

ON EXTENDED FROBENIUS STRUCTURES

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ABSTRACT. A classical result in quantum topology is that oriented 2-dimensional topological quantum field theories (2-TQFTs) are fully classified by commutative Frobenius algebras. In 2006, Turaev and Turner introduced additional structure on Frobenius algebras, forming what are called extended Frobenius algebras, to classify 2-TQFTs in the unoriented case. This work provides a systematic study of extended Frobenius algebras in various settings: over a field, in a monoidal category, and in the framework of monoidal functors. Numerous examples, classification results, and general constructions of extended Frobenius algebras are established.

1. Introduction

The goal of this work is to study extended Frobenius algebras in various settings. Before providing further context, note that linear structures here are over an algebraically closed field $\mathbb{k}$ of characteristic zero, unless stated otherwise. Categories $\mathcal{C}$ are assumed to be locally small, and will have further structure as specified below. We will also read graphical diagrams from top to bottom.

We are motivated by the vast program on producing *topological quantum field theories* (TQFTs), which are categorical constructions that yield topological invariants. Loosely speaking, a TQFT is a (higher) functor from a (higher) category of topological data to a (higher) target category with extra structure. In the 2-dimensional case, 2-TQFTs are symmetric monoidal functors from a symmetric monoidal category of 1-manifolds and 2-cobordisms to a choice of a symmetric monoidal category $\mathcal{C}$. Often, $\mathcal{C}$ is taken to be the symmetric monoidal category $\mathbf{Vec}$ of $\mathbb{k}$ -vector spaces. A classical result is that a 2-TQFT with values in $\mathcal{C}$ is classified by where it sends the circle, which in the oriented setting, is a commutative Frobenius algebra in $\mathcal{C}$; see, e.g., [Koc04]. Turaev and Turner expanded this correspondence in the unoriented setting, by tacking on extra structure to Frobenius algebras to form what are called extended Frobenius algebras [TT06, Section 2].

We thank the anonymous referee for their careful feedback, time, and consideration. We would like to thank Harshit Yadav for sharing some of their graphical arguments in Appendix A. Czenky was partially supported by Simons Collaboration Grant No. 999367. Walton was partially supported by the US NSF grant #DMS-2348833, and by an AMS Claytor-Gilmer Research Fellowship.

Received by the editors 2025-03-12 and, in final form, 2025-11-03.

Transmitted by Kate Ponto. Published on 2025-11-07.

2020 Mathematics Subject Classification: 16K99, 18M15, 18M30, 57R56.

Key words and phrases: extended Frobenius algebra, extended Frobenius monoidal functor, extended Hopf algebra.

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Turaev-Turner’s 2-TQFT Result $(\star)$: Isoclasses of unoriented 2-dimensional TQFTs in $\mathbf{Vec}$ are in bijection with isomorphism classes of commutative extended Frobenius algebras over $\mathbb{k}$.

Since then, extended Frobenius algebras have appeared in many works, such as in an adaptation of $(\star)$ to compute virtual link homologies [Tub14], for an analogue of $(\star)$ for homotopy quantum field theories [Tag12], in a modification of $(\star)$ to examine linearized TQFTs [Cze24], in a categorical expansion of $(\star)$ [Oca24], and in a study of topological invariants of ribbon graphs [CL24].

We expect that extended Frobenius algebras will continue to play a crucial role in the TQFT program. Thus, we focus on the algebraic side of the program and study extended Frobenius algebras in detail—producing numerous examples, classification results, and general constructions.

We begin by taking $\mathcal{C} = \mathbf{Vec}$, hence working over the field $\mathbb{k}$. Consider the terminology below.

1.1. DEFINITION.

- (a) A Frobenius algebra over $\mathbb{k}$ is a tuple $(A, m, u, \Delta, \varepsilon)$, where (A, m, u) is an associative unital $\mathbb{k}$ -algebra, and (A, Δ, ε) is a coassociative counital $\mathbb{k}$ -coalgebra, satisfying the Frobenius law: $(a \otimes 1_A)\Delta(b) = \Delta(ab) = \Delta(a)(1_A \otimes b)$, for all $a, b \in A$. A morphism of Frobenius algebras over $\mathbb{k}$ is a morphism of the underlying $\mathbb{k}$ -algebras and of $\mathbb{k}$ -coalgebras.
- (b) [TT06, Definition 2.5] A Frobenius algebra $(A, m, u, \Delta, \varepsilon)$ is an extended Frobenius algebra over $\mathbb{k}$ if it is equipped with a morphism $\phi : A \rightarrow A$ and an element $\theta \in A$ such that:

- (i) $\phi : A \rightarrow A$ is an involution of Frobenius algebras,
- (ii) $\theta \in A$ satisfies $\phi(\theta a) = \theta a$, for all $a \in A$,
- (iii) $m(\phi \otimes \text{id}_A)\Delta(1_A) = \theta^2$.

A morphism $f : (A, \phi_A, \theta_A) \rightarrow (B, \phi_B, \theta_B)$ of extended Frobenius algebras over $\mathbb{k}$ is a morphism $f : A \rightarrow B$ of $\mathbb{k}$ -Frobenius algebras such that $f\phi_A = \phi_B f$ and $f(\theta_A) = \theta_B$.

- (c) We refer to (ϕ, θ) in part (b) as the extended structure of the underlying Frobenius algebra A , and say that A is extendable when ϕ and θ exist. We also call an extended structure (ϕ, θ) on A ϕ -trivial when $\phi = \text{id}_A$, and call it θ -trivial when $\theta = 0$.

Note that we do not assume that algebras are commutative in our work. Our first main result is the classification of extended structures for well-known examples of Frobenius algebras over $\mathbb{k}$.

1.2. THEOREM. [Propositions 2.9–2.11, 2.12–2.14, 2.16–2.17] *Take $n \geq 2$, and $\omega_n \in \mathbb{k}$ an n -th root of unity. The extended structures for the Frobenius algebras below are classified, recapped as follows.*

- (a) $\mathbb{k}$: *all extensions are ϕ -trivial.*
- (b) $\mathbb{C}$ over $\mathbb{R}$: *all extensions are ϕ -trivial or θ -trivial.*
- (c) $\mathbb{k}[x]/(x^n)$: *all extensions are ϕ trivial when n is odd, and is not extendable when n is even.*
- (d) $\mathbb{k}C_2$: *all extensions are ϕ -trivial or θ -trivial.*
- (e) $\mathbb{k}C_3$: *all extensions are ϕ -trivial or ϕ maps a generator g of C_3 to $\omega_3 g^2$.*
- (f) $\mathbb{k}C_4$: *all extensions are ϕ -trivial, or θ -trivial, or ϕ maps a generator g of C_4 to $\omega_4 g^3$.*
- (g) $\mathbb{k}(C_2 \times C_2)$: *here, ϕ maps g to $\omega_2 g'$, where g, g' are generators of $C_2 \times C_2$.*
- (h) $T_2(-1) := \mathbb{k}\langle g, x \rangle / (g^2 - 1, x^2, gx + xg)$: *all extensions are ϕ -trivial.*

Next, we move to the monoidal setting. See Section 3.1 for background material on monoidal categories $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ and on algebraic structures within $\mathcal{C}$, especially (extended) Frobenius algebras in $\mathcal{C}$. This specializes to the setting above by working in $(\text{Vec}, \otimes_{\mathbb{k}}, \mathbb{k})$. Let $\text{ExtFrobAlg}(\mathcal{C})$ denote the category of extended Frobenius algebras in $\mathcal{C}$ [Definition 3.2]. We first establish monoidal structures on $\text{ExtFrobAlg}(\mathcal{C})$. Namely, if $\mathcal{C}$ is also symmetric, then $\text{ExtFrobAlg}(\mathcal{C})$ is monoidal with $\otimes = \otimes^{\mathcal{C}}$ and $\mathbb{1} = \mathbb{1}^{\mathcal{C}}$ [Proposition 3.6]. Moreover, if $\mathcal{C}$ is additive monoidal, then $\text{ExtFrobAlg}(\mathcal{C})$ is monoidal with $\otimes$ being the biproduct of $\mathcal{C}$ and $\mathbb{1}$ being the zero object of $\mathcal{C}$ [Proposition 3.7].

Now we focus on separability in a monoidal category $\mathcal{C}$. A Frobenius algebra in $\mathcal{C}$ is *separable* if its comultiplication map is a right inverse of its multiplication map [Definition 4.2]. Separability (or *specialness*) is a widely used condition in quantum theory (see, e.g., [M03, RFFS07, HV19]). In particular, it is used to construct *state sum 2-TQFTs* [NR15]. This brings us to the result below.

1.3. PROPOSITION. [Proposition 4.3] *A separable Frobenius algebra in a monoidal category is always extendable.*

Next, we turn our attention to Hopf algebras, which also play a role in quantum theory and TQFTs (see, e.g., [KL01, BBG21, CCC22]). It is well-known that finite-dimensional Hopf algebras over $\mathbb{k}$ (or more generally, Hopf algebras over $\mathbb{k}$ with a certain integral) admit a Frobenius structure. A lesser known result is that in a *symmetric* monoidal category $\mathcal{C}$, *integral Hopf algebras* in $\mathcal{C}$ [Definition 4.6] are Frobenius [Proposition 4.8]. A graphical proof of this result is in Appendix A, which may be of independent interest to the reader. Building on this, we introduce *extended Hopf algebras* in symmetric monoidal categories [Definition 4.13], and obtain the result below.

1.4. PROPOSITION. [Proposition 4.14] *If an integral Hopf algebra in a symmetric monoidal category is extendable, then so is its corresponding Frobenius structure (via Proposition 4.8).*

Finally, we examine functors that preserve extended Frobenius algebras in monoidal categories. To start, take monoidal categories $\mathcal{C}$ and $\mathcal{C}'$, and note that a *Frobenius monoidal functor* $\mathcal{C} \rightarrow \mathcal{C}'$ [Definition 5.2] sends Frobenius algebras in $\mathcal{C}$ to those in $\mathcal{C}'$. It is also known that the separability condition is preserved when such a functor is separable [Proposition 5.4], and that such functors can be used to form higher categorical structures [Remark 5.6]. See also [DP08] and [Bĭ8, Chapter 6] for more details. Our last set of results extends the theory of Frobenius monoidal functors by introducing the notion of an *extended Frobenius monoidal functor* [Definition 5.8]. We establish that this construction satisfies many desirable conditions as discussed below.

1.5. THEOREM. [Propositions 5.9, 5.11, Theorem 5.13, Remark 5.14] *The following statements hold.*

- (a) *A separable Frobenius monoidal functor is extended Frobenius monoidal.*
- (b) *An extended Frobenius monoidal functor preserves extended Frobenius algebras.*
- (c) *The composition of two extended Frobenius monoidal functors is extended Frobenius monoidal.*
- (d) *The collections of monoidal categories and extended Frobenius monoidal functors between them forms a (2-)category (with 2-cells being certain natural transformations).*

Parts (b,c) require intricate arguments (deferred to an appendix only appearing in the ArXiv preprint of this work). Various separable Frobenius monoidal functors appear in the literature; see, e.g., [Szl05, MS10, Mor12, BT15, HLRC23, FHL23, Yad24]. So, parts (a,b) above imply that each of these constructions produce extended Frobenius algebras in monoidal categories. There are also extended Frobenius monoidal functors that are not necessarily separable [Examples 5.17, 5.18].

Organization of the article. In Section 2, we study extended Frobenius algebras over a field, proving Theorem 1.2. In Section 3, we focus on extended Frobenius algebras in a monoidal category $\mathcal{C}$, and introduce graphical calculus diagrams for such structures. We also establish monoidal structures on the category of extended Frobenius algebras in $\mathcal{C}$ in Section 3. In Section 4, we make connections to separable algebras in monoidal categories, and verify Proposition 1.3. We also strengthen ties to Hopf algebras in monoidal categories in Section 4, obtaining Proposition 1.4. The result that integral Hopf algebras are Frobenius is verified in Appendix A via graphical calculus arguments. In Section 5, we introduce extended Frobenius monoidal functors, and establish Theorem 1.5. Portions of the proof of Theorem 1.5 involve lengthy commutative diagram calculations; these are included in Appendix B, appearing only in the ArXiv preprint version of this work.

2. Extended Frobenius algebras over a field

In this section, we study extended Frobenius algebras over a field $\mathbb{k}$ as introduced in Definition 1.1. We provide many examples of, and preliminary results for, such structures in Section 2.1. Then, in Section 2.8, we establish Theorem 1.2 on the classification of extended structures for several Frobenius algebras over $\mathbb{k}$.

The roman numerals (i), (ii), (iii) here will refer to the conditions in Definition 1.1(b).

2.1. PRELIMINARY RESULTS AND EXAMPLES. We begin with some useful preliminary results on extended Frobenius algebras A over $\mathbb{k}$. First, the Frobenius law from Definition 1.1(a) implies that

$$\Delta(a) = a(1_A)^1 \otimes (1_A)^2, \quad \text{for } \Delta(1_A) := (1_A)^1 \otimes (1_A)^2, \quad (1)$$

for $a \in A$. So, $\Delta(1_A)$ determines the Frobenius structure of A .

2.2. LEMMA. *If A is a Frobenius algebra that is a domain, then an extended structure of A (if it exists) must be either ϕ -trivial or θ -trivial.*

PROOF. Suppose that an extended structure (A, ϕ, θ) exists. Then, $\theta\phi(a) = \phi(\theta)\phi(a) = \phi(\theta a) = \theta a$, for all $a \in A$ by condition (i). Hence, $\theta(\phi(a) - a) = 0$ for all $a \in A$, and the result follows from A being a domain. ■

2.3. LEMMA. *Let A be a Frobenius algebra over $\mathbb{k}$, and let (A, ϕ, θ) and (A, ϕ', θ') be two extended structures of A . If $\theta \in \mathbb{k}1_A$ and $\theta \neq \theta'$, then an extended Frobenius algebra morphism from (A, ϕ, θ) to (A, ϕ', θ') does not exist.*

PROOF. Suppose by way of contrapositive that $\theta = \lambda 1_A$ for some $\lambda \in \mathbb{k}$ and there is a morphism $f : (A, \phi, \theta) \rightarrow (A, \phi', \theta')$ of extended Frobenius algebras. Since f is unital and preserves the extended structure, $\theta = \lambda 1_A = \lambda f(1_A) = f(\lambda 1_A) = f(\theta) = \theta'$, as desired. ■

We will see in Proposition 2.12 that Lemma 2.3 fails when $\theta \notin \mathbb{k}1_A$. We now include some examples of extended structures for well-known Frobenius algebras.

2.4. EXAMPLE. Let G be a finite group. Its group algebra $\mathbb{k}G$ has a Frobenius algebra structure determined by $\Delta(e_G) = \sum_{h \in G} h \otimes h^{-1}$. Then,

$$\phi = \text{id}_{\mathbb{k}G}, \quad \theta = \pm \sqrt{|G|} \cdot e_G$$

yields extended structures of $\mathbb{k}G$. Now, conditions (i) and (ii) are trivially satisfied. Condition (iii) holds as $m(\phi \otimes \text{id}_{\mathbb{k}G})\Delta(e_G) = m(\sum_{h \in G} h \otimes h^{-1}) = |G| \cdot e_G = \theta^2$.

2.5. EXAMPLE. Let C_n denote the cyclic group of order $n \geq 2$, and let g denote a generator of C_n . Consider the Frobenius structure on $\mathbb{k}C_n$ as defined in Example 2.4. Then

$$\phi(g) = \omega_n g^{-1}, \quad \theta = \pm \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \omega_n^j g^{-2j}$$

is an extended structure of $\mathbb{k}C_n$ for any n -th root of unity $\omega_n \in \mathbb{k}$. It is a quick check that condition (i) holds. Towards condition (ii), let $a := \sum_{i=0}^{n-1} a_i g^i$ be an element in $\mathbb{k}C_n$. Then,

$$\begin{aligned} \phi(a\theta) &= \pm \frac{1}{\sqrt{n}} \sum_{i,j=0}^{n-1} a_i \omega_n^j \phi(g)^{i-2j} = \pm \frac{1}{\sqrt{n}} \sum_{i,j=0}^{n-1} a_i \omega_n^{i-j} g^{-i+2j} \\ &= \pm \frac{1}{\sqrt{n}} \sum_{i,k=0}^{n-1} a_i \omega_n^k g^{i-2k} = a\theta. \end{aligned}$$

For condition (iii), we compute:

$$\begin{aligned} m(\phi \otimes \text{id}_{\mathbb{k}G})\Delta(e_{C_n}) &= m(\phi \otimes \text{id}_{\mathbb{k}C_n}) \left(\sum_{j=0}^{n-1} g^j \otimes g^{-j} \right) = \sum_{j=0}^{n-1} \omega_n^j g^{-2j} \\ &= \frac{1}{n} \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} \omega_n^k g^{-2k} = \frac{1}{n} \sum_{i,j=0}^{n-1} \omega_n^{i+j} g^{-2(i+j)} \\ &= \frac{1}{n} \left(\sum_{j=0}^{n-1} \omega_n^j g^{-2j} \right)^2 = \theta^2. \end{aligned}$$

2.6. EXAMPLE. Let $\omega := \omega_n$ be a primitive n -th root of unity, for $n \geq 2$. Consider the Taft algebra,

$$T_n(\omega) := \mathbb{k}\langle g, x \rangle / (g^n - 1, x^n, gx - \omega xg),$$

with Frobenius structure determined by

$$\Delta(1_{T_n(\omega)}) = \sum_{j=0}^{n-1} (-\omega^j g^{j+1} \otimes g^{-(j+1)} x + g^j x \otimes g^{-j}).$$

Then, this Frobenius structure on $T_n(\omega)$ can be extended via

$$\phi = \text{id}_{T_n(\omega)}, \quad \theta \in \bigoplus_{j=0, k=1}^{n-1} \mathbb{k} g^j x^k.$$

To show this, we compute: $m(\phi \otimes \text{id}_{T_n(\omega)})\Delta(1) = 0 = \theta^2$, so condition (iii) holds. Conditions (i) and (ii) are trivially satisfied.

2.7. EXAMPLE. Let $\text{Mat}_n(\mathbb{k})$ be the algebra of $n \times n$ matrices over $\mathbb{k}$, with basis $\{E_{i,j}\}_{i,j=1}^n$ of elementary matrices. Consider the Frobenius structure determined by $\Delta(E_{i,j}) = \sum_{\ell=1}^n E_{i,\ell} \otimes E_{\ell,j}$, for all $1 \leq i, j \leq n$. Then,

$$\phi = \text{id}_{\text{Mat}_n(\mathbb{k})}, \quad \theta = \pm \sqrt{n} \cdot I_n$$

give extended structures of $\text{Mat}_n(\mathbb{k})$. Indeed,

$$m(\phi \otimes \text{id}_{\text{Mat}_n(\mathbb{k})})\Delta(I_n) = \sum_{i,\ell=1}^n E_{i,\ell} E_{\ell,i} = n \cdot I_n = \theta^2,$$

so condition (iii) holds. Moreover, conditions (i) and (ii) are trivially satisfied.

2.8. **CLASSIFICATION RESULTS.** Now we proceed to establish Theorem 1.2, starting with the results for the Frobenius algebras: $\mathbb{k}$ over $\mathbb{k}$, $\mathbb{C}$ over $\mathbb{R}$, and the nilpotent algebra $\mathbb{k}[x]/(x^n)$ over $\mathbb{k}$.

2.9. **PROPOSITION.** *The only extended structures of the Frobenius algebra $\mathbb{k}$ where $\Delta_{\mathbb{k}} : \mathbb{k} \xrightarrow{\sim} \mathbb{k} \otimes \mathbb{k}$ are ϕ -trivial, with $\theta = \pm 1_{\mathbb{k}}$. Moreover, these extended Frobenius algebra structures are non-isomorphic.*

PROOF. Suppose ϕ and θ give an extended structure of $\mathbb{k}$. Since $\phi : \mathbb{k} \rightarrow \mathbb{k}$ is a morphism of algebras, the only possible choice is $\phi = \text{id}_{\mathbb{k}}$, which satisfies conditions (i) and (ii) trivially. Condition (iii) implies that $\theta = \pm 1_{\mathbb{k}}$. Lastly, the structures are non-isomorphic by Lemma 2.3. ■

2.10. **PROPOSITION.** *Take the Frobenius algebra $\mathbb{C}$ over $\mathbb{R}$ with $\Delta(1) = 1 \otimes 1 - i \otimes i$. Then,*

- (a) $\phi = \text{id}_{\mathbb{C}}$ and $\theta = \pm\sqrt{2}$, and
- (b) $\phi(z) = \bar{z}$ for all $z \in \mathbb{C}$, and $\theta = 0$,

are all of the extended structures of $\mathbb{C}$, and these extended Frobenius algebras are non-isomorphic.

PROOF. By Lemma 2.2, an extended structure of $\mathbb{C}$ should be ϕ -trivial or θ -trivial. If $\phi = \text{id}_{\mathbb{C}}$, then $\theta^2 = m(\phi \otimes \text{id}_{\mathbb{C}})\Delta(1) = m(1 \otimes 1 - i \otimes i) = 2$, and so $\theta = \pm\sqrt{2}$. On the other hand, if $\theta = 0$, then $0 = m(\phi \otimes \text{id}_{\mathbb{C}})\Delta(1) = 1 - \phi(i)i$. Hence, $\phi(i) = -i$ and it follows that ϕ must be complex conjugation. Now condition (iii) holds, and it is a quick check that conditions (i) and (ii) are satisfied for these choices. Lastly, it follows from Lemma 2.3 that these structures are all non-isomorphic. ■

2.11. **PROPOSITION.** *Consider the algebra $\mathbb{k}[x]/(x^n)$, for $n \geq 2$, with Frobenius structure determined by $\Delta(1) = \sum_{i=0}^{n-1} x^i \otimes x^{n-i-1}$. Then, the following statements hold.*

- (a) *For n even, $\mathbb{k}[x]/(x^n)$ is not extendable.*
- (b) *For n odd, all extended structures of $\mathbb{k}[x]/(x^n)$ are ϕ -trivial, with*

$$\theta = \pm\sqrt{n}x^{\frac{n-1}{2}} + \sum_{j=\frac{n+1}{2}}^{n-1} \theta_j x^j$$

for some $\theta_{\frac{n+1}{2}}, \dots, \theta_{n-1} \in \mathbb{k}$.

PROOF. Suppose that ϕ and θ give an extended structure of $\mathbb{k}[x]/(x^n)$. Then, a routine calculation with ϕ being multiplicative and $\phi^2 = \text{id}$ (from condition (i)) implies that $\phi(x) = \pm x$. So, in the rest of the proof, we look at the cases $\phi = \text{id}$ and $\phi(x) = -x$, and conclude the latter is never possible, while the former is only possible when n is odd.

Suppose first that $\phi = \text{id}$. Then, conditions (i) and (ii) are satisfied trivially. Let $\theta_0, \dots, \theta_{n-1} \in \mathbb{k}$ such that $\theta = \sum_{i=0}^{n-1} \theta_i x^i$. Then, condition (iii) implies that

$$nx^{n-1} = \sum_{i=0}^{n-1} \theta_i^2 x^{2i} + \sum_{i \neq j} \theta_i \theta_j x^{i+j}. \quad (2)$$

From the coefficient of 1, it follows that $\theta_0 = 0$. We can argue by induction that $\theta_i = 0$ for all $0 \leq i \leq \frac{n-1}{2} - 1$ if n is odd, and for all $0 \leq i \leq \frac{n}{2} - 1$ if n is even. It follows that if n is even, then the coefficient of x^{n-1} in (2) leads to the contradiction: $n = 2 \sum_{i=0}^{\frac{n}{2}-1} \theta_i \theta_{n-1-i} = 0$. Thus, $\phi = \text{id}$ is not possible when n is even. On the other hand, if n is odd, then the coefficient of x^{n-1} in (2) yields $n = (\theta_{\frac{n-1}{2}})^2 + 2 \sum_{i=0}^{\frac{n-1}{2}-1} \theta_i \theta_{n-1-i}$, which implies that $\theta_{\frac{n-1}{2}} = \pm \sqrt{n} \cdot 1_{\mathbb{k}}$. So, $\phi = \text{id}$ and $\theta = \pm \sqrt{n} x^{\frac{n-1}{2}} + \sum_{j=\frac{n+1}{2}}^{n-1} \theta_j x^j$ precisely satisfy conditions (i), (ii), and (iii) yielding an extended structure on the Frobenius algebra $\mathbb{k}[x]/(x^n)$ when n is odd.

It remains to look at the case $\phi(x) = -x$. It follows from ϕ being a morphism of coalgebras that this is not possible when n is even, since we get the following contradiction:

$$\sum_{i=0}^{n-1} x^i \otimes x^{n-i-1} = \Delta(\phi(1)) = (\phi \otimes \phi)\Delta(1) = -\sum_{i=0}^{n-1} x^i \otimes x^{n-i-1}.$$

When n is odd, the equalities $\phi(\theta) = \theta$ and $\phi(x\theta) = x\theta$ from condition (ii) yield the equations

$$\sum_{i=0}^{n-1} \theta_i x^i = \sum_{i=0}^{n-1} (-1)^i \theta_i x^i \quad \text{and} \quad \sum_{i=0}^{n-2} \theta_i x^{i+1} = \sum_{i=0}^{n-2} (-1)^{i+1} \theta_i x^{i+1},$$

respectively. Hence $\theta_i = 0$ for $1 \leq i \leq n-2$, and we have that $\theta = \theta_{n-1} x^{n-1}$. But then this would imply $0 = \theta^2 = m(\phi \otimes \text{id})\Delta(1) = x^{n-1}$. Hence, $\phi(x) = -x$ is also not possible when n is odd. ■

For a group G , consider the Frobenius algebra $\mathbb{k}G$ as in Example 2.4. We now provide classification results for the extended structures of $\mathbb{k}G$ when $G = C_2, C_3, C_4$, and $C_2 \times C_2$.

2.12. PROPOSITION. *Let g be a generator of C_2 . The extended structures of $\mathbb{k}C_2$ are:*

- (a) $\phi = \text{id}_{\mathbb{k}C_2}$ and $\theta \in \{\pm\sqrt{2}e_{C_2}, \pm\sqrt{2}g\}$, and
- (b) $\phi(g) = -g$ and $\theta = 0$.

Moreover, $(\mathbb{k}C_2, \text{id}_{\mathbb{k}C_2}, \sqrt{2}g) \cong (\mathbb{k}C_2, \text{id}_{\mathbb{k}C_2}, -\sqrt{2}g)$ as extended Frobenius algebras, and all other structures are non-isomorphic. That is, there are four isomorphism classes of extended Frobenius structures on $\mathbb{k}C_2$.

PROOF. Suppose that ϕ and θ define an extended structure on $\mathbb{k}C_2$, where $\phi(g) = \phi_0 e_{C_2} + \phi_1 g$ and $\theta = \theta_0 e_{C_2} + \theta_1 g$ for $\phi_0, \phi_1, \theta_0, \theta_1 \in \mathbb{k}$. By the counitality of ϕ , we have that $\phi_0 = \varepsilon(\phi(g)) = \varepsilon(g) = 0$, and $\phi_1^2 = \varepsilon(\phi_1^2 g^2) = \varepsilon(\phi(g^2)) = \varepsilon(g^2) = 1$. So, $\phi_1 = \pm 1$. Both choices are involutions and it is a quick check that they satisfy condition (i). We look now at the conditions (ii) and (iii).

When $\phi = \text{id}$, we have that $\theta_0^2 + \theta_1^2 = 2e_{C_2}$ and $2\theta_0\theta_1 = 0$, and so either $\theta = \pm\sqrt{2}e_{C_2}$ or $\theta = \pm\sqrt{2}g$. Both of these satisfy conditions (ii) and (iii). When $\phi(g) = -g$, condition (iii) yields $\theta_0^2 + \theta_1^2 = 0$ and $2\theta_0\theta_1 = 0$. Hence, $\theta = 0$, and condition (ii) is satisfied in this case.

Lastly, it follows from Lemma 2.3 that an isomorphism can only exist between $(\mathbb{k}C_2, \text{id}_{\mathbb{k}C_2}, \sqrt{2}g)$ and $(\mathbb{k}C_2, \text{id}_{\mathbb{k}C_2}, -\sqrt{2}g)$, which are in fact isomorphic via the morphism of extended Frobenius algebras $f : \mathbb{k}C_2 \rightarrow \mathbb{k}C_2$ defined by $g \mapsto -g$. ■

2.13. PROPOSITION. *Let g be a generator of C_3 . The extended structures of $\mathbb{k}C_3$ are:*

- (a) $\phi = \text{id}_{\mathbb{k}C_3}$ and $\theta \in \{\pm\sqrt{3}e_{C_3}, \pm\frac{1}{\sqrt{3}}(e_{C_3} - 2\omega_3g - 2\omega_3^2g^2)\}$,
- (b) $\phi(g) = \omega_3g^2$ and $\theta = \pm\frac{1}{\sqrt{3}}(e_{C_3} + \omega_3g + \omega_3^2g^2)$,

where $\omega_3 \in \mathbb{k}$ is some 3-rd root of unity. These structures are all non-isomorphic.

PROOF. Suppose ϕ and θ define an extended structure of $\mathbb{k}C_3$, where

$$\phi(g) = \phi_0e_{C_3} + \phi_1g + \phi_2g^2, \quad \theta = \theta_0e_{C_3} + \theta_1g + \theta_2g^2,$$

for $\phi_i, \theta_i \in \mathbb{k}$. By condition (i), we get that $\phi = \text{id}$ or $\phi(g) = \omega_3g^2$. We now examine the conditions: $m(\phi \otimes \text{id}_{\mathbb{k}C_3})\Delta(e_{C_3}) = \theta^2$, and $\phi(\theta a) = \theta a$ for $a \in \mathbb{k}C_3$.

When $\phi = \text{id}$, we get $\theta^2 = 3e_{C_3}$. Hence, $\theta_0 \neq 0$, and if $\theta_1 = 0$ or $\theta_2 = 0$, these imply $\theta = \pm\sqrt{3}e_{C_3}$. Else, if $\theta_1, \theta_2 \neq 0$, it follows that $\theta = \pm\frac{1}{\sqrt{3}}(e_{C_3} - 2\omega_3g - 2\omega_3^2g^2)$ for some 3-rd root of unity ω_3 . Condition (ii) is trivially satisfied for these cases. When $\phi(g) = \omega_3g^2$, then condition (iii) implies that $\theta^2 = e_{C_3} + \omega_3g + \omega_3^2g^2$. We also require $\theta = \phi(\theta) = \theta_0e_{C_3} + \theta_1\omega_3g^2 + \theta_2\omega_3^2g$, and thus $\theta_2 = \omega_3\theta_1$. Therefore, we get that $\theta = \pm\frac{1}{\sqrt{3}}(e_{C_3} + \omega_3g + \omega_3^2g^2)$. One can check that these choices satisfy condition (ii); see Example 2.5.

Lastly, any morphism f of extended Frobenius algebras between these possible structures is counital, so $f(g) = cg$ or $f(g) = cg^2$ for some $c \in \mathbb{k}$ such that $c^3 = 1$. From this and Lemma 2.3, we conclude there are no such morphisms between the different extended structures. ■

2.14. PROPOSITION. *Let g be a generator of C_4 . The extended structures of $\mathbb{k}C_4$ are:*

- (a) $\phi = \text{id}_{\mathbb{k}C_4}$ and $\theta \in \{\pm 2e_{C_4}, \pm 2g^2, \pm(1-i)(g + ig^3), \pm(1+i)(g - ig^3)\}$;
- (b) $\phi(g) = -g$ and $\theta = 0$;
- (c) $\phi(g) = \omega_4g^3$ and $\theta \in \left\{\pm\frac{1+\omega_4^2}{2}(e_{C_4} - g^2), \pm i\frac{1+\omega_4^2}{2}(g - g^3)\right\}$,

for any 4-th root of unity $\omega_4 \in \mathbb{k}$. These form eight isomorphism classes of extended structures.

PROOF. Suppose that ϕ and θ define an extended structure on $\mathbb{k}C_4$, where for $\phi_i, \theta_i \in \mathbb{k}$, we have $\phi(g) = \phi_0 e_{C_3} + \phi_1 g + \phi_2 g^2 + \phi_3 g^3$ and $\theta = \theta_0 e_{C_3} + \theta_1 g + \theta_2 g^2 + \theta_3 g^3$. By condition (i), we get that $\phi_2 = 0$ with $\phi(g) = \phi_1 g$ or $\phi(g) = \phi_3 g^3$; else, $\phi_2 \neq 0$ with $\phi_1^2 + \phi_3^2 = 0$. But a routine computation using $\phi^2(g) = g$ and condition (iii) shows that the $\phi_2 \neq 0$ case is not possible. So, either $\phi(g) = \phi_1 g$ or $\phi(g) = \phi_3 g^3$. Since $\phi^2(g) = g$, we obtain $\phi(g) = \pm g$ or $\phi(g) = \omega_4 g^3$ for some $\omega_4 \in \mathbb{k}$.

Suppose that $\phi = \text{id}_{\mathbb{k}C_4}$. Then, condition (ii) is trivially satisfied. Condition (iii) implies that $4e_{C_4} = \theta^2$, and we get the choices for θ in part (a). When $\phi(g) = -g$, condition (ii) implies that $\theta_1 = \theta_3 = 0$. So, by condition (iii), we obtain that $\theta_0^2 + 2\theta_0\theta_2g^2 + \theta_2^2 = 0$, and it follows that $\theta = 0$. This yields the choice in part (b). Lastly, if $\phi(g) = \omega_4 g^3$, then from condition (ii), we know that $\theta_1 = \omega_4^3 \theta_3$. Also from condition (iii), we get that $\theta^2 = (1 + \omega_4^2)e_{C_4} + (\omega_4 + \omega_4^3)g^2$. Solving for θ^2 in $\mathbb{k}C_4$, we get the two choices for θ in part (c). The former coincides with the choice of structure given in Example 2.5. For the latter, it is easy to check that condition (ii) still holds.

We prove now that there are exactly eight isomorphism classes of extended structures. It follows from Lemma 2.3 that three such classes are given by

$$\{(\mathbb{k}C_4, \text{id}_{\mathbb{k}C_4}, 2e_{C_4})\}, \{(\mathbb{k}C_4, \text{id}_{\mathbb{k}C_4}, -2e_{C_4})\}, \{(\mathbb{k}C_4, \phi(g) = -g, 0)\}.$$

Next, there can be no isomorphisms $f : (\mathbb{k}C_4, \text{id}_{\mathbb{k}C_4}, \theta) \rightarrow (\mathbb{k}C_4, \phi(g) = \omega_4 g^3, \theta')$, as this would imply $f(g) = f(\omega_4 g^3)$. Also, the algebra isomorphisms $f, f' : \mathbb{k}C_4 \rightarrow \mathbb{k}C_4$ defined by $f(g) = -g$ and $f'(g) = ig$ imply that

$$\{(\mathbb{k}C_4, \text{id}_{\mathbb{k}C_4}, \pm(1-i)(g+ig^3)), \{(\mathbb{k}C_4, \text{id}_{\mathbb{k}C_4}, \pm(1+i)(g-ig^3))\}, \{(\mathbb{k}C_4, \text{id}_{\mathbb{k}C_4}, \pm 2g^2)\}$$

are isomorphism classes of extended structures. The remaining isomorphism classes are then

$$\{(\mathbb{k}C_4, \phi(g) = \omega_4 g^3, \pm \frac{1+\omega_4^2}{2}(e_{C_4} - g^2))\}, \{(\mathbb{k}C_4, \phi(g) = \omega_4 g^3, \pm i \frac{1+\omega_4^2}{2}(g - g^3))\}$$

by a routine calculation. ■

Given the results in Proposition 2.12, 2.13, 2.14, we propose the following result.

2.15. CONJECTURE. *Let g be a generator of C_n . The following are the only possibilities for the Frobenius automorphism ϕ for an extended structure on $\mathbb{k}C_n$:*

- (a) $\phi(g) = \pm g$ or $\phi(g) = \omega_n g^{-1}$ when n is even,
- (b) $\phi(g) = g$ or $\phi(g) = \omega_n g^{-1}$ when n is odd,

where $\omega_n \in \mathbb{k}$ is any n -th root of unity.

The remainder of Theorem 1.2 is established in the next two results.

2.16. PROPOSITION. *The extended structures of $\mathbb{k}(C_2 \times C_2)$ are:*

- (a) $\phi = \text{id}_{\mathbb{k}(C_2 \times C_2)}$ and $\theta \in \{\pm 2e, \pm 2g_i, \pm(e + g_\ell) \pm (g_i - g_j), \pm(e - g_\ell) \pm (g_i + g_j)\}$;
- (b) $\phi(g_i) = -g_i, \phi(g_j) = -g_j, \phi(g_\ell) = g_\ell$, and $\theta = 0$;
- (c) $\phi(g_i) = g_j, \phi(g_j) = g_i, \phi(g_\ell) = g_\ell$, and $\theta \in \{\pm(e + g_\ell), \pm(g_i + g_j)\}$;
- (d) $\phi(g_i) = -g_j, \phi(g_j) = -g_i, \phi(g_\ell) = g_\ell$, and $\theta \in \{\pm(e - g_\ell), \pm(g_i - g_j)\}$;

where $C_2 \times C_2 = \{e, g_1, g_2, g_3\}$ and $\{i, j, \ell\} = \{1, 2, 3\}$.

PROOF. It follows from ϕ being counital that $\phi(g_i) = a_{i,1}g_1 + a_{i,2}g_2 + a_{i,3}g_3$ for $a_{i,p} \in \mathbb{k}$, for all $1 \leq i, p \leq 3$. Since ϕ is multiplicative, we then get that

$$e = \phi(g_i^2) = \phi(g_i)^2 = (a_{i,1}^2 + a_{i,2}^2 + a_{i,3}^2)e + 2a_{i,1}a_{i,2}g_3 + 2a_{i,1}a_{i,3}g_2 + 2a_{i,2}a_{i,3}g_1.$$

Hence, $\phi(g_i) = \pm g_j$ for some $1 \leq j \leq 3$. But $\phi^2 = \text{id}_{\mathbb{k}(C_2 \times C_2)}$, and thus the remaining possibilities for ϕ are the ones listed in the statement. It remains to find suitable θ for each possible ϕ . Let $\theta_0, \theta_1, \theta_2, \theta_3 \in \mathbb{k}$ such that $\theta = \theta_0e + \theta_1g_2 + \theta_2g_2 + \theta_3g_3$.

We compute $\theta^2 = \phi(e)e + \sum_{i=1}^3 \phi(g_i)g_i$. When $\phi = \text{id}_{\mathbb{k}(C_2 \times C_2)}$, one can check that we get the choices of θ in part (a) by condition (iii). When $\phi(g_i) = -g_i, \phi(g_j) = -g_j$ and $\phi(g_\ell) = g_\ell$ for $\{i, j, \ell\} = \{1, 2, 3\}$, condition (iii) implies $\theta^2 = 0$, so $\theta = 0$; this implies part (b). When $\phi(g_i) = g_j, \phi(g_j) = g_i$ and $\phi(g_\ell) = g_\ell$ for $\{i, j, \ell\} = \{1, 2, 3\}$, conditions (ii) and (iii) yield the choices of θ in part (c). The case $\phi(g_i) = -g_j, \phi(g_j) = -g_i$ and $\phi(g_\ell) = g_\ell$ for $\{i, j, \ell\} = \{1, 2, 3\}$ is analogous. ■

2.17. PROPOSITION. *Consider the Taft algebra $T_2(-1) := \mathbb{k}\langle g, x \rangle / (g^2 - 1, x^2, gx + xg)$ as defined in Example 2.6. All extensions of $T_2(-1)$ are ϕ -trivial, with $\theta \in \mathbb{k}x \oplus \mathbb{k}gx$.*

PROOF. First, note that $\Delta(1) = -g \otimes gx + x \otimes 1 + 1 \otimes x + gx \otimes g$. So, by (1), we get that $\Delta(g) = -1 \otimes gx + gx \otimes 1 + g \otimes x + x \otimes g$, $\Delta(x) = gx \otimes gx + x \otimes x$, and $\Delta(gx) = x \otimes gx + gx \otimes x$. Hence, $\varepsilon(1) = \varepsilon(g) = \varepsilon(gx) = 0$ and $\varepsilon(x) = 1$. Now suppose that $\phi : T_2(-1) \rightarrow T_2(-1)$ and $\theta \in T_2(-1)$ define an extended structure on $T_2(-1)$. Let $a_i, b_i \in \mathbb{k}$ such that $\phi(g) = a_1 + a_2g + a_3x + a_4gx$ and $\phi(x) = b_1 + b_2g + b_3x + b_4gx$. Since ϕ is an algebra morphism, we have that

$$\begin{aligned} 1 &= \phi(g)^2 = a_1^2 + a_2^2 + 2a_1a_2g + 2a_1a_3x + 2a_1a_4gx, \\ 0 &= \phi(x)^2 = b_1^2 + b_2^2 + 2b_1b_2g + 2b_1b_3x + 2b_1b_4gx. \end{aligned}$$

It follows that $\phi(g) = \pm g + a_3x + a_4gx$ and $\phi(x) = b_3x + b_4gx$. On the other hand, since ϕ is counital, we get $0 = \varepsilon(\phi(g)) = a_3$ and $1 = \varepsilon(\phi(x)) = b_3$. So, $\phi(g) = \pm g + a_4gx$ and $\phi(x) = x + b_4gx$. Also, ϕ is an involution, hence $g = \phi(\pm g + a_4gx) = \pm(g + a_4gx) \pm a_4(gx + b_4x)$. It follows that $\phi = \text{id}_{T_2(-1)}$. Lastly, $\theta^2 = m(\phi \otimes \text{id}_{T_2(-1)})\Delta(1) = 0$, and thus $\theta \in \mathbb{k}x \oplus \mathbb{k}gx$. ■

2.18. CONJECTURE. *Consider the Taft algebra,*

$$T_n(\omega) := \mathbb{k}\langle g, x \rangle / (g^n - 1, x^n, gx - \omega xg)$$

from Example 2.6. Then, all extensions of $T_n(\omega)$ are ϕ -trivial, with $\theta \in \bigoplus_{i=0}^{n-1} \mathbb{k}g^i x$.

3. Extended Frobenius algebras in a monoidal category

In this section, we first discuss monoidal categories and algebraic structures in monoidal categories in Section 3.1. There, we generalize Definition 1.1 to the monoidal setting, following [TT06, Section 2.2]; see Definition 3.2. Finally, we put monoidal structures on the category of extended Frobenius algebras in Section 3.5.

3.1. BACKGROUND MATERIAL. For details on algebras in monoidal categories, see, for example, [Koc04, Chapter 3], [TV17, Parts I and II] or [Wal24, Chapters 3 and 4]. The first reference also includes an introduction to Frobenius algebras in monoidal categories. Extended Frobenius algebras in monoidal categories can be found in [TT06, Section 2.2], [Cze24], and [Oca24].

3.1.1. MONOIDAL CATEGORIES. A *monoidal category* consists of a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, a natural isomorphism

$$a := \{a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)\}_{X,Y,Z \in \mathcal{C}},$$

an object $\mathbb{1} \in \mathcal{C}$, and natural isomorphisms

$$\ell := \{\ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X\}_{X \in \mathcal{C}}, \quad r := \{r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X\}_{X \in \mathcal{C}},$$

such that the pentagon and triangle axioms hold.

Unless stated otherwise, by MacLane's strictness theorem, we will assume that all monoidal categories are *strict* in the sense that

$$X \otimes Y \otimes Z := (X \otimes Y) \otimes Z = X \otimes (Y \otimes Z), \quad X := \mathbb{1} \otimes X = X \otimes \mathbb{1},$$

for all $X, Y, Z \in \mathcal{C}$; that is, $a_{X,Y,Z}$, ℓ_X , r_X are identity maps.

A monoidal category $\mathcal{C}$ is *symmetric* if it is equipped with

$$c := \{c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X\}_{X,Y \in \mathcal{C}},$$

a natural isomorphism with $c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}$ for $X, Y \in \mathcal{C}$, such that the hexagon axioms hold. The component $c_{X,Y}$ of c , the $c^2 = \text{id}$ property, the naturality of c at a morphism $f \in \mathcal{C}$, and unit coherence of c are all depicted in Figure 1.

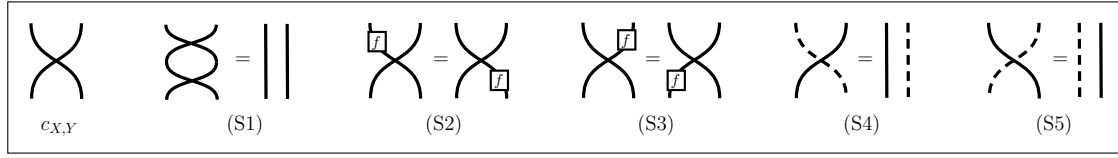


Figure 1: Some axioms for a symmetric monoidal category.

3.1.2. ALGEBRAIC STRUCTURES IN MONOIDAL CATEGORIES. Take a monoidal category $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$.

An *algebra* in $\mathcal{C}$ is an object $A \in \mathcal{C}$, equipped with morphisms $m : A \otimes A \rightarrow A$ and $u : \mathbb{1} \rightarrow A$ in $\mathcal{C}$, subject to associativity and unitality axioms:

$$m(m \otimes \text{id}_A) = m(\text{id}_A \otimes m), \quad m(u \otimes \text{id}_A) = \text{id}_A = m(\text{id}_A \otimes u).$$

These structures form a category, $\text{Alg}(\mathcal{C})$, where a morphism $(A, m_A, u_A) \rightarrow (B, m_B, u_B)$ is a morphism $f : A \rightarrow B$ in $\mathcal{C}$ such that $f m_A = m_B(f \otimes f)$ and $f u_A = u_B$.

A *coalgebra* in $\mathcal{C}$ is an object $A \in \mathcal{C}$, equipped with morphisms $\Delta : A \rightarrow A \otimes A$ and $\varepsilon : A \rightarrow \mathbb{1}$ in $\mathcal{C}$, subject to coassociativity and counitality axioms:

$$(\Delta \otimes \text{id}_A)\Delta = (\text{id}_A \otimes \Delta)\Delta, \quad (\varepsilon \otimes \text{id}_A)\Delta = \text{id}_A = \varepsilon(\text{id}_A \otimes u)\Delta.$$

These structures form a category, $\text{Coalg}(\mathcal{C})$, where $(A, \Delta_A, \varepsilon_A) \rightarrow (B, \Delta_B, \varepsilon_B)$ is a morphism $f : A \rightarrow B$ in $\mathcal{C}$ such that $\Delta_B f = (f \otimes f)\Delta_A$ and $\varepsilon_B f = \varepsilon_A$.

Our main algebraic structures of interest in this article are given as follows.

3.2. DEFINITION. Consider the following entities in a monoidal category $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$.

- (a) A Frobenius algebra in $\mathcal{C}$ is a tuple $(A, m, u, \Delta, \varepsilon)$, where (A, m, u) is an algebra in $\mathcal{C}$, and (A, Δ, ε) is a coalgebra in $\mathcal{C}$, subject to the Frobenius law:

$$(m \otimes \text{id}_A)(\text{id}_A \otimes \Delta) = \Delta m = (\text{id}_A \otimes m)(\Delta \otimes \text{id}_A).$$

A morphism of Frobenius algebras in $\mathcal{C}$ is a morphism of the underlying algebras and coalgebras in $\mathcal{C}$. The above objects and morphisms form a category, $\text{FrobAlg}(\mathcal{C})$.

- (b) An extended Frobenius algebra in $\mathcal{C}$ is a tuple $(A, m, u, \Delta, \varepsilon, \phi, \theta)$, where we have that $(A, m, u, \Delta, \varepsilon)$ is a Frobenius algebra in $\mathcal{C}$, and $\phi : A \rightarrow A$ and $\theta : \mathbb{1} \rightarrow A$ are morphisms in $\mathcal{C}$ such that

- (i) ϕ is a morphism of Frobenius algebras in $\mathcal{C}$, with $\phi^2 = \text{id}_A$;
- (ii) $\phi m(\theta \otimes \text{id}_A) = m(\theta \otimes \text{id}_A)$;
- (iii) $m(\phi \otimes \text{id}_A)\Delta u = m(\theta \otimes \theta)$.

A morphism $f : (A, \phi_A, \theta_A) \rightarrow (B, \phi_B, \theta_B)$ of extended Frobenius algebras in $\mathcal{C}$ is a morphism $f : A \rightarrow B$ of Frobenius algebras in $\mathcal{C}$, such that $f \phi_A = \phi_B f$ and $f \theta_A = \theta_B$. The above objects and morphisms form a category, $\text{ExtFrobAlg}(\mathcal{C})$.

- (c) The morphisms ϕ and θ in part (b) are the extended structure of the underlying Frobenius algebra. When ϕ and θ exist, we say that the underlying Frobenius algebra is extendable.
- (d) An extended structure (ϕ, θ) on a Frobenius algebra A is said to be ϕ -trivial if ϕ is the identity morphism, and is θ -trivial if θ is the zero morphism (when these exist in $\mathcal{C}$).

The structure morphisms for an extended Frobenius algebra in $\mathcal{C}$ are depicted in Figure 2, and the axioms that they satisfy are depicted in Figure 3. Here, we read diagrams from top down.

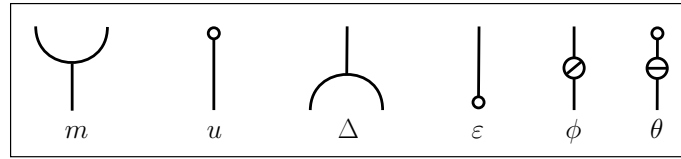


Figure 2: Structure morphisms for an extended Frobenius algebra in $\mathcal{C}$.

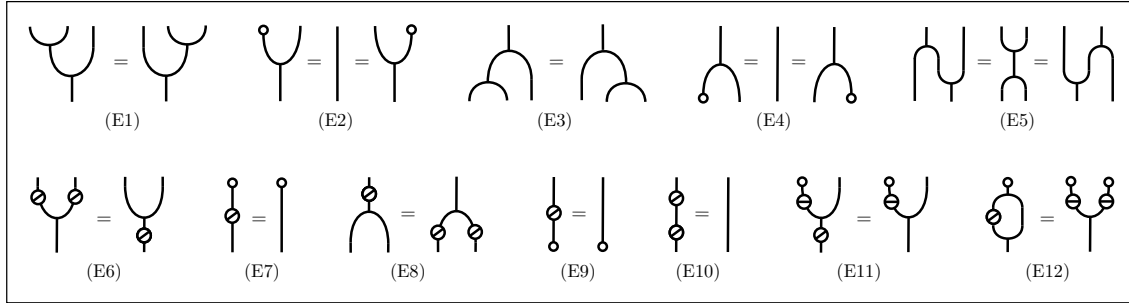


Figure 3: Axioms for an extended Frobenius algebra in $\mathcal{C}$.

One useful lemma is the following, adapted from [TT06, Lemma 2.8] for the monoidal setting.

3.3. LEMMA. *If $(A, m, u, \Delta, \varepsilon, \phi, \theta)$ is an extended Frobenius algebra in $\mathcal{C}$, then*

$$m(\phi \otimes \text{id}_A) \Delta = m(m(\theta \otimes \theta) \otimes \text{id}_A).$$

PROOF. This is proved in Figure 4 with references to Figures 2 and 3. ■

3.4. PROPOSITION. *A morphism of extended Frobenius algebras in $\mathcal{C}$ must be an isomorphism.*

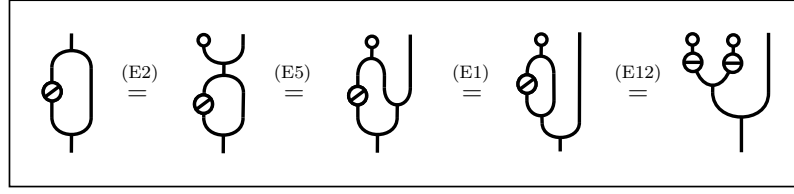
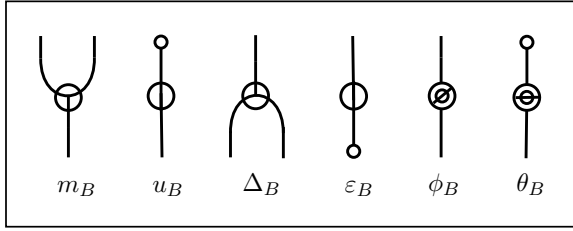
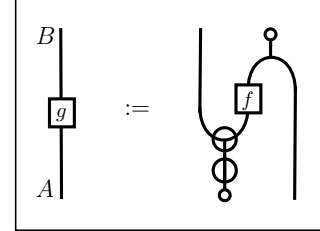
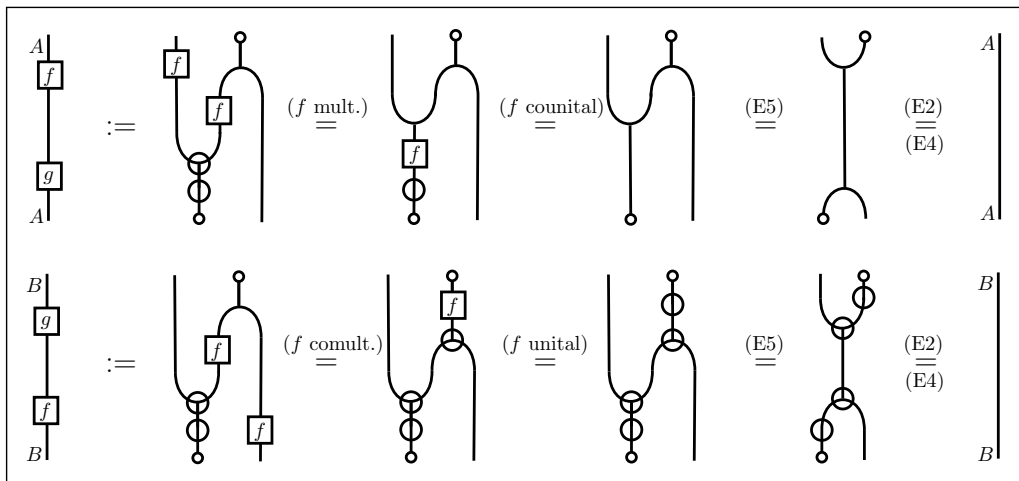


Figure 4: Proof of Lemma 3.3.

PROOF. This follows from the well-known fact that a morphism of Frobenius algebras in $\mathcal{C}$ must be an isomorphism. We repeat the proof here for the reader's convenience. Take a morphism of (extended) Frobenius algebras $f : A \rightarrow B$ in $\mathcal{C}$, that is, f is a morphism of the underlying algebras and coalgebras in $\mathcal{C}$. In graphical calculus, we will denote the (extended) Frobenius structure morphisms on A by those given in Figure 2, and the (extended) Frobenius structure morphisms on B will be denoted according to Figure 5. We then define a morphism $g : B \rightarrow A$ in Figure 6, and show that $gf = \text{id}_A$ and $fg = \text{id}_B$ using graphical calculus in Figure 7.

Figure 5: Extended Frobenius structure on B .Figure 6: Defining g .Figure 7: Proof that $gf = \text{id}_A$ and $fg = \text{id}_B$.

■

3.5. **STRUCTURE RESULTS.** Recall the category $\text{ExtFrobAlg}(\mathcal{C})$ defined in Definition 3.2. We put monoidal structures on this category, using two distinct monoidal products, in the following results.

3.6. **PROPOSITION.** *Let $(\mathcal{C}, \otimes, \mathbb{1}, c)$ be a symmetric monoidal category. Then, the category $\text{ExtFrobAlg}(\mathcal{C})$ is monoidal with $\otimes := \otimes^{\mathcal{C}}$ and $\mathbb{1} := \mathbb{1}^{\mathcal{C}}$.*

PROOF. We first note that $\mathbb{1}^{\mathcal{C}} = (\mathbb{1}^{\mathcal{C}}, \ell_{\mathbb{1}}, \text{id}_{\mathbb{1}}, \ell_{\mathbb{1}}^{-1}, \text{id}_{\mathbb{1}}, \text{id}_{\mathbb{1}}, \text{id}_{\mathbb{1}})$ is an extended Frobenius algebra in $\mathcal{C}$.

Next, we show that the monoidal product of two extended Frobenius algebras is extended Frobenius. Namely, we verify that given extended Frobenius algebras

$$(A, m_A, u_A, \Delta_A, \varepsilon_A, \phi_A, \theta_A) \text{ and } (B, m_B, u_B, \Delta_B, \varepsilon_B, \phi_B, \theta_B),$$

then

$$(A \otimes B, \tilde{m}, \tilde{u}, \tilde{\Delta}, \tilde{\varepsilon}, \tilde{\phi}, \tilde{\theta})$$

is an extended Frobenius algebra, where

$$\begin{aligned} \tilde{m} &:= (m_A \otimes m_B)(\text{id}_A \otimes c_{B,A} \otimes \text{id}_B), & \tilde{\Delta} &:= (\text{id}_A \otimes c_{A,B} \otimes \text{id}_B)(\Delta_A \otimes \Delta_B) \\ \tilde{u} &:= u_A \otimes u_B, & \tilde{\varepsilon} &:= \varepsilon_A \otimes \varepsilon_B, & \tilde{\phi} &:= \phi_A \otimes \phi_B, & \tilde{\theta} &:= \theta_A \otimes \theta_B. \end{aligned}$$

Figure 8 shows what these morphisms look like in graphical calculus, using the symbols from Figure 2 for A and the symbols from Figure 5 for B , as in Proposition 3.4. Recall also the axioms for a symmetric monoidal category from Figure 1.

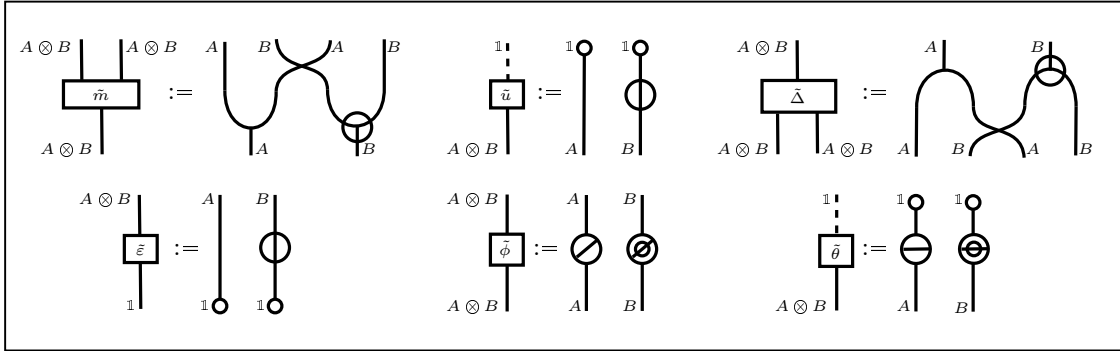


Figure 8: Extended Frobenius structure morphisms for $A \otimes B$.

We then have that $(A \otimes B, \tilde{m}, \tilde{u}, \tilde{\Delta}, \tilde{\varepsilon}) \in \text{FrobAlg}(\mathcal{C})$ by [Koc04, Section 2.4]. To see that this Frobenius algebra is extended via $\tilde{\phi}$ and $\tilde{\theta}$, we verify the three required conditions in Definition 3.2(b).

- (i) It is easy to see that $\tilde{\phi}$ is an involution, since both ϕ_A and ϕ_B are involutions. Moreover, since both ϕ_A, ϕ_B are Frobenius morphisms, so is their monoidal product in $\mathcal{C}$.

(ii) Figure 9 gives that $\tilde{\phi} \tilde{m}(\tilde{\theta} \otimes \text{id}_{A \otimes B}) = \tilde{m}(\tilde{\theta} \otimes \text{id}_{A \otimes B})$.

(iii) Finally, Figure 10 gives that $\tilde{m}(\tilde{\phi} \otimes \text{id}_{A \otimes B}) \tilde{\Delta} \tilde{u} = \tilde{m}(\tilde{\theta} \otimes \tilde{\theta})$.

Thus, we have that $(A \otimes B, \tilde{\phi}, \tilde{\theta}) \in \text{ExtFrobAlg}(\mathcal{C})$, as desired.

Lastly, we note that by taking $\mathbb{1}^{\mathcal{C}}$ as the unit and $\otimes^{\mathcal{C}}$ as the monoidal product in $\text{ExtFrobAlg}(\mathcal{C})$, with extended structures behaving as described above, we obtain that the required pentagon and triangle axioms in $(\text{ExtFrobAlg}(\mathcal{C}), \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$ are all inherited from the same axioms in $(\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$. From this, we can conclude that $(\text{ExtFrobAlg}(\mathcal{C}), \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$ is a monoidal category. ■

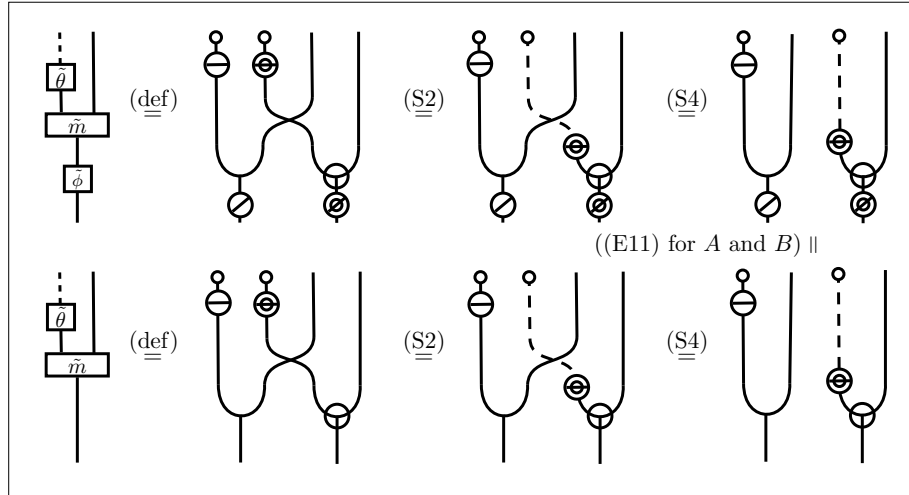
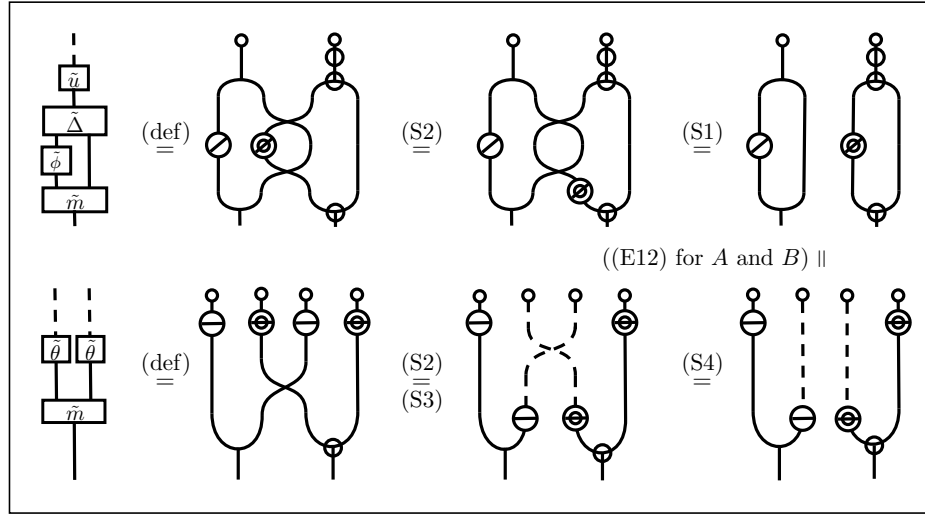


Figure 9: Proof that $A \otimes B$ satisfies Definition 3.2(b)(ii).

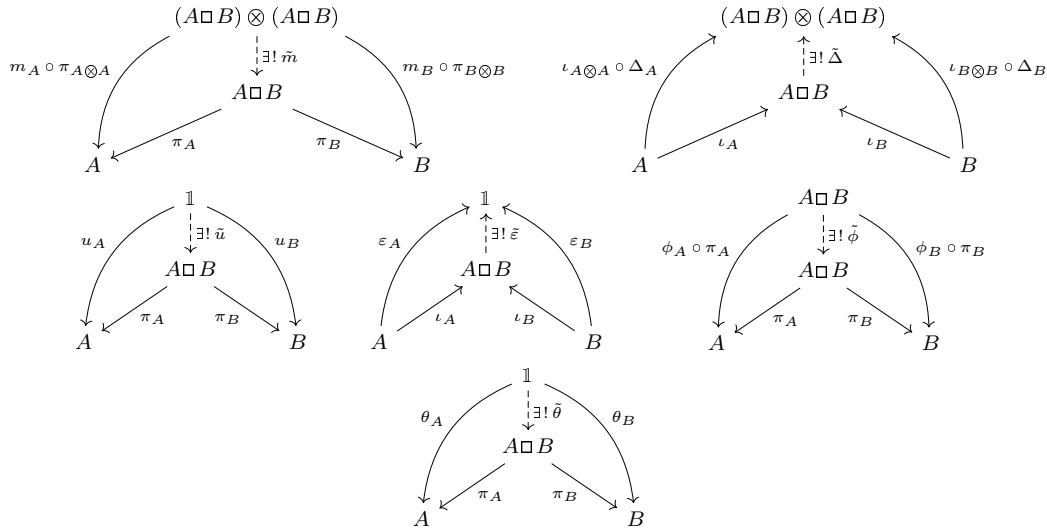
Now we turn our attention to extended Frobenius algebras in additive monoidal categories. See [Wal24, Section 3.1.3] for background material on such categories.

3.7. PROPOSITION. *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be an additive monoidal category. Then, the category $\text{ExtFrobAlg}(\mathcal{C})$ is monoidal with $\otimes$ being the biproduct $\square$, and $\mathbb{1}$ being the zero object 0 .*

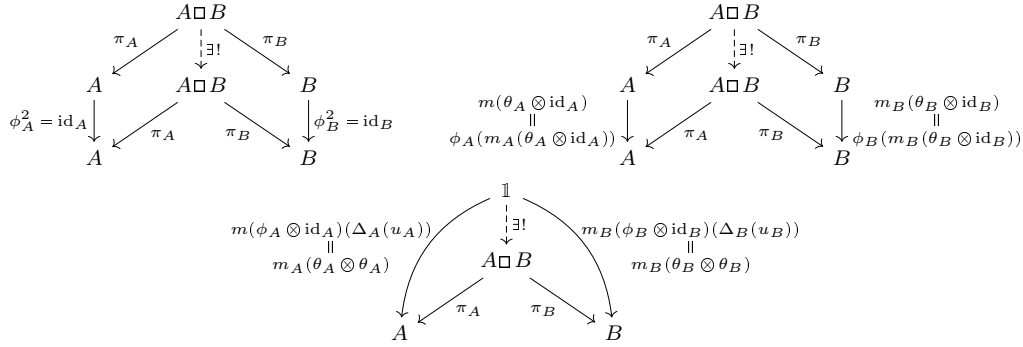
PROOF. We first note that 0 is an extended Frobenius algebra in $\mathcal{C}$, with structure morphisms $m, u, \Delta, \varepsilon$, and θ all being zero morphisms, and $\phi = \text{id}_0$. We next note that similar to the previous proposition, the pentagon and triangle axioms in $(\text{ExtFrobAlg}(\mathcal{C}), \square, 0)$ will be inherited from these same axioms on the strict monoidal category $(\mathcal{C}, \square, 0)$. Hence, to finish the proof, it suffices to show that the biproduct of two extended Frobenius algebras is again extended Frobenius.

Figure 10: Proof that $A \otimes B$ satisfies Definition 3.2(b)(iii).

To do so, let $(A, m_A, u_A, \Delta_A, \varepsilon_A, \phi_A, \theta_A)$ and $(B, m_B, u_B, \Delta_B, \varepsilon_B, \phi_B, \theta_B)$ be two extended Frobenius algebras in $\mathcal{C}$. We will show that $(A \sqcup B, \tilde{m}, \tilde{u}, \tilde{\Delta}, \tilde{\varepsilon}, \tilde{\phi}, \tilde{\theta})$ is an extended Frobenius algebra, where $\tilde{m}$, $\tilde{u}$, $\tilde{\Delta}$, $\tilde{\varepsilon}$, $\tilde{\phi}$, and $\tilde{\theta}$ are defined by the following universal property diagrams.



It is well known that with the above constructions, $(A \sqcup B, \tilde{m}, \tilde{u}, \tilde{\Delta}, \tilde{\varepsilon})$ is a Frobenius algebra. See [Koc04, Exercises 2.2.7 and 2.2.8] for the case where $\mathcal{C} = \mathbf{Vec}$. Thus, we only need to verify that $\tilde{\phi}$ and $\tilde{\theta}$ extend this Frobenius algebra. The three required properties from Definition 3.2(b) can be verified by respectively considering each of the universal property diagrams below.



Using uniqueness of the completing map in each of the diagrams, it follows that (i) $(\tilde{\phi})^2 = id_{A \square B}$, (ii) $\tilde{m}(\tilde{\theta} \otimes id_{A \square B}) = \tilde{\phi}(\tilde{m}(\tilde{\theta} \otimes id_{A \square B}))$, and (iii) $\tilde{m}(\tilde{\phi} \otimes id_{A \square B})(\tilde{\Delta}(\tilde{u})) = \tilde{m}(\tilde{\theta} \otimes \tilde{\theta})$. This completes the proof that $(A \square B, \phi, \theta)$ is an extended Frobenius algebras in $\mathcal{C}$, which in turn gives that $(\text{ExtFrobAlg}(\mathcal{C}), \square, 0)$ is a monoidal category. ■

4. Ties to separable algebras and Hopf algebras

In this section, we study extended Frobenius algebras in (symmetric) monoidal categories $\mathcal{C}$, in the context of separable algebras and Hopf algebras in $\mathcal{C}$; see Sections 4.1 and 4.5, respectively. We also introduce the notion of an extended Hopf algebra in $\mathcal{C}$, and make connections to extended Frobenius algebras in $\mathcal{C}$, in Section 4.12.

4.1. TIE TO SEPARABLE ALGEBRAS. Take $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ to be a monoidal category, and consider the terminology below. See [Bİ8, Chapter 6] and references within for the case when $\mathcal{C} = \text{Vec}$.

4.2. DEFINITION.

- (a) We say that an algebra $A := (A, m, u)$ in $\mathcal{C}$ is separable if there exists a morphism $t : A \rightarrow A \otimes A$ such that $mt = id_A$, and

$$(m \otimes id_A)(id_A \otimes t) = tm = (id_A \otimes m)(t \otimes id_A).$$

- (b) A Frobenius algebra $A := (A, m, u, \Delta, \varepsilon)$ is separable Frobenius if $m\Delta = id_A$.

These structures form full subcategories as indicated below:

$$\text{SepAlg}(\mathcal{C}) \subset \text{Alg}(\mathcal{C}), \quad \text{SepFrobAlg}(\mathcal{C}) \subset \text{FrobAlg}(\mathcal{C}).$$

4.3. PROPOSITION. If A is a separable Frobenius algebra in $\mathcal{C}$, then A is extendable.

PROOF. Suppose that $A := (A, m, u, \Delta, \varepsilon)$ is a separable Frobenius algebra, and take $\phi := id_A$ and $\theta := u$. Then, conditions (i) and (ii) of Definition 3.2(b) clearly hold. Condition (iii) of Definition 3.2(b) holds by the computation below:

$$m(\phi \otimes id_A)\Delta u = m\Delta u = u = m(u \otimes u) = m(\theta \otimes \theta),$$

where the third equality follows from a unitality axiom of A . ■

4.4. **EXAMPLE.** The monoidal unit $\mathbb{1} \in \mathcal{C}$ is a separable Frobenius algebra, with m and Δ identified as $\text{id}_{\mathbb{1}}$, and with $u = \varepsilon = \text{id}_{\mathbb{1}}$. The Frobenius structure is then extended with $\phi = \theta = \text{id}_{\mathbb{1}}$.

4.5. **TIE TO HOPF ALGEBRAS.** Take $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, \text{J})$ to be a symmetric monoidal category. See [Rad12, Chapter 10] and references within for the case when $\mathcal{C} = \mathbf{Vec}$ for the material below.

4.6. **DEFINITION.** Consider the following constructions in $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, c)$.

- (a) A Hopf algebra in $\mathcal{C}$ is a tuple $(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S)$, where (H, m, u) is an algebra in $\mathcal{C}$ and $(H, \underline{\Delta}, \underline{\varepsilon})$ is a coalgebra in $\mathcal{C}$, subject to the bialgebra laws,

$$\underline{\Delta}, \underline{\varepsilon} \in \mathbf{Alg}(\mathcal{C}) \quad (\Leftrightarrow \quad m, u \in \mathbf{Coalg}(\mathcal{C})),$$

and where $S : H \rightarrow H$ (antipode) is a morphism in $\mathcal{C}$ satisfying the antipode axiom,

$$m(S \otimes \text{id}_H) \underline{\Delta} = u \underline{\varepsilon} = m(\text{id}_H \otimes S) \underline{\Delta}.$$

If the antipode S is invertible with inverse $S^{-1} : H \rightarrow H$ in $\mathcal{C}$, then we call the tuple $(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, S^{-1})$ a Hopf algebra with invertible antipode.

- (b) A left integral for a Hopf algebra $(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S)$ is a morphism $\Lambda : \mathbb{1} \rightarrow H$ which satisfies $m(\text{id}_H \otimes \Lambda) = \Lambda \underline{\varepsilon}$. A right cointegral for the Hopf algebra $(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S)$ is a morphism $\lambda : H \rightarrow \mathbb{1}$ satisfying $(\lambda \otimes \text{id}_H) \underline{\Delta} = u \lambda$. If Λ and λ further satisfy $\lambda \Lambda = \text{id}_{\mathbb{1}}$, then Λ and λ are said to be normalized. A Hopf algebra equipped with a normalized (co)integral pair is called an integral Hopf algebra.
- (c) A morphism of integral Hopf algebras $f : H \rightarrow K$ is a morphism, which is both an algebra and coalgebra morphism, and which satisfies $f \Lambda_H = \Lambda_K$ and $\lambda_K f = \lambda_H$.
- (d) We organize the above into a category, $\mathbf{IntHopfAlg}(\mathcal{C})$, whose objects are integral Hopf algebras and whose morphisms are morphisms of integral Hopf algebras as defined above.

See Figures A.12-A.15 in Appendix A for a graphical representation of this definition.

4.7. **REMARK.** If a Hopf algebra is equipped with a normalized integral and cointegral, then the antipode is invertible; see, e.g., [CD20, Lemma 3.5].

Now we show that an integral Hopf algebra in $\mathcal{C}$ admits the structure of a Frobenius algebra in $\mathcal{C}$. A similar argument can also be found in [FS10, Appendix A.2].

4.8. **PROPOSITION.** We have that

$$\begin{aligned} \Psi : \mathbf{IntHopfAlg}(\mathcal{C}) &\rightarrow \mathbf{FrobAlg}(\mathcal{C}) \\ (H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, S^{-1}, \Lambda, \lambda) &\mapsto (H, m, u, \Delta := (m \otimes S)(\text{id}_H \otimes \underline{\Delta} \Lambda), \varepsilon := \lambda) \end{aligned}$$

is a well-defined functor, which acts as the identity on morphisms.

PROOF. This is established in Appendix A via graphical calculus arguments. ■

4.9. **EXAMPLE.** Let G be any finite group. The group algebra $\mathbb{k}G$ is a finite-dimensional Hopf algebra with $\underline{\Delta}(g) = g \otimes g$, $\underline{\varepsilon}(g) = 1$, and $S(g) = g^{-1}$, for all $g \in G$. This Hopf algebra admits a normalized (co)integral pair given by $\Lambda := \sum_{h \in G} h$ and $\lambda(g) := \delta_{e,g} 1_{\mathbb{k}}$. Applying Ψ to this integral Hopf algebra, we obtain the Frobenius structure on $\mathbb{k}G$ described in Example 2.4 and (1), where $\Delta(g) := \sum_{h \in G} gh \otimes h^{-1}$ and $\varepsilon(g) := \lambda(g) = \delta_{e,g} 1_{\mathbb{k}}$, for all $g \in G$.

4.10. **PROPOSITION.** *If $H \in \text{IntHopfAlg}(\mathcal{C})$ is equipped with $\theta : \mathbb{1} \rightarrow H \in \mathcal{C}$ such that $m(\theta \otimes \theta) = u \underline{\varepsilon} \Lambda$, then the Frobenius algebra $\Psi(H)$ from Proposition 4.8 is extendable. In particular, when $\mathcal{C} = \mathbf{Vec}$, the Frobenius algebra $\Psi(H)$ over $\mathbb{k}$ is extendable with $\phi = \text{id}_{\Psi(H)}$ and $\theta = \pm \sqrt{\underline{\varepsilon}(\Lambda(1_{\mathbb{k}}))} u$.*

PROOF. Suppose that the morphism $\theta : \mathbb{1} \rightarrow H$ as in the statement exists. Then, taking $\phi = \text{id}_{\Psi(H)}$, and using this θ , we extend the Frobenius structure. To verify the axioms of Definition 3.2(b), notice that conditions (i) and (ii) hold trivially. Condition (iii) is verified in Figure 11; using notation and axioms from Appendix A. The last statement on the case when $\mathcal{C} = \mathbf{Vec}$ is clear. ■

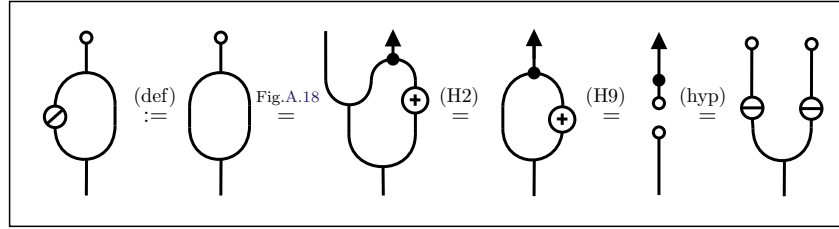


Figure 11: Proof of Definition 3.2(b)(iii) for Proposition 4.10.

4.11. **EXAMPLE.** Let G be a finite group, and recall that the group algebra $\mathbb{k}G$ has a Hopf algebra structure, which induces a Frobenius algebra structure, as described in Example 4.9. In this case, we have that

$$u \underline{\varepsilon}(\Lambda) = u(\underline{\varepsilon}(\sum_{h \in G} h)) = u(\sum_{h \in G} 1_{\mathbb{k}}) = |G| \cdot u(1_{\mathbb{k}}) = |G| \cdot e_G.$$

The above proposition then tells us that the choice $\phi = \text{id}_{\mathbb{k}G}$ and $\theta = \pm \sqrt{|G|} \cdot e_G$ extends the induced Frobenius algebra structure on $\mathbb{k}G$. Note that this is the same extended Frobenius structure as introduced in Example 2.4.

4.12. **EXTENDED HOPF ALGEBRAS.** Continue to let $\mathcal{C}$ be a symmetric monoidal category. Here, we introduce extended Hopf algebras in $\mathcal{C}$.

4.13. **DEFINITION.** *An integral Hopf algebra $(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, S^{-1}, \Lambda, \lambda)$ is called extended if it is equipped with two morphisms $\phi : H \rightarrow H$ and $\theta : \mathbb{1} \rightarrow H$ in $\mathcal{C}$ satisfying the following axioms:*

- (i) ϕ is a morphism of integral Hopf algebras such that $\phi^2 = \text{id}_H$;

$$(ii) \quad \phi m(\theta \otimes \text{id}_H) = m(\theta \otimes \text{id}_H);$$

$$(iii) \quad m(\phi \otimes S)\underline{\Delta}\Lambda = m(\theta \otimes \theta).$$

A morphism of extended Hopf algebras $f : (H, \phi, \theta) \rightarrow (H', \phi', \theta')$ is a morphism of integral Hopf algebras in $\mathcal{C}$ which also satisfies $f\phi = \phi'f$ and $f\theta = \theta'$.

We use the above to define a category $\text{ExtHopfAlg}(\mathcal{C})$. Also, consider the forgetful functor,

$$U : \text{ExtHopfAlg}(\mathcal{C}) \rightarrow \text{IntHopfAlg}(\mathcal{C})$$

$$(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, S^{-1}, \Lambda, \lambda, \phi, \theta) \mapsto (H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, S^{-1}, \Lambda, \lambda).$$

We have the following result.

4.14. PROPOSITION. *Take $H \in \text{ExtHopfAlg}(\mathcal{C})$. Then, the Frobenius algebra $\Psi U(H)$ in $\mathcal{C}$ from Proposition 4.8 is extendable via the morphisms ϕ and θ .*

PROOF. We will verify that ϕ and θ extend the Frobenius algebra $\Psi U(H)$ by checking the axioms of Definition 3.2(b). Since

$$\phi : (H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, S^{-1}, \Lambda, \lambda) \rightarrow (H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, S^{-1}, \Lambda, \lambda)$$

is a morphism of integral Hopf algebras, the functoriality of Ψ and U gives that $\phi : (H, m, u, \Delta, \varepsilon) \rightarrow (H, m, u, \Delta, \varepsilon)$ is a Frobenius algebra morphism. Moreover, we have that $\phi^2 = \text{id}_H$ by Definition 4.13(i). So, condition (i) of Definition 3.2(b) holds. Condition (ii) of Definition 3.2(b) also holds by Definition 4.13(ii) since the multiplication morphism is the same for both the Hopf and Frobenius structures on H . Towards condition (iii) of Definition 3.2(b), we compute:

$$m(\phi \otimes \text{id}_H)\Delta u = m(\phi \otimes S)(m \otimes \text{id}_H)(u \otimes \underline{\Delta})\Lambda = m(\theta \otimes \theta),$$

where the first equality is the definition of Δ and a level exchange, and the second equality is by the unitality of m and u and Definition 4.13(iii). ■

The consequence below is straight-forward.

4.15. COROLLARY. *There is a functor $\underline{\Psi} : \text{ExtHopfAlg}(\mathcal{C}) \rightarrow \text{ExtFrobAlg}(\mathcal{C})$ which sends an extended Hopf algebra $(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, S^{-1}, \Lambda, \lambda, \phi, \theta)$ to the extended Frobenius algebra $(H, m, u, \Delta, \varepsilon, \phi, \theta)$, with Δ and ε defined in Proposition 4.8, and which acts as the identity on morphisms.*

4.16. REMARK. While the above result tells us that every extended Hopf algebra gives rise to an extended Frobenius algebra via the same ϕ and θ , the converse is not true. In particular, given $H \in \text{IntHopfAlg}(\mathcal{C})$, we get that $\Psi(H)$ is in $\text{FrobAlg}(\mathcal{C})$. If $\Psi(H)$ is extendable via $\phi_{\Psi(H)}$ and $\theta_{\Psi(H)}$, it is not necessarily true that $(H, \phi_{\Psi(H)}, \theta_{\Psi(H)})$ is an extended Hopf algebra in $\mathcal{C}$.

For instance, consider the Frobenius algebra structure on $\mathbb{k}C_2$, induced by the Hopf structure, as described in Example 4.9. This Frobenius structure can be extended by taking $\phi(g) = -g$ (where g is a generator of C_2) and $\theta = 0$, as in Proposition 2.12(b). However, this choice of ϕ and θ does not extend the integral Hopf structure on $\mathbb{k}C_2$, since ϕ is not comultiplicative with respect to $\underline{\Delta}$.

5. Extended Frobenius monoidal functors

In this section, we introduce the construction of an extended Frobenius monoidal functor, which preserves extended Frobenius algebras [Proposition 5.11]. Background material is covered in Section 5.1, and the main construction is covered in Section 5.7. Examples are presented in Section 5.16.

5.1. BACKGROUND ON MONOIDAL FUNCTORS. We can move between monoidal categories in several ways. Consider the terminology below, along with the references, [Bĭ8, Chapter 6], [DP08], [TV17, Sections 1.4 and 7.5], [Wal24, Section 3.2], for details about the material in this part.

5.2. DEFINITION. *Take a functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ between monoidal categories $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{C}', \otimes', \mathbb{1}')$.*

- (a) *We say that F is a monoidal functor if it is equipped with a natural transformation $F^{(2)} := \{F_{X,Y}^{(2)} : F(X) \otimes' F(Y) \rightarrow F(X \otimes Y)\}_{X,Y \in \mathcal{C}}$, and a morphism $F^{(0)} : \mathbb{1}' \rightarrow F(\mathbb{1})$ in $\mathcal{C}'$, that satisfy associativity and unitality constraints.*
- (b) *A monoidal functor $(F, F^{(2)}, F^{(0)})$ is said to be strong if $F^{(2)}$ is a natural isomorphism and $F^{(0)}$ is an isomorphism. In this case, denote $F_{X,Y}^{(-2)} := (F_{X,Y}^{(2)})^{-1}$ and $F^{(-0)} := (F^{(0)})^{-1}$.*
- (c) *We say that F is a comonoidal functor if it is equipped with a natural transformation $F_{(2)} := \{F_{(2)}^{X,Y} : F(X \otimes Y) \rightarrow F(X) \otimes' F(Y)\}_{X,Y \in \mathcal{C}}$, and a morphism $F_{(0)} : F(\mathbb{1}) \rightarrow \mathbb{1}'$ in $\mathcal{C}'$, that satisfy coassociativity and counitality constraints.*
- (d) *We say that F is a Frobenius monoidal functor if it is part of a tuple*

$$(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}),$$

where $(F, F^{(2)}, F^{(0)})$ is a monoidal functor, and $(F, F_{(2)}, F_{(0)})$ is a comonoidal functor, subject to the Frobenius conditions, for all $X, Y, Z \in \mathcal{C}$:

$$\begin{aligned} (F_{X,Y}^{(2)} \otimes' \text{id}_{F(Z)}) (\text{id}_{F(X)} \otimes' F_{(2)}^{Y,Z}) &= F_{(2)}^{X \otimes Y, Z} \circ F_{X,Y \otimes Z}^{(2)}, \\ (\text{id}_{F(X)} \otimes' F_{Y,Z}^{(2)}) (F_{(2)}^{X,Y} \otimes' \text{id}_{F(Z)}) &= F_{(2)}^{X, Y \otimes Z} \circ F_{X \otimes Y, Z}^{(2)}. \end{aligned}$$

- (e) A Frobenius monoidal functor $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)})$ is separable if for each $X, Y \in \mathcal{C}$:

$$F_{X,Y}^{(2)} \circ F_{(2)}^{X,Y} = \text{id}_{F(X \otimes Y)}.$$

Moreover, consider the transformations of (co)monoidal functors below.

5.3. DEFINITION. Take monoidal categories $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ and $\mathcal{C}' := (\mathcal{C}', \otimes', \mathbb{1}')$.

- (a) A monoidal natural transformation from a monoidal functor $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \rightarrow \mathcal{C}'$ to a monoidal functor $(G, G^{(2)}, G^{(0)}) : \mathcal{C} \rightarrow \mathcal{C}'$ is a natural transformation $\phi : F \Rightarrow G$ such that

$$\phi_{X \otimes Y} \circ F_{X,Y}^{(2)} = G_{X,Y}^{(2)} \circ (\phi_X \otimes' \phi_Y) \text{ for all } X, Y \in \mathcal{C}, \quad \phi_{\mathbb{1}} \circ F^{(0)} = G^{(0)}.$$

- (b) A comonoidal natural transformation from a comonoidal functor $(F, F_{(2)}, F_{(0)}) : \mathcal{C} \rightarrow \mathcal{C}'$ to a comonoidal functor $(G, G_{(2)}, G_{(0)}) : \mathcal{C} \rightarrow \mathcal{C}'$ is a natural transformation $\phi : F \Rightarrow G$ such that

$$(\phi_X \otimes' \phi_Y) \circ F_{(2)}^{X,Y} = G_{(2)}^{X,Y} \circ \phi_{X \otimes Y} \text{ for all } X, Y \in \mathcal{C}, \quad F_{(0)} = G_{(0)} \circ \phi_{\mathbb{1}}.$$

- (c) A Frobenius monoidal natural transformation is a natural transformation

$$\phi : F \Rightarrow G$$

between Frobenius monoidal functors

$$(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}), \quad (G, G^{(2)}, G^{(0)}, G_{(2)}, G_{(0)})$$

from $\mathcal{C}$ to $\mathcal{C}'$ that is monoidal for the underlying monoidal functor structure and comonoidal for the underlying comonoidal functor structure.

Next, we see in the result below that the various types of functors in Definition 5.2 preserve the corresponding algebraic structures introduced in Section 3.1.2 and Definition 4.2.

5.4. PROPOSITION. [Wal24, Proposition 4.3] [DP08, Corollary 5] [Bĭ8, Lemma 6.10] Take monoidal categories $\mathcal{C}$ and $\mathcal{C}'$.

- (a) A monoidal functor $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \rightarrow \mathcal{C}'$ yields $\text{Alg}(F) : \text{Alg}(\mathcal{C}) \rightarrow \text{Alg}(\mathcal{C}')$, a functor where $\text{Alg}(F)(A, m_A, u_A)$ is defined as

$$(F(A), m_{F(A)} := F(m_A) F_{A,A}^{(2)}, u_{F(A)} := F(u_A) F^{(0)}).$$

- (b) A comonoidal functor $(F, F_{(2)}, F_{(0)}) : \mathcal{C} \rightarrow \mathcal{C}'$ yields a functor $\text{Coalg}(F) : \text{Coalg}(\mathcal{C}) \rightarrow \text{Coalg}(\mathcal{C}')$, where $\text{Coalg}(F)(A, \Delta_A, \varepsilon_A)$ is defined as

$$(F(A), \Delta_{F(A)} := F_{(2)}^{A,A} F(\Delta_A), \varepsilon_{F(A)} := F_{(0)} F(\varepsilon_A)).$$

- (c) Moreover, a Frobenius monoidal functor $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}) : \mathcal{C} \rightarrow \mathcal{C}'$ yields a functor $\text{FrobAlg}(F) : \text{FrobAlg}(\mathcal{C}) \rightarrow \text{FrobAlg}(\mathcal{C}')$ by using the formulas from parts (a) and (b).
- (d) A separable Frobenius monoidal functor $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}) : \mathcal{C} \rightarrow \mathcal{C}'$ yields a functor $\text{SepFrobAlg}(\mathcal{C}) \rightarrow \text{SepFrobAlg}(\mathcal{C}')$ by using the formulas from parts (a) and (b).

One nice feature of the functors here is that they are closed under composition.

5.5. PROPOSITION. [Wal24, Exercise 3.4] [DP08, Proposition 4] [Bĭ8, Exercises 3.10 and 6.4] Take monoidal categories $\mathcal{C}$, $\mathcal{C}'$, and $\mathcal{C}''$.

- (a) Let $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \rightarrow \mathcal{C}'$ and $(G, G^{(2)}, G^{(0)}) : \mathcal{C}' \rightarrow \mathcal{C}''$ be monoidal functors. Then, the composition $GF : \mathcal{C} \rightarrow \mathcal{C}''$ is monoidal, with

$$(GF)_{X,Y}^{(2)} := G(F_{X,Y}^{(2)}) \circ G_{F(X),F(Y)}^{(2)} \quad \forall X, Y \in \mathcal{C}, \quad (GF)^{(0)} := G(F^{(0)}) \circ G^{(0)}.$$

- (b) Let $(F, F_{(2)}, F_{(0)}) : \mathcal{C} \rightarrow \mathcal{C}'$ and $(G, G_{(2)}, G_{(0)}) : \mathcal{C}' \rightarrow \mathcal{C}''$ be comonoidal functors. Then, the composition $GF : \mathcal{C} \rightarrow \mathcal{C}''$ is comonoidal, with

$$(GF)^{X,Y}_{(2)} := G^{F(X),F(Y)}_{(2)} \circ G(F^{X,Y}_{(2)}) \quad \forall X, Y \in \mathcal{C}, \quad (GF)_{(0)} := G_{(0)} \circ G(F_{(0)}).$$

- (c) Let $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}) : \mathcal{C} \rightarrow \mathcal{C}'$ and $(G, G^{(2)}, G^{(0)}, G_{(2)}, G_{(0)}) : \mathcal{C}' \rightarrow \mathcal{C}''$ be Frobenius monoidal functors. Then, the composition $GF : \mathcal{C} \rightarrow \mathcal{C}''$ is Frobenius monoidal by using the formulas from parts (a) and (b).
- (d) The composition of two separable Frobenius monoidal functors is also separable Frobenius monoidal by using the formulas from parts (a) and (b).

5.6. REMARK. It is now straightforward to build the 2-category, MON (resp., COMON, FROBMON, SEPFROBMON), via the data below.

- (a) 0-cells are monoidal categories.
- (b) 1-cells are (resp., co-, Frobenius, separable Frobenius) monoidal functors.
- (c) 2-cells are (resp., co-, Frobenius, Frobenius) monoidal natural transformations.
- (d) The identity 1-cell/2-cell is the identity (resp., co-, Frobenius, Frobenius) monoidal functor/natural transformation.
- (e) Horizontal composition of 1-cells is given in Proposition 5.5.
- (f) Vertical/horizontal composition of 2-cells is given by the standard vertical/ horizontal composition of monoidal and comonoidal natural transformations.

See [Wal24, Section 4.10.3] and references within, and also see [JY21, Exercise 2.7.11].

5.7. MAIN CONSTRUCTION AND RESULTS. Here, we extend the results in Propositions 5.4 and 5.5 to the category $\text{ExtFrobAlg}(\mathcal{C})$. In particular, we will define a type of functor that preserves extended Frobenius algebras, and then show that this type of functor is closed under composition.

5.8. DEFINITION. A Frobenius monoidal functor $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)})$ from $(\mathcal{C}, \otimes, \mathbb{1})$ to $(\mathcal{C}', \otimes', \mathbb{1}')$ is called an extended Frobenius monoidal functor (or is extendable) if there exist a natural transformation $\hat{F} : F \Rightarrow F$ and a morphism $\check{F} : \mathbb{1}' \rightarrow F(\mathbb{1}) \in \mathcal{C}'$ such that the conditions below hold.

- (a) $\hat{F}$ is a Frobenius monoidal natural transformation.
- (b) $F_{\mathbb{1}, \mathbb{1}}^{(2)} \circ (\hat{F}_{\mathbb{1}} \otimes' \text{id}_{F(\mathbb{1})}) \circ F_{(2)}^{\mathbb{1}, \mathbb{1}} \circ F^{(0)} = F_{\mathbb{1}, \mathbb{1}}^{(2)} \circ (\check{F} \otimes' \check{F})$.
- (c) The following are true for each $X, Y \in \mathcal{C}$:
 - (i) $\hat{F}_X \circ \hat{F}_X = \text{id}_{F(X)}$;
 - (ii) $\hat{F}_{\mathbb{1} \otimes X} \circ F_{\mathbb{1}, X}^{(2)} \circ (\check{F} \otimes' \text{id}_{F(X)}) = F_{\mathbb{1}, X}^{(2)} \circ (\check{F} \otimes' \text{id}_{F(X)})$;
 - (iii) $F_{X, Y}^{(2)} \circ (\hat{F}_X \otimes' \text{id}_{F(Y)}) \circ F_{(2)}^{X, Y} = F_{X \otimes Y, \mathbb{1}}^{(2)} \circ (\hat{F}_{X \otimes Y} \otimes' \text{id}_{F(\mathbb{1})}) \circ F_{(2)}^{X \otimes Y, \mathbb{1}}$.

Part (b) is represented by the following commutative diagram.

$$\begin{array}{ccccc}
 \mathbb{1}' & \xrightarrow{F^{(0)}} & F(\mathbb{1}) & \xrightarrow{F_{(2)}^{\mathbb{1}, \mathbb{1}}} & F(\mathbb{1}) \otimes' F(\mathbb{1}) \\
 \check{F} \otimes' \check{F} \downarrow & & & & \downarrow \hat{F}_{\mathbb{1}} \otimes' \text{id}_{F(\mathbb{1})} \\
 & & & & F(\mathbb{1}) \otimes' F(\mathbb{1}) \\
 & & & & \downarrow F_{\mathbb{1}, \mathbb{1}}^{(2)} \\
 F(\mathbb{1}) \otimes' F(\mathbb{1}) & \xrightarrow{F_{\mathbb{1}, \mathbb{1}}^{(2)}} & F(\mathbb{1}) & &
 \end{array}$$

Parts (c)(ii, iii) are represented by the left and right diagrams below, respectively.

$$\begin{array}{ccc}
 \mathbb{1}' \otimes' F(X) & \xrightarrow{\check{F} \otimes' \text{id}_{F(X)}} & F(\mathbb{1}) \otimes' F(X) \\
 \check{F} \otimes' \text{id}_{F(X)} \downarrow & & \downarrow F_{\mathbb{1}, X}^{(2)} \\
 F(\mathbb{1}) \otimes' F(X) & & F(\mathbb{1} \otimes X) \\
 F_{\mathbb{1}, X}^{(2)} \downarrow & & \downarrow \hat{F}_{\mathbb{1} \otimes X} \\
 F(\mathbb{1} \otimes X) & \xrightarrow{\hat{F}_{\mathbb{1} \otimes X}} & F(\mathbb{1} \otimes X)
 \end{array}
 \quad
 \begin{array}{ccc}
 F(X \otimes Y) & \xrightarrow{F_{(2)}^{X \otimes Y, \mathbb{1}}} & F(X \otimes Y) \otimes' F(\mathbb{1}) \\
 F_{(2)}^{X, Y} \downarrow & & \downarrow \hat{F}_{X \otimes Y} \otimes' \text{id}_{F(\mathbb{1})} \\
 F(X) \otimes' F(Y) & & F(X \otimes Y) \otimes' F(\mathbb{1}) \\
 \hat{F}_X \otimes' \text{id}_{F(Y)} \downarrow & & \downarrow F_{X \otimes Y, \mathbb{1}}^{(2)} \\
 F(X) \otimes' F(Y) & \xrightarrow{F_{X, Y}^{(2)}} & F(X \otimes Y)
 \end{array}$$

Extended Frobenius monoidal functors are plentiful. Specifically, we have the following result; compare to Proposition 4.3.

5.9. PROPOSITION. Separable Frobenius monoidal functors admit the structure of extended Frobenius monoidal functors.

PROOF. Let $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)})$ be a separable Frobenius monoidal functor. Then, take $\hat{F} = \text{Id}_F$ and $\check{F} = F^{(0)}$. It is then straightforward to verify that these choices of $\hat{F}$ and $\check{F}$ extend the Frobenius monoidal structure on F . $\blacksquare$

5.10. **EXAMPLE.** Strong monoidal functors are separable with $F_{(2)} := F^{(-2)}$ and $F_{(0)} := F^{(-0)}$, so they are also extended Frobenius monoidal functors.

The next result is the desired extension of Proposition 5.4. See Appendix B.1 for the proof (in the ArXiv preprint version of this article); namely, it involves lengthy commutative diagram arguments to verify that the formulas in the statement below yield an extended Frobenius algebra.

5.11. **PROPOSITION.** *An extended Frobenius monoidal functor*

$$(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}, \hat{F}, \check{F}) : \mathcal{C} \rightarrow \mathcal{C}'$$

induces a functor $\text{ExtFrobAlg}(\mathcal{C}) \rightarrow \text{ExtFrobAlg}(\mathcal{C}')$. For $A \in \text{ExtFrobAlg}(\mathcal{C})$, we get $m_{F(A)}$, $u_{F(A)}$, $\Delta_{F(A)}$, $\varepsilon_{F(A)}$ as in Proposition 5.4(a,b), with $\phi_{F(A)} = F(\phi_A) \hat{F}_A$ and $\theta_{F(A)} = F(\theta_A) \check{F}$.

Since separable Frobenius monoidal functors are extended by Propositions 5.9, we obtain the following corollary of Proposition 5.11.

5.12. **COROLLARY.** *If $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}) : \mathcal{C} \rightarrow \mathcal{C}'$ is a separable Frobenius monoidal functor, then it induces a functor $\text{ExtFrobAlg}(\mathcal{C}) \rightarrow \text{ExtFrobAlg}(\mathcal{C}')$.*

Now that we have defined extended Frobenius monoidal functors, the natural next thing to do is to arrange them into a 2-category. To do this, we need the following result, which extends Proposition 5.5 to extended Frobenius monoidal functors. The proof of this theorem can be found in Appendix B.2 (in the ArXiv preprint version of this article).

5.13. **THEOREM.** *The composition of two extended Frobenius monoidal functors is again an extended Frobenius monoidal functor.*

To prove this, let $(GF)^{(2)}$, $(GF)^{(0)}$, $(GF)_{(2)}$, $(GF)_{(0)}$ be as in Proposition 5.5(a,b). Proposition 5.5(c) then implies that GF is a Frobenius monoidal functor. We also define $\widehat{GF} : GF \Rightarrow GF$ by $\widehat{GF}_X := G(\hat{F}_X) \circ \hat{G}_{F(X)}$ for all $X \in \mathcal{C}$, and define $\widetilde{GF} := G(\check{F}) \circ \check{G} : \mathbb{1}'' \rightarrow GF(\mathbb{1})$.

5.14. **REMARK.** The collection of monoidal categories, extended Frobenius monoidal functors, and Frobenius natural transformations compatible with the extended Frobenius monoidal structures forms a 2-category, EXTFROBMON . Compare to Remark 5.6.

5.15. **REMARK.** One can also obtain Proposition 5.11 as a consequence of Theorem 5.13. Take the monoidal category $\overline{\mathbb{1}}$ consisting of a single object $\mathbb{1}$ and morphism $\text{id}_{\mathbb{1}}$. Then, a Frobenius monoidal functor $(E, E^{(2)}, E^{(0)}, E_{(2)}, E_{(0)}) : \overline{\mathbb{1}} \rightarrow \mathcal{C}$ is extendable if and only if $E(\mathbb{1}) \in \text{ExtFrobAlg}(\mathcal{C})$. So, when $A \in \text{ExtFrobAlg}(\mathcal{C})$, the functor $A^\# : \overline{\mathbb{1}} \rightarrow \mathcal{C}$ with $A^\#(\mathbb{1}) := A$ is extended Frobenius monoidal. Now if $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}, \hat{F}, \check{F}) : \mathcal{C} \rightarrow \mathcal{C}'$ is extended Frobenius monoidal, Theorem 5.13 implies that the functor $FA^\# : \overline{\mathbb{1}} \rightarrow \mathcal{C}'$ is also extended Frobenius monoidal. Hence, $F(A)$ is an extended Frobenius algebra in $\mathcal{C}'$ as in the proof of Proposition 5.11. Compare to [DP08, Corollary 5].

5.16. **EXAMPLES.** Following up with Propositions 3.6 and 3.7, consider the examples of extended Frobenius monoidal functors below.

5.17. **EXAMPLE.** Let $(\mathcal{C}, \otimes, \mathbb{1}, c)$ be a symmetric monoidal category, with an extended Frobenius algebra $B \in \mathbf{ExtFrobAlg}(\mathcal{C})$. Then, the functor $- \otimes B : \mathcal{C} \rightarrow \mathcal{C}$ is extended Frobenius with

$$(- \otimes B)_{X,Y}^{(2)} := (\mathrm{id}_{X \otimes Y} \otimes m_B)(\mathrm{id}_X \otimes c_{B,Y} \otimes \mathrm{id}_B), \quad (- \otimes B)_{(2)}^{X,Y} := (\mathrm{id}_X \otimes c_{Y,B} \otimes \mathrm{id}_B)(\mathrm{id}_{X \otimes Y} \otimes \Delta_B),$$

$(- \otimes B)^{(0)} := u_B$, $(- \otimes B)_{(0)} := \varepsilon_B$, $(\widehat{- \otimes B})_X := \mathrm{id}_X \otimes \phi_B$, $(\widetilde{- \otimes B}) := \theta_B$, for any $X, Y \in \mathcal{C}$. We note further that when B is not a separable Frobenius algebra, the Frobenius functor defined above is not separable.

5.18. **EXAMPLE.** Let $(\mathcal{C}, \otimes, \mathbb{1})$ be an additive monoidal category, with an extended Frobenius algebra $B \in \mathbf{ExtFrobAlg}(\mathcal{C})$. Then, the functor $- \square B : \mathcal{C} \rightarrow \mathcal{C}$ is extended Frobenius with

$$(- \square B)_{X,Y}^{(2)} := \pi_{X \otimes Y} \square (m_B \circ \pi_{B \otimes B}), \quad (- \square B)_{(2)}^{X,Y} := \iota_{X \otimes Y} \square (\iota_{B \otimes B} \circ \Delta_B),$$

$(- \square B)^{(0)} := \mathrm{id}_{\mathbb{1}} \square u_B$, $(- \square B)_{(0)} := \mathrm{id}_{\mathbb{1}} \square \varepsilon_B$, $(\widehat{- \square B})_X := \pi_X \square (\phi_B \circ \pi_B)$, $(\widetilde{- \square B}) := \mathrm{id}_{\mathbb{1}} \square \theta_B$, for any $X, Y \in \mathcal{C}$. Again, when B is not a separable Frobenius algebra, the Frobenius functor defined above is not separable.

A. Graphical proof that integral Hopf implies Frobenius

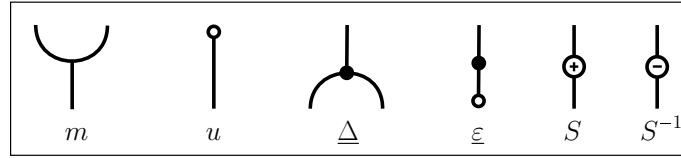
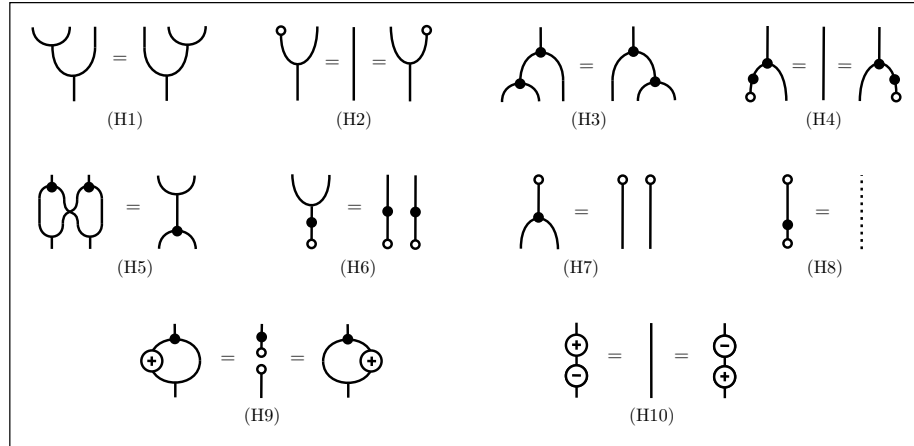
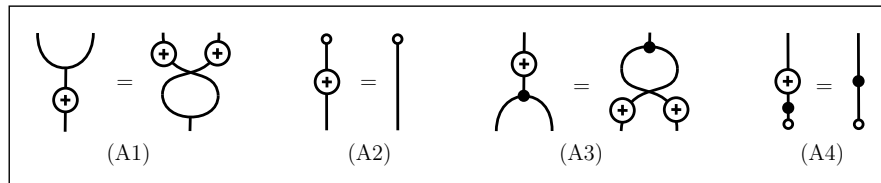
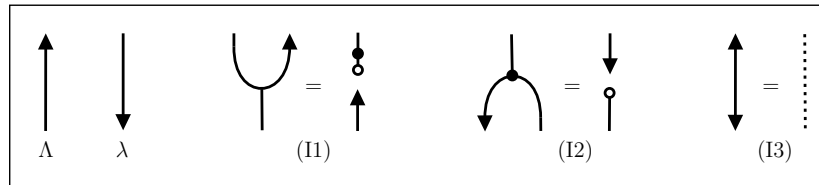
In this section, we give a graphical proof of Proposition 4.8, showing that an integral Hopf algebra in a symmetric monoidal category $\mathcal{C}$ is a Frobenius algebra in $\mathcal{C}$. Recall axioms (S1) - (S5) from Figure 1 in Section 3.1 above.

A.1. **DIAGRAMS FOR INTEGRAL HOPF ALGEBRAS.** Recall from Definition 4.6 that a Hopf algebra with invertible antipode in $\mathcal{C}$ is an object $H \in \mathcal{C}$ equipped with morphisms $m : H \otimes H \rightarrow H$, $u : \mathbb{1} \rightarrow H$, $\underline{\Delta} : H \rightarrow H \otimes H$, $\underline{\varepsilon} : H \rightarrow \mathbb{1}$, $S : H \rightarrow H$ with inverse $S^{-1} : H \rightarrow H$; this is depicted in Figure A.12. These morphisms must satisfy the axioms in Figure A.13. We also have that Hopf algebras with invertible antipode in $\mathcal{C}$ satisfy the identities in Figure A.14. Moreover, an integral and a cointegral of a Hopf algebra H with invertible antipode in $\mathcal{C}$ are given by morphisms $\Lambda : \mathbb{1} \rightarrow H$ and $\lambda : H \rightarrow \mathbb{1}$, respectively, satisfying the axioms depicted in Figure A.15. Now consider the following preliminary result.

A.2. **LEMMA.** *We have the following identities.*

- (a) $(m \otimes S)(\mathrm{id}_H \otimes \underline{\Delta} \Lambda) = (\mathrm{id}_H \otimes m)(\mathrm{id}_H \otimes S \otimes \mathrm{id}_H)(\underline{\Delta} m \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes \Lambda \otimes \mathrm{id}_H) \underline{\Delta}.$
- (b) $\lambda S \Lambda = \mathrm{id}_{\mathbb{1}}.$

PROOF. Part (a) is proved in Figure A.16, and part (b) is proved in Figure A.17. References to Figures 1, A.13, A.14, and A.15 are made throughout. ■

Figure A.12: Structure morphisms for a Hopf algebra in $\mathcal{C}$.Figure A.13: Axioms for a Hopf algebra with invertible antipode in $\mathcal{C}$.Figure A.14: Identities for a Hopf algebra in $\mathcal{C}$.Figure A.15: Normalized (co)integral for a Hopf algebra in $\mathcal{C}$.

A.3. PROOF OF PROPOSITION 4.8. We aim to show that

$$\Psi : \text{IntHopfAlg}(\mathcal{C}) \rightarrow \text{FrobAlg}(\mathcal{C})$$

$$(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, S^{-1}, \Lambda, \lambda) \mapsto (H, m, u, \Delta := (m \otimes S)(\text{id}_H \otimes \underline{\Delta} \Lambda), \varepsilon := \lambda)$$

is a well-defined functor, which acts as the identity on morphisms.

For the assignment of objects under the functor Ψ , the coproduct Δ and counit ε are depicted in Figure A.18. The counitality axioms are then established in Figure A.19; the Frobenius laws are established in Figure A.20; and the coassociativity axiom is established in Figure A.21. References to Figures A.13–A.17 are made throughout. Next, for the assignment of morphisms under Ψ , take a morphism of integral Hopf algebras

$$f : (H, m_H, u_H, \underline{\Delta}_H, \underline{\varepsilon}_H, S_H^{\pm 1}, \Lambda_H, \lambda_H) \rightarrow (K, m_K, u_K, \underline{\Delta}_K, \underline{\varepsilon}_K, S_K^{\pm 1}, \Lambda_K, \lambda_K).$$

We will verify that $\Psi(f) := f$ is a morphism of Frobenius algebras from

$$(H, m_H, u_H, \Delta_H, \varepsilon_H) \rightarrow (K, m_K, u_K, \Delta_K, \varepsilon_K).$$

We have multiplicativity and unitality for free, since the Hopf multiplications and units on H and K are the same as the Frobenius multiplications and units on H and K . Next, we get Frobenius counitality immediately from the fact that f is compatible with the cointegrals of H and K ; namely, the Frobenius counits of H and K are given by $\varepsilon_H = \lambda_H$ and $\varepsilon_K = \lambda_K$. Finally, we have that Frobenius comultiplicativity holds via the commutative diagram below.

$$\begin{array}{ccccc}
 H & & \xrightarrow{f} & & K \\
 \Delta_H \downarrow & \searrow \text{id}_H \otimes \Lambda_H & & \swarrow \text{id}_K \otimes \Lambda_K & \downarrow \Delta_K \\
 & H \otimes H & \xrightarrow{f \otimes f} & K \otimes K & \\
 & \downarrow \text{id}_H \otimes \Delta_H & & \downarrow \text{id}_K \otimes \Delta_K & \\
 & H \otimes H \otimes H & \xrightarrow{f \otimes f \otimes f} & K \otimes K \otimes K & \\
 & \swarrow m_H \otimes S_H & & \searrow m_K \otimes S_K & \\
 H \otimes H & \xrightarrow{f \otimes f} & & K \otimes K &
 \end{array}$$

Here, the left and right regions commute by definition of Δ_H and Δ_K . The top region commutes because f is compatible with the integrals of H and K . The bottom region commutes because f is an algebra map and is compatible with the antipodes of H and K . Finally, the middle region commutes because f is a coalgebra map between the Hopf algebras H and K .

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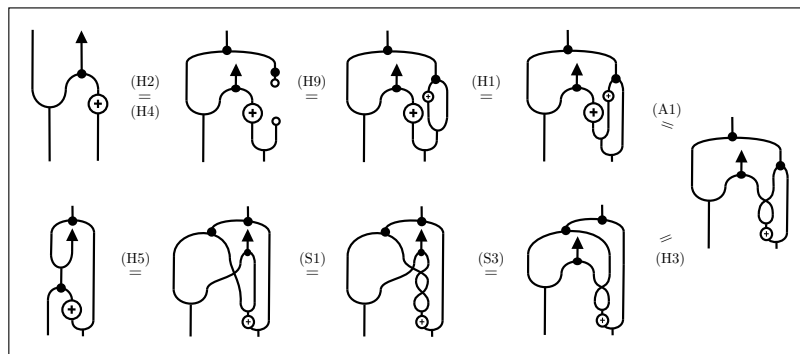


Figure A.16: Proof of Lemma A.2(a).

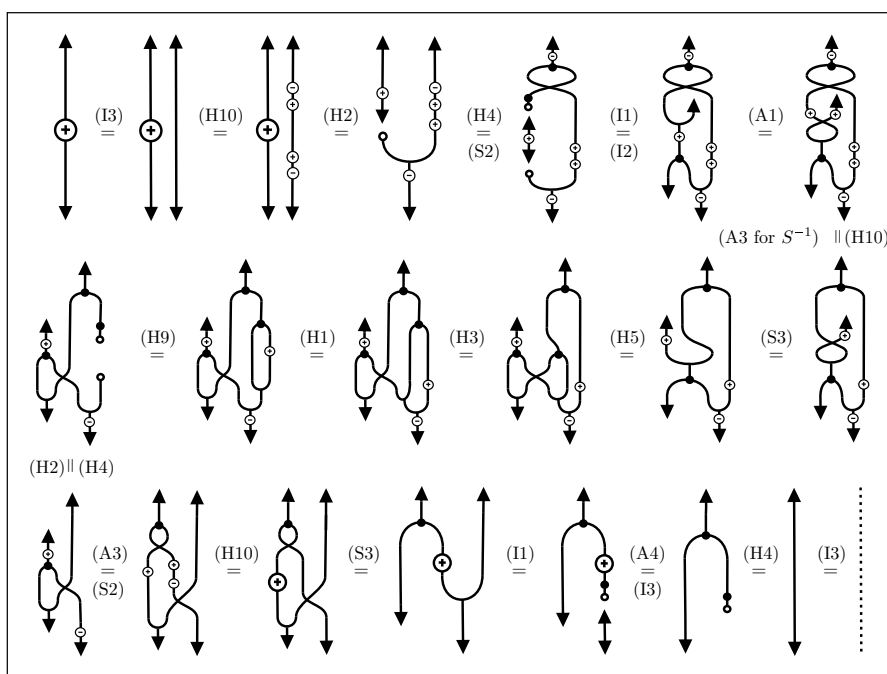
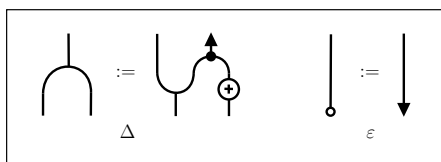


Figure A.17: Proof of Lemma A.2(b).

Figure A.18: Coproduct and counit for the Frobenius-from-Hopf structure in $\mathcal{C}$.

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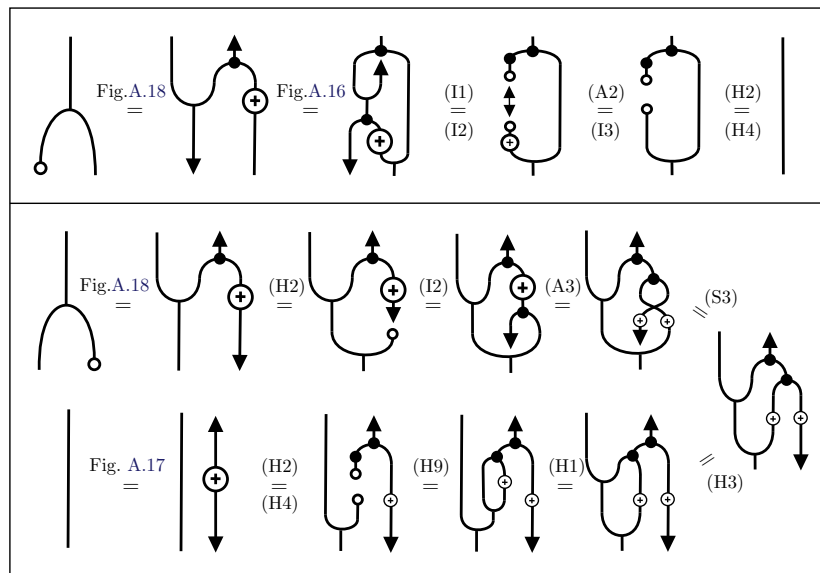


Figure A.19: Proof of counitality for the Frobenius-from-Hopf structure in $\mathcal{C}$.

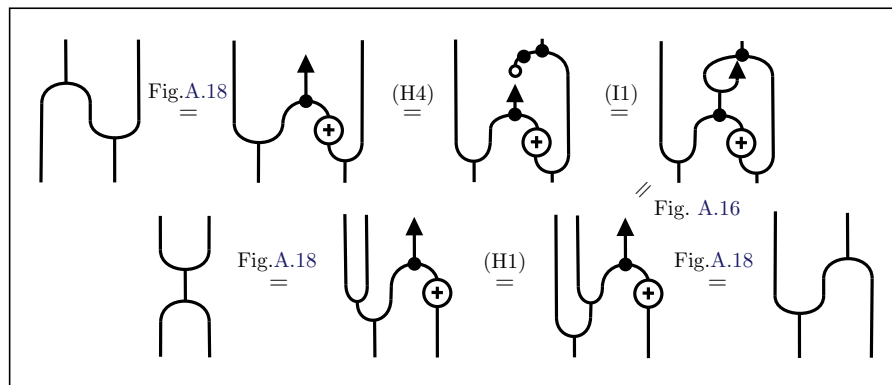


Figure A.20: Proof of the Frobenius laws for the Frob.-from-Hopf struc. in $\mathcal{C}$.

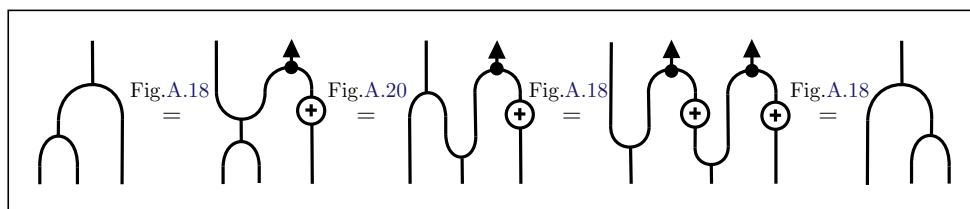


Figure A.21: Proof of coassociativity for the Frob.-from-Hopf struc. in $\mathcal{C}$.

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