

# ON THE LACK OF COLIMITS IN VARIOUS CATEGORIES ARISING IN POINTFREE TOPOLOGY AND ALGEBRAIC LOGIC

MARCO ABBADINI, GURAM BEZHANISHVILI, LUCA CARAI

**ABSTRACT.** We prove that the category of McKinsey-Tarski algebras is not equivalent to a variety of algebras, thus answering a question of Peter Jipsen in the negative. More generally, we show that various categories of BAOs (boolean algebras with an operator), Heyting algebras, and frames with appropriate morphisms between them are not cocomplete. As a consequence, none of these categories is equivalent to a prevariety, let alone a variety.

## 1. Introduction

Boolean algebras and Heyting algebras are some of the most studied classes of algebras in algebraic logic. It is well known that both classes are closed under homomorphic images, subalgebras, and products, hence form a variety. This allows the use of powerful tools from universal algebra in their study. The situation changes when we restrict our attention to the categories **CBA** of complete boolean algebras with complete boolean morphisms and **CHA** of complete Heyting algebras with complete Heyting morphisms. Indeed, it is a classic result from the 1960s that the free complete boolean algebra on countably many generators does not exist [Gai64, Hal64]. De Jongh [dJ80] proved that already the two-generated free complete Heyting algebra does not exist (see also [BK24]). Thus, neither **CBA** nor **CHA** is equivalent to a variety. Natural generalizations of varieties are quasi-varieties (classes of algebras closed under isomorphisms, subalgebras, products, and ultraproducts), which further generalize to prevarieties (classes of algebras closed under isomorphisms, subalgebras, and products). It is well known that each prevariety viewed as a category is complete and cocomplete (see, e.g., [SR99, Thm. IV.2.1.3 and IV.2.2.3]). Therefore, each category that is not cocomplete is not equivalent to a prevariety. Consequently, neither **CBA** nor **CHA** is equivalent to even a prevariety.

There are other natural morphisms to consider between complete boolean algebras and between complete Heyting algebras. For example, we can look at the category of complete boolean algebras with boolean morphisms between them, as well as the category of complete Heyting algebras with Heyting morphisms, or, more generally, bounded lattice morphisms between them. We show that none of these categories is equivalent to a

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prevariety, and further generalize this to prove that the category of Heyting algebras with bounded lattice morphisms between them is also not equivalent to a prevariety.

In addition, we look at BAOs (boolean algebras with an operator  $\diamond$ ) and show that a similar phenomenon occurs there. The morphisms we consider are *stable morphisms* (that is, boolean morphisms  $f: A \rightarrow B$  satisfying  $\diamond f(a) \leq f(\diamond a)$  for each  $a \in A$ ), which play an important role in the study of the finite model property in modal logic [Ghi10, BBI16]. Such morphisms also play an important role in the study of *The Algebra of Topology* of McKinsey and Tarski [MT44, BR23]. Our article is motivated by the question of Peter Jipsen, who asked, at the [Frame Theory Workshop](#) (Chapman University, December 2024), whether the category **MT** of McKinsey-Tarski algebras with complete stable morphisms is (equivalent to) a variety. We answer this question in the negative by showing that **MT** is not equivalent even to a prevariety. In fact, we show that neither **MT** nor the category of McKinsey-Tarski algebras with stable morphisms is equivalent to a prevariety. These results generalize to various categories of complete BAOs with these two types of morphisms, as well as to various categories of BAOs with stable morphisms.

We briefly describe the flavor of our results. In Section 2 we show that **MT** lacks some countable copowers, and hence is not equivalent to a prevariety. We then generalize this result to various categories of complete BAOs with complete stable morphisms. In Section 3 we prove that dropping the completeness assumption from the above notion of morphism results in categories that lack even some binary copowers. In Section 4 we show that dropping the completeness assumption also from the objects results in categories that lack some coequalizers. Finally, in Section 5 we prove that the category of frames with Heyting morphisms lacks some binary copowers, and so does the category of frames with bounded lattice morphisms. Furthermore, we prove that the category of Heyting algebras with bounded lattice morphisms lacks some coequalizers.

In Sections 3 to 5 our main tool is duality theory: Stone duality for boolean algebras, Jónsson-Tarski duality for BAOs, and Esakia and Priestley dualities for Heyting algebras and bounded distributive lattices. Dualizing the fact that prevarieties are cocomplete, it is then sufficient to show that the dual categories in question are not complete. This we do by showing that in these categories either products or equalizers do not exist.

## 2. MT-algebras

In this section, we show that **MT** is not cocomplete. As a consequence, it is not equivalent to a variety, resolving the question of Peter Jipsen in the negative. We also show that the same technique applies to several other categories of complete BAOs with complete stable morphisms, yielding that they are not cocomplete, and hence not equivalent to a prevariety.

We recall (see [JT51]) that an *operator* on a boolean algebra  $B$  is a unary function  $\diamond: B \rightarrow B$  preserving finite joins, and that the pair  $(B, \diamond)$  is called a *boolean algebra with an operator* or simply a *BAO*. A *BAO-morphism* between two BAOs  $A$  and  $B$  is a boolean morphism  $f: A \rightarrow B$  such that  $f(\diamond a) = \diamond f(a)$  for each  $a \in A$ . We let **BAO**

denote the category of BAOs and BAO-morphisms.

We will mainly be interested in *stable* morphisms between BAOs [BBI16]; that is, boolean morphisms  $f: A \rightarrow B$  such that  $\diamond f(a) \leq f(\diamond a)$  for each  $a \in A$ . Stable morphisms are also known as *continuous morphisms* and play an important role in the study of axiomatization, finite model property, and decidability of modal logics [Ghi10, BBI16].

An important class of BAOs is formed by closure algebras of McKinsey and Tarski [MT44]. We recall that a *closure algebra* is a BAO  $(B, \diamond)$  satisfying  $a \leq \diamond a$  and  $\diamond \diamond a \leq \diamond a$  for each  $a \in B$ . Closure algebras play a prominent role in modal logic as they serve as algebraic models of **S4**, one of the most studied modal systems (see, e.g., [RS70]). Because of this, closure algebras are also known as **S4**-algebras. Complete closure algebras provide an alternate pointfree approach to topology, and were coined McKinsey-Tarski algebras in [BR23]. We thus arrive at the main definition of this section:

2.1. DEFINITION. *A McKinsey-Tarski algebra, or simply an MT-algebra, is a complete closure algebra. An MT-morphism between MT-algebras is a complete boolean morphism that is stable. Let **MT** denote the category of MT-algebras and MT-morphisms.*

Our aim is to show that **MT** lacks some countable copowers. For this, we utilize the following well-known result from the 1960s. Let **BA** denote the category of boolean algebras and boolean morphisms, and **CBA** the category of complete boolean algebras and complete boolean morphisms.

2.2. THEOREM. [Gai64, Hal64] *The free countably generated complete boolean algebra does not exist. Thus, **CBA** is not cocomplete, and hence is not equivalent to a prevariety.*

We clearly have the forgetful functor  $\mathcal{U}: \mathbf{MT} \rightarrow \mathbf{CBA}$ . We start by proving that it has both left and right adjoints. Following [Hal56], for each boolean algebra  $B$ , we consider the so-called *simple operator*  $\diamond_s: B \rightarrow B$  given by

$$\diamond_s a = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a \neq 0 \end{cases}$$

for each  $a \in B$ . It is straightforward to see that  $(B, \diamond_s)$  is a closure algebra, and hence an MT-algebra whenever  $B$  is complete.

2.3. THEOREM. *The forgetful functor  $\mathcal{U}: \mathbf{MT} \rightarrow \mathbf{CBA}$  is both a left and right adjoint, and hence preserves both colimits and limits.*

PROOF. We define the functor  $\mathcal{L}: \mathbf{CBA} \rightarrow \mathbf{MT}$  as follows. On objects,  $\mathcal{L}(A) = (A, \diamond_s)$ ; and on morphisms,  $\mathcal{L}$  is the identity. We prove that, for  $A \in \mathbf{CBA}$  and  $B \in \mathbf{MT}$ , we have

$$\text{hom}_{\mathbf{MT}}(\mathcal{L}(A), B) \cong \text{hom}_{\mathbf{CBA}}(A, \mathcal{U}(B))$$

where the bijection is the identity. The inclusion  $\subseteq$  is clear since each **MT**-morphism is a complete boolean morphism. For the reverse inclusion, suppose  $f: A \rightarrow \mathcal{U}(B)$  is a

complete boolean morphism. We claim that it is a stable morphism from  $\mathcal{L}(A)$  to  $B$ , i.e. that  $\diamond f(a) \leq f(\diamond_s a)$ . If  $a = 0$ , then

$$\diamond f(0) = \diamond 0 = 0 = f(0) = f(\diamond_s 0).$$

If  $a \neq 0$ , then

$$\diamond f(a) \leq 1 = f(1) = f(\diamond_s a).$$

Thus,  $f$  is an **MT**-morphism.

We next define the functor  $\mathcal{R}: \mathbf{CBA} \rightarrow \mathbf{MT}$  as follows. On objects,  $\mathcal{R}(A) = (A, \diamond_i)$ , where  $\diamond_i$  is the identity on  $A$ ; and on morphisms,  $\mathcal{R}$  is the identity. Since  $(A, \diamond_i)$  is an **MT**-algebra,  $\mathcal{R}$  is well defined. We prove that, for  $A \in \mathbf{CBA}$  and  $B \in \mathbf{MT}$ , we have

$$\text{hom}_{\mathbf{MT}}(B, \mathcal{R}(A)) = \text{hom}_{\mathbf{CBA}}(\mathcal{U}(B), A).$$

The inclusion  $\subseteq$  is clear because each **MT**-morphism is a complete boolean morphism. For the reverse inclusion, suppose  $g: \mathcal{U}(B) \rightarrow A$  is a **CBA**-morphism. Since  $(B, \diamond)$  is an **MT**-algebra,  $a \leq \diamond a$  for every  $a \in B$ . Therefore,  $g$  is a stable morphism from  $B$  to  $\mathcal{R}(A)$  because, for every  $a \in B$ , we have  $\diamond_i g(a) = g(a) \leq g(\diamond a)$ . Thus,  $g$  is an **MT**-morphism. ■

**2.4. THEOREM.** *Let  $\mathbf{C}$  be a category and  $\mathcal{U}: \mathbf{C} \rightarrow \mathbf{CBA}$  a colimit-preserving functor. If there is  $X \in \mathbf{C}$  such that  $\mathcal{U}(X)$  is the four-element boolean algebra, then  $\mathbf{C}$  lacks some countable copowers.*

**PROOF.** Suppose all countable copowers exist in  $\mathbf{C}$ . Since  $\mathcal{U}: \mathbf{C} \rightarrow \mathbf{CBA}$  preserves colimits, there exists a countably-indexed copower  $C$  of  $\mathcal{U}(X)$  in **CBA**. Because **BA** and **CBA** share finitely generated free objects (see, e.g., [Joh82, p. 33]), the four-element boolean algebra  $\mathcal{U}(X)$  is the free object on one generator in **CBA**, and so  $C$  is the free object on  $\omega$  generators in **CBA**, contradicting Theorem 2.2. ■

As an immediate consequence of Theorems 2.3 and 2.4 we obtain:

**2.5. THEOREM.** ***MT** lacks some countable copowers, and hence is not equivalent to a prevariety.*

Apart from closure algebras, there are other classes of BAOs that play a prominent role in modal logic (see, e.g., [CZ97]). We conclude the section by showing how to extend Theorem 2.5 to those classes.

**2.6. DEFINITION.**

- (1) A **T**-algebra is a BAO  $(B, \diamond)$  such that  $a \leq \diamond a$  for each  $a \in B$ .
- (2) A **K4**-algebra is a BAO  $(B, \diamond)$  such that  $\diamond \diamond a \leq \diamond a$  for each  $a \in B$ .
- (3) A **B**-algebra is a BAO  $(B, \diamond)$  such that  $\diamond \neg \diamond a \leq \neg a$  for each  $a \in B$ .
- (4) A **TB**-algebra is a BAO which is both a **T**-algebra and a **B**-algebra.

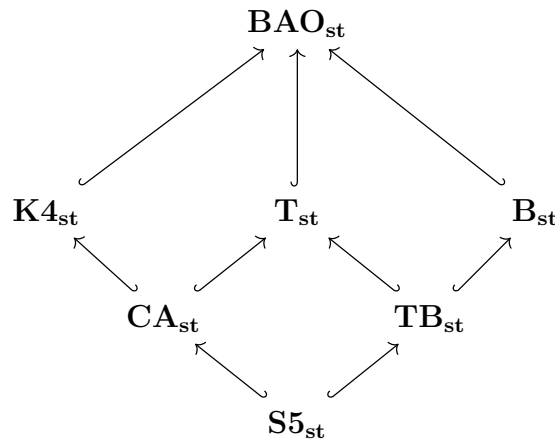
(5) An **S5**-algebra is a BAO which is both a closure algebra and a **B**-algebra.

As the name suggests, **T**-algebras serve as algebraic models of the modal system **T**, and the same applies to **K4**-algebras, **B**-algebras, **TB**-algebras, and **S5**-algebras (the latter were introduced by Halmos [Hal56] under the name of monadic algebras).

2.7. DEFINITION.

- (1) Let  $\mathbf{BAO}_{st}$  denote the category of BAOs and stable morphisms between them.
- (2) Let  $\mathbf{CA}_{st}$  denote the full subcategory of  $\mathbf{BAO}_{st}$  consisting of closure algebras.
- (3) Let  $\mathbf{T}_{st}$ ,  $\mathbf{K4}_{st}$ ,  $\mathbf{B}_{st}$ ,  $\mathbf{TB}_{st}$ , and  $\mathbf{S5}_{st}$  denote the full subcategories of  $\mathbf{BAO}_{st}$  consisting of the algebras introduced in Definition 2.6.

We have the following inclusion relationships between these categories:



We let  $\mathbf{CBAO}_{st}$  denote the category of complete BAOs with complete stable morphisms between them, and define  $\mathbf{CT}_{st}$ ,  $\mathbf{CK4}_{st}$ ,  $\mathbf{CB}_{st}$ ,  $\mathbf{CTB}_{st}$ , and  $\mathbf{CS5}_{st}$  similarly (observe that  $\mathbf{CCA}_{st} = \mathbf{MT}$ ). We have the following analog of Theorem 2.5.

2.8. THEOREM. *Each of the categories  $\mathbf{CBAO}_{st}$ ,  $\mathbf{CT}_{st}$ ,  $\mathbf{CK4}_{st}$ ,  $\mathbf{CB}_{st}$ ,  $\mathbf{CTB}_{st}$ , and  $\mathbf{CS5}_{st}$  lacks some countable copowers. Consequently, none is equivalent to a prevariety.*

PROOF. Let  $\mathbf{C}$  be one of the categories in the statement. Observe that the four-element boolean algebra equipped with  $\diamond_s$  belongs to  $\mathbf{C}$ . Thus, in view of Theorem 2.4, it is sufficient to show that the forgetful functor  $\mathcal{U} : \mathbf{C} \rightarrow \mathbf{CBA}$  is both a left and right adjoint. The same proof as in Theorem 2.3 gives a functor  $\mathcal{L} : \mathbf{CBA} \rightarrow \mathbf{C}$  that is left adjoint to  $\mathcal{U}$ . To define  $\mathcal{R} : \mathbf{CBA} \rightarrow \mathbf{C}$ , we consider cases.

First suppose that  $\mathbf{C}$  is  $\mathbf{CT}_{st}$ ,  $\mathbf{CTB}_{st}$ , or  $\mathbf{CS5}_{st}$ . Then the same proof as in Theorem 2.3 gives a functor  $\mathcal{R} : \mathbf{CBA} \rightarrow \mathbf{C}$  that is right adjoint to  $\mathcal{U}$  (since  $(A, \diamond_i) \in \mathbf{C}$  for any  $A \in \mathbf{CBA}$ ). Next suppose that  $\mathbf{C}$  is  $\mathbf{CBAO}_{st}$ ,  $\mathbf{CB}_{st}$ , or  $\mathbf{CK4}_{st}$ . Then define  $\mathcal{R} : \mathbf{CBA} \rightarrow \mathbf{C}$  by  $\mathcal{R}(A) = (A, \diamond_0)$ , where  $\diamond_0 : A \rightarrow A$  is given by  $\diamond_0 a = 0$  for each  $a \in A$ ,

and on morphisms  $\mathcal{R}$  is the identity. Since  $\mathcal{R}(A)$  is both a **K4**-algebra and a **B**-algebra,  $\mathcal{R}$  is well defined. Moreover, for  $A \in \mathbf{CBA}$  and  $B \in \mathbf{C}$ , we have

$$\mathrm{hom}_{\mathbf{C}}(B, \mathcal{R}(A)) = \mathrm{hom}_{\mathbf{CBA}}(\mathcal{U}(B), A).$$

Indeed, the inclusion  $\subseteq$  is clear since each **C**-morphism is a complete boolean morphism. For the reverse inclusion, suppose  $g: \mathcal{U}(B) \rightarrow A$  is a **CBA**-morphism. Then it is a stable morphism from  $B$  to  $\mathcal{R}(A)$  because, for every  $a \in B$ ,

$$\diamond_0 g(a) = 0 \leq g(\diamond a),$$

yielding that  $g$  is a **C**-morphism. Thus,  $\mathcal{R}: \mathbf{CBA} \rightarrow \mathbf{C}$  is right adjoint to  $\mathcal{U}$ . ■

### 3. MT-algebras with stable morphisms

In this section we show that the situation remains the same if instead of complete stable morphisms we consider all stable morphisms. We do this by showing that binary powers don't exist on the dual side. We start with the following general result.

**3.1. THEOREM.** *Let  $\mathbf{C}$  be a complete category. Suppose  $\mathcal{H}$  is a class of objects of  $\mathbf{C}$  such that every morphism  $f: X \rightarrow Y$  in  $\mathbf{C}$  is an isomorphism if, for each  $Z \in \mathcal{H}$ ,  $f \circ -: \mathrm{hom}_{\mathbf{C}}(Z, X) \rightarrow \mathrm{hom}_{\mathbf{C}}(Z, Y)$  is a bijection. Then the inclusion  $\mathbf{E} \hookrightarrow \mathbf{C}$  of any full subcategory  $\mathbf{E}$  containing all objects of  $\mathcal{H}$  preserves limits.*

**PROOF.** Let  $D: \mathbf{I} \rightarrow \mathbf{E}$  be a diagram and  $(m_i: M \rightarrow D(i))_{i \in \mathbf{I}}$  a limit cone in  $\mathbf{E}$ . Suppose  $(n_i: N \rightarrow D(i))_{i \in \mathbf{I}}$  is the limit in  $\mathbf{C}$ , which exists because  $\mathbf{C}$  is complete. Then there is a unique morphism  $f: M \rightarrow N$  such that for every  $i \in \mathbf{I}$  we have  $m_i = n_i \circ f$ . Since  $(m_i: M \rightarrow D(i))_{i \in \mathbf{I}}$ , considered as a cone in  $\mathbf{C}$ , is obtained by composing the limit cone  $(n_i: N \rightarrow D(i))_{i \in \mathbf{I}}$  with  $f$  and composing a limit cone with an isomorphism gives a limit cone, to prove that  $(m_i: M \rightarrow D(i))_{i \in \mathbf{I}}$  is a limit cone in  $\mathbf{C}$ , it is enough to prove that  $f$  is an isomorphism.

Fix  $Z \in \mathcal{H}$ . The functors  $\mathrm{hom}_{\mathbf{E}}(Z, -): \mathbf{E} \rightarrow \mathbf{Set}$  and  $\mathrm{hom}_{\mathbf{C}}(Z, -): \mathbf{C} \rightarrow \mathbf{Set}$  preserve limits because every representable functor does [AHS06, Cor. 13.9].<sup>1</sup> Therefore,

$$(m_i \circ -: \mathrm{hom}_{\mathbf{E}}(Z, M) \rightarrow \mathrm{hom}_{\mathbf{E}}(Z, D(i)))_{i \in \mathbf{I}}$$

is a limit cone in  $\mathbf{Set}$  over the diagram  $\mathrm{hom}_{\mathbf{E}}(Z, -) \circ D: \mathbf{I} \rightarrow \mathbf{Set}$  and

$$(n_i \circ -: \mathrm{hom}_{\mathbf{C}}(Z, N) \rightarrow \mathrm{hom}_{\mathbf{C}}(Z, D(i)))_{i \in \mathbf{I}}$$

is a limit cone in  $\mathbf{Set}$  over the diagram  $\mathrm{hom}_{\mathbf{C}}(Z, -) \circ D: \mathbf{I} \rightarrow \mathbf{Set}$ . Since  $\mathbf{E}$  is a full subcategory of  $\mathbf{C}$ , for every  $Z \in \mathcal{H}$  we have that  $\mathrm{hom}_{\mathbf{E}}(Z, M) = \mathrm{hom}_{\mathbf{C}}(Z, M)$  and  $\mathrm{hom}_{\mathbf{E}}(Z, D(i)) = \mathrm{hom}_{\mathbf{C}}(Z, D(i))$  for each  $i \in \mathbf{I}$ . Moreover, we have a function

<sup>1</sup>We recall that a representable functor is a functor that up to natural isomorphism is of the form  $\mathrm{hom}(A, -)$  for some object  $A$ .

$f \circ - : \text{hom}_{\mathbf{C}}(Z, M) \rightarrow \text{hom}_{\mathbf{C}}(Z, N)$  that maps  $h : Z \rightarrow M$  to  $f \circ h : Z \rightarrow N$ . We claim that  $f \circ -$  is a bijection. Since  $m_i = n_i \circ f$  for every  $i$ , the function

$$m_i \circ - : \text{hom}_{\mathbf{C}}(Z, M) \rightarrow \text{hom}_{\mathbf{C}}(Z, D(i))$$

is the composite

$$\text{hom}_{\mathbf{C}}(Z, M) \xrightarrow{f \circ -} \text{hom}_{\mathbf{C}}(Z, N) \xrightarrow{n_i \circ -} \text{hom}_{\mathbf{C}}(Z, D(i)).$$

Therefore,  $f \circ -$  is the unique factorization of the limit cone

$$(m_i \circ - : \text{hom}_{\mathbf{C}}(Z, M) \rightarrow \text{hom}_{\mathbf{C}}(Z, D(i)))_{i \in \mathbf{I}}$$

through the limit cone

$$(n_i \circ - : \text{hom}_{\mathbf{C}}(Z, N) \rightarrow \text{hom}_{\mathbf{C}}(Z, D(i)))_{i \in \mathbf{I}}.$$

Because both cones are limit cones over the same diagram,  $f \circ -$  must be an isomorphism in **Set**, hence a bijection. Thus, it is enough to use the assumption to conclude that  $f$  is an isomorphism in **C**. ■

We recall that a *conservative* functor is a functor  $\mathcal{F} : \mathbf{A} \rightarrow \mathbf{B}$  that reflects isomorphisms, i.e. such that, for every morphism  $f$  in **A**, if  $\mathcal{F}(f)$  is an isomorphism in **B** then  $f$  is an isomorphism in **A** (see, e.g., [Rie16, Sec. 5.6]).

**3.2. COROLLARY.** *Let **C** be a category with a conservative functor to **Set** that is represented by an object  $X \in \mathbf{C}$ . The inclusion  $\mathbf{E} \hookrightarrow \mathbf{C}$  of any full subcategory **E** containing  $X$  preserves limits.*

**PROOF.** This is the specialization of Theorem 3.1 to the case of  $\mathcal{H} = \{X\}$ . ■

Let **KHaus** denote the category of compact Hausdorff spaces and continuous maps.

**3.3. THEOREM.** *Let **E** be a full subcategory of **KHaus** containing a singleton. The inclusion  $\mathbf{E} \hookrightarrow \mathbf{KHaus}$  preserves limits.*

**PROOF.** We show that **E** and **KHaus** satisfy the condition of Corollary 3.2 by taking  $\mathbf{C} = \mathbf{KHaus}$ . Since a continuous map between compact Hausdorff spaces is a homeomorphism if and only if it is a bijection, the forgetful functor  $\mathcal{U} : \mathbf{KHaus} \rightarrow \mathbf{Set}$  is conservative. Let  $\{x\} \in \mathbf{E}$  be a singleton space. There is a natural isomorphism between  $\text{hom}_{\mathbf{KHaus}}(\{x\}, -)$  and  $\mathcal{U}$  whose component relative to  $Y \in \mathbf{KHaus}$  sends  $f : \{x\} \rightarrow Y$  to  $f(x)$ . Thus,  $\mathcal{U} : \mathbf{KHaus} \rightarrow \mathbf{Set}$  is a conservative functor represented by  $\{x\}$ . Since **E** contains  $\{x\}$ , the result follows from Corollary 3.2. ■

We recall that a *Stone space* is a zero-dimensional compact Hausdorff space. Let **Stone** denote the full subcategory of **KHaus** consisting of Stone spaces. By the celebrated Stone duality, **BA** is dually equivalent to **Stone**. We call an extremally disconnected compact Hausdorff space an *ED-space*. It is straightforward to see that each ED-space is a Stone space. Let **ED** denote the full subcategory of **Stone** consisting of ED-spaces.

3.4. THEOREM. *The category **ED** lacks some binary powers.*

PROOF. This is an immediate consequence of Theorem 3.3 and the well-known fact in topology that there are ED-spaces whose product is not extremally disconnected (for example,  $\beta\mathbb{N}$  is an ED-space but  $\beta\mathbb{N} \times \beta\mathbb{N}$  is not; see [GJ60, p. 97]). ■

Let **cBA** denote the category of complete boolean algebras with boolean morphisms. Dualizing Theorem 3.4, we get:

3.5. COROLLARY. ***cBA** lacks some binary copowers, and hence is not equivalent to a prevariety.*

PROOF. Since a boolean algebra is complete if and only if its Stone space is extremally disconnected (see, e.g., [GH09, Thm. 39]), restricting Stone duality to complete boolean algebras yields that **cBA** is dually equivalent to **ED**. Therefore, since **ED** lacks some binary powers, **cBA** lacks some binary copowers. Thus, **cBA** is not cocomplete, and hence is not equivalent to a prevariety. ■

As an immediate consequence of Corollary 3.5, we obtain:

3.6. COROLLARY. *A category **C** lacks some binary copowers provided there is a colimit-preserving essentially surjective functor  $\mathcal{U} : \mathbf{C} \rightarrow \mathbf{cBA}$ .*

Let **MT<sub>st</sub>** be the category of MT-algebras and stable morphisms. We have the following version of Theorem 2.5 for **MT<sub>st</sub>**:

3.7. COROLLARY. ***MT<sub>st</sub>** lacks some binary copowers, and hence is not equivalent to a prevariety.*

PROOF. The same argument as in the proof of Theorem 2.3 shows that the forgetful functor  $\mathcal{U} : \mathbf{MT}_{\text{st}} \rightarrow \mathbf{cBA}$  is both a left and right adjoint. The former gives that  $\mathcal{U}$  is colimit-preserving and it is clearly essentially surjective. Thus, **MT<sub>st</sub>** lacks some binary copowers by Corollary 3.6. ■

We finish this section by considering the following analogs of the categories in Definition 2.7. Let **cBAO<sub>st</sub>** denote the category whose objects are complete BAOs and whose morphisms are stable morphisms, and define **cT<sub>st</sub>**, **cK4<sub>st</sub>**, **cB<sub>st</sub>**, **cTB<sub>st</sub>**, and **cS5<sub>st</sub>** similarly (observe that **cCA<sub>st</sub>** = **MT<sub>st</sub>**). The same proof as above yields the following version of Theorem 2.8:

3.8. COROLLARY. *Each of the categories **cBAO<sub>st</sub>**, **cT<sub>st</sub>**, **cK4<sub>st</sub>**, **cB<sub>st</sub>**, **cTB<sub>st</sub>**, and **cS5<sub>st</sub>** lacks some binary copowers. Consequently, none of them is equivalent to a prevariety.*

#### 4. Closure algebras with stable morphisms

In Section 2 we showed that the category **MT** of MT-algebras and MT-morphisms lacks some countable copowers. In Section 3 we relaxed the notion of an MT-morphism by dropping the completeness assumption and showed that the category **MT<sub>st</sub>** of MT-algebras and stable morphisms lacks some binary copowers. In this section, we further drop the requirement of completeness on the object side and show that the category **CA<sub>st</sub>** of closure algebras and stable morphisms lacks some coequalizers. This we do by showing that the corresponding category of Stone spaces equipped with continuous quasi-orders lacks some equalizers and then employing Jónsson-Tarski duality. The same results are proved for several other categories of BAOs with stable morphisms. Consequently, none of these is equivalent to a prevariety.

Let  $X$  and  $X'$  be compact Hausdorff spaces,  $R$  a closed relation on  $X$  (i.e., a closed subset of  $X \times X$ ), and  $R'$  a closed relation on  $X'$ . Following [BBI16, Def. 3.2], we call a continuous map  $f: X \rightarrow X'$  *stable* provided

$$x R y \implies f(x) R' f(y) \quad \forall x, y \in X.$$

4.1. **DEFINITION.** Let **KHausR** denote the category whose objects are pairs  $(X, R)$  where  $X$  is a compact Hausdorff space and  $R$  is a closed relation on  $X$ , and whose morphisms are stable maps.

It is clear that isomorphisms in **KHausR** are continuous bijections that preserve and reflect the binary relations.

4.2. **PROPOSITION.** The category **KHausR** is complete.

**PROOF.** The product of  $\{(X_i, R_i) : i \in I\}$  in **KHausR** is  $(\prod X_i, \prod R_i)$ , where  $\prod X_i$  is the product of  $\{X_i : i \in I\}$  in **KHaus** and  $\prod R_i$  is the product relation given by

$$f\left(\prod R_i\right)g \iff f(i) R_i g(i) \quad \forall i \in I.$$

The equalizer of  $f, g: (X, R) \rightarrow (X', R')$  in **KHausR** is their equalizer

$$\{x \in X : f(x) = g(x)\}$$

in **KHaus** equipped with the restriction of  $R$ . Consequently, **KHausR** is complete. ■

Let  $\mathbf{2} = \{0, 1\}$  denote the two-element chain with its usual (reflexive) order, which we view as an object of **KHausR** by equipping it with the discrete topology.

4.3. **THEOREM.** If **E** is a full subcategory of **KHausR** containing  $\mathbf{2}$  and such that for all  $(X, R) \in \mathbf{E}$  the relation  $R$  is reflexive, then the inclusion  $\mathbf{E} \hookrightarrow \mathbf{KHausR}$  preserves limits.

PROOF. Let  $\mathbf{KHausR}_T$  denote the full subcategory of  $\mathbf{KHausR}$  consisting of those  $(X, R)$  where  $R$  is reflexive.<sup>2</sup> Since the inclusion  $\mathbf{KHausR}_T \hookrightarrow \mathbf{KHausR}$  preserves limits (by the proof of Proposition 4.2), it is enough to prove that the inclusion  $\mathbf{E} \hookrightarrow \mathbf{KHausR}_T$  preserves limits. This can be done by verifying the condition of Corollary 3.2 by showing that

$$\text{hom}_{\mathbf{KHausR}_T}(\mathbf{2}, -): \mathbf{KHausR}_T \rightarrow \mathbf{Set}$$

is conservative. Let  $f: (X, R) \rightarrow (X', R')$  be a morphism in  $\mathbf{KHausR}_T$  whose image

$$f \circ -: \text{hom}_{\mathbf{KHausR}_T}(\mathbf{2}, X) \rightarrow \text{hom}_{\mathbf{KHausR}_T}(\mathbf{2}, X')$$

is a bijection. To prove that  $f$  is an isomorphism in  $\mathbf{KHausR}_T$  it is sufficient to show that it is a bijection that reflects the relation. We first show that  $f$  is a bijection. Let  $y \in X'$ , and consider the map  $c_y: \mathbf{2} \rightarrow X'$  with constant value  $y$ , which is a morphism in  $\mathbf{KHausR}_T$ . Since  $f \circ -$  is onto, there is a morphism  $g: \mathbf{2} \rightarrow X$  in  $\mathbf{KHausR}_T$  such that  $f \circ g = c_y$ . Therefore,  $y$  is in the image of  $f$ , and so  $f$  is onto. Suppose that  $x_1, x_2 \in X$  are such that  $f(x_1) = f(x_2)$  and consider the maps  $c_{x_1}, c_{x_2}: \mathbf{2} \rightarrow X$  with constant values  $x_1, x_2$ . Then  $f \circ c_{x_1} = f \circ c_{x_2}$  because  $f(x_1) = f(x_2)$ . Since  $f \circ -$  is one-to-one,  $c_{x_1} = c_{x_2}$ , and so  $x_1 = x_2$ . Thus,  $f$  is one-to-one, and hence  $f$  is a bijection.

We now show that  $f$  reflects the relation. Let  $x_1, x_2 \in X$  be such that  $f(x_1) R' f(x_2)$ . Define  $h: \mathbf{2} \rightarrow X'$  by  $h(0) = f(x_1)$  and  $h(1) = f(x_2)$ . Because  $\mathbf{2}$  is discrete,  $h$  is continuous. Since  $f(x_1) R' f(x_2)$ , we have that  $h$  is a morphism in  $\mathbf{KHausR}_T$ . Because  $f \circ -$  is onto, there is a morphism  $g: \mathbf{2} \rightarrow X$  such that  $f \circ g = h$ . So,  $f \circ g(0) = h(0) = f(x_1)$  and  $f \circ g(1) = h(1) = f(x_2)$ . Since  $f$  is one-to-one, we conclude that  $g(0) = x_1$  and  $g(1) = x_2$ . Thus,  $x_1 R x_2$  because  $0 < 1$  and  $g$  is a morphism in  $\mathbf{KHausR}_T$ . ■

We recall that a closed relation  $R$  on a compact Hausdorff space  $X$  is *continuous* provided

$$R^{-1}[U] := \{x \in X : x R u \text{ for some } u \in U\}$$

is open for each open  $U \subseteq X$ .

4.4. DEFINITION. Let  $\mathbf{StoneCQ}$  denote the full subcategory of  $\mathbf{KHausR}$  whose objects are those  $(X, R)$  where  $X$  is a Stone space and  $R$  is a continuous quasi-order (reflexive and transitive).

Applying Theorem 4.3 yields:

4.5. COROLLARY. The inclusion  $\mathbf{StoneCQ} \hookrightarrow \mathbf{KHausR}$  preserves limits.

We prove that  $\mathbf{StoneCQ}$  is not complete.

4.6. THEOREM.  $\mathbf{StoneCQ}$  lacks some equalizers.

<sup>2</sup>The subscript  $T$  is motivated by Theorem 4.8.

PROOF. Consider the space  $X$  shown in Figure 1, where  $x_\infty$  is the limit of the sequence  $(x_n)$  of isolated points and  $y_\infty$  is the limit of the sequence  $(y_n)$  of isolated points. Therefore,  $X$  is a Stone space. The relation  $R$  is defined on  $X$  as shown in Figure 1, where circles indicate that  $R$  is reflexive. It is not difficult to see that  $R$  is a continuous relation. Thus,  $(X, R)$  is an object of **StoneCQ**.

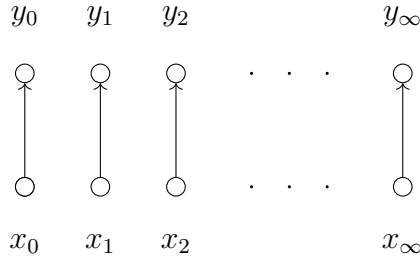


Figure 1: The space  $(X, R)$ .

Let  $X'$  be the space shown in Figure 2, where  $u_\infty$  is the limit of the sequence  $(u_n)$  of isolated points and  $v_\infty$  is the limit of each of the sequences  $(v_n)$  and  $(w_n)$  of isolated points. Therefore,  $X'$  is a Stone space. The relation  $R'$  is defined on  $X'$  as shown in Figure 2, where circles indicate that  $R'$  is reflexive. It is not difficult to see that  $R'$  is a continuous relation. Thus,  $(X', R')$  is an object of **StoneCQ**.

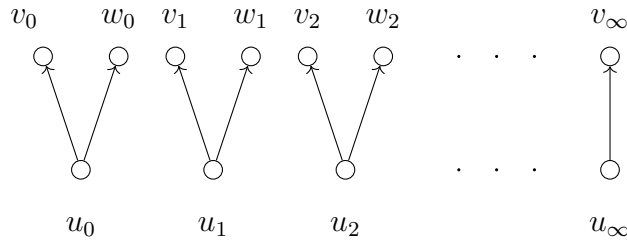


Figure 2: The space  $(X', R')$ .

Define  $f, g: X \rightarrow X'$  as follows:

$$\begin{aligned} f(x_\infty) &= u_\infty = g(x_\infty), \\ f(x_n) &= u_n = g(x_n), \\ f(y_\infty) &= v_\infty = g(y_\infty), \\ f(y_n) &= v_n, \\ &w_n = g(y_n). \end{aligned}$$

In other words,  $f$  and  $g$  only differ in that  $f(y_n) = v_n$  while  $g(y_n) = w_n$ . It is straightforward to see that  $f$  and  $g$  are **KHausR**-morphisms, and that the equalizer of  $f$  and  $g$  in **KHausR** is  $Y = \{x_n : n \in \mathbb{N}\} \cup \{x_\infty, y_\infty\}$  (see the proof of Proposition 4.2). But the restriction of  $R$  to  $Y$  is no longer continuous because  $\{y_\infty\}$  is open in  $Y$ , but  $R^{-1}[\{y_\infty\}] = \{y_\infty, x_\infty\}$  is not. Thus, by Corollary 4.5, the equalizer of  $f$  and  $g$  does not exist in **StoneCQ**. ■

We now apply Jónsson-Tarski duality for BAOs [JT51] (see also [CZ97] or [BdRV01]) to closure algebras, by which a BAO  $(B, \diamond)$  is a closure algebra if and only if in its Jónsson-Tarski dual  $(X, R)$  the relation  $R$  is reflexive and transitive. This together with the fact that stable morphisms between BAOs correspond to stable maps between their dual spaces [BBI16, Lem. 3.3] yields that  $\mathbf{CA}_{\text{st}}$  is dually equivalent to  $\mathbf{StoneCQ}$ . Putting this together with Theorem 4.6 immediately gives:

4.7. COROLLARY.  $\mathbf{CA}_{\text{st}}$  lacks some coequalizers, and hence it is not equivalent to a prevariety.

The table below summarizes the main results of Sections 2 to 4.

Category	Objects	Morphisms	Location
$\mathbf{MT}$	MT-algebras	MT-morphisms	Thm. 2.5
$\mathbf{MT}_{\text{st}}$	MT-algebras	Stable morphisms	Cor. 3.7
$\mathbf{CA}_{\text{st}}$	Closure algebras	Stable morphisms	Cor. 4.7

Table 1: Categories of closure algebras that are not cocomplete.

As in the previous two sections, Corollary 4.7 extends to various categories of BAOs with stable morphisms. To do so, as we did for  $\mathbf{CA}_{\text{st}}$ , we use duality. For this, we recall the following well-known result (see [JT51], [CZ97], or [BdRV01]).

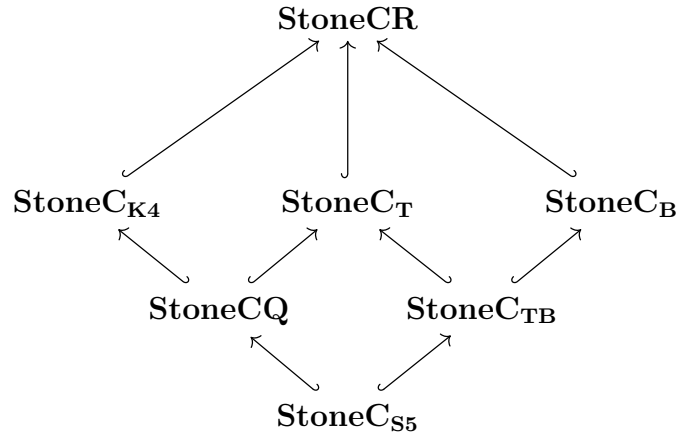
4.8. THEOREM. Let  $(B, \diamond)$  be a BAO and  $(X, R)$  its Jónsson-Tarski dual.

- (1)  $(B, \diamond)$  is a  $\mathbf{T}$ -algebra if and only if  $R$  is reflexive;
- (2)  $(B, \diamond)$  is a  $\mathbf{K4}$ -algebra if and only if  $R$  is transitive;
- (3)  $(B, \diamond)$  is a  $\mathbf{B}$ -algebra if and only if  $R$  is symmetric;
- (4)  $(B, \diamond)$  is a  $\mathbf{TB}$ -algebra if and only if  $R$  is reflexive and symmetric;
- (5)  $(B, \diamond)$  is an  $\mathbf{S5}$ -algebra if and only if  $R$  is an equivalence relation.

We thus have the following counterpart of Definition 2.7.

4.9. DEFINITION. Let  $\mathbf{StoneCR}$  denote the full subcategory of  $\mathbf{KHausR}$  whose objects are those  $(X, R)$  where  $X$  is a Stone space and  $R$  is a continuous relation, and define  $\mathbf{StoneC}_{\mathbf{T}}$ ,  $\mathbf{StoneC}_{\mathbf{K4}}$ ,  $\mathbf{StoneC}_{\mathbf{B}}$ ,  $\mathbf{StoneC}_{\mathbf{TB}}$ , and  $\mathbf{StoneC}_{\mathbf{S5}}$  as the full subcategories of  $\mathbf{StoneCR}$  with the additional condition on  $R$  described in Theorem 4.8.

We have the following inclusion relationship between the categories defined above and  $\mathbf{StoneCQ}$ :



Theorem 4.8 together with the fact that stable morphisms between BAOs correspond to stable maps between their dual spaces yields the following version of Jónsson-Tarski duality and its restrictions:

4.10. THEOREM.

- (1)  $\mathbf{BAO}_{\text{st}}$  is dually equivalent to  $\mathbf{StoneCR}$ .
- (2)  $\mathbf{T}_{\text{st}}$  is dually equivalent to  $\mathbf{StoneC}_T$ .
- (3)  $\mathbf{K4}_{\text{st}}$  is dually equivalent to  $\mathbf{StoneC}_{K4}$ .
- (4)  $\mathbf{B}_{\text{st}}$  is dually equivalent to  $\mathbf{StoneC}_B$ .
- (5)  $\mathbf{TB}_{\text{st}}$  is dually equivalent to  $\mathbf{StoneC}_{TB}$ .
- (6)  $\mathbf{S5}_{\text{st}}$  is dually equivalent to  $\mathbf{StoneC}_{S5}$ .

We now show that, like  $\mathbf{StoneC}_Q$ , all the categories in the diagram above also lack some equalizers. We follow the convention in modal logic (see [CZ97, p. 66]) and denote reflexive points of a binary relation by  $\circ$  and irreflexive points by  $\bullet$ . Let  $\mathbf{1}_\bullet$  denote the irreflexive singleton and  $\mathbf{2}_\bullet$  the irreflexive 2-element chain  $\bullet \rightarrow \bullet$ . We denote the bottom of this chain by 0 and the top by 1. Finally, let  $\mathbf{2}_\bullet^d$  denote the irreflexive doubleton  $\bullet \leftrightarrow \bullet$  and  $\mathbf{2}_\circ^d$  the reflexive doubleton  $\circ \leftrightarrow \circ$ . In each of these cases, the two points (which we also denote by 0 and 1) are both related to each other. We view  $\mathbf{1}_\bullet$ ,  $\mathbf{2}_\bullet$ ,  $\mathbf{2}_\bullet^d$ , and  $\mathbf{2}_\circ^d$  as objects of  $\mathbf{KHausR}$  by equipping them with the discrete topology. The next theorem is an analog of Theorem 4.3.

4.11. THEOREM. Let  $\mathbf{E}$  be a full subcategory of  $\mathbf{KHausR}$  satisfying one of the following conditions.

- (1)  $\mathbf{E}$  contains  $\mathbf{1}_\bullet$  and  $\mathbf{2}_\bullet$ .
- (2) For all  $(X, R) \in \mathbf{E}$  the relation  $R$  is symmetric, and  $\mathbf{E}$  contains  $\mathbf{1}_\bullet$  and  $\mathbf{2}_\bullet^d$ .

(3) For all  $(X, R) \in \mathbf{E}$  the relation  $R$  is reflexive and symmetric, and  $\mathbf{E}$  contains  $\mathbf{2}_\circ^d$ .

Then the inclusion functor  $\mathbf{E} \hookrightarrow \mathbf{KHausR}$  preserves limits.

PROOF. (1). Let  $\mathbf{C} = \mathbf{KHausR}$  and  $\mathcal{H} = \{\mathbf{1}_\bullet, \mathbf{2}_\bullet\}$ . We prove that the condition of Theorem 3.1 is satisfied by showing that, for any morphism  $f: (X, R) \rightarrow (X', R')$  in  $\mathbf{KHausR}$ , if the functions

$$f \circ -: \text{hom}_{\mathbf{KHausR}}(\mathbf{1}_\bullet, X) \rightarrow \text{hom}_{\mathbf{KHausR}}(\mathbf{1}_\bullet, X')$$

and

$$f \circ -: \text{hom}_{\mathbf{KHausR}}(\mathbf{2}_\bullet, X) \rightarrow \text{hom}_{\mathbf{KHausR}}(\mathbf{2}_\bullet, X')$$

are bijections, then  $f$  is an isomorphism. To prove that  $f$  is an isomorphism in  $\mathbf{KHausR}$  it is sufficient to show that it is a bijection that reflects the relation, which we do by adjusting the proof of Theorem 4.3 accordingly.

We first show that  $f$  is a bijection. Let  $y \in X'$ , and consider the map  $c_y: \mathbf{1}_\bullet \rightarrow X'$  with constant value  $y$ , which is a morphism in  $\mathbf{KHausR}$ . Since  $f \circ -$  is onto, there is a morphism  $g: \mathbf{1}_\bullet \rightarrow X$  in  $\mathbf{KHausR}$  such that  $f \circ g = c_y$ . Therefore,  $y$  is in the image of  $f$ , and so  $f$  is onto. Suppose that  $x_1, x_2 \in X$  are such that  $f(x_1) = f(x_2)$  and consider the maps  $c_{x_1}, c_{x_2}: \mathbf{1}_\bullet \rightarrow X$  with constant values  $x_1, x_2$ . Then  $f \circ c_{x_1} = f \circ c_{x_2}$  because  $f(x_1) = f(x_2)$ . Since  $f \circ -$  is one-to-one,  $c_{x_1} = c_{x_2}$ , and so  $x_1 = x_2$ . Thus,  $f$  is one-to-one, and hence  $f$  is a bijection.

We now show that  $f$  reflects the relation. Let  $x_1, x_2 \in X$  be such that  $f(x_1) R' f(x_2)$ . Define  $h: \mathbf{2}_\bullet \rightarrow X'$  by  $h(0) = f(x_1)$  and  $h(1) = f(x_2)$ . Since  $\mathbf{2}_\bullet$  is discrete,  $h$  is continuous. Because  $f(x_1) R' f(x_2)$ , we have that  $h$  is a morphism in  $\mathbf{KHausR}$ . Since  $f \circ -$  is onto, there is a morphism  $g: \mathbf{2}_\bullet \rightarrow X$  such that  $f \circ g = h$ . Therefore,  $fg(0) = h(0) = f(x_1)$  and  $fg(1) = h(1) = f(x_2)$ . Since  $f$  is one-to-one, we conclude that  $g(0) = x_1$  and  $g(1) = x_2$ . Thus,  $x_1 R x_2$  because  $0 < 1$  and  $g$  is a morphism in  $\mathbf{KHausR}$ .

(2). Let  $\mathbf{KHausR}_B$  denote the full subcategory of  $\mathbf{KHausR}$  consisting of those  $(X, R)$  where  $R$  is symmetric. It follows from the proof of Proposition 4.2 that the inclusion  $\mathbf{KHausR}_B \hookrightarrow \mathbf{KHausR}$  preserves limits. Therefore, it is enough to prove that the inclusion  $\mathbf{E} \hookrightarrow \mathbf{KHausR}_B$  preserves limits, which can be done by replacing  $\mathbf{2}_\bullet$  with  $\mathbf{2}_\circ^d$  in the proof of (1).

(3). Let  $\mathbf{KHausR}_{TB}$  denote the full subcategory of  $\mathbf{KHausR}$  consisting of those  $(X, R)$  where  $R$  is reflexive and symmetric. By the proof of Proposition 4.2, the inclusion  $\mathbf{KHausR}_{TB} \hookrightarrow \mathbf{KHausR}$  preserves limits. Thus, it is enough to prove that the inclusion  $\mathbf{E} \hookrightarrow \mathbf{KHausR}_{TB}$  preserves limits, which can be done by replacing  $\mathbf{2}$  by  $\mathbf{2}_\circ^d$  in the proof of Theorem 4.3. ■

4.12. REMARK. Comparing Theorem 4.11 to Theorem 4.3, in the reflexive case we don't need  $\mathbf{E}$  to contain the reflexive singleton because every constant map from  $\mathbf{2}$  to an object  $(X, R)$  with  $R$  reflexive is a morphism of  $\mathbf{KHausR}$ . This does not hold in general if  $R$  is not reflexive: indeed, if  $y$  is an irreflexive point, then the constant map  $c_y: \mathbf{2}_\bullet \rightarrow X$  with constant value  $y$  is not a morphism.

4.13. THEOREM. *Each of the categories  $\mathbf{StoneCR}$ ,  $\mathbf{StoneC}_T$ ,  $\mathbf{StoneC}_{K4}$ ,  $\mathbf{StoneC}_B$ ,  $\mathbf{StoneC}_{TB}$ , and  $\mathbf{StoneC}_{S5}$  lacks some equalizers.*

PROOF. The same proof as for  $\mathbf{StoneCQ}$  gives that  $\mathbf{StoneC}_T$  lacks some equalizers. Let  $\mathbf{E}$  denote one of the remaining categories. We adjust the proof of Theorem 4.6 by taking the equivalence relations generated by the partial orders of the spaces  $X$  and  $X'$  shown in Figures 1 and 2:

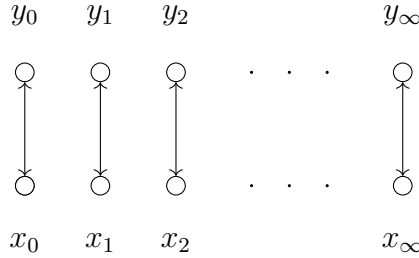


Figure 3: The space  $(X, R)$ .

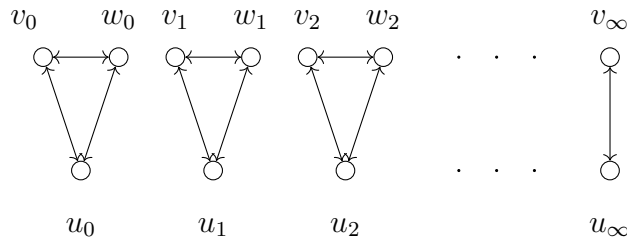


Figure 4: The space  $(X', R')$ .

It is not difficult to see that  $(X, R)$  and  $(X', R')$  are objects of  $\mathbf{StoneC}_{S5}$ , and hence of  $\mathbf{E}$ . Now define  $f, g: X \rightarrow X'$  as in the proof of Theorem 4.6 and observe that the same proof yields that the relation on the equalizer of  $f$  and  $g$  in  $\mathbf{KHausR}$  is no longer continuous. Thus, by Theorem 4.11, the equalizer of  $f$  and  $g$  does not exist in  $\mathbf{E}$ . ■

Putting Theorems 4.10 and 4.13 together immediately gives:

4.14. COROLLARY. *Each of  $\mathbf{BAO}_{st}$ ,  $\mathbf{T}_{st}$ ,  $\mathbf{K4}_{st}$ ,  $\mathbf{B}_{st}$ ,  $\mathbf{TB}_{st}$ , and  $\mathbf{S5}_{st}$  lacks some coequalizers. Consequently, none of them is equivalent to a prevariety.*

### 5. Heyting algebras and frames with lattice morphisms

In this final section we first show that the category of frames with Heyting morphisms lacks some binary copowers. We then utilize Priestley duality for bounded distributive lattices to show that the category of Heyting algebras with bounded lattice morphisms lacks some coequalizers and its full subcategory consisting of frames lacks some binary copowers. Thus, none of these categories is equivalent to a prevariety.

We recall that a bounded distributive lattice  $A$  is a *Heyting algebra* if the set

$$\{x \in A : a \wedge x \leq b\}$$

has a largest element for each  $a, b \in A$ . A *frame* is a complete Heyting algebra (that is, a complete lattice satisfying the join-infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$$

for each  $a \in A$  and  $S \subseteq A$ ).

5.1. DEFINITION. Let  $\mathbf{BDL}$  denote the category of bounded distributive lattices with bounded lattice morphisms, and let  $\mathbf{HA}$  denote the category of Heyting algebras with Heyting morphisms.

We also consider the following categories:

5.2. DEFINITION. Let  $\mathbf{HA}_{\mathbf{BL}}$  denote the full subcategory of  $\mathbf{BDL}$  consisting of Heyting algebras,  $\mathbf{Frm}_{\mathbf{BL}}$  the full subcategory of  $\mathbf{HA}_{\mathbf{BL}}$  consisting of frames, and  $\mathbf{Frm}_{\mathbf{HA}}$  the full subcategory of  $\mathbf{HA}$  consisting of frames.

5.3. REMARK. The category  $\mathbf{HA}_{\mathbf{BL}}$  is different from  $\mathbf{HA}$  in that the morphisms in  $\mathbf{HA}_{\mathbf{BL}}$  are bounded lattice morphisms, which in general do not preserve Heyting implication. Similarly,  $\mathbf{Frm}_{\mathbf{HA}}$  and  $\mathbf{Frm}_{\mathbf{BL}}$  have the same objects, but differ at the morphism level, and both are different from the category of frames and frame morphisms.

By [BD74, Cor. IX.5.4], there is a reflector  $\mathcal{R}: \mathbf{HA} \rightarrow \mathbf{BA}$ , which restricts to a reflector  $\mathcal{R}: \mathbf{Frm}_{\mathbf{HA}} \rightarrow \mathbf{cBA}$  by [BD74, Thm. VIII.4.4]. Since  $\mathbf{cBA}$  is a full subcategory of  $\mathbf{Frm}_{\mathbf{HA}}$ ,  $\mathcal{R}$  is essentially surjective by [BD74, Thm. I.18.4]. Because  $\mathcal{R}$  is a left adjoint, it preserves colimits. Thus, Corollary 3.6 applies, yielding the following:

5.4. THEOREM.  $\mathbf{Frm}_{\mathbf{HA}}$  lacks some binary copowers, and hence is not equivalent to a prevariety.

It remains to prove that  $\mathbf{HA}_{\mathbf{BL}}$  and  $\mathbf{Frm}_{\mathbf{BL}}$  are not cocomplete, which we do by using Priestley duality and its restrictions to Heyting algebras and frames.

5.5. DEFINITION.

- (1) A Priestley space is a Stone space  $X$  equipped with a partial order  $\leq$  such that for every  $x, y \in X$ , if  $x \not\leq y$ , then there is a clopen upset  $U$  such that  $x \in U$  and  $y \notin U$ .
- (2) Let  $\mathbf{Pries}$  denote the category of Priestley spaces and continuous order-preserving maps between them.

It is well known (see, e.g., [Pri84, Prop. 2.6]) that if  $X$  is a Priestley space, then  $\leq$  is a closed relation on  $X$ . Therefore,  $\mathbf{Pries}$  is a full subcategory of  $\mathbf{KHausR}$  because  $\mathbf{Pries}$ -morphisms are precisely stable maps between Priestley spaces viewed as objects of  $\mathbf{KHausR}$ .

5.6. THEOREM. (Priestley duality [Pri70]) **BDL** is dually equivalent to **Pries**.

5.7. DEFINITION.

- (1) An Esakia space is a Priestley space in which the downset of every clopen subset is clopen.
- (2) A localic space (L-space, for short) is a Priestley space in which the closure of any open upset is a clopen upset.
- (3) Let **Esa<sub>st</sub>** and **LSp<sub>st</sub>** denote the full subcategories of **Pries** consisting of Esakia spaces and L-spaces, respectively.

5.8. REMARK. It is well known that each L-space is an Esakia space. This, for example, can be seen by observing that a Priestley space is an Esakia space if and only if the closure of each open upset is an upset (see, e.g., [BGJ13, Lem. 4.2]).

It follows from the results of [Esa74, PS88] that Priestley duality restricts to **HA<sub>BL</sub>** and **Frm<sub>BL</sub>**:

5.9. THEOREM. Priestley duality restricts to a dual equivalence between **HA<sub>BL</sub>** and **Esa<sub>st</sub>** and to a dual equivalence between **Frm<sub>BL</sub>** and **LSp<sub>st</sub>**.

5.10. THEOREM.

- (1) **Esa<sub>st</sub>** is not complete as it lacks some equalizers.
- (2) **LSp<sub>st</sub>** is not complete as it lacks some binary powers.

PROOF. (1). It is sufficient to observe that the spaces considered in the proof of Theorem 4.6 are Esakia spaces, hence the same proof yields that **Esa<sub>st</sub>** lacks some equalizers.

(2). We view each ED-space as an L-space where the relation is the identity relation. We claim that **LSp<sub>st</sub>** lacks the product of  $\beta\mathbb{N}$  with itself. By Theorem 4.3, the inclusion  $\mathbf{LSp}_{\text{st}} \hookrightarrow \mathbf{KHausR}$  preserves limits. Therefore, if the product of  $\beta\mathbb{N}$  with itself existed in **LSp<sub>st</sub>**, it would be the topological product  $\beta\mathbb{N} \times \beta\mathbb{N}$  by the proof of Proposition 4.2, which is a contradiction since  $\beta\mathbb{N} \times \beta\mathbb{N}$  is not extremally disconnected. ■

Putting Theorems 5.9 and 5.10 together yields:

5.11. COROLLARY.

- (1) **HA<sub>BL</sub>** is not cocomplete as it lacks some coequalizers. Thus, **HA<sub>BL</sub>** is not equivalent to a prevariety.
- (2) **Frm<sub>BL</sub>** is not cocomplete as it lacks some binary copowers. Thus, **Frm<sub>BL</sub>** is not equivalent to a prevariety.

The table below summarizes the main results of this section.

Category	Objects	Morphisms	Location
$\mathbf{HA}_{BL}$	Heyting algebras	Bounded lattice morphisms	Cor. 5.11(1)
$\mathbf{Frm}_{BL}$	Frames	Bounded lattice morphisms	Cor. 5.11(2)
$\mathbf{Frm}_{HA}$	Frames	Heyting morphisms	Thm. 5.4

Table 2: Categories of Heyting algebras and frames that are not cocomplete.

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## References

- [AHS06] J. Adámek, H. Herrlich, and G. E. Strecker. Abstract and concrete categories: the joy of cats. *Repr. Theory Appl. Categ.*, (17):1–507, 2006.
- [BBI16] G. Bezhhanishvili, N. Bezhhanishvili, and R. Iemhoff. Stable canonical rules. *J. Symb. Log.*, 81(1):284–315, 2016.
- [BD74] R. Balbes and P. Dwinger. *Distributive lattices*. University of Missouri Press, Columbia, Mo., 1974.
- [BdRV01] P. Blackburn, M. de Rijke, and Y. Venema. *Modal logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 2001.
- [BGJ13] G. Bezhhanishvili, D. Gabelaia, and M. Jibladze. Funayama’s theorem revisited. *Algebra Universalis*, 70(3):271–286, 2013.
- [BK24] G. Bezhhanishvili and A. Kornell. The category of topological spaces and open maps does not have products. *Adv. Math.*, 458:Paper No. 109963, 2024.
- [BR23] G. Bezhhanishvili and R. Raviprakash. McKinsey-Tarski algebras: an alternative pointfree approach to topology. *Topology Appl.*, 339:Paper No. 108689, 2023.
- [CZ97] A. Chagrov and M. Zakharyashev. *Modal logic*. Oxford University Press, New York, 1997.

- [dJ80] D. H. J. de Jongh. A class of intuitionistic connectives. In *The Kleene Symposium (Proc. Sympos., Univ. Wisconsin, Madison, Wis., 1978)*, volume 101 of *Stud. Logic Found. Math.*, pages 103–111. North-Holland, Amsterdam-New York, 1980.
- [Esa74] L. L. Esakia. Topological Kripke models. *Soviet Math. Dokl.*, 15:147–151, 1974.
- [Gai64] H. Gaifman. Infinite Boolean polynomials. I. *Fund. Math.*, 54:229–250, 1964.
- [GH09] S. Givant and P. Halmos. *Introduction to Boolean algebras*. Undergraduate Texts in Mathematics. Springer, New York, 2009.
- [Ghi10] S. Ghilardi. Continuity, freeness, and filtrations. *J. Appl. Non-Classical Logics*, 20(3):193–217, 2010.
- [GJ60] L. Gillman and M. Jerison. *Rings of continuous functions*. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960.
- [Hal56] P. R. Halmos. Algebraic logic. I. Monadic Boolean algebras. *Compositio Math.*, 12:217–249, 1956.
- [Hal64] A. W. Hales. On the non-existence of free complete Boolean algebras. *Fund. Math.*, 54:45–66, 1964.
- [Joh82] P. T. Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1982.
- [JT51] B. Jónsson and A. Tarski. Boolean algebras with operators. I. *Amer. J. Math.*, 73:891–939, 1951.
- [MT44] J. C. C. McKinsey and A. Tarski. The algebra of topology. *Ann. of Math.*, 45:141–191, 1944.
- [Pri70] H. A. Priestley. Representation of distributive lattices by means of ordered Stone spaces. *Bull. London Math. Soc.*, 2:186–190, 1970.
- [Pri84] H. A. Priestley. Ordered sets and duality for distributive lattices. In *Orders: description and roles (L’Arbresle, 1982)*, volume 99 of *North-Holland Math. Stud.*, pages 39–60. North-Holland, Amsterdam, 1984.
- [PS88] A. Pultr and J. Sichler. Frames in Priestley’s duality. *Cah. Topol. Géom. Différ. Catég.*, 29(3):193–202, 1988.
- [Rie16] E. Riehl. *Category theory in context*. Mineola, NY: Dover Publications, 2016.

- [RS70] H. Rasiowa and R. Sikorski. *The mathematics of metamathematics*, volume 41. PWN—Polish Scientific Publishers, Warsaw, third edition, 1970. Monografie Matematyczne.
- [SR99] J. D. H. Smith and A. B. Romanowska. *Post-modern algebra*. Pure and Applied Mathematics. John Wiley & Sons, Inc., New York, 1999.

*Research Institute in Mathematics and Physics, Université Catholique de Louvain  
1348 Ottignies-Louvain-la-Neuve, Belgium*

*Department of Mathematical Sciences, New Mexico State University  
Las Cruces NM 88003, USA*

*Department of Mathematics “Federigo Enriques”, University of Milan  
20133 Milan, Italy*

Email: marco.abbadini@uclouvain.be  
guram@nmsu.edu  
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Kate Ponto, University of Kentucky: [kate.ponto \(at\) uky.edu](mailto:kate.ponto@uky.edu)

Jiri Rosický, Masaryk University: [rosicky@math.muni.cz](mailto:rosicky@math.muni.cz)

Giuseppe Rosolini, Università di Genova: [rosolini@unige.it](mailto:rosolini@unige.it)

Michael Shulman, University of San Diego: [shulman@sandiego.edu](mailto:shulman@sandiego.edu)

Alex Simpson, University of Ljubljana: [Alex.Simpson@fmf.uni-lj.si](mailto:Alex.Simpson@fmf.uni-lj.si)

James Stasheff, University of North Carolina: [jds@math.upenn.edu](mailto:jds@math.upenn.edu)

Tim Van der Linden, Université catholique de Louvain: [tim.vanderlinden@uclouvain.be](mailto:tim.vanderlinden@uclouvain.be)

Christina Vasilakopoulou, National Technical University of Athens: [cvasilak@math.ntua.gr](mailto:cvasilak@math.ntua.gr)