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A TOOLKIT FOR STRUCTURED LIFTS

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Abstract. We develop a general framework for working with structured lifting problems, establishing closure and uniqueness properties of their solutions. In a subsequent paper, we apply these results to axiomatize computation rules of cubical type theory.

Introduction

When working with categorical models of dependent type theory, one often models pattern matching using compatible choices of solutions for a certain class of lifting problems, with the canonical example being the J-eliminator for identity types [3]. This is true verbatim for models such as contextual categories [7] and categories with families [9, 4], and is typically imposed on weaker models, like comprehension categories [13], type-theoretic fibration categories [19], and tribes [15], when describing their corresponding strict analogues.

Such compatible families of lifts arise naturally in the context of *algebraic* weak factorization systems [12, 10] and algebraic model structures [18], a replacement of weak factorization systems and model structures that require that the factorizations and lifts not merely exist, but rather be chosen in a suitably compatible (hence algebraic) manner [6]. As a result, algebraic weak factorization systems have been used to supply models of dependent type theory [5, 2]. When working with the category of all models, however, their usefulness is limited due to the fact that the identity type factorization is in general not functorial [11]. In other words, the problem with using algebraic weak factorization systems for axiomatizing the structure present in the models is the "factorization" part of the weak factorization system.

One could in principle try to build a new algebraic weak factorization system on a model of dependent type theory, but two problems arise. First, the standard tool for constructing algebraic weak factorization systems, the algebraic small object argument, requires the base category to admit certain colimits [10], which general models of type theory do not possess. Second, even if it were possible to run the algebraic small object argument on the set of reflexivity maps, the resulting right class would not in general

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recover dependent projections. All of this to say that algebraic weak factorization systems are simply too strong a framework to speak of models of dependent type theory.

In the present paper, we address these issues by developing the theory of structured lifts without the corresponding factorization requirements, and with minimal assumptions on the base category. This is in contrast with more "heavy duty" conditions required to get the framework of algebraic weak factorization systems off the ground: for example, requiring that the two classes be given by (co-)algebras for a suitable (co-)monad on the base category.

We prove several closure properties of these lifts, for example, their compatibility with base change, composition, and Leibniz transpose. We further investigate the uniqueness properties of these lifts. All of these are done with an eye towards understanding computation rules of dependent type theory in general, and of cubical type theory in particular [8, 1]. Indeed, for the three operations mentioned above: base change corresponds to the stability of eliminators under substitution, while composition and Leibniz transpose describe the compatibility between the filling operation and dependent sums and path types, respectively. The uniqueness properties ensure that the computation rules hold in their expected (propositional) form.

By keeping track of exponentiable objects, all of our constructions can be formulated not only in the framework of universe category models of dependent type theory [21, 17], but also in the framework of Uemura's categories with representable maps [20].

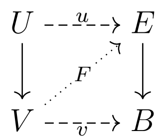
This paper is organized as follows. In Section 1, we introduce the requisite notions of lifts, structured lifts, and restricted lifts. We also explain the relation between structured lifts and categorical models of type theory, formulated as Uemura's categories with representable maps. In the subsequent four sections, we investigate different closure properties of structured lifts. Specifically, in Subsection 2, we show that structured lifts can be restricted and composed; in Subsection 3, we prove their closure under base change; and in Subsection 5, we establish their closure under Leibniz transposes. As a preliminary step towards Subsection 5, in Section 4, we design a way of speaking about pushout-product without appealing to any cocompleteness properties of the base category. Finally, in Section 6, we describe a general framework for showing that two lifts are related, and we instantiate our results in a model category.

1. Structured Lifts

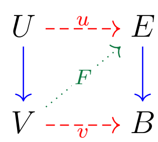
In this section, we will introduce the notion of uniform lifts. At a high level, they are the uniform versions of lifting properties, where the uniformity arise from functoriality of the product. Throughout the rest of this paper, we fix a finitely complete category $\mathbb{C}$ whose internal-Homs, when they exist, we denote by $[-,-]$ and whose internal-Homs, when they exist, in each overslice ${}^{\mathbb{C}}\!/_{C}$ we denote by $[-,-]_C$. The main example we have in mind are Uemura's categories with representable maps (CwRs).

We first start by recalling what is a lifting problem and its associated solution.

1.1. DEFINITION. *A lifting problem of $U \to V$ against $E \to B$ is a pair of dashed maps (u, v) as below making the square commute. A solution to the lifting problem (u, v) is a diagonal filler F to the square making the entire diagram commute.*

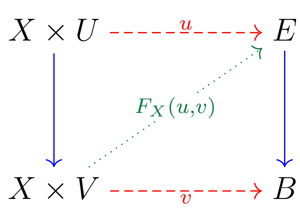


For ease of viewing, we color code by depicting lifting problems in red, the maps being lifted against each other in blue, and lifting solutions in green. Therefore, we would draw the above lifting problem and solution pair as

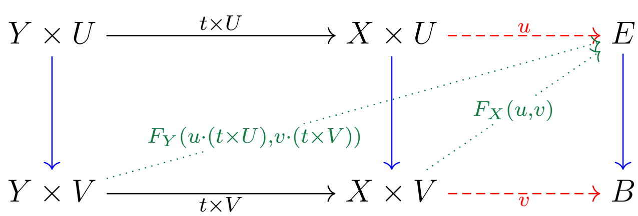


Lifting solutions are dependent on the supplied lifting problem. Therefore, given a family of lifting problems satisfying some form of compatibility conditions, we can require the corresponding family of lifting solutions to also be compatible. In this paper, we are interested in the case when the compatibility conditions arise from functoriality of the product.

1.2. DEFINITION. *Fix maps $i\colon U \to V$ and $p\colon E \to B$. A* family of lifts *is an association taking each object $X \in \mathbb{C}$ and lifting problem (u, v) of $X \times i$ against p to a solution $F_X(u, v)$.*



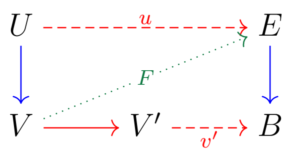
This family F is said to be uniform *when one has $F_Y(u \cdot (t \times U), v \cdot (t \times V)) = F_X(u, v) \cdot (t \times V)$ for any $t\colon Y \to X$.*

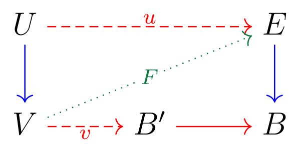


In the above definitions, the dashed horizontal maps u, v can be arbitrary. However, we can also require that they factor through certain specific maps. This gives rise to the notion of *left-restricted* and *right-restricted* lifts and their uniform versions.

1.3. Definition. *A lifting problem of $U \to V$ against $E \to B$* left-restricted *along $V \to V'$ is a pair of dashed maps (u, v') as below making the square on the left commute. A solution to this left-restricted lifting problem (u, v) is a diagonal filler F to the square making the entire diagram on the left commute.*

Dually, lifting problem of $U \to V$ against $E \to B$ right-restricted *along $B' \to B$ is a pair of dashed maps (u, v) as above making the square on the right commute. A solution to this right-restricted lifting problem (u, v) is a diagonal filler F to the square making the entire diagram on the right commute.*

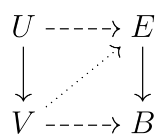




The primary reason for considering restricting along maps $V \to V'$ on the bottom left is due to modelling the filling operation for Path-types of cubical type theory, as we consider in the following example.

1.4. Example. When modelling cubical type theory, one has two classes of maps on $\mathbb{C}$: fibrations and trivial cofibrations as well as a distinguished object $\mathbb{I}$ equipped with two endpoints $\{\mathbb{0}\}, \{\mathbb{1}\} \rightrightarrows \mathbb{I}$ so that when $\mathbb{C}$ is sufficiently cocomplete, one can put $\partial\mathbb{I} \coloneqq \{\mathbb{0}\} \sqcup \{\mathbb{1}\} \hookrightarrow \mathbb{I}$.

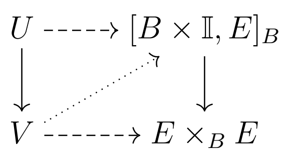
Similar to model categories, the filling operation of cubical type theory manifests as a choice of lifts of trivial cofibrations against the fibrations, as in choices of solutions to lifting problems as follows, where $U \to V$ is a trivial cofibration and $E \to B$ is a fibration.



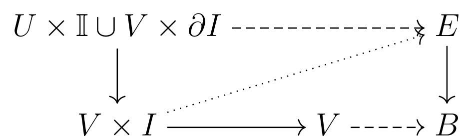
On the other hand, the Path-type of cubical type theory states that for each fibration $E \to B$, the restriction along local-Homs

$$[B \times \mathbb{I}, E]_B \longrightarrow [B \times \partial I, E]_B \cong E \times_B E \in {}^{\mathbb{C}}/_{B}$$

is yet again a fibration. Thus, for cofibrations $U \to V$, one has a choice of lifts



As we will observe in Corollary 5.11, as we are taking *local* Homs, applying the usual Leibniz transpose argument to the immediately preceding lifting problem yields the lifting problem

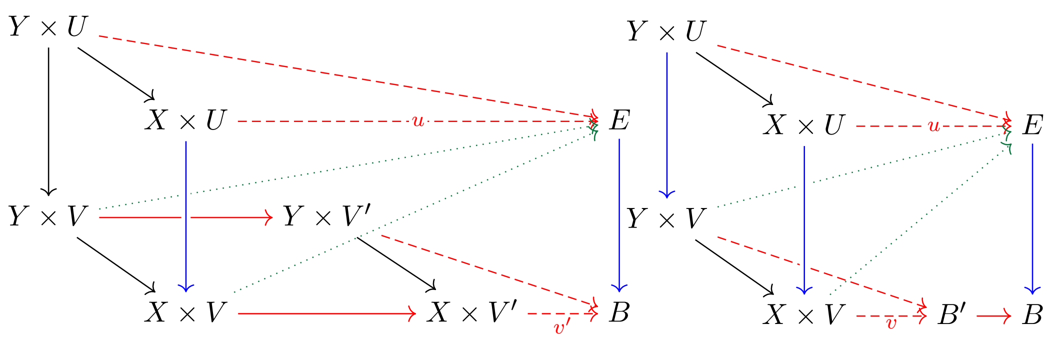


which is an example of a left restricted lifting problem.

Parameterising over the choice of restricted lifting problems, we can similarly define families of their lifting solutions and a corresponding uniformity condition.

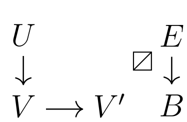
1.5. Definition. *Fix maps* $i\colon U \to V$ *and* $p\colon E \to B$.

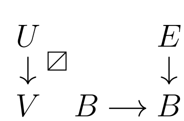
A family of lifts left-restricted *along* $j\colon V \to V'$ *is an association taking each object* $X \in \mathbb{C}$ *and lifting problem* (u, v') *of* $X \times i$ *left-restricted along* $X \times j$ *against* p *to a solution* $F_X(u, v')$. *This family* F *is said to be* uniform *when one has one has* $F_Y(u \cdot (t \times U), v' \cdot (t \times V')) = F_X(u, v') \cdot (t \times V)$ *for any* $t\colon Y \to X$.



Dually, a family of lifts right-restricted *along* $q\colon B' \to B$ *is an association taking each object* $X \in \mathbb{C}$ *and lifting problem* (u, v) *of* $X \times i$ *right-restricted along* q *against* p *to a solution* $F_X(u, v)$. *This family is said to be* uniform *when one has* $F_Y(u \cdot (t \times U), v \cdot (t \times V)) = F_X(u, v) \cdot (t \times V)$ *for any* $t\colon Y \to X$.

The set of all such left- and right-restricted uniform family of lifts, which we refer to as lifting structures *are respectively denoted by*





1.6. Remark. The uniformity conditions of Definition 1.5 can be phrased in terms of naturality conditions.

We first consider the left-restricted uniform case with a map $U \to V$ on the left restricted along $V \to V'$ against $E \to B$ on the right. For each $X \in \mathbb{C}$, the set of all unrestricted lifting problems of $X \times U \to X \times V$ against $E \to B$ is observed to be the

Hom-set of the arrow category $\mathbb{C}^{\to}(X \times U \to X \times V, E \to B)$. Letting X vary, we obtain the presheaf of parameterised lifting problems.

$$\mathbb{C}^{\to}\left(\begin{array}{c} -\times U \\ \downarrow \\ -\times V \end{array}, \begin{array}{c} E \\ \downarrow \\ B \end{array}\right)$$

Similarly, the set of lifting problems restricted along $X \times V \to X \times V'$ is observed to be the Hom-set $\mathbb{C}^{\to}(X \times U \to X \times V \to X \times V', E \to B)$ and letting X vary then gives rise to the presheaf of parameterised restricted lifting problems.

$$\mathbb{C}^{\to}\left(\begin{array}{l} -\times U \\ \quad\downarrow \\ -\times V \to -\times V' \end{array}, \begin{array}{c} E \\ \downarrow \\ B \end{array}\right)$$

Given a restricted lifting problem $(u\colon X \times U \to E, v'\colon X \times V' \to B) \in \mathbb{C}^{\to}(X \times U \to X \times V \to X \times V', E \to B)$, we can always forget the restriction and produce an unrestricted lifting problem $(u\colon X \times U \to E, v'\colon X \times V' \to B) \in \mathbb{C}^{\to}(X \times U \to X \times V, E \to B)$. This operation is observed to be natural in X so once again letting X vary gives rise to a map between the two aforementioned presheaves.

$$\mathbb{C}^{\to}\left(\begin{array}{l} -\times U \\ \quad\downarrow \\ -\times V \to -\times V' \end{array}, \begin{array}{c} E \\ \downarrow \\ B \end{array}\right) \longrightarrow \mathbb{C}^{\to}\left(\begin{array}{c} -\times U \\ \downarrow \\ -\times V \end{array}, \begin{array}{c} E \\ \downarrow \\ B \end{array}\right)$$

On the other hand, a solution to each restricted lifting problem $(u\colon X \times U \to E, v'\colon X \times V' \to B) \in \mathbb{C}^{\to}(X \times U \to X \times V \to X \times V', E \to B)$ is some choice of diagonal map $X \times V \to E$. Letting X vary, the set of parameterised solution candidates is given by the presheaf

$$\mathbb{C}\left(-\times V, E\right)$$

Given such a diagonal map $X \times V \to E$, we can always recover the lifting problem it solves by pre-composing with $X \times U \to X \times V$ and post-composing with $E \to B$ to obtain a map from $X \times U \to X \times V$ to $E \to B$. Therefore, we obtain a map taking each family of lifting solution candidate to the corresponding family of lifting problems it tries to solve.

$$\mathbb{C}\left(-\times V, E\right) \longrightarrow \mathbb{C}^{\to}\left(\begin{array}{c} -\times U \\ \downarrow \\ -\times V \end{array}, \begin{array}{c} E \\ \downarrow \\ B \end{array}\right)$$

So, to say that a lifting solution candidate indeed solves the lifting problem which it is assigned to, one requires a horizontal dotted map in the following. Therefore, the set of uniform left-restricted lifts is just the set of natural transformations in the following slice category in $\widehat{\mathbb{C}}$

$$\begin{array}{ccc}
\mathbb{C}^{\to}\left(\begin{array}{c} -\times U \\ \downarrow \\ -\times V \longrightarrow -\times V' \end{array}, \begin{array}{c} E \\ \downarrow \\ B \end{array}\right) & \cdots\cdots\cdots\cdots\cdots\to & \mathbb{C}\left(-\times V, E\right) \\
& \searrow \qquad \swarrow & \\
& \mathbb{C}^{\to}\left(\begin{array}{c} -\times U \\ \downarrow \\ -\times V \end{array}, \begin{array}{c} E \\ \downarrow \\ B \end{array}\right) &
\end{array}$$

Dually, given $B' \to B$, denote the set of right-restricted families of lifting problems as the presheaf

$$\mathbb{C}^{\to}\left(\begin{array}{c} -\times U \\ \downarrow \\ -\times V \end{array}, \begin{array}{c} E \\ \downarrow \\ B' \to B \end{array}\right) := \left(X \mapsto \left\{ \left(\begin{array}{c} X\times U \overset{u}{\dashrightarrow} E \\ X\times V \underset{v}{\dashrightarrow} B' \end{array} \right) \;\middle|\; \begin{array}{ccccc} X\times U & & \overset{u}{\dashrightarrow} & & E \\ \downarrow & & & & \downarrow \\ X\times V & \underset{v}{\dashrightarrow} & B' & \longrightarrow & B \end{array} \right\} \right) \in \widehat{\mathbb{C}}$$

Then, the set of uniform right-restricted lifts along $B' \to B$ is the set of natural transformations in the following slice category in $\widehat{\mathbb{C}}$.

$$\begin{array}{ccc}
\mathbb{C}^{\to}\left(\begin{array}{c} -\times U \\ \downarrow \\ -\times V \end{array}, \begin{array}{c} E \\ \downarrow \\ B' \to B \end{array}\right) & \cdots\cdots\cdots\cdots\cdots\to & \mathbb{C}^{\to}\left(-\times V, E\right) \\
& \searrow \qquad \swarrow & \\
& \mathbb{C}^{\to}\left(\begin{array}{c} -\times U \\ \downarrow \\ -\times V \end{array}, \begin{array}{c} E \\ \downarrow \\ B \end{array}\right) &
\end{array}$$

We observe that when $\mathbb{C}$ is locally cartesian closed, the set of uniform (restricted) lifts can be internalised using representability arguments as in the following Lemmas 1.7 and 1.9.

1.7. Lemma. *Suppose one has maps $U \to V$ and $E \to B$ and $V \to V'$ between exponential objects. Then, one has representability of:*

(1) Restricted lifting problems.

$$\mathbb{C}(-, [U,E] \times_{[U,B]} [V',B]) \longleftarrow\!\cong\!\longrightarrow \mathbb{C}^{\to}\left(\begin{array}{c} -\times U \\ \downarrow \\ -\times V \longrightarrow -\times V' \end{array}, \begin{array}{c} E \\ \downarrow \\ B \end{array}\right)$$

(2) Unrestricted lifting problems.

$$\mathbb{C}(-, [U,E] \times_{[U,B]} [V,B]) \xleftarrow{\cong} \mathbb{C}^{\to}\left(\begin{array}{c}- \times U\\ \downarrow \\ - \times V\end{array}, \begin{array}{c}E\\ \downarrow\\ B\end{array}\right)$$

(3) Lifting solutions.

$$\mathbb{C}(-, [V,E]) \xleftarrow{\cong} \mathbb{C}\left(- \times V, E\right)$$

PROOF. By continuity of the Yoneda embedding,

$$\begin{aligned}\mathbb{C}(-, [U,E] \times_{[U,B]} [V',B]) &\cong \mathbb{C}(-, [U,E]) \times_{\mathbb{C}(-,[U,B])} \mathbb{C}(-, [V',B])\\ &\cong \mathbb{C}(- \times U, E) \times_{\mathbb{C}(-\times U,B)} \mathbb{C}(- \times V', B)\end{aligned}$$

This is exactly the commutative squares from $- \times U \to - \times V'$ to $E \to B$, so one has the isomorphism in the first item. A similar argument gives the isomorphism in the second and third items. ∎

1.8. COROLLARY. *Suppose one has maps $U \to V$ and $E \to B$ and $V \to V'$ where U, V, V' are exponentiable objects. Then, the set of left-restricted uniform lifts is just the Hom-set in the slice category.*

$$\begin{array}{c}U\\ \downarrow\\ V \to V'\end{array} \boxslash \begin{array}{c}E\\ \downarrow\\ B\end{array} \cong \mathbb{C}/_{[U,E] \times_{[U,B]} [V,B]}([U,E] \times_{[U,B]} [V',B], [V,E])$$

PROOF. Immediately by Lemma 1.7. ∎

Dually, we have an internalisation result on the right.

1.9. LEMMA. *Suppose one has maps $U \to V$ and $E \to B$ and $B' \to B$ where U, V are exponentiable objects. Then, one has representability of:*

(1) Restricted lifting problems.

$$\mathbb{C}(-, [U,E] \times_{[U,B]} [V,B']) \xleftarrow{\cong} \mathbb{C}^{\to}\left(\begin{array}{c}- \times U\\ \downarrow \\ - \times V\end{array}, \begin{array}{c}E\\ \downarrow\\ B' \to B\end{array}\right)$$

(2) Unrestricted lifting problems.

$$\mathbb{C}(-, [U,E] \times_{[U,B]} [V,B]) \xleftarrow{\cong} \mathbb{C}^{\to}\left(\begin{array}{c}- \times U\\ \downarrow \\ - \times V\end{array}, \begin{array}{c}E\\ \downarrow\\ B\end{array}\right)$$

(3) Lifting solutions.

$$\mathbb{C}(-,[V,E]) \xleftarrow{\cong}\!\!\rightarrow \mathbb{C}^{\to}\left(\begin{matrix} \\ -\times V \end{matrix} \;,\; \begin{matrix} E \\ \\ \end{matrix} \right)$$

Proof. Identical argument by representability as in Lemma 1.7. ∎

1.10. Corollary. *Suppose one has maps $U \to V$ and $E \to B$ and $B' \to B$ where U, V are exponentiable objects. Then, the set of left-restricted uniform lifts is just the Hom-set in the slice category.*

$$\begin{matrix} U \\ \downarrow \\ V \end{matrix} \boxslash \begin{matrix} & E \\ & \downarrow \\ B' \to & B \end{matrix} \cong \mathbb{C}/_{[U,E]\times_{[U,B]}[V,B]}\big([U,E]\times_{[U,B]}[V,B'],[V,E]\big)$$

As an immediate consequence of internalisation, we obtain the following result on the existence of stable lifting structures.

1.11. Theorem. *Let $\mathbb{C}$ a locally cartesian closed category model category in which the pullback-power of (trivial cofibration, fibration)- or (cofibration, trivial fibration)-pairs admit sections. Then, the sets of structured lifts*

$$\begin{matrix} U & & & E \\ \downarrow & & \boxslash & \downarrow \\ V & \longrightarrow & V' & B \end{matrix} \qquad\qquad \begin{matrix} U & & & E \\ \downarrow & \boxslash & & \downarrow \\ V & B' & \longrightarrow & B \end{matrix}$$

and are non-empty as long as $(U \to V, E \to B)$ is either a (trivial cofibration, fibration)- or a (cofibration, trivial fibration)-pair.

Proof. By Lemmas 1.7 and 1.9. ∎

Lifting Structures in CwRs. In Uemura's framework of categories with representable maps (CwRs), a type theory is *defined* to be a CwR [20, Definition 3.2.3]. The internalisation results of Corollaries 1.8 and 1.10 now allows one to freely extend a CwR $\mathbb{T}$ with a lifting structure.

In order to state this construction, we recall some preliminaries about CwRs from [20, 14].

1.12. Definition. [14, Definition 2.3.1], cf. [20, Definition 4.2] *A* category with representable maps (CwR) *is a category $\mathbb{C}$ with finite limits equipped with a replete wide subcategory of pullback-stable class $\mathsf{R}_{\mathbb{C}}$ of exponentiable maps called the* representable maps.

A map of CwRs is a map between their underlying categories preserving finite limits, representable maps, and pushforwards along representable maps.

Denote by CwR *the 2-category of (small) CwRs, maps of CwRs, and natural transformations whose square at representable maps are pullbacks.*

1.13. Definition. *Denote by* $\mathsf{Cat_m}$ *the 2-category of (small) categories with marked maps and squares.*

That is, a marked category is a category $\mathbb{C}$ *equipped with two specific choices of replete wide subcategories* $\mathcal{M}_{\mathbb{C}} \hookrightarrow \mathbb{C}$ *and* $\mathcal{S}_{\mathbb{C}} \hookrightarrow \mathbb{C}^{\to}$*. A 1-cell between marked categories is exactly a functor between underlying categories sending marked maps and squares to marked maps and squares. A 2-cell between 1-cells of marked categories is a natural transformation whose naturality square at marked arrows are also marked squares.*

Often, we denote the marked maps by the arrow $\nrightarrow$*.*

The representable maps of a CwR gives it the structure of a marked category by taking the marked maps as representable maps and marked squares as pullback squares. Conversely, each marked category freely gives rise to a CwR due to the following result by Jelínek.

1.14. Theorem. [14, Corollaries 3.2.16 and 3.2.17] *The forgetful 2-functor* $|-|\colon \mathsf{CwR} \to \mathsf{Cat_m}$ *has a left biadjoint* $\langle - \rangle \colon \mathsf{Cat_m} \to \mathsf{CwR}$ *and* CwR *has all bicolimits.*

We now use this result to freely extend CwRs with lifting structures, which amounts to formally adjoining a map between objects in a slice category.

1.15. Construction. *Let* $\mathbb{C}$ *be a CwR. Fix two maps* $\mathbb{I} \to \mathbb{J}$ *and* $\mathbb{I} \to |\mathbb{C}|$ *of marked categories. Then, we write* $\mathbb{C} \cup_{\mathbb{I}} \mathbb{J}$ *for the following bipushout in* CwR

$$\begin{array}{ccc} \langle \mathbb{I} \rangle & \longrightarrow & \mathbb{C} \\ \big\downarrow & \ulcorner & \big\downarrow \\ \langle \mathbb{J} \rangle & \longrightarrow & \mathbb{C} \cup_{\mathbb{I}} \mathbb{J} \end{array}$$

where the map $\langle \mathbb{I} \rangle \to \mathbb{C}$ *in the top row is the* $(\langle - \rangle \dashv |-|)$*-transpose of the map* $\mathbb{I} \to |\mathbb{C}|$ *and map in the left row is the image of* $\mathbb{I} \to \mathbb{J}$ *under* $\langle - \rangle$*, as per [14, Corollaries 3.2.16 and 3.2.17].*

1.16. Construction. *Let* $\mathbb{C}$ *be a CwR and fix maps* $U \to V \to V'$ *between exponential objects and* $E \to B$ *in* $\mathbb{C}$*. Define the CwR obtained from* $\mathbb{C}$ *by freely extending with a lifting structure of* $U \to V$ *against* $E \to B$ *restricted along* $V \to V'$ *to be the following CwR obtained by applying Construction 1.15*

$$\mathbb{C}\left[\begin{array}{ccc} U & & E \\ \downarrow & \boxslash & \downarrow \\ V \to V' & & B \end{array}\right] \coloneqq \mathbb{C} \cup_{\{\bullet \to \bullet \leftarrow \bullet\}} \left\{ \begin{array}{ccc} [U,E] \times_{[U,B]} [V',B] & \longrightarrow & [V,E] \\ \downarrow & \swarrow & \\ [U,E] \times_{[U,B]} [V,B] & & \end{array} \right\}$$

This bipushout freely adds a map $[U,E] \times_{[U,B]} [V',B] \to [V,E]$ *over* $[U,E] \times_{[U,B]} [V',B]$ *to* $\mathbb{C}$ *in* CwR*.*

Dually, if $U \to V$ is a map between exponentiable objects in $\mathbb{C}$ and one has maps $E \to B$ along with $B' \to B$ then the CwR with a formal lifting structure of $U \to V$ against $E \to B$ restricted along $B' \to B$ is obtained by the bipushout in CwR *obtained by Construction 1.15.*

$$\mathbb{C}\left[\begin{array}{ccc} U & \boxslash & E \\ \downarrow & & \downarrow \\ V & B' \to & B \end{array}\right] \coloneqq \mathbb{C} \cup_{\{\bullet \to \bullet \leftarrow \bullet\}} \left\{ \begin{array}{c} [U,E] \times_{[U,B]} [V,B'] \to [V,E] \\ \downarrow \swarrow \\ [U,E] \times_{[U,B]} [V,B] \end{array} \right\}$$

1.17. THEOREM. *Let $\mathbb{C}$ be a CwR and fix maps $U \to V \to V'$ between representable objects (i.e. objects whose maps into the terminal object are representable) and $E \to B$ in $\mathbb{C}$. Then, isomorphism classes of maps as on the left below correspond bijectively to isomorphism classes of maps $F \colon \mathbb{C} \to \mathbb{D} \in$* CwR *equipped with a choice of a lifting structure of $FU \to FV$ against $FE \to FB$ restricted along $FV \to FV'$.*

$$\mathbb{C}\left[\begin{array}{ccc} U & \boxslash & E \\ \downarrow & & \downarrow \\ V \to V' & & B \end{array}\right] \to \mathbb{D} \in \mathsf{CwR} \qquad \mathbb{C}\left[\begin{array}{ccc} U & \boxslash & E \\ \downarrow & & \downarrow \\ V & B' \to & B \end{array}\right] \to \mathbb{D} \in \mathsf{CwR}$$

Dually, given $B' \to B$ in $\mathbb{C}$, isomorphism classes of maps as on the right above correspond to isomorphism classes of maps $F \colon \mathbb{C} \to \mathbb{D} \in$ CwR *equipped with a choice of a lifting structure of $FU \to FV$ against $FE \to FB$ restricted along $FB' \to FB$.*

PROOF. We show the case for left restricted lifting structures. By the universal property of the bipushout, isomorphism classes of maps

$$\mathbb{C} \cup_{\{\bullet \to \bullet \leftarrow \bullet\}} \left\{ \begin{array}{c} [U,E] \times_{[U,B]} [V',B] \to [V,E] \\ \downarrow \swarrow \\ [U,E] \times_{[U,B]} [V,B] \end{array} \right\} \to \mathbb{D} \in \mathsf{CwR}$$

correspond uniquely to isomorphism classes of maps $F \colon \mathbb{C} \to \mathbb{D}$ equipped with a choice of a map $F([U,E] \times_{[U,B]} [V',B]) \to F([V,E])$ over $F([U,E] \times_{[U,B]} [V,B])$. But F is left exact and preserves all pushforwards along representable maps, so F sends the cospan

$$[U,E] \times_{[U,B]} [V',B] \to [U,E] \times_{[U,B]} [V,B] \leftarrow [V,E] \in \mathbb{C}$$

to

$$[FU,FE] \times_{[FU,FB]} [FV',FB] \to [FU,FE] \times_{[FU,FB]} [FV,FB] \leftarrow [FV,FE] \in \mathbb{D}$$

Because $F \colon \mathbb{C} \to \mathbb{D}$ is a map of CwRs and $U \to V \to V'$ are maps between representable objects the map $FU \to FV \to FV'$ is again a map between representable, and thus exponentiable, objects. Thus, the lifting structures of $FU \to FV$ against $FE \to FB$ restricted along $FV \to FV'$ are exactly maps $[FU,FE] \times_{[FU,FB]} [FV',FB] \to [FV,FE]$ over $[FU,FE] \times_{[FU,FB]} [FV,FB]$ by Corollary 1.8. ∎

2. Restricting and Composing Structured Lifts

We now give various ways of restricting lifting structures to construct new lifting structures from old. We first fix the left map and put on various restrictions on the right map in Constructions 2.1 and 2.2. We also show that structured lift against composable pairs of right maps induce structured lifts against the composite in Construction 2.3. Then, we fix the right map and put on various restrictions on the left map in Constructions 2.4 and 2.6.

First, we show that restricting an existing structured lift on the right gives a restricted structured lift.

2.1. Construction. *Fix maps $U \to V$ and $E \to B$ and $q\colon B' \to B$. We define a map as on the left*

$$\begin{pmatrix} U & & E \\ \downarrow & \boxslash & \downarrow \\ V & & B \end{pmatrix} \longrightarrow \begin{pmatrix} U & & & E \\ \downarrow & \boxslash & & \downarrow \\ V & & B' \xrightarrow[q]{} & B \end{pmatrix} \qquad \begin{array}{ccccc} X \times U & \overset{u}{\dashrightarrow} & & & E \\ \downarrow & & F(u,qv) & & \downarrow \\ X \times V & \underset{v}{\dashrightarrow} & B' & \xrightarrow[q]{} & B \end{array}$$

where each structured lift F of $U \to V$ against $E \to B$ is taken to the structured lift that solves each lifting problem (u, v) with the solution $F(u, qv)$.

We next note that the restriction on the right can be removed simply by pulling back the right map.

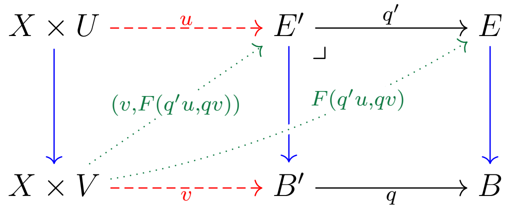
2.2. Construction. *Suppose one has a map $U \to V$ along with a pullback $E' \to B'$ of $E \to B$ along $q\colon B' \to B$.*

$$\begin{array}{ccc} E' & \xrightarrow{q'} & E \\ \downarrow & \lrcorner & \downarrow \\ B' & \xrightarrow[q]{} & B \end{array}$$

Then, we define a bijection as follows

$$\begin{pmatrix} U & & & B \\ \downarrow & \boxslash & & \downarrow \\ V & & B' \xrightarrow[q]{} & B \end{pmatrix} \xrightarrow{\cong} \begin{pmatrix} U & & E' \\ \downarrow & \boxslash & \downarrow \\ V & & B' \end{pmatrix}$$

It sends each restricted F of $U \to V$ against $E \to B$ restricted along $B' \to B$ to the lifting structure which solves lifting problems (u, v) with the solution $(v, F(q'u, qv))$.

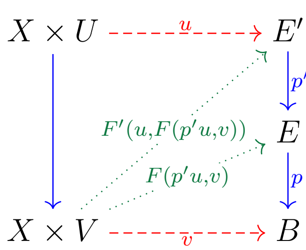


In the next construction, we show how structured lifts against composable pairs of right maps induce structured lifts against their composite.

2.3. Construction. *Fix maps $U \to V$ and a composable pair of maps $E' \xrightarrow{p'} E \xrightarrow{p} B$. Then, we define a map*

$$\begin{pmatrix} U & & E' \\ \downarrow & \boxslash & \downarrow p' \\ V & & E \end{pmatrix} \times \begin{pmatrix} U & & E \\ \downarrow & \boxslash & \downarrow p \\ V & & B \end{pmatrix} \longrightarrow \begin{pmatrix} & & E' \\ U & & \downarrow p' \\ \downarrow & \boxslash & E \\ V & & \downarrow p \\ & & B \end{pmatrix}$$

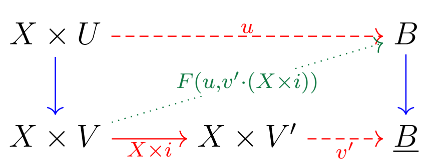
that sends a pair $F' \in (U \to V) \boxslash (E' \to E)$ and $F \in (U \to V) \boxslash (E \to B)$ to the lifting structure which solves lifting problems (u, v) with the consecutive lift $F'(u, F(p'u, v))$.



We now fix the right map and put on various restrictions on the left map. First, we show the left analogue of Construction 2.1.

2.4. Construction. *Fix maps $U \to V$ and $i \colon V \to V'$ and $E \to B$. We construct a map as on the left*

$$\begin{pmatrix} U & & E \\ \downarrow & \boxslash & \downarrow \\ V & & B \end{pmatrix} \longrightarrow \begin{pmatrix} U & & & E \\ \downarrow & & \boxslash & \downarrow \\ V \xrightarrow[i]{} V' & & & B \end{pmatrix}$$



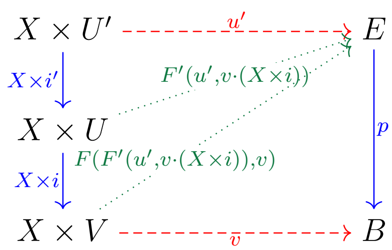
where the image of $F \in (U \to V) \boxslash (E \to B)$ solves each lifting problem (u, v') with the solution $F(u, v' \cdot (X \times i))$.

We also have a left analogue of Construction 2.3.

2.5. Construction. *Fix a composable pair of maps $U' \xrightarrow{i'} U \xrightarrow{i} V$ and a map $E \to B$. Then, we define a map*

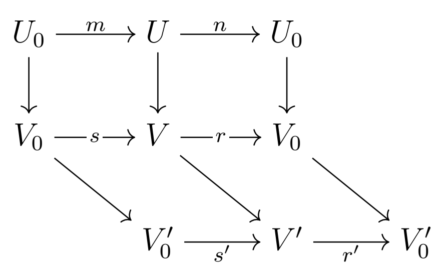
$$\begin{pmatrix} U' & & E \\ {\scriptstyle i'}\downarrow & \boxslash & \downarrow \\ U & & B \end{pmatrix} \times \begin{pmatrix} U & & E \\ {\scriptstyle i}\downarrow & \boxslash & \downarrow \\ V & & B \end{pmatrix} \longrightarrow \begin{pmatrix} U' & & \\ {\scriptstyle i'}\downarrow & & E \\ U & \boxslash & \downarrow \\ {\scriptstyle i}\downarrow & & B \\ V & & \end{pmatrix}$$

that sends a pair $F' \in (U' \to U) \boxslash (E \to B)$ and $F \in (U \to V) \boxslash (E \to B)$ to the lifting structure which solves lifting problems (u', v) with the consecutive lift $F(F'(u', v \cdot (X \times i)), v)$.



We next show the structured version of the fact that the left maps to lifting problems are closed under retracts.

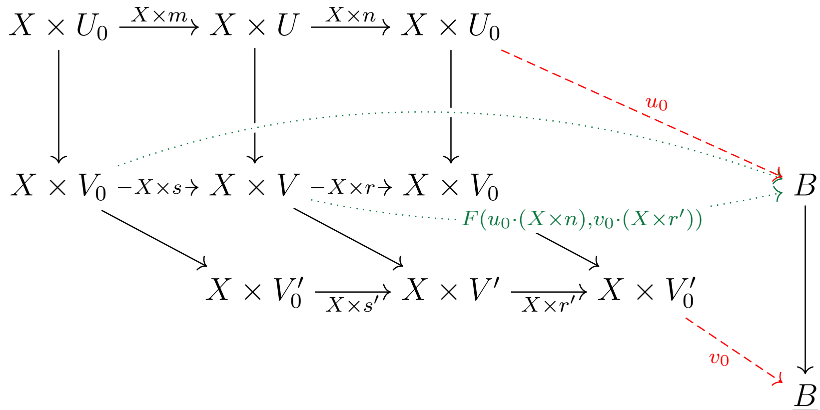
2.6. Construction. *Suppose one has a retract of composable pairs of maps.*



Then, we define a map

$$\begin{pmatrix} U & & & E \\ \downarrow & & \boxslash & \downarrow \\ V & \longrightarrow V' & & B \end{pmatrix} \longrightarrow \begin{pmatrix} U_0 & & & E \\ \downarrow & & \boxslash & \downarrow \\ V_0 & \longrightarrow V'_0 & & B \end{pmatrix}$$

sending each lifting structure F from the left to the lifting structure that solves each lifting problem (u_0, v_0) as below with the solution $F(u_0 \cdot (X \times n), v_0 \cdot (X \times r')) \cdot (X \times s)$.



By chaining Construction 2.1 and 2.2, we can observe that if $U \to V$ lifts uniformly on the left against $E \to B$ then $U \to V$ also lifts uniformly on the left against any pullback $E' \to B'$ of $E \to B$. Dually, one may wonder if one can induce uniform lifting structures on pullbacks of the left map $U \to V$.

First, recall that an exponentiable map $p\colon C \to D$ is a map where the pullback functor $p^*\colon \mathbb{C}/D \to \mathbb{C}/C$ admits both the left and right adjoint, respectively given by postcomposition and pushforwards.

$$\mathbb{C}/D \xrightarrow{p^*} \mathbb{C}/C, \qquad p_! \dashv p^* \dashv p_*$$

We also recall the result from model category that if fibrations are stable under pushforwards along fibrations then trivial cofibrations are stable under pullback along fibrations. By reproducing this result in a structured setting, we provide a construction that induces uniform lifting structures on pullbacks of the left map.

2.7. Construction. *Fix a left map $i\colon U \to V$ and a right map $E \to B$ along with an exponentiable map $t\colon W \to V$ and a pullback $t^*i\colon t^*U \to W$ of $U \to V$ along t.*

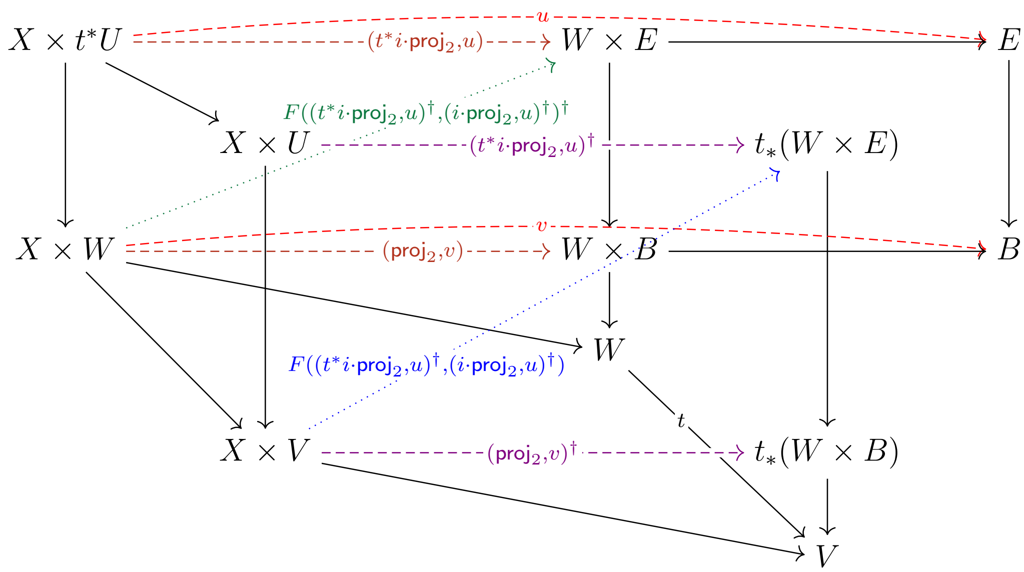
$$\begin{array}{ccc} t^*U & \longrightarrow & U \\ {\scriptstyle t^*i}\downarrow & \lrcorner & \downarrow{\scriptstyle i} \\ W & \xrightarrow[t]{} & V \end{array}$$

We construct a map

$$\begin{pmatrix} U & & t_*(W \times E) \\ \downarrow & \boxslash & \downarrow \\ V & & t_*(W \times B) \end{pmatrix} \longrightarrow \begin{pmatrix} t^*U & & E \\ \downarrow & \boxslash & \downarrow \\ V' & & B \end{pmatrix}$$

That is, we show that the pullback $t^*U \to W$ lifts uniformly against a right map $E \to B$ when the left map $U \to V$ lifts uniformly against the map $t_*(W \times E) \to t_*(W \times B)$ obtained from right map $E \to B \in \mathbb{C}/1$ by applying the polynomial functor of $1 \leftarrow W \to V \to 1$.

Let F be a lifting structure of $U \to V$ against $t_*(W \times E) \to t_*(W \times B)$. Now assume that we are given a lifting problem (u, v) of $X \times t^*U \to X \times W$ against $E \to B$ as in the curved back face.



Then, we produce the required lifting solution by the following procedure:

(1) *Induce a lifting problem given by $((t^*i \cdot \mathsf{proj}_2, u), (\mathsf{proj}_2, v))$ of $X \times t^*U \to X \times W$ against $W \times E \to W \times B$.*

(2) *Because $X \times -$ preserves pullbacks, the pullback of $X \times U \to X \times V \in \mathbb{C}/V$ under $t \colon W \to V$ is $X \times t^*U \to X \times W \in \mathbb{C}/W$. So we can transpose $((t^*i \cdot \mathsf{proj}_2, u), (\mathsf{proj}_2, v))$ to get a lifting problem $((t^*i \cdot \mathsf{proj}_2, u)^\dagger, (\mathsf{proj}_2, v)^\dagger)$ to get a solution.*

(3) *By the original solution F, we obtain a solution $F((t^*i \cdot \mathsf{proj}_2, u)^\dagger, (\mathsf{proj}_2, v)^\dagger)$.*

(4) *Transposing it back we obtain the required solution $F((t^*i \cdot \mathsf{proj}_2, u)^\dagger, (\mathsf{proj}_2, v)^\dagger)^\dagger$.*

Because the transpose operation is natural, the uniformity structure of F shows that this indeed induces a structured lift.

As a consequence, we obtain the following by combining various constructions above.

2.8. Theorem. *Fix a pullback $t^*U \to W \in \mathbb{C}/C$ of a map $U \to V \in \mathbb{C}/C$ along a map $t \colon W \to V \in \mathbb{C}/C$.*

$$\begin{array}{ccc} t^*U & \longrightarrow & U \\ \downarrow & \lrcorner & \downarrow \\ W & \xrightarrow[t]{} & V \end{array} \qquad\qquad \begin{array}{ccc} t_*(W \times_C E) & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow \\ t_*(W \times_C B) & \longrightarrow & B \end{array}$$

If $U \to V \in \mathbb{C}$ lifts uniformly against a map $E \to B \in \mathbb{C}$ and the pushed forwards map $t_*(W \times E) \to t_*(W \times B)$ occurs as a pullback of $E \to B$ then $t^*U \to W \in \mathbb{C}$ lifts uniformly against $E \to B \in \mathbb{C}$ as well.

Proof. By Constructions 2.7, 2.1 and 2.2. ∎

3. Rebasing Lifting Structures

We now work with lifting structures in slice categories and show that they are stable under the pullback and post-composition change of base operations.

We begin with spelling out what it means to be a lifting structure in a slice category.

3.1. Definition. *Fix an object $C \in \mathbb{C}$. Given maps $U \to V$ and $V \to V'$ and $E \to B$ in the slice $\mathbb{C}/C$, the set of structured lifts in $\mathbb{C}/C$ of $U \to V$ restricted on the left along $V \to V'$ against $E \to B$ is denoted as on the left below.*

$$\begin{array}{cccc} U & & & E \\ \downarrow & & \boxslash_C & \downarrow \\ V & \longrightarrow V' & & B \end{array} \qquad\qquad \begin{array}{cccc} U & & & E \\ \downarrow & \boxslash_C & & \downarrow \\ V & & B' \longrightarrow & B \end{array}$$

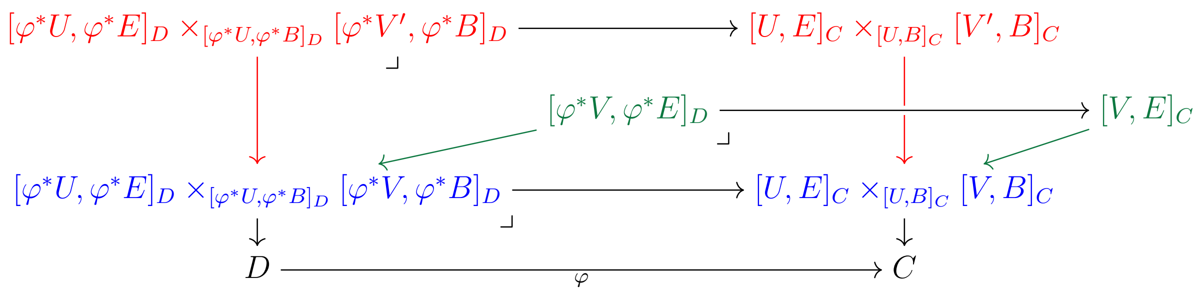
Dually, given $U \to V$ and $B' \to B$ and $E \to B$ in the slice $\mathbb{C}/C$, the set of structured lifts in $\mathbb{C}/C$ of $U \to V$ restricted on the right along $B' \to B$ against $E \to B$ is denoted as on the right above.

Because lifting structures are inherently a limiting, or a "right-sided concept", it is unsurprising they are stable under pullback.

3.2. Construction. *Fix a map $\varphi \colon D \to C$ along with maps $U \to V \to V' \in \mathbb{C}/C$ and $E \to B \in \mathbb{C}/C$ where U, V, V' are exponential objects in the slice over C so that by the Beck-Chevalley condition, $\varphi^*U, \varphi^*V, \varphi^*V'$ are exponential objects in the slice over D. We define a map*

$$\left(\begin{array}{cccc} U & & & E \\ \downarrow & & \boxslash_C & \downarrow \\ V & \longrightarrow V' & & B \end{array}\right) \longrightarrow \left(\begin{array}{cccc} \varphi^*U & & & \varphi^*E \\ \downarrow & & \boxslash_D & \downarrow \\ \varphi^*V & \longrightarrow \varphi^*V' & & \varphi^*B \end{array}\right)$$

by nothing that the pullback functor $\varphi^* \colon \mathbb{C}/C \to \mathbb{C}/D$ preserves limits and exponentials so, one has the pullbacks



The required map is then given by Lemma 1.7.

Similarly, for maps $U \to V$ and $B' \to B \leftarrow E$ where U, V are exponentiable objects in $\mathbb{C}/C$, we can construct a map

$$\begin{pmatrix} U & & E \\ \downarrow \ \boxslash_C & & \downarrow \\ V & B' \longrightarrow & B \end{pmatrix} \longrightarrow \begin{pmatrix} \varphi^* U & & \varphi^* E \\ \downarrow \ \boxslash_D & & \downarrow \\ \varphi^* V & \varphi^* B' \longrightarrow & \varphi^* B \end{pmatrix}$$

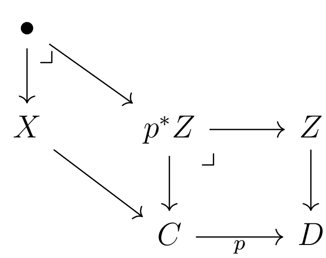
We next proceed to show that the rebasing by post-composition also preserves certain forms of lifting structures. To do so, recall that the pushforward realises the external left adjoint to the pullback functor as also the internal left adjoint, made precise in the following sense.

3.3. Lemma. *Fix an exponentiable map $p \colon C \to D$. Then,*

(1) For any $X \to C \in \mathbb{C}/C$ and $Z \to D \in \mathbb{C}/D$, one has $p_!(X \times_C p^* Z) \cong p_! X \times_D Z$, where $p_!$ is the left adjoint to the pullback functor.

(2) For any exponentiable $X \to C \in \mathbb{C}/C$ and $Y \to D \in \mathbb{C}/D$, one has $p_*[X, p^* Y]_C \cong [p_! X, Y]_D$, where p_* is the pushforward (i.e. right adjoint the pullback) along p.

Proof. ((1)) is directly by the pullback lemma:



((2)) is now by ((1)) and representability, because for each $Z \to D \in \mathbb{C}/D$, one has

$$\mathbb{C}/D(Z, p_*[X, p^* Y]_C) = \mathbb{C}/C(p^* Z, [X, p^* Y]_C)$$

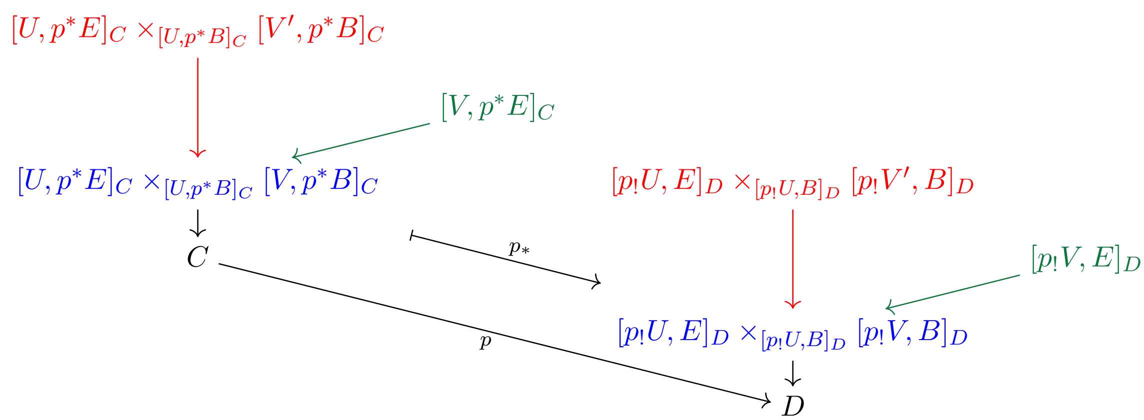
$$
\begin{aligned}
&= \mathbb{C}/_C(p^*Z \times_C X, p^*Y) \\
&= \mathbb{C}/_D(p_!(p^*Z \times_C X), Y) \\
&= \mathbb{C}/_D(Z \times_D p_! X, Y) \\
\mathbb{C}/_D(Z, p_*[X, p^*Y]_C) &= \mathbb{C}/_D(Z, [p_! X, Y]_D)
\end{aligned}
$$

∎

3.4. Construction. *Fix a map $p\colon C \to D$. Let there be maps $U \to V$ and $V \to V'$ in $\mathbb{C}/_C$ along with a map $E \to B \in \mathbb{C}/_D$, where U, V, V' are exponentiable in the slice category over C. We construct a map*

$$
\begin{pmatrix} U & & p^*E \\ \downarrow & \boxslash_C & \downarrow \\ V \longrightarrow V' & & p^*B \end{pmatrix} \longrightarrow \begin{pmatrix} p_!U & & E \\ \downarrow & \boxslash_D & \downarrow \\ p_!V \longrightarrow p_!V' & & B \end{pmatrix}
$$

by Lemma 3.3, which states that right adjoint $p_*\colon \mathbb{C}/_C \to \mathbb{C}/_D$ maps



For the same reason, if there are maps $U \to V$ in $\mathbb{C}/_C$ where U, V are exponentiable in the slice over C and maps $B' \to B$ and $E \to B$ in $\mathbb{C}/_D$ then one also has a map

$$
\begin{pmatrix} U & & p^*E \\ \downarrow \boxslash_C & & \downarrow \\ V & p^*B' \longrightarrow & p^*B \end{pmatrix} \longrightarrow \begin{pmatrix} p_!U & & E \\ \downarrow \boxslash_D & & \downarrow \\ p_!V & B' \longrightarrow & B \end{pmatrix}
$$

We also show a local version of Construction 2.7. Because Construction 2.7 mentions about pushforwards, we need to recall that pushforwards in slice categories are just computed as pushforwards in the ambient category.

3.5. Lemma. *Fix $C \in \mathbb{C}$ and a map $t\colon U \to V \in \mathbb{C}/_C$. Then the pushforwards along t in $\mathbb{C}/_C$ is the pushforwards along t in the ambient category $\mathbb{C}$.*

Proof. This is because pullbacks in the slice category are exactly pullbacks in the ambient category. ∎

Therefore, we have a local version of Construction 2.7.

3.6. Construction. *Fix an object $C \in \mathbb{C}$ and a left map $i\colon U \to V \in {}^{\mathbb{C}}/_{C}$ and a right map $E \to B \in {}^{\mathbb{C}}/_{C}$, where U, V are exponentiable in the slice, along with an exponentiable map $t\colon W \to V$ and a pullback $t^*U \to W \in {}^{\mathbb{C}}/_{C}$ of $U \to V$ along t.*

Then, we construct a map

$$\left(\begin{array}{ccc} U & & t_*(W \times_C E) \\ \downarrow & \boxslash_C & \downarrow \\ V & & t_*(W \times_C B) \end{array}\right) \longrightarrow \left(\begin{array}{ccc} t^*U & & E \\ \downarrow & \boxslash_C & \downarrow \\ V' & & B \end{array}\right)$$

by applying Construction 2.7 and using Lemma 3.5 to note that pushforwards in the slice ${}^{\mathbb{C}}/_{C}$ is just the pushforwards in $\mathbb{C}$.

As a consequence, we obtain the following local version of Theorem 2.8.

3.7. Theorem. *Fix an object $C \in \mathbb{C}$ and a pullback $t^*U \to W \in {}^{\mathbb{C}}/_{C}$ of a map $U \to V \in {}^{\mathbb{C}}/_{C}$ between exponentiable objects along an exponentiable map $t\colon W \to V \in {}^{\mathbb{C}}/_{C}$.*

$$\begin{array}{ccc} t^*U & \longrightarrow & U \\ \downarrow & \lrcorner & \downarrow \\ W & \xrightarrow[t]{} & V \end{array} \qquad\qquad \begin{array}{ccc} t_*(W \times_C E) & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow \\ t_*(W \times_C B) & \longrightarrow & B \end{array}$$

If $U \to V \in {}^{\mathbb{C}}/_{C}$ lifts uniformly against a map $E \to B \in {}^{\mathbb{C}}/_{C}$ and the pushed forwards map $t_*(W \times_C E) \to t_*(W \times_C B)$ occurs as a pullback of $E \to B$ then $t^*U \to W \in {}^{\mathbb{C}}/_{C}$ lifts uniformly against $E \to B \in {}^{\mathbb{C}}/_{C}$ as well.

Proof. By Constructions 3.6, 2.1 and 2.2. ∎

Slice Lifting Structures in CwRs. In order to categorically axiomatise pattern matching operations such as J-elimination in the framework CwRs, we require the construction of CwRs freely extended with a formal lifting structure in slice categories. The idea is the same as Construction 1.16.

3.8. Construction. *Let $\mathbb{C}$ be a CwR and fix maps $U \to V \to V'$ between exponential objects and $E \to B$ the slice over a fixed object $C \in \mathbb{C}$. Define the CwR obtained from $\mathbb{C}$ by freely extending with a lifting structure of $U \to V$ against $E \to B$ restricted along $V \to V'$ in the slice over C as the following bipushout in* CwR *using Construction 1.15.*

$$\mathbb{C}\left[\begin{array}{ccc} U & & E \\ \downarrow & \boxslash_C & \downarrow \\ V \to V' & & B \end{array}\right] \coloneqq \mathbb{C} \cup_{\{\bullet \to \bullet \leftarrow \bullet\}} \left\{\begin{array}{ccc} [U,E]_C \times_{[U,B]_C} [V',B]_C & \to & [V,E]_C \\ \downarrow & \swarrow & \\ [U,E]_C \times_{[U,B]_C} [V,B]_C & & \end{array}\right\}$$

Dually, if $U \to V$ is a map between exponentiable objects in $^{\mathbb{C}}/_{C}$ and one has maps $E \to B$ along with $B' \to B$ in $^{\mathbb{C}}/_{C}$ then the CwR with a formal lifting structure of $U \to V$ against $E \to B$ restricted along $B' \to B$ in the slice over C is obtained by the following bipushout in CwR *using Construction 1.15.*

$$\mathbb{C}\left[\begin{array}{ccc} U & & E \\ \downarrow & \boxslash_C & \downarrow \\ V & B' \to & B \end{array}\right] \coloneqq \mathbb{C} \cup_{\{\bullet\to\bullet\leftarrow\bullet\}} \left\{ \begin{array}{ccc} [U,E]_C \times_{[U,B]_C} [V,B']_C & \to & [V,E]_C \\ \downarrow & \swarrow & \\ [U,E]_C \times_{[U,B]_C} [V,B]_C & & \end{array} \right\}$$

3.9. Theorem. *Let $\mathbb{C}$ be a CwR and fix maps $U \to V \to V'$ between representable objects (i.e. objects whose maps into the terminal object are representable) and $E \to B$ in $\mathbb{C}$. Then, isomorphism classes of maps as on the left below correspond bijectively to isomorphism classes of maps $F\colon \mathbb{C} \to \mathbb{D} \in$ CwR equipped with a choice of a lifting structure of $FU \to FV$ against $FE \to FB$ restricted along $FV \to FV'$ over the slice FC.*

$$\mathbb{C}\left[\begin{array}{ccc} U & & E \\ \downarrow & \boxslash_C & \downarrow \\ V \to V' & & B \end{array}\right] \to \mathbb{D} \in \mathsf{CwR} \qquad\qquad \mathbb{C}\left[\begin{array}{ccc} U & & E \\ \downarrow & \boxslash_C & \downarrow \\ V & B' \to & B \end{array}\right] \to \mathbb{D} \in \mathsf{CwR}$$

Dually, given $B' \to B$ in $\mathbb{C}$, isomorphism classes of maps as on the right above correspond to isomorphism classes of maps $F\colon \mathbb{C} \to \mathbb{D} \in$ CwR equipped with a choice of a lifting structure of $FU \to FV$ against $FE \to FB$ restricted along $FB' \to FB$ over the slice FC.

Proof. Identical to Theorem 1.17. ∎

4. Structured Approximations of Pushout-Product

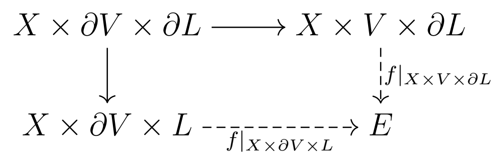
In preparation for the structured version of Leibniz transposes in the subsequent Subsection 5, we first motivate and introduce a structured approximation of the pushout-product.

As motivation, suppose we are given two maps, which we think of as "boundary inclusions" and denote their domains and codomains using the boundary symbol, $\partial V \to V$ and $\partial L \to L$. Usually, the boundary $\partial(V \times L)$ of the product $V \times L$ is constructed by the pushout-product $\partial V \times L \cup V \times \partial L \to V \times L$. However, because the pushout-product relies on the left concept of pushouts, and we only assume our universe of discourse $\mathbb{C}$ to be locally cartesian closed, these pushouts may not exist in the first place. Therefore, we would like to have a concept of maps that behaves sufficiently like such a pushout-product.

The usual product boundary $\partial V \times L \cup V \times \partial L$, being a pushout, has the universal property that maps $\partial V \times L \cup V \times \partial L \to E$ are in bijective correspondence with pairs of maps $\partial V \times L \to E$ and $V \times \partial L \to E$. But because in a cartesian closed category, products are left adjoint to exponential, taking products preserve pushouts. This means that for each $X \in \mathbb{C}$, one in fact has $X \times (\partial V \times L \cup V \times \partial L \to E) \cong X \times \partial V \times L \cup X \times V \times \partial L$.

Therefore, we would like to say that an object E believes an object $\partial(V \times L)$ behaves like the product boundary when, for each $X \in \mathbb{C}$, one has a natural bijective correspondence between maps f and undercones $(f|_{X\times\partial V\times L}, f|_{X\times V\times\partial L})$ as follows.

$$X \times \partial(V \times L) \xrightarrow{f} E$$

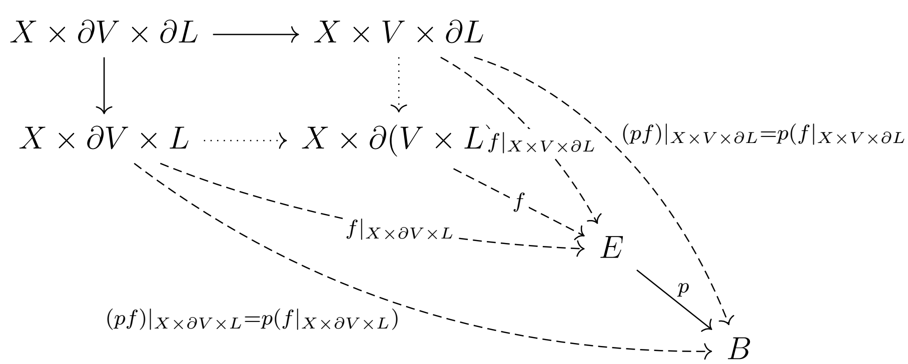


Such parameterised beliefs can be packaged into the following definition using cartesian closeness.

4.1. Definition. *Given maps $\partial V \to V$ and $\partial L \to L$ between exponentiable objects in $\mathbb{C}$, a* product boundary approximation structure *on an exponentiable object $\partial(V \times L)$ relative to an object E is an isomorphism*

$$[\partial V \times L, E] \times_{[\partial V \times \partial L, E]} [V \times \partial L, E] \xrightarrow{\cong} [\partial(V \times L), E]$$

Next, suppose E and B both believe $\partial(V \times L)$ behaves like the product boundary. Then, E thinks that each map $f \colon X \times \partial(V \times L) \to E$ is completely determined by the two components $f|_{X\times\partial V\times L}$ and $f|_{X\times V\times\partial L}$. Now, given a map $p \colon E \to B$, we can post-compose to obtain a map $pf \colon X \times \partial(V \times L) \to E \to B$. Then, B also thinks pf is completely determined by the two components $(pf)|_{X\times\partial V\times L}$ and $(pf)|_{X\times V\times\partial L}$. We want to say that composing with p does not destroy the hallucinations E and B have about $\partial(V \times L) \to V \times L$ when the hallucinated component restrictions $(pf)|_{X\times\partial V\times L}$ and $(pf)|_{X\times V\times\partial L}$ of B are respectively obtained by composing p with the hallucinated component restrictions $f|_{X\times\partial V\times L}$ and $f|_{X\times V\times\partial L}$ of E.

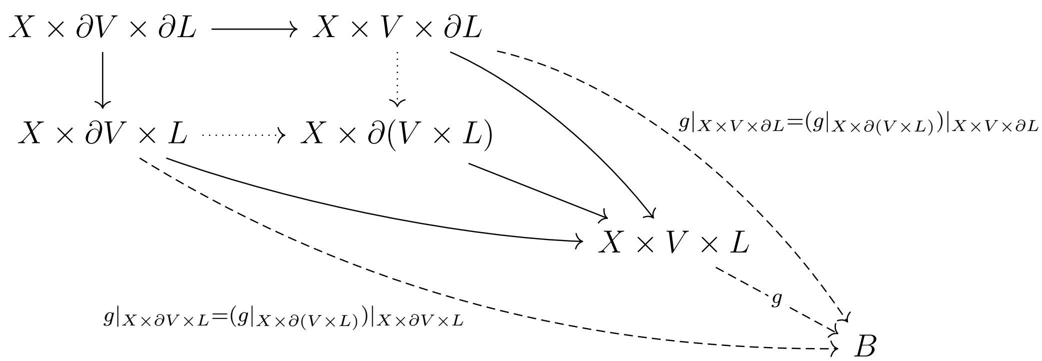


We once again can package such hallucinogenic maps using internal-Homs.

4.2. Definition. *Fix maps $\partial V \to V$ and $\partial L \to L$ between exponentiable objects in $\mathbb{C}$. Suppose that an exponentiable object $\partial(V \times L)$ is equipped with product approximation structures s_E and s_B respectively relative to E and B. A map p* preserves boundary approximation structures *when the following diagram commutes.*

$$\begin{array}{ccc} [\partial V \times L, E] \times_{[\partial V \times \partial L, E]} [V \times \partial L, E] & \overset{\cong}{\longleftrightarrow} & [\partial(V \times L), E] \\ \downarrow & & \downarrow \\ [\partial V \times L, B] \times_{[\partial V \times \partial L, B]} [V \times \partial L, B] & \underset{\cong}{\longleftrightarrow} & [\partial(V \times L), B] \end{array}$$

We have now defined what is means for an *object* $\partial(V \times L)$ to behave like the *boundary object* of $V \times L$. In the usual pushout-product definition, such boundary objects $\partial V \times L \cup V \times \partial L$ are equipped with inclusions into the actual product object $V \times L$. We can also say that a map $\partial(V \times L) \to V \times L$ behaves like a pushout-product inclusion, but again only relative to some object B in a state of delusion. When we have such a map $\partial(V \times L) \to V \times L$ like this, given any map $g \colon X \times V \times L \to B$, we can restrict it to get a map $g|_{X \times \partial(V \times L)} \colon X \times \partial(V \times L) \to B$. By hallucinations, $g|_{X \times \partial(V \times L)}$ is completely determined by its further restrictions $(g|_{X \times \partial(V \times L)})|_{X \times \partial V \times L}$ and $(g|_{X \times \partial(V \times L)})|_{X \times V \times \partial L}$. On the other hand, we could have just gotten to these restrictions in one step using $g|_{X \times \partial V \times L}$ and $g|_{X \times V \times \partial L}$ without passing through $X \times \partial(V \times L) \to X \times V \times L$. A sufficiently well-crafted fake $\partial(V \times L) \to V \times L$ should deceive B to make it into thinking these two ways of restricting and amalgamation are the same.



Once again, using internal-Homs we have a very concise definition of the above phenomenon.

4.3. Definition. *Fix maps $\partial V \to V$ and $\partial L \to L$ between exponentiable objects. A map $\partial(V \times L) \to V \times L$ between exponentiable objects realises a product boundary structure $[\partial V \times L, B] \times_{\partial V \times \partial L, E} [V \times \partial L, B] \xrightarrow{\cong} [\partial(V \times L), B]$ on $\partial(V \times L)$ relative to B as a* pushout-product approximation structure *when the following triangle commutes*

$$\begin{array}{ccc} [\partial V \times L, B] \times_{[\partial V \times \partial L, B]} [V \times \partial L, B] & \xrightarrow{\quad\cong\quad} & [\partial(V \times L), B] \\ & \nwarrow \qquad\qquad \nearrow & \\ & [V \times L, B] & \end{array}$$

We can now combine everything together to say when a map thinks another map behaves like the pushout product.

4.4. Definition. *Given maps $\partial V \to V$ and $\partial L \to L$ and $E \to B$ where $\partial V, V, \partial L, L$ are exponentiable objects in $\mathbb{C}$.*

A $(\partial V \to V) \mathbin{\widehat{\times}} (\partial L \to L)$-approximation structure on $\partial(V \times L) \to V \times L$ relative to *$E \to B$ consists of product boundary approximation structures s_E and s_B on the object $\partial(V \times L)$, which has to be exponentiable, relative to E2 and B such that*

- *$E \to B$ preserves the product boundary approximation structures; and*
- *$\partial(V \times L) \to V \times L$ realises s_B as a $(\partial V \to V) \mathbin{\widehat{\times}} (\partial L \to L)$ approximation structure relative to B.*

In other words, one requires black isomorphisms as below making the diagram commute.

$$\begin{array}{ccc}
[\partial V \times L, E] \times_{[\partial V \times \partial L, E]} [V \times \partial L, E] & \xleftarrow{\cong} \!\!\rightarrow & [\partial(V \times L), E] \\
\downarrow & & \downarrow \\
[\partial V \times L, B] \times_{[\partial V \times \partial L, B]} [V \times \partial L, B] & \xleftarrow{\cong} \!\!\rightarrow & [\partial(V \times L), B] \\
\uparrow & & \uparrow \\
[V \times L, B] & \xleftarrow{=} \!\!\rightarrow & [V \times L, B]
\end{array}$$

(pushout-product-approx)

Throughout the rest of this section, we establish analogues of constructions from Subsection 3. Specifically, we construct, like in Construction 3.2, that structured approximations of the pushout-product are stable under pullback. We also construct, like in Construction 3.4, that certain forms of structured approximations of the pushout-product are stable under the left adjoint to the pullback functor (i.e. the post-composition map).

We start with the analogue of Construction 3.2.

4.5. Lemma. *Fix map $\varphi\colon D \to C$ along with maps $\partial J \to J$ and $\partial V \to V$ and $E \to B$ all in $\mathbb{C}/C$, where $\partial J, J, \partial V, V$ are exponentiable objects in $\mathbb{C}/C$.*

Suppose

$$\partial(J \times_C V) \to J \times_C V$$

has a $(\partial V \to V) \mathbin{\widehat{\times}}_C (\partial J \to J)$-approximation structure relative to $E \to B$ in $\mathbb{C}/C$. Then, the pullback

$$\partial(J \times_C V) \to \varphi^* J \times_D \varphi^* V$$

has a $(\varphi^*(\partial V) \to \varphi^* V) \mathbin{\widehat{\times}}_D (\varphi^*(\partial J) \to \varphi^* J)$-approximation structure relative to the rebased map $\varphi^* E \to \varphi^* B$ in $\mathbb{C}/D$.

Proof. This follows from the fact that the pullback functor preserves both limits and internal-Homs, much like in Construction 3.2. ∎

We also have an analogue of Construction 3.4.

4.6. Lemma. *Fix map $\varphi\colon C \to D$ along with maps $\partial J \to J$ in ${}^{\mathbb{C}}/_{C}$ and $\partial V \to V$ and $E \to B$ in ${}^{\mathbb{C}}/_{D}$ where $\partial J, J, \partial V, V$ are exponentiable in their respective slice categories.*

Suppose

$$\partial(V \times_C p^*J) \to V \times_C p^*J$$

has a $(\partial V \to V) \mathbin{\widehat{\times}}_C (p^*(\partial J) \to p^*J)$-approximation structure relative to $p^*E \to p^*B$ in ${}^{\mathbb{C}}/_{C}$. Then, under the $p_!$ postcomposition functor

$$p_!(\partial(V \times_C p^*J)) \to p_!(V \times_C p^*J)$$

inherits a $(p_!(\partial V) \to p_!V) \mathbin{\widehat{\times}}_D (\partial J \to J)$-approximation structure relative to $E \to B$ in ${}^{\mathbb{C}}/_{D}$.

Proof. Much like Construction 3.4 by using Lemma 3.3. For example, by assumption, one has that $[\partial V \times_C p^*J, p^*B] \times_{[\partial V \times_C p^*(\partial J), p^*B]} [V \times p^*(\partial J), p^*B] \cong [\partial(V \times_C p^*J), p^*B]$, and the image of this isomorphism under the right adjoint $p_*\colon {}^{\mathbb{C}}/_{C} \to {}^{\mathbb{C}}/_{D}$ is $[p_!(\partial V) \times_D J, B] \times_{[p_!(\partial V) \times_D \partial J, B]} [p_!V \times \partial J, B] \cong [p_!\partial(V \times_C p^*J), B]$. ∎

We finish by showing that approximations of pushout-products are associative.

4.7. Theorem. *Suppose that one has three maps $\partial U \to U$ and $\partial V \to V$ and $\partial W \to W$ between exponentiable objects along with two approximations of pushout-products $\partial(U \times V) \to U \times V$ approximating $(\partial U \to U) \mathbin{\widehat{\times}} (\partial V \to V)$ and $\partial(V \times W) \to V \times W$ approximating $(\partial V \to V) \mathbin{\widehat{\times}} (\partial W \to W)$ with respect to a map $E \to B$ between exponentiable objects.*

A map $\partial(U \times (V \times W)) \to U \times V \times W$ approximates the pushout-product

$$(\partial U \to U) \mathbin{\widehat{\times}} (\partial(V \times W) \to V \times W)$$

relative to $E \to B$ if and only if it also approximates the pushout-product

$$(\partial(U \times V) \to U \times V) \mathbin{\widehat{\times}} (\partial W \to W)$$

relative to $E \to B$.

Proof. The $\Rightarrow$ and $\Leftarrow$ directions are symmetric. We show the $\Rightarrow$ implication.

By assumption, one has horizontal isomorphisms

$$\begin{array}{ccc}
[\partial(U \times V) \times W, E] \times_{[\partial(U \times V) \times \partial W, E]} [U \times V \times \partial W, E] & \overset{\cong}{\longleftrightarrow} & [\partial((U \times V) \times W), E] \\
\downarrow & & \downarrow \\
[\partial(U \times V) \times W, B] \times_{[\partial(U \times V) \times \partial W, B]} [U \times V \times \partial W, B] & \overset{\cong}{\longleftrightarrow} & [\partial((U \times V) \times W), B] \\
\uparrow & & \uparrow \\
[U \times V \times W, B] & = & [U \times V \times W, B]
\end{array}$$

and we must show there exists dashed isomorphisms

$$\begin{array}{ccc}
[\partial U \times V \times W, E] \times_{[\partial U \times \partial(V \times W), E]} [U \times \partial(V \times W), E] & \overset{\cong}{\leftarrow\!\!-\!\!\rightarrow} & [\partial((U \times V) \times W), E] \\
\downarrow & & \downarrow \\
[\partial U \times V \times W, B] \times_{[\partial U \times \partial(V \times W), B]} [U \times \partial(V \times W), B] & \overset{\cong}{\leftarrow\!\!-\!\!\rightarrow} & [\partial((U \times V) \times W), B] \\
\uparrow & & \uparrow \\
[U \times V \times W, B] & = & [U \times V \times W, B]
\end{array}$$

We show that the left columns of the two diagrams above are isomorphic. Because $\partial(U \times V) \to U \times V$ approximates $(\partial U \to U) \widehat{\times} (\partial V \to V)$ and $\partial(V \times W) \to V \times W$ approximates $(\partial V \to V) \widehat{\times} (\partial W \to W)$ with respect to $E \to B$, we have that

$$[\partial(U \times V), B] \cong [\partial U \times V, B] \times_{[\partial U \times \partial V, B]} [U \times \partial V, B]$$
$$[\partial(V \times W), B] \cong [\partial V \times W, B] \times_{[\partial V \times \partial W, B]} [V \times \partial W, B]$$

Thus, applying $[X, -]$ for any $X \in \mathbb{C}$, we have

$$[\partial(U \times V) \times X, B] \cong [\partial U \times V \times X, B] \times_{[\partial U \times \partial V \times X, B]} [U \times \partial V \times X, B]$$
$$[X \times \partial(V \times W), B] \cong [X \times \partial V \times W, B] \times_{[X \times \partial V \times \partial W, B]} [X \times V \times \partial W, B]$$

So, $[\partial U \times V \times W, B] \times_{[\partial U \times \partial(V \times W), B]} [U \times \partial(V \times W), B]$ and $[\partial(U \times V) \times W, B] \times_{[\partial(U \times V) \times \partial W, B]} [U \times V \times \partial W, B]$ are both limits of the following diagram.

$$\begin{array}{ccccc}
[\partial U \times V \times W, B] & \longrightarrow & [\partial U \times V \times \partial W, B] & = & [\partial U \times V \times \partial W, B] \\
\downarrow & & \downarrow & & \| \\
[\partial U \times \partial V \times W, B] & \longrightarrow & [\partial U \times \partial V \times \partial W, B] & \longleftarrow & [\partial U \times V \times \partial W, B] \\
\uparrow & & \uparrow & & \uparrow \\
[U \times \partial V \times W, B] & \longrightarrow & [U \times \partial V \times \partial W, B] & \longleftarrow & [U \times V \times \partial W, B]
\end{array}
\qquad
\begin{array}{c}
[\partial U \times V \times W, B] \\
\downarrow \\
[\partial U \times \partial(V \times W), B] \\
\uparrow \\
[U \times \partial(V \times W), B]
\end{array}$$

$$[\partial(U \times V) \times W, B] \longrightarrow [\partial(U \times V) \times \partial W, B] \longleftarrow [U \times V \times \partial W, B]$$

So $[\partial U \times V \times W, B] \times_{[\partial U \times \partial(V \times W), B]} [U \times \partial(V \times W), B] \cong [\partial(U \times V) \times W, B] \times_{[\partial(U \times V) \times \partial W, B]} [U \times V \times \partial W, B]$. The same argument applies to show the required isomorphism with B replaced with E. ■

4.8. Remark. Associativity of approximations of pushout-products are also the reason that in Definition 4.4 we do not require do not require that $\partial(V \times L)$ be part of a cocone under $\partial V \times L \leftarrow \partial V \times \partial L \to V \times \partial L$ and that the horizontal maps of (PUSHOUT-PRODUCT-APPROX) are those induced from composing with the cocone legs, as one might expect.

Had we required this, then in the context of Theorem 4.7, one would have that $\partial(U \times (V \times W))$ is part of a cocone square as follows

$$\begin{array}{ccc} \partial U \times \partial(V \times W) & \longrightarrow & U \times \partial(V \times W) \\ \downarrow & & \downarrow \\ \partial U \times (V \times W) & \longrightarrow & \partial(U \times (V \times W)) \end{array}$$

and one must show that one has the following dotted maps giving rise to a cocone square as follows

$$\begin{array}{ccc} \partial(U \times V) \times \partial W & \longrightarrow & (U \times V) \times \partial W \\ \downarrow & & \vdots \\ \partial(U \times V) \times W & \cdots\rightarrow & \partial(U \times (V \times W)) \end{array}$$

However, as we lack the usual universal colimiting properties of the boundaries of products, to actually the induce the required dotted maps, we would further require that $\partial(U \times (V \times W))$ itself thinks that $\partial(U \times V)$ and $\partial(V \times W)$ behave like boundaries of products, thus leading to further technicalities.

5. Structured Leibniz Transpose

Recall the result from Joyal-Tierney calculus [16, Appendix] that given maps $\partial V \to V$ and $\partial L \to L$, the pushout-product $\partial V \times L \cup V \times \partial L \to V \times L$ lifts against a map $E \to B$ exactly when $\partial V \to V$ lifts against the pullback-power $[L, E] \to [\partial L, E] \times_{[\partial L, B]} [L, B]$.

$$\begin{array}{ccc} \partial V \times L \cup V \times \partial L & \dashrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ V \times L & \dashrightarrow & B \end{array} \qquad \begin{array}{ccc} \partial V & \dashrightarrow & [L, E] \\ \downarrow & \nearrow & \downarrow \\ V & \dashrightarrow & [\partial L, E] \times_{[\partial L, B]} [L, B] \end{array}$$

In the previous Section 4, we defined a notion of approximations of pushout-products, and in Section 1, we defined a notion of structured restricted lifts. Therefore, we would like to apply these concepts for Joyal-Tierney calculus, by replacing the pushout-product $\partial V \times L \cup V \times \partial L \to V \times L$ with an approximation $\partial(V \times L) \to V \times L$ and the lifting problem above with structured restricted versions. This arrives us at the following result.

5.1. Theorem. *Let there be maps $\partial V \to V$ and $\partial L \to L$ and $E \to \underline{E}$ and $L \to L'$ where $\partial V, V, \partial L, L, L'$ are exponentiable objects.*

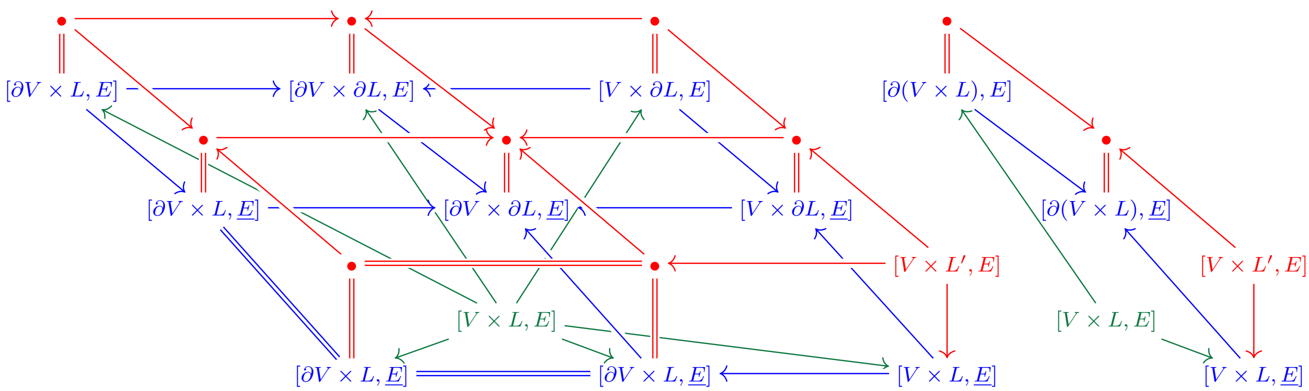
Suppose that $\partial(V \times L) \to V \times L$ structurally approximates the pushout-product $(\partial V \to V) \mathbin{\widehat{\times}} (\partial L \to L)$ relative to the map $E \to \underline{E}$. Then, we have a bijection

$$\left(\begin{array}{cccc} \partial(V \times L) & & & E \\ \downarrow & & \boxslash & \downarrow \\ V \times L & \longrightarrow & V \times L' & \underline{E} \end{array} \right) \xrightarrow{\cong} \left(\begin{array}{cccc} \partial V & & & [L, E] \\ \downarrow & \boxslash & & \downarrow \\ V & [\partial L, E] \times_{[\partial L, \underline{E}]} [L', \underline{E}] & \longrightarrow & [\partial L, E] \times_{[\partial L, \underline{E}]} [L, \underline{E}] \end{array} \right)$$

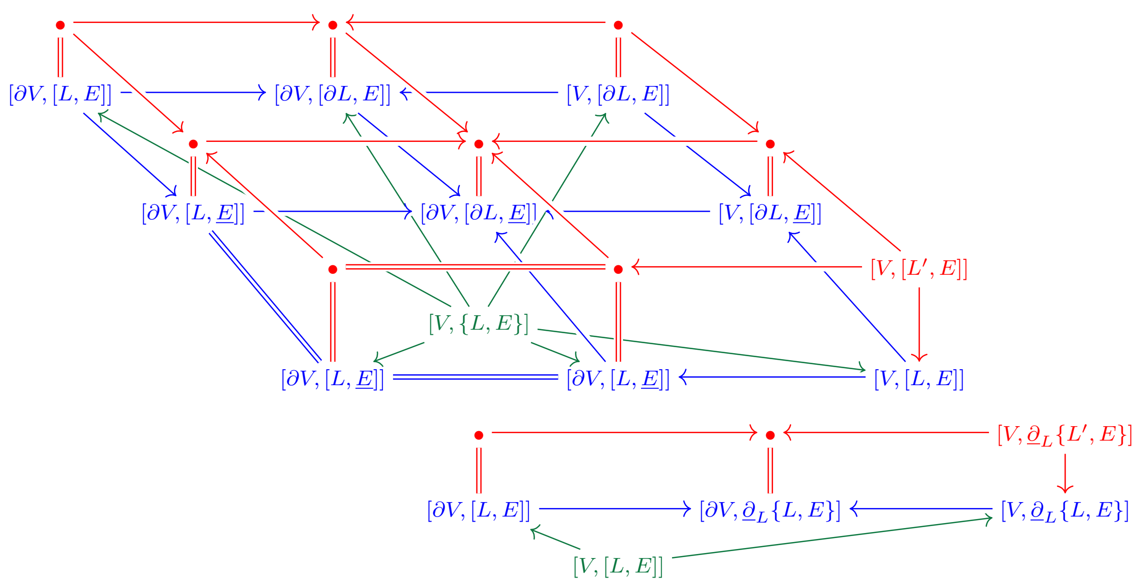
Proof. We set $\underline{\partial}_L\{L', E\} \to \underline{\partial}_L\{L, E\} \leftarrow [L, E]$ as the span $[\partial L, E] \times_{[\partial L, \underline{E}]} [L', E] \to [\partial L, E] \times_{[\partial L, \underline{E}]} [L, E] \leftarrow [L, E]$ on the right above. Then, by Lemmas 1.7 and 1.9, it suffices to show that we have the horizontal isomorphisms as below.

$$\begin{array}{ccc}
[\partial(V \times L), E] \times_{[\partial(V\times L), \underline{E}]} [V \times L', \underline{E}] & \xleftrightarrow{\cong} & [\partial V, \{L, E\}] \times_{[\partial V, \underline{\partial}_L\{L,E\}]} [V, \underline{\partial}_L\{L', E\}] \\
\downarrow & & \downarrow \\
[\partial(V \times L), E] \times_{[\partial(V\times L), \underline{E}]} [V \times L, \underline{E}] & \xleftrightarrow{\cong} & [\partial V, \{L, E\}] \times_{[\partial V, \underline{\partial}_L\{L,E\}]} [V, \underline{\partial}_L\{L, E\}] \\
\uparrow & & \uparrow \\
[V \times L, E] & \xleftrightarrow{\cong} & [V, \{L, E\}]
\end{array} \tag{LTRANS}$$

By the assumption that $\partial(V \times L) \to V \times L$ approximates $(\partial V \to V) \mathbin{\widehat{\times}} (\partial L \to L)$ relative to $E \to \underline{E}$, in the diagram below, the limit of each row is isomorphic to the object on the right of the corresponding row; and the maps between the objects on the right are induced by the maps between their respective rows.



Applying the exponential transpose to each object of the diagram above, one observes that the above diagram is isomorphic to the following diagram.



But the internal-Hom functor is continuous, so one observes that the limit of each column is isomorphic to the object on the front of the corresponding column; and the maps between the objects on the front are induced by the maps between their respective columns.

The result then follows because the limits of the top, middle and bottom layer objects in the of the previous two diagrams are exactly the objects in the corresponding layer in (LTRANS). In other words, the result follows because when viewed as objects of $\mathbb{C}^{\bullet\to\bullet\leftarrow\bullet}$ the left and right cospans on the left and right columns of (LTRANS) are both the limits of the same $\begin{pmatrix} \bullet \longrightarrow \bullet \longleftarrow \bullet \\ \downarrow \quad\;\; \downarrow \quad\;\; \downarrow \\ \bullet \longrightarrow \bullet \longleftarrow \bullet \\ \uparrow \quad\;\; \uparrow \quad\;\; \uparrow \\ \bullet \longrightarrow \bullet \longleftarrow \bullet \end{pmatrix}$-shaped diagram in $\mathbb{C}^{\bullet\to\bullet\leftarrow\bullet}$. ∎

The above proof gives a procedure for transposing structure lifts.

5.2. Construction. *Using representability and examining the use of the exponential transpose in the proof of Theorem 5.1, we see that given a structured lift* $F \in \left(\begin{smallmatrix} \partial(V\times L) \\ \downarrow \\ V\times L \longrightarrow V\times L' \end{smallmatrix}\right) \boxslash \left(\begin{smallmatrix} E \\ \downarrow \\ \underline{E} \end{smallmatrix}\right)$ *its transposed pointwise structured lift* $F^\sharp \in \left(\begin{smallmatrix} \partial V \\ \downarrow \\ V \end{smallmatrix}\right) \boxslash \left(\begin{smallmatrix} [L,E] \\ \downarrow \\ [\partial L, E]\times_{[\partial L,\underline{E}]}[L',E] \longrightarrow [\partial L, E]\times_{[\partial L,\underline{E}]}[L,E] \end{smallmatrix}\right)$ *solves a lifting problem*

$$\begin{array}{ccccc} X\times\partial V & & \overset{f}{\dashrightarrow} & & [L,E] \\ \downarrow & & \nearrow_{F^\sharp_X(f,(g,h))} & & \downarrow \\ X\times V & \underset{(g,h)}{\dashrightarrow} & [\partial L,E]\times_{[\partial L,\underline{E}]}[L',\underline{E}] & \longrightarrow & [\partial L,E]\times_{[\partial L,\underline{E}]}[L,\underline{E}] \end{array}$$

by the following procedure:

(1) Take the exponential transposes $f^\dagger\colon X\times\partial V\times L\to E$ *and* $g^\dagger\colon X\times V\times\partial L\to E$ *and* $h^\dagger\colon X\times V\times L'\to E$

(2) Because E *believes* $X\times\partial(V\times L)$ *is the product-boundary* $X\times\partial V\times L\cup X\times V\times\partial L$*, one has a common extension* $\langle f^\dagger, g^\dagger\rangle\colon X\times\partial(V\times L)\to E$ *of* $f^\dagger$ *and* $g^\dagger$*.*

(3) By commutativity of the original lifting problem and the fact that $\underline{E}$ *also thinks* $X\times\partial(V\times L)$ *is* $X\times\partial V\times L\cup X\times V\times\partial L$*, we can form the transposed lifting problem, whose commutativity stems from the fact that* $E\to\underline{E}$ *preserves the belief of product-boundary approximation.*

$$\begin{array}{ccccc} X\times\partial(V\times L) & & \overset{\langle f^\dagger,g^\dagger\rangle}{\dashrightarrow} & & E \\ \downarrow & & \nearrow_{F_X(\langle f^\dagger,g^\dagger\rangle,h^\dagger)} & & \downarrow \\ X\times V\times L & \longrightarrow & X\times V\times L' & \underset{h^\dagger}{\dashrightarrow} & \underline{E} \end{array}$$

(4) F provides a solution $F_X(\langle f^\dagger, g^\dagger\rangle, h^\dagger) \colon X \times V \times L \to E$ to this lifting problem, whose exponential transpose is taken to be the solution to the original lifting problem.

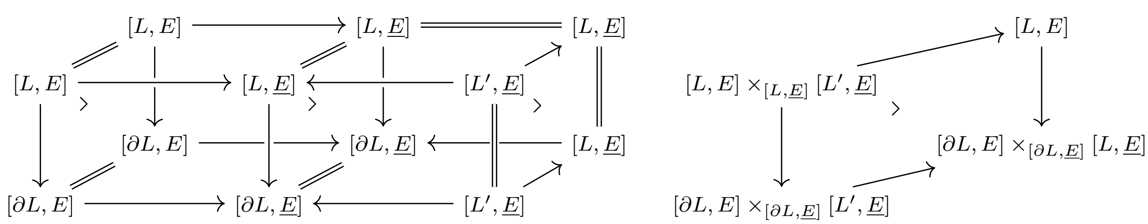
$$F^\sharp_X(f, (g, h)) \coloneqq F_X(\langle f^\dagger, g^\dagger\rangle, h^\dagger)^\dagger$$

We then next try to use Construction 2.2 to get rid of the restriction on the right of Theorem 5.1. For this, we need to first compute the following pullback.

5.3. Lemma. *Let there be maps $\partial L \to L \to L'$ and $E \to \underline{E}$ where $\partial L, L, L'$ are exponentiable objects. One has the following pullback*

$$\begin{array}{ccc} [L,E] \times_{[L,\underline{E}]} [L',\underline{E}] & \longrightarrow & [L,E] \\ \downarrow & & \downarrow \\ [\partial L,E] \times_{[\partial L,\underline{E}]} [L',\underline{E}] & \longrightarrow & [\partial L,E] \times_{[\partial L,\underline{E}]} [L,\underline{E}] \end{array}$$

Proof. The objects in the above diagram are the limits of the cospans as follows while the maps above are the maps between the limits induced by the maps of the respective cospans. The result then follows by the fact that limits commute with limits.



■

5.4. Proposition. *Let there be maps $\partial V \to V$ and $\partial L \to L$ and $E \to \underline{E}$ and $L \to L'$ where $\partial V, V, \partial L, L, L'$ are exponentiable objects.*

Suppose that $\partial(V \times L) \to V \times L$ structurally approximates $(\partial V \to V) \mathbin{\widehat{\times}} (\partial L \to L)$ relative to $E \to \underline{E}$. Then, we have a bijection

$$\left(\begin{array}{ccc} \partial(V \times L) & & E \\ \downarrow & & \boxslash \downarrow \\ V \times L & \longrightarrow V \times L' & \underline{E} \end{array}\right) \cong \left(\begin{array}{cc} \partial V & [L,E] \times_{[L,\underline{E}]} [L',\underline{E}] \\ \downarrow \boxslash & \downarrow \\ V & [\partial L,E] \times_{[\partial L,\underline{E}]} [L',\underline{E}] \end{array}\right)$$

Proof. Apply Theorem 5.1, Construction 2.2 and Lemma 5.3. ■

5.5. Corollary. *Let there be maps* $E \to \underline{E} \in \mathbb{C}$ *and* $\partial L \to L \to L' \in \mathbb{C}$ *where* $\partial L, L, L'$ *are exponentiable.*

Fix an object $C \in \mathbb{C}$ *along with a map* $\partial V \to V \in \mathbb{C}/C$ *and assume that* $\partial(L \times V) \to L \times V \in \mathbb{C}/C$ *approximates* $(\partial V \to V) \mathbin{\widehat{\times}_C} (\partial L \times C \to L \times C) \in \mathbb{C}/C$ *relative to* $E \times C \to \underline{E} \times C \in \mathbb{C}/C$. *Then,*

$$\begin{pmatrix} \partial(V \times L) & & E \times C \\ \downarrow & \boxslash_C & \downarrow \\ V \times L \longrightarrow V \times L' & & \underline{E} \times C \end{pmatrix} \cong \begin{pmatrix} \partial V & & ([L, E] \times_{[L,\underline{E}]} [L', \underline{E}]) \times C \\ \downarrow & \boxslash_C & \downarrow \\ V & & ([\partial L, E] \times_{[\partial L,\underline{E}]} [L', \underline{E}]) \times C \end{pmatrix}$$

Proof. By Proposition 5.4 we have that

$$\begin{pmatrix} \partial(V \times L) & & E \times C \\ \downarrow & \boxslash_C & \downarrow \\ V \times L \longrightarrow V \times L' & & \underline{E} \times C \end{pmatrix} \cong \begin{pmatrix} \partial V & & [L \times C, E \times C]_C \times_{[L\times C,\underline{E}\times C]_C} [L' \times C, \underline{E} \times C]_C \\ \downarrow & \boxslash_C & \downarrow \\ V & & [\partial L \times C, E \times C]_C \times_{[\partial L\times C,\underline{E}\times C]_C} [L' \times C, \underline{E} \times C]_C \end{pmatrix}$$

The result then follows from the fact that limits commute with limits and pullbacks preserve exponentials. ■

Structured Leibniz Transposes for Path Objects. We now focus on a specific case of structured Leibniz transposes involving interval and path objects. In cofibrantly-generated model structures, one often makes use of an interval object to generate the trivial cofibrations. In particular, an interval object I is some good cylinder object for the terminal object and the trivial cofibrations are usually generated in such a way that the pushout-product of the interval boundary inclusion $\partial I \hookrightarrow I$ with a trivial cofibration $K \to L$ is remains a trivial cofibration. Then, by the Leibniz transpose operation, this allows one to construct fibred path objects as the pullback-power.

In model categories, being a fibration requires the mere existence of lifting solutions and one has access to colimiting concepts such as pushout-product. However, in our setting, we wish to use the framework developed in Section 1 to enforce some sort of uniformity on the lifts, and we wish to use the framework developed in Section 4 to reduce dependency on colimiting concepts for applicability in type-theoretic settings.

Therefore, to reproduce the path object construction via the Leibiniz transpose in a structured manner amenable to type-theoretic applications, the idea is to use Proposition 5.4 and the associated Corollary 5.5 to phrase this structured version of the path object construction by way of the Leibiniz transpose involving the interval object. The formal statement of this is expressed in Corollary 5.11.

However, to build up to this, we need to first axiomatise the fibred path object constructed from an interval object.

5.6. Definition. *Let $\partial I \to I \in \mathbb{C}$ be a map between exponentiable objects. For each X, we denote its* I-path object *as the exponential* $P^I_{\mathbb{C}}(X) \coloneqq [I, X]$ *and the* coboundary of the I-path object *is the exponential* $\partial P^I_{\mathbb{C}}(X) \coloneqq [\partial I, X]$ *so that there is a* endpoint evaluation map $\mathsf{ev}_\partial \colon P^I_{\mathbb{C}}(X) \to \partial P^I_{\mathbb{C}}(X)$.

Given a map $E \to B \in \mathbb{C}$, we also write $\widehat{P}^I_{\mathbb{C}}(E \to B) \colon P^I_{\mathbb{C}}(E) \to \partial P^I_{\mathbb{C}}(E) \times_{\partial P^I_{\mathbb{C}}(B)} P^I_{\mathbb{C}}(B)$ for the pullback-Hom of $\partial I \to I$ with $E \to B$.

For each $E \to B \in {}^{\mathbb{C}}/_{C}$ and a map $\partial I \to I \in \mathbb{C}$ we write $P^I_C(E)$ and $\partial P^I_C(E)$ and $\widehat{P}^I_C(E \to B)$ to mean the respective constructions $P^{I\times C}_{\mathbb{C}/C}(E)$ and $\partial P^{I\times C}_{\mathbb{C}/C}(E)$ and $\widehat{P}^{I\times C}_{\mathbb{C}/C}(E \to B)$ in the slice category.

We observe that the local exponential used in defining I-path objects in slices occur as pullbacks of I-path objects in the ambient category. This follows from the following more general fact.

5.7. Lemma. *For any $E \to B \in {}^{\mathbb{C}}/_{B}$, and an exponentiable $A \in \mathbb{C}$, the local exponential $[A \times B, E]_B \cong B \times_{[A,B]} [A, E]$ is just the pullback-Hom of $A \to 1$ with $E \to B$ in $\mathbb{C}$.*

$$\begin{array}{ccc} [A \times B, E]_B & \longrightarrow & [A, E] \\ \big\downarrow & \lrcorner & \big\downarrow \\ B & \longrightarrow & [A, B] \end{array}$$

Proof. We argue using representability, by showing that $[A \times B, E]_B \to B \in {}^{\mathbb{C}}/_{B}$ and $B \times_{[A,B]} [A, E] \to B \in {}^{\mathbb{C}}/_{B}$ represent the same presheaf over ${}^{\mathbb{C}}/_{B}$.

We notice that, in the diagram on the left, for each $x \colon X \to B \in {}^{\mathbb{C}}/_{B}$, the local Hom-set ${}^{\mathbb{C}}/_{B}(A \times X, E)$ are precisely those maps $A \times X \to E$ whose post-composition with $E \to B$ is exactly $A \times X \xrightarrow{\mathsf{proj}_2} X \xrightarrow{x} B$, which one observes as follows.

$$\begin{array}{ccccc} A \times X & \to & A \times B & \dashrightarrow & E \\ \downarrow & & \downarrow & \swarrow & \\ X & \longrightarrow & B & & \end{array}$$

Therefore, we have the constructive pullback squares as follows.

$$\begin{array}{ccccccc} {}^{\mathbb{C}}/_{B}(A \times X, E) & \to & \mathbb{C}(X, B \times_{[A,B]} [A, E]) & \to & \mathbb{C}(A \times X, E) & \xrightarrow{\cong} & \mathbb{C}(X, [A, E]) \\ \big\downarrow & \lrcorner & \big\downarrow & \lrcorner & \big\downarrow & \lrcorner & \big\downarrow \\ 1 & \xrightarrow{\quad x \quad} & \mathbb{C}(X, B) & \longrightarrow & \mathbb{C}(A \times X, B) & \xrightarrow[\cong]{} & \mathbb{C}(X, [A, B]) \end{array}$$

Analogously, the fibre of $\mathbb{C}(X, B \times_{[A,B]} [A, E]) \to \mathbb{C}(X, B)$ over x is exactly ${}^{\mathbb{C}}/_{B}(X, B \times_{[A,B]} [A, E])$ as well. This proves ${}^{\mathbb{C}}/_{B}(A \times X, E) \cong {}^{\mathbb{C}}/_{B}(X, B \times_{[A,B]} [A, E])$ so that $[A \times B, E]_B \cong B \times_{[A \times B]} [A, E]$. ∎

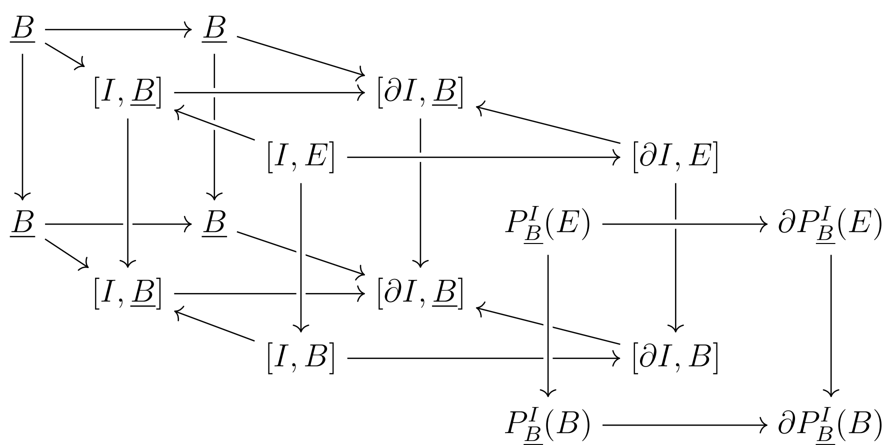
As from the name, we wish to take the I-path objects as a path object construction in slices. They indeed serve this goal in model categories when I is chosen to be an interval. Our notion of an interval is made precise using the notion of good cylinder objects, which we now recall.

5.8. Definition. *In a model category* $\mathbb{C}$, *a* good *cylinder object for an object* A *is the middle object* $\mathsf{Cyl}(A)$ *of any factorisation of the codiagonal* $A \sqcup A \to A$ *as* $A \sqcup A \hookrightarrow \mathsf{Cyl}(A) \xrightarrow{\sim} A$ *into a cofibration followed by a weak equivalence. It is* very good *when* $\mathsf{Cyl}(A) \xrightarrow{\sim} A$ *is a trivial fibration.*

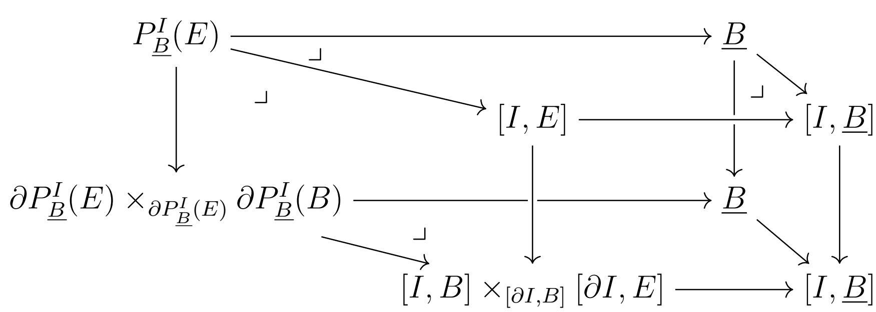
5.9. Theorem. *Let* $\mathbb{C}$ *be locally cartesian closed and equipped with a model structure in which* I *is a good, but not necessarily very good, cylinder object for the terminal object and the pullback-power of a cofibration with a fibration is a fibration.*

Then, by taking $\partial I := 1 \sqcup 1$, *for any* $E \twoheadrightarrow B \in \mathbb{C}/\underline{B}$ *fibration, the pullback-Hom* $\widehat{P}^I_{\underline{B}}(E \to B)\colon P^I_{\underline{B}}(E) \twoheadrightarrow \partial P^I_{\underline{B}}(E) \times_{\partial P^I_{\underline{B}}(E)} \partial P^I_{\underline{B}}(B)$ *is a fibration.*

Proof. Using Lemma 5.7 four times and yet another 3-by-3 argument



we see that the pullback of the front face $\partial P^I_{\underline{B}}(E) \times_{\partial P^I_{\underline{B}}(E)} \partial P^I_{\underline{B}}(B)$ can be computed by first taking the pullbacks of each layer $\underline{B} \to [I, \underline{B}] \leftarrow [I, B] \times_{[\partial I, B]} [\partial I, E]$ then taking the limit. Thus, the map $\widehat{P}^I_{\underline{B}}(E \to B)\colon P^I_{\underline{B}}(E) \twoheadrightarrow \partial P^I_{\underline{B}}(E) \times_{\partial P^I_{\underline{B}}(E)} \partial P^I_{\underline{B}}(B)$ fits into the following cube whose top, bottom and right face are pullbacks.



This means that, $\widehat{P}^I_{\underline{B}}(E \to B)$ occurs as a pullback of $[I, E] \to [I, B] \times_{[\partial I, B]} [\partial I, E]$. The result then follows by the pullback-power assumption. ■

In view of Theorem 5.9, we now axiomatise the structures of an interval object so that Theorem 5.9 can be reproduced in a type-theoretic setting in Corollary 5.11.

5.10. Definition. *A* pre-interval *structure on an exponentiable object I consists of two points $\{0\}, \{1\} \rightrightarrows I$ along with a map $\partial I \hookrightarrow I$ between exponentiable objects such that for the pullback $\{0\} \cap \{1\}$ defined as the pullback on the left, one has a factorisation on the right.*

$$\begin{array}{ccc} \{0\} \cap \{1\} & \hookrightarrow & \{0\} \\ \downarrow & \lrcorner & \downarrow \\ \{1\} & \hookrightarrow & I \end{array} \qquad\qquad \begin{array}{ccccc} \{0\} \cap \{1\} & \hookrightarrow & \{0\} & & \\ \downarrow & & \downarrow & \searrow & \\ \{1\} & \hookrightarrow & \partial I & & \\ & \searrow & & \searrow & \\ & & & & I \end{array}$$

A pre-interval structure on an object I becomes an interval structure relative to a set of objects $\mathcal{B}$ *when for each $B \in \mathcal{B}$, one further has that $[\{0\} \cap \{1\}, B] \cong 1$ and that $B \cong [\{0\}, B] \to [\partial I, B] \leftarrow [\{1\}, B] \cong B$ is a product span (i.e. $[\partial I, B] \cong B \times B$).*

5.11. Corollary. *Fix a map $E \to B$ along with an exponentiable object I equipped with an interval structure $(\{0\}, \{1\} \rightrightarrows I, \partial I \to I)$ relative to $\{E, B\}$.*

Further take maps $\partial V \to V \in \mathbb{C}/C$ between exponentiable objects in a slice and a map $\partial(V \times I) \to V \times I \in \mathbb{C}/C$ between exponentiable objects that structurally approximates $(\partial V \to V) \mathbin{\widehat{\times}_C} (\partial I \times C \to I \times C)$ relative to the map $E \times C \to B \times C \in \mathbb{C}/C$. Then, we have a bijection

$$\left(\begin{array}{ccc} \partial(V \times I) & & E \times C \\ \downarrow & \boxslash_C & \downarrow \\ V \times I \longrightarrow V & & B \times C \end{array}\right) \cong \left(\begin{array}{ccc} \partial V & & P^I_B(E) \times C \\ \downarrow & \boxslash_C & \downarrow \\ V & & (E \times_B E) \times C \end{array}\right)$$

Proof. Immediate by Corollary 5.5 and Lemma 5.7. ■

In anticipation of working with slice categories, we note several base change properties for interval structures. These are analogues of Lemma 4.5 and Construction 3.2 but for interval and path objects.

5.12. Lemma. *Let there be a map $\varphi\colon D \to C$. Then, a pre-interval structure $(\{0\}, \{1\} \rightrightarrows I, \partial I \to I)$ on an exponentiable object $I \to C \in {}^{\mathbb{C}}\!/_{C}$ gives rise to a pre-interval structure $(\varphi^*\{0\}, \varphi^*\{1\} \rightrightarrows \varphi^* I, \varphi^*(\partial I) \to \varphi^* I)$ on $\varphi^* I \to D \in {}^{\mathbb{C}}\!/_{C}$.*

If further this pre-interval structure on $I \to C$ is an interval structure relative to an object $B \to C \in {}^{\mathbb{C}}\!/_{C}$ then the pre-interval structure on $\varphi^* I \to D$ is also an interval structure relative to $\varphi^* B \to D$.

Likewise, for maps $E \to B, \partial I \to I \in {}^{\mathbb{C}}\!/_{C}$, the image under the pullback functor $\varphi^*\colon {}^{\mathbb{C}}\!/_{C} \to {}^{\mathbb{C}}\!/_{D}$ of the fibred path object, the endpoint evaluation maps and the pullback-Hom constructed from $\partial I \to I \in {}^{\mathbb{C}}\!/_{C}$ as on the left column are respectively isomorphic to the fibred path object, the endpoint evaluation maps and the pullback-Hom constructed from the interval object $\varphi^*(\partial I) \to \varphi^* I \in {}^{\mathbb{C}}\!/_{D}$ as on the right column.

$$\begin{array}{ccc}
{}^{\mathbb{C}}\!/_{C} & \xrightarrow{\quad\varphi^*\quad} & {}^{\mathbb{C}}\!/_{D} \\
P^I_{\mathbb{C}/C}(E) & \mapsto & P^{\varphi^* I}_{\mathbb{C}/D}(\varphi^* E) \\
P^I_{\mathbb{C}/C}(B) & \mapsto & P^{\varphi^* I}_{\mathbb{C}/D}(\varphi^* B) \\
\left(P^I_{\mathbb{C}/C}(E) \to \partial P^I_{\mathbb{C}/C}(E)\right) & \mapsto & \left(P^{\varphi^* I}_{\mathbb{C}/D}(E) \to \partial P^{\varphi^* I}_{\mathbb{C}/D}(E)\right) \\
\left(P^I_{\mathbb{C}/C}(B) \to \partial P^I_{\mathbb{C}/C}(B)\right) & \mapsto & \left(P^{\varphi^* I}_{\mathbb{C}/D}(B) \to \partial P^{\varphi^* I}_{\mathbb{C}/D}(B)\right) \\
\widehat{P}^I_{\mathbb{C}/C}(E \to B) & \mapsto & \widehat{P}^{\varphi^* I}_{\mathbb{C}/C}(\varphi^* E \to \varphi^* B)
\end{array}$$

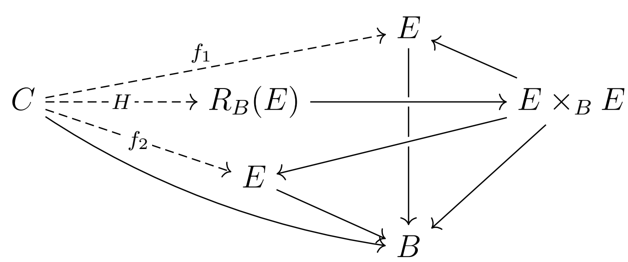
Proof. Straightforward by using the fact that the pullback functor preserves internal-Homs and limits. ■

6. Relating Lifts

In orthogonal factorisation systems, lifts are unique and in model categories, these lifts are homotopically unique. In other words, two lifts to the same lifting problem in these systems are related in some way: by the equality relation in the former case and by the homotopy equivalence relation in the latter. On the other hand, in Section 1, we introduced the concept of lifting structures as specific choices of lifts to lifting problems satisfying some uniformity conditions. Therefore, we would like to combine these two concepts of relating two lifts and uniform choices of lifts to talk about two uniform lifts being uniformly related to each other. In particular, we show in Theorem 6.5 that in certain model categories, the homotopies involved in the homotopy uniqueness of lifts can be chosen in a suitably uniform manner.

In order to do so, we first fix an abstraction of a “relation” and a “witness” of said relation, which is an internalisation of a binary relation to slice categories.

6.1. Definition. *Let there be a map $E \to B$. A* fibrewise relation *is a map $R_B(E) \to E \times_B E$. Given $C \to B$ and two maps $f_1, f_2\colon C \to E$ over B, we say H* witnesses that f_1 and f_2 agree up to relation $R_B(E)$ *when $H\colon C \to R_B(E)$ is a factorisation of $(f_1, f_2)\colon C \to E \times_B E$ via $R_B(E) \to E \times_B E$.*

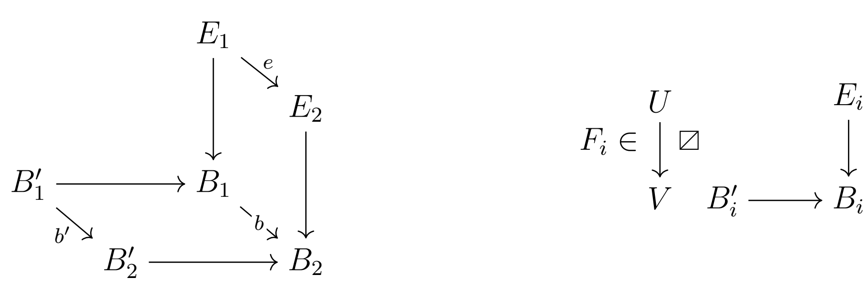


6.2. Example. Our two main sources of fibrewise relation are those giving rise to an equality and a right homotopy equivalence.

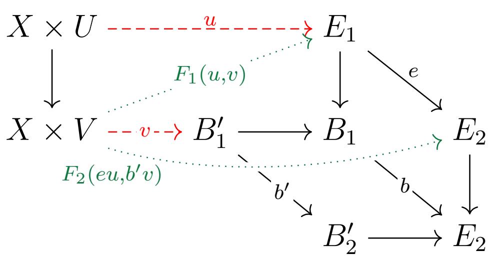
(1) If one takes $R_B(E)$ to be the diagonal $E \hookrightarrow E \times_B E$ witnesses H are unique and f_1 agrees with f_2 precisely when they are exactly equal.

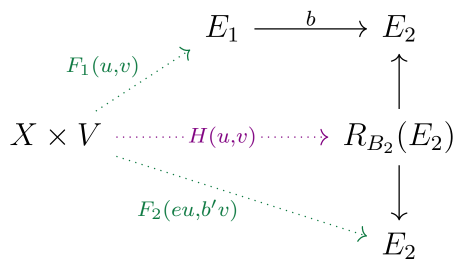
(2) In model categories, if $R_B(E)$ is taken to be the path object of E as an object of the slice over B then a witness H is precisely a right homotopy between f_1 and f_2.

We will now try to motivate how these fibrewise relations are used to uniformly relate lifts. Consider a map $U \to V$ and a map of spans (e, b, b') and two lifting structures F_i for $i = 1, 2$



Then, for $X \in \mathbb{C}$ fixed, given two lifting problems (u, v) of $X \times U \to X \times V$ against $E_1 \to B_1$ restricted along $B'_1 \to B_1$ as below, one can produce a solution $F_1(u, v)$. On the other hand, one can also induce a new lifting problem $(eu, b'v)$ against $E_2 \to B_2$ restricted along $B'_2 \to B_2$ and get a solution $F_2(eu, b'v)$. This gives two maps $u \cdot F_1(u, v), F_2(eu, b'v)\colon X \times U \rightrightarrows E_2$ over B_2 as on the left below, where the triangle involving the lifts and the map e does not necessarily commute.





A witness $H(u,v)$ that $u \cdot F_1(u,v)$ and $F_2(eu,b'v)$ are related up to relation $R_{B_2}(E_2) \to E_2 \times_{B_2} E_2$ is then a factorisation H as on the right above.

Instantiating with Example 6.2, we see that if $R_{B_2}(E_2) \to E_2 \times_{B_2} E_2$ is the diagonal then such a map $H(u,v)$ exists precisely when $e \cdot F_1(u,v) = F_2(eu,b'v)$ and if $R_{B_2}(E_2) \to E_2 \times_{B_2} E_2$ is the fibred path object then $H(u,v)$ is a right homotopy $H(u,v)\colon e \cdot F_1(u,v) \simeq F_2(eu,b'v)$ over E_2. As a further special case, if (e,b,b') are taken to be all identities, then F_1, F_2 both solve the same lifting problems, so the above two cases reduce respectively to saying F_1, F_2 produce the same lifting solution or right homotopically identical lifting solutions.

Now fix another map $t\colon Y \to X$ so that uniformity condition gives

$$F_1(u,v) \cdot (t \times V) = F_1(u \cdot (t \times U), v \cdot (t \times V))$$
$$F_2(eu,b'v) \cdot (t \times V) = F_2(eu \cdot (t \times U), b'v \cdot (t \times V))$$

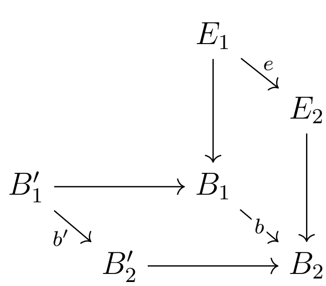
Then, using $H(u,v)$, one can produce a witness $H(u,v) \cdot (t \times V)$ saying that $e \cdot F_1(u \cdot (t \times U), v \cdot (t \times V))$ and $F_2(eu \cdot (t \times U), b'v \cdot (t \times V)$.

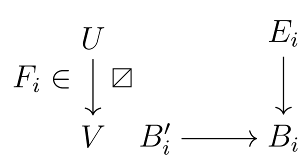
Therefore, if we let (u,v) vary and associate a family of witnesses $H(u,v)$, a natural uniformity condition is to require that

$$H(u,v) \cdot (t \times V) = H(u \cdot (t \times U), v \cdot (t \times V))$$

In view of the internalisation results of Lemma 1.9, we see that the uniformity condition of witnesses can be formalised as follows using the locally cartesian closed structure.

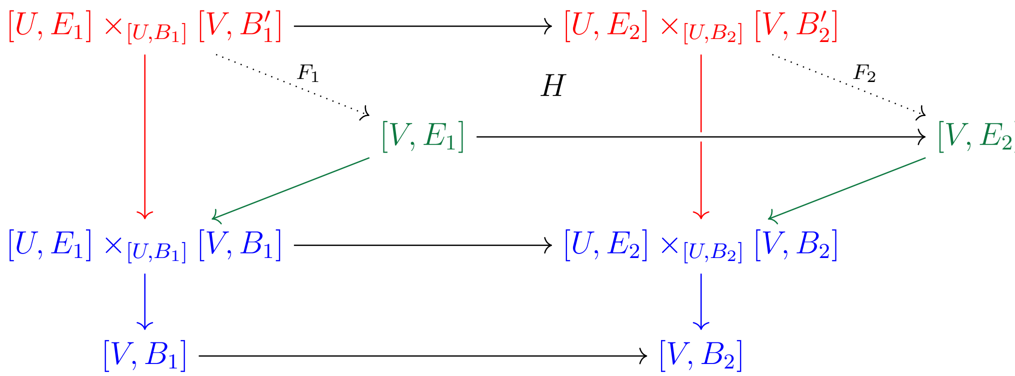
6.3. Definition. *Fix a map $U \to V$ between exponentiable objects and (e,b,b') a map of spans and two lifting structures F_i for $i = 1,2$*





Let $R_{B_2}(E_2) \to E_2 \times_{B_2} E_2$ be a fibrewise relation.

A structured $R_{B_2}(E_2)$-witness H from F_1 to F_2 via (e, b, b') *is a witness H that the top square of the following diagram commutes up to the fibrewise relation $[V, R_{B_2}(E_2)] \to [V, E_2 \times_{B_2} E_2] \cong [V, E_2] \times_{[V,B_2]} [V, E_2]$.*



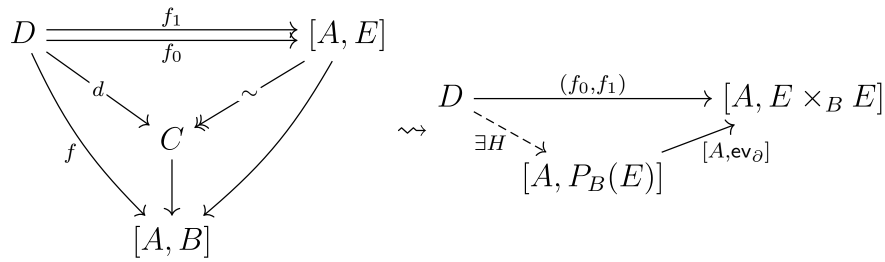
If (e, b, b') is the identity then we simply say H is a structural $R_{B_2}(E_2)$-witness between F_1 and F_2.

With this formal definition in place, we show that in model categories, when the fibrewise relation is taken to be the fibrewise path object and when the structured lifting problem is either (trivial cofibration, fibration) or (cofibration, trivial fibration) then there always exist structured lifting solutions and moreover any two such structured lifting solutions are structurally homotopic. In other words, we will show in Theorem 6.5 structured lifts are structurally homotopically unique if they exist. But for that we, first need the following technical lemma.

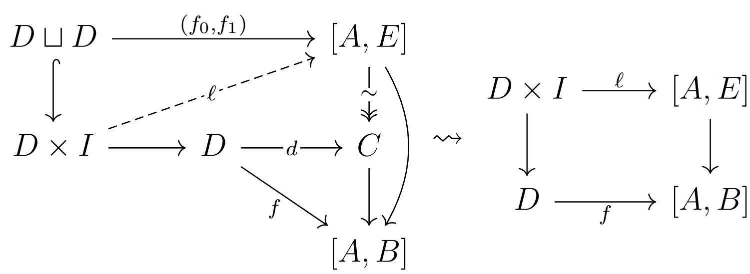
6.4. Lemma. *Let $\mathbb{C}$ be locally cartesian closed and equipped with a model structure in which:*

- *I is a good, but not necessarily very good, cylinder object for the terminal object.*
- *The cofibrations are preserved by products.*

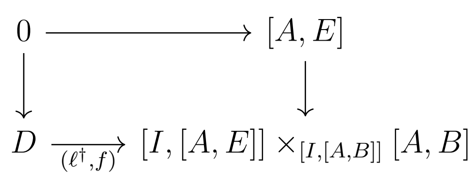
Further fix a map $E \to B$ and an object A along with a factorisation of $[A, E] \to [A, B]$ into $[A, E] \xrightarrow{\sim} C \to [A, B]$ where the first map is a trivial fibration. Take $P^I_B(E) \coloneqq [I, E] \times_{[I,B]} B$ to be the fibred I-path object of $E \to B$ constructed from I so that one has the endpoint evaluation map $\mathsf{ev}_\partial \colon P^I_B(E) \to E \times_B E$. Then, for any $d \colon D \to C$ and pair of maps $f_0, f_1 \colon D \to [A, B]$ over C as below on the left we have a factorisation H as below *on the right.*



Proof. Because $\mathbb{C}$ is cartesian closed, products preserve colimits, so in particular $1 \sqcup 1$ is preserved by products. Also, by assumption, cofibrations are preserved by products. Thus, the image of cofibration $1 \sqcup 1 \hookrightarrow I$ under the product with D is the cofibration $D \sqcup D \hookrightarrow D \times I$. But also because $[A, E] \xrightarrow{\sim} C$ is assumed to be a trivial fibration, we can solve the following lifting problem by some solution ℓ as on the left so we get a diagram on the right.

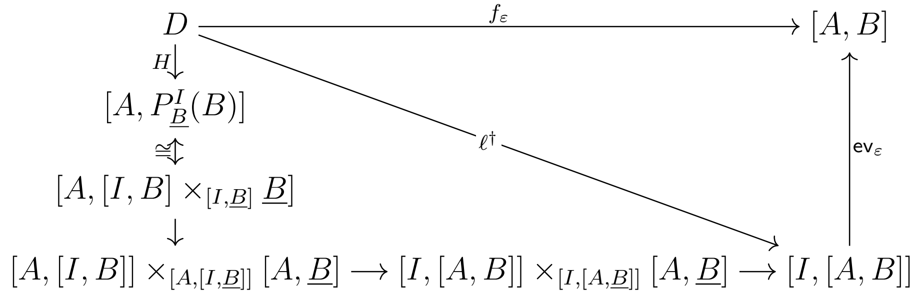


Because $D \times I \to D$ is the pushout-product of $0 \to D$ with $I \to 1$, Leibniz transpose of the right diagram above gives



It follows that one may choose $H \colon D \to [A, P^I_B(E)]$ as $H \coloneqq D \xrightarrow{(\ell^\dagger, f)} [I, [A, E]] \times_{[I,[A,B]]} [A, B] \cong [A, [I, E]] \times_{[A,[I,B]]} [A, B] \cong [A, [I, E] \times_{[I,B]} B] \cong [A, P^I_B(E)]$.

We conclude by verifying that this choice of H indeed factors $(f_0, f_1) \colon D \to [A, E \times_B E]$. This follows because for $\varepsilon = 0, 1$ one has



■

With the tools in place, we now show that structured lifts are structurally homotopically unique if they exist.

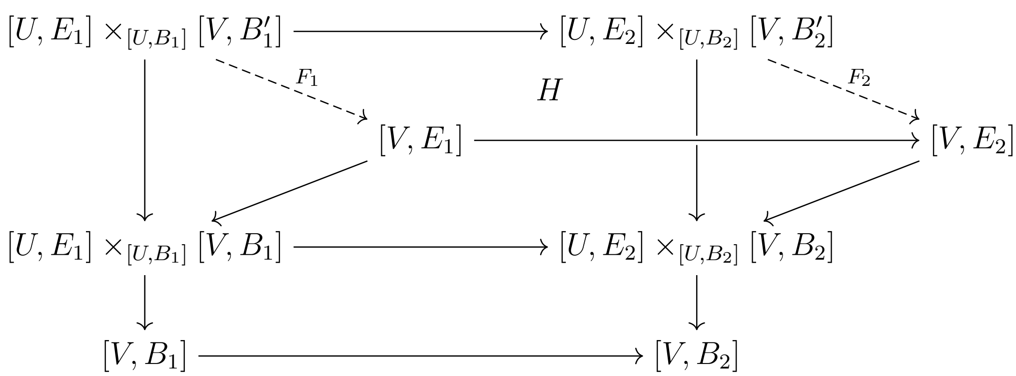
6.5. Theorem. *Let $\mathbb{C}$ be locally cartesian closed and equipped with a model structure in which:*

- *I is a good, but not necessarily very good, cylinder object for the terminal object.*
- *Cofibrations are preserved by products.*
- *The pullback-power of (trivial cofibration, fibration)- and (cofibration, trivial fibration)-pairs are trivial fibrations.*

Import the data $U \to V$ and (e, b, b') and F_1, F_2 from Definition 6.3 where the fibrewise relation $R_{B_2}(E_2) \to E_2 \times_{B_2} E_2$ is taken to be the boundary evaluation map of the fibred I-path object $P^I_{B_2}(E_2) = [B_2 \times I, E_2]_{B_2} \to E_2 \times_{B_2} E_2$.

Assume that $(U \to V, E_2 \to B_2)$ is either a (trivial cofibration, fibration) or (cofibration, trivial fibration) pair. Then, there exists a structured $P^I_{B_2}(E_2)$-witness H from F_1 to F_2 via (e, b, b').

Proof. We must show that the top square below commutes up to the fibrewise homotopy relation $P^I_{B_2}(E_2)$ with some witness H.



But the condition that $(U \to V, E_2 \to B_2)$ is either a (trivial cofibration, fibration) or (cofibration, trivial fibration) pair ensure that $[V, E_2] \xrightarrow{\sim} [U, E_2] \times_{[U,B_2]} [V, B_2]$ is a trivial fibration, so the result follows by Lemma 6.4. ■

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