g"] \arrow[d, "\lambda_Y"] & Z \arrow[d, "\lambda_Z"] \\ \mathscr{s}X \arrow[r, "\mathscr{s}f"] & \mathscr{s}Y \arrow[r, "\mathscr{s}g"] & \mathscr{s}Z \arrow[loop right, "\mathscr{s}\lambda_Z"] \end{tikzcd}$$

(2) Again, it is sufficient to show that Γ preserves composition and identities. For an MT-algebra M, let I_M be the identity in **MT** and 1_M the identity in $\mathbf{MT_P}$. Then $\Gamma I_M = I_M \circ 1_M = 1_M$. Let $f : M_1 \to M_2$ and $g : M_2 \to M_3$ be MT-mophisms. Then, for $a \in M_1$,

$$\Gamma(g \circ f)(a) = g(f(1_{M_3}(a))) = \bigvee \{g(f(x)) \mid x \in \mathscr{LC}M,\, x \leq a\} = (g \star f)(a).$$

Thus, Γ is a functor. ■

6.12. LEMMA. *For a continuous map* $f : X \to \mathscr{s}Y$, *the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "\varepsilon_X"] \arrow[d, "f"'] & \mathit{at}\,\mathscr{P}\,X \arrow[r, "\lambda_{\mathit{at}\mathscr{P}X}"] \arrow[d, "\mathit{at}\mathscr{P}f"] & \mathscr{s}\,\mathit{at}\,\mathscr{P}\,X \arrow[d, "\mathscr{s}\mathit{at}\mathscr{P}f"] \\ \mathscr{s}Y \arrow[r, "\varepsilon_{\mathscr{s}Y}"'] & \mathit{at}\,\mathscr{P}\,\mathscr{s}Y \arrow[r, "\lambda_{\mathit{at}\mathscr{P}\mathscr{s}Y}"'] & \mathscr{s}\,\mathit{at}\,\mathscr{P}\,\mathscr{s}Y \end{tikzcd}$$

PROOF. The left square commutes because $\varepsilon : 1_{\mathbf{Top}} \to \mathit{at}\,\mathscr{P}$ is a natural transformation (see Theorem 2.11). The right square commutes because applying the functor $\mathit{at}\,\mathscr{P}$ to the natural transformation $\lambda : 1_{\mathbf{Top}} \to \mathscr{s}$ yields a natural transformation $\lambda \circ (\mathit{at}\,\mathscr{P}) : \mathit{at}\,\mathscr{P} \to \mathscr{s}\,\mathit{at}\,\mathscr{P}$. ■

6.13. Lemma. *Let $f : X \to \mathscr{s}Y$ be a continuous map.*

(1) $\mathit{at}_{\mathrm{S}}\, \mathscr{P}_{\mathrm{P}}\, f(x) = \mathit{at}\, \mathscr{P}\, f(x)$ *for all* $x \in \mathit{at}\, \mathscr{P}\, X$*.*

(2) $\mathscr{s}\, \mathit{at}_{\mathrm{S}}\, \mathscr{P}_{\mathrm{P}}\, f = \mathscr{s}\, \mathit{at}\, \mathscr{P}\, f$*.*

Proof. Since (2) follows from (1), it is sufficient to prove (1). Let $x \in \mathit{at}\, \mathscr{P}\, X$. Then $x = \varepsilon(x') = \{x'\}$ for a unique $x' \in X$. Thus, $\mathit{at}\, \mathscr{P}\, f(x) = \varepsilon f(x')$. Moreover, $\mathscr{P}_{\mathrm{P}}\, f(a) = f^{-1}\Omega(\lambda_{\mathscr{s}Y})^{-1}(a) = f^{-1}(a)$ for every $a \in \mathscr{O}\, \mathscr{P}\, \mathscr{s}Y$ since $\lambda_{\mathscr{s}Y}$ is the identity on $\mathscr{s}Y$. Hence,

$$\begin{aligned}
\mathit{at}_{\mathrm{S}}\, \mathscr{P}_{\mathrm{P}}\, f(x) &= \{\eta_M(a) \mid a \in \mathscr{O}\, \mathscr{P}\, \mathscr{s}Y,\ x \le \mathscr{P}_{\mathrm{P}}\, f(a)\} \\
&= \{\eta_M(a) \mid a \in \mathscr{O}\, \mathscr{P}\, \mathscr{s}Y,\ x \le f^{-1}(a)\} \\
&= \{\eta_M(a) \mid a \in \mathscr{O}\, \mathscr{P}\, \mathscr{s}Y,\ \{x'\} \subseteq f^{-1}(a)\} \\
&= \{\eta_M(a) \mid a \in \mathscr{O}\, \mathscr{P}\, \mathscr{s}Y,\ f(x') \in a\} \\
&= \varepsilon f(x').
\end{aligned}$$

Consequently, $\mathit{at}_{\mathrm{S}}\, \mathscr{P}_{\mathrm{P}}\, f(x) = \varepsilon f(x') = \mathit{at}\, \mathscr{P}\, f(x)$. ∎

6.14. Theorem. $\mathbf{Top_S}$ *is equivalent to* $\mathbf{SMT_P}^{\mathrm{op}}$.

Proof. We first define $\widehat{\varepsilon} : 1_{\mathbf{Top_S}} \to \mathit{at}_{\mathrm{S}}\, \mathscr{P}_{\mathrm{P}}$ by setting $\widehat{\varepsilon}_X = \lambda_{\mathit{at}\, \mathscr{P}\, X} \circ \varepsilon_X$ for each $X \in \mathbf{Top_S}$. By Proposition 6.11(1), $\widehat{\varepsilon}_X = \Lambda\varepsilon_X : X \rightsquigarrow \mathit{at}\, \mathscr{P}\, X$ is a $\mathbf{Top_S}$-isomorphism since ε_X is a homeomorphism (see Remark 6.3). We show that $\widehat{\varepsilon}$ is a natural transformation by showing that the following diagram on the left commutes in $\mathbf{Top_S}$. Using the identification $\mathscr{s} = \mathscr{s}\mathscr{s}$, this is equivalent to showing that the diagram on the right commutes in $\mathbf{Top}$:

$$\begin{tikzcd}
X \arrow[r, rightsquigarrow, "\widehat{\varepsilon}_X"] \arrow[d, rightsquigarrow, "f"'] & \mathit{at}\, \mathscr{P}\, X \arrow[d, rightsquigarrow, "\mathit{at}_{\mathrm{S}}\, \mathscr{P}_{\mathrm{P}}\, f"] \\
Y \arrow[r, rightsquigarrow, "\widehat{\varepsilon}_Y"'] & \mathit{at}\, \mathscr{P}\, Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\varepsilon_X"] \arrow[d, "f"'] & \mathit{at}\, \mathscr{P}\, X \arrow[r, "\lambda_{\mathit{at}\mathscr{P}X}"] & \mathscr{s}\, \mathit{at}\, \mathscr{P}\, X \arrow[d, "\mathscr{s}\mathit{at}_{\mathrm{S}}\mathscr{P}_{\mathrm{P}}f"] \\
\mathscr{s}Y \arrow[r, "\varepsilon_{\mathscr{s}Y}"'] & \mathit{at}\, \mathscr{P}\, \mathscr{s}Y \arrow[r, "\lambda_{\mathit{at}\mathscr{P}\mathscr{s}Y}"'] & \mathscr{s}\, \mathit{at}\, \mathscr{P}\, \mathscr{s}Y
\end{tikzcd}$$

By Lemma 6.12, it suffices to show that $\mathscr{s}\, \mathit{at}\, \mathscr{P}\, f = \mathscr{s}\, \mathit{at}_{\mathrm{S}}\, \mathscr{P}_{\mathrm{P}}\, f$, which is given by Lemma 6.13.

We next define $\widehat{\eta} : 1_{\mathbf{SMT_P}^{\mathrm{op}}} \to \mathscr{P}_{\mathrm{P}}\, \mathit{at}_{\mathrm{S}}$ by setting $\widehat{\eta}_M = \eta_M \circ 1_M$ for each $M \in \mathbf{SMT_P}$. By Proposition 6.11(2), $\widehat{\eta}_M = \Gamma\eta_M : M \to \mathscr{P}\, \mathit{at}\, M$ is a proximity isomorphism since η_M is an isomorphism of MT-algebras (because M is spatial; see Theorem 2.11). We show that $\widehat{\eta}$ is a natural transformation by showing that the following diagram commutes in $\mathbf{SMT_P}$:

$$\begin{tikzcd}
M \arrow[r, "\widehat{\eta}_M"] \arrow[d, "g"'] & \mathscr{P}\, \mathit{at}\, M \arrow[d, "\mathscr{P}_{\mathrm{P}}\mathit{at}_{\mathrm{S}}g"] \\
N \arrow[r, "\widehat{\eta}_N"'] & \mathscr{P}\, \mathit{at}\, N
\end{tikzcd}$$

By Lemma 6.9, it is enough to show that the diagram commutes on open elements. Let $u \in \mathcal{O}M$. By (♠),

$$\begin{aligned}\mathscr{P}_{\mathrm{P}}\, \mathit{at}_{\mathrm{S}}\, g \star \widehat{\eta}_M(u) &= \mathscr{P}_{\mathrm{P}}\, \mathit{at}_{\mathrm{S}}\, g \circ \widehat{\eta}_M(u)\\ &= \mathscr{P}_{\mathrm{P}}\, \mathit{at}_{\mathrm{S}}\, g(\eta_M(u))\\ &= \{y \in \mathit{at}\, N \mid \eta_M(u) \in \mathit{at}_{\mathrm{S}}\, g(y)\}.\end{aligned}$$

Moreover,

$$\eta_N \star g(u) = \eta_N \circ g(u) = \{y \in \mathit{at}\, N \mid y \leq g(u)\}.$$

Thus, it is enough to recall from Lemma 6.4 that

$$\mathit{at}_{\mathrm{S}}\, g(y) = \{\eta_M(u) \mid u \in \mathcal{O}M,\, y \leq g(u)\}.$$

Hence, $\mathbf{Top_S}$ is equivalent to $\mathbf{SMT_P}^{\mathrm{op}}$. ■

Let $\mathbf{TDTop_S}$ be the full subcategory of $\mathbf{Top_S}$, and let $\mathbf{STDMT_P}$ be the full subcategory of $\mathbf{SMT_P}$ consisting of T_D-algebras. We have:

6.15. Corollary. *$\mathbf{TDTop_S}$ is equivalent to $\mathbf{STDMT_P}^{\mathrm{op}}$.*

Proof. By Theorem 2.13(2), $X \in \mathbf{TDTop_S}$ implies $\mathscr{P}_{\mathrm{P}}\, X = \mathscr{P}\, X \in \mathbf{STDMT_P}^{\mathrm{op}}$, and $M \in \mathbf{STDMT_P}^{\mathrm{op}}$ implies $\mathit{at}_{\mathrm{S}}\, M = \mathit{at}\, M \in \mathbf{TDTop_S}$. Thus, the equivalence of Theorem 6.14 restricts to an equivalence between $\mathbf{TDTop_S}$ and $\mathbf{STDMT_P}^{\mathrm{op}}$. ■

Let **TD-SFrm** be the full subcategory of **Frm** consisting of T_D-spatial frames. The equivalence of Theorem 4.19(2) restricts to yield:

6.16. Proposition. ***TD-SFrm** is equivalent to $\mathbf{STDMT_P}$.*

Proof. It suffices to show that $L \in \mathbf{TD\text{-}SFrm}$ implies $\mathscr{F} L \in \mathbf{STDMT_P}$, and $M \in \mathbf{STDMT_P}$ implies $\mathcal{O}M \in \mathbf{TD\text{-}SFrm}$. If $L \in \mathbf{TD\text{-}SFrm}$ then there exists a T_D-space X such that $L \cong \Omega X$. Since X is a T_D-space, $\mathscr{P}\, X$ is a T_D-algebra by Theorem 2.13(2), and hence $\mathscr{F}\Omega X \cong \mathscr{P}\, X$ by Proposition 4.17. Thus, $\mathscr{F} L \cong \mathscr{F}\Omega X \cong \mathscr{P}\, X$ is a spatial T_D-algebra, and hence $\mathscr{F} L \in \mathbf{STDMT_P}$. If $M \in \mathbf{STDMT_P}$ then $M \cong \mathscr{P}\, X$ for some T_D-space X, and hence $\mathcal{O}M = \Omega X \in \mathbf{TD\text{-}SFrm}$. ■

Putting together Theorem 6.14, Corollary 6.15, and Proposition 6.16, we arrive at the following commutative diagram:

$$\begin{tikzcd}
\mathbf{Top_S} \arrow[r, leftrightarrow, "\cong"] & \mathbf{SMT_P}^{\mathrm{op}} & \\
\mathbf{TDTop_S} \arrow[u, hook, "\text{full}"] \arrow[r, leftrightarrow, "\cong"] & \mathbf{STDMT_P}^{\mathrm{op}} \arrow[u, hook, "\text{full}"] \arrow[r, leftrightarrow, "\cong"] & \mathbf{TD\text{-}SFrm}^{\mathrm{op}}
\end{tikzcd}$$

Tables of relevant categories

For the reader's convenience, we conclude by listing the categories considered in this article, indicating the objects, morphisms, and where the categories appear for the first time in the body of the text.

Category	**Objects**	**Morphisms**	
Top	topological spaces	continuous maps	Section 2
Sob	sober spaces	continuous maps	Section 2
T0Top	T_0-spaces	continuous maps	Section 2
TDTop	T_D-spaces	continuous maps	Section 2
$\mathbf{Top_{LC}}$	topological spaces	locally closed maps	Definition 5.11
$\mathbf{T0Top_{LC}}$	T_0-spaces	locally closed maps	Section 5
$\mathbf{Top_{S}}$	topological spaces	sober maps	Definition 6.2
$\mathbf{TDTop_{S}}$	TD-spaces	sober maps	Corollary 6.15

Table 1: Categories of topological spaces

Category	**Objects**	**Morphisms**	
Frm	frames	frame morphisms	Section 2
$\mathbf{Frm_{D}}$	frames	D-morphisms	Section 2
SFrm	spatial frames	frame morphisms	Section 2
TD-SFrm	T_D-spatial frames	frame morphisms	Section 2
$\mathbf{TD\text{-}SFrm_{D}}$	T_D-spatial frames	D-morphisms	Section 2

Table 2: Categories of frames

Category	**Objects**	**Morphisms**	
MT	MT-algebras	MT-morphisms	Definition 2.9
$\mathbf{MT_{P}}$	MT-algebras	proximity morphisms	Theorem 3.13
$\mathbf{MT_{D}}$	MT-algebras	D-morphisms	Definition 5.14
TDMT	T_D-algebras	MT-morphisms	Definition 2.12
$\mathbf{TDMT_{P}}$	T_D-algebras	proximity morphisms	Definition 4.18
T0MT	T_0-algebras	MT-morphisms	Definition 2.12
SMT	spatial MT-algebras	MT-morphisms	Section 2
$\mathbf{SMT_{D}}$	spatial MT-algebras	D-morphisms	Definition 5.14
$\mathbf{SMT_{P}}$	spatial MT-algebras	proximity morphisms	Section 6
STDMT	spatial T_D-algebras	MT-morphisms	Section 2
$\mathbf{STDMT_{P}}$	spatial T_D-algebras	proximity morphisms	Corollary 6.15
ST0MT	spatial T_0-algebras	MT-morphisms	Section 2
$\mathbf{ST0MT_{D}}$	spatial T_0-algebras	D-morphisms	Definition 5.14

Table 3: Categories of MT-algebras

Category	**Objects**	**Morphisms**	
Bool	boolean algebras	boolean homomorphisms	Definition 4.2
DLat	bdd distr lattices	bdd lattice homomorphisms	Definition 4.2
HLat	Heyting algebras	bdd lattice homomorphisms	Definition 4.2
Heyt	Heyting algebras	Heyting homomorphisms	Definition 4.2
Int	interior algebras	int alg morphisms	Definition 2.7
Ess	essential algebras	int alg morphisms	Definition 4.1
$\mathbf{Int_C}$	interior algebras	continuous morphisms	Definition 2.7
$\mathbf{Ess_C}$	essential algebras	continuous morphisms	Definition 4.1
Cons	constructible algebras	constructible morphisms	Definition 4.4

Table 4: Other categories

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