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THE SIERPINSKI CARPET AS A FINAL COALGEBRA

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Abstract. We advance the program of connections between final coalgebras as sources of circularity in mathematics and fractal sets of real numbers. In particular, we are interested in the Sierpinski carpet, taking it as a fractal subset of the unit square. We construct a category of *square metric spaces* and an endofunctor on it which corresponds to the operation of gluing eight copies of a given square metric space along segments, as in the Sierpinski carpet. We show that the initial algebra and final coalgebra exists for our functor, and that the final coalgebra is bilipschitz equivalent to the Sierpinski carpet. Along the way, we make connections to topics such as the iterative construction of initial algebras as ω-colimits, corecursive algebras, and the classic treatment of fractal sets due to Hutchinson.

1. Introduction

This paper continues work on fractal sets modeled as final coalgebras. It builds on a line of work that began with Freyd's result [8] that the unit interval $[0, 1]$ is the final coalgebra of a certain endofunctor on the category of *bi-pointed* sets. Leinster's paper [12] is a far-reaching generalization of Freyd's result. It represents many of what would be intuitively called *self-similar* spaces using (a) bimodules (also called profunctors or distributors); (b) an examination of non-degeneracy conditions on functors of various sorts; (c) a construction of final coalgebras for the types of functors of interest using a notion of resolution. In addition to the characterization of fractal sets as sets, his seminal paper also characterizes them as topological spaces.

In a somewhat different direction, work related to Freyd's Theorem continues with development of *tri-pointed sets* [6] and the proof that the Sierpinski gasket $\mathbb{SG}$ is related to the final coalgebra of a functor modeled on that of Freyd [8]. (Please note that the *gasket* is different from the *carpet*.) Although it might seem that this result is but a special case of the much better results in Leinster [12], the work on tri-pointed sets was carried out in the setting of metric spaces rather than topological spaces (and so it re-proved Freyd's result in that setting, too). Work in the metric setting is unfortunately more complicated. It originates in Hasuo, Jacobs, and Niqui [10], a paper which emphasized algebras in addition to coalgebras, and proposed endofunctors defined using quotient metrics. Following this, Bhattacharya et al. [6] show that for the unit interval, the initial algebra of Freyd's functor is also interesting, being the metric space of dyadic rationals,

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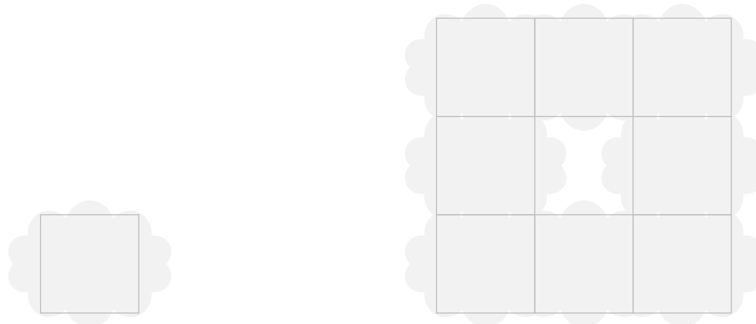
and thus the unit interval itself is its Cauchy completion. For the Sierpinski gasket, the initial algebra of the functor on tripointed sets is connected to the finite addresses used in building the gasket as a fractal; its completion again turns out to be the final coalgebra; and while the gasket itself is *not* the final coalgebra, the two metric spaces are bilipschitz equivalent.

In this paper, we take the next step in this area by considering the Sierpinski carpet $\mathbb{S}$. The difference between this and the gasket (or the unit interval) is that the gluing of spaces needed to define the functor involves *gluing along line segments*, not just at points. This turns out to complicate matters at every step. The main results of the paper are analogs of what we saw for the gasket: we have a category of metric spaces with additional structure that we call *square metric spaces*, an endofunctor $M \otimes -$ which takes a space to 8 scaled copies of itself glued along segments (the notation recalls Leinster's paper, and again we are in the metric setting), a proof that the initial algebra and final coalgebra exist, and that the latter is the completion of the former, and a verification that the actual Sierpinski carpet $\mathbb{S}$ is bilipschitz equivalent to the final coalgebra. Along the way, we need to consider a different functor $N \otimes -$ which is like $M \otimes -$ but involves 9 copies (no "hole"). The final coalgebra of $N \otimes -$ turns out to be the unit square with the taxicab metric. Moreover, in much of this work we have found it convenient to work with *corecursive algebras* as a stepping stone to the final coalgebra; the unit square with the taxicab metric turns out to be a corecursive algebra for $N \otimes -$ on square metric spaces. The Sierpinski carpet $\mathbb{S}$ turns out to be a corecursive algebra for the endofunctor $M \otimes -$, but it is not a final coalgebra for that endofunctor.

1.1. Outline. The paper begins with a discussion of the Sierpinski carpet $\mathbb{S}$ in classical terms, reviewing the results from Hutchinson [11] that we need. What we need most is that $\mathbb{S}$ is the fixed point of certain contractive map σ on the space of non-empty compact subsets of the unit square. The first leading idea in the paper is that the action of σ can be generalized to give an endofunctor $F\colon \mathcal{C} \longrightarrow \mathcal{C}$ on a category $\mathcal{C}$. But it is not immediate what that $\mathcal{C}$ and F are. The category $\mathcal{C}$ is defined in Section 3; we call it the category SquaMS of *square metric spaces*, and the functor F in Section 4 is written $X \mapsto M \otimes X$. A square metric space is metric space X together with a map $S_X : M_0 \longrightarrow X$, where M_0 is the boundary of the unit square. In pictures, it would look like the space on the left in Figure 1. The mapping S_X needs to be injective and satisfy some natural metric properties.

For technical reasons, Met in this paper is the category of metric spaces with distances bounded by 2 (not by 1, since we need M_0 to be an object). On the right in the figure, we indicate $M \otimes X$. We go into detail on this functor $M \otimes -$ in Section 4, and this will take a fair amount of preparation.

The second leading idea is that $\mathbb{S}$ should be related to the *final coalgebra* of $M \otimes -$. Indeed, this explains the title of this paper. We have the intuition that this should be so from previous work on the unit interval [9] and the Sierpinski gasket [6], and from the general treatment of self-similar sets [12]. However, as we remarked above, this paper involves a great deal more work than in those earlier works; we are not giving



(a) X (b) $M \otimes X$

Figure 1

a straightforward generalization of them. For example, Section 5 constructs the initial algebra of $M \otimes -$, and this already is more difficult than in previous work because the morphisms of the initial-algebra chain of $M \otimes -$ are not isometric embeddings. Still, $M \otimes -$ does have an initial algebra, and its completion is the final coalgebra of this functor. This and other results are proved in Section 6. We find it useful to bring in the concept of a *corecursive algebra*, and so the results of that section should be of independent interest. The paper ends in Section 7 with a proof that $\mathbb{S}$ is bilipschitz equivalent to the final coalgebra of the functor $M \otimes -$.

The paper as a whole contains a mixture of geometric ideas that crop up in the study of square metric spaces and our functor $M \otimes -$, and also very general facts about colimits of chains in various categories and facts about corecursive algebras. We hope that readers interested in one or the other of these kinds of work will come away from our paper with interest in the other kind, and that the mixture of ideas here will be useful in the category-theoretic treatment of other fractal sets.

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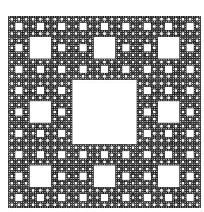
2. The Sierpinski carpet

The main object of interest in this paper is the Sierpinski carpet.



We will begin by recalling the definition of the Sierpinski carpet $\mathbb{S}$ (shown above) in terms of contractions of the unit square U_0, as in Hutchinson's work [11].

2.1. Review of Hutchinson's theorem. Let (X, d) be a complete metric space, and let Com be the set of non-empty compact subsets of X, with the *Hausdorff metric* d_H. Here is how this is defined. Given compact $A, B \subseteq X$, $d_H(A, B)$ is the supremum of distances of points of one of the sets to the other one. This is defined by

$$d_H(A, B) = \max(\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)). \tag{2.1}$$

In both cases, the distance from a point to a set is given by infima:

$$d(a, B) = \inf_{b \in B} d(a, b).$$

and similarly for $d(A, b)$.

Let M be a finite index set and suppose that for each $m \in M$, we have a contracting map $\sigma_m \colon X \longrightarrow X$. We extend each σ_m setwise to a function on (compact) sets by taking images: for $A \subseteq X$, $\sigma_m(A) = \{\sigma_m(x) : x \in A\}$. This map σ_m is a contraction of (Com, d_H). Moreover, we define $\sigma \colon \mathrm{Com} \longrightarrow \mathrm{Com}$ by

$$\sigma(A) = \bigcup_{m \in M} \sigma_m(A)$$

Again σ is a contracting map, and we let K be its unique (non-empty) fixed point. K is called the *invariant set* determined by the family $\{\sigma_m : m \in M\}$.

2.1.1. Definition. Fix $A \in \mathrm{Com}$ and contractions σ_m for $m \in M$. For each finite sequence $\vec{m} = m_1 m_2 \cdots m_k$ of elements of M, we define a set $A_{\vec{m}}$ by recursion on k, starting with $k = 0$ and the empty sequence ε:

$$\begin{array}{lcl} A_\varepsilon & = & A \\ A_{m_1 m_2 \cdots m_k m_{k+1}} & = & \sigma_{m_1}(A_{m_2 m_3 \cdots m_{k+1}}) \end{array}$$

2.1.2. Proposition. [Hutchinson [11]] *We have the following facts about the invariant set K:*

1. *If A is a non-empty compact, then diam$(A_{m_1 \ldots m_p}) \to 0$ as $p \to \infty$, where diam$(B) = \sup\{d(x, y) : x, y \in B\}$.*
2. *For every infinite sequence $\overline{m} = m_1, m_2, \ldots, m_p, \ldots$ in M,*

$$K_\varepsilon \supseteq K_{m_1} \supseteq K_{m_1 m_2} \supseteq \cdots \supseteq K_{m_1 m_2 \cdots m_p} \supseteq \cdots \tag{2.2}$$

and $\bigcap_{p=1}^{\infty} K_{m_1 \ldots m_p}$ is a singleton whose member is denoted $k_{\overline{m}}$. K is the union of these singletons.

3. *If A is a non-empty compact set, then $d(A_{m_1 \dots m_p}, k_{\overline{m}}) \to 0$ as $p \to \infty$. In particular, $\sigma^p(A) \to K$ as $p \to \infty$ in the Hausdorff metric.*

2.2. The Sierpinski carpet. Now we apply the general results in the last section to define the Sierpinski carpet $\mathbb{S}$ as a subset of $U_0 = [0,1]^2$. Throughout this paper, we will be working with (U_0, d_{Taxi}), where

$$d_{\mathsf{Taxi}}((x,y),(x_1,y_1)) = |x - x_1| + |y - y_1| \tag{2.3}$$

is the taxicab metric.

Most typically, we would view $\mathbb{S}$ as a subset of U_0 with the Euclidean metric, d_{Euc}. However, we will see that we can use the taxicab metric in our characterization of $\mathbb{S}$.

2.2.1. Definition. Two metric spaces A and B are *bilipschitz equivalent* if there is a bijection $f : A \longrightarrow B$ and a number $K \geq 1$ such that

$$\tfrac{1}{K} d_A(x,y) \leq d_B(f(x), f(y)) \leq K d_A(x,y)$$

for all $x, y \in A$.

2.2.2. Proposition. *(U_0, d_{Taxi}) is bilipschitz equivalent to (U_0, d_{Euc}).*

Proof. Our bijection will be the identity map. Let $K = 2$ and let $(x,y), (x_1,y_1) \in U_0$. Then

$$\begin{aligned}
\tfrac{1}{2} d_{\mathsf{Euc}}((x,y),(x_1,y_1)) &\leq \tfrac{1}{2}(d_{\mathsf{Euc}}((x,y),(x_1,y)) + d_{\mathsf{Euc}}((x_1,y),(x_1,y_1))) \\
&= \tfrac{1}{2}(|x - x_1| + |y - y_1|) \\
&\leq d_{\mathsf{Taxi}}((x,y),(x_1,y_1)) \\
&= |x - x_1| + |y - y_1| \\
&= \sqrt{(x - x_1)^2} + \sqrt{(y - y_1)^2} \\
&\leq \sqrt{(x - x_1)^2 + (y - y_1)^2} + \sqrt{(x - x_1)^2 + (y - y_1)^2} \\
&= 2 d_{\mathsf{Euc}}((x,y),(x_1,y_1))
\end{aligned}$$

∎

2.2.3. Corollary. *$C \subset U_0$ is a closed set with respect to d_{Taxi} if and only if it is a closed set with respect to d_{Euc}.*

Let $\mathcal{C}$ denote the collection of non-empty closed subsets of U_0 (with respect to either metric). In order to apply Hutchinson's work to define $\mathbb{S}$, we need to recall the general definition of the Hausdorff metric on compact sets from (2.1). In our setting, let us introduce some notation:

$$d_{He}(A,B) = \max(\sup_{a \in A} d_{\mathsf{Euc}}(a,B), \sup_{b \in B} d_{\mathsf{Euc}}(A,b))$$

and

$$d_{Ht}(A,B) = \max(\sup_{a\in A} d_{\mathsf{Taxi}}(a,B), \sup_{b\in B} d_{\mathsf{Taxi}}(A,b)).$$

2.2.4. PROPOSITION. *$(\mathcal{C}, d_{He})$ is bilipschitz equivalent to $(\mathcal{C}, d_{Ht})$.*

PROOF. This follows from Proposition 2.2.2

$$\begin{array}{rcl}
\frac{1}{2}d_{He}(A,B) & = & \frac{1}{2}\max(\sup_{a\in A} d_{\mathsf{Euc}}(a,B), \sup_{b\in B} d_{\mathsf{Euc}}(A,b)) \\
& = & \max(\sup_{a\in A} \frac{1}{2}d_{\mathsf{Euc}}(a,B), \sup_{b\in B} \frac{1}{2}d_{\mathsf{Euc}}(A,b)) \\
& \leq & \max(\sup_{a\in A} d_{\mathsf{Taxi}}(a,B), \sup_{b\in B} d_{\mathsf{Taxi}}(A,b)) \\
& = & d_{Ht}(A,B)
\end{array}$$

and similarly, $d_{Ht}(A,B) \leq 2d_{He}(A,B)$. ■

So from here on, we will consider (U_0, d_{Taxi}) and define $\mathbb{S}$ as a subset of U_0 with respect to the taxicab metric.

For the remainder of the section, we may write d_{U_0} or simply d to denote d_{Taxi}.

2.2.5. DEFINITION.

1. M is $\{0,1,2\}^2 \setminus \{(1,1)\}$.
2. For each $m = (i,j) \in M$, let $\mathsf{shrink}(m) \in U_0$ be given by
$$\mathsf{shrink}(m) = (\tfrac{1}{3}i, \tfrac{1}{3}j).$$
3. For a subset $A \subseteq U_0$, we define $\sigma_m\colon \mathrm{Com} \longrightarrow \mathrm{Com}$ by
$$\sigma_m(A) = \mathsf{shrink}(m) + \tfrac{1}{3}(A).$$

Finally, let $\sigma\colon \mathrm{Com} \longrightarrow \mathrm{Com}$ be $\sigma(A) = \bigcup_{m\in M} \sigma_m(A)$.

Since we are scaling by a factor of $\frac{1}{3}$, it is routine to verify that σ is a contraction on Com with respect to d_{Ht}. Indeed, it easy to verify that it is also a contracting map with respect to d_{He}.

2.2.6. DEFINITION. The Sierpinski carpet $\mathbb{S}$ is the unique fixed point of $\sigma\colon \mathrm{Com} \longrightarrow \mathrm{Com}$. That is, it is the unique non-empty compact (with respect to d_{Taxi}) subset of U_0 fixed by σ.

When we consider $\mathbb{S}$ as a metric space, we primarily take the metric to be the one inherited from (U_0, d_{Taxi}). For example, the distance between $(0,0)$ and $(1,1)$ is 2. But because they are bilipschitz equivalent, if we had defined $\mathbb{S}$ with respect to the Euclidean metric, we would get the exact same fixed point.

Indeed, $\mathbb{S}$ is the unique non-empty compact (with respect to either metric) subset of $\mathbb{R}^2$ fixed by σ. But this is not relevant for us, and we prefer to work with subsets of the unit square U_0.

3. The category of square metric spaces

We start by defining SquaMS, the category of *square metric spaces*. Though some of the arguments in the following sections will apply more generally, our work will primarily focus on this category. Our goal is to find an endofunctor F on this category and an F-coalgebra which is bilipschitz equivalent to the Sierpinski carpet.

3.0.1. Definition. Let

$$M_0 = \{(r,s) : r \in \{0,1\}, s \in [0,1]\} \cup \{(r,s) : r \in [0,1], s \in \{0,1\}\} \tag{3.1}$$

be the boundary of the unit square.

A *square set* is a set X with with an injective map $S_X : M_0 \to X$. The idea is that S_X designates the 4 sides of the square. Let $\mathsf{SquaSet}$ denote the category whose objects are square sets, and whose morphisms preserve S_X. That is, for square sets X and Y and $f : X \to Y$, for $(r,s) \in M_0$, we must have $f(S_X((r,s))) = S_Y((r,s))$.

3.0.2. Example. Here are some examples of square sets:

- M_0 with $S_{M_0} = id$.
- $X = [0,1]^2$, where S_X is the inclusion map.
- The Sierpinski carpet $\mathbb{S}$, where S_X is the inclusion map.

We are interested in square sets which are metric spaces.

3.0.3. Definition. (X, S_X) is a *square metric space* if X is a metric space bounded by 2, and the boundary indicated by S_X satisfies the following:

(SQ_1) For $i \in \{0,1\}$ and $r, s \in [0,1]$,

$$d_X(S_X((i,r)), S_X((i,s))) = |s-r|$$

and

$$d_X(S_X((r,i)), S_X((s,i))) = |s-r|.$$

That is, along each side of the square, distances coincide with distances on the unit interval.

(SQ_2) For $(r,s), (t,u) \in M_0$,

$$d_X(S_X((r,s)), S_X((t,u))) \geq d_{\mathsf{Taxi}}(S_{M_0}((r,s)), S_{M_0}((t,u))) = |r-t| + |s-u|.$$

This is a non-degeneracy requirement, which prevents our squares from "collapsing". For example, we want to avoid the case when opposite corners are less than distance 1 from each other.

Note that we do not require the metric on the boundary of the square to coincide with the Euclidean metric. Specifically, we are not requiring that opposite corners have distance $\sqrt{2}$. In fact, we will be interested in a *path metric* around the square. That is, we will determine the distance between points by the shortest path around the square (described in more detail below).

3.0.4. Example. Here are examples of square metric spaces:

- The unit square $([0,1]^2, S)$ where S is the inclusion map, with the taxicab metric.
- (M_0, id) with the path metric: for $x, y \in M_0$, if they are on the same side, their distance coincides with the unit interval, if they are on adjacent sides which share a corner C, $d(x,y) = d(x,C) + d(C,y)$, and if they are on opposite sides, $d(x,y)$ is the minimum (between the two sides) of $d(x, C_1) + 1 + d(C_2, y)$ where C_1, C_2 are endpoints of a side not containing either x or y, with C_1 on the side containing x and C_2 on the side containing y. Note that these distances are all bounded by 2 (the distance between opposite corners is 2). Unless otherwise stated, when we use the notation M_0, it is for the boundary of the unit square with the path metric.
- (M_0, id) with the taxicab metric (the metric inherited from $([0,1]^2, S)$ above). Note that the distance between points on opposite sides in this metric is almost always less than the distance in the path metric. It will be important to distinguish the taxicab and path metrics on the set M_0.

3.0.5. Definition. Let X and Y be metric spaces. A map $f : X \longrightarrow Y$ is *short* if for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2).$$

Other names for this notion are *non-expanding* or *non-distance-increasing* map. When we consider metric spaces as a category MS, we are using short maps as the morphisms.

3.0.6. Proposition. *If (X, S_X) is a square metric space, then $S_X : M_0 \longrightarrow X$ is a short map.*

Proof. Let $x, y \in M_0$. If x and y are on the same side of M_0, then $d_X(S_X(x), S_X(y)) = d_{M_0}(x,y)$, by ($\mathrm{SQ}_1$). If x and y are on adjacent sides, let C be the corner between them. Then using the triangle inequality in (X, S_X) and what we have just seen,

$$\begin{aligned} d_X(S_X(x), S_X(y)) &\leq d_X(S_X(x), S_X(C)) + d_X(S_X(C), S_X(y)) \\ &= d_{M_0}(x, C) + d_{M_0}(C, y) \\ &= d_{M_0}(x, y). \end{aligned}$$

Finally, we have the case when x and y are on opposite sides of the square. Let C_1, C_2 be the endpoints of the side which provides the shortest path from x to y in M_0. Then

$$\begin{aligned} & d_X(S_X(x), S_X(y)) \\ \leq\ & d_X(S_X(x), S_X(C_1)) + d_X(S_X(C_1), S_X(C_2)) + d_X(S_X(C_2), S_X(y)) \\ =\ & d_{M_0}(x, C_1) + 1 + d_{M_0}(C_2, y) \\ =\ & d_{M_0}(x, y) \end{aligned}$$

■

3.0.7. Definition. Let SquaMS be the category whose objects are square metric spaces (bounded by 2) whose morphisms $f : (X, S_X) \longrightarrow (Y, S_Y)$ are short maps which preserve S: $S_Y = f \circ S_X$.

Proposition 3.0.8 provides a characterization of SquaMS.[1]

3.0.8. Proposition. SquaMS *is the full subcategory of the slice category* M_0/MS *determined by the objects* $(X, S_X : M_0 \longrightarrow X)$ *with the property that* S_X *is short and* (X, S_X) *satisfies* (SQ_1) *and* (SQ_2)*. The initial object in* SquaMS *is* (M_0, id) *with the path metric.*

3.0.9. Proposition. *Monomorphisms in* SquaMS *are the morphisms which are one-to-one.*

Proof. Let $f : X \to Y$ be a monomorphism. Let $Z = M_0 \cup \{z\}$ be the boundary of the unit square with the path metric and one extra point z such that $d_Z(z, (r, s)) = 2$ for all $(r, s) \in M_0$. This is an object in SquaMS. Let $x_0, x_1 \in X$ and suppose $f(x_0) = f(x_1)$. Define $g_i : Z \to X$ by $(r, s) \mapsto S_X((r, s))$ for $(r, s) \in M_0$ and $z \mapsto x_i$ for $i = 0, 1$. These clearly preserve S_Z, and are short maps since

$$d_X(g_i((r, s)), g_i((t, u))) \leq d_Z(S_Z((r, s)), S_Z((t, u)))$$

for $(r, s), (t, u) \in M_0$ by the same argument as the previous proposition, and

$$d_X(g_i(z), g_i((r, s))) \leq 2 = d_Z(z, (r, s))$$

for $(r, s) \in M_0$. Now $f \circ g_0 = f \circ g_1$, since $f(g_i((r, s))) = S_Y((r, s))$, and $f(g_0(z)) = f(x_0) = f(x_1) = f(g_1(z))$. So since f is a monomorphism, $g_0 = g_1$, which means that $x_0 = g_0(z) = g_1(z) = x_1$.

For the other direction, suppose f is an injective morphism and $g_0, g_1 : Z \to X$ are morphisms from an arbitrary object Z such that $f \circ g_0 = f \circ g_1$. Then for $z \in Z$, $f(g_0(z)) = f(g_1(z))$. Since f is injective, $g_0(z) = g_1(z)$. Hence, $g_0 = g_1$. ■

3.0.10. Proposition. SquaMS *has no final object.*

[1] We are grateful to an anonymous referee for this observation.

Proof. As in the previous proposition, let Z be the boundary of the unit square, M_0, with the path metric and a single point z defined to be distance 2 from every point in M_0. This is an object in SquaMS, via the inclusion $M_0 \to Z$.

Let Y be an object in the category. Then consider $f_0 : Z \to Y$ defined by $(r, s) \mapsto S_Y((r, s))$ for $(r, s) \in M_0$ and $z \mapsto S_Y((0, 0))$, and $f_1 : Z \to Y$ defined by $(r, s) \mapsto S_Y((r, s))$ for $(r, s) \in M_0$ and $z \mapsto S_Y((1, 1))$. As in the previous proposition, these maps are both morphisms. So since there are two distinct morphisms from Z to Y, Y cannot be a final object in SquaMS. ∎

4. The functors $M \otimes -$ and $N \otimes -$

In this section we will define a functor

$$M \otimes - : \mathsf{SquaMS} \longrightarrow \mathsf{SquaMS}$$

which, when applied to the initial square metric space M_0 (using the path metric), will give us objects which correspond to iterations of the Sierpinski carpet. The idea is that M will be a set of indices indicating positions to place scaled copies of X, and $M \otimes X$ also indicates identifications that turn the metric space $M \times X$ into a square metric space. In detail, $M \otimes X$ will contain 8 copies of X arranged in a 3×3 grid, but without the central copy. We have mentioned this functor in the Introduction. In Figure 1 we showed caricatures of square spaces and the action of M. For a square space X, we want $M \otimes X$ to look like eight copies scaled by a factor of $\frac{1}{3}$ with appropriate gluings on edges of the squares, and with a "hole" in the middle.

Later in the paper, we will iterate this functor in order to form a chain, beginning with M_0, the boundary of the unit square. Then we take the colimit of this chain, and finally take the completion of the colimit. As we shall see, we obtain a space bilipschitz equivalent to the Sierpinski carpet; this is the main result in the paper. We will also define a different functor $N \otimes -$. The difference between $M \otimes X$ and $N \otimes X$ is that $N \otimes X$ uses 9 copies instead of 8; it has no central "hole." This functor shares properties with $M \otimes -$. To obtain the desired results about $M \otimes -$ it is useful to also use results on $N \otimes -$.

4.1. A general discussion of quotient metrics on sets.

In this section, we work at a high level of generality so that we can obtain results which we then apply to the main functors on SquaMS of interest in this paper.[2] As mentioned above, those functors are called $M \otimes -$ and $N \otimes -$, but they are not defined until Sections 4.2 and 4.5 respectively.

Let L be a finite set. We call the elements of L *indices*. The idea is that these will indicate positions in which we will place scaled copies of a given square metric space X. We shall endow the product set $L \times X$ with a metric space structure in (4.2). We subsequently define $L \otimes X$ using the quotient metric (Definition 4.1.4, via a certain equivalence relation

[2] We will do this work for the category SquaMS, though it should be noted that the results of this section can be adapted to apply to a broad collection of categories, such as the bipointed or tripointed metric spaces in [6, 8].

E). Our work is rather general. We will give requirements on L and E which will guarantee that $L \otimes X$ is a metric space.

We are not, however, going to show that $L \otimes -$ is a functor on SquaMS. Indeed, our requirements on L and $\sim$ will not guarantee that $L \otimes X$ is in SquaMS, and they are not enough allow us to define $L \otimes f$ for morphisms f in SquaMS. The intention here is to work at a level of generality such that we can use the metric space result towards showing that $M \otimes -$ and $N \otimes -$ are functors.

Let E be an equivalence relation on $L \times M_0$. Later in the paper, given an object X, the pairs in E will identify places where we "glue copies of X" by a procedure which we will specify shortly. Of course, the set E is defined independently of X; it is simply an equivalence relation on $L \times M_0$.

For a fixed object X in SquaMS, define a relation $\approx$ on $L \times X$ as follows: For $m, n \in L$ and $(r, s), (t, u) \in M_0$,

$$(m, S_X((r, s))) \approx (n, S_X((t, u))) \text{ if and only if } ((m, (r, s)), (n, (t, u))) \in E. \tag{4.1}$$

Let $\sim$ be the symmetric, reflexive, and transitive closure of $\approx$ on $L \times X$. In more detail, if E is symmetric, then so is $\approx$. If E is transitive, then again so is $\approx$. But even if E is reflexive, $\approx$ need not be reflexive, since S_X is almost never surjective. So this is why we must in general extend $\approx$ to get the relation $\sim$.

In the following definition, we wish to characterize equivalence relations which suit our needs later on, but are sufficiently general to apply to a broader class of similar constructions.

As we said, the big idea is that we will "glue copies of X" together, specifically along sides of the image of M_0 under S_X. We need to do this in such a way that we set ourselves up to view the resulting object as a metric space.

4.1.1. Definition. Let $D = \{B, \ell, R, T\}$ be a set denoting the bottom, left, right, and top sides of M_0. That is,

$$\begin{array}{rclcrcl} B &=& \{(r, 0) : r \in [0, 1]\}, & \qquad & R &=& \{(1, s) : s \in [0, 1]\}, \\ \ell &=& \{(0, s) : s \in [0, 1]\}, & & T &=& \{(r, 1) : r \in [0, 1]\}. \end{array}$$

An equivalence relation E on $L \times M_0$ is *quotient suitable* if the following data exist, and if E is characterized in terms of them as mentioned below:

First, an injective partial function $\kappa : L \times D \to L \times D$ such that

- For all $m \in L$ and $Y \in D$, there is no $Z \in D$ such that $\kappa(m, Y) = (m, Z)$.
- The domain and image of κ are disjoint.
- If (m, Y) is in the domain of κ and $\kappa(m, Y) = (n, Z)$, then for all (m, Y') in the domain of κ with $Y' \neq Y$, $\kappa(m, Y') \neq (n, Z')$ for any $Z' \in D$.

Second, for each (m, Y) in the domain of κ, an isometry $f_{m,Y} : Y \to Z$ (where $(n, Z) = \kappa(m, Y)$).

Observe that we may view each side of M_0 as an isometric copy of $[0, 1]$, so the only possible isometries $f_{m,Y}$ are either the identity or the map $r \mapsto 1 - r$.

So if $\kappa(m, Y) = (n, Z)$, we also have an isometry $f_{n,Z} : Z \to Y$ where $f_{n,Z} = f^{-1}_{m,Y}$.

And our requirement about all of this is that E is the symmetric, transitive, and reflexive closure of

$$\bigcup_{(m,Y)\in dom(\kappa)} \{((m, y),(n, f_{m,Y}(y))) : y \in Y, \text{for some } Z,\ \kappa(m, Y) = (n, Z)\}.$$

The big idea is that E comes from matching sides of M_0 to sides in different copies of it. The first requirement on κ tells us that in a single copy of M_0, none of the sides are equivalent to each other. The second requirement along with the fact that κ is an injective function tells us that if we fix one copy and one side, it is matched with at most one other side in one other copy. The third requirement tells us that between two copies, we cannot have multiple sides which are equivalent. Geometrically, we may view the maps $f_{m,Y}$ as preserving a side, or reflecting it.

When E is a quotient suitable relation, we have a few nice properties of the induced equivalence relation on $L \times X$ for an arbitrary $X \in \mathsf{SquaMS}$. When we refer to sides in $S_X[M_0]$, we mean the image of the corresponding sides in M_0 under S_X. Since S_X is injective, the sides are disjoint except at their shared corners.

4.1.2. Lemma. *Let E be a quotient suitable relation on $L \times M_0$ and let $X \in \mathsf{SquaMS}$. Let $\sim$ be the equivalence relation on $L \times X$ described below (4.1).*

1. *If $(m, x) \neq (n, y)$ in $L \times X$, then $(m, x) \sim (n, y)$ implies that $x, y \in S_X[M_0]$.*
2. *$\sim$ relates corners to corners. That is, if $(r, s) \in M_0$ is such that $r, s \in \{0, 1\}$, and $m, n \in L$ and $(t, u) \in M_0$ are such that $(m, S_X((r, s))) \sim (n, S_X((t, u)))$, then (t, u) is a corner (that is, $t, u \in \{0, 1\}$).*
3. *Suppose x is in $S_X[M_0]$ but is not a corner and y is on the same side of $S_X[M_0]$. If there are $m, n \in L$ and $x' \in X$ are such that $m \neq n$ and $(m, x) \sim (n, x')$, then there is some $y' \in M_0$ on the same side as x' such that $(m, y) \sim (n, y')$.*

   Furthermore, $d_X(x, y) = d_X(x', y')$.
4. *Suppose x is not a corner in $S_X[M_0]$ and that there are $m, n \in L$ and $x' \in X$ such that $m \neq n$ and $(m, x) \sim (n, x')$. Suppose further that y is on the same side as x in $S_X[M_0]$ and is also not a corner, and that for some $l \in L$ with $l \neq m$ and $y' \in X$, $(m, y) \sim (l, y')$. Then we must have $l = n$ and y' is on the same side as x' in $S_X[M_0]$.*

3. and 4. can be thought of as existence and uniqueness in some sense. The idea is that if we have one point on a side related to another side in another copy, its entire side is related to that other side in that other copy as well, and furthermore, we cannot relate any other sides between these two copies of X.

PROOF.

1. Immediate from the definition of $\sim$.

2. Follows from the fact that the only isometries between sides will map corners to corners, and taking the symmetric, reflexive, and transitive closure will still only relate corners to corners.

3. Start with $x \in S_X[M_0]$ which is not a corner, and let $y \in S_X[M_0]$ be on the same side as x. Let $(r,s),(t,u) \in M_0$ be such that $S_X((r,s)) = x$ and $S_X((t,u)) = y$, and let $Y \in D$ be the side containing (r,s) and (t,u).

   Suppose there are $m,n \in L$ and $x' \in X$ such that $m \neq n$ and $(m,x) \sim (n,x')$. Then by part 1., there is $(r',s') \in M_0$ such that $x' = S_X((r',s'))$, and by part 2., (r',s') is not a corner. So there is a single side $Z \in D$ containing (r',s'). Then from the definition of quotient suitable, we must have $(r',s') = f_{m,Y}((r,s))$, where $f_{m,Y} : Y \to Z$ is the appropriate isometry.

   Then by our definition of E, we know that $((m,(t,u)),(n,f_{m,Y}((t,u)))) \in E$, so let $(t',u') = f_{m,Y}((t,u))$, which is on the same side as (r',s'). Thus, $y' = S_X((t',u'))$ is such that $(m,y) \sim (n,y')$, and is on the same side as x'.

   Furthermore, since S_X is an isometry between points on the same side,

$$\begin{aligned} d_X(x,y) &= d_X(S_X((r,s)),S_X((t,u)) \\ &= d_{M_0}((r,s),(t,u)) \\ &= d_{M_0}(f_{m,Y}((r,s)),f_{m,Y}((t,u))) \\ &= d_{M_0}((r',s'),(t',u')) \\ &= d_X(S_X((r',s')),S_X((t',u'))) \\ &= d_X(x',y') \end{aligned}$$

4. Suppose x is not a corner in $S_X[M_0]$ and there are $m,n \in L$ and $x' \in X$ such that $m \neq n$ and $(m,x) \sim (n,x')$. Then $x = S_X((r,s))$ for some $(r,s) \in M_0$, and by part 1., $x' = S_X((r',s'))$ for some $(r',s') \in M_0$.

   Suppose further that $y \in S_X[M_0]$ is also not a corner and is on the same side as x, so $y = S_X((t,u))$ for some $(t,u) \in M_0$ on the same side as (r,s). Assume for some $l \in L$ with $l \neq m$ and $y' \in X$, $(m,y) \sim (l,y')$. Then by part 1., $y' = S_X((t',u'))$ for some $(t',u') \in M_0$.

   Let $Y \in D$ be the unique side containing (r,s) and (t,u), and let $Z \in D$ be the unique side containing (r',s'). (Since neither (r,s) nor (t,u) is a corner, each is only on one side). Then since $((m,(r,s)),(n,(r',s'))) \in E$, by the definition of quotient suitable, $(r',s') = f_{m,Y}((r,s))$.

   Even after taking the symmetric, reflexive, and transitive closures, the only elements of the equivalence class of $(m,(t,u))$ under E are itself and $(n,f_{m,Y}((t,u)))$. Thus, since $((m,(t,u)),(l,(t',u'))) \in E$ and $l \neq m$, we must have $(n,f_{m,Y}((t,u))) = (l,(t',u'))$, so

$l = n$ and $(t', u') = f_{m,Y}((t, u))$ which is on the side Z. Thus, $y' = S_X((t', u'))$ is on the same side as x'.

■

The Quotient Space and Quotient Metric Recall that every object in SquaMS has distances bounded by 2. Ultimately we will define $L \otimes X$, in which we will consider a quotient of $L \times X$, and show that this is a metric space. As a stepping stone, we consider a metric d on $L \times X$ defined by

$$d_{L\times X}((m, x), (n, y)) = \left\{ \begin{array}{ll} \frac{1}{3} d_X(x, y) & \text{if } m = n \\ 2 & \text{otherwise} \end{array} \right\} \tag{4.2}$$

So the distance is scaled by $\frac{1}{3}$ inside of each copy of X, and otherwise, it is 2 (the maximum distance). The constant $\frac{1}{3}$ comes from the particular sets M and N to which we apply the construction in the next sections.

We see right away that $d_{L\times X}$ is bounded by 2, since points in the same copy of X will be at most $\frac{2}{3} < 2$ from each other, and points in different copies will be 2 away from each other.

Fix a quotient suitable equivalence relation E on $L \times M_0$.

4.1.3. Definition. Let X be a square metric space. The space $L \otimes X$ is the quotient of $L \times X$ by the equivalence relation $\sim$ (described below (4.1)):

$$\begin{array}{lcl} L \otimes X & = & (L \times X)/\!\sim \\ m \otimes x & \text{denotes} & \text{the equivalence class of } (m, x) \text{ in } L \otimes X. \end{array}$$

(Note that our notations $L \otimes X$ and $m \otimes x$ do not include $\sim$, but this is to unburden the notation. All our work uses $\sim$.)

In order to define a metric on $L \otimes X$, we will need the following notions:

4.1.4. Definition.

1. For $(m, x), (n, y) \in L \times X$, a *path* from (m, x) to (n, y) is a finite list of elements of $L \times X$, $(m_0, x_0), \ldots, (m_k, x_k)$, such that $(m_0, x_0) = (m, x)$ and $(m_k, x_k) = (n, y)$.

2. The *score* of the path $(m_0, x_0), \ldots, (m_k, x_k)$ is

$$\sum_{i=0}^{k-1} \widehat{d}((m_i, x_i), (m_{i+1}, x_{i+1}))$$

where

$$\widehat{d}((m_i, x_i), (m_{i+1}, x_{i+1})) = \left\{ \begin{array}{ll} 0 & \text{if } (m_i, x_i) \sim (m_{i+1}, x_{i+1}) \\ d_{L\times X}((m_i, x_i), (m_{i+1}, x_{i+1})) & \text{otherwise} \end{array} \right\}$$

3. For $m \otimes x$ and $n \otimes y$ let $d_{L\otimes X}(m \otimes x, n \otimes y)$ denote the infimum over all paths from (m, x) to (n, y) of the score. We will refer to this as the *quotient metric.*

Note that, as it is defined, $d_{L\otimes X}$ is a pseudo-metric: clearly $d_{L\otimes X}$ is symmetric, the distance between any point and itself is 0, and it will satisfy the triangle inequality since the concatenation of two paths is a path. We will show that $d_{L\otimes X}$ is in fact a metric: distinct points will have positive distance. To achieve this, we will show that the distance is actually witnessed by the score of some particular finite path; it is not just an infimum of the scores of an infinite set of paths.

4.1.5. Definition. For $(m, x), (n, y) \in L \times X$, an *alternating path in $L \times X$* from (m, x) to (n, y) is a path from (m, x) to (n, y) such that every other element is related by $\sim$, and those elements not related by $\sim$ are distinct and share the same first entry m_i. The relation by $\sim$ can start with the first or second entry, and either the last pair is related by $\sim$ or the pair just before the last is related by $\sim$. In other words, it is a sequence of the form

$$(m, x) = (m_0, x_0), (m_0, x_0') \sim (m_1, x_1), (m_1, x_1') \sim \ldots \sim (m_p, x_p), (m_p, x_p') = (n, y) \quad (4.3)$$

where (m_0, x_0) or (m_p, x_p') (or both) might be omitted. (If the first is omitted, then x belongs to $S_X[M_0]$, and similarly with the last and y.)

4.1.6. Remark. Note, if (m, x) and (n, y) have an alternating path between them in which $p = 0$, either $(m, x) = (n, y)$ or $m = n$. When we say that there is an alternating path from $m\otimes x$ to $n\otimes y$, we mean there is an alternating path from (m, x) to (n, y). Note that the choice of representatives of the equivalences classes of $m \otimes x$ and $n \otimes y$ are not important, since if we have an alternating path between two representatives, by adding one more entry on each end with $\sim$ or replacing the first or last entry as appropriate, we have an alternating path between any two representatives of $m\otimes x$ and $n\otimes y$ respectively.

4.1.7. Lemma. *For (m, x) and (n, y) in $L \times X$, either every path from (m, x) to (n, y) has score 2, or for any path from (m, x) to (n, y) there is an alternating path from (m, x) to (n, y) with smaller or equal score*

Proof. If it exists, take a path from (m, x) to (n, y), say

$$(m, x) = (m_0, x_0), (m_1, x_1), (m_2, x_2), \ldots, (m_p, x_p) = (n, y) \quad (4.4)$$

with score strictly less than 2.

If any adjacent pair on the path, say (m_k, x_k) and (m_{k+1}, x_{k+1}) has $m_k \neq m_{k+1}$ and the pair is not related by $\sim$, then this pair contributes 2 to the score, which is impossible since we assumed the score is < 2. We thus assume that this case does not arise in what follows.

We may take our path (4.4) and shorten any chain of $\sim$ relations. This is because $\sim$ is transitive. Thus, we can assume that no three adjacent pairs are related by $\sim$. In other words, we never have $(m_k, x_k) \sim (m_{k+1}, x_{k+1}) \sim (m_{k+2}, x_{k+2})$.

At this point, we argue by induction on p that for every path (4.4) which meets all of the assumptions so far, there is an alternating path with smaller or equal score. If $p = 0$, then $(m, x) = (n, y)$ so $(m, x) = (m_0, x_0), (m_0, x'_0) = (n, y)$ is an alternating path with score 0 (so, equal to the score of the original).

Assume our result for paths of length $< p$, and consider a path as in (4.4) of length p. If this path is not alternating, then there must be $(m_k, x_k) \not\sim (m_{k+1}, x_{k+1}) \not\sim (m_{k+2}, x_{k+2})$, since we have assumed we cannot have 3 or more entries in a row related by $\sim$. In this case, what we said at the end of the first paragraph implies that $m_k = m_{k+1} = m_{k+2}$. So we may shorten our path by deleting (m_{k+1}, x_{k+1}). By the triangle inequality (in X) and the definition of the metric on $L \times X$ in (4.2), the score does not increase with this deletion. And then applying our induction hypothesis to the shortened path proves our result. ∎

We need a few more technical lemmas about shortening alternating paths.

4.1.8. Lemma. *Consider an alternating path*

$$(m, x) = (m_0, x_0), (m_0, x'_0) \sim (m_1, x_1), (m_1, x'_1) \sim \ldots \sim (m_p, x_p), (m_p, x'_p) = (n, y) \quad (4.5)$$

with $p \geq 0$.

Just in the context of this lemma, say a bad configuration *in an alternating path (4.5) is a number k such that one of the following holds:*

- *$k = 0$, x_0 and x'_0 are on the same side of $S_X[M_0]$, and x'_0 is not a corner,*
- *$0 < k < p$, x_k and x'_k are on the same side of $S_X[M_0]$, and at least one of x_k or x'_k is not a corner,*
- *$k = p$, x_p and x'_p are on the same side of $S_X[M_0]$, and x_p is not a corner.*

Note that if (m_0, x_0) (or (m_p, x_p)) is omitted because x (or respectively y) is in $S_X[M_0]$, then the only possible bad configuration is the second of the three cases above.

Then: from (4.5) we can find an alternating path with strictly fewer entries, a smaller or equal score than the original path, and with no bad configurations.

Proof. First suppose our alternating path (4.5) has exactly one bad configuration. Without loss of generality, we will suppose the bad configuration is k such that $0 < k < p$ and x'_k is not a corner. The cases when $k = 0$ or p, as well as the case when $0 < k < p$ and x_k is not a corner (but x'_k might be) are all similar.

This assumption tells us that x_k and x'_k are on the same side of $S_X[M_0]$. Our hypothesis in this lemma implies that $(m_k, x'_k) \sim (m_{k+1}, x_{k+1})$.

By Lemma 4.1.2(3), there exists $\widehat{x}_k$ on the same side as x_{k+1} such that $(m_k, x_k) \sim (m_{k+1}, \widehat{x}_k)$. (Note: when we say "on the same side", there is no ambiguity. Since x'_k is not a corner, x_{k+1} is also not a corner by Lemma 4.1.2(1), which means that there is in fact only one side of $S_X[M_0]$ containing it.) By the definition of the metric on $L \times X$, the triangle inequality in X, and Lemma 4.1.2(3),

$$\begin{array}{rl} & d_{L\times X}((m_{k+1},\widehat{x}_k),(m_{k+1},x'_{k+1})) \\ = & \frac{1}{3}d_X(\widehat{x}_k,x'_{k+1}) \\ \leq & \frac{1}{3}d_X(\widehat{x}_k,x_{k+1})+\frac{1}{3}d_X(x_{k+1},x'_{k+1}) \\ = & \frac{1}{3}d_X(x_k,x'_k)+\frac{1}{3}d_X(x_{k+1},x'_{k+1}) \\ = & d_{L\times X}((m_k,x_k),(m_k,x'_k))+d_{L\times X}((m_{k+1},x_{k+1}),(m_{k+1},x'_{k+1})). \end{array}$$

Thus, we can replace this section of the path:

$$(m_{k-1},x_{k-1}),(m_{k-1},x'_{k-1})\sim(m_k,x_k),(m_k,x'_k)\sim(m_{k+1},x_{k+1}),(m_{k+1},x'_{k+1})$$

with the path just below, which has strictly fewer entries:

$$(m_{k-1},x_{k-1}),(m_{k-1},x'_{k-1})\sim(m_{k+1},\widehat{x}_k),(m_{k+1},x'_{k+1}).$$

We are using that $(m_{k-1},x'_{k-1})\sim(m_k,x_k)\sim(m_{k+1},\widehat{x}_k)$, and that $\sim$ is transitive. Hence, we get a path with fewer entries and a smaller or equal score. Furthermore, since we assumed that there was only one bad configuration, we only need to make sure that $\widehat{x}_k$ and x'_{k+1} are not on the same side, but we know this holds since $\widehat{x}_k$ is on the same side as x_{k+1} and is not a corner, so it cannot be on the same side as x'_{k+1}.

Then we proceed by induction on the number of bad configurations in the alternating path. Use the process described to "remove" the first (leftmost in the indexing) bad configuration, then apply the induction hypothesis. ■

On a related note, if we have two entries in our path which are *strictly on the same side of* $S_X[M_0]$ (that is, neither are corners) in the same copy of X, then we can replace one of those entries with a corner such that we do not have two entries which are strictly on the same side of M_0.

4.1.9. Lemma. *For (m,x) and (n,y) in $L\times X$, either every path from (m,x) to (n,y) has score 2, or for any path from (m,x) to (n,y) in $L\times X$, there exists a path (4.6) with shorter or equal score, and at most as many entries as the original,*

$$(m,x)=(m_0,x_0),(m_0,x'_0)\sim(m_1,x_1)\ldots(m_{p-1},x'_{p-1})\sim(m_p,x_p),(m_p,x'_p)=(n,y) \quad (4.6)$$

such that

- *The new path (4.6) is an alternating path,*
- *For $0\leq i\leq p$, if x_i and x'_i are on the same side of $S_X[M_0]$, then they are both corners,*
- *For j and k with $0\leq j<k\leq p$, if $m_j=m_k$, and x'_j and x_k are on the same side of $S_X[M_0]$, then at least one of these points x'_j or x_k is a corner.*

Proof. By Lemma 4.1.7 and Lemma 4.1.8 we may start with an alternating path

$$(m,x)=(m_0,x_0),(m_0,x_0')\sim(m_1,x_1)\dots(m_{p-1},x_{p-1}')\sim(m_p,x_p),(m_p,x_p')=(n,y) \quad (4.7)$$

satisfying our requirements such that if x_i and x_i' are on the same side of $S_X[M_0]$, then at least one of them is a corner for $0\leq i\leq p$.

From here on, we will assume we have such a path.

Say a pair of indices j and k with $0\leq j<k\leq p$ is a *bad configuration* (in this proof) if

- $m_j=m_k$,
- x_j' and x_k are on the same side of $S_X[M_0]$,
- Neither of x_j' and x_k are corners.

Note that if our alternating path in (4.7) has a bad configuration, it will fail to satisfy the third condition in our lemma.

We will prove by induction on the number of bad configurations that we may adapt our path (4.7) such that it will satisfy all three requirements, and so that it has at most as many entries as (4.7) and a score at most that of the original.

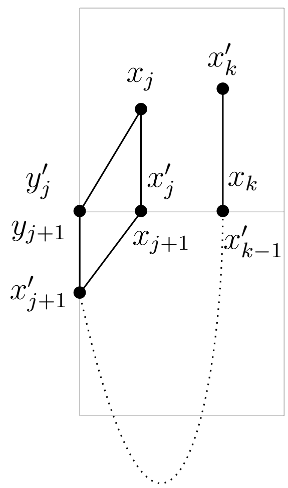
Suppose that we have exactly one bad configuration, so there are j and k with $0\leq j<k\leq p$ such that $m_j=m_k$, and x_j' and x_k are both not corners and are on the same side of $S_X[M_0]$.

Suppose that j and k are only one index apart, that is, $k=j+1$. Then our path looks like $(m_j,x_j),(m_j,x_j')\sim(m_k,x_k),(m_k,x_k')$. But then since $m_j=m_k$, we may eliminate $(m_j,x_j')\sim(m_k,x_k)$, and still have an alternating path with less or equal score, and which no longer has a bad configuration.

Otherwise, by Lemma 4.1.2(4), we must have $m_{j+1}=m_{k-1}$, and since x_j' and x_k are not corners, x_{j+1} and x_{k-1}' are on the same side.

By assumption, there are no other bad configurations, which means that if x_{j+1}' and x_{k-1} are both not corners, then they cannot be on the same side of $S_X[M_0]$. In other words, they cannot both be non-corners and on the opposite side of x_{j+1} and x_{k-1}'. So at least one of them must be on an adjacent side.

To better understand the situation, assume without loss of generality that x_{j+1} and x_{k-1}' are on the top of $S_X[M_0]$, and that x_{j+1}' is on the left side. The other cases are similar. We have the following picture.



In this picture, x_j and x'_k are actually somewhere on $S_X[M_0]$, but it is not important where. We can replace x'_j with y'_j and x_{j+1} with y_{j+1}, the corner between x_{j+1} and x'_{j+1}, since we assumed they are on adjacent sides. Note that $(m_j, y'_j) \sim (m_{j+1}, y_{j+1})$.

Currently, the portion of the path from (m_j, x_j) to (m_{j+1}, x'_{j+1}) contributes

$$d_{L\times X}((m_j, x_j), (m_j, x'_j)) + d_{L\times X}((m_{j+1}, x_{j+1}), (m_{j+1}, x'_{j+1}))$$

to the score. So using the definition of the metric on $L \times X$, we get

$$\begin{aligned}
& d_{L\times X}((m_j, x_j), (m_j, x'_j)) + d_{L\times X}((m_{j+1}, x_{j+1}), (m_{j+1}, x'_{j+1})) \\
= \;& \tfrac{1}{3} d_X(x_j, x'_j) + \tfrac{1}{3} d_X(x_{j+1}, x'_{j+1}) \\
\geq \;& \tfrac{1}{3} d_X(x_j, x'_j) + \tfrac{1}{3} d_X(x_{j+1}, y_{j+1}) + \tfrac{1}{3} d_X(y_{j+1}, x'_{j+1}) && (1) \\
= \;& \tfrac{1}{3} d_X(x_j, x'_j) + \tfrac{1}{3} d_X(x'_j, y'_j) + \tfrac{1}{3} d_X(y_{j+1}, x'_{j+1}) && (2) \\
\geq \;& \tfrac{1}{3} d_X(x_j, y'_j) + \tfrac{1}{3} d_X(y_{j+1}, x'_{j+1}) && (3) \\
= \;& d_{L\times X}((m_j, x_j), (m_j, y'_j)) + d_{L\times X}((m_{j+1}, y_{j+1}), (m_{j+1}, x'_{j+1}))
\end{aligned}$$

where (1) is by (SQ$_2$) and Corollary 3.0.6, (2) is by Lemma 4.1.2(3), and (3) is by the triangle inequality in X. Thus, we may replace $(m_j, x'_j) \sim (m_{j+1}, x_{j+1})$ in our path with $(m_j, y'_j) \sim (m_{j+1}, y_{j+1})$ to obtain a path with the same number of entries, a score which is shorter or equal to that of the original, and such that j and k are no longer a bad configuration. Note that our new path is still an alternating path, and since the replacement entry is a corner, performing this process cannot create a path which violates the second or third points in the statement of the lemma.

Thus, we may proceed by induction on the number of bad configurations. Perform the process described above to remove one instance of a bad configuration, then apply the induction hypothesis. ■

We have one more technical lemma involving elements of an alternating path sharing sides of $S_X[M_0]$.

4.1.10. LEMMA. *Consider an alternating path from (m, x) to (n, y) in $L \times X$,*

$$(m, x) = (m_0, x_0), (m_0, x'_0) \sim (m_1, x_1), \ldots, (m_{p-1}, x_{p-1}) \sim (m_p, x_p), (m_p, x'_p) = (n, y). \tag{4.8}$$

Suppose that there are i and j with $0 \leq i < j \leq p$ such that $m_i = m_j$. Further suppose that one of the following holds:

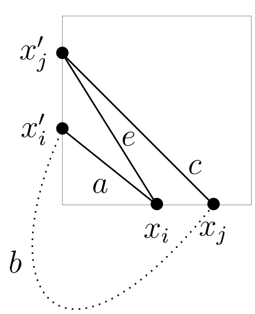
- *x_i and x_j are on the same side of $S_X[M_0]$ as each other, and x'_i and x'_j are on the same side of $S_X[M_0]$ as each other,*
- *x_i and x'_j are on the same side of $S_X[M_0]$ as each other, and x'_i and x_j are on the same side of $S_X[M_0]$ as each other.*

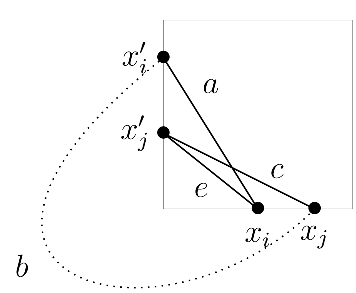
Then we can delete the entries strictly between (m_i, x_i) and (m_j, x'_j) in our alternating path to obtain another alternating path with a smaller or equal score to that of the original.

PROOF. First, suppose x_i and x_j are on the same side of $S_X[M_0]$. Without loss of generality, suppose they are on the bottom and x_i is to the left of x_j (in case it is the opposite, we may just reverse the order of the path).

We will consider the cases when x'_i and x'_j are on the left side of M_0 (the right side is similar) and when they are on the top of M_0.

In the first case, $x'_i = S_X((0, s))$ and $x'_j = S_X((0, s'))$ for some $s, s' \in [0, 1]$. Either $s < s'$ or $s' < s$. First, suppose that $s < s'$. We consider the diagram on the left below.

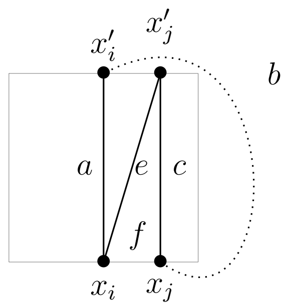




The entries in the path from (m_i, x_i) to (m_j, x'_j) contribute $a + b + c$ to the score. Since the metric on M_0 is the path metric, and S_X acts isometrically on adjacent sides by (SQ$_2$) and Corollary 3.0.6, $e < c$, so $e < a + b + c$, meaning we can delete the entries between $(m_i, x_i), (m_j, x'_j)$ to obtain an alternating path (since $m_i = m_j$) with a smaller or equal score and with strictly fewer entries.

The case when $s' < s$ is similar. The picture is on the right above. The entries from (m_i, x_i) to (m_j, x'_j) contribute $a + b + c$ to the score of the path. Again, since the metric on M_0 is the path metric and S_X acts isometrically on points on adjacent sides of M_0, $e < c$. So $e < a + b + c$. Thus, we can delete the entries between $(m_i, x_i), (m_j, x'_j)$ to get an alternating path with strictly fewer entries whose score is less than or equal to that of the original.

Finally, we consider the case when x'_i and x'_j are on the top of M_0. It does not matter which is leftmost.



The entries from (m_i, x_i) to (m_j, x'_j) contribute $a+b+c$ to the score, and note that $a \geq 1$ and $c \geq 1$ by (SQ$_2$). By the triangle inequality, $e \leq f + c$. Since x_i and x_j are strictly on the bottom of the image of M_0 under S_X, by (SQ$_1$), $f < 1$. So $e < f + c < 1 + c \leq a + c \leq a + b + c$. Thus, we may delete the entries between (m_i, x_i) to (m_j, x'_j) to get a path with strictly fewer entries and a smaller or equal score than that of the original.

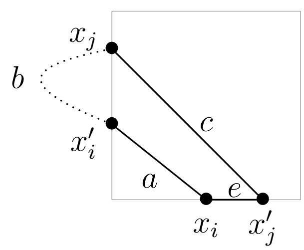
So this completes the first case.

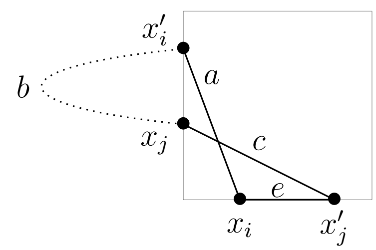
Now suppose x_i and x'_j are on the same side. Note that we cannot simply apply Lemma 4.1.9 because x_i and x'_j could be corners.

As before, we will assume without loss of generality that x_i and x'_j are on the bottom and consider the cases when x'_i and x_j are on the left (the right is similar) and when they are on the top.

Again, $x'_i = S_X((0, s))$ and $x_j = S_X((0, s'))$ for some $s, s' \in [0, 1]$. Either $s < s'$ or

$s' < s$. First, suppose that $s < s'$. We consider the diagram on the left below.

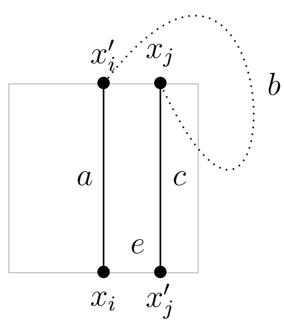




The entries in the path from (m_i, x_i) to (m_j, x_j') contribute $a + b + c$ to the score. By ($\textsc{sq}_2$), $e \leq c$, so $e \leq a+b+c$, meaning we can delete the entries between $(m_i, x_i), (m_j, x_j')$ to obtain an alternating path (since $m_i = m_j$) with a smaller or equal score and with strictly fewer entries.

The case when $s' < s$ is similar. The diagram is on the right above. The entries from (m_i, x_i) to (m_j, x_j') contribute $a + b + c$ to the score of the path. Again, $e \leq c$ by ($\textsc{sq}_2$). Thus, we can delete the entries between $(m_i, x_i), (m_j, x_j')$ to get an alternating path with strictly fewer entries whose score is less than or equal to that of the original.

Finally, we consider the case when x_i' and x_j are on the top of M_0. It does not matter which is leftmost. Here is a picture:



The entries from (m_i, x_i) to (m_j, x_j') contribute $a+b+c$ to the score. Note that $a \geq 1$ and $c \geq 1$ by ($\textsc{sq}_2$). Since x_i and x_j' are on the same side, by ($\textsc{sq}_1$), $e \leq 1$. Thus, $e < a+b+c$, so we may delete the entries between (m_i, x_i) to (m_j, x_j') to get a path with strictly fewer entries and a smaller or equal score than that of the original. ∎

4.1.11. Lemma. *There exists a positive integer K such that for every alternating path*

$$(m, x) = (m_0, x_0), (m_0, x') \sim (m_1, x_1), (m_1, x_1') \sim \ldots (m_p, x_p), (m_p, x_p') = (n, y)$$

with $p > K$, there exists an alternating path from (m, x) to (n, y) with smaller or equal score with strictly fewer entries.

Proof. Fix $K = 36|L| + 1$, which is finite since L is finite. We will see the justification for this choice of K in the proof.

Suppose that we have an alternating path of the form above with $p > K$.

First, we may assume that it does not have any repetitions, since if there are two entries which are equal (not just equivalent, but actually equal), then we can delete every entry between those and one of the two repeated entries to obtain a path with strictly fewer entries and a score which is less than or equal to that of the original.

By Lemma 4.1.2(1), all of the entries in an alternating path except for the first and last must be of the form $(m, S_X((r, s)))$ for some $(r, s) \in M_0$. So by ignoring the first and last entries (if necessary), we have a path

$$(m_0, x'_0) \sim (m_1, x_1), (m_1, x'_1) \sim \ldots (m_p, x_p),$$

where each x_i and x'_i are in the image of $S_X[M_0]$.

Furthermore, by Lemma 4.1.8, we may assume that if x_i and x'_i are not both corners, then they are not on the same side of $S_X[M_0]$. By Lemma 4.1.9, we may further assume that if $0 \leq i < j \leq p$ and $m_i = m_j$, then x'_i is not on the same side as x_j.

Since $p > 36|L| + 1$, the list $m_1, \ldots, m_{p-1}$ has at least $36|L| + 1$ entries. So by the pigeonhole principle, there is $m \in L$ such that for at least 37 many k (with $1 \leq k \leq p-1$), $m_k = m$.

Then consider the set $\{x_k \mid m_k = m, 1 \leq k \leq p-1\}$. Since we assumed that there are no identical entries in our path, every element of this set is distinct, so this set has at least 37 elements. We use the pigeonhole principle again. Since $S_X[M_0]$ has four sides, there is a side of $S_X[M_0]$ containing at least 10 of the x_k's from this set. Fix this side. Again, since the x_k's under discussion are all distinct, at most two of these are corners, so this side contains at least 8 x_k's which are not corners. Let $x_{k_1}, \ldots, x_{k_8}$ be 8 of them. Consider the corresponding points $x'_{k_1}, \ldots, x'_{k_8}$. Since they are all distinct, there are at least four which are not corners. By our assumptions about our path, they cannot be on the same side as $x_{k_1}, \ldots, x_{k_8}$. Thus, we have 3 sides where these four entries can be, so there must be one of the four sides such that some x'_{k_i} and x'_{k_j} are on the same side (and are both not corners). For ease of notation, we will just refer to these indices as i and j. From here on out, we will assume without loss of generality that x_i and x_j are on the bottom side of the image of M_0 under S_X, that is, $\{(r, 0) : 0 < r < 1\}$.

Arrange x_i and x_j so that $i < j$. Then, by Lemma 4.1.10, we may delete some portion of our path to obtain an alternating path with strictly fewer entries whose score is at most that of the original path. ∎

We are ready to prove that the quotient metric can be calculated as the score of some particular finite path, not just an infimum over a set of paths. The assumption in Theorem 4.1.12 below is very mild; the idea is that the distance between points in the same copy of X cannot be made shorter by going outside of X on some other path. It plays a key role in our connection of the Sierpinski carpet with iterations of a functor on square sets.[3] As an example of why we need to make this assumption (in addition to our the requirement that E be quotient suitable), consider the following example: Let $L = \{a, b, c\}$ and let $X = M_0$. Then define E on $L \times M_0$ by

$$\begin{array}{ccc} (a,(r,1)) & E & (b,(r,0)) \\ (b,(0,r)) & E & (c,(1,r)) \\ (c,(0,r)) & E & (a,(1-r,0)) \end{array}$$

for $r \in [0, 1]$. We can visualize this as in Figure 2.

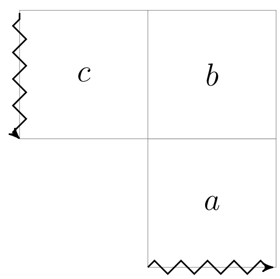


Figure 2: Justification for the assumption in Theorem 4.1.12

Then $(a, (1, 0)), (a, ((0, 1))$ would be an alternating path with score 2. In addition

$$(a,(1,0)) \sim (c,(0,0)),(c,(1,0)) \sim (a,(0,1))$$

is also an alternating path with score 1. The big idea is that with squares we do not need to consider situations where we allow gluing which requires "twisting" copies of M_0. In fact, gluing with a twist would create a situation where we could find shorter distances by going through different copies of the square. We resolve this via the hypotheses in the following theorem. (Incidentally, we have not defined the sets M and N yet to which we shall apply all of this general theory, but when we do define them, we will see that the hypotheses of the theorem just below are indeed satisfied by both M and N.) The reason that we do not address this at the level of defining quotient suitability is that there may exist other examples (such as triangles in the Sierpinski Gasket (see, e.g. [6])) where we would need to allow for this.

4.1.12. Theorem. *Suppose that L is a finite index set and E is a quotient suitable equivalence relation on $L \times M_0$. Further, suppose that for any $x, y \in X$ and $m \in L$, for*

[3] To understand the assumption, it might help to look ahead to Sections 4.2 and 4.5 for the definitions of $L \otimes X$ for the sets M and N that we want most to take for L and for the set E underlying an equivalence relation $\sim$.

any path $(m,x) = (m_0, x_0), \ldots, (m_p, x_p) = (m,y)$,

$$\tfrac{1}{3} d_X(x,y) \leq \sum_{k=0}^{p-1} d_{L\times X}((m_k, x_k), (m_{k+1}, x_{k+1})).$$

That is, given two points (m,x) *and* (m,y) *in the same scaled copy* $\{m\} \times X$ *of* X*, their distance in* $L \otimes X$ *is at least* $\frac{1}{3} d_X(x,y)$*. (Equivalently, our assumption is that there is no path in* $L \times X$ *from* (m,x) *to* (m,y) *with a score smaller than* $\frac{1}{3} d_X(x,y)$*.) Then for all* $m \otimes x$ *and* $n \otimes y$*, either*

1. $d_{L\otimes X}(m \otimes x, n \otimes y) = 2$*, or*
2. *For some alternating path from* (m,x) *to* (n,y)*,*

$$d_{L\otimes X}(m \otimes x, n \otimes y) = \sum_{k=0}^{p} d_{L\times X}((m_k, x_k), (m_k, x'_k)),$$

Proof. Let $m \otimes x$ and $n \otimes y$ in $L \otimes X$ be given. If every path between them has score 2, then $d_{L\otimes X}(m \otimes x, n \otimes y) = 2$. Otherwise, consider an alternating path from (m,x) to (n,y). Lemma 4.1.7 shows that the distance from $m \otimes x$ to $n \otimes y$ is the infimum of the scores of alternating paths. The point is that any path which is not alternating gives rise to a alternating path with score that is at most the score of the original.

By Lemma 4.1.11, since E is quotient suitable, there is a finite K such that we only need to consider alternating paths $(m,x) = (m_0, x_0) \ldots (m_p, x'_p) = (n,y)$ with $p \leq K$. Since there are only finitely many tuples from L of length $\leq K+1$, we need only show that for each $p \leq K$ and each fixed tuple $m = m_0, m_1, \ldots, m_p = n$, the infimum of the scores of paths involving this tuple (allowing the x's to vary) is attained.

For $0 \leq i \leq p-1$, let

$$C_i = (\{m_i\} \times S_X[M_0]) \times (\{m_{i+1}\} \times S_X[M_0]).$$

Each C_i is a compact set: M_0 is compact, and S_X is continuous (since it is a short map by Corollary 3.0.6), so the image $S_X[M_0]$ is compact. And thus, so is each set $\{m_i\} \times S_X[M_0]$. So the following set C^* is also compact:

$$C^* = \{(m,x)\} \times C_0 \times C_1 \times \cdots \times C_{p-1} \times \{(n,y)\}$$

Each element of C^* is a tuple, and each gives us a path as in (4.3). In more detail, we can write an element of C^* as

$$((m_0, x_0), ((m_0, x'_0), (m_1, x_1)), \ldots, ((m_{p-1}, x'_{p-1}), (m_p, x_p)), (m_p, x'_p)) \tag{4.9}$$

where $(m_0, x_0) = (m,x)$ and $(m_p, x'_p) = (n,y)$ Again, the m's are the ones which we fixed above, and the x's belong to $S_X[M_0]$.

The path corresponding to this is the one with the same notation as in (4.3). Moreover, every path as in (4.3) comes from an element of our set C^*. Consider the function which takes an element of C^* to the score of its corresponding path.

This function is continuous, so we have a continuous function $C^* \longrightarrow \mathbb{R}$. Since C^* is compact, this function indeed attains its minimum value at some point, just as we want. ∎

4.1.13. DEFINITION. Let $m \otimes x$ and $n \otimes y$ be points in $L \otimes X$. A *witness path* from $m \otimes x$ to $n \otimes y$ is an alternating sequence of points

$$(m, x) = (m_0, x_0), (m_0, x_0') \sim (m_1, x_1), (m_1, x_1') \sim \ldots, (m_p, x_p), (m_p, x_p')$$

such that

$$d_{L \otimes X}(m \otimes x, n \otimes y) = \sum_{k=0}^{p} d_{M \times X}((m_k, x_k), (m_k, x_k')).$$

Our previous work shows us that the distances in $L \otimes X$ which are below the maximum distance 2 are witnessed by a single finite path, not just an infimum of an infinite set of paths. This gives us the following:

4.1.14. COROLLARY. *Under the same assumption as in Theorem 4.1.12,*

1. *$L \otimes X$ is a metric space.*
2. *For each $m \in L$, the function $x \mapsto m \otimes x : X \to L \otimes X$ is an injection.*
   Moreover, for $x, y \in X$, $d_{L \otimes X}(m \otimes x, m \otimes y) = \frac{1}{3} d_X(x, y)$.

PROOF. For the first assertion, assume that $d_{L \otimes X}(m \otimes x, n \otimes y) = 0$. Then there exists a witness path whose score is equal to 0. The adjacent entries $(m_k, x_k), (m_{k+1}, x_{k+1})$ not related by $\sim$ must then contribute 0 to the score. This only happens when $m_k = m_{k+1}$ and $x_k = x_{k+1}$. In this case, the entire path is a sequence completely related by $\sim$. So we have $(m, x) \sim (n, y)$. Thus, $m \otimes x = n \otimes y$.

The second point is immediate from the assumption in Theorem 4.1.12. ∎

As mentioned at the beginning of this section, we are not aiming to show that $L \otimes -$ is a functor (indeed, for $X \in \mathsf{SquaMS}$, $L \otimes X$ may not be a square set). However, we will show here that for a morphism $f : X \to Y$ in SquaMS, that we may define a function $L \otimes f : L \otimes X \to L \otimes Y$ by $f(m \otimes x) = m \otimes f(x)$ and that this is well-defined. We want to reiterate though that the function $L \otimes f$ generally will not have the properties required to be a morphism (e.g., it may not be a short map, and if there is a Square Set structure, might not preserve it).

4.1.15. LEMMA. *For $X, Y \in \mathsf{SquaMS}$ and $f : X \to Y$ a morphism in SquaMS, the function $L \otimes f : L \otimes X \to L \otimes Y$ given by $L \otimes f(m \otimes x) = m \otimes f(x)$ is well defined.*

PROOF. Let $X, Y \in \mathsf{SquaMS}$ and $f : X \to Y$ be a morphism in SquaMS. Let $m \otimes x = n \otimes y$ in $L \otimes X$. Since $m \otimes x = n \otimes y$, either $(m, x) = (n, y)$, in which case $f(m \otimes x) = f(n \otimes y)$, or $(m, x) \sim (n, y)$. By Lemma 4.1.2 (1), $x, y \in S_X[M_0]$, so $x = S_X((r, s))$ and $y = S_X((t, u))$ for some $(r, s), (t, u) \in M_0$. Since E does not depend on X, since $(m, S_X((r, s))) \sim (n, S_X((t, u)))$, we must have $(m, S_Y((r, s))) \sim (n, S_Y((t, u)))$ in $L \otimes Y$ as well.

Since f is a morphism in SquaMS, it preserves S_X, so $f(x) = S_Y((r, s))$ and $f(y) = S_Y((t, u))$. Thus, $L \otimes f(m \otimes x) = m \otimes f(x) = m \otimes f(S_X((r, s))) = m \otimes S_Y((r, s)) \sim n \otimes S_Y((t, u)) = n \otimes f(S_X((t, u))) = n \otimes f(y) = L \otimes f(n \otimes y)$.

■

4.2. DEFINING $M \otimes -$ FOR SQUARE METRIC SPACES. The last section dealt with properties of the operation $X \mapsto L \otimes X$ which were presented in an abstract fashion. Now it is time to be more concrete. We take L to be a particular set M in this section, and also define a relation E on $M \times M_0$ and show that it is quotient suitable. Then we will verify the other hypotheses used in the results of the last section for this M and E. In a subsequent section, we do the same thing for a different set N and a different relation E.

Let $M = \{0, 1, 2\}^2 \setminus \{(1, 1)\}$. Each $m = (i, j) \in M$ will indicate a (column, row) entry in the 3×3 grid, except that $(1, 1)$ is missing.

$(0,2)$	$(1,2)$	$(2,2)$
$(0,1)$		$(2,1)$
$(0,0)$	$(1,0)$	$(2,0)$

The idea is that $m \in M$ will tell us where a scaled copy of an object X in SquaMS will go. Our goal is to show that $X \mapsto M \otimes X$ is a functor on SquaMS. We will use the results of the previous section to establish that $M \otimes X$ is a metric space.

We will obtain $M \otimes X$ as a quotient space of $M \times X$. Let E be the equivalence relation generated by the following relation on $M \times M_0$ for $r \in [0, 1]$:

$$\begin{array}{lcl@{\qquad}lcl}
((0,0),(r,1)) & E & ((0,1),(r,0)) & ((2,2),(r,0)) & E & ((2,1),(r,1)) \\
((0,1),(r,1)) & E & ((0,2),(r,0)) & ((2,1),(r,0)) & E & ((2,0),(r,1)) \\
((0,2),(1,r)) & E & ((1,2),(0,r)) & ((2,0),(0,r)) & E & ((1,0),(1,r)) \\
((1,2),(1,r)) & E & ((2,2),(0,r)) & ((1,0),(0,r)) & E & ((0,0),(1,r))
\end{array} \tag{4.10}$$

For any X in SquaMS we then define $\approx$ using (4.1). Finally, we take the equivalence relation generated by $\approx$ and call it $\sim$, just as in our more general work in the previous section.

4.2.1. LEMMA. *E on $M \times M_0$ is quotient suitable (Definition 4.1.1).*

PROOF. If we define $\kappa \subset (M \times D)^2$ by $\kappa((m,Y)) = (n,Z)$ if and only if there are $y \in Y$ and $z \in Z$ such that $(m,y)E(n,z)$ and m appears before n in the lexicographic order on M, we see quickly that κ satisfies the conditions in the definition of quotient suitable, and that the relation E described in the definition coincides precisely with our relation E on $M \times M_0$. ■

Next, we will see that $M \otimes X$ is a square set (and ultimately, a square metric space). Recall that square sets come with a function $S_X : M_0 \longrightarrow X$.

Define $S_{M\otimes X} : M_0 \longrightarrow M \otimes X$ by

$$S_{M\otimes X}((0,r)) = \left\{ \begin{array}{ll} (0,0) \otimes S_X((0,3r)) & 0 \leq r \leq \frac{1}{3} \\ (0,1) \otimes S_X((0,3r-1)) & \frac{1}{3} \leq r \leq \frac{2}{3} \\ (0,2) \otimes S_X((0,3r-2)) & \frac{2}{3} \leq r \leq 1 \end{array} \right\} \tag{4.11}$$

$$S_{M\otimes X}((r,1)) = \left\{ \begin{array}{ll} (0,2) \otimes S_X((3r,1)) & 0 \leq r \leq \frac{1}{3} \\ (1,2) \otimes S_X((3r-1,1)) & \frac{1}{3} \leq r \leq \frac{2}{3} \\ (2,2) \otimes S_X((3r-2,1)) & \frac{2}{3} \leq r \leq 1 \end{array} \right\} \tag{4.12}$$

$$S_{M\otimes X}((1,r)) = \left\{ \begin{array}{ll} (2,0) \otimes S_X((1,3r)) & 0 \leq r \leq \frac{1}{3} \\ (2,1) \otimes S_X((1,3r-1)) & \frac{1}{3} \leq r \leq \frac{2}{3} \\ (2,2) \otimes S_X((1,3r-2)) & \frac{2}{3} \leq r \leq 1 \end{array} \right\} \tag{4.13}$$

$$S_{M\otimes X}((r,0)) = \left\{ \begin{array}{ll} (0,0) \otimes S_X((3r,0)) & 0 \leq r \leq \frac{1}{3} \\ (1,0) \otimes S_X((3r-1,0)) & \frac{1}{3} \leq r \leq \frac{2}{3} \\ (2,0) \otimes S_X((3r-2,0)) & \frac{2}{3} \leq r \leq 1 \end{array} \right\} \tag{4.14}$$

The idea is that each new side consists of 3 copies of the corresponding side from X. So far, $(M \otimes X, S_{M\otimes X})$ is a square *set* (we know $S_{M\otimes X}$ is well-defined because of the identified segments in $\sim$ on $M \times X$). As before in (4.2), the metric $d_{M\times X}$ is

$$d_{M\times X}((m,x),(n,y)) = \left\{ \begin{array}{ll} \frac{1}{3}d_X(x,y) & \text{if } m = n \\ 2 & \text{otherwise} \end{array} \right\} \tag{4.15}$$

So the distance is scaled by $\frac{1}{3}$ inside of each copy of X, and otherwise, it is 2 (the maximum distance). Then we define the quotient metric on $M \otimes X$ as we did in the previous section, and so far, we know that it is a pseudo metric. To apply Theorem 4.1.12 and Corollary 4.1.14 to show that $M \otimes X$ is a metric space, we need more details about the quotient metric. In particular, we need to see that $d_{M\otimes X}(m \otimes x, m \otimes y) = \frac{1}{3}d_X(x,y)$ for $x, y \in X$ and $m \in M$. To achieve this, we will describe the paths in $M \otimes X$ in more detail.

4.3. CLASSIFICATION OF REGULAR WITNESS PATHS. Let X be a square metric space, and let $x, y \in M \otimes X$. Recall the definition of a *witness path* in Definition 4.1.13. Such a path is an *alternating path* from x to y; it does not contain superfluous visits to any entry, and its score is minimal over all paths from x to y.

4.3.1. DEFINITION. We say that such a witness path is *regular* if, in addition, its length (as a sequence of points) is minimal over all witness paths from x to y.

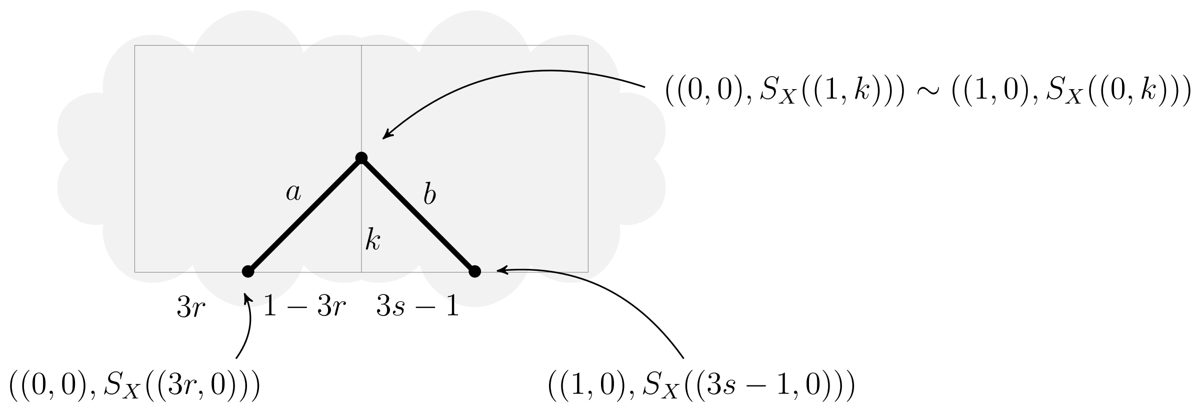
It will be helpful to have a classification of regular witness paths. But before that we will consider a few illustrative examples.

4.3.2. EXAMPLE. Let $0 \le r \le \frac{1}{3}$ and $\frac{1}{3} < s \le \frac{2}{3}$, and let $k \in [0,1]$. Consider the following path in $M \times X$:

$$\begin{array}{ll} & ((0,0), S_X((3r,0))), ((0,0), S_X((1,k))) \\ \sim & ((1,0), S_X((0,k))), ((1,0), S_X((3s-1,0))) \end{array} \tag{4.16}$$

We check that as k ranges over $[0,1]$, the score of this path is minimized when $k = 0$, and the minimum score of such a path is $|r - s|$.

Here is the reasoning. Let us draw a picture and introduce some notation. In the figure below, a, b, and k represent distances in $M \times X$ along the evident line segments.



The path under discussion is shown. It has score $a+b$. Now the left endpoint of the path and the midpoint have the same first component, $(0,0)$. By (4.2), the distance between them is

$$\begin{array}{rcl} a & = & d_{M\times X}(((0,0), S_X((3r,0))), ((0,0), S_X((1,k)))) \\ & = & \frac{1}{3} d_X(S_X((3r,0)), S_X((1,k))) \\ & \ge & \frac{1}{3} - r + \frac{1}{3}k. \end{array}$$

At the end we used the fact that X is a square metric space: by (SQ_2), the distance above is at least the taxicab distance in the unit square between the corresponding points, and this is $1 - 3r + k$.

Similar work shows us that $b \ge s - \frac{1}{3} + \frac{1}{3}k$. Thus, the score of our path is $\ge s - r + \frac{2}{3}k$. As a function of k, this is obviously minimized when $k = 0$. When $k = 0$, the score is

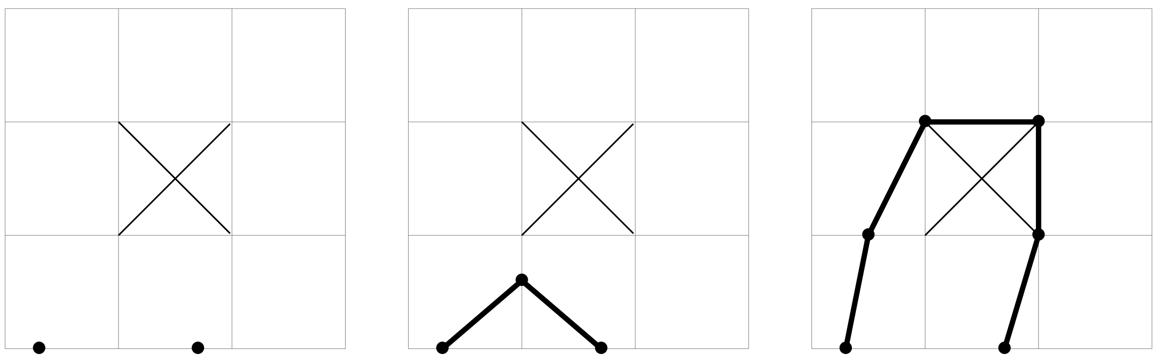
$$\tfrac{1}{3}|1-3r| + \tfrac{1}{3}|3s-1| = \tfrac{1}{3}(|1-3r| + |3s-1|) = \tfrac{1}{3}|1-3r+3s-1| = |r-s|.$$

4.3.3. Example. Let $0 \le r \le \frac{1}{3}$ and $\frac{1}{3} < s \le \frac{2}{3}$, and note that

$$\begin{array}{rcl} S_{M\otimes X}((r,0)) & = & (0,0)\otimes S_X((3r,0)) \\ S_{M\otimes X}((s,0)) & = & (1,0)\otimes S_X((3s-1,0)). \end{array}$$

Then we claim that

$$d_{M\otimes X}(S_{M\otimes X}((r,0)), S_{M\otimes X}((s,0))) = |r-s|.$$



Moreover, this same formula holds for points in the top (or bottom, or left, or right) edges of suitably neighboring squares in $M \otimes X$.

Here is the reason. The points involved are shown as on the left above. To evaluate the distance in $M \otimes X$, we return to $M \times X$ and consider paths between the points. Since we want to minimize the score, by Lemma 4.1.7, we can consider alternating paths. The most obvious such path would be as in the middle, where we add in a third point as shown. Then the work we did in Example 4.3.2 shows that the score of such a path is at least $|r-s|$, and moreover that we can get a path with exactly this score by taking the third point to be the corner. But in this result, we need to consider other alternating paths besides this "obvious one." To have a path of minimal score, we should not repeat points in the edges. Indeed, we will see in our proof of Lemma 4.3.4 below, we cannot even repeat the elements of M.

One representative path would be the one shown on the right. In this path, the elements of M (starting with the point in the bottom left) are

$$(0,0),(0,1),(0,2),(1,2),(2,2),(2,1),(0,1)$$

But the score of this path is greater than the score of the path in the middle: each time one crosses a square from side to side, the score adds $\frac{1}{3}$ by (SQ_2). So the score is at least $\frac{5}{3}$. And $|r-s| \le \frac{2}{3}$.

So in fact, the path (4.16) is a regular witness path, since it witnesses the distance, and we would not be able to obtain a shorter path since they are in different copies (so alternating path between them must have at least four entries).

Now we will prove the conditions required to apply Theorem 4.1.12.

4.3.4. LEMMA. *For any $x, y \in X$ and $m \in M$, for any path*

$$(m, x) = (m_0, x_0), \ldots, (m_p, x_p) = (m, y), \tag{4.17}$$

we have

$$\tfrac{1}{3} d_X(x, y) \leq \sum_{k=0}^{p-1} d_{M \times X}((m_k, x_k), (m_{k+1}, x_{k+1})). \tag{4.18}$$

PROOF. Fix m throughout this proof. By Lemma 4.1.7, since the path $(m, x), (m, y)$ has score $\leq \frac{2}{3} < 2$, we may assume that our path is an alternating path, and we write it as

$$(m, x) = (m_0, x_0), (m_0, x_0') \sim \ldots (m_{p-1}, x_{p-1}') \sim (m_p, x_p), (m_p, x_p') = (m, y).$$

By Lemma 4.1.8, 4.1.9, and 4.1.10, we may also assume the following for i, j such that $0 \leq i < j \leq p$:

- If x_i and x_i' are on the same side of $S_X[M_0]$, then they are both corners,
- If x_i' and x_j are on the same side of $S_X[M_0]$, then at least one of them is a corner,
- If x_i and x_j are on the same side of $S_X[M_0]$, then x_i' and x_j' are *not* on the same side of $S_X[M_0]$,
- If x_i and x_j' are on the same side of $S_X[M_0]$, then x_i' and x_j are *not* on the same side of $S_X[M_0]$.

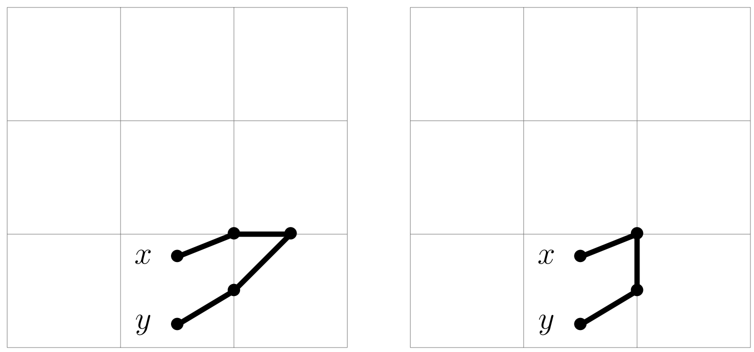
We show by strong induction on the natural number $k \geq 1$ that for every path as in (4.17) between points whose first coordinate is m, if $|\{i : m_i = m\}| = k$, then the estimate in (4.18) holds. So we fix $k \geq 1$, assume our result for numbers $< k$, and then show it for k. We argue by cases on k. If $k = 1$, we have $p = 0$, and the path from (m, x) to (m, y) is just $(m, x), (m, y)$. Its length is $\frac{1}{3} d_X(x, y)$, by (4.15). When $k \geq 3$, the path in (4.17) has $m_\ell = m$ for some $0 < \ell < p$. We cut this path into two subpaths, the part between (m_0, x_0) and (m_ℓ, x_ℓ), and the part from (m_ℓ, x_ℓ) to (m_p, x_p'). In both subpaths, the number of j such that $m_j = m$ is $< k$. So the induction hypothesis applies to the subpaths. By this and the triangle inequality, we show the desired inequality.

$$\tfrac{1}{3} d_X(x, y) \leq \tfrac{1}{3} d_X(x, x_\ell) + \tfrac{1}{3} d_X(x_\ell, y) \leq \sum_{i=0}^{p-1} d_{M \times X}((m_i, x_i), (m_{i+1}, x_{i+1}))$$

The remaining case is when $k = 2$. Thus, we may assume that the only pairs (m_i, x_i) in our path with $m_i = m$ are m_0 and m_p. By the conditions listed at the beginning of the proof, x_0' and x_p can only be on the same side of $S_X[M_0]$ if at least one of them is a corner. However, by examining such a path, we can use a triangle inequality argument with (SQ_1) to shorten the path.

As an example, we take $m = (1,0)$ and consider the following path (where x, y are arbitrary elements of X, not necessarily in $S_X[M_0]$):

$$\begin{array}{rcl} ((1,0),x),((1,0),S_X((1,1))) & \sim & ((2,1),S_X((0,0))), \\ ((2,1),S_X((\frac{1}{2},0))) & \sim & ((2,0),S_X((\frac{1}{2},1))), \\ ((2,0),S_X((0,\frac{1}{2}))) & \sim & ((1,0),S_X((1,\frac{1}{2}))),((1,0),y) \end{array}$$



By (SQ$_1$),

$$\begin{array}{rl} & d_{M\times X}(((2,1),S_X((0,0))),((2,1),S_X((\frac{1}{2},0)))) \\ = & d_{M\times X}(((2,0),S_X((0,1))),((2,0),S_X((\frac{1}{2},1)))) \end{array}$$

Next, by the triangle inequality in X and the definition of $d_{M\times X}$,

$$\begin{array}{rl} & d_{M\times X}(((2,0),S_X((0,1))),((2,0),S_X((0,\frac{1}{2})))) \\ \leq & d_{M\times X}(((2,0),S_X((0,1))),((2,0),S_X((\frac{1}{2},1)))) \\ + & d_{M\times X}(((2,0),S_X((\frac{1}{2},1))),((2,0),S_X((0,\frac{1}{2})))) \end{array}$$

So we have the path pictured on the right above and shown below:

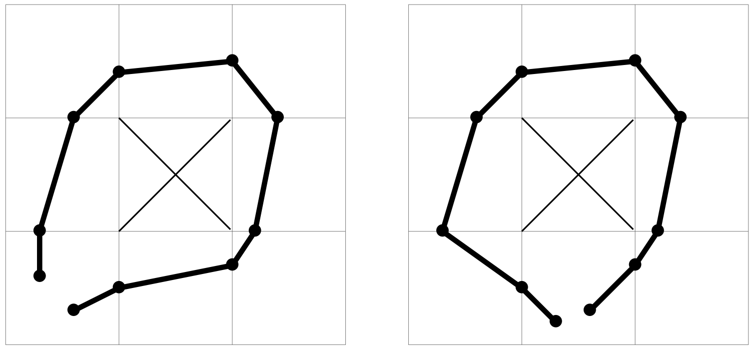
$$\begin{array}{rcl} ((1,0),x),((1,0),S_X((1,1))) & \sim & ((2,0),S_X((0,1))), \\ ((2,0),S_X((0,\frac{1}{2}))) & \sim & ((1,0),S_X((1,\frac{1}{2}))),((1,0),y) \end{array}$$

It has fewer entries and score at most that of the original. And then we use the same type of argument again. By (SQ$_1$),

$$\begin{array}{rl} & d_{M\times X}(((2,0),S_X((0,1))),((2,0),S_X((0,\frac{1}{2})))) \\ = & d_{M\times X}(((1,0),S_X((1,1))),((1,0),S_X((1,\frac{1}{2})))) \end{array}$$

So using the triangle inequality twice, we get that $d_{M\times X}(((1,0),x),((1,0),y))$ is less than or equal to the score of the path, as required.

We also need to consider paths which enter and exit the m-copy of X on (strictly) different sides. Such a path will have at least as many entries as one of the two following possibilities (up to rotation):



In both of these cases, due to (SQ_2), each of these paths will have score ≥ 1, since each time we have a segment which goes between opposite sides of a copy of X, we contribute at least $\frac{1}{3}$ to the score. So the score of a path of this form will be at least $1 \geq \frac{1}{3}d_X(x, y)$. ∎

We rephrase the result as follows:

4.3.5. Corollary. *For any $x, y \in X$ and $m \in M$, $d_{M \otimes X}(m \otimes x, m \otimes y) = \frac{1}{3}d_X(x, y)$.*

Proof. By Lemma 4.3.4 and the fact that $(m, x), (m, y)$ is a path. ∎

4.3.6. Corollary. *$M \otimes X$ is a metric space.*

Proof. This follows from Corollary 4.1.14. ∎

Now we turn our attention to understanding the quotient metric in more detail.

4.3.7. Theorem. *Let X be a square metric space. Let $x, y \in M \otimes X$. Then there exists a regular witness path between x and y, and every regular witness path between x and y looks like one of the paths shown in Figure 3.*

4.3.8. Remark. Theorem 4.3.7 is stated somewhat loosely, but we believe that a patient reader could make it completely precise, and also that it is more comprehensible to state it the way we do. Here is a bit more about what we mean. We are aiming at a classification of all of the regular witness paths between pairs of points in $M \otimes X$. The first case is where x and y are in the same copy of X; that is, there is some $m \in M$ such that $x, y \in m \otimes X := \{m \otimes x : x \in X\}$, which is what we proved in Corollary 4.3.5. In this case, our result is that every regular witness path stays inside $m \otimes X$. The second case is when x and y are in adjacent copies. In this case, our result is very similar to what was shown in Example 4.3.3. The next case is when x and y lie in diagonally connected squares, such as $(0, 1) \otimes X$ and $(1, 0) \otimes X$. In this case, the result is that the only regular witness paths are the ones that go through the shared corner, as shown.

Continuing, we have pairs of points in squares related by "a move of the chess knight". In this case, there are two possible "shapes" that a regular witness path could have, indicated by the two paths from x to y. Both go through the upper-left corner of the "hole", but they differ after that. For different X, x, and y, a regular witness path might

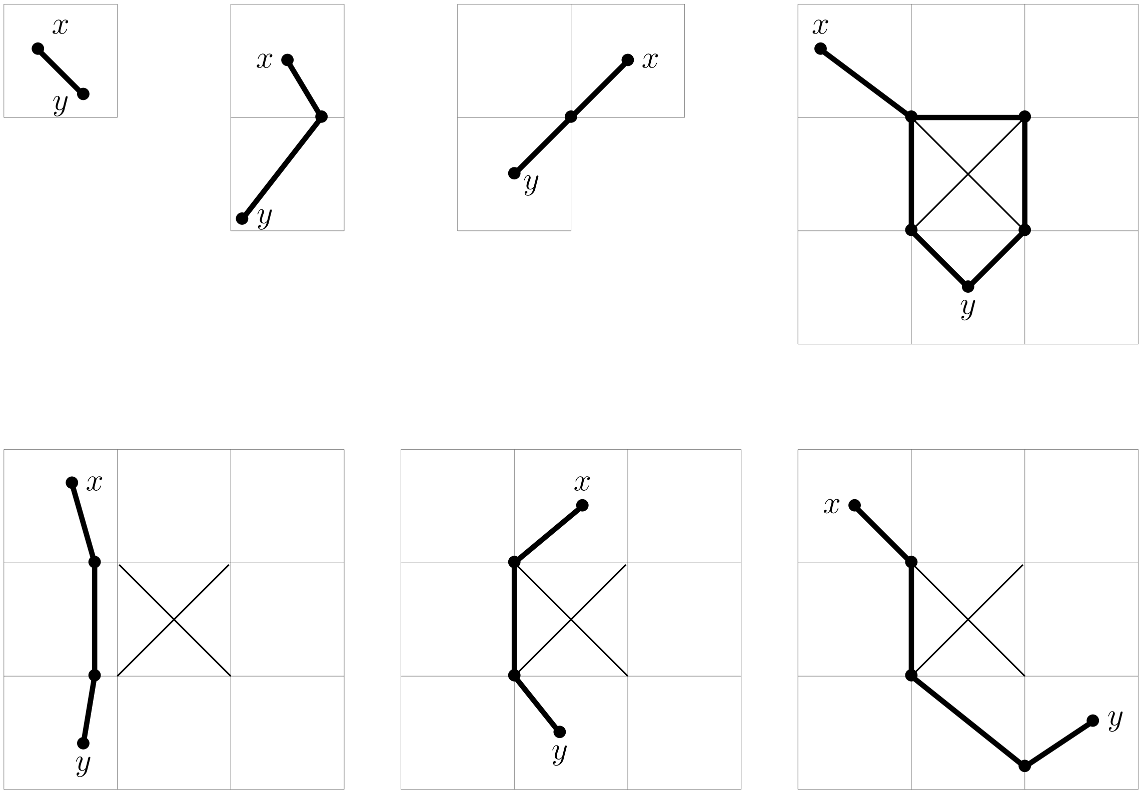


Figure 3: Regular witness paths in $M \otimes X$.

look like one or the other of these paths; in general, we do not have enough information to tell. And in some sense, we do not need to tell. We only need a classification of what the minimal witness paths look like, and this is the topic of our theorem.

Then we have the case of squares on opposite sides, such as $(0,0) \otimes X$ and $(2,0) \otimes X$, or $(1,0) \otimes X$ and $(2,0) \otimes X$. The interesting point here is that this case splits into two subcases, depending on whether one must "navigate around the central hole" due to the fact that $(1,1) \notin X$. Finally, we have the case of squares on opposite corners: $(0,0) \otimes X$ and $(2,2) \otimes X$, or $(2,0) \otimes X$ and $(0,2) \otimes X$. In this case, there is no need to indicate another path around the hole, since we are only working "up to rotation/reflection", and the other path is a rotation of the path shown.

Proof. First note that for any $m \otimes x$ and $n \otimes y$ in $M \otimes X$, there exists an alternating path between then by the way that E is defined. Consider such a path,

$$(m, x) = (m_0, x_0), (m_0, x_0') \sim \ldots \sim (m_p, x_p), (m_p, x_p') = (n, y).$$

We will show by induction on p that there is a regular witness path of one of the forms indicated in Figure 3 whose score is less than or equal to that of the path.

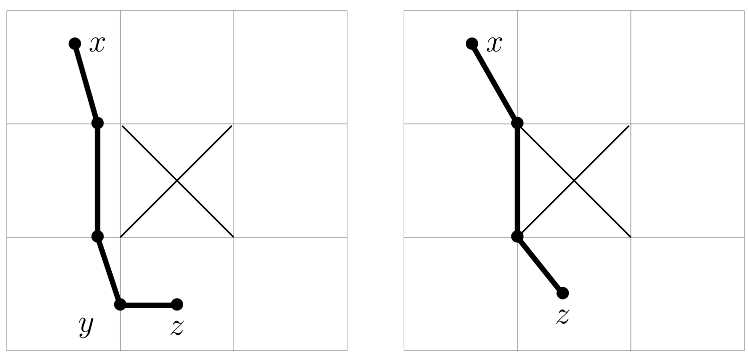
For $p = 0$, by Remark 4.1.6, $(m, x) = (n, y)$, or $(m, x) = (m_0, x_0), (m_0, x_0') = (n, y)$, so by Corollary 4.3.5, the regular witness path is $(m, x), (m, y)$, which is the first entry in Figure 3.

Let us assume our result for p and prove it for $p + 1$.

Before we do this, we will check by inspection that in each of the cases in the figure, if x or y is in $m \otimes S_X[M_0]$ (that is, is on the boundary of its copy of X), and we add one more point in an adjacent copy, then we obtain another case from Figure 3, or we can obtain a path with a smaller score by replacing it with one of the other cases in Figure 3.

For example, in the first entry in the figure, if $y = m \otimes (r, 0)$, that is, it is on the bottom boundary of its copy of X, and we add on z in the copy below, then we get an instance of the second entry in Figure 3.

For a more involved example, in the bottom left entry, suppose $y = (0, 0) \otimes (1, \frac{1}{2})$, so it is on the right boundary of its copy of X. Then suppose that we add on another entry $z = (1, 0) \otimes (\frac{1}{2}, \frac{1}{2})$.



We can replace the path on the left with the path on the right, and using an argument similar to that in Example 4.3.3, we see that this has a smaller or equal score. The path on the right is an instance of one of the paths in the top right entry of Figure 3. The rest of the cases are similar.

From here, if we consider a path with $p + 1$ entries and remove one entry, by the induction hypothesis, we can replace it with one of the paths from Figure 3 without increasing the score. Then when we add it back, either we obtain one of the paths from the figure, or, as we argued, we can find a path from Figure 3 with a shorter or equal score. ∎

4.4. $M \otimes X$ AS A FUNCTOR ON SQUARE METRIC SPACES. We restate Corollary 4.3.5 with a little more information which will be useful later on when we apply $M \otimes -$ k many times. We will always denote this repeated application by $M^k \otimes -$, and similarly for N, defined later on in Section 4.5.

4.4.1. COROLLARY. *For an object X in* SquaMS, *$m \in M$, and $x, y \in X$,*

$$d_{M \otimes X}(m \otimes x, m \otimes y) = \tfrac{1}{3} d_X(x, y).$$

In particular, $d_{M\otimes X}(m\otimes x, m\otimes y) \leq \frac{2}{3}$*, and for all* k*,*

$$d_{M^k\otimes X}(m_1\otimes\ldots\otimes m_k\otimes x, m_1\otimes\ldots\otimes m_k\otimes y) \leq \tfrac{2}{3^k}.$$

Corollary 4.3.6 tells us that for every object X in SquaMS, $M\otimes X$ is a metric space, and we have defined $S_{M\otimes X}: M_0 \to M\otimes X$ ((4.11)-(4.14)). We next check that $S_{M\otimes X}$ satisfies the non-degeneracy requirements.

4.4.2. Lemma. *$M\otimes X$ with $S_{M\otimes X}$ satisfies* (SQ$_1$) *and* (SQ$_2$)*.*

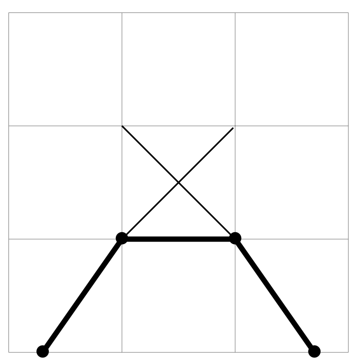
Proof. To show (SQ$_1$), without loss of generality, we will examine $S_{M\otimes X}((r,0))$ and $S_{M\otimes X}((s,0))$ for $r,s\in[0,1]$ (since all of the sides will behave the same way). We show

$$d_{M\otimes X}(S_{M\otimes X}((r,0)), S_{M\otimes X}((s,0))) = |s-r|. \tag{4.19}$$

First, suppose that $S_{M\otimes X}((r,0))$ and $S_{M\otimes X}((s,0))$ are in the same copy of X. (This means that there is at least one $m\in M$ such that these points belong to $m\otimes X$. Our result follows immediately from Corollary 4.4.1 and (4.15).

Next, suppose that $S_{M\otimes X}((r,0))$ and $S_{M\otimes X}((s,0))$ are in adjacent copies of X. One way that this could happen would be when $0\leq r\leq\frac{1}{3}$ and $\frac{1}{3}<s\leq\frac{2}{3}$. In this case, Example 4.3.3 shows that the distance is $|r-s|$. The details in all other cases are similar, and we omit them.

It remains to check that this holds for $S_{M\otimes X}((r,0))$ and $S_{M\otimes X}((s,0))$ in non-adjacent copies of X (that is, $0\leq r\leq\frac{1}{3}$ and $\frac{2}{3}\leq s\leq 1$). We note again that we have a path with length $|s-r|$ via the bottom corners of the $(1,0)$ copy of X. Another option that stays "on the bottom" is shown below:

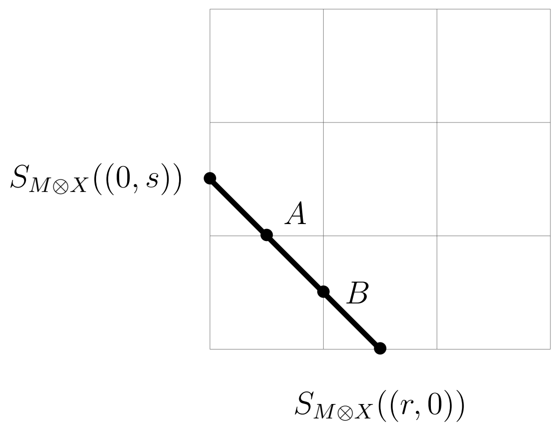


But then an argument using the fact that X satisfies (SQ$_2$) shows that the score on the path shown above is at least as large as the score mentioned above, $|r-s|$.

If we go the other way around $M\otimes X$, our path will have score greater than 1 because of the non-degeneracy requirement, and using a triangle inequality argument similar to the adjacent copies case, we see that this is minimized by going through the corners.

Now to check (SQ$_2$), let (r,s) and (t,u) in M_0 be given, and consider a witness path between $S_{M\otimes X}((r,s))$ and $S_{M\otimes X}((t,u))$. First note that each pair of entries in the path contributing positively to the score will be on the sides of a copy of X, so we can take advantage of (SQ_2) in X. We will show that the sum of horizontal and vertical components of each entry of our path will be at least the sum of the horizontal and vertical components of the distance between (r, s) and (t, u), and thus, our distance will be bounded below by the taxicab metric.

The different cases for relative placement of (r, s) and (t, u) are similar, so we will examine $S_{M\otimes X}((0, s))$ and $S_{M\otimes X}((r, 0))$ with $\frac{1}{3} < r, s < \frac{2}{3}$ in detail.



By examining cases, we can show that a shortest path will be of the form pictured (though we may have $A = B = (0, 0) \otimes S_X((1, 1)))$. Then by ($\mathrm{SQ}_2$) in X, we see that its length is

$$d_{M\otimes X}(S_{M\otimes X}((0, s)), A) + d_{M\otimes X}(A, B) + d_{M\otimes X}(B, S_{M\otimes X}((r, 0))).$$

Note that $S_{M\otimes X}((0, s)) = (0, 1) \otimes S_X((0, 3s - 1))$ and $S_{M\otimes X}((r, 0)) = (1, 0) \otimes S_X((3r - 1, 0))$. Let

$$\begin{array}{rclcl} A & = & (0,1) \otimes S_X((t,0)) & = & (0,0) \otimes S_X((t,1)) \\ B & = & (0,0) \otimes S_X((1,u)) & = & (1,0) \otimes S_X((0,u)) \end{array}$$

Then the distance is

$$\begin{array}{rl} & \frac{1}{3}d_X(S_X((0,3s-1)), S_X((t,0))) + \frac{1}{3}d_X(S_X((t,1)), S_X((1,u))) \\ & \quad +\frac{1}{3}d_X(S_X((0,u)), S_X((3r-1,0))) \\ \geq & \frac{1}{3}(|3s-1-0| + |t-0| + |1-t| + |1-u| + |u-0| + |3r-1-0|) \\ = & \frac{1}{3}(3|s-0| + 3|r-0|) \\ = & |s-0| + |r-0|, \end{array}$$

as required. In the first inequality, we used the fact that X is an object in SquaMS. ■

At this point we know that $M \otimes X$ is an object in SquaMS. That is, we know how the functor $M \otimes -$ works on objects of SquaMS. Now let $f : X \to Y$ be a morphism in SquaMS, and define $M \otimes f : M \otimes X \to M \otimes Y$ by $M \otimes f(m \otimes x) = m \otimes f(x)$. By Lemma 4.1.15, we know that $M \otimes f$ is well defined. To check that $M \otimes f : M \otimes X \to M \otimes Y$ is a morphism, first note that for (r, s) on the boundary of the unit square,

$$(M \otimes f)(S_{M\otimes X}((r,s))) = (M \otimes f)(m \otimes S_X((r',s')))$$

for some $m \in M$ and $(r', s') \in M_0$, by the definition of $S_{M\otimes X}$. This equals

$$\begin{array}{rll} & m \otimes f(S_X((r',s'))) & \\ = & m \otimes S_Y((r',s')) & \text{since } f \text{ preserves } S_X \\ = & S_{M\otimes Y}((r,s)) & \end{array}$$

The last equality holds since $S_{M\otimes Y}$ is defined using the same scheme as $S_{M\otimes X}$. So $M \otimes f$ preserves $S_{M\otimes X}$.

To see that $M\otimes f$ is a short map, let $m\otimes x, n\otimes y \in M\otimes X$. If $d_{M\otimes X}(m\otimes x, n\otimes y) = 2$, then $d_{M\otimes X}(m \otimes x, n \otimes y) = 2 \geq d_{M\otimes Y}(M \otimes f(m \otimes x), M \otimes f(n \otimes y))$. Otherwise, let

$$m \otimes x = m_0 \otimes x_0, \dots, m_p \otimes x_p = n \otimes y$$

be a witness path between them (this is shorthand, each entry is the equivalence class of adjacent entries which are related by $\sim$).

Then if $d_{M\otimes X}(m_k \otimes x_k, m_{k+1} \otimes x_{k+1}) \neq 0$, it is because $m_{k+1} \otimes x_{k+1} = m_k \otimes x'_{k+1}$ for some $x'_{k+1} \in X$.

So

$$\begin{array}{rll} & d_{M\otimes X}(m_k \otimes x_k, m_{k+1} \otimes x_{k+1}) & \\ = & d_{M\times X}((m_k, x_k), (m_k, x'_{k+1})) & \\ = & \frac{1}{3} d_X(x_k, x'_{k+1}) & \\ \geq & \frac{1}{3} d_Y(f(x_k), f(x'_{k+1})) & \text{since } f \text{ is a short map} \\ = & d_{M\times Y}((m_k, f(x_k)), (m_k, f(x'_{k+1}))) & \\ = & d_{M\otimes Y}((M \otimes f)(m_k \otimes x_k), (M \otimes f)(m_{k+1} \otimes x_{k+1})) & \end{array}$$

Thus, since there is a path in $M \otimes Y$ from $(M \otimes f)(m \otimes x)$ to $(M \otimes f)(m \otimes y)$ whose score is bounded above by the score of a shortest path in $M \otimes X$, $M \otimes f$ is a short map.

Finally, note that $M \otimes -$ preserves compositions and identity maps, as required.

4.4.3. Theorem. *$M \otimes -$ is a functor on* SquaMS.

Finally, we want to take advantage of the following lower bound on paths. Recall that

$$d_{U_0}((x,y),(x_1,y_1)) = |x - x_1| + |y - y_1| \tag{4.20}$$

is the taxicab metric on the unit square.

4.4.4. Proposition. *Let B be an object in* SquaMS *and consider $m \otimes S_B((r,s))$ and $n \otimes S_B((t,u))$ in $M \otimes B$. Then*

$$d_{M\otimes B}(m \otimes S_B((r,s)), n \otimes S_B((t,u))) \geq d_{M\otimes U_0}(m \otimes S_{U_0}((r,s)), n \otimes S_{U_0}((t,u))).$$

PROOF. By Theorem 4.1.12 there is a witness path in $M \otimes B$ of the form

$$m \otimes S_B((r,s)) = m_0 \otimes S_B((r_0,s_0)), m_0 \otimes S_B((r'_0,s'_0)) \sim m_1 \otimes S_B((r_1,s_1)),$$
$$\ldots, m_{p-1} \otimes S_B((r_{p-1},s_{p-1})),$$
$$m_{p-1} \otimes S_B((r'_{p-1},s'_{p-1})) \sim m_p \otimes S_B((r_p,s_p)) = n \otimes S_B((t,u))$$

Then note that

$$\begin{array}{rl} & d_{M\otimes B}(m_k \otimes S_B((r_k,s_k)), m_k \otimes S_B((r'_k,s'_k))) \\ = & \frac{1}{3} d_B(S_B((r_k,s_k)), S_B((r'_k,s'_k))) \\ \geq & \frac{1}{3} d_{U_0}(S_{U_0}((r_k,s_k)), S_{U_0}((r'_k,s'_k))) \\ = & d_{M\otimes U_0}(m_k \otimes S_{U_0}((r_k,s_k)), m_k \otimes S_{U_0}((r'_k,s'_k))) \end{array}$$

where the inequality follows from (SQ_2). So the score of the shortest path in $M \otimes B$ is bounded below by the score of the corresponding path in $M \otimes U_0$, which is an upper bound of the distance between the corresponding points in $M \otimes U_0$. (But there may be a shorter path in $M \otimes U_0$, and this is why our result has an inequality.) ■

4.5. DEFINING $N \otimes -$ IN SquaMS. It will be useful for us to augment M in the following way. Let $N = \{0,1,2\}^2 = M \cup \{(1,1)\}$, which will correspond to the full 3×3 grid. We aim to expand the work from the previous section to show that $N \otimes -$ is also a functor. We will use this in later sections.

$(0,2)$	$(1,2)$	$(2,2)$
$(0,1)$	$(1,1)$	$(2,1)$
$(0,0)$	$(1,0)$	$(2,0)$

In our pictures of $N \otimes X$, we do not show an X over the square $(1,1)$ the way we did with $M \otimes X$.

First we want to apply Corollary 4.1.14 to see that $N \otimes X$ is in fact a metric space with the quotient metric. The majority of the work for us is done. The definition of $S_{N\otimes X}$ will coincide with $S_{M\otimes X}$, and we will need to expand E to include

$$\begin{array}{l} ((0,1),(1,r)) \approx ((1,1),(0,r)) \\ ((1,2),(r,0)) \approx ((1,1),(r,1)) \\ ((2,1),(0,r)) \approx ((1,1),(1,r)) \\ ((1,0),(r,1)) \approx ((1,1),(r,0)) \end{array}$$

for $r \in [0,1]$. Call this relation $\widehat{E}$.

4.5.1. LEMMA. *$\widehat{E}$ is quotient suitable on $N \times M_0$ (Definition 4.1.1).*

The proof is the same as that of Lemma 4.2.1

4.5.2. LEMMA. *Let X be any object in* SquaMS. *Let $n \in N$ and let $x, y \in X$. Let $(n_0, x_0), \ldots, (n_p, x_p)$ be a path, where $(n_0, x_0) = (n, x)$ and $(n_p, x_p) = (n, y)$. (Notice that the same n is used in both the start and end of the path.) Then*

$$\tfrac{1}{3} d_X(x, y) \leq \sum_{k=0}^{p-1} d_{N\times X}((n_k, x_k), (n_{k+1}, x_{k+1})).$$

PROOF. Fix n throughout this proof. By Lemma 4.1.7, since the path $(n, x), (n, y)$ has score $\leq \frac{2}{3} < 2$, we may assume that our path is an alternating path,

$$(n, x) = (n_0, x_0), (n_0, x_0') \sim \ldots (n_{p-1}, x_{p-1}') \sim (n_p, x_p), (n_p, x_p') = (n, y).$$

By Lemma 4.1.8, Lemma 4.1.9, and Lemma 4.1.10, we may also assume the following for i, j such that $0 \leq i < j \leq p$

- If x_i and x_i' are on the same side of $S_X[M_0]$, then they are both corners,
- If x_i' and x_j are on the same side of $S_X[M_0]$, then at least one of them is a corner,
- If x_i and x_j are on the same side of $S_X[M_0]$, then x_i' and x_j' are *not* on the same side of $S_X[M_0]$,
- If x_i and x_j' are on the same side of $S_X[M_0]$, then x_i' and x_j are *not* on the same side of $S_X[M_0]$.

As in Lemma 4.3.4, we will proceed by induction on the natural number k where $k = |\{i : m_i = m\}|$.

When $k = 1$, $p = 0$, and the path from (n, x) to (n, y) is just $(n, x), (n, y)$, whose length is $\frac{1}{3} d_X(x, y)$. When $k \geq 3$, our path has $n_l = n$ for some $0 < l < p$. We cut this path into two subpaths, the part between (n_0, x_0) and (n_l, x_l), and the part from (n_l, x_l') to (n_p, x_p'). In both subpaths, the number j such that $n_j = n$ is $< k$, so the induction hypothesis applies to the subpaths. By this and the triangle inequality, we show the desired inequality.

$$\frac{1}{3} d_X(x, y) \leq \frac{1}{3} d_X(x, x_\ell) + \frac{1}{3} d_X(x_\ell, y) \leq \sum_{i=0}^{p-1} d_{N\times X}((n_i, x_i), (n_{i+1}, x_{i+1})).$$

The remaining case is when $k = 2$. Up until this point, the proof as been the same as Lemma 4.3.4, and the remaining part is similar, but we have a few more cases to consider since we include the center copy of X. There are three possible cases we must consider, up to rotation and reflection: when our two points are in a corner copy of X (a copy indexed by $(0, 0)$, $(0, 2)$, $(2, 0)$, or $(2, 2)$); when they are in a copy of X around the outside

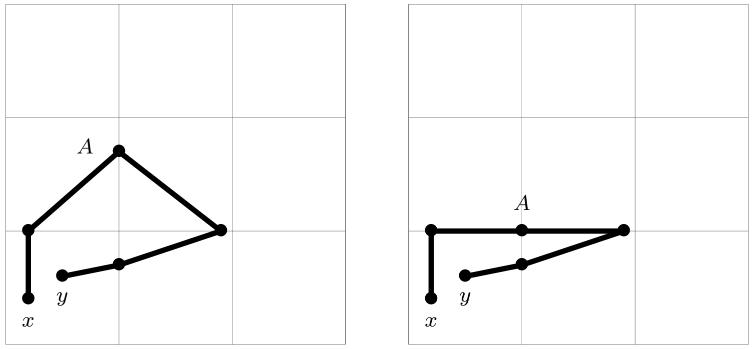


Figure 4: The first two figures in this proof, called (a) and (b) in this proof.

which is not a corner (a copy indexed by $(0,1)$, $(1,0)$, $(2,0)$, or $(0,2)$); and when they are in the middle copy of X (indexed by $(1,1)$).

First we will consider the case when both points are, without loss of generality, in the $(0,0)$ copy of X. Then we need not consider paths which contain points in the $(0,2)$, $(1,2)$, $(2,2)$, $(2,1)$, or $(2,0)$ copies of X, since any such path (like the one pictured at the end of the proof of Lemma 4.3.4) would contribute $\frac{2}{3}$ to the score by (SQ_2), by crossing two copies of X (one going out and one coming back). So we need only consider the case which is shown in Figure 4(a). (This case did not come up in the proof of Lemma 4.3.4 because we did not have a middle copy of X.) Using the same argument as in Example 4.3.2, we can find a path with lesser or equal score by moving the point A to the shared bottom corner of the $(0,1)$ and $(1,1)$ copies of X as in Figure 4(b). But then the path $(n,x), A, (n,y)$ has score which is less than or equal to the score of this path, and again, by the triangle inequality (since we are now entirely in the $(0,0)$ copy of X), $(n,x), (n,y)$ is a path with smaller or equal score, which is $\frac{1}{3}d_X(x,y)$.

Next, without loss of generality, we consider the case shown in Figure 5(c), when $n = (1,0)$. Again, we need not consider paths which contain points in the $(0,2)$, $(1,2)$ or $(2,2)$ copies of X, since these will contribute $\frac{2}{3}$ to the score. So we have two subcases. The first subcase is the same as the case we examined when $n = (0,0)$ (but shifted to the right). The second subcase, shown in Figure 5(d), can be shorted in a similar fashion by moving A and B to the respective corner points (see Figure 5(e)), as we saw in the proof of Lemma 4.4.2. Again, this gives us a path with another point in $(1,0) \otimes X$, so we see that the score is bounded below by the score of $(n,x), (n,y)$, which is $\frac{1}{3}d_X(x,y)$.

Finally, consider the case when $n = (1,1)$. We can split this into two subcases. First, when the path exits and enters the $(1,1)$ copy of X from adjacent sides, say the top and the right. The path contains points in eight of the nine copies of X, as shown in Figure 6(f). In this subcase, by (SQ_2), the score is greater than $\frac{2}{3}$. So we do not have a witness path in this case. The second subcase is when the path contains points in only four copies of X, as shown in Figure 6(g). This is the same as the case we examined when $n = (0,0)$. So in both of these subcases, the score is bounded below by $\frac{1}{3}d_X(x,y)$.

The final case is shown in Figure 6(h), when the path exits and enters the $(1,1)$ copy

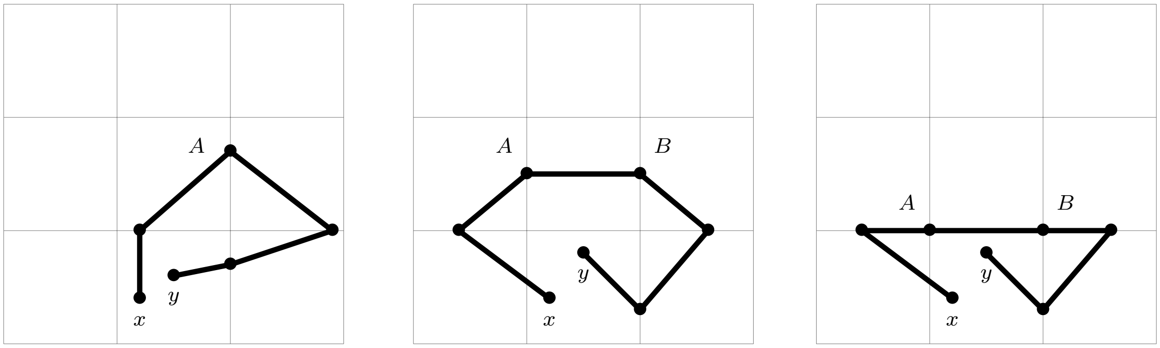


Figure 5: The next three figures in this proof, called (c), (d), and (e).

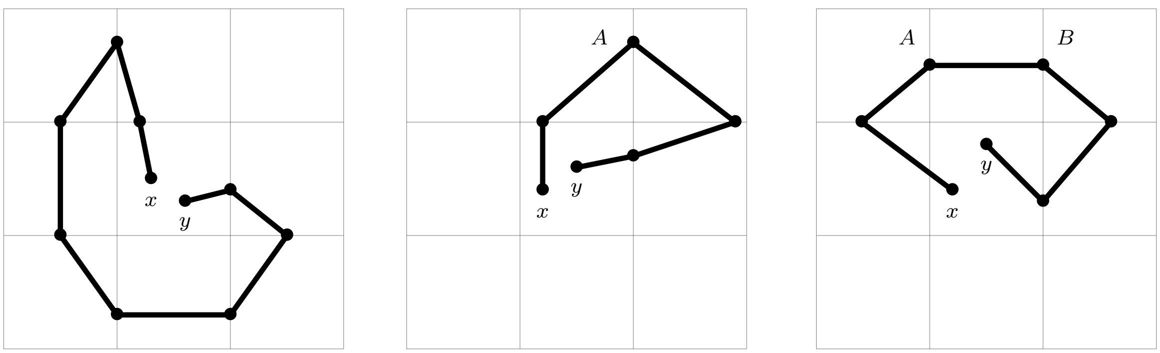


Figure 6: The last three figures in this proof, called (f), (g) and (h).

of X on opposite sides.

But this is essentially the same as the case we examined when $n = (1, 0)$.

By examining cases, we have shown that every path between points in the same copy of X has a score which is bounded below by $\frac{1}{3}d_X(x, y)$. ■

As with M, we have the following useful corollary for $N \otimes -$.

4.5.3. Corollary. *For an object X in* SquaMS*, $n \in N$, and $x, y \in X$,*

$$d_{N\otimes X}(n \otimes x, n \otimes y) = \frac{1}{3}d_X(x, y).$$

In particular, $d_{N\otimes X}(n \otimes x, n \otimes y) \leq \frac{2}{3}$, and if we apply $N \otimes -$ to X k many times, we get $d_{N^k\otimes X}(n_1 \otimes \ldots \otimes n_k \otimes x, n_1 \otimes \ldots \otimes n_k \otimes y) \leq \frac{2}{3^k}$.

So by Corollary 4.1.14, $N \otimes X$ is a metric space. The proof that $N \otimes X$ satisfies (SQ_1) is the same as that for $M \otimes X$, and (SQ_2) follows for $N \otimes X$ from (SQ_2) in X, in the same way as it does for $M \otimes X$ in Lemma 4.4.2.

Morphisms will be preserved by $N \otimes -$ just as they are by $M \otimes -$. Hence, we have the following:

4.5.4. PROPOSITION. *$N \otimes -$ is a functor on* SquaMS.

We also have an analogous lower bound on distances between boundary points to Proposition 4.4.4. The proof is the same.

4.5.5. LEMMA. *Let B be an object in* SquaMS *and consider $m \otimes S_B((r,s))$ and $n \otimes S_B((t,u))$ in $N \otimes B$. Then*

$$d_{N\otimes B}(m \otimes S_B((r,s)), n \otimes S_B((t,u))) \geq d_{N\otimes U_0}(m \otimes S_{U_0}((r,s)), n \otimes S_{U_0}((t,u))).$$

4.6. DISTANCES BETWEEN CORNER POINTS IN ITERATES OF $N \otimes -$ ON M_0 AND U_0. In much of this paper, we are going to be interested in iterating the functor $M \otimes -$ on the unit square U_0, or on the initial square space M_0, or more generally on square spaces B which admit a morphism $B \longrightarrow M \otimes B$. But at this point, we need some results about the iteration of $N \otimes -$ on such spaces. In fact, results on $M \otimes -$ often go through results on $N \otimes -$, partly because this latter functor is easier to study.

4.6.1. DEFINITION. Let B be either M_0 or U_0. The set CP_k of *corner points* of $N^k \otimes B$ is defined as follows:

$$\begin{array}{rcl} CP_0 & = & \{(0,0),(0,1),(1,0),(1,1)\} \\ CP_{k+1} & = & \{n \otimes x \mid n \in N, x \in CP_k\} \end{array}$$

Let $f_k \colon CP_k \longrightarrow U_0$ be (as expected): f_0 is the inclusion, and $f_{k+1}(n \otimes x) = \frac{1}{3}n + \frac{1}{3}f_k(x)$.

Later on in Definition 6.1.5, we will define $\alpha_N : N \otimes U_0 \to U_0$ similarly, so that $f_{k+1}(n \otimes x) = \alpha_N(n \otimes f_k(x))$.

We regard CP_k as a metric space with distances inherited from $N^k \otimes B$.

The main result in this section shows that it does not matter whether we take B to be M_0 or U_0 in Definition 4.6.1: the distances between corner points are the same.

4.6.2. DEFINITION. For any sequence of p numbers $i_1, \ldots, i_p \in \{0,1,2\}$ and for any $r \in [0,1]$, we define the number $|i_1, \ldots, i_p; r| \in [0,1]$ in the following way.

$$\begin{array}{rcl} |r| & = & r \\ |i_1, i_2, \ldots, i_p; r| & = & \frac{1}{3}i_1 + \frac{1}{3}|i_2, \ldots, i_p; r| \end{array}$$

In a more explicit presentation,

$$|i_1, i_2, \ldots, i_p; r| = \frac{r}{3^p} + \sum_{m=1}^{p} \frac{i_m}{3^m}$$

4.6.3. Lemma. *For all* $i_1, \ldots, i_p \in \{0,1,2\}$ *and all* $r \in [0,1]$,

$$\begin{array}{rcl} S_{N^p \otimes B}((|i_1, i_2, \ldots, i_p; r|, 0)) & = & (i_1, 0) \otimes \ldots \otimes (i_p, 0) \otimes S_B((r,0)) \\ S_{N^p \otimes B}((|i_1, i_2, \ldots, i_p; r|, 1)) & = & (i_1, 2) \otimes \ldots \otimes (i_p, 2) \otimes S_B((r,1)) \end{array}$$

Proof. We prove this by induction on p. For $p = 0$, the result is clear. Assume our result for p, and fix r and $i_1, \ldots, i_p, i_{p+1}$. Let $r^* = |i_2, \ldots, i_p; r|$. By induction hypothesis,

$$S_{N^p \otimes B}((r^*, 0)) = (i_2, 0) \otimes \ldots \otimes (i_{p+1}, 0) \otimes S_B((r,0)),$$

and similarly for $S_{N^p \otimes B}((r^*, 1))$. To save on a little notation, write r^{**} for $|i_1, i_2, \ldots, i_p; r|$. When $i_1 = 0$, $r^{**} = \frac{1}{3} r^*$. Using (4.14), (4.12), and the induction hypothesis,

$$\begin{array}{rcl} S_{N \otimes (N^p \otimes B)}((r^{**}, 0)) & = & (0,0) \otimes S_{N^p \otimes B}((r^*, 0)) \\ & = & (i_1, 0) \otimes \ldots \otimes (i_{p+1}, 0) \otimes S_B((r,0)) \\ S_{N \otimes (N^p \otimes B)}((r^{**}, 1)) & = & (0,2) \otimes S_{N^p \otimes B}((r^*, 1)) \\ & = & (i_1, 2) \otimes \ldots \otimes (i_{p+1}, 2) \otimes S_B((r,1)) \end{array}$$

If $i_1 = 1$, we have $r^{**} = \frac{1}{3} r^* + \frac{1}{3}$, and if $i_1 = 2$, $r^{**} = \frac{1}{3} r^* + \frac{2}{3}$. In all of these cases, the verifications are similar. ∎

Here is another fact about this notation.

4.6.4. Lemma. *For all* $p \geq 1$ *and all* $i_1, \ldots, i_p; r$, $\frac{i_1}{3} \leq |i_1, \ldots, i_p; r| \leq \frac{i_1+1}{3}$. *The only way to have* $|i_1, \ldots, i_p; r| = 0$ *is for* $i_1, \ldots, i_p; r = 0, \ldots, 0; 0$. *The only way to have* $|i_1, \ldots, i_p; r| = 1$ *is for* $i_1, \ldots, i_p; r = 2, \ldots, 2; 1$.

Proof. By induction on p. When $p = 1$, this is clear from the definition of $|i_1; r|$. Assume our result for p, and take $i_1, \ldots, i_{p+1}; r$. By induction hypothesis, $0 \leq |i_2, \ldots, i_{p+1}; r| \leq 1$.

So since $|i_1, \ldots, i_{p+1}; r| = \frac{1}{3} i_1 + \frac{1}{3} |i_2, \ldots, i_{p+1}; r|$,

$$\begin{array}{ccccc} 0 & \leq & |i_2, \ldots, i_{p+1}; r| & \leq & 1 \\ 0 & \leq & \frac{1}{3}|i_2, \ldots, i_{p+1}; r| & \leq & \frac{1}{3} \\ \frac{i_1}{3} & \leq & \frac{i_1}{3} + \frac{1}{3}|i_2, \ldots, i_{p+1}; r| & \leq & \frac{i_1}{3} + \frac{1}{3} \\ \frac{i_1}{3} & \leq & |i_1, \ldots, i_{p+1}; r| & \leq & \frac{i_1+1}{3} \end{array}$$

The last assertions in our result are easy to check by induction on p. ∎

The last few definitions allowed r and s to be any numbers in $[0,1]$. For the next main results we need to restrict to corner points (see Definition 4.6.1). The key result in this section, Lemma 4.6.7 below, is false without the restriction to corner points.

Next we turn our attention to the possible witness paths between elements of $N \otimes X$. In Lemma 4.6.5, we will show that (up to rotation and reflection) we can always find a path which goes "up and to the right" between copies of X. In it, we use an ordering $\triangleleft$ on $\{0,1,2\} \times \{0,1,2\}$. That ordering is the strict part of the product ordering determined by the natural order $0 < 1 < 2$ on $\{0,1,2\}$. In other words, $(i,j) \triangleleft (k,\ell)$ iff $i \leq k$ and $j \leq \ell$ and at least one of these inequalities is strict.

Here is a preliminary observation on this notation. Consider $\{0,1,2\} \times \{0,1,2\}$ as a graph G, where there is an edge from (i,j) to (k,ℓ) iff either ($i = k$ and $|j-\ell| = 1$) or else ($j = \ell$ and $|i-k| = 1$). Suppose that $(a,b) \triangleleft (c,d)$. Then there is a geodesic (a path of minimal length) in G from (a,b) to (c,d) consisting of points which "goes up" in the order $\triangleleft$. (For example, the distance in G from $(0,0)$ to $(2,2)$ is 4, and we have a path which "goes up" in $\triangleleft$: $(0,0) \triangleleft (1,0) \triangleleft (1,1) \triangleleft (1,2) \triangleleft (2,2)$.)

Lemma 4.6.5 just below is an analogous fact, but not for the graph G but instead for a square space of the form $N \otimes B$.

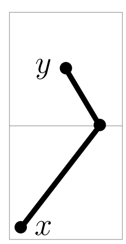
4.6.5. Lemma. *Let B be in* SquaMS *and $x, y \in N \otimes B$. Suppose that $x = m_0 \otimes x_0$ and $y = m_k \otimes y_0$ with $m_0 \triangleleft m_k$ with respect to this partial order (we can rotate or reflect if necessary). Then there is a witness path of the form*

$$x = m_0 \otimes x_0, m_0 \otimes S_B((r_0, s_0)), m_1 \otimes S_B((r_1, s_1)), \ldots,$$

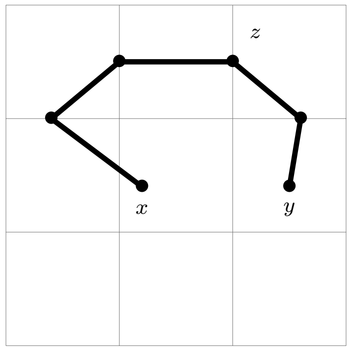
$$m_{k-1} \otimes S_B((r_{k-1}, s_{k-1})), m_k \otimes y_0 = y$$

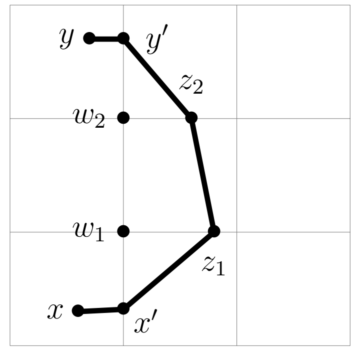
where each for each $0 \leq i < k$, $m_i \otimes S_B((r_i, s_i)) = m_{i+1} \otimes S_B((r'_i, s'_i))$ for some $(r'_i, s'_i) \in M_0$, and $m_i \triangleleft m_{i+1}$.

Proof. We have several cases. The first is when m_0 and m_k are neighbors in G (for example, $(0,2)$ and $(1,2)$). In this case, the work which we did in the proof of Theorem 4.3.7 adapts easily to give the result which we want. That theorem dealt with the functor $M \otimes -$ and not $N \otimes -$, but for this case the work there shows that the witness paths from x to y look like



The one case which we need to add is when we have a path that uses the middle square and looks like the path from x to y in the picture on the left below:





But here the part from x to z may be shortened: since x and z lie in neighboring squares, this is the content of our previous observations. And once we shorten this path so that it does not take the long trip by starting out going left from x, it is then isomorphic to a path in $M \otimes B$, we are in a position to use our previous work to produce from it a witness path with the appropriate feature: the first components go up in $\triangleleft$.

Next, let us consider the case when m_0 and m_k have distance 2 in G. The classification of regular witness paths for $M \otimes -$ which we saw in Figure 3 applies except for small changes. We need an addition for a situation as on the right above. We have a path from x to y going through the middle square, as shown. It is x, x', z_1, z_2, y', y. This path is not what we want because at the end we have a pair with $\neg((1,2) \triangleleft (0,2))$. However, let us consider the path x, w_1, w_2, y. This new path is increasing in $\triangleleft$. We claim that its score is at most that of the original path We have $d(w_1, w_2) = \frac{1}{3}$, by ($\textsc{sq}_1$), and $d(z_1, z_2) \geq \frac{1}{3}$, by ($\textsc{sq}_2$). Moreover,

$$d(x, w_1) \leq d(x, x') + d(x', w_1) \leq d(x, x') + d(x', z_1),$$

using the triangle inequality and ($\textsc{sq}_2$). The same calculations apply on the other end of the path, and we put things together to see that indeed the score of the new path is at most the score of the old.

There are very similar arguments when m_0 and m_k have distance 3 or 4 in G. Indeed, the cases which we have considered make the arguments short in these cases. We omit the details. ∎

The following is a lemma about distances between points in $N \otimes B$ (where B is an arbitrary square metric space) in different copies of B.

4.6.6. Lemma. *Let B be a square metric space and let $x, y \in B$. Consider the points $(0,0) \otimes x$, $(0,1) \otimes y$, $(0,2) \otimes y$, and $(1,1) \otimes y$ in $N \otimes B$.*

1. *There is a witness path from $(0,0) \otimes x$ to $(0,1) \otimes y$ of the form*

$$(0,0) \otimes x, (0,0) \otimes S_B((r,1)) = (0,1) \otimes S_B((r,0)), (0,1) \otimes y,$$

where $r \in [0,1]$.

2. *There is a witness path from* $(0,0)\otimes x$ *to* $(0,2)\otimes y$ *of the form*

$$\begin{array}{c}(0,0)\otimes x,(0,0)\otimes S_B((r_1,1))=(0,1)\otimes S_B((r_1,0)),\\ (0,1)\otimes S_B((r_2,1))=(0,2)\otimes S_B((r_2,0)),(0,2)\otimes y,\end{array}$$

where $r_1, r_2 \in [0,1]$.

3. *There is a witness path from* $(0,0)\otimes x$ *to* $(1,1)\otimes y$ *of the form*

$$(0,0)\otimes x,(0,0)\otimes S_B((1,1))=(1,1)\otimes S_B((0,0)),(1,1)\otimes y.$$

4. *More generally, by rotating or reflecting, we get the analogous results for copies of* B *which are in the same row or column (1. and 2.), or which share a corner (3.).*

Proof. Parts 1. and 2. follow from Lemma 4.6.5. In part 3., we know from Lemma 4.6.5, we know that there is a path of the form

$$\begin{array}{c}(0,0)\otimes x,(0,0)\otimes S_B((r_1,1))=(0,1)\otimes S_B((r_1,0)),\\ (0,1)\otimes S_B((1,s_2))=(1,1)\otimes S_B((0,s_2)),(1,1)\otimes y\end{array}$$

or of the form

$$\begin{array}{c}(0,0)\otimes x,(0,0)\otimes S_B((1,s_1))=(1,0)\otimes S_B((0,s_1)),\\ (1,0)\otimes S_B((r_2,1))=(1,1)\otimes S_B((r_2,0)),(1,1)\otimes y.\end{array}$$

Without loss of generality, suppose it is the former. We are going to use the triangle inequality and (SQ_2) to show that the score of such a path is minimized when

$$(0,1)\otimes S_B((r_1,0))=(0,1)\otimes S_B((1,0))=(0,1)\otimes S_B((1,s_2)).$$

That is, our path has the smallest score when it is of the form

$$(0,0)\otimes x,(0,0)\otimes S_B((1,1))=(1,1)\otimes S_B((0,0)),(1,1)\otimes y,$$

as required.

For ease of notation, let

$$\begin{array}{rcl}a&=&d_{N\otimes B}((0,0)\otimes x,(0,0)\otimes d_B((1,1)))\\ b&=&d_{N\otimes B}((1,1)\otimes S_B((0,0)),(1,1)\otimes y).\end{array}$$

Our goal is to show that the score of the proposed path is $\geq a+b$. Let

$$\begin{array}{rcl}c&=&d_{N\otimes B}((0,0)\otimes x,(0,0)\otimes S_B((r_1,1))),\\ e&=&d_{N\otimes B}((0,1)\otimes S_B((r_1,0)),(0,1)\otimes S_B((1,s_2))),\\ f&=&d_{N\otimes B}((1,1)\otimes S_B((0,s_2)),(1,1),y).\end{array}$$

So we want to show $c+e+f \geq a+b$. Observe that $e \geq (1-r_1)+s_2$ by ($\textsc{sq}_2$). In addition, $c+(1-r_1) \geq a$ and $s_2+f \geq b$ by the triangle inequality. Thus,

$$c+e+f \geq a-(1-r_1)+(1-r_1)+s_2+b-s_2 = a+b,$$

as required. ■

4.6.7. Lemma. *Let B be a square metric space, and let $p \geq 0$.*

1. *For all $x, y \in N^p \otimes B$ of the form*

$$\begin{array}{rcl} x & = & (i_1, j_1) \otimes (i_2, j_2) \otimes \ldots \otimes (i_p, j_p) \otimes S_B((r,s)) \\ y & = & (k_1, \ell_1) \otimes (k_2, \ell_2) \otimes \ldots \otimes (k_p, \ell_p) \otimes S_B((t,u)) \end{array} \tag{4.21}$$

we have the following distance formula:[4]

$$d_{N^p \otimes B}(x,y) \quad \geq \quad |i^* - k^*| + |j^* - \ell^*|, \tag{4.22}$$

where

$$\begin{array}{rclcrcl} i^* & = & |i_1, i_2, \ldots, i_p; r|, & & k^* & = & |k_1, k_2, \ldots, k_p; t|, \\ j^* & = & |j_1, j_2, \ldots, j_p; s|, & & \ell^* & = & |\ell_1, \ell_2, \ldots, \ell_p; u|. \end{array}$$

2. *Assume that B is either M_0 or U_0 and that x and y are corner points. Then we may improve (4.22) to a "taxicab-like" formula:*

$$d_{N^p \otimes B}(x,y) \quad = \quad |i^* - k^*| + |j^* - \ell^*|.$$

Proof. By induction on p.

The base case is $p = 0$. Part (1) is just the statement of ($\textsc{sq}_2$). We turn to part (2). This is where we use the assumption that we are dealing with corner points and the overall space B is either M_0 or U_0. That is, the distance among points $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$ may be calculated as if we were using the taxicab metric, even though the space M_0 uses the path metric; the formula in this lemma is in general false and holds mainly for the corner points.

Let us check both (1) and (2) for $p+1$, assuming them for p. The argument breaks into cases depending on which copy of $N^p \otimes B$ our points x and y belong to.

The first case is when x and y are in the same copy of $N^p \otimes B$. That is, $(k_1, \ell_1) = (i_1, j_1)$. We are going to check (2); the argument for (1) is similar. So x and y are corner points, and B is M_0 or U_0. Let

$$\begin{array}{rcl} x' & = & (i_2, j_2) \otimes \ldots \otimes (i_p, j_p) \otimes S_B((r,s)) \\ y' & = & (k_2, \ell_2) \otimes \ldots \otimes (k_p, \ell_p) \otimes S_B((t,u)) \end{array}$$

[4] The notation $|i_1, i_2, \ldots, i_p; r|$ was introduced in Definition 4.6.2.

So $x = (i_1, j_1) \otimes x'$ and $y = (i_1, j_1) \otimes y'$. In this case, x' and y' are corner points as well. By induction hypothesis, $d_{N^p \otimes B}(x', y') = |i_2^* - k_2^*| + |j_2^* - \ell_2^*|$, where $i_2^* = |i_2, i_3, \ldots, i_p; r|$, and similarly for j_2^*, k_2^*, and l_2^* (note that these start with second entry of the non-subscripted version, hence the 2). Now $i^* = \frac{1}{3}i_1 + \frac{1}{3}i_2^*$, and similarly for the others. By Corollary 4.5.3,

$$d_{N^{p+1} \otimes B}(x, y) = \frac{1}{3} d_{N^p \otimes B}(x', y') = \left| \frac{1}{3} i_2^* - \frac{1}{3} k_2^* \right| + \left| \frac{1}{3} j_2^* - \frac{1}{3} \ell_2^* \right| = |i^* - k^*| + |j^* - \ell^*|.$$

using $(k_1, \ell_1) = (i_1, j_1)$ in the last step.

This concludes our work for (2) in this first case of the induction step, and as we said, (not-necessarily-corner) is similar.

The other cases are when x and y are in different copies of $N^p \otimes B$. We are going to give full details for the case when x and y are in copies which share an edge. Concretely, we shall work with the assumption $(i_1, j_1) = (0, 0)$ and $(k_1, l_1) = (0, 1)$. Let x and y be as in (4.21), but with $p + 1$ terms (i, j) or (k, ℓ) instead of p. Let x' and y' be as shown below, where we reiterated x and y for convenience:

$$\begin{array}{rcl} x & = & (0,0) \otimes (i_2, j_2) \otimes \ldots \otimes (i_{p+1}, j_{p+1}) \otimes S_B((r, s)) \\ x' & = & (0,0) \otimes (i_2, 2) \otimes \ldots \otimes (i_{p+1}, 2) \otimes S_B((r, 1)) \\ & = & (0,1) \otimes (i_2, 0) \otimes \ldots \otimes (i_{p+1}, 0) \otimes S_B((r, 0)) \\ y & = & (0,1) \otimes (k_2, \ell_2) \otimes \ldots \otimes (k_{p+1}, \ell_{p+1}) \otimes S_B((t, u)) \\ y' & = & (0,1) \otimes (k_2, 0) \otimes \ldots \otimes (k_{p+1}, 0) \otimes S_B((t, 0)) \\ & = & (0,0) \otimes (k_2, 2) \otimes \ldots \otimes (k_{p+1}, 2) \otimes S_B((t, 1)). \end{array}$$

We check (1) first. For this, take any witness path from x to y.

It follows from Lemma 4.6.6 that we may find such a path consisting of x and y connected by an element $z = (0, 0) \otimes z' = (0, 1) \otimes z''$, where $z' = S_{N^p \otimes B}((r', 1))$ and $z'' = S_{N^p \otimes B}((r', 0))$ for some $r' \in [0, 1]$.

Before showing the full details, here is the idea. Consider the points x, x', and z. These all lie in one and the same copy of $N^p \otimes B$, and so we may drop the outermost $(0, 0)$ from their expressions and apply part (1) of the induction hypothesis and also Corollary 4.5.3. We can also take y, y', and z and drop the outermost $(0, 1)$ from their expressions and use the induction hypothesis. Further, x' and y' each have two expressions, and we can use the induction hypothesis. So in this way, we may get lower bounds on $d(x, z)$, $d(y, z)$, and $d(x', y')$. Adding these gives a lower bound on the score of the path from x to y using z. We will see that it is $\geq |i^* - k^*| + |j^* - \ell^*|$. Part (2) in this lemma concerns the case when all the points involved are corner points. In this case, we can make a judicious choice of z (namely either x' or y') and match this lower bound. This is how we verify the exact formula for $d(x, y)$ in this case.

4.6.8. CLAIM. *We have* $i^* = \frac{1}{3}i_2^*$, $j^* = \frac{1}{3}j_2^*$, $k^* = \frac{1}{3}k_2^*$, *and* $\ell^* = \frac{1}{3} + \frac{1}{3}\ell_2^*$. *Moreover, the following hold:*

$$\begin{array}{rcl} d_{N^{p+1}\otimes B}(x,x') & \geq & \frac{1}{3} - j^* \\ d_{N^{p+1}\otimes B}(y,y') & \geq & \frac{1}{3}\ell_2^* \\ d_{N^{p+1}\otimes B}(x',y') & \geq & \frac{1}{3}|i_2^* - k_2^*| = |i^* - k^*| \end{array}$$

PROOF. The first assertions are easy from the definitions of the $*$ notation; in the last one, we use the fact that $\ell_1 = 1$. All remaining assertions are proved similarly, and so we only go into details about the first assertion. Let $w, w' \in N^p \otimes B$ be as below, so that $x = (0,0) \otimes w$, and $x' = (0,0) \otimes w'$.

$$\begin{array}{rcl} w & = & (i_2, j_2) \otimes \ldots \otimes (i_{p+1}, j_{p+1}) \otimes S_B((r,s)) \\ w' & = & (i_2, 2) \otimes \ldots \otimes (i_{p+1}, 2) \otimes S_B((r,1)) \end{array}$$

By our induction hypothesis on p,

$$\begin{array}{rcl} d_{N^p\otimes B}(w,w') & \geq & |i_2^* - i_2^*| + |j_2^* - (\frac{1}{3^p} + \sum_{i=1}^p \frac{2}{3^i})| \\ & = & 0 + |j_2^* - 1| \\ & = & 1 - j_2^*. \end{array}$$

The first inequality follows from the induction hypothesis and Definition 4.6.2. The second line is because

$$\frac{1}{3^p} + \sum_{i=1}^{p} \frac{2}{3^i} = \frac{1}{3^p} + \frac{2}{3}\left(\frac{1 - \frac{1}{3^p}}{1 - \frac{1}{3}}\right) = \frac{1}{3^p} + \frac{3^p - 1}{3^p} = 1.$$

Finally, $j_2^* \leq 1$ by the same calculation, since $j_i \leq 2$ and $s \leq 1$.

Because x and x' lie in the same copy of $N^p \otimes B$, we may use Corollary 4.5.3 to get the first inequality:

$$d_{N^{p+1}\otimes B}(x,x') = \frac{1}{3}d_{N^p\otimes B}(w,w') \geq \frac{1}{3}(1 - j_2^*) = \frac{1}{3} - j^*.$$

The proofs of the other two parts of this claim are similar applications of the induction hypothesis. ∎

Using the claim,

$$\begin{array}{rcl} d_{N^{p+1}\otimes B}(x,x') + d_{N^{p+1}\otimes B}(y,y') & \geq & \frac{1}{3} - j^* + \frac{1}{3}\ell_2^* \\ & = & (\frac{1}{3} + \frac{1}{3}\ell_2^*) - j^* \\ & = & \ell^* - j^* \\ & = & |j^* - \ell^*| \end{array} \tag{4.23}$$

At the end, we used Lemma 4.6.4: $\ell^* \geq \frac{1}{3} \geq \frac{1}{3}j_2^* = j^*$, which again uses the fact that $j_2^* \leq 1$.

Recall that we had a point $z = (0,0) \otimes z' = (0,1) \otimes z''$. We need some estimates concerning $d(x,z)$ and $d(z,y)$. Let us introduce notation for z' and z'':

$$\begin{array}{rcl} z' & = & (u_2,2)\otimes\ldots\otimes(u_{p+1},2)\otimes S_B((v,1)) \\ z'' & = & (u_2,0)\otimes\ldots\otimes(u_{p+1},0)\otimes S_B((v,0)) \end{array}$$

Our induction hypothesis applies to $w, z' \in N^p \otimes B$. Since $x = (0,0)\otimes w$ and $z = (0,0)\otimes z'$, we have

$$d(x,z) \quad = \quad \tfrac{1}{3}d(w,z') \quad \geq \quad \tfrac{1}{3}|u_2^* - i_2^*| + (\tfrac{1}{3} - \tfrac{1}{3}j_2^*).$$

Similarly,

$$d(y,z) \quad \geq \quad \tfrac{1}{3}|u_2^* - k_2^*| + \tfrac{1}{3}\ell_2^*.$$

Recall that for any real numbers, $|a-b| + |b-c| \geq |a-c|$. We get a lower estimate for the score of our witness path:

$$d(x,z) + d(z,y) \quad \geq \quad \tfrac{1}{3}|i_2^* - k_2^*| + |j^* - \ell^*| \quad = \quad |i^* - k^*| + |j^* - \ell^*|.$$

We also used the calculations which we saw in (4.23). Since $d(x,y)$ is the score of some witness path, by Lemma 4.6.6 we see that indeed

$$d(x,y) \geq |i^* - k^*| + |j^* - \ell^*|. \tag{4.24}$$

We continue with our work under the assumption $(i_1, j_1) = (0,0)$ and $(k_1, l_1) = (0,1)$, turning to part (2). In this case, x and y are corner points. It follows that x' and y' are also corner points. We restate Claim 4.6.8, adding to the assumptions that x, y, x', and y' are corner points, and strengthening the conclusions by replacing $\leq$ with $=$ throughout. The proof goes through because w and w' are again corner points, so we are entitled to use (2) for p on them. In particular, $d(x',y') = |i^* - k^*|$. We then infer an additional fact: $d(x,x') + d(y,y') = |j^* - \ell^*|$. This is shown exactly as in (4.23), but with the $\geq$ assertion replaced by equality. Then by the triangle inequality,

$$\begin{array}{rcl} d(x,y) & \leq & d(x,x') + d(x',y') + d(y',y) \\ & = & d(x,x') + d(y',y) + d(x',y') \quad = \quad |i^* - k^*| + |j^* - \ell^*|. \end{array}$$

By (4.24), we have equality. This shows part (2) in the case that $(i_1, j_1) = (0,0)$ and $(k_1, l_1) = (0,1)$. Similar work applies in the other cases when x and y are in copies of $N^p \otimes B$ which share an edge.

The other cases in this induction step are similar. ∎

We have the following proposition; it will be more important for us going forward than the formula in Lemma 4.6.7.

4.6.9. Proposition. *For all k:*

1. $f_k : CP_k \to U_0$ *(below Definition 4.6.1) is an isometric embedding.*

2. For $m_1, \ldots, m_k, n_1, \ldots, n_k \in N$ and $x, y \in M_0 \subset U_0$ which are corner points,

$$\begin{aligned} & d_{N^k \otimes M_0}(m_1 \otimes \ldots \otimes m_k \otimes x, n_1 \otimes \ldots \otimes n_k \otimes y) \\ = \; & d_{N^k \otimes U_0}(m_1 \otimes \ldots \otimes m_k \otimes x, n_1 \otimes \ldots \otimes n_k \otimes y). \end{aligned}$$

That is, corresponding corner points have the same distance whether we are viewing them in $N^k \otimes M_0$ or $N^k \otimes U_0$.

4.7. The natural transformation ι. Recall that as a set, M is a subset of N. We are next interested in the relation between the two functors $M \otimes -$ and $N \otimes -$.

4.7.1. Proposition. *There is a natural transformation $\iota \colon (M \otimes -) \longrightarrow (N \otimes -)$.*

Proof. For a space X, ι_X is the inclusion of spaces $M \otimes X \longrightarrow N \otimes X$. This is a short map because every witness path in $M \otimes X$ between points is a path between the same points in $N \otimes X$. Naturality is the assertion that the diagram below commutes:

$$\begin{array}{ccc} M \otimes X & \xrightarrow{M \otimes f} & M \otimes Y \\ {\scriptstyle \iota_X}\downarrow & & \downarrow{\scriptstyle \iota_Y} \\ N \otimes X & \xrightarrow[N \otimes f]{} & N \otimes Y \end{array}$$

For each $m \otimes x \in M \otimes X$, the upper passage gives $m \otimes f(x)$, and this is exactly what the lower passage gives. ■

4.8. The Cauchy completion functor. To obtain the final $M \otimes -$ and $N \otimes -$ coalgebras, we will use the technique in [6] of using the completion of the initial algebra. Here we recall some facts about C, the Cauchy completion functor.

Consider a category $\mathcal{C}$ of metric spaces whose morphisms are short maps, and for X an object in $\mathcal{C}$, let CX be its Cauchy completion, where we identify equivalent Cauchy sequences (that is, $(x_i)_i$ and $(y_i)_i$ such that $d_X(x_i, y_i)$ tends to 0). For Cauchy sequences $(x_i)_i$ and $(y_i)_i$ from an object X in $\mathcal{C}$, $d_{CX}((x_i)_i, (y_i)_i) = \lim_{i \to \infty} d_X(x_i, y_i)$, which is well-defined (as it will be 0 for equivalent Cauchy sequences). If $(x_i)_i$ and $(y_i)_i$ are not equivalent, then $d_{CX}((x_i)_i, (y_i)_i) > 0$. For $f : X \to Y$ a morphism in $\mathcal{C}$, let $Cf : CX \to CY$ be defined by $(x_i)_i \mapsto (f(x_i))_i$. Since $(x_i)_i$ is a Cauchy sequence in X and f is a short map, $(f(x_i))_i$ is a Cauchy sequence in Y; this, too, is well-defined. We assume that $\mathcal{C}$ is closed under C and that Cf is a morphism in $\mathcal{C}$ whenever f is. This defines C as a functor on $\mathcal{C}$. Finally, each space X embeds in CX by taking constant sequences, and we have a natural transformation $i : Id \longrightarrow C$.

We specialize all of this to the case when $\mathcal{C}$ is SquaMS.

4.8.1. Lemma. SquaMS *is closed under C. C may be considered as an endofunctor on* SquaMS. *As such, $i : Id \longrightarrow C$ is a natural transformation.*

PROOF. Let X be an object in SquaMS and consider CX. CX is a metric space bounded by 2, since $d_{CX}((x_i)_i,(y_i)_i) = \lim_{i\to\infty} d_X(x_i,y_i) \leq 2$ since $d(x_i,y_i) \leq 2$ for all i.

We endow the set CX with the square set structure $S_{CX} = i_X \circ S_X$. Since i_X and S_X are injective, so is S_{CX}.

Since i_X is an isometric embedding, CX is not only a square set, it is a square metric space.

For example, to verify the first requirement of (SQ$_1$) in Definition 3.0.3, let $i \in 0,1$ and $r,s \in [0,1]$.

$$\begin{array}{rcl} d_{CX}(S_{CX}((i,r)),S_{CX}((i,s))) & = & d_{CX}(i_X(S_X((i,r))),i_X(S_X((i,s)))) \\ & = & d_{CX}((S_X((i,r)))_k,(S_X((i,s)))_k) \\ & = & \lim_{k\to\infty} d_X(S_X((i,r)),S_X((i,s))) \\ & = & d_X(S_X((i,r)),S_X((i,s))) \\ & = & |s-r|. \end{array}$$

The other condition in (SQ$_1$) and the requirements of (SQ$_2$) follow from a similar argument.

If $f : X \longrightarrow Y$ is a morphism of square spaces, then $f \circ S_X = S_Y$. And so

$$Cf \circ S_{CX} = Cf \circ i_X \circ S_X = i_Y \circ f \circ S_X = i_Y \circ S_Y = S_{CY}.$$

We are using the naturality of i between endofunctors on $\mathcal{C}$. Thus, C is an endofunctor on SquaMS. The same calculation shows that $i : Id \longrightarrow C$ is a natural transformation between functors on square spaces. ∎

We aim to show that up to isomorphism, $M\otimes -$ and $N\otimes -$ commute with C. We will show the result for $M\otimes -$, but the proof for $N\otimes -$ is the same. For any object X in $\mathcal{C}$, consider

$$M\otimes C(X) \xrightarrow{\delta_X^M} C(M\otimes X) \xrightarrow{\rho_X^M} M\otimes C(X).$$

given by

$$\begin{array}{rcl} \delta_X^M(m\otimes(x_0,x_1,\ldots)) & = & (m\otimes x_0, m\otimes x_1,\ldots) \\ \rho_X^M((m_k\otimes x_k)_k) & = & m^*\otimes(x_{k_0},x_{k_1},\ldots), \end{array}$$

where m^* is the first index in M (via some order of the finite set M) which occurs infinitely many times in $(m_k\otimes x_k)_k$ and $x_{k_0}, x_{k_1},\ldots$ are the corresponding elements of X.

4.8.2. LEMMA. *δ^M and ρ^M are natural isomorphisms.*

PROOF. It is routine to check that for all X in $\mathcal{C}$, δ_X^M and ρ_X^M are well-defined, that they are inverse functions (modulo equivalence of Cauchy sequences), that they are short maps, and thus, isometries.

We need to check that δ_X^M and ρ_X^M preserve S_X. For δ_X^M, let $(r,s) \in M_0$ and consider $S_{M\otimes C(X)}((r,s))$. Then for some m and (r',s') which do not depend on $C(X)$, $S_{M\otimes C(X)}((r,s)) = m\otimes S_{C(X)}((r',s'))$. $S_{C(X)}((r',s'))$ can be viewed as the limit of the constant sequence $(S_X((r',s')))$. So $\delta_X^M(S_{M\otimes C(X)}((r,s))) = \delta_X^M(m\otimes(S_X((r',s')))) =$

$(m \otimes S_X((r', s')))$, which is equal to the constant sequence $(S_{M\otimes X}((r, s)))$, whose limit is $S_{C(M\otimes X)}((r, s))$, as required.

For ρ^M_X, if $(r, s) \in M_0$, we can view $S_{C(M\otimes X)}((r, s))$ as the limit of the constant sequence $(S_{M\otimes X}((r, s)))$, which is equal to the constant sequence $(m \otimes S_X((r', s')))$ for some $m \in M$ and $(r', s') \in M_0$ only depending on (r, s). Then $\rho^M_X(S_{C(M\otimes X)}((r, s))) = m \otimes (S_X((r', s'))) = m \otimes S_{C(X)}((r', s')) = S_{M\otimes C(X)}((r, s))$. ■

We get analogous natural isomorphisms δ^N and ρ^N for $N \otimes -$ defined in the same way. Thus, we have the following.

4.8.3. PROPOSITION. *For X in* SquaMS*, $\delta^M_X : M \otimes C(X) \to C(M \otimes X)$ and $\delta^N_X : N \otimes C(X) \to C(N \otimes X)$ are isomorphisms.*

5. The initial algebra of $M \otimes -$ obtained as the colimit of its initial algebra ω-chain

The overall message of this paper is that the Sierpinski carpet as a metric space is bilipschitz equivalent to a final coalgebra of the endofunctor $M \otimes -$ on the category of square metric spaces. However, to show this, we need a lot of material on a dual concept, *initial algebras*. It turns out that in our setting the final coalgebra is the Cauchy completion of the initial algebra.

5.0.1. DEFINITION. Let $\mathcal{A}$ be a category and $F : \mathcal{A} \longrightarrow \mathcal{A}$ an endofunctor. An *algebra for F* is a pair (A, f), where A is an object, and $f : FA \longrightarrow A$ is a morphism. We call A the *carrier* and f the *structure (morphism)*. A *pre-fixed point* of F is an algebra whose structure is a monomorphism.

Let (A, f) and (B, g) be algebras for F. An *algebra morphism* from (A, f) to (B, g) is a morphism $\phi : A \longrightarrow B$ in $\mathcal{A}$ such that $\phi \circ f = g \circ F\phi$:

$$\begin{array}{ccc} FA & \xrightarrow{f} & A \\ {\scriptstyle F\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ FB & \xrightarrow[g]{} & B \end{array}$$

This gives a category $\mathsf{Alg}\,F$ of F-algebras, and an *initial algebra* is an initial object in $\mathsf{Alg}\,F$. As expected, if such an algebra exists at all, it is unique up to isomorphism in $\mathsf{Alg}\,F$.

We recall a standard result in category theory, *Lambek's Lemma*: if (A, f) is an initial algebra, then f is an isomorphism in the base category $\mathcal{A}$.

5.1. A PRE-FIXED POINT OF $M \otimes -$. The main result of this section is the existence of an initial algebra $M \otimes G \to G$ in SquaMS. Before we start in on that, we exhibit a pre-fixed point related to the topic of this paper. Let $U_0 = [0, 1]^2$, equipped with the taxicab metric d_{U_0}, where

$$d_{U_0}((x,y),(x_1,y_1)) = |x - x_1| + |y - y_1|.$$

Define $\alpha_M : M \otimes U_0 \to U_0$ by

$$(i,j) \otimes (r,s) \mapsto (\tfrac{1}{3}(i+r), \tfrac{1}{3}(j+s)). \tag{5.1}$$

The following result is not immediate because the metrics are different in U_0 and $M \otimes U_0$.

5.1.1. LEMMA. *The map* $\alpha_M : M \otimes U_0 \to U_0$ *is a monomorphism of* SquaMS. *Thus,*

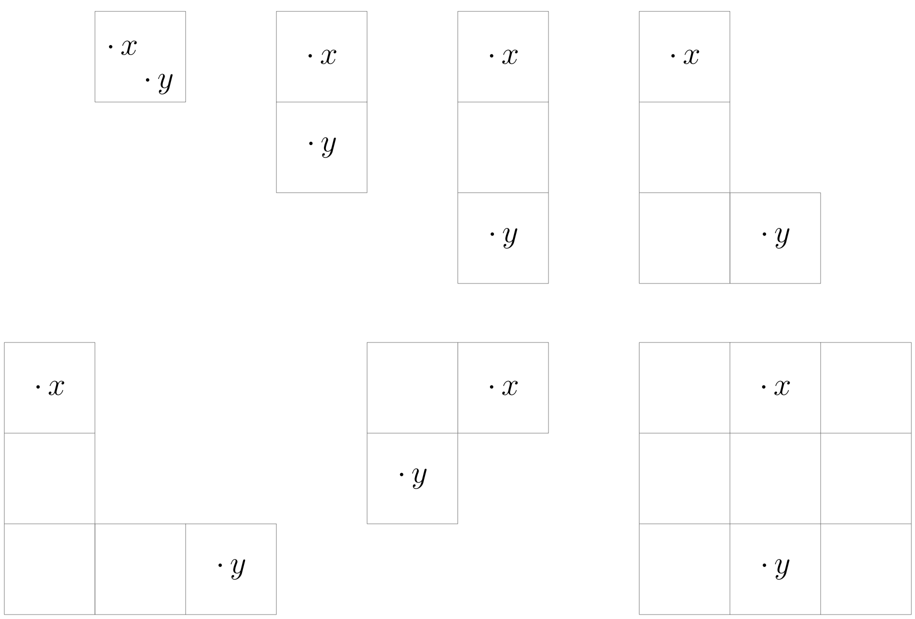
$$(U_0, \alpha_M : M \otimes U_0 \to U_0)$$

is a pre-fixed point of $M \otimes -$.

PROOF. First, it is easy to verify using the equivalences in E that α_M preserves $S_{M \otimes U_0}$.

We next show that α_M is injective. To begin, if $\alpha_M((i,j) \otimes (r,s)) = \alpha_M((k,l) \otimes (t,u))$, when $(i,j) = (k,l)$, we must have $(r,s) = (t,u)$. Otherwise, by examining cases we check that for any possible combination of (i,j) and (k,l), this equality forces (r,s) and (t,u) to be such that $(i,j) \otimes (r,s)$ and $(k,l) \otimes (t,u)$ are equal under the equivalence relation E.

We next check that α_M is a short map. Let $x = (i,j) \otimes (r,s)$ and $y = (k,l) \otimes (t,u)$ in $M \otimes U_0$. Then x and y fall into one of the following cases (up to possible rotation and reflection).



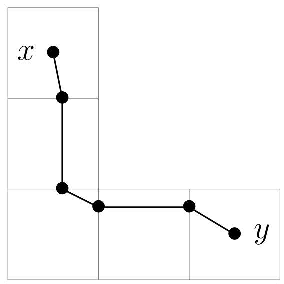
In each case it is reasonably routine to verify that $d_{M \otimes U_0}(x,y) \geq d_{U_0}(\alpha_M(x), \alpha_M(y))$, but we will examine one of these cases carefully, the one indicated in the lower-left corner.

Suppose that $(i,j) = (0,2)$ and $(k,l) = (2,0)$, as shown. Note that

$$d_{U_0}(\alpha_M(x), \alpha_M(y)) = \tfrac{1}{3}|(i+r) - (k+t)| + \tfrac{1}{3}|(j+s) - (l+u)|.$$

Then, without loss of generality, the shortest path in $M \otimes U_0$ between x and y is of the following form:

$$\begin{array}{l} x = (0,2) \otimes (r,s), (0,2) \otimes (r_1, 0) = (0,1) \otimes (r_2, 1), \\ (0,1) \otimes (r_3, 0) = (0,0) \otimes (r_4, 1), (0,0) \otimes (1, s_1) = (1,0) \otimes (0, s_2), \\ (1,0) \otimes (1, s_3) = (2,0) \otimes (0, s_4), (2,0) \otimes (t,u) = y \end{array}$$



We estimate the score of this path. First, we consider the horizontal components from each scaled copy of U_0. Their contribution to the score is

$$\begin{array}{rl} \geq & \frac{1}{3}|r_1 - r| + \frac{1}{3}|r_2 - r_1| + \frac{1}{3}|1 - r_2| + \frac{1}{3}|1 - 0| + \frac{1}{3}|t - 0| \\ \geq & \frac{1}{3}|2 + (t - r)| \\ = & \frac{1}{3}|(i + r) - (k + t)| \end{array}$$

(The last equality holds because $i = 0$ and $k = 2$.) Similarly for the vertical components. Thus, $d_{M \otimes U_0}(x,y) \geq d_{U_0}(\alpha_M(x), \alpha_M(y))$.

The other cases are similar.

To conclude the proof, we recall that by Proposition 3.0.9, injective functions give rise to monomorphisms in SquaMS. ■

5.2. Colimits of ω-chains. We apply Theorem 5.2.2, a well-known result in category theory, to construct an initial algebra by taking *the colimit of a certain ω-chain* and verifying that the functor preserves this colimit. We thus begin with a review of the definitions. Even though we are mainly interested in square metric spaces, we find it convenient to work somewhat more generally and also to study the situation in several related categories.

Let $\mathcal{A}$ be a category. An *ω-chain* in $\mathcal{A}$ is a functor from $(\omega, \leq)$ as a category into $\mathcal{A}$. It is determined by an infinite sequence of objects and morphisms of $\mathcal{A}$ indexed by ω:

$$A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} \cdots \qquad A_k \xrightarrow{a_k} A_{k+1} \qquad \cdots \tag{5.2}$$

To turn this into a functor from $(\omega, \leq)$, we must specify *connecting morphisms* $a_{k,\ell}$ for

$k \leq \ell$. We obviously take $a_{k,k} = \mathsf{id}_{A_k}$, and then for $k < \ell$ we take $a_{k,\ell}$ to be the composition $a_{\ell-1} \circ a_{\ell-2} \circ \cdots \circ a_k$.

A *cocone* of (5.2) is a pair $(B, (b_k)_k)$ consisting of an object B together with morphisms $b_k \colon A_k \longrightarrow B$ so that that $b_k = b_\ell \circ a_{k,\ell}$ when $k \leq \ell$. Sometimes we abuse notation slightly and write a cocone as $b_k \colon A_k \longrightarrow B$, but technically a cocone is an object together with a family of morphisms. A *colimit* of the chain (5.2) is a cocone $(C, (c_k)_k)$ with the property that for every cocone $(B, (b_k)_k)$ there is a unique morphism $f \colon C \longrightarrow B$ so that $b_k = f \circ c_k$ for all $k \in \omega$.

5.2.1. Definition. Consider an ω-chain as in (5.2) with connecting morphisms $a_{k,\ell}$. Let $(C, (c_k)_k)$ be a colimit. We say that F *preserves this colimit* if the chain FA_k with connecting morphisms $Fa_{k,\ell}$ has $(FC, (Fc_k)_k)$ as a colimit.

Here is the reason that this is of interest in this paper.

5.2.2. Theorem. [Adámek [1]] *Let $\mathcal{A}$ be a category with initial object 0. Let $F : \mathcal{A} \to \mathcal{A}$ be an endofunctor. Consider the initial-algebra chain*

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2 !} \cdots F^k 0 \xrightarrow{F^k !} F^{k+1} 0 \cdots \tag{5.3}$$

Suppose the colimit $G = colim_{k<\omega} F^k 0$ exists, and write $g_k : F^k 0 \to G$ for the cocone morphism. Suppose that F preserves this colimit. Let $a : FG \to G$ be the unique morphism so that $a \circ Fg_k = g_{k+1}$ for all k. Then (G, a) is an initial algebra.

We are especially concerned with the case $\mathcal{A} = \mathsf{SquaMS}$, $0 = M_0$, and $! = S_{M \otimes M_0} : M_0 \longrightarrow M \otimes M_0$. We shall show that with those choices, the colimit of the initial algebra ω-chain exists, calling on much more general results. Then we shall prove that the functor $M \otimes -$ preserves this colimit.

At various points in this paper we are going to need colimits of other ω-chains in SquaMS. For every $M \otimes -$ coalgebra (B, β), we need the chain below and its colimit.

$$B \xrightarrow{\beta} M \otimes B \xrightarrow{M \otimes \beta} M^2 \otimes B \xrightarrow{M^2 \otimes \beta} \cdots M^k \otimes B \xrightarrow{M^k \otimes \beta} M^{k+1} \otimes B \cdots, \tag{5.4}$$

We need the same colimit with N replacing M, too. We shall prove that the colimit of (5.4) exists and that it is preserved by the functor. For this, we combine general facts about colimits in *pseudo-metric* spaces with facts about the functors $M \otimes -$ and $N \otimes -$ which we have already seen.

We thus make a digression to study colimits of ω-chains in greater generality. We want to explore the colimits in sets, pseudo-metric spaces, metric spaces, square sets, and square metric spaces. In each case, we characterize colimits of ω-chains.

5.3. Colimits of ω-chains in Sets. Suppose that we have an ω-chain in Set

$$A_0 \longrightarrow A_1 \longrightarrow \cdots \tag{5.5}$$

with connecting maps $a_{k,\ell}\colon A_k \longrightarrow A_\ell$. Suppose that we have a set C and a cocone $(c_k)_{k\in\omega}$, where $c_k\colon A_k \longrightarrow C$. Assume the following two properties:

($\mathsf{Set1}$) $C = \bigcup_k c_k[A_k]$, and

($\mathsf{Set2}$) Given $k \in \omega$ and elements $x, y \in A_k$ with $c_k(x) = c_k(y)$, there exists $\ell \geq k$ in ω such that $a_{k,\ell}(x) = a_{k,\ell}(y)$.

Note that ($\mathsf{Set2}$) implies a stronger form of the same statement: if $x \in A_k$ and $y \in A_\ell$ with $k \leq \ell$ and $c_k(x) = c_\ell(y)$, then there is $p \geq \max(\ell, k)$ such that $a_{k,p}(x) = a_{\ell,p}(y)$. Here is how we see this. Notice that $a_{k,\ell}(x) \in A_\ell$. Apply ($\mathsf{Set2}$) to $a_{k,\ell}(x)$ and y as elements of A_ℓ to get some $p \geq \ell$ so that $a_{\ell,p}(a_{k,\ell}(x)) = a_{\ell,p}(y)$. But $a_{\ell,p}(a_{k,\ell}(x)) = a_{k,p}(x)$.

We claim that C with the morphisms $c_k : A_k \to C$ is a colimit. Indeed, suppose that we are given a cocone $b_k\colon A_k \longrightarrow B$. We need to define a cocone morphism $f\colon C \longrightarrow B$ and to prove that it is unique. We define $f(c_k(x)) = b_k(x)$ for all $k \in \omega$ and $x \in A_k$. This is a well-defined function due to our observation in the previous paragraph. It is defined on all of C, by ($\mathsf{Set1}$). It is a cocone morphism by definition. And it is the unique such, since the condition $f \circ c_k = b_k$ gives the definition of f.

Construction To prove the existence of a colimit of (5.5), we only need to find a set C and a cocone $(c_k)_k$ with ($\mathsf{Set1}$) and ($\mathsf{Set2}$). Take the disjoint union $\sum_k A_k$, then take the relation $\equiv$ given by

$$(x, k) \equiv (y, \ell) \quad \text{iff} \quad a_{k,p}(x) = a_{\ell,p}(y) \text{ for some } p \geq k, \ell$$

In fact, this relation is an equivalence relation. The quotient $C = (\sum_k A_k)/\!\equiv$ is then the colimit, with maps $c_k = A_k \longrightarrow \sum_k A_k \longrightarrow C$. Conditions ($\mathsf{Set1}$) and ($\mathsf{Set2}$) are immediate.

5.4. Colimits of ω-chains in Pseudo-Metric Spaces. A pseudo-metric on a set X is a *distance function* $d : X \times X \longrightarrow [0, \infty]$ with the following properties: $d(x, x) = 0$, $d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$. However, $d(x, y) = 0$ need not imply $x = y$. A 2-*bounded* space has all distances bounded by 2. Let us consider the category Pseu of 2-bounded *pseudo-metric spaces*. As with the metric space categories in this paper, we take the morphisms in Pseu to be the short maps (also called non-expanding functions). Let $\mathcal{U}\colon \mathsf{Pseu} \longrightarrow \mathsf{Set}$ be the forgetful functor. As mentioned in [3] for the case of 1-bounded spaces, Pseu is also cocomplete. That is, it has all colimits, not just colimits of ω-chains. We only need a special case of this, the result for colimits of ω-chains.

Characterization of colimits of ω-chains in Pseu. Consider a chain

$$(A_0, d_0) \longrightarrow (A_1, d_1) \longrightarrow \cdots \tag{5.6}$$

with connecting short maps $a_{k,\ell}\colon A_k \longrightarrow A_\ell$. Suppose that we have a Pseu-object (C, d_C) with short maps $c_k\colon A_k \longrightarrow C$. Assume the following three properties:

($\mathsf{Pseu1}$) As sets, $C = \bigcup_k c_k[A_k]$, and

(Pseu2) Given $n \in \omega$ and elements $x, y \in A_k$ with $c_k(x) = c_k(y)$, there exists $\ell \geq k$ in ω such that $a_{k,\ell}(x) = a_{k,\ell}(y)$.

(Pseu3) For all k and all $x, y \in A_k$, $d_C(c_k(x), c_k(y)) = \inf_{p \geq k} d_p(a_{k,p}(x), a_{k,p}(y))$.

We claim that (C, d_C) is the colimit of (5.6) in Pseu. Due to (Pseu1) and (Pseu2), the underlying set C is a colimit of the ω-chain in Set obtained by forgetting the pseudo-metric. So, given a cocone $b_k \colon A_k \longrightarrow B$ in Pseu, we have a Set map $f \colon C \longrightarrow B$ (from above) given by $f(c_k(x)) = b_k(x)$. We need only check that this map is short. First take a fixed k and elements $x, y \in A_k$. We want to show that

$$d_B(f(c_k(x)), f(c_k(y))) \leq d_C(c_k(x), c_k(y)).$$

This means that we want

$$d_B(b_k(x), b_k(y))) \leq \inf_{p \geq k} d_p(a_{k,p}(x), a_{k,p}(y)).$$

For this, we can show that for all $p \geq k$,

$$d_B(b_k(x), b_k(y))) \leq d_p(a_{k,p}(x), a_{k,p}(y)).$$

Now $b_p \colon A_p \longrightarrow B$ is short, and $b_p \circ a_{k,p} = b_k$ due to the cocone property. So

$$d_p(a_{k,p}(x), a_{k,p}(y)) \geq d_B(b_p(a_{k,p}(x)), b_p(a_{k,p}(y))) = d_B(b_k(x), b_k(y)).$$

More generally, we need to consider $\ell \leq k$ and elements $x \in A_k$ and $y \in A_\ell$. In this case, $a_{\ell,k}(y) \in A_k$. So by what we just did,

$$d_B(f(c_k(x)), f(c_k(a_{\ell,k}(y)))) \leq d_C(c_k(x), c_k(a_{\ell,k}(y))).$$

But $c_k \circ a_{\ell,k} = c_\ell$. So we have

$$d_B(f(c_k(x)), f(c_\ell(y))) \leq d_C(c_k(x), c_\ell(y)).$$

Construction To prove the existence of a colimit of (5.6), we need only find a space with properties (Pseu1) – (Pseu3) above. Take the colimit C in Set. This ensures (Pseu1) and (Pseu2). Endow this set with the pseudo-metric

$$d^*(x, y) = \inf\{d_k(x', y') : k < \omega, x', y' \in A_k, c_k(x') = x, \text{ and } c_k(y') = y\}.$$

This ensures (Pseu3).

5.5. Colimits of ω-chains in Metric Spaces. Let MS denote the category of 2-bounded metric spaces with short maps as morphisms, and suppose we have a chain in MS:

$$(A_0, d_0) \longrightarrow (A_1, d_1) \longrightarrow \cdots \tag{5.7}$$

with connecting short maps $a_{k,\ell}\colon A_k \longrightarrow A_\ell$. Suppose that we have a *metric space* (C, d_C) with short maps $c_k\colon A_k \longrightarrow C$. Assume the following properties:

(MS1) As sets, $C = \bigcup_k c_k[A_k]$, and

(MS2) For all k and all $x, y \in A_k$, $d_C(c_k(x), c_k(y)) = \inf_{p \geq k} d_p(a_{k,p}(x), a_{k,p}(y))$.

We claim that (C, d_C) is the colimit of (5.7) in MS. Suppose that we have a cocone $b_k\colon A_k \longrightarrow B$ in MS. We want to define $f\colon C \longrightarrow B$ as before, by $f(c_k(x)) = b_k(x)$. To prove that f is well-defined in Set or Pseu, we had used a condition that we do not assume here, so the argument is different.

Suppose that we have k and $x, y \in A_k$ with $c_k(x) = c_k(y)$. We want to show that $b_k(x) = b_k(y)$. (The more general case of having ℓ, k and $x \in A_k$, $y \in A_\ell$ with $c_k(x) = c_\ell(y)$ is treated similarly.) By condition (MS2),

$$\inf_{p \geq k} d_p(a_{k,p}(x), a_{k,p}(y)) = 0 = d_C(c_k(x), c_k(y)).$$

Fix $\varepsilon > 0$. There is some $p \geq k$ so that $d_p(a_{k,p}(x), a_{k,p}(y)) \leq \varepsilon$. Since $b_p\colon A_p \longrightarrow B$ is short,

$$d_B(b_k(x), b_k(y)) = d_B(b_p(a_{k,p}(x)), b_p(a_{k,p}(y))) \leq d_p(a_{k,p}(x), a_{k,p}(y)) \leq \varepsilon.$$

This holds for all $\varepsilon > 0$. So $d_B(b_k(x), b_k(y)) = 0$. Since B is a metric space, $b_k(x) = b_k(y)$.

This proves that f is well-defined. The same argument which we gave for Pseu shows that it is the colimit map in MS.

Construction To prove the existence of a colimit of (5.7), we need only find a metric space with properties (MS1) and (MS2) above. Take the colimit C in Set, and endow it with the same pseudo-metric from before

$$d^*(x, y) = \inf\{d_k(x', y') : k < \omega, x', y' \in A_k, i_k(x') = x, \text{ and } i_k(y') = y\}.$$

Then let $x \sim y$ iff $d^*(x, y) = 0$. This is an equivalence relation, and so we can take the quotient $C/{\sim}$. This quotient is (importantly) a metric space. The natural map $C \longrightarrow C/{\sim}$ does not change any non-zero distances. From this, (MS1) and (MS2) follow easily.

5.5.1. Example. Let $X_n = \{u_n, v_n\}$ be the space with two points and $d_n(u_n, v_n) = 2^{-n}$. Let $a_n : X_n \longrightarrow X_{n+1}$ be the short map given by $a_n(u_n) = u_{n+1}$, and $a_n(v_n) = v_{n+1}$. We thus have an ω-chain of metric spaces, and we take the colimit in Pseu and in MS. In Pseu, the colimit is a pseudo-metric space consisting of two points of distance 0. This is not a metric space. In MS, the colimit is a single point. These examples motivate the difference between the three conditions (Pseu1) –(Pseu3) and the two conditions (MS1)–(MS2).

5.6. Colimits of ω-sequences in SquaSet. Suppose that we have a chain

$$A_0 \longrightarrow A_1 \longrightarrow \cdots$$

in SquaSet. Suppose that we have a cocone $a_k\colon A_k \longrightarrow C$ in SquaSet, and assume (Set1) and (Set2). Then we claim that our cocone is the colimit in SquaSet. To see this, we need only endow C with a square set structure $S_C\colon M_0 \longrightarrow C$ and also show that given a cocone $b_k\colon A_k \longrightarrow B$ in SquaSet, the colimit map $f\colon C \longrightarrow B$ preserves this structure.

We define S_C by $c_0 \circ S_{A_0} : M_0 \longrightarrow A_0 \longrightarrow C$. To see that this works, note that since $b_0\colon A_0 \longrightarrow B$ is a square space map, $b_0 \circ S_{A_0} = S_B$. Thus

$$f \circ S_C = f \circ c_0 \circ S_{A_0} = b_0 \circ S_{A_0} = S_B.$$

Since S_B is injective and $f \circ S_C = S_B$, S_C is also injective.

5.7. Colimits of ω-sequences in SquaMS. Consider next a chain

$$(A_0, d_0) \longrightarrow (A_1, d_1) \longrightarrow \cdots$$

in SquaMS. Suppose that we have a cocone $a_k\colon A_k \longrightarrow C$ in SquaMS, and assume (MS1) and (MS2). Then we claim that our cocone is the colimit in SquaMS.

We know how to take the colimit C in SquaSet, endowing C with a SquaSet structure. We also know how to take the colimit in MS. So the only remaining point is to check the non-degeneracy requirements (SQ_1) and (SQ_2). To check (SQ_1), let $r, s \in [0,1]$ and consider $S_C((r,0))$ and $S_C((s,0))$ (the other cases are similar). Then

$$d_C(S_C((r,0)), S_C((s,0))) = \inf_{k<\omega} d_k(S_{A_k}((r,0)), S_{A_k}((s,0))) = |r-s|,$$

since each A_k is an object in SquaMS. Similarly, to check (SQ_2), let $(r,s),(t,u) \in M_0$. Then

$$d_C(S_C((r,s)), S_C((t,u))) = \inf_{k<\omega} d_k(S_{A_k}((r,s)), S_{A_k}((t,u))) \geq |r-t| + |s-u|,$$

5.8. $M \otimes -$ preserves colimits of ω-chains. We next show that the functor $M \otimes -$ preserves colimits of ω-chains. This result is used in Section 5.9, where we apply Adámek's Theorem 5.2.2 to construct the initial algebra of this functor.

5.8.1. Lemma. *The endofunctor $M \otimes -$ preserves colimits of ω-chains.*

Proof. Consider a chain

$$(A_0, d_0) \longrightarrow (A_1, d_1) \longrightarrow \cdots$$

in SquaMS, and let its colimit be the space (C, d_C) with colimit cocone $(c_k)_k$, where $c_k : A_k \longrightarrow C$. We are going to show that the colimit of

$$(M \otimes A_0, d_{M\otimes A_0}) \longrightarrow (M \otimes A_1, d_{M\otimes A_1}) \longrightarrow \cdots \tag{5.8}$$

is $(M \otimes C, (M \otimes c_k)_k)$. To begin, we already know that the cocone $(C, (c_k)_k)$ has properties (MS1) and (MS2) for the original chain. We need only check that $(M \otimes C, (M \otimes c_k)_k)$ has these same properties (MS1) and (MS2) for the chain in (5.8).

For (MS1), it is clear that as sets,

$$M \otimes C = M \otimes \bigcup_k c_k[A_k] = \bigcup_k \Big(M \otimes c_k[A_k] \Big) = \bigcup_k (M \otimes c_k)[A_k].$$

For (MS2), we want to show that for all $k \in \omega$ and all $x, y \in A_k$, and all $m, n \in M$,

$$d_{M\otimes C}(m \otimes c_k(x), n \otimes c_k(y)) = \inf_{p \geq k} d_{M \otimes A_p}(m \otimes a_{k,p}(x), n \otimes a_{k,p}(y)). \tag{5.9}$$

We first consider the case when $n = m$. In this case,

$$\begin{aligned} d_{M\otimes C}(m \otimes c_k(x), m \otimes c_k(y)) &= \tfrac{1}{3} d_C(c_k(x), c_k(y)) \\ &= \tfrac{1}{3} \inf_{p \geq k} d_p(a_{k,p}(x), a_{k,p}(y)) \\ &= \inf_{p \geq k} \frac{1}{3} d_p(a_{k,p}(x), a_{k,p}(y)) \\ &= \inf_{p \geq k} d_{M\otimes A_p}(m \otimes a_{k,p}(x), m \otimes a_{k,p}(y)). \end{aligned}$$

With this special case done, we consider the general case. We use the fact from Theorem 4.1.12 that in $M \otimes C$, there is a fixed path that attains the distance between our points $m \otimes c_k(x)$ and $n \otimes c_k(y)$. This path has finitely many sub-paths (at most 5 in fact), and each subpath is in one and the same copy of C. It follows from our first observation that (5.9) holds.

This concludes the proof. ∎

We also have a result exactly like Lemma 5.8.1 but for the functor $N \otimes -$. The details are basically the same.

5.9. Using colimits to obtain the initial algebras of $M \otimes -$ and $N \otimes -$. At this point, we recall Adámek's Theorem (Theorem 5.2.2), and apply this to SquaMS, with F either $M \otimes -$ or $N \otimes -$. As we know, colimits of all ω-chains exist in our category. We are of course interested in the colimit of the initial-algebra chain (5.3). The functors preserve this colimit, since they preserves all colimits of ω-chains. Thus, there is are initial algebras. We write these as

$$\begin{array}{l} (G, \eta\colon M \otimes G \longrightarrow G) \\ (W, \lambda\colon N \otimes W \longrightarrow W) \end{array} \tag{5.10}$$

In both cases, the algebra structures are isometries, by Lambek's Lemma.

Further, the colimit morphisms are given by the natural equivalence relations. For example, consider M (the functor where we need this remark). We have

$$g_k\colon M^k \otimes M_0 \to G \tag{5.11}$$

given by $g_k(x) = [x]$, where the equivalence relation involved here relates, for $l \leq m$, $y \in M^\ell \otimes M_0$ with $z \in M^m \otimes M_0$ iff $a_{\ell,m}(y) = z$, where $a_{\ell,m}\colon M^\ell \otimes M_0 \longrightarrow M^m \otimes M_0$ is the evident map.

6. Final coalgebras for $N \otimes -$ and $M \otimes -$

This section discusses final coalgebras for the two main functors in this paper, $N \otimes -\colon \mathsf{SquaMS} \longrightarrow \mathsf{SquaMS}$, and $M \otimes -\colon \mathsf{SquaMS} \longrightarrow \mathsf{SquaMS}$. The main results are that the unit square U_0 with the taxicab metric is a final coalgebra for $N \otimes -\colon \mathsf{SquaMS} \longrightarrow \mathsf{SquaMS}$, and that this coalgebra is the Cauchy completion of the initial algebra. Turning to $M \otimes -$, we show that again the Cauchy completion of the initial algebra is the final coalgebra. It would have been pleasing if this final coalgebra had been the Sierpinski carpet $\mathbb{S}$. But this is not to be: the bijective map $\mathbb{S} \longrightarrow M \otimes \mathbb{S}$ is not a short map. Nevertheless, we shall prove later than $\mathbb{S}$ is bilipschitz equivalent to the carrier of final coalgebra of $M \otimes -$. In a different direction, forgetting the metric, $\mathbb{S}$ is the final coalgebra of our functor on $\mathsf{SquaSet}$.

6.0.1. Definition. Let $H\colon \mathcal{A} \longrightarrow \mathcal{A}$ be an endofunctor on any category. A *coalgebra for* H is a pair (A, a), where $a\colon A \longrightarrow HA$. Given two coalgebras (A, a) and (B, b) for this functor, a *coalgebra morphism* is a morphism $h\colon A \longrightarrow B$ in $\mathcal{A}$ such that $b \circ h = Hh \circ a$:

$$\begin{array}{ccc} A & \xrightarrow{\;a\;} & HA \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Hh} \\ B & \xrightarrow[\;b\;]{} & HB \end{array}$$

(A, a) is a *final coalgebra* if for every coalgebra (X, e) there is a unique coalgebra morphism $e^\dagger : X \to A$. Equivalently, it is a final object in the category of coalgebras.

Final coalgebras need not exist, but when they do, they are unique up to isomorphism. Moreover, if (C, γ) is a final coalgebra, then by Lambek's Lemma (the dual of the form that we stated earlier), γ is an isomorphism in the base category $\mathcal{A}$.

6.1. Corecursive algebras. Our work on final coalgebras involves a secondary notion: *corecursive algebras*. We bring corecursive algebras into the paper because they generalize final coalgebras and because the Sierpinski carpet turns out to be a corecursive algebra in SquaMS.

6.1.1. Definition. Let $H\colon \mathcal{A} \longrightarrow \mathcal{A}$ be an endofunctor on any category. An algebra $a\colon HA \longrightarrow A$ is *corecursive* if for every coalgebra $e\colon X \longrightarrow HX$ there is a unique *coalgebra-*

to-algebra morphism $e^\dagger \colon X \longrightarrow A$. This means that $e^\dagger = a \circ He^\dagger \circ e$:

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & HX \\ {\scriptstyle e^\dagger}\downarrow & & \downarrow{\scriptstyle He^\dagger} \\ A & \xleftarrow[\;a\;]{} & HA \end{array}$$

The map $e^\dagger$ is also called *the solution to* e *in the algebra* (A, a).

The following is the dual form for Proposition 7 in [7].

6.1.2. Proposition. *If a corecursive H-algebra (A, a) has an invertible structure map a, then (A, a^{-1}) is a final coalgebra for the same functor. If (A, a) is a final coalgebra, then (A, a^{-1}) is a corecursive algebra.*

6.1.3. Lemma. *Let $e\colon X \longrightarrow HX$ and $f\colon Y \longrightarrow HY$ be coalgebras, and let $h\colon X \longrightarrow Y$ be a coalgebra morphism. Let $a\colon HA \longrightarrow A$ be a corecursive algebra. Then $e^\dagger = f^\dagger \circ h$.*

The proof of this may be found in Example 3.2 in [5].

Recall that $N = \{0, 1, 2\}^2$, and that $U_0 = [0, 1]^2$. We are going to consider the functor $H_0 X = N \times X$ on Set.

Recall from the previous section our definition of $\alpha_M : M \otimes U_0 \to U_0$, which we proved was an injective morphism in Lemma 5.1.1. Here we will introduce some notation towards defining an analogous morphism $\alpha_N : N \otimes U_0 \to U_0$.

Let $\mathsf{shrink}\colon N \longrightarrow U_0$ be given by

$$\mathsf{shrink}(i, j) = (\tfrac{1}{3}i, \tfrac{1}{3}j).$$

We have an H_0-algebra structure $\alpha_0\colon N \times U_0 \longrightarrow U_0$ given by

$$\alpha_0((i, j), (x, y)) = \mathsf{shrink}(i, j) + \tfrac{1}{3}(x, y).$$

6.1.4. Lemma. $(U_0, \alpha_0\colon N \times U_0 \longrightarrow U_0)$ *is a corecursive algebra for H_0 on* Set.

Proof. Although it is possible to give a self-contained elementary proof, this result also follows from Corollary 2.11 in [4] (see also [2, Example 7.3.10]). We must check a few hypotheses to apply that result. We discuss these one-by-one.

Let CMS be the category of complete metric spaces with distances bounded by 2. We have a forgetful functor $U : \mathsf{CMS} \longrightarrow \mathsf{Set}$.[5] We verify three hypotheses.

First, the functor $H_0 = N \times X : \mathsf{Set} \longrightarrow \mathsf{Set}$ lifts to CMS. This has nothing to do with our specific set N, it holds for all sets N. Here is what this means. Consider N as a discrete space with distance 2 between all points. Then we have a functor $H_1\colon \mathsf{CMS} \longrightarrow \mathsf{CMS}$ given

[5] A forgetful functor is standardly denoted by U. For us, this has an unfortunate clash with our notation U_0 for the unit square. Bringing this to the reader's attention should help avoid any confusion.

by $H_1 X = N \times X$, with the metric defined as follows:

$$d(((i,j),(x,y)),((i',j'),(x',y'))) = \begin{cases} 2 & \text{if } (i,j) \neq (i',j') \\ \frac{1}{3} d_{U_0}((x,y),(x',y')) & \text{if } (i,j) = (i',j') \end{cases}$$

H_1 works as expected on morphisms. The lifting property is that $U \circ H_1 = H_0 \circ U$, and this is easy to check.

Second, this lifted functor H_1 is *locally contracting*. Indeed, for all "parallel pairs" of CMS-morphisms $f, g : X \longrightarrow Y$, $d(H_1 f, H_1 g) = \frac{1}{3} d(f,g)$. This is a routine verification using the supremum metric on function spaces and the distance formula above.

Finally, the Set-morphism $\alpha_0 \colon N \times U_0 \longrightarrow U_0$ also is a CMS-morphism $\alpha_0 \colon H_1 U_0 \longrightarrow U_0$. This means that α_0 is short. To check this, take two elements of $H_1 U_0$, say $p = ((i,j),(x,y))$ and $p' = ((i',j'),(x',y'))$. If $(i,j) \neq (i',j')$, then their distance in N is 2, and hence the distance between p and p' is also 2. But the distance between $\alpha(p)$ and $\alpha(p')$ is at most 2. In the other case, $(i,j) = (i',j')$. In this case,

$$d_{H_1 U_0}(\alpha(p), \alpha(p')) = \tfrac{1}{3} d_{U_0}((x,y),(x',y')) = d_{U_0}(\tfrac{1}{3}(x,y), \tfrac{1}{3}(x',y')).$$

These hypotheses then imply that (U_0, α_0) is a corecursive algebra for H_0 on Set. ∎

Lemma 6.1.4 was a preliminary result; the main point is Lemma 6.1.8, its adaptation for the category $\mathsf{SquaSet}$ of square sets.

6.1.5. Definition. Let $\alpha_N \colon N \otimes U_0 \longrightarrow U_0$ be given by

$$\alpha_N(n \otimes z) \quad = \quad \mathsf{shrink}(n) + \tfrac{1}{3}(z).$$

Notice that $n \in N$ here is a pair; earlier we had written it as (i,j). Similarly, $z \in U_0$; earlier we wrote it as (r,s). It takes quite a few routine elementary calculations to check that α_N is well-defined. That is, we must check that if $(n,z) \approx (n',z')$, then $\mathsf{shrink}(n) + \frac{1}{3}(z) = \mathsf{shrink}(n') + \frac{1}{3}(z')$. For example, we have $((0,0),(r,1)) \approx ((0,1),(r,0))$. And

$$\mathsf{shrink}(0,0) + \tfrac{1}{3}(r,1) = (\tfrac{r}{3}, \tfrac{1}{3}) = \mathsf{shrink}(0,1) + \tfrac{1}{3}(r,0).$$

Furthermore, it is easy to verify that α_N preserves $S_{N \otimes U_0}$, so α_N is a $\mathsf{SquaSet}$ morphism.

6.1.6. Lemma. *In* SquaMS*,* $\alpha_N \colon N \otimes U_0 \longrightarrow U_0$ *is an isomorphism: it maps* $N \otimes U_0$ *one-to-one onto* U_0*, and it is an isometry.*

Proof. Clearly α_N is surjective: given $(r,s) \in U_0$, let $(i,j) \in N$ be the greatest in the lexicographic order such that $\frac{1}{3} i \leq r$ and $\frac{1}{3} j \leq s$. Then $\alpha_N((i,j) \otimes (3r-i, 3s-j)) = (r,s)$.

To see that α_N is injective, we will show that it an isometry.

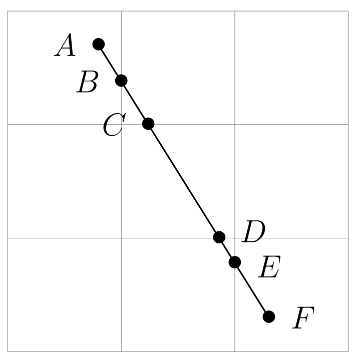
First, let us check that α_N is a short map. Taking $p = 1$ and $B = U_0$ in (4.22), we see that

$$\begin{array}{rl} & d_{N\otimes U_0}((i,j)\otimes S_{U_0}((r,s)),(k,\ell)\otimes S_{U_0}((t,u))) \\ \geq & \big||i;r|-|k;t|\big| - \big||j;s|-|\ell;u|\big| \\ = & \left|\frac{1}{3}(i+r)-\frac{1}{3}(k+t)\right| - \left|\frac{1}{3}(j+s)-\frac{1}{3}(\ell+u)\right| \\ = & d_{\mathsf{Taxi}}((\frac{1}{3}(i+r),\frac{1}{3}(j+s)),(\frac{1}{3}(k+t),\frac{1}{3}(\ell+u))) \\ = & d_{\mathsf{Taxi}}(\alpha_N((i,j)\otimes(r,s)),\alpha_N((k,\ell)\otimes(t,u))). \end{array}$$

Now to see that this is an isometry, consider $\alpha_N(x)$ and $\alpha_N(y)$ in U_0. The idea is that we can introduce a grid to U_0 which corresponds to the boundaries of copies of U_0 in $N\otimes U_0$, and look at the intersections of a segment between $\alpha_N(x)$ and $\alpha_N(y)$ with that grid. We then use this to construct a path in $N\otimes U_0$ between x and y whose distance is equal to that between $\alpha_N(x)$ and $\alpha_N(y)$ in U_0, and this will be an upper bound of the distance between x and y in $N\otimes U_0$.

Rather than work through the thick notation of a general case, we will present the following illustrative example.

We consider



These are points in $N\otimes U_0$, shown explicitly on the left below. The column on the right gives their images under α_N, as elements of U_0.

$$\begin{array}{lllllllll} A & = & (0,2)\otimes(.8,.7) & & & \alpha_N(A) & = & (.267,.9) \\ B & = & (0,2)\otimes(1,.38) & = & (1,2)\otimes(0,.38) & \alpha_N(B) & = & (.333,.793) \\ C & = & (1,2)\otimes(.2375,0) & = & (1,1)\otimes(.2375,1) & \alpha_N(C) & = & (.4125,.667) \\ D & = & (1,1)\otimes(.8625,0) & = & (1,0)\otimes(.8625,1) & \alpha_N(D) & = & (.621,.333) \\ E & = & (1,0)\otimes(1,.78) & = & (2,0)\otimes(0,.78) & \alpha_N(E) & = & (.667,.26) \\ F & = & (2,0)\otimes(.3,.3) & & & \alpha_N(F) & = & (.767,.1) \end{array}$$

The way we got these was to find the line between $\alpha_N(A)$ and $\alpha_N(F)$, then to find the intersection points of this line with the relevant grid lines, and finally to find the preimages under α_N. For B, C, D, and E, we have two preimages.

We are going to verify that

$$d_{U_0}(\alpha_N(A), \alpha_N(F)) \geq d_{N\otimes U_0}(A, F).$$

Recall, we are using the taxicab metric in U_0 (2.3). So

$$d_{U_0}(\alpha_N(A), \alpha_N(F)) = |.767 - .267| + |.1 - .9| = .5 + .8 = 1.3.$$

To show that $d_{N\otimes U_0}(A, F) \geq 1.3$, we find an alternating path (a sequence of points in $N \times U_0$ as described in Definition 4.1.5) from A to F and check that the score of this path is again 1.3. The alternating path we want is suggested by A, ..., F. It is

$$\begin{array}{l} ((0,2),(.8,.7)),((0,2),(1,.38)) \sim ((1,2),(0,.38)), \\ ((1,2),(.2375,0)) \sim ((1,1),(.2375,1)), \\ ((1,1),(.8625,0)) \sim ((1,0),(.8625,1)), \\ ((1,0),(1,.78)) \sim ((2,0),(0,.78)),((2,0),(.3,.3)). \end{array}$$

The score of this alternating path is

$$\tfrac{1}{3}(.2 + .32 + .2375 + .38 + .625 + 1 + .1375 + .22 + .3 + .48).$$

The relationship between this and our calculation of $d_{U_0}(\alpha_N(A), \alpha_N(F))$ is clarified if we separate the horizontal and vertical contributions. Our score above is

$$\begin{array}{rl} & \tfrac{1}{3}((.2 + .2375 + .625 + .1375 + .3) + (.32 + .38 + 1 + .22 + .48)) \\ = & \tfrac{1}{3}(1.5 + 2.4) \\ = & 1.3 \end{array}$$

This is as desired. This all is merely an example, but the general case is similar. We conclude that $d_{N\otimes U_0}(x, y) \leq d_{U_0}(\alpha_N(x), \alpha_N(y))$.

Thus, α_N is an isometry, so it is injective, and hence an isomorphism in SquaMS. ∎

For every SquaSet B there is a canonical quotient map in Set, $\nu_B : N \times B \longrightarrow N \otimes B$. It is given by $\alpha_X((n, x)) = n \otimes x$.

6.1.7. Proposition. *For every* SquaSet *morphism $f : B \longrightarrow C$ the evident "naturality square" commutes:* $(N \otimes f) \circ \nu_B = \nu_C \circ (N \times f)$.

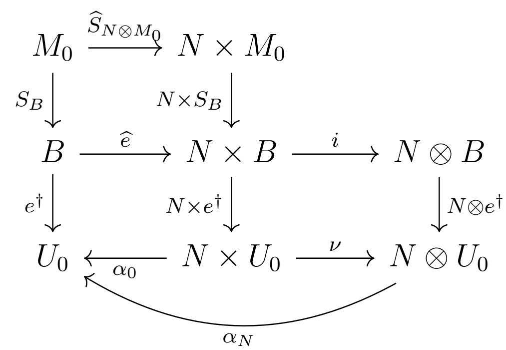
Proof. The for $(n, x) \in N \times X$,

$$(\eta_Y \circ (N \times f))((n, x)) = n \otimes f(x)((N \otimes f) \circ \eta_X)(n, x).$$

∎

6.1.8. Lemma. $(U_0, \alpha_N : N \otimes U_0 \longrightarrow U_0)$ *is a corecursive algebra for $N \otimes X$ on* SquaSet.

Proof. We are given a coalgebra $e : B \longrightarrow N \otimes B$, and it is our task to show that there is a unique $e^\dagger : B \longrightarrow U_0$ in SquaSet such that $e^\dagger = \alpha_N \circ (N \otimes e^\dagger) \circ e$. We first make a diagram in Set:



S_B and α_N are SquaSet morphisms, and so what we mean above is the same maps in Set. The maps ν_B and ν_{U_0} are the ones we saw in Proposition 6.1.7. We will discuss the map $N \otimes e^\dagger$ later, after we define $e^\dagger$ and verify that it is a square set morphism.

Please note that we have changed the notation on the one of the "hat" map, writing $\widehat{S}_{N\otimes M_0}$ instead of $S_{\widehat{N\otimes M_0}}$.

The morphisms $\widehat{e}$ and $\widehat{S}_{N\otimes M_0}$ are defined in a canonical way, as follows. Fix an ordering $<$ on N, say the lexicographic order. First, consider $\widehat{e}$. Let $\widehat{e}(b)$ be any pair $(n, b') \in N \times B$ such that $e(b) = [\widehat{e}(b)]$ and n is $<$-least in N such that some b' exists with this property. This defines n uniquely, and it is easy to see that b' is also unique. This is because if $(n, b) \approx (n, b')$, then $b = b'$. We see easily that $e = i \circ \widehat{e}$. The morphism $\widehat{S}_{N\otimes M_0}$ is similar. As an example, $\widehat{S}_{N\otimes M_0}(1/3, 0) = ((0, 0), (1, 0))$. By the way, despite the notation, $\widehat{S}_{N\otimes M_0}$ is a morphism in Set here.

By Proposition 6.1.4 applied to $\widehat{e}$, we get $(\widehat{e})^\dagger$, making the square in the corner commute. We will shorten this to $e^\dagger$, as this will turn out to be the SquaSet morphism we want. The definitions of $\widehat{e}$ and $\widehat{S}_{N\otimes M_0}$ and the fact that e is a SquaSet morphism imply that the square in the upper-left commutes. For example, consider $(\frac{1}{3}, 0) \in M_0$. Now $(N \times S_B) \circ \widehat{S}_{N\otimes M_0}((\frac{1}{3}, 0)) = N \times S_B(((0, 0), (1, 0))) = ((0, 0), S_B((1, 0)))$ because $((0, 0), S_B((1, 0)))$ is the first representative of the class $S_{N\otimes B}((\frac{1}{3}, 0))$ according to the lexicographic order on N. Similarly, $\widehat{e} \circ S_B((\frac{1}{3}, 0)) = \widehat{e}(S_B((\frac{1}{3}, 0))) = ((0, 0), S_B((1, 0)))$ since $(0, 0) \otimes S_B((1, 0))$ is the first representation of $S_{N\otimes B}((\frac{1}{3}, 0))$ according to the lexicographic ordering on N (as opposed to $(1, 0) \otimes S_B((0, 0))$). So we get a coalgebra morphism for the functor $N \times -$. By Lemma 6.1.3, $\widehat{S}^\dagger_{N\otimes M_0} = e^\dagger \circ S_B$.

We claim that $\widehat{S}^\dagger_{N\otimes M_0} = S_{U_0}$. That is, we claim that S_{U_0} satisfies the corecursive algebra condition which uniquely defines $\widehat{S}^\dagger_{N\otimes M_0}$:

$$S_{U_0} = \alpha_0 \circ (N \times S_{U_0}) \circ \widehat{S}_{N\otimes M_0}$$

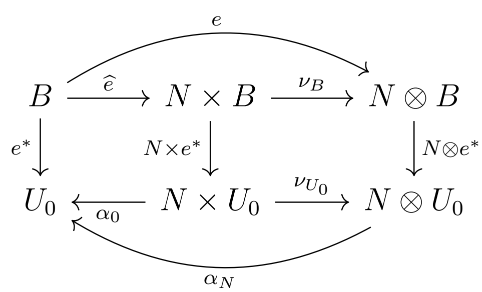
We verify an example. For example, for $1/3 < r \leq 2/3$,

$$
\begin{aligned}
(\alpha_0 \circ (N \times S_{U_0}) \circ \widehat{S}_{N\otimes M_0})(r,0) &= \alpha_0 \circ (N \times S_{U_0})((1,0),(3r-1,0)) \\
&= \alpha_0((1,0), S_{U_0}((3r-1,0))) \\
&= \alpha_0((1,0),(3r-1,0)) \\
&= (r,0) \\
&= S_{U_0}((r,0))
\end{aligned}
$$

All of the other cases are similar. By uniqueness of solutions, S_{U_0} *is* the solution. The upshot is that at this point we know that $e^\dagger \circ S_B = \widehat{S}^\dagger_{N\otimes M_0} = S_{U_0}$, and hence that $e^\dagger$ is a SquaSet morphism.

Now that we know that $e^\dagger$ is a SquaSet morphism, we use the functor $N\otimes -$ on SquaSet to get $N \otimes e^\dagger \colon N \otimes B \longrightarrow N \otimes U_0$; recall that this is defined by $(N \otimes e^\dagger)(n \otimes b) = n \otimes e^\dagger(b)$. The square in the bottom commutes by Proposition 6.1.7. The region on the bottom commutes: $\alpha_N \circ \nu = \alpha_0$. Recalling that $e = \nu_B \circ \widehat{e}$, a diagram chase shows that we have the desired equality $e^\dagger = \alpha_N \circ (N \otimes e^\dagger) \circ e$.

We also check that $e^\dagger$ is the unique solution of e in SquaSet. Suppose that we have a SquaSet morphism e^* so that $e^* = \alpha_N \circ (N \otimes e^*) \circ e$. We show that $e^* = e^\dagger$. Consider the diagram below:

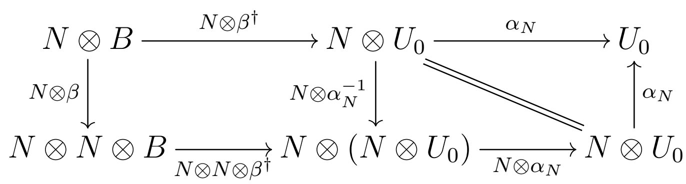


Since e^* is a morphism in SquaSet, we are entitled to write $N \otimes e^*$, as shown. But the diagram above is in Set. The top and bottom commute, as we have seen. The square on the right commutes, easily. The verification here is similar to what we saw in the first part of the proof. And now a diagram chase shows that the square on the left commutes as well. But this means that e^* is a solution to the $N \times -$ coalgebra $(B, \widehat{e})$. And so by uniqueness of solutions in (U_0, α_0), $e^* = e^\dagger$. ∎

The next main result is that (U_0, α_N^{-1}) is a final $N \otimes -$coalgebra in square *metric spaces*. Here the metric on U_0 is the taxicab metric. We need a few preliminary lemmas. In these, we fix an $(N \otimes -)$-coalgebra in SquaMS, $(B, \beta : B \to N \otimes B)$. We already know that there is a unique SquaSet morphism $\beta^\dagger \colon B \longrightarrow U_0$ such that $\beta^\dagger = \alpha_N \circ (N \otimes \beta^\dagger) \circ \beta$. Also, α_N is an isometry (see Lemma 6.1.6) hence α_N^{-1} is short. Our main work in this section shows that $\beta^\dagger$ is short (on all of B), of course using that β is a short map. The surprising feature of our proof is that we must consider other coalgebras in order to prove the shortness of $\beta^\dagger$. Notice that $(N \otimes B, N \otimes \beta)$ is also an $(N \otimes -)$-coalgebra. Furthermore, $\beta \colon B \longrightarrow N \otimes B$ is a coalgebra morphism.

6.1.9. LEMMA. $(N \otimes \beta)^{\dagger} = \alpha_N \circ (N \otimes \beta^{\dagger})$.

PROOF. Consider the diagram below in $\mathsf{SquaSet}$. It makes sense because α_N is invertible (see Lemma 6.1.6).



The triangles commute. The square on the right (rotated 90^o and reflected) then shows that $(N \otimes \alpha_N^{-1})^{\dagger} = \alpha_N$. The square on the left commutes because when we remove N and turn the arrow on the right around (from α_N^{-1} to α_N), we have the definition of $\beta^{\dagger}$. That square thus shows that $N \otimes \beta^{\dagger}$ is a coalgebra morphism. Applying Lemma 6.1.3 to it, we see that

$$(N \otimes \beta)^{\dagger} = (N \otimes \alpha_N^{-1})^{\dagger} \circ (N \otimes \beta^{\dagger}) = \alpha_N \circ (N \otimes \beta^{\dagger}).$$

■

6.1.10. DEFINITION. Let $Z \subseteq B$. We say that $\beta^{\dagger}$ is *short on* Z if for all $b, c \in Z$, $d_{U_0}(\beta^{\dagger}(b), \beta^{\dagger}(c)) \leq d_B(b, c)$.

Also, we write $N \otimes Z$ for $\{n \otimes b : n \in N \text{ and } b \in Z\}$.

6.1.11. LEMMA. *Let $Z \subseteq B$ be any set that includes the image $S_B[M_0]$. If $\beta^{\dagger}$ is short on Z, then $(N \otimes \beta)^{\dagger}$ is short on $N \otimes Z$.*

PROOF. Let $b, c \in Z$ and $n_1, n_2 \in N$. We may assume that $n_1 \neq n_2$, since if $n_1 = n_2$ this follows easily from the fact that α_N is a short map and $\beta^{\dagger}$ is short on Z. There are $(r_1, s_1), (r_2, s_2) \in M_0$ such that a witness path in $N \otimes B$ from $n_1 \otimes b$ to $n_2 \otimes c$ contains $n_1 \otimes S_B((r_1, s_1))$ and $n_2 \otimes S_B((r_2, s_2))$. We are going to write S for S_B to save on some notation. We have

$$\begin{aligned}
& d_{N \otimes B}(n_1 \otimes b, n_2 \otimes c) \\
= \;& d(n_1 \otimes b, n_1 \otimes S((r_1, s_1))) + d(n_1 \otimes S((r_1, s_1)), n_2 \otimes S((r_2, s_2))) \\
& + d(n_2 \otimes S((r_2, s_2)), n_2 \otimes c) \qquad (1) \\
= \;& \tfrac{1}{3} d_B(b, S((r_1, s_1))) + d_{N \otimes B}(n_1 \otimes S((r_1, s_1))), n_2 \otimes S((r_2, s_2)) \\
& + \tfrac{1}{3} d_B(S((r_2, s_2)), c) \qquad (2) \\
\geq \;& \tfrac{1}{3} d_{U_0}(\beta^{\dagger}(b), (r_1, s_1)) + d_{N \otimes U_0}(n_1 \otimes (r_1, s_1), n_2 \otimes (r_2, s_2)) \\
& + \tfrac{1}{3} d_{U_0}((r_2, s_2), \beta^{\dagger}(c)) \qquad (3) \\
\geq \;& d_{U_0}(\mathsf{shrink}(n_1) + \tfrac{1}{3}\beta^{\dagger}(b), \mathsf{shrink}(n_1) + \tfrac{1}{3}(r_1, s_1)) \\
& + d_{U_0}(\mathsf{shrink}(n_1) + \tfrac{1}{3}(r_1, s_1), \mathsf{shrink}(n_2) + \tfrac{1}{3}(r_2, s_2))
\end{aligned}$$

$$
\begin{aligned}
& + d_{U_0}(\mathsf{shrink}(n_2) + \tfrac{1}{3}(r_2, s_2), \mathsf{shrink}(n_2) + \tfrac{1}{3}\beta^\dagger(c)) && (4)\\
\geq \quad & d_{U_0}(\mathsf{shrink}(n_1) + \tfrac{1}{3}\beta^\dagger(b), \mathsf{shrink}(n_2) + \tfrac{1}{3}\beta^\dagger(c)) && (5)\\
= \quad & d_{U_0}(\alpha_N((N \otimes \beta^\dagger)(n_1, b)), \alpha_N((N \otimes \beta^\dagger)(n_2, c))) && (6)\\
= \quad & d_{U_0}((N \otimes \beta)^\dagger(n_1 \otimes b), (N \otimes \beta)^\dagger(n_2 \otimes c)) && (7)
\end{aligned}
$$

In (1), the distances are in $N \otimes B$. (1) holds by the choice of (r_1, s_1) and (r_2, s_2) (such that $n_1 \otimes S_B((r_1, s_1))$ and $n_2 \otimes S_B((r_2, s_2))$ are on a witness path from $n_1 \otimes b$ to $n_2 \otimes c$). In (2), we are using Corollary 4.1.14, the result on distances in a single copy of X inside of $N \otimes X$. (3) uses the assumption that $\beta^\dagger$ is short on Z, and the fact that $\beta^\dagger \circ S_B((r_i, s_i)) = (r_i, s_i)$. It also uses Lemma 4.5.5 in the middle.

(4) uses two facts about distances in U_0. Let $x, y, z \in U_0$. First, $c \cdot d(x, y) = d(c \cdot x, c \cdot y)$ when $0 \leq c \leq 1$. Second, $d(x, y) = d(x + z, y + z)$, provided $x + z$ and $y + z$ belong to U_0. And in the middle summand of (4), we used the fact that $\alpha_N \colon N \otimes U_0 \longrightarrow U_0$ is a short map, and the definition of α_N.

(5) uses the triangle inequality in U_0. (6) uses the definition of α_N and $N \otimes \beta^\dagger$. (7) is by Lemma 6.1.9.

This completes the proof. ∎

6.1.12. Lemma. *Let $(B, \beta \colon B \longrightarrow N \otimes B)$, and let $k \in \omega$. There is a coalgebra*

$$(C, \gamma \colon C \longrightarrow N \otimes C),$$

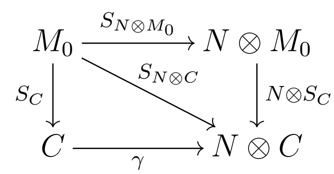
a coalgebra morphism $g \colon B \longrightarrow C$, and a set $Z \subseteq C$ so that

1. *$S_C[M_0] \subseteq Z$.*
2. *$\gamma^\dagger$ is short on Z.*
3. *For every $c_1 \in C$ there is some $c_2 \in Z$ such that $d_C(c_1, c_2) \leq \frac{2}{3^k}$, and also $d_{U_0}(\gamma^\dagger(c_1), \gamma^\dagger(c_2)) \leq \frac{2}{3^k}$.*

Proof. By induction on k. For $k = 0$, we take $(C, \gamma) = (B, \beta)$, $g = \mathsf{id}_B$, and $Z = S_B[M_0]$. Every point in B is at a distance ≤ 2 from $S_B((0, 0))$, and every point in U_0 is distance at most 2 from every other point. $\gamma^\dagger = \beta^\dagger$ is short on $Z = M_0$ because of (SQ_2), which requires that distances on the boundary are bounded below by the distances determined by the taxicab metric.

Assume our result for k, and fix (C, γ), g, and Z with the required properties. The map γ is a coalgebra morphism $\gamma \colon C \longrightarrow N \otimes C$. Consider $(N \otimes C, N \otimes \gamma)$, $\gamma \circ g$ and $N \otimes Z$.

We check that $S_{N \otimes C}[M_0] \subseteq N \otimes Z$. Let $(r, s) \in M_0$. Recall the $\mathsf{SquaSet}$ structure $S_{N \otimes M_0} \colon M_0 \longrightarrow N \otimes M_0$. It is a general feature of how $N \otimes -$ works as a functor that the diagram below commutes:



Write $S_{N\otimes M_0}((r,s))$ as $n \otimes (r',s')$, where $(r',s') \in M_0$ and $n \in N$. Then

$$S_{N\otimes C}((r,s)) = (N \otimes S_C)(n \otimes (r',s')) = n \otimes S_C((r',s')) \in N \otimes S_C[M_0] \subseteq N \otimes Z.$$

By Lemma 6.1.11, $(N \otimes \gamma)^\dagger$ is short on $N \otimes Z$.

Finally, we verify the last point. Fix a point $n \otimes c_1 \in N \otimes C$. Let $c_2 \in Z$ be such that $d(c_1,c_2) \leq \frac{2}{3^k}$, and $d_{U_0}(\gamma^\dagger(c_1), \gamma^\dagger(c_2)) \leq \frac{2}{3^k}$. Then $n \otimes c_2 \in N \otimes Z$, and

$$d_{N\otimes C}(n \otimes c_1, n \otimes c_2) = \tfrac{1}{3} d_C(c_1,c_2) \leq \frac{2}{3^{k+1}}.$$

(We are using the same n as chosen at the start of this paragraph.) Recall that $(N\otimes\gamma)^\dagger = \alpha_N \circ (N \otimes \gamma^\dagger)$ by Lemma 6.1.9. And

$$\begin{array}{rl}
 & d_{U_0}((N\otimes\gamma)^\dagger(n\otimes c_1), (N\otimes\gamma)^\dagger(n\otimes c_2)) \\
= & d_{U_0}(\alpha_N \circ (N\otimes\gamma^\dagger)(n\otimes c_1), \alpha_N \circ (N\otimes\gamma^\dagger)(n\otimes c_2)) \\
= & d_{U_0}(\alpha_N(n\otimes\gamma^\dagger(c_1)), \alpha_N(n\otimes\gamma^\dagger(c_2))) \\
= & d_{U_0}(\mathsf{shrink}(n) + \frac{1}{3}\gamma^\dagger(c_1), \mathsf{shrink}(n) + \frac{1}{3}\gamma^\dagger(c_2)) \\
= & \frac{1}{3} d_{U_0}(\gamma^\dagger(c_1), \gamma^\dagger(c_2)) \\
\leq & \frac{2}{3^{k+1}}
\end{array}$$

This completes the proof. ∎

6.1.13. Lemma. *$\beta^\dagger \colon B \longrightarrow U_0$ is short.*

Proof. Fix $\varepsilon > 0$. Let $b_1, b_2 \in B$. Let k be large enough so that $2/3^k < \varepsilon/4$. Let C, g, Z, c_1 and c_2 be as in Lemma 6.1.12 so that $c_1, c_2 \in Z$, $d_C(g(b_i), c_i) \leq \varepsilon/4$, and also $d_{U_0}(\gamma^\dagger(g(b_i)), \gamma^\dagger(c_i)) \leq \varepsilon/4$ for $i = 1,2$. Then $d_C(c_1,c_2) \leq d_C(g(b_1), g(b_2)) + \varepsilon/2$. And

$$\begin{array}{rlr}
 & d_{U_0}(\beta^\dagger(b_1), \beta^\dagger(b_2)) & \\
= & d_{U_0}(\gamma^\dagger(g(b_1)), \gamma^\dagger(g(b_2))) & (1) \\
\leq & d_{U_0}(\gamma^\dagger(g(b_1)), \gamma^\dagger(c_1)) + d_{U_0}(\gamma^\dagger(c_1), \gamma^\dagger(c_2)) + d_{U_0}(\gamma^\dagger(c_2), \gamma^\dagger(g(b_2))) & \\
\leq & \varepsilon/4 + d_C(c_1,c_2) + \varepsilon/4 & (2) \\
\leq & \varepsilon/2 + (d_C(g(b_1), g(b_2)) + \varepsilon/2) & \\
\leq & \varepsilon + d_B(b_1,b_2) & (3)
\end{array}$$

Point (1) uses Lemma 6.1.3. Point (2) uses the shortness of $\gamma^\dagger$ on Z. Point (3) uses the shortness of g. This for all $\varepsilon > 0$ proves our result. ∎

6.1.14. THEOREM. *(U_0, α_N) is a corecursive algebra for $N \otimes -$ on SquaMS, and (U_0, α_N^{-1}) is a final coalgebra for this same functor.*

PROOF. We already know that if we forget the metric, (U_0, α_N) is a corecursive algebra for $N \otimes -$ on $\mathsf{SquaSet}$. In the case that we have a short coalgebra structure, (B, β), the unique $\mathsf{SquaSet}$ map $\beta^\dagger$ is short, by Lemma 6.1.13. The forgetful functor $\mathsf{SquaMS} \longrightarrow \mathsf{SquaSet}$ is faithful, and so $\beta^\dagger$ is the unique coalgebra-to-algebra map in SquaMS. This shows the first assertion in our result. The second follows since α_N is invertible (see Lemma 6.1.6). ■

6.2. U_0 IS ISOMORPHIC TO THE COMPLETION OF THE INITIAL ALGEBRA FOR $N \otimes -$. Recall from (5.10) that the initial algebra of $N \otimes -$ on SquaMS is denoted $(W, \lambda \colon N \otimes W \longrightarrow W)$. Recall also that in Definition 6.1.5 we saw an algebra $\alpha_N \colon N \otimes U_0 \longrightarrow U_0$. By initiality there is a unique $(N \otimes -)$-algebra morphism

$$\psi \colon W \longrightarrow U_0$$

In addition, for the same functor $N \otimes -$, λ^{-1} is a coalgebra and U_0 is corecursive, and therefore $(\lambda^{-1})^\dagger = \psi$. This discussion is in SquaMS, and so ψ is a short map. Recall also that W is the colimit of the initial sequence

$$M_0 \overset{! = S_{N \otimes M_0}}{\longrightarrow} N \otimes M_0 \overset{N \otimes !}{\longrightarrow} N^2 \otimes M_0 \overset{N^2 \otimes !}{\longrightarrow} N^3 \otimes M_0 \overset{N^3 \otimes !}{\longrightarrow} \cdots N^k \otimes M_0 \overset{N^k \otimes !}{\longrightarrow} N^{k+1} \otimes M_0 \cdots \quad (6.1)$$

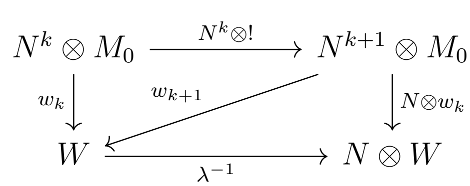
We write $w_k \colon N^k \otimes M_0 \longrightarrow W$ for the colimit injection.

For all k, let $\ell_k \colon N^k \otimes M_0 \longrightarrow U_0$ be given by $\ell_k = \psi \circ w_k$.

Recall the sets CP_k from Definition 4.6.1, and also the maps $f_k \colon CP_k \longrightarrow U_0$, which satisfy the equations $f_0((r, s)) = (r, s)$, and $f_{k+1}(n \otimes x) = \alpha_N(n \otimes f_k(x))$.

6.2.1. PROPOSITION.

1. *The family $(\ell_k)_k$ is a cocone of the inital sequence: for all k, $\ell_k = \ell_{k+1} \circ (N^k \otimes !)$.*
2. *For all k, the diagram below commutes:*



3. *f_k is the restriction of the map $\ell_k \colon N^k \otimes M_0 \longrightarrow U_0$ to CP_k.*

PROOF.

1. This is a consequence of the general fact that if we post-compose all maps in a given cocone by the same morphism, we again have a cocone.

2. The triangles commute because W is the colimit of the initial-algebra chain and $N\otimes -$ preserves the colimit. So the square commutes.

3. We show by induction on k that for $r, s \in \{0, 1\}$, and $n_1, \ldots, n_k \in N$,

$$\ell_k(n_1 \otimes \ldots \otimes n_k \otimes (r, s)) = f_k(n_1 \otimes \ldots \otimes n_k \otimes (r, s)).$$

For $k = 0$, $\ell_0((r, s)) = (r, s) = f_0((r, s))$, since $\ell_0 = \psi \circ w_0 = \psi \circ S_{U_0}$ is a morphism in SquaMS and thus preserves M_0.

Assume our result for k, and fix r, s, and $n_1, \ldots, n_k, n_{k+1} \in N$. To save on notation, write x for $n_2 \otimes \ldots \otimes n_{k+1} \otimes (r, s)$. (In case $k = 1$, x is (r, s).) This point x belongs to CP_k. Then

$$\begin{array}{lll} & \ell_{k+1}(n_1 \otimes x) & \\ = & \psi(w_{k+1}(n_1 \otimes x)) & \text{by definition of } \ell_{k+1} \\ = & (\psi \circ \lambda)((N \otimes w_k)(n_1 \otimes x)) & \text{by part (2), } \lambda \circ (N \otimes w_k) = w_{k+1} \\ = & (\alpha_N \circ (N \otimes \psi))((N \otimes w_k)(n_1 \otimes x)) & \psi \text{ is an } (N \otimes -)\text{-algebra morphism} \\ = & \alpha_N(n_1 \otimes \psi(w_k(x))) & \text{by definition of } N \otimes \psi \text{ and } N \otimes w_k \\ = & \alpha_N(n_1 \otimes \ell_k(x)) & \text{by definition of } \ell_k \\ = & \alpha_N(n_1 \otimes f_k(x)) & \text{by induction hypothesis} \\ = & f_{k+1}(n \otimes x) & \text{by definition of } f_{k+1} \end{array}$$

This completes the proof. ∎

In the result below and in the sequel, we use the notation $w_{j,k}$ when $j \leq k$ for the connecting morphism of the initial-algebra chain (6.1):

$$w_{j,k} \colon N^j \otimes M_0 \longrightarrow N^k \otimes M_0.$$

In a more general setting (using different notation) we discussed these below (5.2).

6.2.2. Proposition. *Concerning the maps $w_{j,k}$ when $j \leq k$ and the sets of corner points:*

1. $w_{j,k}[CP_j] \subseteq CP_k$.

2. The restriction of $w_{j,k}$ to CP_j is an isometric embedding.

Proof. The first part is an easy induction.

For the second part, let z and z' belong to CP_j.

$$\begin{array}{lll} & d(z, z') & \\ = & d_{U_0}(f_j(z), f_j(z')) & (1) \\ = & d_{U_0}(\ell_j(z), \ell_j(z')) & (2) \\ = & d_{U_0}(\ell_k \circ w_{j,k}(z), \ell_k \circ w_{j,k}(z')) & (3) \\ = & d_{U_0}(f_k \circ w_{j,k}(z), f_k \circ w_{j,k}(z')) & (4) \\ = & d(w_{j,k}(z), w_{j,k}(z')) & (5) \end{array}$$

Lines (1) and (5) hold by Proposition 4.6.9 applied to both f_j and f_k. (2) and (4) hold because f_j is the restriction of ℓ_j, and (3) is proved by an easy induction on $k \geq j$, using Proposition 6.2.1. ■

With these preliminaries done, we now return to the topic of this section.

Recall from (5.10) that $(W, \lambda\colon N \otimes W \to W)$ is an initial $N \otimes -$ algebra and that λ is an isomorphism. With C the Cauchy completion functor on the category, we have another algebra which we will call $(V, \theta\colon N \otimes V \longrightarrow V)$, where $V = CW$ is a square metric space whose underlying metric is complete, and θ is an isomorphism. (The map θ is $C\lambda \circ \delta_W^N$, where $\delta_W^N\colon N \otimes CW \longrightarrow C(N \otimes W)$ is the isomorphism which we have seen in Proposition 4.8.3.)

6.2.3. LEMMA. *Let* $CP = \bigcup_k w_k[CP_k]$.

1. *For* $x, y \in CP$, *let* j *be such that there are* $x', y' \in N^j \otimes M_0$ *with* $w_j(x') = x$ *and* $w_j(y') = y$. *Then* $d_{N^j \otimes M_0}(x', y') = d_W(x, y)$.

2. CP *is a dense subset of* W.

3. *The restriction of* ψ *to* CP *is an isometry.*

4. ψ *is an isometry.*

5. ψ *extends to an isomorphism* $\overline{\psi}\colon V \longrightarrow U_0$.

PROOF.

1. First note that such a j exists, since if $x \in w_l[CP_l]$ and $y \in w_j[CP_j]$ for some $l \leq j$, then let $\widehat{x} \in CP_l$ be such that $w_l(\widehat{x}) = x$ and let $x' = w_{l,j}(\widehat{x})$. Then $w_j(x') = x$, as required.

   By Proposition 6.2.2(2), for any $k \geq j$, $d_{N^k \otimes M_0}(w_{j,k}(x'), w_{j,k}(y')) = d_{N^j \otimes M_0}(x', y')$.

   So $d_W(x, y) = \inf_{k \geq j} d_{N^k \otimes M_0}(w_{j,k}(x'), w_{j,k}(y')) = d_{N^j \otimes M_0}(x', y')$.

2. Let $\epsilon > 0$ be given and choose K such that $\frac{2}{3^K} < \epsilon$. Let $x \in W$ and let $k \geq K$ be such that there is $x' \in N^k \otimes M_0$ with $w_k(x') = x$. Then there are $n_1, \ldots, n_k \in N$ and $(r, s) \in M_0$ such that $x' = n_1 \otimes \ldots \otimes n_k \otimes (r, s)$. Let $c = n_1 \otimes \ldots \otimes n_k \otimes (0, 0) \in CP_k$, and note that $w_k(c) \in CP$. Then $d_{N^k \otimes M_0}(x', c) \leq \frac{2}{3^k} < \epsilon$ by Corollary 4.5.3, so since d_W is the infimum of the distances in $N^k \otimes M_0$, $d_W(x, w_k(c)) \leq d_{N^k \otimes M_0}(x', c) < \epsilon$. Hence, CP is dense in W.

3. Let $x, y \in CP$ and let k be such that there are $x', y' \in CP_k$ with $w_k(x') = x$ and $w_k(y') = y$. Note that $\psi(x) = \psi \circ w_k(x') = \ell_k(x') = f_k(x')$ and similarly, $\psi(y) = f_k(y')$. Then

$$\begin{aligned} d_{U_0}(\psi(x), \psi(y)) &= d_{U_0}(f_k(x'), f_k(y')) \\ &= d_{N^k \otimes M_0}(x', y') && \text{by Proposition 4.6.9} \\ &= d_W(x, y) && \text{by 1.} \end{aligned}$$

4. This follows from parts 2 and 3.

5. For this it will be enough to show that the image of CP is dense in U_0. Let $(x,y) \in U_0$ be given. It is a standard fact that every real number has a ternary representation; see also [2, Example 7.3.10(2)] for a corecursive algebra proof of the related fact that real numbers have binary representations. We can choose (i_k, j_k) in N such that $(x,y) = \left(\sum_{k=0}^{\infty} \frac{i_k}{3^{k+1}}, \sum_{k=0}^{\infty} \frac{j_k}{3^{k+1}}\right)$. For $\epsilon > 0$, choose K such that $\frac{2}{3^K} < \epsilon$. Let

$$c = (i_0, j_0) \otimes \ldots \otimes (i_{K-1}, j_{K-1}) \otimes (0,0) \in CP_k,$$

and note that $f_k(c) = \ell_k(c) = \psi(w_k(c)) = \overline{\psi}(w_k(c))$. So since $w_k(c) \in CP$, this is in the image of CP. Then

$$d_{U_0}(f_K(c), (x,y)) = \left|\sum_{k=K+1}^{\infty} \frac{i_k}{3^{k+1}}\right| + \left|\sum_{k=K+1}^{\infty} \frac{j_k}{3^{k+1}}\right| \leq \frac{2}{3^K} < \epsilon.$$

Thus, the image of CP under ψ is dense in U_0, as required.

■

6.2.4. Theorem. *$(V, \theta\colon N \otimes V \longrightarrow V)$ is a corecursive $(N \otimes -)$-algebra, and therefore $(V, \theta^{-1}\colon V \longrightarrow N \otimes V)$ is a final $(N \otimes -)$-coalgebra.*

Proof. Let $(B, \beta\colon B \longrightarrow N \otimes B)$ be a coalgebra. Consider the metric space V^B, and note that since V is complete, V^B is also complete. The subspace of V^B of short maps which preserve the square space structure is a closed subset since limits of structure-preserving short maps will be short and will preserve the structure. Crucially, the set of such maps is non-empty. This is because we have SquaMS morphism $\beta^{\dagger}\colon B \longrightarrow U_0$ by Lemma 6.1.13 (this map has nothing to do with β in this proof) and an isomorphism $\overline{\psi}^{-1}: U_0 \longrightarrow V$ by Lemma 6.2.3. We also have a $\frac{1}{3}$-contracting map $\Phi\colon V^B \longrightarrow V^B$ given by $\Phi(f) = \theta \circ (N \otimes f) \otimes \beta$.

Thus, Φ has a unique fixed point. The fixed points of Φ are exactly the coalgebra to algebra morphisms $B \longrightarrow V$. Thus, there is a unique such morphism from $B \longrightarrow V$. This proves that (V, θ) is a corecursive algebra. Since θ is invertible, (V, θ^{-1}) is a final coalgebra; see Proposition 6.1.2. ■

6.2.5. Corollary. *V, the Cauchy completion of the initial $(N \otimes -)$-algebra, is isomorphic to U_0 with the taxicab metric.*

Proof. Since V and U_0 are both final $(N \otimes -)$-coalgebras (Theorem 6.1.14 and Theorem 6.2.4), they are isomorphic. ■

6.3. THE SIERPINSKI CARPET IS A CORECURSIVE $M\otimes -$ ALGEBRA. For our next result on this topic, recall that we have an isometry $\alpha_N : N \otimes U_0 \longrightarrow U_0$ (see Lemma 6.1.6).

Let τ be the restriction of α_N to $M \otimes \mathbb{S}$. Recall the maps σ_m from Definition 2.2.5, and also σ. Then note that for $m \otimes s \in M \otimes \mathbb{S}$,

$$\tau(m \otimes s) = \mathsf{shrink}(m) + \frac{1}{3}s = \sigma_m(s) \in \mathbb{S}. \tag{6.2}$$

And for $s \in \mathbb{S} = \sigma(\mathbb{S}) = \bigcup_{m \in M} \sigma_m(\mathbb{S})$, there are $s' \in \mathbb{S}$ and $m \in M$ such that $s = \sigma_m(s') = \tau(m \otimes s')$. So $\tau : M \otimes \mathbb{S} \to \mathbb{S}$ is a bijection.

This map τ is not an isometry, so it has no inverse in SquaMS, but it still is an isomorphism in SquaSet.

6.3.1. PROPOSITION. *The diagram below commutes in* SquaMS*:*

$$\begin{array}{ccc} M \otimes \mathbb{S} & \xrightarrow{\quad\tau\quad} & \mathbb{S} \\ {\scriptstyle M\otimes i}\downarrow & & \downarrow{\scriptstyle i} \\ M \otimes U_0 & \xrightarrow[\alpha_M = \alpha_N \circ \iota_{U_0}]{} & U_0 \end{array}$$

Here i is the inclusion, and the natural transformation ι is from Proposition 4.7.1.

PROOF. Let $m \otimes x \in M \otimes \mathbb{S}$ be given. Then $i \circ \tau(m \otimes x) = i(\alpha_N(m \otimes x)) = \alpha_N(m \otimes x)$ and $\alpha_M \circ M \otimes i(m \otimes x) = \alpha_M(m \otimes x) = \alpha_N \circ \iota_{U_0}(m \otimes x) = \alpha_N(m \otimes x)$. ∎

Let $(B, \beta : B \longrightarrow M \otimes B)$ be a coalgebra. By postcomposing with the inclusion $\iota_B : M \otimes B \longrightarrow N \otimes B$, we get an $N \otimes -$ coalgebra $\iota_B \circ \beta : B \to N \otimes B$. So we have $(\iota_B \circ \beta)^\dagger : B \longrightarrow U_0$. We aim to show that for all $b \in B$, $(\iota_B \circ \beta)^\dagger(b) \in \mathbb{S}$. Before presenting the proof, we will walk the reader through the ideas. We will assume that B is enumerated without repeats as $b_1, b_2, \ldots, b_k, \ldots$, and also that our coalgebra β is given by

$$\beta(b_i) = m_i \otimes b_{i+1}$$

(Please note that we are not saying that all coalgebras look like this; we are only making an example. In fact, a general coalgebra for this functor would be a family of an arbitrary set of disjoint versions of this example, together with an arbitrary set of finite coalgebras; these would be eventually periodic. None of this really matters in this paper.) The m_i can be chosen in M, not just in N. Then the solution $(\iota_B \circ \beta)^\dagger : B \longrightarrow U_0$ corresponds to elements $r_1, r_2, \ldots$ in U_0 such that

$$\begin{array}{rcl} r_1 & = & \alpha_N(m_1 \otimes r_2) \\ r_2 & = & \alpha_N(m_2 \otimes r_3) \\ r_3 & = & \alpha_N(m_3 \otimes r_4) \\ & \vdots & \end{array}$$

Again, we would like to show that each r_i belongs to $\mathbb{S}$. It is clear that

$$r_1 \in \alpha_N(m_1 \otimes U_0) = \sigma_{m_1}(U_0)$$

The notation $m_1 \otimes U_0$ and similar notation below is from Remark 4.3.8. A little more thought shows that

$$r_1 \in \alpha_N(m_1 \otimes \alpha_N(m_2 \otimes U_0)) = \sigma_{m_1}(\sigma_{m_2}(U_0))$$

and then

$$r_1 \in \alpha_N(m_1 \otimes \alpha_N(m_2 \otimes \alpha_N(m_3 \otimes U_0))) = \sigma_{m_1}(\sigma_{m_2}(\sigma_{m_3}(U_0)))$$

In the notation of Hutchinson's Theorem (Proposition 2.1.2), $r_1 \in (U_0)_{m_1m_2m_3\ldots m_p}$ for all p. Since all of the m's belong to M, Proposition 2.1.2 parts (2) and (3) tell us that $r_1 \in \mathbb{S}_{m_1m_2\ldots} \subset \mathbb{S}$. Similarly, we can argue for each i, $r_i = (\iota_B \circ \beta)^\dagger(b_i) \in \mathbb{S}_{m_im_{i+1}\ldots} \subset \mathbb{S}$.

Most of the work in the proof of our next result is in managing the notation (and changing it a little) and then filling in the details in the sketch above.

6.3.2. Proposition. *For all* $b \in B$, $(\iota_B \circ \beta)^\dagger(b) \in \mathbb{S}$.

Proof. As in the proof of Lemma 6.1.8, fix an associate $\widehat{\beta}\colon B \longrightarrow M \times B$. Define maps $u_k\colon B \longrightarrow B$ for $k \geq 0$ and $v_k\colon B \longrightarrow M$ for $k \geq 1$:

$$\begin{array}{rcl} u_0(b) & = & b \\ \widehat{\beta}(u_k(b)) & = & (v_{k+1}(b), u_{k+1}(b)) \end{array}$$

We claim that for all $k \geq 0$: $u_k(u_1(b)) = u_{k+1}(b)$. The proof is by induction on k. For $k = 0$, our result is clear. Assume that $u_k(u_1(b)) = u_{k+1}(b)$. Then

$$(v_{k+1}(u_1(b)), u_{k+1}(u_1(b))) = \widehat{\beta}(u_k(u_1(b))) = \widehat{\beta}(u_{k+1}(b)) = (v_{k+2}(b), u_{k+2}(b)). \tag{6.3}$$

So $u_{k+1}(u_1(b)) = u_{k+2}(b)$. This establishes our claim. And from this claim we repeat (6.3) to see that for all $k \geq 1$, $v_k(u_1(b)) = v_{k+1}(b)$.

For each $b \in B$, we have an infinite sequence of elements of M

$$v_1(b), v_2(b), \ldots, v_k(b), \ldots \tag{6.4}$$

Moreover, we will show by induction on k that

$$(\iota_B \circ \beta)^\dagger(b) \in (U_0)_{v_1(b), v_2(b), \ldots, v_k(b)} \tag{6.5}$$

for all $b \in B$. For $k = 0$, $(\iota_B \circ \beta)^\dagger(b) \in U_0 = (U_0)_\varepsilon$. Fix $k \geq 0$, and assume that for all $b \in B$, $(\iota_B \circ \beta)^\dagger(b) \in (U_0)_{v_1(b),\ldots,v_k(b)}$. Now fix b. So $\widehat{\beta}(b) = (v_1(b), u_1(b))$. To save on notation, we will write b' for $u_1(b)$. By our assumption,

$$(\iota_B \circ \beta)^\dagger(b') \in (U_0)_{v_1(b'),\ldots,v_k(b')} = (U_0)_{v_2(b),\ldots,v_{k+1}(b)}.$$

(Notice that we used a fact from above to write $v_i(b') = v_i(u_1(b)) = v_{i+1}(b)$.) And then

$$(\iota_B \circ \beta)^\dagger(b) = \alpha_N(v_1(b) \otimes (\iota_B \circ \beta)^\dagger(b')) \in \sigma_{v_1(b)}((U_0)_{v_2(b),\ldots,v_{k+1}(b)}) = (U_0)_{v_1(b),v_2(b),\ldots,v_{k+1}(b)}.$$

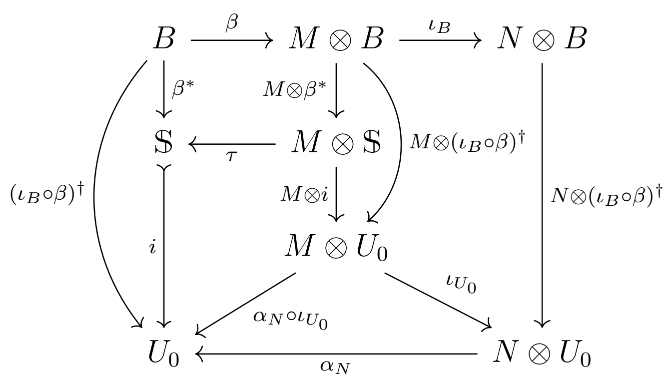
This completes the induction. Since the sequence in (6.4) comes from M, by (6.5) and Proposition 2.1.2, we get that $(\iota_B \circ \beta)^\dagger(b) \in \mathbb{S}$. ∎

As a result of Proposition 6.3.2, we regard $(\iota_B \circ \beta)^\dagger$ as a morphism with codomain $\mathbb{S}$. That is, $(\iota_B \circ \beta)^\dagger \colon B \longrightarrow U_0$ factors through the inclusion $i \colon \mathbb{S} \longrightarrow U_0$. So we have a map $\beta^* \colon B \longrightarrow \mathbb{S}$ such that

$$(\iota_B \circ \beta)^\dagger = i \circ \beta^*. \tag{6.6}$$

6.3.3. Theorem. *$(\mathbb{S}, \tau)$ is a corecursive algebra for $M \otimes - \colon$ SquaMS $\longrightarrow$ SquaMS.*

Proof. Let (B, β) be a coalgebra. Consider the following diagram in SquaMS:



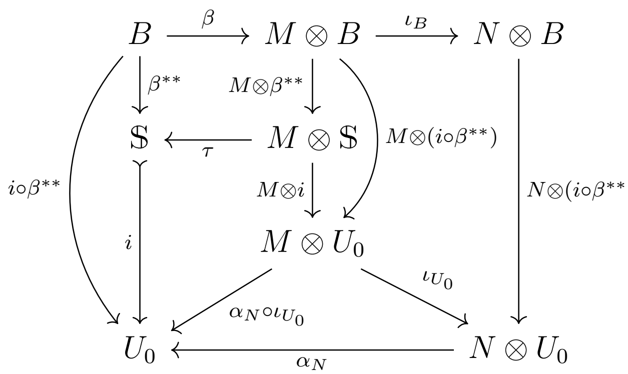
We need to show that the top left corner commutes. We are using the natural transformation $\iota : (M\otimes-)\longrightarrow(N\otimes-)$ from Proposition 4.7.1. We get $(\iota_B\circ\beta)^\dagger$ by Theorem 6.1.14, and the outside of the diagram commutes. We have seen in (6.6) that the small region in the center commutes.

The region on the far right commutes by the naturality of ι.

The region in the lower-left commutes by Proposition 6.3.1. The bottom commutes trivially. Thus, all of the inside parts commute. A diagram chase shows that $i \circ \beta^* = i \circ \tau \circ (M \circ \beta^*) \circ \beta$. Since i is monic, $\beta^* = \tau \circ (M \circ \beta^*) \circ \beta$. This shows that β^* is a coalgebra-to-algebra map.

For the uniqueness of β^*, suppose that $\beta^{**} \colon B \longrightarrow \mathbb{S}$ satisfies $\beta^{**} = \tau \circ (M \circ \beta^{**}) \circ \beta$.

Consider the diagram below:



At first glance, the maps are different from those in the previous diagram. All of the inside parts of this diagram commute: the part on the left by definition, the part on the right by naturality, and the remaining parts for the same reasons as in the previous diagram. Thus, the ouside commutes. This implies that $i \circ \beta^{**}$ is a coalgebra-to-algebra morphism for $\iota_B \circ \beta$. By the uniqueness part of Theorem 6.1.14, $i \circ \beta^{**} = (\iota_B \circ \beta)^\dagger = i \circ \beta^*$. Since i is monic, $\beta^{**} = \beta^*$. ∎

Unfortunately τ^{-1} is not a short map, so it is not a morphism in SquaMS. However, it is an isomorphism in SquaSet, so we do get the following.

6.3.4. Corollary. $(\$, \tau^{-1})$ *is a final coalgebra for* $M \otimes -$: SquaSet ⟶ SquaSet.

Proof. First, let us show that $(\$, \tau)$ is a corecursive algebra for $M\otimes-$: SquaSet⟶SquaSet. Let (B, β) be a coalgebra. Endow B with the following metric: For $(r, s), (t, u) \in M_0$, let

$$d_B(S_B((r, s)), S_B((t, u))) = d_{U_0}((r, s), (t, u)),$$

and for $x, y \notin S_B[M_0]$, let $d_B(x, y) = 2$ and $d_B(x, S_B((r, s))) = 2$. It is easy to verify that this is an object in SquaMS.

Then β is automatically short. By Theorem 6.3.3, there is a unique solution $\beta^\dagger$. This same morphism is a solution in SquaSet, of course. For the uniqueness, note that every morphism from the discrete space B to $\$$ is automatically short.

The morphism τ is a bijection, and so it is invertible in SquaSet. So we are done by Proposition 6.1.2. ∎

6.4. The final $(M \otimes -)$-coalgebra $(Q, \gamma : Q \to M \otimes Q)$. Recall $(G, \eta : M \otimes G \to G)$, the initial algebra. By Lambek's Lemma, η is an isomorphism. Let $Q = CG$, the Cauchy completion. Consider the map below:

$$\gamma : Q \to M \otimes Q \qquad\qquad Q = CG \xrightarrow{C\eta^{-1}} C(M \otimes G) \xrightarrow{\rho_G} M \otimes CG = M \otimes Q\,. \tag{6.7}$$

The morphism ρ^M_G is the isomorphism from Proposition 4.8.2. In this section we will show that (Q, γ) is the final $(M \otimes -)$-coalgebra.

For the remainder of the paper, let

$$i : G \hookrightarrow Q \tag{6.8}$$

denote the inclusion map from G into Q, and note that G is dense in Q.

Let $(B, \beta\colon B \longrightarrow M \otimes B)$ be a coalgebra. The main task at this point is to exhibit a short map $h\colon B \to Q$. We will use the short map $(\iota_B \circ \beta)^\dagger\colon B \longrightarrow U_0$ in our definition, but our use will not be what one might at first expect. Instead, to get h we will need to *go via* $M^n \otimes U_0$ (in some appropriate sense that we shall discuss). Even if we wanted to use $(\iota_B \circ \beta)^\dagger$ directly, there is an issue which arises in considering a map from $\mathbb{S}$ (as a subset of U_0) to Q: the most natural and direct map will not be short. For example, consider points $(\frac{1}{2}, \frac{1}{3})$ and $(\frac{1}{2}, \frac{2}{3})$ in U_0. These have distance $\frac{1}{3}$ in the taxicab metric. However, these correspond to the top and bottom of the "hole" at $(1,1)$ in Q, that is, if we view Q as $M \otimes Q$, the points are $(1,0) \otimes S_Q((\frac{1}{2}, 1))$ and $(1,2) \otimes S_Q((\frac{1}{2}, 0))$, so their distance under the quotient metric will be $\frac{2}{3}$ (to navigate around the hole). So the obvious bijective correspondence between a subset of U_0 and Q will not be a short map, and indeed, not an isometry. However, we navigate around this difficulty, going a different way. We will consider corner points as we did for $N \otimes -$, but note that the density of corner points in the relevant subset of U_0 is not going to help us: again, the map from the appropriate subset of U_0 to Q is not a short map.

Corner Points for $(M \otimes -)$ We will start by adapting the definition of corner points for the $(N \otimes -)$ functor.

6.4.1. Definition. The set CP_k^M of *corner points* of $M^k \otimes M_0$ is defined as follows:

$$\begin{aligned} CP_0^M &= \{(0,0), (0,1), (1,0), (1,1)\} \\ CP_{k+1}^M &= \{m \otimes x \mid m \in M, x \in CP_k^M\} \end{aligned}$$

We can also refer to corner points in $M^k \otimes U_0$ via the inclusion $M^k \otimes S_{U_0}(CP_k^M)$, and this is a bijective correspondence. Right away we see that the distance between corner points in CP_k^M (as a subset of $M^k \otimes M_0$) is bounded below by the distance between their images in $M^k \otimes U_0$, because $M^k \otimes S_{U_0}$ is a short map.

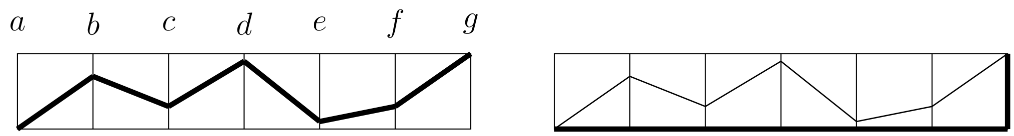
In the next lemma and corollary, we will prove that $M^k \otimes S_{U_0}$ restricted to CP_k^M is in fact an isometry.

6.4.2. Lemma. *Let x and y be corner points in $M^k \otimes U_0$. Then there exists a witness path from x to y consisting entirely of corner points in $M^k \otimes U_0$.*

Proof. The idea is to take any path p from x to y and to modify p, obtaining a path p' from x to y with a score at most that of p and with at least one fewer node which is not a corner point. (The score of a path was defined near the beginning of Section 4.1.) So in effect we are arguing by induction on the number of non-corner-points that the score can drop by replacing such a point by a corner point, and perhaps making further modifications.

Our path may be written as a path in the 8^k copies of U_0. That is, a witness path (see Definition 4.1.13) in $M^k \otimes U_0$ is most naturally presented as a path of "segments", each from $M^{k-1} \otimes U_0$. But this is not the way we want to view it here. We want to say that our path is a path of length $\leq 8^k$ in copies of U_0 with the taxicab metric. We know that 8^k is an upper bound on the number of segments in our path, since if a copy of U_0 is visited twice, then by Corollary 4.4.1 we could find a smaller score by removing the cycle.

The first thing to do is to modify p on behalf of all edges which connect two non-corner points. In the picture on the left below is a suggestive example. We are going to work with this rather than the general case. The edges that connect two non-corner points are the ones shown, except for the first and last.



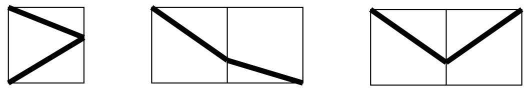
Every edge which connects two non-corner points is part of a maximal sub-path q of such edges. This is because the first and last points on p are corner points, and p itself is finite. Then we replace the sub-path q as on the right above. It is important to note that making this replacement still gives us a path in $M^k \otimes U_0$. (That is, we do not step out of $M^k \otimes U_0$ into $N^k \otimes U_0$ by making it. This is because we remain within the copies of U_0 used in the original path, so none of our new segments will fall in one of the "holes" determined by M.) And different maximal sub-paths may be replaced simultaneously. We check that the bold path on the left represents a longer subpath than the one on the right. Let the coordinates of a be (x_a, y_a), and similarly for b, c, …, g. Then the length of the path on the left is

$$\begin{array}{rl} & |x_b - x_a| + |y_b - y_a| + \cdots + |x_g - x_f| + |y_g - y_f| \\ \geq & 6 + |y_b - y_a| + |y_d - y_c| + |y_f - y_e| + |y_g - y_f| \\ \geq & 6 + 1 = 7 \end{array}$$

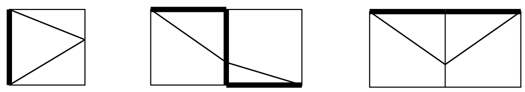
The idea is that each $|x_b - x_a|$ is at least 1 since they are on opposite sides of a copy of U_0, so these will cumulatively contribute at least 6 to the score. Similarly, in order to transit from y_a to y_f, we must contribute at least 1 to the score, since they are on opposite sides (of a row of adjacent copies) of U_0.

The length of the bold path on the right is 7. The same argument would work for a sub-path which was like this but rotated 90°. There is a second kind of replacement which is similar to what we just saw but where the sub-path's two endpoints have the same y-coordinate. This second kind is easier to handle, since a sequence of horizontal segments works.

After these two kinds of replacements our path p might contain non-corner points, but edges which contain non-corner points also contain a corner point. These edges come in pairs of three possible forms:



Then each of these sub-paths may be replaced by one using only corner points, with the overall score not increasing, as shown below:



In each case, it is clear that the new sub-path has a length at most that of the old; this is most interesting in the middle case, where we use the fact that the metric in U_0 is the taxicab metric.

In this way, we have taken a path p in $M^k \otimes U_0$ between corner points and modified it to a path between the same points in $M^k \otimes M_0$ without increasing the length. ■

Throughout the remainder of this section, we will adopt the following notation: for $\overline{m} \in M^k$ and $x \in X$, $\overline{m} \otimes x$ is $m_1 \otimes \ldots \otimes m_k \otimes x \in M^k \otimes X$, where $\overline{m} = (m_1, \ldots, m_k)$.

6.4.3. Corollary. *Let $r, s, t, u \in \{0, 1\}$. Then for $\overline{m}, \overline{n} \in M^k$,*

$$d_{M^k \otimes U_0}(\overline{m} \otimes S_{U_0}((r,s)), \overline{n} \otimes S_{U_0}((t,u))) = d_{M^k \otimes M_0}(\overline{m} \otimes S_{M_0}((r,s)), \overline{n} \otimes S_{M_0}((t,u))).$$

That is, the distance between corners in $M^k \otimes U_0$ coincides with the distance in $M^k \otimes M_0$.

Proof. By the previous lemma, there is a witness path in $M^k \otimes U_0$ such that every entry is a corner. So for each pair contributing positively to the score, if they are adjacent corners, they contributes $(\frac{1}{3})^k$, and if they are opposite corners, they contribute $(\frac{2}{3})^k$ to the score.

So consider the corresponding path in $M^k \otimes M_0$. This score will be the same. Thus, the distance in $M^k \otimes U_0$ is bounded above the distance of the corresponding points in $M^k \otimes M_0$. However, we know that the distance in $M^k \otimes M_0$ is bounded above by its image in $M^k \otimes U_0$ under $M^k \otimes S_{U_0}$, since this is a short map. Thus, these distances are equal. ■

The map h Let $(B, \beta\colon B \longrightarrow M \otimes B)$ be a coalgebra. Our final task is to find a morphism $h : B \to Q$ in SquaMS: once we know that the set of morphisms from B to Q is non-empty, we can use a fixed-point argument like the one we saw in Theorem 6.2.4 to show that $(Q, \gamma : Q \to M \otimes Q)$ is the final $M \otimes -$coalgebra in SquaMS.

We will start by defining functions $h_k : B \to M^k \otimes M_0$ which are *not short maps*, but are approximately short in some technical sense described below. We will need our work on corner points and the short map $M^k \otimes (\iota_B \circ \beta)^\dagger : M^k \otimes B \to M^k \otimes U_0$ to show that the h_k maps satisfy our approximate shortness property. Then for a fixed $x \in B$, this gives a sequence $[h_k(x)]_k$ in G, which we will show is a Cauchy sequence, and thus, has a limit in

Q. This limit is what h will map x to. Furthermore, we will show that h preserves S_B, and thus, is a SquaMS morphism.

For $x \in B$, define infinite sequences $m_1(x), m_2(x), \ldots \in M$ and $b_0(x), b_1(x), \ldots \in B$ as follows: let $b_0(x) = x$, and for $k \geq 1$, given $b_0(x), \ldots, b_{k-1}(x)$ and $m_1(x), \ldots, m_{k-1}(x)$, *choose* $m_k(x) \in M$ and $b_k(x) \in B$ such that

$$\beta(b_{k-1}(x)) = m_k(x) \otimes b_k(x). \tag{6.9}$$

Note that there may be more than one choice for $m_k(x)$ and $b_k(x)$. The point is that we are fixing a particular selection.

Here is how our notation works:

$$B \xrightarrow{\beta} M \otimes B \xrightarrow{M \otimes \beta} M^2 \otimes B \cdots \xrightarrow{M^{k-1} \otimes \beta} M^k \otimes B \xrightarrow{M^k \otimes \beta} \cdots$$

$$x \qquad m_1(x) \otimes b_1(x) \qquad m_1(x) \otimes m_2(x) \otimes b_2(x) \qquad \overline{m}(x) \otimes b_k(x)$$

For a given $x \in B$, we have indicated notation for the images of x under the maps shown. When the context is clear, we abbreviate $m_1(x) \otimes \ldots \otimes m_k(x)$ by $\overline{m}(x)$. (However, we should be careful to note that $\overline{m}$ is not the name of any function.)

Let $h_k : B \to M^k \otimes M_0$ be given by

$$h_k(x) = m_1(x) \otimes \ldots \otimes m_k(x) \otimes (0, 0).$$

Note that h_k is not a short map. The idea is that as k increases, the distances between elements of $M^k \otimes B$ (and indeed, $M^k \otimes M_0$) depend less and less on the element of B (or M_0) and more on $m_1 \otimes \ldots \otimes m_k$, so we will use these h_k's to approximate h, the main map in this section. Even though each h_k is not short, we do have an approximate notion of shortness which it satisfies.

6.4.4. Definition. A map $f : X \to Y$ is *ϵ-short* if for $x, y \in X$,

$$d_Y(f(x), f(y)) \leq d_X(x, y) + \epsilon.$$

6.4.5. Lemma. *$h_k : B \to M^k \otimes M_0$ is $\frac{4}{3^k}$-short.*

PROOF. Let $x, y \in B$ be given, and for ease of notation, let $\overline{m}(x) = m_1(x) \otimes \ldots \otimes m_k(x)$, $\overline{m}(y) = m_1(y) \otimes \ldots \otimes m_k(y)$, $x' = b_k(x)$, and $y' = b_k(y)$. We have:

$$\begin{array}{lll}
& d_{M^k \otimes M_0}(h_k(x), h_k(y)) & \\
= & d_{M^k \otimes M_0}(\overline{m}(x) \otimes (0,0), \overline{m}(y) \otimes (0,0)) & (1)\\
= & d_{M^k \otimes U_0}(\overline{m}(x) \otimes (0,0), \overline{m}(y) \otimes (0,0)) & (2)\\
\leq & d_{M^k \otimes U_0}(\overline{m}(x) \otimes (0,0), \overline{m}(x) \otimes (\iota_B \circ \beta)^\dagger(x')) & (3)\\
& +d_{M^k \otimes U_0}(\overline{m}(x) \otimes (\iota_B \circ \beta)^\dagger(x'), \overline{m}(y) \otimes (\iota_B \circ \beta)^\dagger(y')) & \\
& +d_{M^k \otimes U_0}(\overline{m}(y) \otimes (\iota_B \circ \beta)^\dagger(y'), \overline{m}(y) \otimes (0,0)) & \\
\leq & d_{M^k \otimes U_0}(\overline{m}(x) \otimes (\iota_B \circ \beta)^\dagger(x'), \overline{m}(y) \otimes (\iota_B \circ \beta)^\dagger(y')) + \frac{4}{3^k} & (4)\\
\leq & d_{M^k \otimes B}(\overline{m}(x) \otimes x', \overline{m}(y) \otimes y') + \frac{4}{3^k} & (5)\\
\leq & d_B(x,y) + \frac{4}{3^k}. & (6)
\end{array}$$

Equality (1) is by the definition of the maps h_k and the values $\overline{m}(x)$ and $\overline{m}(y)$. (2) is by Corollary 6.4.3. (3) is by the triangle inequality. (4) is by Corollary 4.4.1. That is, for a fixed $m^* \in M^k$, $d_{M^k \otimes U_0}(m^* \otimes u, m^* \otimes v) < \frac{2}{3^k}$ for all $u, v \in U_0$. In particular,

$$d_{M^k \otimes U_0}(\overline{m}(x) \otimes (0,0), \overline{m}(x) \otimes (\iota_B \circ \beta)^\dagger(x')) \leq \frac{2}{3^k},$$

and similarly for y. (5) follows from the fact that $(\iota_B \circ \beta)^\dagger : B \to U_0$ is a short map, which implies that $M^k \otimes (\iota_B \circ \beta)^\dagger : M^k \otimes B \to U_0$ is also a short map. Finally, (6) is because $(M^{k-1} \otimes !) \circ \ldots \circ !$ is a short map, and because (as indicated in our diagram below (6.9)), $(M^{k-1} \otimes !) \circ \ldots \circ !(x) = \overline{m}(x) \otimes x'$ (and similarly for y). ∎

6.4.6. LEMMA. *Let $x \in B$ be given. $[h_k(x)]_k$ is a Cauchy sequence in G, the initial $(M \otimes -)$-algebra.*

PROOF. Let $\epsilon > 0$ be given and choose K sufficiently large so that $\frac{2}{3^K} < \epsilon$. Let $k, j > K$ be given, and suppose $k > j$. We use $\overline{m} \otimes (0,0)$ as an abbreviation for $m_1(x) \otimes \ldots \otimes m_k(x) \otimes (0,0)$, and $\overline{\overline{m}} \otimes (0,0)$ as an abbreviation for $m_1(x) \otimes \ldots \otimes m_j(x) \otimes (0,0)$.

With this notation,

$$\begin{array}{lll}
h_k(x) & = & \overline{m} \otimes (0,0) \in M^k \otimes M_0,\\
h_j(x) & = & \overline{\overline{m}} \otimes (0,0) \in M^j \otimes M_0.
\end{array}$$

Since $\beta(0,0) = (0,0) \otimes (0,0)$, we see that

$$(M^{k-1} \otimes \beta) \circ \ldots \circ \beta(h_j(b)) = \overline{\overline{m}} \otimes \overbrace{(0,0) \otimes \ldots \otimes (0,0)}^{k-j+1},$$

This belongs to the equivalence class $[h_j(b)]$ in G. So since d_G is the infimum of distances between representatives coming from the sets $M^k \otimes M_0$,

$$\begin{array}{ll} & d_G([h_k(x)],[h_j(x)]) \\ \leq & d_{M^k \otimes M_0}(\overline{m} \otimes (0,0), \overline{\overline{m}} \otimes (0,0) \otimes \ldots \otimes (0,0)) \\ \leq & \frac{2}{3^j} \\ \leq & \frac{2}{3^K}. \end{array}$$

We are using Corollary 4.4.1. ∎

Since Q is the completion of G, we can define $h : B \to Q$ by letting $h(x)$ be the limit of the Cauchy sequence $[i(h_k(x))]_k$.

6.4.7. Proposition. *$h : B \to Q$ is a short map.*

Proof. Let $x, y \in B$ be given. For ease of notation, let $m_i = m_i(x)$, $x_i = b_i(x)$, $n_i = m_i(y)$ and $y_i = b_i(y)$. That is, for all k,

$$\begin{array}{lcl} h_k(x) & = & m_1 \otimes \ldots \otimes m_k \otimes (0,0) \\ (M^{k-1} \otimes \beta) \circ \ldots \circ \beta(x) & = & m_1 \otimes \ldots \otimes m_k \otimes x_k \end{array}$$

We have similar equations for y, but using the elements $n_i \in M$ instead of m_i.

Let $\epsilon > 0$ be given. Our aim is to show that $d_B(x,y) + \epsilon \geq d_Q(h(x), h(y))$. This, for all $\epsilon > 0$ will yield our result. Choose k sufficiently large so that

$$\frac{4}{3^k} < \frac{\epsilon}{2} \tag{6.10}$$

and

$$|d_G(h_k(x), h_k(y)) - d_Q(h(x), h(y))| < \frac{\epsilon}{2}. \tag{6.11}$$

This is possible, since $h(x)$ and $h(y)$ are limits of the sequences $[h_k(x)]_k$ and $[h_k(y)]_k$ respectively. Then

$$\begin{array}{rcll} d_Q(h(x), h(y)) & \leq & d_G(h_k(x), h_k(y)) + \frac{\epsilon}{2} & \text{by (6.11)} \\ & \leq & d_B(x,y) + \frac{4}{3^k} + \frac{\epsilon}{2} & \text{by Lemma 6.4.5} \\ & \leq & d_B(x,y) + \epsilon & \text{by (6.10)} \end{array}$$

as required. ∎

6.4.8. Lemma. *$h : B \to Q$ is a morphism in* SquaMS.

Proof. Since we know that h is a short map, it only remains to show that it preserves S_B to see that it is a SquaMS morphism.

Let $(r,s) \in M_0$ be given, and first note that

$$S_Q((r,s)) = i(S_G((r,s))) = i([S_{M^k \otimes M_0}((r,s))]) \tag{6.12}$$

for all k, where $i : G \hookrightarrow Q$ is the inclusion in (6.8), since the morphisms $M^k \otimes !$ preserve M_0.

Let $m_i = m_i(S_B((r,s))) \in M$ and $x_i = b_i(S_B((r,s))) \in B$. (Here b_i is from (6.9), with i for k.) For all k,

$$((M^{k-1} \otimes \beta) \circ \ldots \circ \beta)(S_B((r,s))) = m_1 \otimes \ldots \otimes m_k \otimes x_k.$$

In particular, since $(M^{k-1} \otimes \beta) \circ \ldots \circ \beta$ is a SquaMS morphism, we have that $x_k = S_B((r_k, s_k))$ for some $(r_k, s_k) \in M_0$.

We also have

$$h_k(S_B((r,s)) = m_1 \otimes \ldots \otimes m_k \otimes (0,0).$$

Next we need to show that $S_{M^k \otimes M_0}((r,s)) = m_1 \otimes \ldots \otimes m_k \otimes (r_k, s_k)$. Note that the following diagram commutes:

$$\begin{array}{ccccccccccc}
M_0 & \xrightarrow{!} & M \otimes M_0 & \xrightarrow{M\otimes !} & M^2 \otimes M_0 & \xrightarrow{M^2\otimes !} & \cdots & \xrightarrow{M^{k-1}\otimes !} & M^k \otimes M_0 & \xrightarrow{M^k\otimes !} & \cdots \\
\downarrow{\scriptstyle S_B} & & \downarrow{\scriptstyle M\otimes S_B} & & \downarrow{\scriptstyle M^2\otimes S_B} & & & & \downarrow{\scriptstyle M^k\otimes S_B} & & \\
B & \xrightarrow{\beta} & M \otimes B & \xrightarrow{M\otimes\beta} & M^2 \otimes B & \xrightarrow{M^2\otimes\beta} & \cdots & \xrightarrow{M^{k-1}\otimes\beta} & M^k \otimes B & \xrightarrow{M^k\otimes\beta} & \cdots
\end{array}$$

Let $n_0, \ldots, n_k \in M$ and $(t_k, u_k) \in M_0$ be such that

$$n_1 \otimes \ldots \otimes n_k \otimes (t_k, u_k) = S_{M^k \otimes M_0}((r,s)), \tag{6.13}$$

and note that this is equal to $(M^{k-1} \otimes !) \circ \ldots \circ !((r,s))$. We would get the same result by starting with (r,s) in M_0 and going across the top of the diagram and then down to $M^k \otimes B$ via $M^k \otimes S_B$, or by going down to B via S_B first and then across the bottom of the diagram. Thus, we have

$$\begin{array}{rcl} M^k \otimes S_B(n_1 \otimes \ldots \otimes n_k \otimes (t_k, u_k)) & = & n_1 \otimes \ldots \otimes n_k \otimes S_B((t_k, u_k)) \\ & = & m_1 \otimes \ldots \otimes m_k \otimes S_B((r_k, s_k)). \end{array} \tag{6.14}$$

So these must be equivalent under E. Since E does not depend on B, we must also have

$$n_1 \otimes \ldots \otimes n_k \otimes S_{M_0}((t_k, u_k)) = m_1 \otimes \ldots \otimes m_k \otimes S_{M_0}((r_k, s_k)).$$

Thus, $S_{M^k \otimes M_0}((r,s)) = m_1 \otimes \ldots \otimes m_k \otimes (r_k, s_k)$.

Now we will show that for all $\epsilon > 0$,

$$d_Q(h(S_B((r,s))), S_Q((r,s))) < \epsilon,$$

and this gives our result. Let $\epsilon > 0$ be given and choose k sufficiently large so that $\frac{2}{3^k} < \frac{\epsilon}{2}$ and

$$d_Q(h(S_B((r,s))), i([h_k(S_B((r,s)))])) < \frac{\epsilon}{2}. \tag{6.15}$$

For this k,

$$
\begin{array}{rll}
& d_Q(h(S_B((r,s))), S_Q((r,s))) & \\
\leq & d_Q(h(S_B((r,s))), i([h_k(S_B((r,s)))])) + d_Q(i([h_k(S_B((r,s)))]), S_Q((r,s))) & (1) \\
\leq & d_G([h_k(S_B((r,s)))], S_G((r,s))) + \frac{\epsilon}{2} & (2) \\
\leq & d_G([h_k(S_B((r,s)))], [S_{M^k \otimes M_0}((r,s))]) + \frac{\epsilon}{2} & (3) \\
\leq & d_{M^k \otimes M_0}(h_k(S_B((r,s))), S_{M^k \otimes M_0}((r,s))) + \frac{\epsilon}{2} & \\
= & d_{M^k \otimes M_0}(m_1 \otimes \ldots \otimes m_k \otimes (0,0), m_1 \otimes \ldots \otimes m_k \otimes (r_k, s_k)) + \frac{\epsilon}{2} & (4) \\
\leq & \frac{2}{3^k} + \frac{\epsilon}{2} & (5) \\
< & \epsilon &
\end{array}
$$

(1) is by the triangle inequality. (2) and (3) are by (6.15), (6.12), and the fact that $i : G \hookrightarrow Q$ is an isometric embedding. (4) is by (6.13) and (6.14). (5) is by Corollary 4.4.1.

Thus, $h(S_B((r,s))) = S_Q((r,s))$ for all $(r,s) \in M_0$. So h is a SquaMS morphism. ∎

6.4.9. THEOREM. *$(Q, \gamma\colon Q \longrightarrow M \otimes Q)$ is the final $M \otimes -$: SquaMS $\to$ SquaMS coalgebra.*

PROOF. The proof is the same as that of Theorem 6.2.4, except that Lemma 6.4.8 is used to show that every coalgebra has a morphism into Q, instead of Lemmas 6.1.13 and 6.2.3. ∎

By the same proof as in Corollary 6.3.4, we get the following.

6.4.10. COROLLARY. *$(Q, \gamma : Q \to M \otimes Q)$ is the final $M \otimes -$: SquaSet $\to$ SquaSet coalgebra.*

7. Bilipschitz equivalence

Our concluding task in this paper is to show that even though the Sierpinski carpet $\mathbb{S}$ is not isomorphic to (Q, γ), the final $(M \otimes -)$-coalgebra, the two are bilipschitz equivalent. We begin by recalling the definitions. A function $f : A \longrightarrow B$ between metric spaces is *bilipschitz continuous* if there is a number $K \geq 1$ so that

$$\frac{1}{K} d_A(x,y) \leq d_B(f(x), f(y)) \leq K d_A(x,y)$$

for all $x, y \in A$. In addition A and B are are *bilipschitz equivalent* if there is a bilipschitz continuous bijection $f : A \longrightarrow B$.

We remind the reader that the metric on $\mathbb{S}$ is the metric induced from the taxicab metric on U_0 (see just above Definition 2.2.5). Recall that, by Proposition 2.2.4, $\mathbb{S}$ with the taxicab metric is bilipschitz equivalent to $\mathbb{S}$ with the Euclidean metric, so we obtain the result by considering $\mathbb{S}$ with the taxicab metric.

As we have seen in Theorem 6.3.3, $(\mathbb{S}, \tau : M \otimes \mathbb{S} \to \mathbb{S})$ is a corecursive algebra for $M \otimes -$ in SquaMS. By Corollary 6.3.4, $(\mathbb{S}, \tau^{-1} : \mathbb{S} \to M \otimes \mathbb{S})$ is a final coalgebra in SquaSet, and in particular, it is a coalgebra. In addition, since $(Q, \gamma : Q \to M \otimes Q)$ is a coalgebra, we have a unique coalgebra-to-algebra morphism $\gamma^\dagger : Q \to \mathbb{S}$. And since (Q, γ)

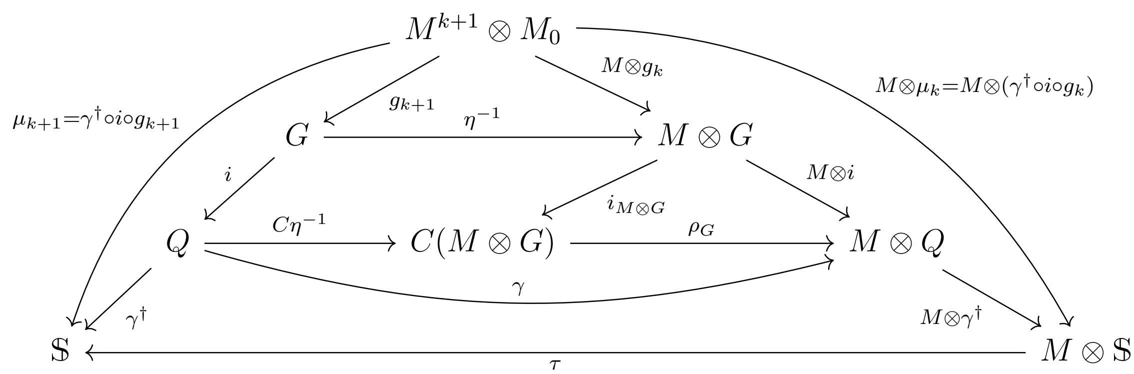
is a final $M \otimes -$ coalgebra (see Corollary 6.4.10), there is a unique SquaSet morphism $(\tau^{-1})^\dagger : \mathbb{S} \to Q$. By finality,

$$\begin{aligned}(\tau^{-1})^\dagger \circ \gamma^\dagger &= \mathsf{id}_Q,\\ \gamma^\dagger \circ (\tau^{-1})^\dagger &= \mathsf{id}_{\mathbb{S}}.\end{aligned}$$

Hence, $\gamma^\dagger$ is a bijection. However, the inverse of $\gamma^\dagger$ is not a short map, so $\gamma^\dagger$ is not a SquaMS isomorphism. We are going to prove that $\gamma^\dagger$ is a bilipschitz bijection.

Since $\gamma^\dagger$ is a short map, we need only find $K \geq 1$ such that $\frac{1}{K} d_Q(x, y) \leq d_{\mathbb{S}}(\gamma^\dagger(x), \gamma^\dagger(y))$. We shall show that $K = 2$ works. To accomplish this, we will first consider maps from $M^k \otimes M_0$ to U_0. The inclusion $\mathbb{S} \hookrightarrow U_0$ is an isometric embedding, by our definition of the metric on $\mathbb{S}$. We prefer to use U_0 in most of this section because it is easier to visualize $M \otimes U_0$ than $M \otimes \mathbb{S}$.

Recall from (6.8) that we also have an isometric embedding $i : G \hookrightarrow Q$. So for each $k < \omega$, we have a morphism $\mu_k = \gamma^\dagger \circ i \circ g_k : M^k \otimes M_0 \to \mathbb{S}$, as in the diagram below:



The top triangle commutes by the definition of the maps η and g_k (see (5.10) and (5.11)). The square below it commutes since i is the component of the natural transformation $Id \longrightarrow C$ which we saw in Lemma 4.8.1. We set aside for a moment the commutativity of the triangle next to this square. The map γ was defined in (6.7) to be $\rho_G \circ C\eta^{-1}$. The bottom commutes by definition of $\gamma^\dagger$.

It remains to consider the triangle in the middle of the figure. Consider $m \otimes x \in M \otimes G$. Using our definitions, we have the desired equation

$$\rho_G(i_{M \otimes G}(m \otimes x)) = \rho_G(m \otimes x, m \otimes x, \ldots) = m \otimes (x, x, \ldots) = (M \otimes i)(m \otimes x).$$

Thus the triangle commutes. The overall figure shows that for every k,

$$\mu_{k+1} = \tau \circ (M \otimes \mu_k). \tag{7.1}$$

We will examine the relationship between distances in $M^k \otimes M_0$ and between corresponding points in $\mathbb{S} \subset U_0$, and then use this to obtain the result. We start with the following fact about points in $M^k \otimes M_0$ whose images under μ_k are on a horizontal or vertical segment.

Throughout we will be using the fact that $d_{\mathbb{S}}$ is the taxicab metric on $\mathbb{S}$ as a subset of U_0 (see (2.3)).

7.0.1. Lemma. *Let $k \geq 0$ and $x, y \in M^k \otimes M_0$ be such that $\mu_k(x)$ and $\mu_k(y)$ share either an x-coordinate or a y-coordinate. Then*

$$d_{M^k \otimes M_0}(x, y) \leq 2d_{\mathbb{S}}(\mu_k(x), \mu_k(y)).$$

Proof. We will show this for $x, y \in M^k \otimes M_0$ such that $\mu_k(x)$ and $\mu_k(y)$ share a y-coordinate; the other case is proved similarly. So we will show that for all $k \geq 0$, if $x, y \in M^k \otimes M_0$ and $\mu_k(x) = (r, s)$, $\mu_k(y) = (t, s)$ for some $(r, s), (t, s) \in [0, 1]^2$, then $d_{M^k \otimes M_0}(x, y) \leq 2d_{\mathbb{S}}(\mu_k(x), \mu_k(y))$. We prove this by induction on k.

If $k = 0$, since μ_0 is a SquaMS morphism, $\mu_0(x) = \mu_0(S_{M_0}(x)) = S_{\mathbb{S}}(x) = x$ (since S_{M_0} is the identity on M_0 and $S_{\mathbb{S}}$ is the inclusion of $M_0 \hookrightarrow \mathbb{S}$), and similarly, $\mu_0(y) = y$. We are going to consider the case $r = 0$ and $t = 1$; the other cases are either similar or easier. So $x = (0, s)$ and $y = (1, s)$. Recall, as in Example 3.0.4, the distance in M_0 is the path metric. So the distance in M_0 from x to y is $1+2s$ when $s \leq \frac{1}{2}$, and it is $1+2(1-s) = 3-2s$ when $s \geq \frac{1}{2}$. In either case, this is ≤ 2. By (SQ$_2$) $d_{\mathbb{S}}(x, y) \geq |t - r| + |s - s| = 1$, so we have $d_{M_0}(x, y) \leq 2d_{\mathbb{S}}(x, y) = 2d_{\mathbb{S}}(\mu_0(x), \mu_0(y))$.

Now assume the result for k and suppose $x, y \in M^{k+1} \otimes M_0$. Let us write $x = m \otimes x'$ and $y = n \otimes y'$, where m and n belong to M, and $x', y' \in M^k \otimes M_0$. (We emphasize that n denotes an element of M, not a number.) We argue by cases on m and n.

Our first case is when $m = n$. We thus assume that $\mu_{k+1}(m \otimes x') = (r, s)$ and $\mu_{k+1}(m \otimes y') = (t, s)$. By (7.1), $\tau(m \otimes \mu_k(x')) = (r, s)$ and $\tau(m \otimes \mu_k(y')) = (t, s)$.

Now τ works the same way as α_M (it is a domain-codomain restriction of α_M, see (6.2) and (5.1)). And so we see easily that $\mu_k(x')$ and $\mu_k(y')$ have the same y-coordinate. So

$$\begin{aligned}
d_{M^{k+1} \otimes M_0}(x, y) &= \tfrac{1}{3} d_{M^k \otimes M_0}(x', y') \\
&\leq \tfrac{1}{3} \cdot 2d_{\mathbb{S}}(d(\mu_k(x'), \mu_k(y'))) && \text{by induction hypothesis} \\
&= 2d_{M \otimes \mathbb{S}}(m \otimes \mu_k(x'), m \otimes \mu_k(y')) && \text{by (6.2)} \\
&= 2d_{\mathbb{S}}(\tau(m \otimes \mu_k(x')), \tau(m \otimes \mu_k(y'))) && \text{see below} \\
&= 2d_{\mathbb{S}}(\mu_{k+1}(x), \mu_{k+1}(y))
\end{aligned}$$

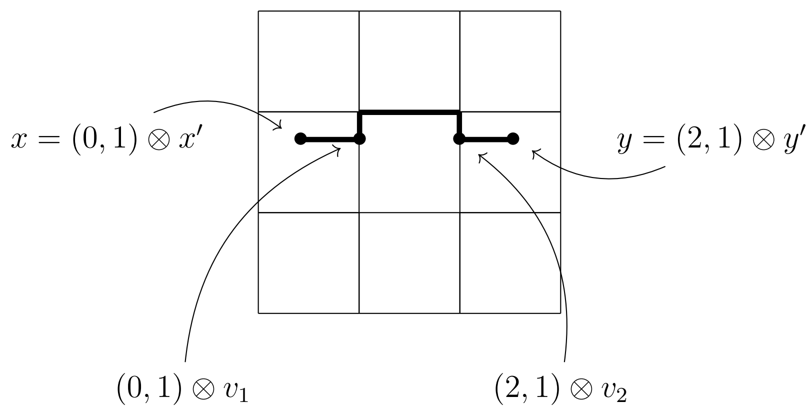
For the "see below" line, we use the fact that within a particular copy $m \otimes \mathbb{S}$, the restriction of τ is an isometric embedding.

Indeed, for $z_1, z_2 \in \mathbb{S}$ and $m \in M$,

$$\begin{aligned}
d_{\mathbb{S}}(\tau(m \otimes z_1), \tau(m \otimes z_2)) &= d_{\mathbb{S}}(\tfrac{1}{3}m + \tfrac{1}{3}z_1, \tfrac{1}{3}m + \tfrac{1}{3}z_2) \\
&= \tfrac{1}{3} d_{\mathbb{S}}(z_1, z_2) \\
&= d_{M \otimes \mathbb{S}}(m \otimes z_1, m \otimes z_2).
\end{aligned}$$

Our second case is when m and n are adjacent squares in M. (For example, we could have $m = (0,0)$, and $n = (1,0)$ or $n = (0,1)$.) The argument in this case is a small elaboration of what we saw in the first case. Our work below on a more complicated case subsumes this one, and so we shall pass over this particular case. The same holds for our third case, when we have $m = (0,0)$, and $n = (0,2)$, or another pair which is a rotation or reflection of this one. The main case which is *not* handled is when $m = (0,1)$ and $n = (2,1)$, or some rotation or reflection of this. In such cases, the shortest path in $M^{k+1} \otimes M_0$ from x to y must "navigate around the central hole."

Without loss of generality, suppose that $s \geq \frac{1}{2}$ (see below, $s < \frac{1}{2}$ is similar). Then there is a path in $M^{k+1} \otimes M_0$ from x to y of the following form:



where $v_1 = S_{M^k \otimes M_0}((1, 3s-1))$ and $v_2 = S_{M^k \otimes M_0}((0, 3s-1))$. Then $\mu_{k+1}((0,1) \otimes v_1) = (\frac{1}{3}, s)$ and $\mu_{k+1}((2,1) \otimes v_2) = (\frac{2}{3}, s)$.

The picture suggests going around the top of the middle square: This is because we assume $s \geq \frac{1}{2}$. (If $s < \frac{1}{2}$, then we get an analogous shorter path going around the bottom of the middle square.) Then since the distance in $M^{k+1} \otimes M_0$ is the score of the shortest path, we have

$$\begin{aligned} d_{M^{k+1} \otimes M_0}(x,y) &\leq \tfrac{1}{3} d_{M^k \otimes M_0}(x', v_1) + (\tfrac{2}{3} - s) + \tfrac{1}{3} + (\tfrac{2}{3} - s) + \tfrac{1}{3} d_{M^k \otimes M_0}(v_2, y') \\ &\leq \tfrac{1}{3}(d_{M^k \otimes M_0}(x', v_1) + 2 + d_{M^k \otimes M_0}(v_2, y')) \end{aligned}$$

since $s \geq \frac{1}{2}$.

Let $(r', s') = \mu_k(x')$ and note that $\mu_k(v_1) = (1, s')$. By the induction hypothesis, $d_{M^k \otimes M_0}(x', v_1) \leq 2(1 - r')$. Further note that

$$(r, s) = (\tfrac{1}{3}(r'), \tfrac{1}{3}(1 + s')).$$

Thus, $d_{M^k \otimes M_0}(x', v_1) \leq 2(1 - 3r)$. Similarly, $d_{M^k \otimes M_0}(v_2, y') \leq 2(3t - 2)$. So, using our calculation above,

$$\begin{array}{rcl} d_{M^{k+1}\otimes M_0}(x,y) & \leq & \frac{1}{3}(2(1-3r)+2+2(3t-2)) \\ & = & 2(t-r) \\ & = & 2d_{\$}((r,s),(t,s)) \\ & = & 2d_{\$}(\mu_{k+1}(x),\mu_{k+1}(y)) \end{array}$$

as required.

This covers all of the possible cases in which $\mu_k(x)$ and $\mu_k(y)$ share a y-coordinate. ∎

Next, we need to show that the distance between the images $\gamma^\dagger(x)$ and $\gamma^\dagger(y)$ can be calculated as the sum of horizontal and vertical segments between endpoints *in the image of* μ_k. This is what will allow us to compare the distance in $M^k \otimes M_0$ to the distance in \$.

7.0.2. Lemma. *For $k \geq 0$, given $x, y \in M^k \otimes M_0$, the distance between $\mu_k(x)$ and $\mu_k(y)$ in \$ is the sum of the lengths of at most four horizontal or vertical segments whose endpoints are in the image of μ_k.*

Proof. Let $x, y \in M^k \otimes M_0$ be given, and let $\mu_k(x) = (r,s)$, $\mu_k(y) = (t,u)$. Without loss of generality, suppose that $r \leq t$ and $s \leq u$ (the other cases are similar). Consider the point (t,s).

Case 1: (t,s) is in the image of $M^k \otimes M_0$ under $\mu_k : M^k \otimes M_0 \to \$$, let $z \in M^k \otimes M_0$ be such that $\mu_k(z) = (t,s)$. Then

$$d_{\$}(\mu_k(x),\mu_k(y)) = d_{\$}(\mu_k(x),\mu_k(z)) + d_{\$}(\mu_k(z),\mu_k(y)).$$

In this case, we are done.

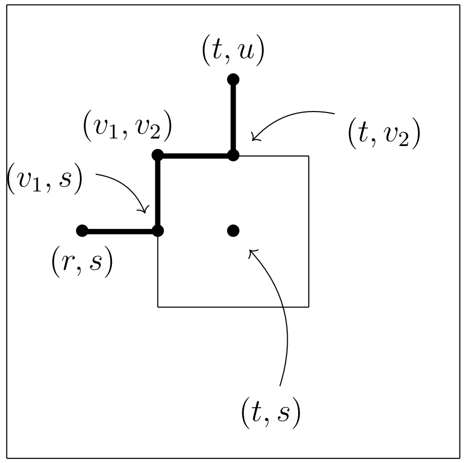
Case 2: (t,s) is not in the image of $M^k \otimes M_0$ under μ_k. That is, (t,s) occurs in a "hole" which we will need to navigate around. Again, we are restricting our attention to the case when $r \leq t$ and $s \leq u$ (the other cases are analogous).

7.0.3. Claim. *For every $k \geq 0$, if $x, y \in M^k \otimes M_0$ and $\mu_k(x) = (r,s)$ and $\mu_k(y) = (t,u)$ and (t,s) is not in the image of μ_k, then there exist $z, z_1, z_2 \in M^k \otimes M_0$ such that*

$$\begin{array}{rcl} d_{\$}(\mu_k(x),\mu_k(y)) & = & d_{M^k\otimes M_0}(\mu_k(x),\mu_k(z_1)) + d_{M^k\otimes M_0}(\mu_k(z_1),\mu_k(z)) \\ & + & d_{M^k\otimes M_0}(\mu_k(z),\mu_k(z_2)) + d_{M^k\otimes M_0}(\mu_k(z_2),\mu_k(y)) \end{array} \quad (7.2)$$

The idea is indicated in the picture below (which may not be to scale, the "hole" may be much smaller and off to one side). The points $z, z_1, z_2 \in M^k \otimes M_0$ are such that

$\mu_k(z_1) = (v_1, s)$, $\mu_k(z) = (v_1, v_2)$, and $\mu_k(z_2) = (t, v_2)$.



Here is the relation of the picture to (7.2). On the right of (7.2), each term comes from a horizontal or vertical segment in U_0. In particular, $\mu_k(x)$ and $\mu_k(z_1)$ share y-coordinates, $\mu_k(z_1)$ and $\mu_k(z)$ share x-coordinates, $\mu_k(z)$ and $\mu_k(z_2)$ share y-coordinates, and $\mu_k(z_2)$ and $\mu_k(y)$ share x-coordinates.

Now we prove the claim by induction on k. When $k = 0$, we must have $(r, s), (t, u) \in M_0$, so since we have $r \le t$ and $s \le u$, the only case in which $(t, s) \notin M_0$ is if $r = 0$ and $u = 1$. But in this case, we can let $z = z_1 = z_2 = (0, 1)$.

Assume the claim for some fixed $k \ge 0$ and let $x, y \in M^{k+1} \otimes M_0$. We will consider two cases for (t, s): when it appears in the center "hole", that is, in $(\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3})$, and when it does not.

First suppose it does not. We will consider the particular case when $(t, s) \in [\frac{2}{3}, 1] \times [0, \frac{1}{3}]$, the bottom right corner. The rest of the cases are similar.

If $\mu_{k+1}(x)$ is also in this corner, let x' be such that $x = (2, 0) \otimes x'$. Otherwise, let $x' = S_{M^k \otimes M_0}((0, 3s))$. Similarly, if $\mu_{k+1}(y)$ is in this bottom right corner, let y' be such that $y = (2, 0) \otimes y'$. Otherwise let $y' = S_{M^k \otimes M_0}((3t - 2, 1))$. Note that $(3t - 2, 3s)$ is not in the image of μ_k, or else we could have z such that $\mu_k(z) = (3t - 2, 3s)$, and thus, we would have

$$\mu_{k+1}((2, 0) \otimes z) = \tau \circ M \otimes \mu_k((2, 0) \otimes z) = \tau((2, 0) \otimes (3t - 2, 3s)) = (t, s),$$

a contradiction to our assumption. So by the induction hypothesis, there are $z_1, z, z_2 \in M^k \otimes M_0$ such that $\mu_k(z_1) = (v_1, 3s)$, $\mu_k(z) = (v_1, v_2)$, and $\mu_k(z_2) = (3t - 2, v_2)$. Then

$$\begin{array}{rcl} \mu_{k+1}((2, 0) \otimes z_1) & = & (\frac{1}{3}(2 + v_1), s) \\ \mu_{k+1}((2, 0) \otimes z) & = & (\frac{1}{3}(2 + v_1), \frac{1}{3}(v_2)) \\ \mu_{k+1}((2, 0) \otimes z_2) & = & (t, \frac{1}{3}(v_2)) \end{array}$$

These are as required since the successive segments they determine are horizontal or vertical; see the picture above.

Finally, suppose (t, s) is in $(\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3})$. Then we must have $\mu_{k+1}(x) \in [0, \frac{1}{3}] \times [\frac{1}{3}, \frac{2}{3}]$ and $\mu_{k+1}(y) \in [\frac{1}{3}, \frac{2}{3}] \times [\frac{2}{3}, 1]$. Let

$$\begin{array}{rcl} z_1 & = & (0,1) \otimes S_{M^k \otimes M_0}((1, 3t-1)) \\ z & = & (0,1) \otimes S_{M^k \otimes M_0}((1,1)) \\ z_2 & = & (1,2) \otimes S_{M^k \otimes M_0}((3t-1, 0)) \end{array}$$

Then $\mu_{k+1}(z_1) = (\frac{1}{3}, s)$, $\mu_{k+1}(z) = (\frac{1}{3}, \frac{2}{3})$, and $\mu_{k+1}(z_2) = (t, \frac{2}{3})$. These again are as required in our claim.

This concludes our induction proof of the claim. Applying it, we can express the distance $d_{\mathbb{S}}(\mu_k(x), \mu_k(y))$ as the sum of the lengths of at most 4 horizontal and vertical segments with endpoints in the image of μ_k. ■

Putting the last lemmas in this section together, we get the following:

7.0.4. Proposition. *For $k \geq 0$ and $x, y \in M^k \otimes M_0$,*

$$d_{M^k \otimes M_0}(x, y) \leq 2 d_{\mathbb{S}}(\mu_k(x), \mu_k(y)).$$

Proof. Let $x, y \in M^k \otimes M_0$, and let $\mu_k(x) = (r, s)$ and $\mu_k(y) = (t, u)$. As in Lemma 7.0.2, assume without loss of generality that $r \leq t$ and $s \leq u$ (the other cases are similar).

If (t, s) is in the image of $M^k \otimes M_0$ under μ_k (as in Case 1 in the proof of Lemma 7.0.2), let z be such that $\mu_k(z) = (t, s)$, and let $z_1 = z_2 = z$.

Otherwise, if (t, s) is not in the image of $M^k \otimes M_0$ under μ_k, let $z_1, z, z_2 \in M^k \otimes M_0$ be as in Claim 7.0.3 of Case 2 in the proof of Lemma 7.0.2. Then in either case,

$$d_{M^k \otimes M_0}(x, y) \leq d_{M^k \otimes M_0}(x, z_1) + d_{M^k \otimes M_0}(z_1, z) + d_{M^k \otimes M_0}(z, z_2) + d_{M^k \otimes M_0}(z_2, y).$$

We use the fact that these are each horizontal or vertical segments in $\mathbb{S}$, and also Lemmas 7.0.1 and 7.0.2 to see that

$$\begin{array}{rl} & d_{M^k \otimes M_0}(x, z_1) + d_{M^k \otimes M_0}(z_1, z) + d_{M^k \otimes M_0}(z, z_2) + d_{M^k \otimes M_0}(z_2, y) \\ \leq & 2(d_{\mathbb{S}}(\mu_k(x), \mu_k(z_1)) + d_{\mathbb{S}}(\mu_k(z_1), \mu_k(z)) + d_{\mathbb{S}}(\mu_k(z), \mu_k(z_2)) + d_{\mathbb{S}}(\mu_k(z_2), \mu_k(y))) \\ = & 2 d_{\mathbb{S}}(\mu_k(x), \mu_k(y)) \end{array}$$

■

Next, we need a version of Proposition 7.0.4 for G. This comes almost immediately from the fact that for any $x \in G$, there exists k and $x' \in M^k \otimes M_0$ such that $g_k(x') = [x'] = x$.

7.0.5. Proposition. *For $x, y \in G$,*

$$d_G(x, y) \leq 2 d_{\mathbb{S}}(\gamma^{\dagger} \circ i(x), \gamma^{\dagger} \circ i(y)).$$

PROOF. Let $x, y \in G$ and $\epsilon > 0$ be given. Choose $k \geq 0$ sufficiently large and $x', y' \in M^k \otimes M_0$ with $[x'] = x$ and $[y'] = y$, and $|d_G(x,y) - d_{M^k \otimes M_0}(x', y')| < \frac{\epsilon}{2}$. Then $\gamma^\dagger \circ i(x) = \gamma^\dagger \circ i \circ g_k(x') = \mu_k(x')$ and similarly for y. Hence

$$\begin{array}{rcll} d_G(x,y) & \leq & d_{M^k\otimes M_0}(x',y') + \frac{\epsilon}{2} & \\ & \leq & 2(d_{\$}(\mu_k(x'), \mu_k(y')) + \frac{\epsilon}{2}) & \text{by Proposition 7.0.4} \\ & = & 2d_{\$}(\gamma^\dagger \circ i(x), \gamma^\dagger \circ i(y)) + \epsilon & \end{array}$$

Since $\epsilon > 0$ was arbitrary, we get the required inequality. ■

Finally, we will use the following very general fact along with the fact that G is dense in Q to get our result.

7.0.6. PROPOSITION. *Let $f : A \to B$ be a short map between metric spaces and let D be dense in A. If $K \geq 1$ is such that*

$$d_A(x,y) \leq K d_B(f(x), f(y))$$

for all $x, y \in D$, then this same inequality holds for all $x, y \in A$.

PROOF. Let $\epsilon > 0$ and $x, y \in A$ be given. Choose $x', y' \in D$ such that $d_A(x, x') < \frac{\epsilon}{4K}$ and $d_A(y, y') < \frac{\epsilon}{4K}$. Since f is a short map,

$$\begin{array}{rcll} d_A(x,y) & \leq & d_A(x',y') + 2(\frac{\epsilon}{4K}) & \\ & \leq & K d_B(f(x'), f(y')) + \frac{\epsilon}{2K} & \\ & \leq & K(d_B(f(x), f(y)) + 2(\frac{\epsilon}{4K})) + \frac{\epsilon}{2K} & \text{see below} \\ & = & K d_B(f(x), f(y)) + \frac{\epsilon}{2} + \frac{\epsilon}{2K} & \\ & \leq & K d_B(f(x), f(y)) + \epsilon & \end{array}$$

In the line marked "see below", we use the fact that f is short to see that $d_B(f(x), f(x')) < \frac{\epsilon}{4K}$ and similarly for y. Since $\epsilon > 0$ was arbitrary, $d_A(x,y) \leq K d_B(f(x), f(y))$ for all $x, y \in A$, as required. ■

7.0.7. THEOREM. *The metric space Q is bilipschitz equivalent to the Sierpinski carpet $\$$ as a subset of the plane with the taxicab metric, and thus, the Euclidean metric.*

PROOF. We use $\gamma^\dagger : Q \longrightarrow \$$. This is a short bijection, it has the additional property that $d_Q(x,y) \leq 2d_{\$}(\gamma^\dagger(x), \gamma^\dagger(y))$. In this last estimate, we use Proposition 7.0.6, taking A to be Q and the dense set D to be the image of G under the isometric embedding i. We also use Proposition 7.0.5. ■

8. Conclusion

Stepping back, the main point of this paper has been to further the interaction between the subject of coalgebra broadly considered (including corecursive algebras) and continuous mathematics. The questions that we asked in this paper concerned the relationship between very natural and very concrete fractal sets on the one hand, and more abstract ideas like initial algebras and final coalgebras on the other. We came to this work in order to explore these general issues. What we found in the exploration was a set of ideas connecting category-theoretic and analytic concepts such as colimits in metric spaces, short maps approximated by non-short maps, and corecursive algebras as an alternative to infinite sums. We hope that the results in this paper further these connections.

Here are two general next steps in this line of research. First, it would be desirable to merge the ideas here with the general categorical framework for self-similarity developed in Leinster [12]. This would mean taking assumptions on our category SquaMS (such as $(\textsc{sq}_1)$ and $(\textsc{sq}_2)$) and also assumptions on the functor (see Theorem 4.1.12) and incorporating them as additional assumptions in Leinster's framework, in addition to the requirements needed there, such as the non-degeneracy requirements. In a different direction, one would want to know which aspects of the classical theory of fractals may be derived from the universal properties which we have established.

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