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A NOTE ON ROBERT PARÉ'S NOTION OF UNIVERSAL COVERING CATEGORY

Dedicated to Professor Robert Paré on the occasion of his 80th birthday

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Abstract. We present what seems to be the simplest and most natural way to show that Robert Paré's definition of *universal covering category* fully agrees with the one that arises in abstract Galois theory.

1. Introduction

This paper is devoted to Robert Paré's notion of *universal covering category*, which is actually defined in [4] as an explicit construction, independently of any general notion of a covering. We present what seems to be the simplest and most natural way to show that this notion fully agrees with the one that arises in abstract Galois theory.

We begin by recalling Paré's construction (Section 2), and then by recalling the notion of covering morphism of categories, which we define as a functor that is a discrete fibration and a discrete opfibration at the same time (see Section 3, where we also explain where this definition came from).

It is well known that the category $\mathrm{Cov}(\mathbf{I})$ of coverings of a connected category $\mathbf{I}$ is equivalent to the category $\mathbf{Set}^G$ of G-sets, where G is the fundamental group of $\mathbf{I}$, which gives several equivalent definitions of a (the) universal covering of $\mathbf{I}$. Specifically, we can say that a covering morphism $P\colon \mathbf{E} \longrightarrow \mathbf{I}$ with connected $\mathbf{E}$ and $\mathbf{I}$ is a universal covering of $\mathbf{I}$ if (and only if) it satisfies one of the following equivalent conditions:

(i) for every covering morphism $F\colon \mathbf{A} \longrightarrow \mathbf{I}$, the pullback projection

$$\mathrm{pr}_1\colon \mathbf{E} \times_{\mathbf{I}} \mathbf{A} \longrightarrow \mathbf{E}$$

is a trivial covering, that is, it is isomorphic over $\mathbf{E}$ to the codiagonal morphism from a copower of $\mathbf{E}$ to $\mathbf{E}$;

(ii) $\mathbf{E}$ is simply connected, although the notion of simple connectedness has several equivalent definitions itself and some of these equivalences use Galois-theoretic arguments;

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(iii) $(\mathbf{E}, P)$ corresponds (up to an isomorphism) to G equipped with the multiplication G-action under the equivalence $\mathrm{Cov}(\mathbf{I}) \sim \mathbf{Set}^G$;

(iv) for every covering $F\colon \mathbf{A} \longrightarrow \mathbf{I}$ with non-empty $\mathbf{A}$ there exists a (not necessarily unique) functor $G\colon \mathbf{E} \longrightarrow \mathbf{A}$ with $FG = P$.

In fact there are even more equivalent conditions, e.g. involving descriptions of all possible functors $G\colon \mathbf{E} \longrightarrow \mathbf{A}$ of condition (iv) above and/or requiring connectedness of $\mathbf{A}$.

We choose to prove that Paré's construction is a covering which satisfies condition (iv). Note that simple connectedness *in a certain sense* of Paré's universal covering category is mentioned in [4], but using it would require a longer discussion on equivalences of different definitions of simple connectedness.

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2. Recalling Paré's construction

Section 1 of [4] begins with categories and functors that one can display as the diagram

$$\begin{tikzcd} \mathbf{Gpd} \arrow[r, "\text{Inclusion}"] & \mathbf{Cat} \arrow[l, bend right, "\pi_1"'] \arrow[l, bend left, "\text{Iso}"] \arrow[r, bend left, "\pi_0"] \arrow[r, bend right, "\text{Ob}"'] & \mathbf{Set} \arrow[l, "D"'] \arrow[l, bend left=70, "C"] \end{tikzcd} \tag{1}$$

in which $\pi_1 \dashv \text{Inclusion} \dashv \text{Iso}$ and $\pi_0 \dashv D \dashv \text{Ob} \dashv C$ in obvious notation. In particular, given a (small) category $\mathbf{I}$, the fundamental groupoid $\pi_1(\mathbf{I})$, which is the same as the fraction groupoid of $\mathbf{I}$, has the same objects as $\mathbf{I}$ and morphisms $I \longrightarrow J$ represented as equivalence classes of diagrams of the form

$$I = I_0 \xrightarrow{i_1^{\varepsilon_1}} I_1 \xrightarrow{i_2^{\varepsilon_2}} \cdots \xrightarrow{i_n^{\varepsilon_n}} I_n = J, \tag{2}$$

where:

- $\varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\}$;
- if $\varepsilon_k = 1$, then i_k is a morphism in $\mathbf{I}$ from I_{k-1} to I_k;
- if $\varepsilon_k = -1$, then i_k is a morphism in $\mathbf{I}$ from I_k to I_{k-1}.

The canonical functor $\mathbf{I} \longrightarrow \pi_1(\mathbf{I})$, carrying $i\colon I \longrightarrow J$ to the class of $i^1\colon I \longrightarrow J$, is denoted in [4] by Q.

Then, for a connected category $\mathbf{I}$ and a fixed object I_0 in $\mathbf{I}$, the universal covering category $U\mathbf{I}$ of $\mathbf{I}$ is defined in [4] as follows:

- an object of $U\mathbf{I}$ is a morphism $p\colon I_0 \longrightarrow I$ in $\pi_1(\mathbf{I})$;
- a morphism $i\colon p \longrightarrow q$ in $U\mathbf{I}$ is a morphism i of $\mathbf{I}$ making the diagram

$$\begin{tikzcd} & I_0 \arrow[dl, "p"'] \arrow[dr, "q"] & \\ I \arrow[rr, "Q(i)"'] & & J \end{tikzcd} \tag{3}$$

  commute.

This is precisely the comma category $(I_0 \downarrow Q)$. We will rather present objects of $U\mathbf{I}$ as pairs (I, p) and write $i\colon (I,p) \longrightarrow (J,q)$ instead of $i\colon p \longrightarrow q$. In particular, it is convenient to say that the canonical functor $U\mathbf{I} \longrightarrow \mathbf{I}$ denoted in [4] by P is defined by

$$P(i\colon (I,p) \longrightarrow (J,q)) = (i\colon I \longrightarrow J). \tag{4}$$

Note also that $U\mathbf{I}$ is obviously connected; moreover, as observed in [4], it is simply connected in the sense that its fundamental groupoid is equivalent to the trivial group.

3. Coverings of categories

Covering morphisms, in the sense of G. Janelidze's Galois theory (see e.g. [1] and references therein), of small categories with respect to the connected component functor were first characterized by S. Lack (unpublished). According to this characterization, we can define them as follows:

3.1. Definition. A functor $F\colon \mathbf{A} \longrightarrow \mathbf{I}$ is said to be a *covering* (morphism of categories) if it is a discrete fibration and a discrete opfibration at the same time, that is, if it satisfies the following conditions:

(a) for every morphism $i\colon I \longrightarrow J$ in $\mathbf{I}$ and every object B in $\mathbf{A}$ with $F(B) = J$ there exists a unique morphism $\alpha\colon A \longrightarrow B$ with $F(\alpha) = i$;

(b) for every morphism $i\colon I \longrightarrow J$ in $\mathbf{I}$ and every object A in $\mathbf{A}$ with $F(A) = I$ there exists a unique morphism $\alpha\colon A \longrightarrow B$ with $F(\alpha) = i$.

Note that this definition also agrees with the definitions of a covering morphism of simplicial sets and of groupoids given e.g. in [3] (for the groupoid case see also [2]). The definition can be re-phrased by saying that, in the situation

$$\begin{tikzcd} A \arrow[r, dotted, "\alpha"] \arrow[d, "F"'] & B \arrow[d, "F"] \\ I \arrow[r, "i"'] & J \end{tikzcd} \tag{5}$$

(with any fixed i and $F(\alpha) = i$), the object A uniquely determines the pair (B, α) and the object B uniquely determines the pair (A, α). In particular, this gives:

3.2. LEMMA. *Every covering* $F\colon \mathbf{A} \longrightarrow \mathbf{I}$ *determines a functor* $\overline{F}\colon \mathbf{I} \longrightarrow \mathrm{Iso}(\mathbf{Set})$ *having* $\overline{F}(I) = F^{-1}(I)$ *and* $\overline{F}(i)(A) = B \Leftrightarrow \exists_{\alpha\colon A \longrightarrow B}\, F(\alpha) = i$ *for each morphism* i *in* $\mathbf{I}$.

Then, by the universal property of the canonical functor $Q\colon \mathbf{I} \longrightarrow \pi_1(\mathbf{I})$, the functor $\overline{F}$ factors through it, and we obtain:

3.3. COROLLARY. *Every covering* $F\colon \mathbf{A} \longrightarrow \mathbf{I}$ *determines a functor* $\tilde{F}\colon \pi_1(\mathbf{I}) \longrightarrow \mathrm{Iso}(\mathbf{Set})$ *having* $\tilde{F}(I) = F^{-1}(I)$ *and* $\tilde{F}Q(i)(A) = B \Leftrightarrow \exists_{\alpha\colon A \longrightarrow B}\, F(\alpha) = i$ *for each morphism* i *in* $\mathbf{I}$.

4. P is the universal covering

Recall again that the ambient category $\mathbf{I}$ is always assumed to be connected. This section is devoted to

4.1. THEOREM. $P\colon U\mathbf{I} \longrightarrow \mathbf{I}$ *is the universal covering of* $\mathbf{I}$.

PROOF. First we will show that $P\colon U\mathbf{I} \longrightarrow \mathbf{I}$ is a covering. Let $i\colon I \longrightarrow J$ be a morphism in $\mathbf{I}$, and let (J, q) be an object in $U\mathbf{I}$ so that $P(J, q) = J$. Then $i\colon (I, (Qi)^{-1}q) \longrightarrow (J, q)$ is the unique morphism in $U\mathbf{I}$ with codomain (J, q) such that $P(i) = i$. Similarly, given (I, p) in $U\mathbf{I}$, $i\colon (I, p) \longrightarrow (J, (Qi)p)$ is the unique morphism with domain (I, p) such that $P(i) = i$. Therefore P is a covering of $\mathbf{I}$ by Definition 3.1.

Now it remains to show that for every covering $F\colon \mathbf{A} \longrightarrow \mathbf{I}$ with non-empty $\mathbf{A}$ there exists a functor $G\colon U\mathbf{I} \longrightarrow \mathbf{A}$ with $FG = P$.

First we observe that there exists an object A_0 in $\mathbf{A}$ with $F(A_0) = I_0$. Indeed, take any object A; since $\mathbf{I}$ is connected, there exists a morphism $s\colon F(A) \longrightarrow I_0$ in $\pi_1(\mathbf{I})$, and we can take $A_0 = \tilde{F}(s)(A)$.

Then we fix such an object A_0 and we can define $G\colon U\mathbf{I} \longrightarrow \mathbf{A}$ by

$$G(i\colon (I, p) \longrightarrow (J, q)) = (\alpha\colon \tilde{F}(p)(A_0) \longrightarrow \tilde{F}(q)(A_0)), \tag{6}$$

where α is the unique morphism in $\mathbf{A}$ with the domain $\tilde{F}(p)(A_0)$ and $F(\alpha) = i$, well defined since:

- $F(\tilde{F}(p)(A_0)) = (\text{codomain of } p) = I$;
- as follows from Corollary 3.3, the codomain of α is $\tilde{F}Q(i)\tilde{F}(p)(A_0)$, but

  $$\tilde{F}Q(i)\tilde{F}(p)(A_0) = \tilde{F}(Q(i)p)(A_0) = \tilde{F}(q)(A_0),$$

  since i being a morphism from (I, p) to (J, q) means that $Q(i)p = q$.

The fact that G preserves identity morphisms and composition immediately follows from the uniqueness of α above, and we have $FG = P$ by our definition of G. ■

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