Theory and Applications of Categories, Vol. 45, No. 29, 2026, pp. 1201–1226.

INVARIANTS FROM DETERMINANTS ON TRACED IDEALS OF PIVOTAL CATEGORIES

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Abstract. In a monoidal category, there are generally two types of ideals: thick and tensor ideals. The current paper's main focus is on ideals of the former type, which are defined on the objects of the ambient category. In the setting of a semisimple pivotal category, we introduce a particular class of pseudo-negligible morphisms and establish some related results. Moreover, given a one-sided thick ideal $\mathcal{I}$ of a semisimple pure pivotal category, we call $\mathcal{I}$ admissible when it is equipped with a one-sided trace function whose associated modified dimension is invertible on the simple objects therein. We then introduce and study a determinant function on such ideals and establish analogous properties to those of the usual determinants. As an application, we construct new topological invariants of planar graphs from these determinant functions on traced ideals.

1. Introduction

Monoidal categories with compatible duality structures allow for a consistent theory of categorical traces and dimensions [Ponto and Shulman (2014)]. These notions have significant applications to low-dimensional topology, representation theory and other fields. They serve as powerful tools to construct invariants of knots, links, 3-manifolds and other topological objects. However, in many interesting cases, they yield only trivial invariants. To address this limitation, a theory of modified traces and dimensions has been developed, where these traces are defined on ideals (on objects) of various suitable monoidal categories; we refer to [Geer et al. (2011), Geer et al. (2013), Geer et al. (2021)]. The modified dimension can be non-zero even when the categorical dimension vanishes. Establishing the existence and uniqueness of such trace functions is generally a challenging task. Special cases where this problem is resolved is through ambidextrous objects and the ideals of projective objects by [Geer et al. (2013)]. The existence of ambidextrous objects has been shown to hold, for example, in factorizable finite pivotal tensor categories, see [Gainutdinov and Runkel (2020)].

Given a semisimple ribbon category, *quantum determinant functions* on its endomorphisms were introduced and studied in [Choulli et al. (2022)], where the relationship with classical determinants was established. These functions were later generalized within the framework of semisimple pivotal categories, see [Draoui et al. (2023)] where, left and

Received by the editors 2025-12-13 and, in final form, 2026-05-12.

Transmitted by Kate Ponto. Published on 2026-05-19.

2020 Mathematics Subject Classification: 15A99, 18A05, 18E05.

Key words and phrases: Diagram invariant, categorical trace, quantum determinants, pivotal category, one-sided ideal, semisimplicity .

right versions were derived and shown to coincide when the category satisfies the extra sphericity condition making it into a spherical category [Barrett and Westbury (1999)].

Roughly speaking, there are two types of ideals. The first type consists of ideals defined on the objects of the ambient category (sometimes referred to as thick ideals in the literature) which are subclasses closed under tensor products and retracts [Draoui]. The second type consists of ideals defined on the morphisms of a preadditive monoidal category (commonly termed tensor ideals), which are subclasses closed under sums, tensor products and compositions [Heidersdorf and Wenzl (2022)].

An interesting example of a tensor ideal is the class of negligible morphisms, which is used to construct semisimple (quotient) categories, a process referred to in [Etingof and Ostrik (2022)] as semisimplification. A significant number of works in the literature is devoted to the study of tensor ideals and their applications, for example, see [Heidersdorf and Wenzl (2022), Street (1995)]. However, ideals defined on objects remain less explored in comparison to tensor ideals, but getting increasing interest in low-dimensional topology thanks to their role in constructing (non trivial) invariants. One notable example of thick ideals is provided by the class of projective objects which coincides, in a pivotal category, with that of the injective ones, see [Geer et al. (2013)].

Given a semisimple (in the sense of [Turaev (2016)]) pivotal category $\mathcal{C}$, we introduce a class of morphisms (between two arbitrary objects) termed *pseudo-negligible* and show that it includes the class of negligible morphisms (Proposition 3.2). It is well known that traces are cyclic; that is, for composable morphisms f and g, we have $Tr(f \circ g) = Tr(g \circ f)$. However, this property does not generally hold for left (or right) quantum determinants. Nevertheless, in the particular case of two endomorphisms of the same object of $\mathcal{C}$, cyclicity is retained due to multiplicativity of these determinants. In this work, we show that cyclicity also holds for morphisms $f : V \to V^\vee$ and $g : V^\vee \to V$, where $V^\vee = V^*$ or V^{**} (Lemma 3.3). In addition, we prove that the class of pseudo-negligible morphisms is closed under composition and tensor products (on both sides). Closedness under passage to duals holds for endomorphisms and follows from cyclicity of quantum determinants in the preceding special case.

The main purpose of this paper is to contribe to the construction of new topological invariants of graphs [Turaev (2016)]. The general approach is as follows. One starts by a suitable category $\mathcal{C}$, for example a semisimple pivotal or ribbon category, then constructs the corresponding category $\mathcal{G}_\mathcal{C}$ of diagrams using the associated graphical interpretation of the morphisms of $\mathcal{C}$ (known as Penrose, or string, diagrams). This gives rise to a functor $F : \mathcal{G}_\mathcal{C} \longrightarrow \mathcal{C}$ which maps any interpretation of a given graph, knot or link diagram in $\mathcal{G}_\mathcal{C}$ to a morphism of $\mathcal{C}$. Using the axioms of $\mathcal{C}$, this machinery creates topological invariants.

During the last decade, there has been growing attention to the construction of modified trace functions on ideals of suitable monoidal categories, such as ribbon categories ([Geer et al. (2021)]), pivotal categories ([Geer et al. (2013)]), unimodular and non unimodular pivotal categories ([Costantino et al. (2020), Geer et al. (2011)]), factorizable finite tensor categories ([Gainutdinov and Runkel (2020)]), and other categorical and algebraic contexts. This development is partly motivated by the need to overcome the obstruction that, in certain non-semisimple settings, the resulting Reshetikhin-Turaev invariants of links ([Turaev (2016)]) constructed from categories arising in representation theory (of finite groups, Lie algebras, quantum groups, etc) may vanish due to the presence of zero quantum dimensions.

Given a semisimple pivotal category $\mathcal{C}$, a one-sided ideal $\mathcal{I}$ of $\mathcal{C}$ is called t-*traced* when we are given a one-sided *trace function* t on $\mathcal{I}$ which is, as introduced in [Geer et al. (2011)], a family of linear maps indexed by objects of $\mathcal{I}$ and evaluated on their endomorphisms, such that these maps are compatible with the tensor product and composition. In other words, they satisfy cyclicity and compatibility with one-sided partial traces of $\mathcal{C}$. A t-traced ideal will be termed *admissible* provided that the modified dimension of any simple object of $\mathcal{I}$ is invertible (see Definition 5.6). For any such admissible traced ideal, we define a *determinant function* on the endomorphisms of its objects (see Definition 5.7). Subsequently, we exhibit several properties of these functions, including multiplicativity, and establish the relationship with the classical determinants when the trace t is tensor-multiplicative (see Theorem 5.8). As an application, we use these determinant functions to construct new *isotopy invariants* of one-sided $\mathcal{I}$-admissible $\mathcal{C}$-colored 1-1-ribbon planar graphs ([Geer et al. (2013)]). The main results can be resumed in the following theorems which are, respectively, Theorems 5.8 and 5.13 in this paper.

1.1. Theorem. *Let $\mathcal{I}$ be an admissible left (resp., right) t-traced ideal and $X \in \mathcal{I}$ an object dominated by a family $\{V_{i(r)}, \varepsilon_r, \mu_r\}_{r\in[n]}$, such that any isomorphic objects of this family are identical. Assume that t is $\otimes$-multiplicative. Then, for every $f \in \mathrm{End}_{\mathcal{C}}(X)$, we have*

$$\mathsf{det}_X(f) = det(M^f)$$

where, the right hand side determinnt is the usual one, $M^f = (a^f_{r,s})_{r,s\in[n]}$, and for all $r, s \in [n]$:

$$a^f_{r,s} := \mathsf{t}(\varepsilon_r f \mu_s)\, \mathsf{d}^{-1}(V_{i(r)}).$$

This establishes the connection of the introduced determinants on admissible traced ideals with the classical ones. Now let $\mathcal{I}$ be an admissible left (resp., right) t-traced ideal. For any left (resp., right) $\mathcal{I}$-admissible left (resp., right) cutting presentation T of a planar $\mathcal{C}$-colored ribbon graph Γ, we set

$$F_{\mathsf{det}}(\Gamma) := \mathsf{det}_{X^\varepsilon}(F(T)).$$

1.2. Theorem. *Let $\mathcal{I}$ be an admissible left (resp., right) t-traced ideal. Then, F_{det} is an isotopy invariant of left (resp., right) $\mathcal{I}$-admissible $\mathcal{C}$-colored 1-1-ribbon graphs.*

2. Preliminaries

2.1. Notation. Throughout the sequel, $\mathcal{C}$ will always denote a (strict) monoidal category with unit I and tensor product $\otimes$, except when otherwise mentioned. We will use capital letters such as U, V, X, Y, … to denote the objects of $\mathcal{C}$, and we use lowercase and

Greek letters f, g, h, α, β, ... to denote the morphisms of $\mathcal{C}$. The classes of objects and morphisms of $\mathcal{C}$ will be denoted by $\mathcal{C}_0$ and $\mathcal{C}_1$, respectively. However, we will simply write $X \in \mathcal{C}$ to mean that X is an object of $\mathcal{C}$ and $f \in \mathcal{C}$ will mean that f is a morphism of $\mathcal{C}$. The class of morphisms $X \to Y$ will be denoted by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, and $\mathrm{Hom}_{\mathcal{C}}(X, X)$ will be written $\mathrm{End}_{\mathcal{C}}(X)$. For brevity, we will usually write fg to denote the composite $f \circ g$, and 1_X to denote the identity endomorphism on $X \in \mathcal{C}$. The tensor product $X^{\otimes^n} := X \otimes \cdots \otimes X$ of X, n times, will be denoted simply by X^n. For any endomorphism $k : I \to I$ and morphism $f \in \mathcal{C}$, the notation kf will designate the tensor product $k \otimes f$. Similarly, $fk := f \otimes k$. The set $\{1, \ldots, n\}$ will be denoted by $[n]$ for every integer $n \in \mathbb{N}^*$.

2.2. Monoidal categories. For the reminder of this section, we refer to [Kassel (1995), Mac-Lane (2013), Turaev and Wenzl (1997), Turaev and Virelizier (2017)].

A *monoidal category* $(\mathcal{C}, \otimes, I, \alpha, l, r)$ is the following given data :

- A (small) category $\mathcal{C}$.
- A tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$.
- A unit object I.
- Natural isomorphisms $\alpha : (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W)$, $l : I \otimes U \longrightarrow U$, and $r : U \otimes I \longrightarrow U$ for all objects U, V, and W of $\mathcal{C}$, called *associativity*, *left unitarity*, and *right unitality* constraints, respectively, such that the pentagon and triangle axioms hold ([Mac-Lane (2013)]).

When α, l, and r are identities, $\mathcal{C}$ is called *strict*. We will assume throughout this paper that $\mathcal{C}$ is strict, due to Mac Lane's Coherence Theorem ([Mac-Lane (2013)]).

2.3. Example. The category of left modules over a commutative ring R and the R-linear morphisms between them is a fundamental example of a monoidal category. The monoidal structure is given by the usual tensor product of modules over R, the unit object is R, and the isomorphisms α, l, and r are the obvious ones. Note that this category is not strict.

A *strong monoidal functor* $(\mathcal{C}, \otimes, I) \longrightarrow (\mathcal{D}, \otimes', I')$ between strict monoidal categories is a triplet $(\mathcal{U}, \mathcal{U}_0, \mathcal{U}_2)$, where $\mathcal{U} : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor and $\mathcal{U}_2 : \mathcal{U}(U) \otimes' \mathcal{U}(V) \to \mathcal{U}(U \otimes V)$, $\mathcal{U}_0 : I' \to \mathcal{U}(I)$ are isomorphisms in $\mathcal{D}$, for all objects U, $V \in \mathcal{C}$, satisfying the associativity, left and right unitality constraints ([Turaev (2016), Turaev and Virelizier (2017)]).

A category $\mathcal{C}$ is called *preadditive* (or an *Ab-category*) if $\mathrm{Hom}_{\mathcal{C}}(U, V)$ is an additive group for all objects U, $V \in \mathcal{C}$, and the tensor product and composition of morphisms are bilinear. The endomorphisms class $\mathrm{Hom}_{\mathcal{C}}(I, I)$ is denoted by $\mathtt{K}_{\mathcal{C}}$ and referred to as the ground ring of $\mathcal{C}$.

2.4. Proposition. *Let $\mathcal{C}$ be a preadditive monoidal category. Then, $(\mathtt{K}_{\mathcal{C}}, +, \circ)$ is a commutative ring, where the composition coincides with the tensor product.*

PROOF. For all $k, k' \in \mathtt{K}_{\mathcal{C}}$, we have $k \circ k' = (1_I \otimes k') \circ (k \otimes 1_I) = k' \circ k$ and $k \circ k' = (k \otimes 1_I) \circ (1_I \otimes k') = (k \circ 1_I) \otimes (1_I \circ k') = k \otimes k'$. ■

In general, for any objects U and V of a preadditive monoidal category $\mathcal{C}$, the abelian group $\mathrm{Hom}_{\mathcal{C}}(U, V)$ is a left $\mathtt{K}_{\mathcal{C}}$-module by the action $k \cdot f := k \otimes f$, for all $k \in \mathtt{K}_C$ and $f \in \mathrm{Hom}_{\mathcal{C}}(U, V)$. In this case, the composition is bilinear. In fact

$$(k \cdot f) \circ g = (k \otimes f)(1_I \otimes g) = k \otimes f \circ g = k \cdot (f \circ g).$$

Analogously, one obtains $f \circ (k \cdot g) = k \cdot (f \circ g)$.

From the associativity of the tensor product, we immediately deduce that the tensor product is $\mathtt{K}_{\mathcal{C}}$-linear with respect to the first factor. However, the $\mathtt{K}_{\mathcal{C}}$-linearity with respect to the second factor does not hold in general. In fact, it is equivalent to the fact that $k \otimes 1_V = 1_V \otimes k$, for all $k \in \mathtt{K}_{\mathcal{C}}$ and $V \in \mathcal{C}$.

2.5. DEFINITION. [Turaev and Virelizier (2017), page 14]. *A monoidal category $\mathcal{C}$ is pure provided that for all $k \in \mathtt{K}_{\mathcal{C}}$ and morphism $f \in \mathcal{C}$, we have $k \cdot f = f \cdot k$.*

It directly follows from the above definition that, in a pure preadditive category $\mathcal{C}$, the tensor product of morphisms is $\mathtt{K}_{\mathcal{C}}$-bilinear.

2.6. PIVOTAL CATEGORIES. Let $(\mathcal{C}, \otimes, I)$ be a (strict) monoidal category. Following [Kassel (1995), Turaev (2016)], a *left* (resp., *right*) *duality* for $\mathcal{C}$ consists of a left (resp., right) duality for every object V in $\mathcal{C}$; that is, the existence of an object V^* (resp., ${}^*V$) in $\mathcal{C}$, called its *left* (resp., *right*) *dual*, along with morphisms

$$d_V : V^* \otimes V \longrightarrow I \qquad \big(\text{resp.,} \quad d'_V : V \otimes {}^*V \longrightarrow I\,\big),$$

called the *evaluation*, and

$$b_V : I \longrightarrow V \otimes V^* \qquad \big(\text{resp.,} \quad b'_V : I \longrightarrow {}^*V \otimes V\,\big),$$

called the *coevaluation*, such that the following triangular identities hold

$$(1_V \otimes d_V)(b_V \otimes 1_V) = 1_V \qquad \big(\text{resp.,} \quad (d'_V \otimes 1_V)(1_V \otimes b'_V) = 1_V\,\big), \tag{1}$$

$$(d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V) = 1_{V^*} \qquad \big(\text{resp.,} \quad (1_{{}^*V} \otimes d'_V)(b'_V \otimes 1_{{}^*V}) = 1_{{}^*V}\,\big). \tag{2}$$

In this case, $\mathcal{C}$ is called *left* (resp., *right*) *rigid*. A monoidal category $\mathcal{C}$ is called *rigid* if it is both left and right rigid. Note that a left (resp., right) dual is unique, up to isomorphism. Note also that left and right duals of a given object of a rigid category do not need to coincide.

For every morphism $f : U \longrightarrow V$ of $\mathcal{C}$ between left dualizable objects, its *left dual* $V^* \longrightarrow U^*$ is is given by

$$f^* := (d_V \otimes 1_{U^*})(1_{V^*} \otimes f \otimes 1_{U^*})(1_{V^*} \otimes b_U).$$

The *right dual* ${}^*f$ of f is analogously defined when U and V are right dualizable objects.

For every object $V \in \mathcal{C}$ and composable morphisms f and g of a left rigid category $\mathcal{C}$, the following identities are immediate: $(f \circ g)^* = g^* \circ f^*$ and $1_{V^*} = (1_V)^*$.

2.7. Definition. [Turaev and Virelizier (2017), Sect. 1.7.1, page 26]. *A pivotal category is a rigid category $\mathcal{C}$ equipped with distinguished left duality (V^*, d_V, b_V) and right duality (V^*, d'_V, b'_V) structures for every object $V \in \mathcal{C}$, such that the induced left and right dual functors coincide. In other words, this means that we have*

- $d'_V = d_V : I \longrightarrow I^*$.
- *For every $f \in \mathrm{Hom}_{\mathcal{C}}(U, V)$, the following induced duals coincide*

$$(d_V \otimes 1_{U^*})(1_{V^*} \otimes f \otimes 1_{U^*})(1_{V^*} \otimes b_U) : V^* \longrightarrow U^*.$$

$$(1_{U^*} \otimes d'_V)(1_{U^*} \otimes f \otimes 1_{V^*})(b'_U \otimes 1_{V^*}) : V^* \longrightarrow U^*.$$

- *The following induced isomorphisms coincide*

$$(d_V \otimes 1_{(U\otimes V)^*})(1_{V^*} \otimes d_U \otimes 1_{V\otimes(U\otimes V)^*})(1_{V^*\otimes U^*} \otimes b_{U\otimes V}) : V^* \otimes U^* \longrightarrow (U \otimes V)^*.$$

$$(1_{(U\otimes V)^*} \otimes d'_V)(1_{(U\otimes V)^*\otimes U} \otimes d'_U \otimes 1_{U^*})(b'_{U\otimes V} \otimes 1_{V^*\otimes U^*}) : V^* \otimes U^* \longrightarrow (U \otimes V)^*.$$

Note that the triangular identities (1) and (2) corresponding to the duality structures (V^*, d_V, b_V) and (V^*, d'_V, b'_V) imply that we also have $b'_V = b_V : I^* \longrightarrow I$. The family

$$\Phi = \left\{ \ \Phi_V = (d'_V \otimes 1_{V^{**}})(1_V \otimes b_{V^*}) : V \longrightarrow V^{**} \ \right\}_{V \in \mathcal{C}}$$

is a (monoidal) natural isomorphism, and it is referred to as the *pivotal structure*. Note here that for every endomorphism $f \in \mathrm{End}_{\mathcal{C}}(V)$, we have $f^{**} = \Phi_V f \Phi_V^{-1}$.

Since in a pivotal category $\mathcal{C}$ every object has distinguished left and right duality structures, an endomorphism of $\mathcal{C}$ will consequently have distinguished left and right trace formulas as defined in the subsequent subsection. Of course, the left and right trace formulas may coincide; the pivotal category in such case is called *spherical*, see [Barrett and Westbury (1999)]. For further exposition on rigid (in particular, pivotal) categories, one may refer to [Draoui and Choulli (2025)].

2.8. Trace and dimension. Pivotal (resp., spherical) categories admit a consistent theory of traces and dimensions.

Let $\mathcal{C}$ be a pivotal category. For any object $V \in \mathcal{C}$ and endomorphism $f \in \mathrm{End}_{\mathcal{C}}(V)$, the *left trace* of f is the element of $\mathtt{K}_{\mathcal{C}}$ given by

$$Tr_l(f) := d_V(1_{V^*} \otimes f)b'_V.$$

Likewise, the *right trace* of f is the following element of $\mathtt{K}_{\mathcal{C}}$

$$Tr_r(f) := d'_V(f \otimes 1_{V^*})b_V.$$

When left and right traces coincide, we simply write $Tr(f)$ and refer to as the trace of f.

The main properties of left and right traces are gathered in the following theorem.

2.9. Theorem. [Turaev and Virelizier (2017), Sect. 2.6]. *For all objects U and V of a pivotal category $\mathcal{C}$, the following assertions hold.*

1. $Tr_l(fg) = Tr_l(gf)$, *for all* $f : U \longrightarrow V$ *and* $g : V \longrightarrow U$.
2. $Tr_l(fk) = kTr_l(f)$ *and* $Tr_r(kf) = kTr_r(f)$, *for all* $k \in \mathtt{K}_{\mathcal{C}}$ *and* $f \in \mathrm{End}_{\mathcal{C}}(V)$.
3. $Tr_l(f \otimes g) = Tr_l\big(Tr_l(f)g\big)$ *and* $Tr_r(f \otimes g) = Tr_r\big(fTr_r(g)\big)$, *for all* $f \in \mathrm{End}_{\mathcal{C}}(V)$ *and* $g \in \mathrm{End}_{\mathcal{C}}(U)$.
4. $Tr_l(f^*) = Tr_r(f)$ *and* $Tr_r(f^*) = Tr_l(f)$, *for every* $f \in \mathrm{End}_{\mathcal{C}}(V)$.
5. $Tr_l(f^{**}) = Tr_l(f)$ *and* $Tr_r(f^{**}) = Tr_r(f)$, *for every* $f \in \mathrm{End}_{\mathcal{C}}(V)$.
6. $Tr_l(k) = Tr_r(k) = k$ *for every* $k \in \mathtt{K}_{\mathcal{C}}$.

For every object V of a pivotal category $\mathcal{C}$, the *left* and *right dimensions* of V are the elements defined, respectively, by

$$dim_l(V) := Tr_l(1_V) \quad \text{and} \quad dim_r(V) := Tr_r(1_V).$$

Moreover, the following properties are satisfied.

1. $dim_l(I) = dim_r(I) = 1_I$.
2. For every object $V \in \mathcal{C}$, we have $dim_r(V^{**}) = dim_l(V^*) = dim_r(V)$. Similar equalities hold for the right dimension.
3. Any isomorphic objects have equal left (resp. right) dimensions.

Let now $\mathcal{C}$ be a pivotal (resp., spherical) pure category. In addition to the aforementioned properties, the left trace is $\otimes$-multiplicative. That is, $Tr_l(f \otimes g) = Tr_l(f)\,Tr_l(g)$ for all morphisms f, $g \in \mathcal{C}$. This follows from Theorem 2.9(2), (3) and (6). Similar result hold for the right trace.

There are also other variants of left and right traces, namely the *left* and *right partial traces* for endomorphisms $f \in \mathrm{End}_{\mathcal{C}}(U \otimes V)$ of a tensor product of objects, which are defined, respectively, by

- $pTr_l(f) := (d_U \otimes 1_V)(1_{U^*} \otimes f)(b'_U \otimes 1_V) \in \mathrm{End}_{\mathcal{C}}(V)$.
- $pTr_r(f) := (1_U \otimes d'_V)(f \otimes 1_{V^*})(1_U \otimes b_V) \in \mathrm{End}_{\mathcal{C}}(U)$.

The terminology "partial trace" is justified by the following identities. For every endomorphism $f \in \mathrm{End}_{\mathcal{C}}(U \otimes V)$, we have

- $Tr_l\big(pTr_l(f)\big) = Tr_l(f)$.
- $Tr_r\big(pTr_r(f)\big) = Tr_r(f)$.

For further details and properties of partial traces, we refer to [Geer et al. (2013)].

2.10. Tensor ideals and negligibility. Let $\mathcal{C}$ be a pivotal category and X, Y two objects of $\mathcal{C}$. A morphism $f \in \mathrm{Hom}_C(X,Y)$ is called *left negligible* provided that for every $g \in \mathrm{Hom}_C(Y,X)$, $Tr_l(fg) = 0$. *Right negligibility* is defined similarly by replacing the left trace by the right trace in the previous identity. A morphism is *negligible* when it is both left and right negligible. Denote by $\mathrm{Neg}(U,V)$ the set of negligible morphisms $U \to V$.

The previous notions of left and right negligible morphisms are generally distinguished in pivotal categories. However, they coincide for a spherical category; in particular, for a ribbon category.

Let $\mathcal{C}$ be a preadditive monoidal category. A subclass $\mathcal{I} \subseteq \mathcal{C}_1$ is called a *left tensor ideal* if $\mathcal{I}$ is closed under sums, compositions on the left, and tensor products on the left. Explicitly, $\mathcal{I}$ is a left tensor ideal if, for all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \cap \mathcal{I}$, $g \in \mathrm{Hom}_{\mathcal{C}}(A,X')$, and $h \in \mathrm{Hom}_{\mathcal{C}}(Y',B)$, where X, X', Y, Y', A, $B \in \mathcal{C}$, the following conditions hold.

1. $f + f' \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \cap \mathcal{I}$ for every $f' \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \cap \mathcal{I}$.
2. $hf \in \mathrm{Hom}_{\mathcal{C}}(X,B) \cap \mathcal{I}$ whenever $X = X'$.
3. $h \otimes f \in \mathrm{Hom}_{\mathcal{C}}(Y' \otimes X, B \otimes Y) \cap \mathcal{I}$.

A *right tensor ideal* is defined analogously, with composition and tensor product are taken on the right. A *tensor ideal* is a subclass of morphisms that is both a left and a right tensor ideal.

2.11. Example. Let $\mathcal{C}$ be a pivotal preadditive category. Then, left negligible, right negligible, and negligible morphisms of $\mathcal{C}$ form, respectively, a left tensor ideal, a right tensor ideal, and a tensor ideal of $\mathcal{C}$ [Turaev (2016), Lemma 4.1.1].

Recall that an object X of a monoidal category $\mathcal{C}$ is called *left negligible* if every endomorphism $f \in \mathrm{End}_{\mathcal{C}}(X)$ is left negligible. Likewise, a *right negligible* object can be defined using right negligibility on endomorphisms of X. *Negligible* objects are then those that are simultaneously left and right negligible. Let $\mathcal{N}^l(\mathcal{C})$, $\mathcal{N}^r(\mathcal{C})$, and $\mathcal{N}(\mathcal{C})$ denote the classes of left negligible, right negligible, and negligible objects of $\mathcal{C}$, respectively.

2.12. Semisimplicity. The following definition follows [Turaev (2016)] (without restriction to ribbon categories).

2.13. Definition. *Let $\mathcal{C}$ be a preadditive pivotal category. An object $V \in \mathcal{C}$ is called simple if the map $\mathtt{K}_{\mathcal{C}} \longrightarrow \mathrm{End}_{\mathcal{C}}(V)$, sending $k \mapsto k \otimes 1_V$, is a bijection.*

It follows from the above definition that the unit object I is simple, and if V is a simple object and V^* is a dual of it, then V^* is also simple. Moreover, any object isomorphic to V is simple as well.

An object V of $\mathcal{C}$ is said to be dominated by a family of simple objects if there exists a finite family $\{V_{i(r)}\}_{r\in[n]}$ of simple objects of $\mathcal{C}$, for some $n \in \mathbb{N}^*$, and morphisms

$\varepsilon_r : V \to V_{i(r)}$ and $\mu_r : V_{i(r)} \to V$, for every $r \in [n]$, such that $\sum_r \mu_r\varepsilon_r = 1_V$. Seen as a map, $i : \mathbb{N} \to J$ is injective. The category $\mathcal{C}$ is said to be dominated by a family $\{V_i\}_{i\in J}$ of its simple objects, where J is an index set, if every object V of $\mathcal{C}$ is dominated by a finite subfamily of $\{V_i\}_{i\in J}$ in the above sense. Throughout the sequel, such a family, together with the associated morphisms, will be denoted by $\{V_{i(r)}, \varepsilon_r, \mu_r\}_{r\in[n]}$ and referred to as a dominating family of V.

A semisimple category is often defined in the literature as an abelian category whose objects split as direct sums of simple ones. Here, we adopt a (slightly) different terminology due to [Turaev (2016)]. Following [Turaev (2016), page 99] and [Turaev and Wenzl (1997), page 416], and without restriction to ribbon preadditive categories, we adopt the following definition.

2.14. Definition. *A semisimple category is a pair $(\mathcal{C}, \{V_i\}_{i\in J})$, consisting of a preadditive pivotal category $\mathcal{C}$ and a family $\{V_i\}_{i\in J}$ of simple objects of $\mathcal{C}$, such that the following axioms are satisfied.*

1. *There is some $i_0 \in J$ such that $V_{i_0} = I$ (**Normalization axiom**).*
2. *For every $i \in J$, there is some $i^* \in J$ such that $V_{i^*} \simeq V_i^*$ (**Duality axiom**).*
3. *$\mathcal{C}$ is dominated by the family $\{V_i\}_{i\in J}$ (**Domination axiom**).*
4. *For any non isomorphic simple objects V_k and V_l of $\{V_i\}_{i\in J}$, $\mathrm{Hom}_{\mathcal{C}}(V_k, V_l) = \{0\}$ (**Schur's axiom**).*

If the maps ε_r and μ_s of the dominating family $\{V_{i(r)}, \varepsilon_r, \mu_r\}_{r\in[n]}$ satisfy:

$$\varepsilon_r\mu_s = \begin{cases} 1_{V_{i(r)}} & \text{if } r = s, \\ 0 & \text{otherwise;} \end{cases} \tag{3}$$

for all r, $s \in [n]$, then we have the next result.

2.15. Lemma. *Let $\mathcal{C}$ be a semisimple category and $V \in \mathcal{C}$ a simple object dominated by $\{V_{i(r)}, \varepsilon_r, \mu_r\}_{r\in[n]}$, where the morphisms ε_r and μ_s satisfy (3). Then, $dim_l(V)$ and $dim_r(V)$ are invertible.*

Proof. This holds by combining [Turaev and Virelizier (2017), Lemmata 4.2 and 4.3]. ∎

2.16. Example. [Draoui et al. (2023), Example 3.10]. Let R be a commutative ring with identity, (G, e) a multiplicative abelian group, and $c : G \otimes G \to R^\times$ a bilinear map, where $R^\times$ denotes the set of units of R. Let $\mathcal{S}$ be the category whose objects are elements of G and the morphisms $g \to h$ are given by

$$\mathrm{Hom}_{\mathcal{S}}(g, h) = \begin{cases} R & \text{if } g = h, \\ \{0\} & \text{if } g \neq h, \end{cases}$$

for all $g, h \in G$. The unit object is e, the tensor product of objects of $\mathcal{S}$ is given by the product of G, and the composition and tensor product of morphisms of $\mathcal{S}$ is given by the product of R. The category $\mathcal{S}$ is semisimple, where the family $\{V_i\}_{i\in J}$ consists of all of its objects since they are all simple. In particular, every object of $\mathcal{S}$ is dominated by itself.

Throughout, we will simply write $\mathcal{C}$ to designate $(\mathcal{C}, \{V_i\}_{i\in J})$ for some choice of a family of simple objects, eventually, the class of all simple objects of $\mathcal{C}$. Moreover, whenever a semisimple pivotal category is considered, the condition (3) will be assumed to hold for all of its objects. Note that if $\mathcal{C}$ is a $\Bbbk$-linear pre-fusion[1] category, then $\mathcal{C}$ is a semisimple pivotal category, where the condition (3) is satisfied.

2.17. Quantum determinants. Let $\mathcal{C}$ be a semisimple pivotal category and $V \in \mathcal{C}$ an object dominated by a family $\{V_{i(r)}, \varepsilon_r, \mu_r\}_{r\in[n]}$ of simple objects, satisfying (3). Let $[n] = I_1 \cup \cdots \cup I_m$ be the partition of $[n]$ "parallel" to a partition of $\{V_{i(r)}\}_{r\in[n]}$ into classes of isomorphic simple objects, and W_j a/any representative of the class of simple objects of indices in I_j. Denote by n_j the cardinal of I_j.

Recall from [Draoui et al. (2023)] that the *left quantum determinant* of a given endomorphism $f \in \mathrm{End}_{\mathcal{C}}(V)$ is the element

$$ldet_V^n(f) := Tr_l(f^{\otimes^n} l\Lambda_V^n)$$

where, $l\Lambda_V^n$ is the endomorphism of $V^{\otimes^n}$ given by

$$l\Lambda_V^n = \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)\ dim_l^{-n_1}(W_1)\ D_\sigma^1 \otimes \cdots \otimes \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)\ dim_l^{-n_m}(W_m)\ D_\sigma^m$$

with D_σ^j is the endomorphism of $V^{\otimes^{n_j}}$ given by

$$D_\sigma^j = \mu_{s_1^j}\varepsilon_{\sigma(s_1^j)} \otimes \cdots \otimes \mu_{s_{n_j}^j}\varepsilon_{\sigma(s_{n_j}^j)} \tag{4}$$

where, $I_j = \{s_1^j, \ldots, s_{n_j}^j\}$, such that $[n] = \dot{\bigcup}_{1\le j\le m} I_j$.

Explicitly, $ldet_V^n(f)$ is given by

$$ldet_V^n(f) = \prod_{j=1}^{m}\Big(\sum_{\sigma\in\mathfrak{S}_{n_j}} \varepsilon(\sigma)\ dim_l^{-n_j}(W_j) \prod_{k_j\in I_j} Tr_l\big(\varepsilon_{k_j} f \mu_{\sigma(k_j)}\big)\Big) \tag{5}$$

where, $\mathfrak{S}_{n_j}$ denotes the set of all permutations of the set I_j and $\varepsilon(\sigma)$ the signature of σ. Likewise, the right quantum determinant of f is the element

$$rdet_V^n(f) := Tr_r(f^{\otimes^n} r\Lambda_V^n)$$

[1] This is a $\Bbbk$-linear ($\Bbbk$ is a field) monoidal category which admits a set J of simple objects containing the unit objects, satisfies the axiom of Schur for J, and each of its objects splits as a direct sum of a finite family of elements of J [Turaev and Virelizier (2017)].

where, $r\Lambda_V^n$ is defined as in the left version, by simply replacing "l" by "r" therein.

Let $f \in \mathrm{End}_{\mathcal{C}}(V)$. We will sometimes write $ldet_V(f)$ (resp., $rdet_V(f)$) to denote $ldet_V^n(f)$ (resp., $rdet_V^n(f)$) when n is understood. Moreover, there corresponds two matrices lM and rM. The matrix lM is given by

$$lM = \big(la_{r,s}\big)_{r,s\in[n]} := \Big(Tr_l(\varepsilon_r f \mu_s)\, dim_l^{-1}(V_{i(s)})\Big)_{r,s\in[n]}. \tag{6}$$

Similarly, the "right" version defines the matrix $rM = \big(ra_{r,s}\big)_{r,s\in[n]}$ by changing "l" to "r".

The following result was proved for ribbon categories [Choulli et al. (2022), Theorem 5.3(3) and Corollary 5.6(a)]. However, it holds also for a pure pivotal (resp., spherical) category.

2.18. Proposition. *Let $\mathcal{C}$ be a semisimple pure pivotal category. Then, for every $f \in$ $\mathrm{End}_{\mathcal{C}}(V)$, the map*

$$\begin{aligned} \varphi_l : \mathrm{End}_{\mathcal{C}}(V) &\longrightarrow M_n(\mathtt{K}_{\mathcal{C}}) \\ f &\longmapsto lM \end{aligned} \tag{7}$$

is a morphism of $\mathtt{K}_{\mathcal{C}}$-algebras, which turns out to be an isomorphism of $\mathtt{K}_{\mathcal{C}}$-algebras if moreover (3) is satisfied. A similar statement holds for the map φ_r defined by changing "l" to "r", where $\varphi_l = \varphi_r$ if $\mathcal{C}$ is spherical.

Proof. The same proof as in [Choulli et al. (2022), Theorem 5.3(3) and Corollary 5.6(a)] is still valid, where the following identity was essential

$$Tr_l(kg) = kTr_l(g), \quad \text{for all } g \in \mathrm{End}_{\mathcal{C}}(V) \text{ and } k \in \mathtt{K}_{\mathcal{C}}. \tag{8}$$

In the present case of a pivotal category, this identity (8) remains valid for the right trace (the first claim of Theorem 2.9(2)). However, it does not hold automatically for φ_l, as illustrated by the second claim of Theorem 2.9(2). It is nevertheless ensured under the assumption that $\mathcal{C}$ is pure, in view of identity (8). ∎

2.19. Proposition. [Draoui et al. (2023), Proposition 5.2]. *Let $\mathcal{C}$ be a semisimple pure pivotal category and $V \in \mathcal{C}$ an object dominated by a family of cardinal n. Then*

$$ldet_V^n(f) = det(lM) \quad \text{and} \quad rdet_V^n(f) = det(rM) \tag{9}$$

where, the right hand sides denote the usual determinant of square matrices.

3. Pseudo-negligibility in semisimple categories

In this section, $\mathcal{C}$ will denote a semisimple pure pivotal category.

3.1. Pseudo-negligible morphisms. We start by introducing the main notion of this subsection together with its variants, and fix the corresponding notation in order to freely use in the sequel without further mention.

Let U and V be two objects of $\mathcal{C}$. Consider the following class, which we term the class of *right l-pseudo-negligible* morphisms $U \to V$:

$$\mathrm{pNeg}_r^l(U,V) := \Big\{ f \in \mathrm{Hom}_{\mathcal{C}}(U,V) \mid ldet_V(fg) = 0, \text{ for every } g \in \mathrm{Hom}_{\mathcal{C}}(V,U) \Big\}. \quad (10)$$

The prefix "right" refers to composition on the right, and "l" refers to left quantum determinants. In particular, when $U = V$, this class is simply denoted by $\mathrm{pNeg}_r^l(V)$. The class of all right l-pseudo-negligible morphisms of $\mathcal{C}$ is denoted by $\mathrm{pNeg}_r^l(\mathcal{C})$. Likewise, the class $\mathrm{pNeg}_r^r(U,V)$ of *right r-pseudo-negligible* morphisms $U \to V$ is defined using right quantum determinants.

Analogously, by instead composing on the left, one defines the classes $\mathrm{pNeg}_l^l(U,V)$ and $\mathrm{pNeg}_l^r(U,V)$ of *left l-pseudo-negligible* and *left r-pseudo-negligible* morphisms $U \to V$, respectively. Now, set

$$\mathrm{pNeg}^l(U,V) := \mathrm{pNeg}_l^l(U,V) \cap \mathrm{pNeg}_r^l(U,V),$$
$$\mathrm{pNeg}^r(U,V) := \mathrm{pNeg}_l^r(U,V) \cap \mathrm{pNeg}_r^r(U,V),$$
$$\mathrm{pNeg}(U,V) := \mathrm{pNeg}^l(U,V) \cap \mathrm{pNeg}^r(U,V),$$

and refer to them, respectively, as the classes of l-pseudo-negligible, r-pseudo-negligible, and pseudo-negligible morphisms $U \to V$.

Clearly, the zero morphism $0 \in \mathrm{pNeg}(U,V)$ for all U, $V \in \mathcal{C}$. Moreover, we have

$$\mathrm{pNeg}^l(I,I) = \mathrm{pNeg}^r(I,I) = \mathrm{pNeg}(I,I) = \mathrm{Neg}(I,I) = \{0\}.$$

The following proposition justifies the terminology "pseudo-negligible".

3.2. Proposition. *For all objects U, $V \in \mathcal{C}$, the following assertions hold.*

1. $\mathrm{Neg}^l(U,V) \subseteq \mathrm{pNeg}^l(U,V)$.
2. $\mathrm{Neg}^r(U,V) \subseteq \mathrm{pNeg}^r(U,V)$.
3. $\mathrm{Neg}(U,V) \subseteq \mathrm{pNeg}(U,V)$.

Proof. 1. Let $f \in \mathrm{Neg}^l(U,V)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(V,U)$. Since f is left negligible, fg is also left negligible, and then so are as well $(fg)^{\otimes^n}$ and $(fg)^{\otimes^n}\Lambda_V^n$, were n is the cardinal of the dominating family on V. Then

$$ldet_V^n(fg) := Tr_l\big((fg)^{\otimes^n}\Lambda_V^n\big) = 0.$$

This shows that $f \in \mathrm{pNeg}_r^l(U,V)$. With similar arguments, cyclicity of left traces (i.e., $Tr_l(\alpha\beta) = Tr_l(\beta\alpha)$ for all composable morphisms α and β) allows one to show that $f \in \mathrm{pNeg}_l^l(U,V)$. By definition of $\mathrm{pNeg}^l(U,V)$, it follows that $f \in \mathrm{pNeg}^l(U,V)$.
2. The proof is similar to that of (1).
3. This is, by definition of $\mathrm{pNeg}(U,V)$, a direct consequence of (1) and (2). ∎

Note that cyclicity of left (resp., right) quantum determinants does not hold; that is, for composable morphisms f and g, we generally have

$$ldet_V(fg) \neq ldet_U(gf) \quad \big(\text{resp., } rdet_V(fg) \neq rdet_U(gf)\big).$$

However, this holds for composable endomorphisms [Draoui et al. (2023), Theorem 4.14(d)]. Moreover, for morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(V, V^\vee)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(V^\vee, V)$, where $V^\vee$ is either V^* or V^{**}, cyclicity holds also in this case.

3.3. Lemma. *Let $V \in \mathcal{C}$ be an object dominated by a family $\{V_{i(r)}, \varepsilon_r, \mu_r\}_{r\in[n]}$. For any morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(V, V^\vee)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(V^\vee, V)$, with $V^\vee = V^*$ or V^{**}, we have*

$$ldet^n_V(gf) = ldet^n_{V^\vee}(fg) \quad \big(\text{resp., } rdet^n_V(gf) = rdet^n_{V^\vee}(fg)\big).$$

Proof. We will show the left version since the right one is similar. Let $V^\vee = V^*$, and notice that V^* is dominated by $\{V^*_{i(r)}, \mu^*_r, \varepsilon^*_r\}_{r\in[n]}$ which also satisfies the condition (3). Consider the following matrices

$$M^f = \left(a^f_{r,s}\right)_{r,s\in[n]} = \left(Tr_l(\mu^*_r f \mu_s)\; dim_l^{-1}(V_{i(r)})\right)_{r,s\in[n]},$$

$$M^g = \left(a^g_{r,s}\right)_{r,s\in[n]} = \left(Tr_l(\varepsilon_r g \varepsilon^*_s)\; dim_l^{-1}(V_{i(r)})\right)_{r,s\in[n]},$$

$$M^{gf} = \left(a^{gf}_{r,s}\right)_{r,s\in[n]} = \left(Tr_l(\varepsilon_r g f \mu_s)\; dim_l^{-1}(V_{i(r)})\right)_{r,s\in[n]},$$

$$M^{fg} = \left(a^{fg}_{r,s}\right)_{r,s\in[n]} = \left(Tr_l(\mu^*_r f g \varepsilon^*_s)\; dim_l^{-1}(V_{i(r)})\right)_{r,s\in[n]}.$$

For all $r, s \in [n]$, we have

$$\begin{aligned}
a^{gf}_{r,s} &= Tr_l\Big(\varepsilon_r g f \mu_s\Big)\, dim_l^{-1}(V_{i(r)}) \\
&= Tr_l\Big(\varepsilon_r g\Big(\sum_{l=1}^{n} \varepsilon^*_l \mu^*_l\Big) f \mu_s\Big)\, dim_l^{-1}(V_{i(r)}) \\
&= \sum_{l=1}^{n} Tr_l\Big(\varepsilon_r g \varepsilon^*_l\ \mu^*_l f \mu_s\Big)\, dim_l^{-1}(V_{i(r)}) \\
&= \sum_{l=1}^{n} Tr_l\Big((k_{r,l} \otimes 1_{V_{i(r)}})\ \mu^*_l f \mu_s\Big)\, dim_l^{-1}(V_{i(r)}) \\
&= \sum_{l=1}^{n} Tr_l\Big(k_{r,l} \otimes \mu^*_l f \mu_s\Big)\, dim_l^{-1}(V_{i(r)}) \\
&= \sum_{l=1}^{n} Tr_l\Big(k_{r,l}\Big)\, Tr_l\Big(\mu^*_l f \mu_s\Big)\, dim_l^{-1}(V_{i(r)})
\end{aligned}$$

$$= \sum_{l=1}^{n} Tr_l\Big(k_{r,l} \otimes 1_{V_{i(r)}}\Big)\, dim_l^{-1}(V_{i(r)})\, Tr_l\Big(\mu_l^* f \mu_s\Big)\, dim_l^{-1}(V_{i(r)})$$
$$= \sum_{l=1}^{n} Tr_l\Big(\varepsilon_r g \varepsilon_l^*\Big)\, dim_l^{-1}(V_{i(r)})\, Tr_l\Big(\mu_l^* f \mu_s\Big)\, dim_l^{-1}(V_{i(r)}) = \sum_{l=1}^{n} a^g_{r,l}\, a^f_{l,s}$$

for some unique $k_{r,l}$ in $\mathtt{K}_C$, since $\varepsilon_r g \varepsilon_l^*$ is an endomorphism of a simple object $V_{i(r)}$. Hence, $M^{gf} = M^g M^f$. Likwise, one can show that $M^{fg} = M^f M^g$. Thus, by commutativity of $\mathtt{K}_C$, we get

$$\begin{aligned} ldet^n_V(fg) &= det(M^{fg}) \\ &= det(M^f)\, det(M^g) \\ &= det(M^g)\, det(M^f) \\ &= det(M^{gf}) \\ &= ldet^n_{V^*}(gf). \end{aligned}$$

The case $V^\vee = V^{**}$ holds analogously as one can similarly define the corresponding matrices observing that V^{**} is dominated by $\{V^{**}_{i(r)}, \varepsilon_r^{**}, \mu_r^{**}\}_{r\in[n]}$ which also satisfies (3). ■

Important notable facts of pseudo-negligible morphisms are the following.

3.4. Lemma. *The following assertions hold in $\mathcal{C}$.*

1. *The composition, on the right, of any right l-pseudo-negligible morphism with an arbitrary (composable) morphism is again right l-pseudo-negligible.*
2. *The tensor product, on both sides, of any right l-pseudo-negligible morphism with an arbitrary morphism is again right l-pseudo-negligible.*
3. *An endomorphism is right r-pseudo-negligible if and only if its dual is right l-pseudo-negligible.*

Proof. 1. Clear from the definition.
2. We prove the left version. Let U, V, U', $V' \in \mathcal{C}$, $f \in \mathrm{pNeg}^l_r(U,V)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(U',V')$. Then, for all $g \in \mathrm{Hom}_{\mathcal{C}}(V\otimes V', U\otimes U')$, and using Item (1), we have

$$\begin{aligned} ldet_{V\otimes V'}\big((f\otimes h)g\big) &= ldet_{V\otimes V'}\big(f(1_U\otimes d'_{V'})(1_U\otimes h\otimes 1_{(V')^*})(g\otimes 1_{(V')^*})(1_V\otimes b_{V'})\big) \\ &= 0 \end{aligned}$$

Likewise, tensoring by h on the left holds.
3. Let $V\in\mathcal{C}$ and $f \in \mathrm{pNeg}^r_r(V)$. Then, for any endomorphism $g \in \mathrm{End}_{\mathcal{C}}(V^*)$, we have

$$\begin{aligned} ldet^n_{V^*}(f^*g) &= rdet^n_{V^{**}}\big((f^*g)^*\big) \\ &= rdet^n_{V^{**}}\big(g^* f^{**}\big) \end{aligned}$$

$$\begin{aligned}&= rdet^n_{V^{**}}\big(g^*\Phi_V f\Phi_V^{-1}\big)\\ &= rdet^n_V\big(f\Phi_V^{-1}g^*\Phi_V\big)\\ &= 0\end{aligned}$$

where, $\Phi_U : U \to U^{**}$ is the pivotal structure. The first passage is by [Draoui et al. (2023), Theorem 4.14(d)], the fourth one is by Lemma 3.3 and the last one is by hypothesis on f.

Conversely, let $f \in \mathrm{End}_{\mathcal{C}}(V)$ and assume that $f^* \in \mathrm{pNeg}^l_r(V^*)$. Then, for every $g \in \mathrm{End}_{\mathcal{C}}(V)$, we have

$$\begin{aligned}rdet_V(fg) &= rdet^n_V\big(\Phi_V^{-1}f^{**}\Phi_V g\big)\\ &= rdet^n_{V^{**}}\big(f^{**}\Phi_V g\Phi_V^{-1}\big)\\ &= 0\end{aligned}$$

where, the second passage follows from Lemma 3.3, and the last one is by hypothesis along with the previous direct implication (that is, since $f^* \in \mathrm{pNeg}^l_r(V^*)$, it follows that $f^{**} \in \mathrm{pNeg}^r_r(V^{**})$). ■

Note that Lemma 3.4 holds analogously when "l" is replaced by "r". Moreover, it also holds for left l-pseudo-negligible morphisms, where in Item (1), the composition is taken on the left.

3.5. Remark. The class of right (resp., left) l-pseudo-negligible morphisms is not closed under sums since left quantum determinants are not additive in general, which is the only missing requirement (in view of Lemma 3.4) for these classes to form a right (resp., left) tensor ideal. However, if $U, V \in \mathcal{C}$ are simple objects, the stability under sums holds [Draoui et al. (2023), Corollary 4.15].

The motivation for considering the class of l(resp., r)-pseudo-negligible morphisms is to pass to a quotient in which such morphisms may vanish. Unfortunately, this class is not closed under sums, and hence fails to form a tensor ideal. However, since this stability holds when restricted to simple objects, one can define a "conditional" equivalence relation as follows, where we restrict to l-pseudo-negligible morphisms as similar results can be obtained for r-pseudo-negligible ones. Let $\mathcal{S}(\mathcal{C})$ denote the full subcategory of simple objects of $\mathcal{C}$. For all morphisms $f, g : U \to V$ in $\mathcal{C}$, we define

$$f \sim g \quad \Longleftrightarrow \quad \begin{cases} f-g \in \mathrm{pNeg}^l(U,V) & \text{if } U, V \in \mathcal{S}(\mathcal{C}),\\ f = g & \text{otherwise.}\end{cases}$$

It is not difficult to check that is an equivalence relation in light of Remark 3.5(3).

Denote by $\widehat{\mathcal{C}}$ the category whose objects are the same as those of $\mathcal{C}$ and whose hom-sets are given as follows

$$\mathrm{Hom}_{\widehat{\mathcal{C}}}(U,V) := \mathrm{Hom}_{\mathcal{C}}(U,V)\ /\sim \quad \text{for all } U, V \in \mathcal{C}.$$

The above construction gives rise to a covariant functor

$$\mathcal{R} : \mathcal{C} \longrightarrow \widehat{\mathcal{C}}$$

which is identical on objects and sends every morphism to its class modulo l-pseudo-negligible morphisms. This functor induces the monoidal structure of $\widehat{\mathcal{C}}$ from that of $\mathcal{C}$. We refer to this process as *pseudo-purification*.

A category $\mathcal{C}$ will be called *l-pseudo-pure* (resp., *r-pseudo-pure*) if every l-pseudo-negligible (resp., r-pseudo-negligible) morphism between simple objects is zero. Thus, $\mathcal{C}$ is *pseudo-pure* when it is both l-pseudo-pure and r-pseudo-pure.

3.6. Theorem. *The category $\widehat{\mathcal{C}}$ is a pseudo-pure monoidal category .*

Proof. The composition and tensor product in $\widehat{\mathcal{C}}$ are induced from those of $\mathcal{C}$ and are well defined since $\mathrm{pNeg}^l(\mathcal{C})$ (resp., $\mathrm{pNeg}^r(\mathcal{C})$) is, respectively, closed under composition on both sides and under tensor products on both sides, together with stability under sums when restricted to simple objects. The unit object is the same as that of $\mathcal{C}$. Thus, $\widehat{\mathcal{C}}$ inherits a monoidal structure from $\mathcal{C}$, and $\widehat{\mathcal{C}}$ is pseudo-pure as it is , by construction, l-pseudo-pure (resp., r-pseudo-pure). ∎

3.7. Pseudo-negligible objects. Recall that $\mathcal{N}^l(\mathcal{C})$, $\mathcal{N}^r(\mathcal{C})$ and $\mathcal{N}(\mathcal{C})$ denote the classes of left negligible, right negligible and negligible objects of $\mathcal{C}$, respectively. Let $p\mathcal{N}^l(\mathcal{C})$ denote the class of l-pseudo-negligible objects, that is

$$p\mathcal{N}^l(\mathcal{C}) := \Big\{ V \in \mathcal{C} \;\mid\; \text{every } f \in \mathrm{End}_{\mathcal{C}}(V) \text{ is l-pseudo-negligible} \Big\}.$$

Likewise, one can define the class $p\mathcal{N}^r(\mathcal{C})$ of r-pseudo-negligible objects of $\mathcal{C}$. Set then $p\mathcal{N}(\mathcal{C}) := p\mathcal{N}^l(\mathcal{C}) \cap p\mathcal{N}^r(\mathcal{C})$ the class of pseudo-negligible objects of $\mathcal{C}$.

3.8. Proposition. *We have $\mathcal{N}^l(\mathcal{C}) \subseteq p\mathcal{N}^l(\mathcal{C})$, $\mathcal{N}^r(\mathcal{C}) \subseteq p\mathcal{N}^r(\mathcal{C})$ and $\mathcal{N}(\mathcal{C}) \subseteq p\mathcal{N}(\mathcal{C})$.*

Proof. Immediate from Proposition 3.2. ∎

The assignments $X \mapsto X^*$ induce a bijection between $p\mathcal{N}^l(\mathcal{C})$ and $p\mathcal{N}^r(\mathcal{C})$.

3.9. Proposition. *The following assertions hold in $\mathcal{C}$.*

1. *Let $X \in \mathcal{C}$. Then, $X \in p\mathcal{N}^l(\mathcal{C})$ if and only if $X^* \in p\mathcal{N}^r(\mathcal{C})$.*
2. *$p\mathcal{N}^l(\mathcal{C})$ and $p\mathcal{N}^r(\mathcal{C})$ are closed under biduality.*

Proof. 1. Let $X \in p\mathcal{N}^l(\mathcal{C})$ and $f \in \mathrm{End}_{\mathcal{C}}(X^*)$. Then, for every $h \in \mathrm{End}_{\mathcal{C}}(X^*)$, we have

$$rdet_{X^*}(fh) = ldet_{X^{**}}\big((fh)^*\big) = ldet_{X^{**}}\big((fh)^*\Phi_X\Phi_X^{-1}\big) = ldet_X\big(\Phi_X^{-1}(fh)^*\Phi_X\big) = 0.$$

The first equality follows from [Draoui et al. (2023), Theorem 4.7(c)]. The second is immediate by invertibility of the pivotal structure. The third follows from Lemma 3.3. The last equality holds by assumption as $\Phi_X^{-1}(fh)^*\Phi_X \in \mathrm{End}_{\mathcal{C}}(X)$ and $X \in p\mathcal{N}^l(\mathcal{C})$.

Conversely, let $X \in \mathcal{C}$ such that $X^* \in p\mathcal{N}^r(\mathcal{C})$, and $f \in \mathrm{End}_{\mathcal{C}}(X)$. Then, for every $h \in \mathrm{End}_{\mathcal{C}}(X)$, we have

$$ldet_X(fh) = rdet_{X^*}\big((fh)^*\big) = 0.$$

The first equality follows from [Draoui et al. (2023), Theorem 4.7(c)], while the second holds by assumption as $(fh)^* \in \mathrm{End}_{\mathcal{C}}(X^*)$ and $X^* \in p\mathcal{N}^r(\mathcal{C})$.
2. Let $X \in p\mathcal{N}^l(\mathcal{C})$. Then, by Item (1), we have $X^* \in p\mathcal{N}^r(\mathcal{C})$, and hence $X^{**} \in p\mathcal{N}^l(\mathcal{C})$. Similar arguments apply to the right version. ■

4. One-sided traces on one-sided ideals: a generalization

Let $\Bbbk$ be a field, $\mathcal{C}$ a pivotal $\Bbbk$-category, and $\mathcal{I}$ a left (resp., right) ideal. Recall from [Geer et al. (2013)] that a *left* (resp., *right*) *trace* t on $\mathcal{I}$ is a family of $\Bbbk$-linear maps

$$\big\{\mathsf{t}_X : \mathrm{End}_{\mathcal{C}}(X) \longrightarrow \Bbbk\big\}_{X\in\mathcal{I}},$$

compatible with the tensor product and composition as follows.

1. For all $X, Y \in \mathcal{I}$, $f : X \to Y$, and $g : Y \to X$, we have
$$\mathsf{t}_Y(fg) = \mathsf{t}_X(gf);$$
2. For all $X \in \mathcal{I}$, $Y \in \mathcal{C}$, and $f \in \mathrm{End}_{\mathcal{C}}(Y \otimes X)$ $\big($resp., $f \in \mathrm{End}_{\mathcal{C}}(X \otimes Y)\big)$, we have
$$\mathsf{t}_{Y\otimes X}(f) = \mathsf{t}_X(pTr_l(f)) \quad \big(\text{resp., } \mathsf{t}_{X\otimes Y}(f) = \mathsf{t}_X(pTr_r(f))\big).$$

A *trace* on an ideal $\mathcal{I}$ is a two-sided trace on $\mathcal{I}$.

There is also attached a dimension notion termed the *modified dimension*, which is a family of maps defined on objects $X \in \mathcal{I}$ by $\mathsf{d}(X) := \mathsf{t}_X(1_X)$.

We fix now a preadditive monoidal category $\mathcal{C}$ and an ideal $\mathtt{I}$ of $\mathtt{K}_{\mathcal{C}}$. The above definition of trace can be generalized as follows.

4.1. Definition. *Let $\mathcal{I}$ be a left (resp., right) ideal of $\mathcal{C}$. A left (resp., right) $\mathtt{I}$-trace on $\mathcal{I}$ is a family t of $\mathtt{K}_{\mathcal{C}}$-linear maps*

$$\mathsf{t} = \big\{\mathsf{t}_X : \mathrm{End}_{\mathcal{C}}(X) \longrightarrow \mathtt{I}\big\}_{X\in\mathcal{I}},$$

satisfying the same previous conditions for the usual trace.
An $\mathtt{I}$-trace on an ideal $\mathcal{I}$ is a two-sided $\mathtt{I}$-trace on $\mathcal{I}$.

Thus defined, a trace on an ideal $\mathcal{I}$ is just a $\mathtt{K}_{\mathcal{C}}$-trace on $\mathcal{I}$.

As in the usual case, one can immediately derive a corresponding $\mathtt{I}$-dimension function.

4.2. Remark. Let $\mathtt{I}'$ be an ideal of $\mathtt{K}_{\mathcal{C}}$.

1. If $\mathtt{I} \subseteq \mathtt{I}'$, then every left (resp., right) $\mathtt{I}$-trace is a left (resp., right) $\mathtt{I}'$-trace.
2. If $\mathtt{t}$ is a left (resp., right) $\mathtt{I}$-trace and a left (resp., right) $\mathtt{I}'$-trace, then $\mathtt{t}$ is a left (resp., right) $\mathtt{I} \cap \mathtt{I}'$-trace.

4.3. Proposition. *Let $\mathcal{I}$ be an ideal of $\mathcal{C}$. Then, a family $\mathtt{t}$ of $\mathtt{K}_{\mathcal{C}}$-linear maps is a left (resp., right) $\mathtt{I}$-trace on $\mathcal{I}$ if and only if $\mathtt{t}^*$ is a right (resp., left) $\mathtt{I}$-trace on $\mathcal{I}^*$, where $\mathtt{t}^*$ is the family of maps defined for every $f \in \mathrm{End}_{\mathcal{C}}(X)$, $X \in \mathcal{I}$, by $\mathtt{t}^*_X(f) = \mathtt{t}_{X^*}(f^*)$.*

Proof. Let's consider the left version (the right one deduces similarly). It suffices to check that for any left $\mathtt{I}$-trace $\mathtt{t}$ on $\mathcal{I}$, $\mathtt{t}^*$ is a well defined right $\mathtt{I}$-trace on $\mathcal{I}^*$. For all $X \in \mathcal{I}^*$ and $f \in \mathrm{End}_{\mathcal{C}}(X)$, we have $\mathtt{t}^*_X(f) := \mathtt{t}_{X^*}(f^*) \in \mathtt{I}$. The proof of conditions (1) and (2) of Definition 4.1 is the same as the usual case, see [Geer et al. (2013), Lemma 3(c)]. ■

4.4. Proposition. *Let $\mathtt{t}$ be a left (resp., right) $\mathtt{I}$-trace on a left (resp., right) ideal $\mathcal{I}$. Then, for every $X \in \mathcal{I}$ and $f \in \mathrm{End}_{\mathcal{C}}(X)$, we have*

1. $\mathtt{t}_{X^{**}}(f^{**}) = \mathtt{t}_X(f)$.
2. *If $\mathcal{I}$ is an ideal and $\mathtt{t}$ an $\mathtt{I}$-trace, then* $\mathtt{t}_{X^*}(f^*) = \mathtt{t}_X(f)$.

Proof. This holds from [Geer et al. (2013), Lemma 3(a) and (b)]. ■

5. Determinants and diagram invariants

By a *left* (resp., *right*) $\mathtt{t}$-*traced ideal* $\mathcal{I}$, we mean a left (resp., right) ideal $\mathcal{I}$ equipped by a left (resp., right) trace $\mathtt{t}$ on it. A $\mathtt{t}$-*traced ideal* will be then a class $\mathcal{I}$ which is both a left and right $\mathtt{t}$-traced ideal.

5.1. One-sided tensor multiplicativity. In what follows, we investigate the partial trace in the general setting of endomorphisms of the tensor product of finitely many objects. We then establish its connection with the trace on an ideal in order to examine its tensor multiplicativity.

Let $f \in \mathrm{End}_{\mathcal{C}}(V_1 \otimes \cdots \otimes V_n)$, with $V_1, \ldots, V_n \in \mathcal{C}$ and $n \geq 2$. For every $1 \leq k \leq n-1$, denote by $pTr_l^{(n-k,k)}(f)$, the $(n-k,k)$-left partial trace of f, which is the endomorphism given by the following composite

$$\big(d_{V_1 \otimes \cdots \otimes V_k} \otimes 1_{V_k \otimes \cdots \otimes V_n}\big)\big(1_{(V_1 \otimes \cdots \otimes V_k)^*} \otimes f\big)\big(b'_{V_1 \otimes \cdots \otimes V_k} \otimes 1_{V_k \otimes \cdots \otimes V_n}\big).$$

Thus defined

$$pTr_l^{(n-k,k)}(f) : V_k \otimes \cdots \otimes V_n \longrightarrow V_k \otimes \cdots \otimes V_n,$$

and satisfies

$$Tr_l(f) = Tr_l\big(pTr_l^{(n-k,k)}(f)\big). \tag{11}$$

In particular, for $n=2$, $pTr_l^{(1,1)}(f)$ coincides with the partial trace $pTr_l(f)\in \mathrm{End}_{\mathcal{C}}(V_2)$ as defined in the preliminary.

Likewise, the $(k,n-k)$-right partial trace of f can be defined and will be denoted by $pTr_r^{(k,n-k)}(f)$. Thus defined

$$pTr_r^{(n-k,k)}(f): V_1\otimes\cdots\otimes V_k \longrightarrow V_1\otimes\cdots\otimes V_k,$$

and satisfies

$$Tr_r(f) = Tr_r\big(pTr_r^{(k,n-k)}(f)\big).$$

5.2. Lemma. *Let $\mathcal{I}$ be a left (resp., right) t-traced ideal. Let $U,V\in\mathcal{I}$, $f\in\mathrm{End}_{\mathcal{C}}(U)$ and $g\in\mathrm{End}_{\mathcal{C}}(V)$. Then*

$$\mathsf{t}_{U\otimes V}(f\otimes g) = Tr_l(f)\,\mathsf{t}_V(g) \quad \big(\text{resp., } \mathsf{t}_{U\otimes V}(f\otimes g) = \mathsf{t}_U(f)Tr_r(g)\,\big).$$

Proof. Using the graphical calculus, it is not difficult to see that $pTr_l(f\otimes g) = Tr_l(f)\otimes g$. On the other hand, we have $\mathsf{t}_{U\otimes V}(f\otimes g) = \mathsf{t}_V\big(pTr_l(f\otimes g)\big)$. Therefore, $\mathsf{t}_{U\otimes V}(f\otimes g) = Tr_l(f)\,\mathsf{t}_V(g)$. The right version holds analogously. ∎

5.3. Corollary. *Let $\mathcal{I}$ be a left (resp., right) t-traced ideal, $n\geq 2$, and $f_1\otimes\cdots\otimes f_n\in \mathrm{End}_{\mathcal{C}}(V_1\otimes\cdots\otimes V_n)$, where $V_1,\dots,V_n\in\mathcal{I}$. Then*

$$\mathsf{t}_{V_1\otimes\cdots\otimes V_n}(f_1\otimes\cdots\otimes f_n) = Tr_l(f_1)\cdots Tr_l(f_{n-1})\ \mathsf{t}_V(f_n),$$

respectively

$$\mathsf{t}_{V_1\otimes\cdots\otimes V_n}(f_1\otimes\cdots\otimes f_n) = \mathsf{t}_U(f_1)\ Tr_r(f_2)\cdots Tr_r(f_n).$$

Proof. It suffices to use the previous discussion on the $(n-k,k)$-left (resp., right) partial traces. For the left version of the corollary, we have

$$\begin{aligned}
\mathsf{t}_{V_1\otimes\cdots\otimes V_n}(f_1\otimes\cdots\otimes f_n) &= \mathsf{t}_{V_n}\big(pTr_l(f_1\otimes\cdots\otimes f_n)\big)\\
&= Tr_l(f_1\otimes\cdots\otimes f_{n-1})\ \mathsf{t}_{V_n}(f_n)\\
&= Tr_l\big(pTr_l^{n-2,1}(f_1\otimes\cdots\otimes f_{n-1})\big)\ \mathsf{t}_{V_n}(f_n)\\
&= Tr_l(f_1\otimes\cdots\otimes f_{n-2})\big)\ Tr_l(f_{n-1})\ \mathsf{t}_{V_n}(f_n)\\
&= Tr_l\big(pTr_l^{n-3,1}(f_1\otimes\cdots\otimes f_{n-2})\big)\ Tr_l(f_{n-1})\ \mathsf{t}_{V_n}(f_n)\\
&\cdots\cdots\\
&= Tr_l(f_1)\cdots Tr_l(f_{n-1})\ \mathsf{t}_V(f_n)
\end{aligned}$$

The right version holds similarly. ∎

Note that Corollary 5.3 can alternatively be proved by induction on n; that is, without invoking the $(n-k,k)$-left (resp., right) partial trace notation.

The basic properties of the modified dimension are collected in the following corollary.

5.4. COROLLARY. *Let $\mathcal{I}$ be a left (resp., right) t-traced ideal and d the associated modified dimension. Then, for any objects $U,V\in\mathcal{I}$, the following assertions hold.*

1. $\mathsf{d}(V)=\mathsf{d}(V^{**})$.
2. *If $\mathcal{I}$ is a t-traced ideal, then $\mathsf{d}(V)=\mathsf{d}(V^{*})$.*
3. $\mathsf{d}(U\otimes V)=dim_l(U)\ \mathsf{d}(V)\quad \big(\text{resp., } \mathsf{d}(U\otimes V)=\mathsf{d}(U)\ dim_r(V)\big)$.
4. *If $U\simeq V$, then $\mathsf{d}(U)=\mathsf{d}(V)$.*

PROOF. 1. Clear from Proposition 4.4(1).
2. This holds from Proposition 4.4(2).
3. By Lemma 5.2, we have

$$\mathsf{d}(U\otimes V)=Tr_l(1_U)\ \mathsf{t}_V(1_V)=dim_l(U)\ \mathsf{d}(V);$$

respectively

$$\mathsf{d}(U\otimes V)=\mathsf{t}_U(1_U)\ Tr_r(1_V)=\mathsf{d}(U)\ dim_r(V).$$

4. Immediate from cyclicity of t (i.e., its compatibility with composition). ■

5.5. DETERMINANTS ON TRACED IDEALS. Throughout the sequel, we fix a semisimple pure pivotal category $\mathcal{C}$.

5.6. DEFINITION. *A left (resp., right) t-traced ideal $\mathcal{I}$ is called admissible if the modified dimension $\mathsf{d}(W)$ of W is invertible, for every simple object $W\in\mathcal{I}$.*

5.7. DEFINITION. *Let $\mathcal{I}$ be an admissible left (resp., right) t-traced ideal, and d the associated modified dimension. A left (resp., right) determinant on $\mathcal{I}$ is the family of maps*

$$\mathsf{det}=\big\{\mathsf{det}_X:\mathrm{End}_{\mathcal{C}}(X)\longrightarrow \mathtt{K}_{\mathcal{C}}\big\}_{X\in\mathcal{I}}$$

given by

$$\mathsf{det}_X(f):=\mathsf{t}_{X^n}(f^{\otimes^n}\lambda_X^n),$$

for every $X\in\mathcal{I}$, dominated by $\{V_{i(r)},\varepsilon_r,\mu_r\}_{r\in[n]}$, and $f\in\mathrm{End}_{\mathcal{C}}(X)$, where λ_X^n is the endomorphism of X^n given by

$$\lambda_X^n:=\sum_{\sigma\in\mathfrak{S}_{n_1}}\varepsilon(\sigma)\ \mathsf{d}^{-n_1}(W_1)\ D_\sigma^1\otimes\cdots\otimes\sum_{\sigma\in\mathfrak{S}_{n_m}}\varepsilon(\sigma)\ \mathsf{d}^{-n_m}(W_m)\ D_\sigma^m,$$

with D_σ^j is the endomorphism of X^{n_j} as defined by (4) and W_j a representative object associated to each class I_j, for every $j\in[m]$.

Note that det is well defined since λ_X^n is well defined. Indeed, for every $j \in [m]$, we have $W_j \in \mathcal{I}$ by the closedness under retractions of $\mathcal{I}$, as $W_j \in \{V_{i(r)}\}_r$ and, for every $r \in [n]$, $(V_{i(r)} \lneq X, \mu_r, \varepsilon_r)$ holds by (3). Notice also that the assumption of invertibility of the modified dimensions of the simple objects in $\mathcal{I}$ can be weakened in case the ground ring $\mathsf{K}_{\mathcal{C}}$ is a field. That is, by [Geer et al. (2013), Lemma 18], if $\mathcal{I}$ is the ideal of projective objects, which is notably the most interesting example of a t-traced ideal in a pivotal category, then a sufficient condition for a simple object $W \in \mathcal{I}$ to have an invertible modified dimension is that its corresponding evaluations d_W and $d_W^{'}$ are epimorphisms.

5.8. THEOREM. *Let $\mathcal{I}$ be an admissible left (resp., right) t-traced ideal, and $X \in \mathcal{I}$ an object dominated by a family $\{V_{i(r)}, \varepsilon_r, \mu_r\}_{r\in[n]}$ such that any isomorphic objects of this family are identical. Assume that t is $\otimes$-multiplicative. Then, for every $f \in \operatorname{End}_{\mathcal{C}}(X)$, we have*

$$\mathsf{det}_X(f) = det(M^f) \tag{12}$$

where $M^f = (a^f_{r,s})_{r,s\in[n]}$, and for all $r, s \in [n]$:

$$a^f_{r,s} := \mathsf{t}(\varepsilon_r f \mu_s)\ \mathsf{d}^{-1}(V_{i(r)}).$$

PROOF. The condition on the objects of the dominating family is to guarantee that $\mathsf{t}(\varepsilon_r f \mu_s)$ is well defined. Indeed, if $V_{i(r)} \simeq V_{i(s)}$, then $\varepsilon_r f \mu_s \in \operatorname{End}_{\mathcal{C}}(V_{i(r)})$; otherwise, by the Schur's axiom, $\varepsilon_r f \mu_s = 0$. Now, to prove (12), it suffices to assume that $V_{i(r)} = V_{i(s)}$ for all $r, s \in [n]$ since the matrix M^f is, in the general case, a block diagonal matrix, hence its determinant will be the product of the determinants of matrices on the diagonal, which are square matrices satisfying the assumed condition for the simple objects investigated by their entries, see [Draoui et al. (2023), Lemma 4.11] and its proof. Hence, using $\otimes$-multiplicativity of t, we have

$$\begin{aligned}
det(M^f) &= \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\ a^f_{\sigma(1),1}\cdots a^f_{\sigma(n),n} \\
&= \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\ \mathsf{t}(\varepsilon_{\sigma(1)} f \mu_1)\ \mathsf{d}^{-1}(V_{i(1)})\cdots \mathsf{t}(\varepsilon_{\sigma(n)} f \mu_n)\ \mathsf{d}^{-1}(V_{i(n)}) \\
&= \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\ \mathsf{d}^{-n}(V_{i(n)})\, \mathsf{t}(f\mu_1\varepsilon_{\sigma(1)})\cdots \mathsf{t}(f\mu_n\varepsilon_{\sigma(n)}) \\
&= \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\ \mathsf{d}^{-n}(V_{i(n)})\ \mathsf{t}(f\mu_1\varepsilon_{\sigma(1)} \otimes\cdots\otimes f\mu_n\varepsilon_{\sigma(n)}) \\
&= \mathsf{t}\Big(\sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\ \mathsf{d}^{-n}(V_{i(n)})\ f^{\otimes^n} D^n_\sigma\Big) = \mathsf{t}\Big(f^{\otimes^n}\lambda^n_X\Big) = \mathsf{det}_X(f).
\end{aligned}$$

∎

Note that the assumption that every isomorphic objects of the dominating family must be identical is not a restriction since we can always construct, from a given dominating family $\{V_{i(r)}, \varepsilon_r, \mu_r\}_{r\in[n]}$, another dominating family $\{V'_{i(r)}, \varepsilon'_r, \mu'_r\}_{r\in[n]}$ of X of the same cardinal and satisfying that assumption and the condition (3) [Draoui et al. (2023), Section 4].

5.9. LEMMA. *With the same hypotheses and notations of Theorem 5.8, we have*

1. $M^{gf} = M^g M^f$, *for every* $g \in \mathrm{End}_{\mathcal{C}}(X)$.
2. $M^{k\cdot f} = k^n M^f$, *for every* $k \in \mathsf{K}_{\mathcal{C}}$.
3. *If* $\mathcal{I}$ *is a* t*-traced ideal, then* $M^{f^*} = \left(M^f\right)^T$ *where, the right hand side denotes the transpose matrix.*

PROOF. 1. Following similar steps as in the proof of Lemma 3.3, for all $r, s \in [n]$, we have

$$\begin{aligned} a^{gf}_{r,s} = \mathsf{t}\Big(\varepsilon_r\ gf\ \mu_s\Big)\,\mathsf{d}^{-1}(V_{i(r)}) &= \sum_{l=1}^{n} \mathsf{t}\Big(\varepsilon_r g \mu_l\ \varepsilon_l f \mu_s\Big)\,\mathsf{d}^{-1}(V_{i(r)}) \\ &= \sum_{l=1}^{n} \mathsf{t}\Big((k_{r,l} \otimes 1_{V_{i(r)}})\ \varepsilon_l f \mu_s\Big)\,\mathsf{d}^{-1}(V_{i(r)}) \\ &= \sum_{l=1}^{n} \mathsf{t}\Big(k_{r,l}\Big)\,\mathsf{t}\Big(\varepsilon_l f \mu_s\Big)\,\mathsf{d}^{-1}(V_{i(r)}) \\ &= \sum_{l=1}^{n} \mathsf{t}\Big(\varepsilon_r f \mu_l\Big)\,\mathsf{d}^{-1}(V_{i(r)})\,\mathsf{t}\Big(\varepsilon_l g \mu_s\Big)\,\mathsf{d}^{-1}(V_{i(r)}) = \sum_{l=1}^{n} a^{g}_{r,l}\, a^{f}_{l,s} \end{aligned}$$

for some unique $k_{r,l}$ in K_C, since $\varepsilon_r g \mu_l$ is an endomorphism of a simple object $V_{i(r)}$. Thus, $M^{gf} = M^g M^f$.
2. Clear by K_C-linearity of the trace t.
3. First, note that X is dominated by $\{V_{i(r)}, \varepsilon_r, \mu_r\}_{r\in[n]}$, if and only if, X^* is dominated by $\{V^*_{i(r)}, \mu^*_r, \varepsilon^*_r\}_{r\in[n]}$ which also satisfies (3). On the other hand, we have $\mathsf{d}(V^*_{i(r)}) = \mathsf{d}(V_{i(r)})$ for every simple object $V_{i(r)}$ by Corollary 5.4(2). Then, the result holds by the following computations

$$\mathsf{t}_{X^*}\Big(\mu^*_r f^* \varepsilon^*_s\Big)\mathsf{d}^{-1}(V^*_{i(r)}) = \mathsf{t}_{X^*}\Big((\varepsilon_s f \mu_r)^*\Big)\mathsf{d}^{-1}(V_{i(r)}) = \mathsf{t}_X\Big(\varepsilon_s f \mu_r\Big)\mathsf{d}^{-1}(V_{i(s)}).$$

∎

5.10. COROLLARY. *Under the same hypotheses of Theorem 5.8,* det *satisfies:*

1. $\mathsf{det}_X(1_X) = 1_I$.
2. $\mathsf{det}_X(fg) = \mathsf{det}_X(gf) = \mathsf{det}_X(f)\,\mathsf{det}_X(g)$, *for every* $g \in \mathrm{End}_{\mathcal{C}}(X)$.
3. $\mathsf{det}_X(k \cdot f) = k^n\ \mathsf{det}_X(f)$, *for every* $k \in \mathsf{K}_{\mathcal{C}}$.
4. *If* $\mathcal{I}$ *is a* t*-traced ideal, then* $\mathsf{det}_{X^*}(f^*) = \mathsf{det}_X(f)$.

PROOF. 1. Since $\mathsf{t}_X(1_X)\,\mathsf{d}^{-1}(X) = 1_I$, then using Theorem 5.8, we obtain $\mathsf{det}_X(1_X) = det(M^{1_X}) = 1_I$.
2. By Lemma 5.9(1.), for every $g \in \mathrm{End}_{\mathcal{C}}(X)$, we have

$$\begin{aligned}\mathsf{det}_X(fg) = det(M^f M^g) &= det(M^f)\, det(M^g) \\ &= \mathsf{det}_X(f)\, \mathsf{det}_X(g) \\ &= \mathsf{det}_X(g)\, \mathsf{det}_X(f) = \mathsf{det}_X(gf)\end{aligned}$$

where, the fourth passage holds by commutativity of $\mathtt{K}_{\mathcal{C}}$.
3. This holds by Theorem 5.8 and Lemma 5.9(2).
4. Immediate from Theorem 5.8 and Lemma 5.9(4). ∎

5.11. DIAGRAM INVARIANTS FROM DETERMINANTS ON TRACED IDEALS. To every semisimple pivotal category $\mathcal{C}$, there corresponds a pivotal category $\mathcal{G}_{\mathcal{C}}$ whose objects are finite sequences of pairs (X, ε), where $X \in \mathcal{C}$ and $\varepsilon = \pm$. Morphisms of $\mathcal{G}_{\mathcal{C}}$ are isotopy classes of $\mathcal{C}$-colored ribbon graphs in $\mathbb{R} \times [0, 1]$. For the explicit structures of $\mathcal{G}_{\mathcal{C}}$, see for example [Geer et al. (2013)] and [Turaev and Virelizier (2017)].

By means of the graphical calculus language (string diagrams) there is associated a functor, called the Reshetikhin-Turaev functor ([Turaev (2016)]):

$$F : \mathcal{G}_{\mathcal{C}} \longrightarrow \mathcal{C}$$

sending any object $\big((X_1, \varepsilon_1), \ldots, (X_n, \varepsilon_n)\big)$ of $\mathcal{G}_{\mathcal{C}}$ to the object $X_1^{\varepsilon_1} \otimes \cdots \otimes X_n^{\varepsilon_n}$ such that for every $i \in [n]$:

$$X_i^{\varepsilon_i} = \begin{cases} X_i & \text{if } \varepsilon_i = + \\ X_i^* & \text{if } \varepsilon_i = - \end{cases} .$$

Following [Geer et al. (2013)], a $\mathcal{C}$-colored 1-1-ribbon graph T is an endomorphism of an object (X, ε), called the section of T. A left (resp., right) cutting presentation of a planar (a graph in $\mathbb{R}^2$) $\mathcal{C}$-colored ribbon graph Γ is a $\mathcal{C}$-colored 1-1-ribbon graph T, such that $\Gamma = Tr_l(F(T))$ (resp., $\Gamma = Tr_r(F(T))$). A $\mathcal{C}$-colored 1-1-ribbon graph T is called left (resp., right) $\mathcal{I}$-admissible provided that its section (X, ε) satisfies $X^\varepsilon \in \mathcal{I}$.

5.12. DEFINITION. *Let $\mathcal{I}$ be an admissible left (resp., right) $\mathtt{t}$-traced ideal. For any left (resp., right) $\mathcal{I}$-admissible left (resp., right) cutting presentation T of a planar $\mathcal{C}$-colored ribbon graph Γ, define*

$$F_{\mathsf{det}}(\Gamma) := \mathsf{det}_{X^\varepsilon}(F(T)).$$

5.13. THEOREM. *Let $\mathcal{I}$ be an admissible left (resp., right) $\mathtt{t}$-traced ideal. Then, F_{det} is an isotopy invariant of left (resp., right) $\mathcal{I}$-admissible $\mathcal{C}$-colored 1-1-ribbon graphs.*

Proof. Let T_1 and T_2 be two left (resp., right) $\mathcal{I}$-admissible left (resp., right) cutting presentations of a planar $\mathcal{C}$-colored ribbon graph Γ, with sections (X_1, ε_1) and (X_2, ε_2) respectively. We have to show that $\mathsf{det}_{V^\varepsilon}(F(T_1)) = \mathsf{det}_{V^\varepsilon}(F(T_2))$. It suffices then to show that

$$\mathsf{t}_{(X_1^{\varepsilon_1})^n}\Big(F(T_1)^{\otimes^n}\lambda^n_{X_1^{\varepsilon_1}}\Big) = \mathsf{t}_{(X_2^{\varepsilon_2})^n}\Big(F(T_2)^{\otimes^n}\lambda^n_{X_2^{\varepsilon_2}}\Big),$$

or again, by definition of the covariant functor F (preservation of tensor products) :

$$\mathsf{t}_{(X_1^{\varepsilon_1})^n}\Big(F(T_1^{\otimes^n})\lambda^n_{X_1^{\varepsilon_1}}\Big) = \mathsf{t}_{(X_2^{\varepsilon_2})^n}\Big(F(T_2^{\otimes^n})\lambda^n_{X_2^{\varepsilon_2}}\Big).$$

For every endomorphism $f \in \mathrm{End}_{\mathcal{C}}(X)$, where $X \in \mathcal{I}$, one can always consider any $\mathcal{C}$-colored ribbon graph Γ' with any left (resp., right) $\mathcal{I}$-admissible left (resp., right) cutting presentation T', whose color is f; namely, $F(T') = f$. Hence, for the endomorphisms $\lambda^n_{X_1^{\varepsilon_1}} \in \mathrm{End}_{\mathcal{C}}((X_1^{\varepsilon_1})^n)$ and $\lambda^n_{X_2^{\varepsilon_2}} \in \mathrm{End}_{\mathcal{C}}((X_2^{\varepsilon_2})^n)$, where $(X_1^{\varepsilon_1})^n, (X_2^{\varepsilon_2})^n \in \mathcal{I}$, one can fix a $\mathcal{C}$-colored ribbon graph Γ_0 with two left (resp., right) $\mathcal{I}$-admissible left (resp., right) cutting presentations T_1', T_2' whose colors are $\lambda^n_{X_1^{\varepsilon_1}}$, $\lambda^n_{X_2^{\varepsilon_2}}$, respectively; namely, $F(T_1') = \lambda^n_{X_1^{\varepsilon_1}}$ and $F(T_2') = \lambda^n_{X_2^{\varepsilon_2}}$. On the other hand, we have

$$\mathsf{t}_{(X_1^{\varepsilon_1})^n}\Big(F(T_1^{\otimes^n})\lambda^n_{X_1^{\varepsilon_1}}\Big) = \mathsf{t}_{(X_1^{\varepsilon_1})^n}\Big(F(T_1^{\otimes^n})F(T_1')\Big) = \mathsf{t}_{(X_1^{\varepsilon_1})^n}\Big(F(T_1^{\otimes^n}T_1')\Big).$$

Likewise, we obtain

$$\mathsf{t}_{(X_2^{\varepsilon_2})^n}\Big(F(T_2^{\otimes^n})\lambda^n_{X_2^{\varepsilon_2}}\Big) = \mathsf{t}_{(X_2^{\varepsilon_2})^n}\Big(F(T_2^{\otimes^n}T_2')\Big).$$

But obviously, $T_1^{\otimes^n}T_1'$ and $T_2^{\otimes^n}T_2'$ are two left (resp., right) $\mathcal{I}$-admissible left (resp., right) cutting presentations of the same $\mathcal{C}$-colored ribbon graph, with sections (X_1, ε_1) and (X_2, ε_2), respectively. By [Geer et al. (2013), Theorem 5], the trace t does not depend on the left (resp., right) $\mathcal{I}$-admissible left (resp., right) cutting presentations. It follows that $\mathsf{t}_{(X_1^{\varepsilon_1})^n}\Big(F(T_1^{\otimes^n}T_1')\Big) = \mathsf{t}_{(X_2^{\varepsilon_2})^n}\Big(F(T_2^{\otimes^n}T_2')\Big)$, and thus $\mathsf{det}_{X^\varepsilon}(F(T_1)) = \mathsf{det}_{X^\varepsilon}(F(T_2))$. ∎

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