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IDEALLY REGULAR CATEGORIES

SANDRA MANTOVANI AND MARIANO MESSORA

ABSTRACT. In this note, we propose a generalisation of G. Janelidze's notion of an ideally exact category beyond the Barr exact setting. We define an *ideally regular category* as a regular, Bourn protomodular category with finite coproducts in which the unique morphism $0 \to 1$ is *effective for descent*. As in the ideally exact case, ideally regular categories support a notion of ideal that classifies regular quotients. Moreover, they admit a characterisation in terms of monadicity over a *homological* category (rather than a semi-abelian one, as in the exact setting). Examples include Bourn protomodular quasivarieties of universal algebra in which $0 \to 1$ is effective for descent (such as the category of torsion-free unital rings), all Bourn protomodular topological varieties with at least one constant (such as topological rings), and all semi-localisations of ideally exact categories.

Introduction

In analogy with the classical theory of ideals of unital rings, G. Janelidze introduced *ideally exact categories* in [9] as a class of categories equipped with a well-behaved notion of 'ideal' capable of classifying quotient objects. Formally, ideally exact categories are defined as Barr exact, Bourn protomodular ([4]) categories with finite coproducts such that the unique morphism $0 \to 1$ is a regular epimorphism. For an ideally exact category $\mathscr{A}$, the pullback functor along $p\colon 0 \to 1$ between the corresponding slice categories

$$p^*\colon \mathscr{A} \simeq (\mathscr{A} \downarrow 1) \to (\mathscr{A} \downarrow 0)$$

is monadic, and $(\mathscr{A} \downarrow 0)$ is semi-abelian (in the sense of [11]) – in fact, ideally exact categories can be equivalently characterised as Barr exact categories with finite coproducts that are monadic over a semi-abelian category. A fundamental feature of such a category $\mathscr{A}$ is that for any object $A \in \mathscr{A}$, the *ideals of* A – defined as normal subobjects of $p^*(A)$ – are in bijective correspondence with the regular quotients of A. A paradigmatic example is indeed the category Ring of unital rings, which is in fact ideally exact. The corresponding functor $p^*\colon \mathsf{Ring} \to (\mathsf{Ring} \downarrow 0)$ (with $0 = \mathbb{Z}$) coincides, up to equivalence, with the forgetful functor $\mathsf{Ring} \to \mathsf{Rng}$ to the semi-abelian category of non-unital rings, thereby recovering the usual notion of an ideal of a ring. Ideally exact categories provide a wide generalisation of this classical situation, and they encompass many commonly encountered examples,

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including all Bourn protomodular varieties with at least one constant, the dual category of any elementary topos, and all coslices of semi-abelian categories.

The aim of the present work is to extend this framework beyond the Barr exact setting. A leading motivation is the attempt to capture examples from the world of Bourn protomodular quasivarieties and topological varieties ([16, 2]), as these are generally not Barr exact categories. To this end, we propose the notion of an *ideally regular category*, defined as a regular, Bourn protomodular category with finite coproducts in which the unique morphism $0 \to 1$ is *effective for descent* (see for example [12]). As effective descent morphisms coincide with regular epimorphisms in the Barr exact context, one immediately sees that every ideally exact category is, in particular, ideally regular. Nevertheless, we show that this broader notion still satisfies analogues of many of the key properties of ideally exact categories established in [9] – although their validity outside the exact setting requires non-trivial reworking of the original arguments. Notably, we prove that ideally regular categories still support a notion of *ideal* – defined in the same way as in the ideally exact setting – such that the ideals of an object are in bijective correspondence with its regular quotients (Theorem 3.2). Moreover, we find that ideally regular categories turn out to be precisely those regular categories with finite coproducts that are monadic over a *homological* ([1]) category with finite coproducts (Theorem 2.2), thereby paralleling the characterisation of ideally exact categories in terms of monadicity over a semi-abelian category. Finally, we show that, as desired, this broader context encompasses many new examples, explored in Section 1. Notable instances include all Bourn protomodular quasivarieties in which $0 \to 1$ is effective for descent (such as the category of torsion-free rings), all Bourn protomodular topological varieties with at least one constant (such as the category of topological rings), as well as all semi-localisations of ideally exact categories and all coslices of homological categories having finite coproducts. We also provide an example of a Bourn protomodular quasivariety where $0 \to 1$ is a regular epimorphism which is not effective for descent. In this case we show that ideals fail to classify quotients, thus proving that even in a well-behaved setting, the effective descent assumption is not automatic and remains essential to the theory.

1. Definition and examples

In this section, we formally state an (intrinsic) definition of *ideally regular category* and we provide a list of illustrative examples. A more detailed justification of this definition will follow from the general properties established in the subsequent sections.

Here and throughout, for any category $\mathscr{A}$ we use $0_{\mathscr{A}}$ and $1_{\mathscr{A}}$ to denote the initial and terminal objects of $\mathscr{A}$, respectively (dropping the subscript '$\mathscr{A}$' when the ambient category is clear for context), and for any object $X \in \mathscr{A}$, we write $!_X$ and $!^X$ for the unique morphisms $0 \to X$ and $X \to 1$, respectively.

For the background on Bourn protomodular and homological categories, we refer the reader to [1]. For material on descent theory, see for example [12].

1.1. Definition. We say that a category $\mathscr{A}$ is *ideally regular* if it is a regular, Bourn protomodular category with finite coproducts such that the unique morphism $0 \to 1$ in $\mathscr{A}$ is effective for descent.

1.2. Example. Every ideally exact category is ideally regular, and an ideally regular category is ideally exact if and only if it is Barr exact.

1.3. Example. Every homological category with finite coproducts is ideally regular, and an ideally regular category is homological if and only if it is pointed.

1.4. Example. Quasivarieties of universal algebra are ideally regular precisely when they are Bourn protomodular and the morphism $0 \to 1$ is effective for descent. Bourn protomodular quasivarieties are characterised in the same way as Bourn protomodular varieties, that is by having, for some natural number n, constants $e_1, \dots, e_n$, binary terms $t_1, \dots, t_n$ and an $(n+1)$-ary term t such that $t(x, t_1(x,y), \dots, t_n(x,y)) = y$ and $t_k(x,x) = e_k$ for $k = 1, \dots, n$ (see [7, 16]). In particular, quasivarieties whose theory contains a group operation are Bourn protomodular.

Whether the map $0 \to 1$ is effective for descent in a quasivariety is more subtle. There exist various criteria in the literature for detecting when a morphism is effective for descent (see again [12] and the references therein). In the case of a quasivariety $\mathscr{Q}$, it is well-known that there exists a variety $\mathscr{V}$ such that $\mathscr{Q}$ is a full subcategory of $\mathscr{V}$ closed under finite limits. If $p\colon 0_{\mathscr{Q}} \to 1_{\mathscr{Q}}$ is a regular epimorphism in $\mathscr{Q}$, then p is also a regular epimorphism in $\mathscr{V}$, since in both categories regular epimorphisms are simply surjective maps ([15]); hence p is effective for descent in $\mathscr{V}$. By applying [12, Corollary in 2.7] together with the fact that $1_{\mathscr{Q}}$ is terminal in both $\mathscr{Q}$ and $\mathscr{V}$, one finds that p is effective for descent in $\mathscr{Q}$ if and only if for every $A \in \mathscr{V}$ such that $0_{\mathscr{Q}} \times A \in \mathscr{Q}$ (with '$\times$' denoting the product in $\mathscr{V}$), one has $A \in \mathscr{Q}$. Using this criterion, we obtain the following examples.

(a) Let $\mathscr{Q}$ be the (Bourn protomodular) quasivariety of unital torsion-free rings, that is unital rings satisfying $(n \cdot x = 0 \to x = 0)$ for all positive integers n. Here, $0 = \mathbb{Z}$ and the map $0 \to 1$ is clearly surjective. Take $\mathscr{V}$ to be the variety of all unital rings, so that $\mathscr{Q}$ is a full subcategory of $\mathscr{V}$ closed under limits. Suppose $R \in \mathscr{V}$ is such that $\mathbb{Z} \times R$ is torsion-free. If $x \in R$ is such that $n \cdot x = 0$ for some positive integer n, then $n \cdot (0, x) = 0$ in $\mathbb{Z} \times R$, hence $(0, x) = 0$, and so $x = 0$. Thus R is torsion-free and $0 \to 1$ is effective for descent. We conclude that the quasivariety of torsion-free unital rings is ideally regular. Note however that $\mathscr{Q}$ is not a Barr exact category.

(b) In a similar fashion, one checks that the quasivariety of reduced unital rings (i.e. unital rings satisfying $(x^2 = 0 \to x = 0)$) and the quasivariety of unital torsion-free algebras over a fixed commutative unital ring R (i.e. unital R-algebras satisfying $(r \cdot x = 0 \to x = 0)$ for all non-zero-divisors $r \in R$) are also ideally regular.

(c) Consider instead the (Bourn protomodular) quasivariety $\mathscr{Q}$ of rings of characteristic 0 (defined by the implications $(n \cdot 1 = 0 \to 1 = 0)$ for all positive integers n). The initial object of $\mathscr{Q}$ is $\mathbb{Z}$ with the usual operations, and once again the morphism

$0_{\mathscr{Q}} \to 1_{\mathscr{Q}}$ is surjective. Note that $\mathscr{Q}$ is a full subcategory closed under limits of the variety Ring of unital rings, and observe that, for all rings $A \in$ Ring, the product ring $\mathbb{Z} \times A$ is always a ring of characteristic 0. We conclude that the regular epimorphism $0_{\mathscr{Q}} \to 1_{\mathscr{Q}}$ is *not* effective for descent, showing that $\mathscr{Q}$ is not ideally regular.

1.5. Example. A semi-localisation (as in [14]) of an ideally exact category is ideally regular. Indeed, if $\mathscr{A}$ is a semi-localisation of a Barr exact, Bourn protomodular category $\mathscr{B}$, by [6] $\mathscr{A}$ is regular and Bourn protomodular, and effective descent morphisms in $\mathscr{A}$ coincide with regular epimorphisms. If moreover $\mathscr{B}$ has finite coproducts, then so does $\mathscr{A}$, being reflective in $\mathscr{B}$. Writing $F\colon \mathscr{B} \to \mathscr{A}$ for the left adjoint to the inclusion $\mathscr{A} \hookrightarrow \mathscr{B}$, we have $F(0_{\mathscr{B}} \to 1_{\mathscr{B}}) = (0_{\mathscr{A}} \to 1_{\mathscr{A}})$ since $\mathscr{A}$ is closed under limits in $\mathscr{B}$ and F preserves colimits. Furthermore, if $0_{\mathscr{B}} \to 1_{\mathscr{B}}$ is a regular epimorphism, then so is $0_{\mathscr{A}} \to 1_{\mathscr{A}}$, since F preserves regular epimorphisms; hence it is also effective for descent. As an example, the category of unital torsion-free rings (in the sense of Example 1.4.(a)) is a (non-exact) semi-localisation of the category of commutative unital rings. This follows from the fact that the torsion elements of any ring form an ideal.

Note however that not all ideally regular categories are semi-localisations of ideally exact categories. For instance, consider the (Bourn protomodular) quasivariety $\mathscr{Q}$ of abelian groups or unital rings satisfying $(4 \cdot x = 0 \to 2 \cdot x = 0)$ (example borrowed from [12, 15]). The morphism $0_{\mathscr{Q}} \to 1_{\mathscr{Q}}$ is effective for descent (this is obvious in the case of abelian groups; for rings, use the criterion in Example 1.4), and hence $\mathscr{Q}$ is ideally regular. However, as shown in [12, 15], the canonical morphism $\mathbb{Z} \to \mathbb{Z}_2$ is a regular epimorphism which is not effective for descent, and hence $\mathscr{Q}$ cannot be a semi-localisation of a Barr exact category.

1.6. Example. Let $\mathbb{T}$ be an *ideally exact algebraic theory*, i.e. a Bourn protomodular algebraic theory with at least one constant. Then the category $\mathscr{A} = \mathrm{Alg}_{\mathbb{T}}(\mathsf{Top})$ of models of $\mathbb{T}$ in the category of topological spaces is ideally regular. Indeed, we know from [2] that $\mathscr{A}$ is regular and Bourn protomodular. Effective descent morphisms in $\mathscr{A}$ coincide with regular epimorphisms and consist of open surjective maps (see for instance [13, 8]). Since $0_{\mathscr{A}}$ is non-empty (as $\mathbb{T}$ has at least one constant) and $1_{\mathscr{A}}$ is always a singleton, it follows that the morphism $0_{\mathscr{A}} \to 1_{\mathscr{A}}$ is always open and surjective. Note moreover that $\mathscr{A}$ is in general not Barr exact, since an effective equivalence relation on an object A needs to be equipped with the topology induced by the product topology on $A \times A$. This is not the case for a general internal equivalence relation in $\mathrm{Alg}_{\mathbb{T}}(\mathsf{Top})$.

2. Equivalent characterisations

In this section we aim to characterise ideally regular categories in terms of properties of a monadic functor over a homological category, in analogy with results from [9].

Before doing so, we fix some notation concerning slice categories and recall some related facts from [9] which will be needed in what follows.

Given a category $\mathscr{A}$ and an object $X \in \mathscr{A}$, objects of the comma category $(\mathscr{A} \downarrow X)$ will be written as pairs (A, α), with $\alpha\colon A \to X$, and morphisms in $(\mathscr{A} \downarrow X)$ will be written as morphisms of the underlying objects. When $\mathscr{A}$ has pullbacks, for any morphism $p\colon E \to B$ in $\mathscr{A}$, we will denote by

$$p_! \dashv p^*\colon (\mathscr{A} \downarrow B) \to (\mathscr{A} \downarrow E)$$

the *adjunction associated to* p, given, for any $(D, \delta) \in (\mathscr{A} \downarrow E)$ and $(A, \alpha) \in (\mathscr{A} \downarrow B)$, by

$$p_!(D, \delta) = (D, p \cdot \delta), \qquad p^*(A, \alpha) = (P, \pi_1),$$

with $\pi_1\colon P \to A$ denoting the projection on A of the pullback of α along p. Let us now recall the following theorem from [9].

2.1. Theorem. [9, Theorem 2.4] Let $\mathbb{T} = (T, \eta, \mu)$ denote the monad associated to the above adjunction. The following properties hold.

- $\mathbb{T}$ is *cartesian*, meaning that T preserves pullbacks, and all the naturality squares associated to η and μ are pullbacks;
- when $\mathscr{A}$ is regular, T preserves regular epimorphisms and kernel pairs;
- when $\mathscr{A}$ is Bourn protomodular and has finite coproducts, $\mathbb{T}$ is *essentially nullary*, meaning that for all objects $X \in \mathscr{A}$, the morphism

$$(T(!_X), \eta_X)\colon T(0) + X \to T(X)$$

  is a strong epimorphism.

We are now ready to state the main result of this section.

2.2. Theorem. Let $\mathscr{A}$ be any category. The following are equivalent.

(a) $\mathscr{A}$ is ideally regular;

(b) $\mathscr{A}$ is regular, has finite coproducts, and there exists a monadic functor $\mathscr{A} \to \mathscr{X}$ where $\mathscr{X}$ is homological and admits finite coproducts;

(c) there exists a monadic functor $\mathscr{A} \to \mathscr{X}$ such that both $\mathscr{A}$ and $\mathscr{X}$ have finite coproducts, $\mathscr{X}$ is homological, and the underlying functor of the corresponding monad preserves regular epimorphisms and kernel pairs.

Proof. Let p denote the unique morphism $0 \to 1$ in $\mathscr{A}$, and let $p_! \dashv p^*$ be the associated adjunction.

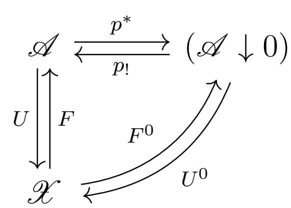
When (a) holds, since p is effective for descent, the functor

$$p^*\colon \mathscr{A} \simeq (\mathscr{A} \downarrow 1) \to (\mathscr{A} \downarrow 0)$$

is monadic, with $(\mathscr{A} \downarrow 0)$ obviously pointed. Moreover, since $\mathscr{A}$ is regular and Bourn protomodular, so is its slice $(\mathscr{A} \downarrow 0)$, which is hence homological. Lastly, once again using the regularity of $\mathscr{A}$, it follows from Theorem 2.1 that $p^*p_!$ preserves regular epimorphisms and kernel pairs. This proves that (a) $\Longrightarrow$ (c).

Next, if (c) holds, then $\mathscr{A}$ is regular since it is equivalent to the category of algebras over a monad whose underlying functor preserves regular epimorphisms (see for example [17]). Hence, (b) follows.

Finally, if (b) holds, assume that $U\colon \mathscr{A} \to \mathscr{X}$ is a monadic functor where $\mathscr{A}$ is regular, $\mathscr{X}$ is homological, and both categories admit finite coproducts. We immediately deduce that $\mathscr{A}$ is Bourn protomodular. Let F be a left adjoint to U, and for simplicity write $F(0) = 0$. We obtain the following diagram of adjunctions,



where $F^0 \dashv U^0$ are given by:

$$F^0(X) = (F(X), F(!^X)), \qquad U^0(A, \alpha) = \ker(U(\alpha))$$

(see [10, 9]). In the proof of [9, Theorem 2.6] it is shown that U^0 reflects isomorphisms, only relying on the Bourn protomodularity of $\mathscr{X}$. Since $U \cong U^0 p^*$ is monadic and U^0 is conservative, it follows from [3, Propositions 4 and 5] that p^* is also monadic, and hence p is effective for descent. ∎

With Theorem 2.2 in place, we can make the following observations. In particular, Corollary 2.3 is based on [9, Theorem 3.3.b], Remark 2.5 on [9, Remark 3.8] and Example 2.6 on [9, Example 3.5].

2.3. Corollary. Let $\mathscr{A}$ be an ideally regular category. Then there exists a monadic functor $\mathscr{A} \to \mathscr{X}$ such that $\mathscr{X}$ is homological and has finite coproducts, the corresponding monad is cartesian, essentially nullary and such that its underlying functor preserves regular epimorphisms and and kernel pairs.

Proof. It suffices to use the same monadic functor p^* as in the proof of Theorem 2.2 and recall Theorem 2.1. ∎

2.4. Remark. Note that, in Theorem 2.2, unlike in the analogous Theorem 3.1 in [9], the existence of finite coproducts must be assumed explicitly even in Item (c). Moreover, as far as we are aware, ideally regular categories do not generally admit finite colimits – unlike ideally exact categories – even when they are monadic over a homological category that does (cf. [9, Theorem 3.3.a]).

2.5. Remark. For an ideally regular category $\mathscr{A}$, the monadic adjunction $\mathscr{A} \rightleftarrows \mathscr{X}$ of Theorem 2.2.(b) and (c) coincides, up to equivalence, with the adjunction associated to $0_{\mathscr{A}} \to 1_{\mathscr{A}}$ if and only if the unit of the adjunction is cartesian. This follows from [9, Theorems 2.4 and 2.6].

2.6. Example. (a) If $\mathscr{X}$ is homological with finite coproducts and $X \in \mathscr{X}$, then the coslice $(X \downarrow \mathscr{X})$ is ideally regular. This follows from the fact that the codomain functor $(X \downarrow \mathscr{X}) \to \mathscr{X}$ is monadic. Furthermore, from Theorem 2.3, we know that every ideally regular category is equivalent to a category of algebras $\mathscr{X}^{\mathbb{T}}$ over an essentially nullary monad $\mathbb{T}$ on a homological category $\mathscr{X}$ – and [5, Theorem 1.1] tells us that $\mathbb{T}$ is nullary if and only if $\mathscr{X}^{\mathbb{T}}$ is canonically equivalent to some $(X \downarrow \mathscr{X})$.
(b) A slice $(\mathscr{X} \downarrow X)$ of a homological category with finite coproducts is ideally regular if and only if $X = 0$.

2.7. Remark. For any category $\mathscr{A}$, the following are equivalent: (a) $\mathscr{A}$ is finitely complete and Bourn protomodular, it has an initial object, and the morphism $0 \to 1$ in $\mathscr{A}$ is effective for descent; (b) there exists a monadic functor $\mathscr{A} \to \mathscr{X}$ with $\mathscr{X}$ pointed, finitely complete and Bourn protomodular. This follows by arguing as in the proof of Theorem 2.2.

3. Ideals and quotients

In this section, we show that ideally regular categories support a notion of ideal that classifies regular quotients, in direct analogy with the ideally exact case. To this end, we will make use of the following property of essentially nullary monads, recalled from [9].

3.1. Proposition. [9, Corollary 4.2] Let $\mathbb{T}$ be an essentially nullary monad on a category with finite coproducts, and let (X, ξ) be a $\mathbb{T}$-algebra. Every reflexive relation

$$R \underset{r_1}{\overset{r_0}{\underset{\longleftarrow e \longrightarrow}{\longrightarrow}}} X$$

on X admits a (unique) $\mathbb{T}$-algebra structure making r_0, r_1 and e morphisms of $\mathbb{T}$-algebras.

We can now establish the following result.

3.2. Theorem. Let $\mathscr{A}$ be an ideally regular category. Fix a monadic functor $U\colon \mathscr{A} \to \mathscr{X}$ such that

- $\mathscr{X}$ is homological and has finite coproducts;

- the monad $\mathbb{T}$ corresponding to U is essentially nullary;
- the functor T underlying $\mathbb{T}$ preserves regular epimorphisms and kernel pairs

(the existence of such a functor is guaranteed by Corollary 2.3). For every object A in $\mathscr{A}$, the following (possibly large) posets are canonically isomorphic to each other:

(a) $\mathrm{Quot}_{\mathscr{A}}(A)$, consisting of quotient objects of A;

(b) $\mathrm{EfRel}_{\mathscr{A}}(A)$, consisting of effective equivalence relations on A;

(c) $\mathrm{EfRel}_{\mathscr{X}}(U(A))$, consisting of effective equivalence relations on $U(A)$;

(d) $\mathrm{Quot}_{\mathscr{X}}(U(A))$, consisting of quotient objects of $U(A)$;

(e) $\mathrm{NSub}_{\mathscr{X}}(U(A))$, consisting of normal subobjects of $U(A)$.

Proof. The isomorphism between the posets in (a) and (b) is known, as are those between the posets in (c), (d) and (e). To prove that the posets in (b) and (c) are isomorphic, let us identify $\mathscr{A}$ with the Eilenberg-Moore category $\mathscr{X}^{\mathbb{T}}$ of algebras over $\mathbb{T}$, so that $U\colon \mathscr{X}^{\mathbb{T}} \to \mathscr{X}$ is the canonical forgetful functor. Under this identification, any object $A \in \mathscr{A}$ corresponds to some algebra $(X, \xi) \in \mathscr{X}^{\mathbb{T}}$, with $\xi\colon T(X) \to X$ in $\mathscr{X}$. Since U is a right adjoint, it induces an injective morphism

$$\mathrm{EfRel}_{\mathscr{X}^{\mathbb{T}}}((X,\xi)) \longrightarrow \mathrm{EfRel}_{\mathscr{X}}(X).$$

Now, in $\mathscr{X}$, consider an effective equivalence relation on X together with its coequaliser:

$$R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X \xrightarrow{\;q\;} Q.$$

By Proposition 3.1, R admits a $\mathbb{T}$-algebra structure $\rho\colon TR \to R$ making r_0 and r_1 morphisms of $\mathbb{T}$-algebras. We aim to show that $((R,\rho), r_0, r_1)$ is an effective relation in $\mathscr{X}^{\mathbb{T}}$. Consider the solid arrows in the following diagram.

$$\begin{array}{ccccc}
R & \underset{r_1}{\overset{r_0}{\rightrightarrows}} & X & \xrightarrow{\;q\;} & Q \\
\uparrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \xi} & & \vdots{\scriptstyle \chi} \\
T(R) & \underset{T(r_1)}{\overset{T(r_0)}{\rightrightarrows}} & T(X) & \xrightarrow[T(q)]{} & T(Q)
\end{array}$$

Since T preserves kernel pairs and regular epimorphisms, $T(q)$ is the coequaliser of $T(r_0)$ and $T(r_1)$. Now, $q \cdot \xi \cdot T(r_0) = q \cdot \xi \cdot T(r_1)$ and hence there exists a unique $\chi\colon T(Q) \to Q$ such that $q \cdot \xi = \chi \cdot T(q)$. Using that q and $T(T(q))$ are epimorphisms, one easily checks that (Q, χ) is a $\mathbb{T}$-algebra, and by construction, q is a morphism of $\mathbb{T}$-algebras from (X, ξ) to (Q, χ). Since monadic functors reflect and create limits, we conclude that $((R, \rho), r_0, r_1)$ is the kernel pair of $q\colon (X, \xi) \to (Q, \chi)$ in $\mathscr{X}^{\mathbb{T}}$, concluding the proof. ∎

3.3. Remark. In a homological category, the poset of normal subobjects of a given object forms a meet-semilattice, with the meet of two normal subobjects given by their pullback. Consequently, all the posets of Theorem 3.2 are in fact meet-semilattices. Moreover, the poset of normal subobjects of an object becomes a lattice as soon as the ambient category also admits pushouts of regular epimorphisms along regular epimorphisms. This condition is automatically satisfied when the category is Barr exact (see [1]).

In light of Theorem 3.2, the definition of an *ideal* from [9] extends naturally to the context of ideally regular categories. We point out, however, that in [9, Theorem 4.3], the functor underlying the monad associated to U is not required to preserve regular epimorphisms and kernel pairs, unlike in our Threorem 3.2.

3.4. Definition. In the situation of Theorem 3.2, given an object A in $\mathscr{A}$, a normal monomorphism in $\mathscr{X}$ with codomain $U(A)$ will be called a *U-ideal* of A. In particular, if U is the pullback functor $\mathscr{A} \to (\mathscr{A} \downarrow 0)$ along $0 \to 1$, then U-ideals of A will be simply called *ideals* of A.

3.5. Example. When $\mathscr{A}$ is, for instance, the category of rings of characteristic 0, the category of unital torsion-free rings (as in Example 1.4.(a)) or the category of unital topological rings, then the corresponding functor $p^*\colon \mathscr{A} \to (\mathscr{A} \downarrow 0)$ coincides, up to equivalence, with the obvious forgetful functor to the category of non-unital rings, non-unital torsion-free rings and non-unital topological rings, respectively. In these cases, we therefore recover the natural notion of ideal one would expect.

3.6. Remark. In any category $\mathscr{A}$ with finite limits and an initial object, one can still define the ideals of an object $A \in \mathscr{A}$ as the normal subobjects of $p^*(A)$, with $p\colon 0 \to 1$. However, such ideals will generally fail to classify regular quotients, even under considerably strong assumptions on $\mathscr{A}$. Consider for instance the quasivariety $\mathscr{Q}$ of Example 1.4.(c) of rings of characteristic 0. We have seen that $\mathscr{Q}$ is a regular, Bourn protomodular category with finite coproducts such that $p\colon 0 = \mathbb{Z} \to 1$ is a regular epimorphism but not an effective descent morphism. One can show that, up to equivalence, the corresponding functor p^* coincides with the obvious forgetful functor to the category of non-unital rings. Yet the ring $\mathbb{Z}$ has no non-trivial quotients of characteristic 0, while still admitting infinitely many ideals.

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Department of Mathematics, University of Milan, Via Cesare Saldini 50, 20133 Milan, Italy
Email: sandra.mantovani@unimi.it
mariano.messora@unimi.it

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