Theory and Applications of Categories, Vol. 45, No. 32, 2026, pp. 1321–1356.

COMMUTATORS AND CROSSED MODULES OF COLOR HOPF ALGEBRAS

ANDREA SCIANDRA

Abstract. In a previous paper, we showed that the category of cocommutative color Hopf algebras is semi-abelian in case the group G is abelian and finitely generated and the characteristic of the base field is different from 2 (not needed if G is finite of odd cardinality). Here we describe the commutator of cocommutative color Hopf algebras and we explain the Hall's criterion for nilpotence and the Zassenhaus Lemma. Furthermore, we introduce the category of color Hopf crossed modules and we explicitly show that this is equivalent to the category of internal crossed modules in the category of cocommutative color Hopf algebras and to the category of simplicial cocommutative color Hopf algebras with Moore complex of length 1.

1. Introduction

The notion of semi-abelian category was introduced in [29] in order to give a categorical "generalization" of the category of groups, as abelian categories "generalize" abelian groups. Among the classical examples of semi-abelian categories there are the categories of groups, Lie algebras and associative rings.

In [39, Theorem 6.1] we showed that the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ of cocommutative color Hopf algebras is semi-abelian if the abelian group G is finitely generated and the characteristic of the base field $\Bbbk$ is different from 2 (not needed if G is finite of odd cardinality). This result generalizes [24, Theorem 2.10] given for the category of ordinary cocommutative Hopf algebras over a field $\Bbbk$ of arbitrary characteristic, which can be seen as a special case of our result by considering G equal to the trivial group. Moreover, taking $G = \mathbb{Z}_2$ and $\mathrm{char}\Bbbk \neq 2$, we obtain that the category of cocommutative super Hopf algebras, extensively used in Mathematics and Physics, is semi-abelian.

Semi-abelian categories provide a good categorical framework to develop an approach to commutator theory, and they present natural notions of semi-direct product [13], internal action [7] and crossed module [28]. Moreover, some famous theorems for groups have categorical generalizations in a semi-abelian category, like the Hall's criterion for nilpotence [25] and the Zassenhaus Lemma [38]. Semi-abelian categories are also suitable for studying the (co)homology of non-abelian structures [21] and the Moore complex

---

Received by the editors 2023-12-01 and, in final form, 2026-05-13.

Transmitted by Giuseppe Metere. Published on 2026-05-25.

2020 Mathematics Subject Classification: Primary 18E13, 16T05; Secondary 18D40, 18G45, 18N50, 16S40.

Key words and phrases: semi-abelian categories, color Hopf algebras, commutators, semi-direct products, crossed modules, simplicial Hopf algebras, Zassenhaus Lemma, Hall's criterion.

structure [10, 22]. Therefore, we will study some of these properties for the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, under the previous assumptions on G and $\Bbbk$.

The structure of the paper is the following. First, for the convenience of the reader, we include preliminaries about the categorical-algebraic properties which we will explore for the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. In Section 3 we give an explicit description of commutators in the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, using the action ξ defined in [39]. Then we explain the Hall's criterion for nilpotence and the Zassenhaus Lemma for the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ in Section 4. In Section 5 we revise the equivalence between internal actions and split extensions in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and, by introducing the notion of color Hopf crossed module, we show the equivalence between the categories of color Hopf crossed modules and internal crossed modules in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. Finally, in Section 6, we study the equivalence between the categories of color Hopf crossed modules and simplicial cocommutative color Hopf algebras with Moore complex of length one. Observe that generalizations of the equivalences in Sections 5 and 6 can be found in the very general results given in [4] and [5], but here we give an explicit description of these in reference to the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ which can also be used to investigate further other topics like crossed squares and 2-crossed modules of cocommutative color Hopf algebras.

Notations and conventions. Sometimes, identity morphisms in a category $\mathcal{C}$ are denoted by 1. If the category $\mathcal{C}$ is pointed, the zero object is denoted by $\mathbf{0}$ and the zero morphism by 0. Given a morphism $f : A \longrightarrow B$ in $\mathcal{C}$, the kernel and the cokernel of f, if they exist, are denoted by $\ker(f) : \mathrm{Ker}(f) \longrightarrow A$ and $\mathrm{coker}(f) : B \longrightarrow \mathrm{Coker}(f)$ or $\mathrm{coker}(f) : B \longrightarrow B/A$. All vector spaces are understood to be over the field $\Bbbk$ and by linear maps we mean $\Bbbk$-linear maps. The unadorned tensor product $\otimes$ stands for $\otimes_{\Bbbk}$. All linear maps whose domain is a tensor product will usually be defined on generators and understood to be extended by linearity. Algebras over $\Bbbk$ will be associative and unital.

2. Preliminaries

In this section, we recall some categorical-algebraic properties, which we will explore for the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. A category $\mathcal{C}$ is *semi-abelian* if it is pointed, finitely cocomplete, (Barr)-exact and protomodular [29]. Thus, $\mathcal{C}$ has zero object, all finite colimits and limits, so that the protomodularity is equivalent to the fact that the Split Short Five Lemma holds in $\mathcal{C}$ (see [6, Proposition 3.1.2]). Moreover, every morphism in $\mathcal{C}$ factorizes as a regular epimorphism (i.e., a coequalizer of a pair of morphisms in $\mathcal{C}$) followed by a monomorphism and this factorization is stable under pullbacks (i.e., $\mathcal{C}$ is regular) and any equivalence relation is the kernel pair of a morphism in $\mathcal{C}$. We refer the reader to [6] for further details about semi-abelian categories.

The Huq commutator. In any pointed category $\mathcal{C}$ with binary products, one says that two subobjects $i : X \longrightarrow A$ and $j : Y \longrightarrow A$ of the same object A *commute* (in the sense of Huq [27]) if there exists an arrow p that makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "{\langle 1,0\rangle}"] \arrow[dr, "i"'] & X \times Y \arrow[d, dashed, "p"] & Y \arrow[l, "{\langle 0,1\rangle}"'] \arrow[dl, "j"] \\
& A &
\end{tikzcd} \tag{1}$$

If the category $\mathcal{C}$ is *unital* [8], such arrow p is unique. This holds in particular when $\mathcal{C}$ is protomodular. In a protomodular category $\mathcal{C}$ with finite limits, a subobject $u : S \longrightarrow A$ of A is called *normal* if it is normal to an equivalence relation on A (see [6, Definitions 3.2.1 and 3.2.9]) and, if $\mathcal{C}$ is pointed, exact and protomodular, then u is normal if and only if it is the kernel of a morphism in $\mathcal{C}$, as shown in [6, Proposition 3.2.20]. In particular, the latter holds for a semi-abelian category. In a semi-abelian category $\mathcal{C}$, the *Huq commutator* of two normal subobjects $i : X \longrightarrow A$ and $j : Y \longrightarrow A$ is the smallest normal subobject $g : B \longrightarrow A$ of A such that its cokernel $q := \mathrm{coker}(g) : A \longrightarrow A/B$ has the property that the images $q(X)$ and $q(Y)$ by q commute in the quotient, as shown in the following diagram:

$$\begin{tikzcd}
& & & & q(X) \arrow[r] \arrow[ddr] & q(X) \times q(Y) \arrow[dd, dashed] & q(Y) \arrow[l] \arrow[ddl] \\
& X \arrow[dr, "i"] & & Y \arrow[dl, "j"'] & & & \\
B \arrow[rr, "g"'] & & A \arrow[rrr, "q"'] & & & A/B &
\end{tikzcd}$$

Usually, it is denoted by $[X, Y]_{\mathrm{Huq}} \longrightarrow A$. Recall also that an object A in $\mathcal{C}$ is called *nilpotent* [27] if there exists a non-negative integer n such that $\gamma_A^n(A) = 0$ where, denoted by $\mathrm{Norm}(A)$ the class of normal subobjects of A, the map $\gamma_A : \mathrm{Norm}(A) \longrightarrow \mathrm{Norm}(A)$ sends a normal subobject X of A to $[A, X]_{\mathrm{Huq}}$. The least of such n is the nilpotency class of A.

If a semi-abelian category $\mathcal{C}$ is also algebraically coherent [18] then a generalization of the Hall's criterion for nilpotence holds:

2.1. THEOREM. [cf. [25, Theorem 3.4]] *Let $\mathcal{C}$ be an algebraically coherent semi-abelian category and $p : E \longrightarrow B$ be a regular epimorphism where B is a nilpotent object in $\mathcal{C}$. If the kernel of p is contained in the Huq commutator $[N, N]_{\mathrm{Huq}}$ of a nilpotent normal subobject N of E, then E is nilpotent. Furthermore, if N is of nilpotency class c and B is of nilpotency class d, then E is of nilpotency class at most $\frac{c(c+1)}{2}(d-1)+c$.*

Semi-direct products. Recall that, given a morphism $v : W \longrightarrow Y$ in a semi-abelian category $\mathcal{C}$, the inverse image functor $v^* : \mathrm{Pt}_Y(\mathcal{C}) \longrightarrow \mathrm{Pt}_W(\mathcal{C})$ of the fibration of points is monadic, see [13] and [6, Theorem 5.1.13]. From [13] and [6, Definition 5.2.8] also recall that, given a semi-abelian category $\mathcal{C}$ and an object G in $\mathcal{C}$, a *G-algebra* is an algebra for the monad $\mathbf{T}_G$ corresponding to the monadic functor $\alpha_G^* : \mathrm{Pt}_G(\mathcal{C}) \longrightarrow \mathcal{C}$, where $\alpha_G : \mathbf{0} \longrightarrow G$

is the unique arrow from the zero object $\mathbf{0}$ of $\mathcal{C}$ to G. Given a G-algebra (X,ξ), the *semi-direct product* $(X,\xi)\rtimes G$ of (X,ξ) and G is the domain part H of the point $H \underset{s}{\overset{p}{\rightleftarrows}} G$ corresponding to (X,ξ) through the equivalence $\mathrm{Pt}_G(\mathcal{C})\cong \mathcal{C}^{\mathbf{T}_G}$.

Reflexive multiplicative graphs and internal groupoids. Recall from [15] that a *reflexive-multiplicative graph* in a category with pullbacks $\mathcal{C}$ is a diagram

$$\begin{tikzcd} A_1\times_{A_0}A_1 \arrow[r, "m"] & A_1 \arrow[r, shift left=2, "\delta"] \arrow[r, shift right=2, "\gamma"'] & A_0 \arrow[l, "i" description] \end{tikzcd} \tag{2}$$

where $(A_1\times_{A_0}A_1,\pi_1,\pi_2)$ is the pullback of the pair (δ,γ) in $\mathcal{C}$, such that $\delta\circ i=\gamma\circ i=\mathrm{Id}_{A_0}$ (i.e., it is a reflexive graph) and $m\circ(\mathrm{Id}_{A_1}, i\circ\delta)=\mathrm{Id}_{A_1}=m\circ(i\circ\gamma,\mathrm{Id}_{A_1})$, where $(\mathrm{Id}_{A_1}, i\circ\delta): A_1\longrightarrow A_1\times_{A_0}A_1$ and $(i\circ\gamma,\mathrm{Id}_{A_1}): A_1\longrightarrow A_1\times_{A_0}A_1$ are induced by the universal property of the pullback $A_1\times_{A_0}A_1$. Usually δ is called the *domain morphism* or *source morphism*, γ the *codomain morphism* or *target morphism*, i the *identity morphism* and m the *multiplication*.

A morphism of reflexive-multiplicative graphs is given by a pair of morphisms $(f_1: A_1'\longrightarrow A_1, f_0: A_0'\longrightarrow A_0)$ in $\mathcal{C}$ such that the four squares of corresponding arrows of the following diagram commute:

$$\begin{tikzcd} A_1'\times_{A_0'}A_1' \arrow[r, "m'"] \arrow[d, "f_1\times f_1"'] & A_1' \arrow[r, shift left=2, "\delta'"] \arrow[r, shift right=2, "\gamma'"'] \arrow[d, "f_1"] & A_0' \arrow[l, "i'" description] \arrow[d, "f_0"] \\ A_1\times_{A_0}A_1 \arrow[r, "m"'] & A_1 \arrow[r, shift left=2, "\delta"] \arrow[r, shift right=2, "\gamma"'] & A_0 \arrow[l, "i" description] \end{tikzcd}$$

A reflexive-multiplicative graph is called an *internal category* if the multiplication m further satisfies

$$\delta\circ m=\delta\circ\pi_2,\quad \gamma\circ m=\gamma\circ\pi_1 \ \text{ and } \ m\circ(\mathrm{Id}_{A_1}\times_{A_0}m)=m\circ(m\times_{A_0}\mathrm{Id}_{A_1}).$$

Moreover, an internal category is called *internal groupoid* if there exists a morphism $\iota: A_1\longrightarrow A_1$ such that

$$\delta\circ\iota=\gamma,\quad \gamma\circ\iota=\delta,\quad m\circ(\iota,\mathrm{Id}_{A_1})=i\circ\delta \ \text{ and } \ m\circ(\mathrm{Id}_{A_1},\iota)=i\circ\gamma.$$

We denote by $\mathrm{RMG}(\mathcal{C})$ the category of reflexive-multiplicative graphs in $\mathcal{C}$ with morphisms of reflexive-multiplicative graphs and by $\mathrm{Grpd}(\mathcal{C})$ the category of internal groupoids in $\mathcal{C}$, with morphisms of reflexive-multiplicative graphs. It is shown in [15] that if $\mathcal{C}$ is Mal'tsev then the forgetful functor $F:\mathrm{Grpd}(\mathcal{C})\longrightarrow\mathrm{RMG}(\mathcal{C})$ is an isomorphism of categories and the groupoid structure on a reflexive graph is unique whenever it exists. In [28] G. Janelidze introduced the category $\mathrm{XMod}(\mathcal{C})$ of internal crossed modules in a semi-abelian category $\mathcal{C}$, which is equivalent to the category $\mathrm{Grpd}(\mathcal{C})$.

For any two equivalence relations over the same object of a category $\mathcal{C}$, there is a notion of connector [11] and, when such a connector exists, one says that the two equivalence relations *centralize* (in the sense of Smith). If two equivalence relations centralize in the sense of Smith, then their associated normal subobjects commute in the sense of Huq, while the converse is not true in general (see [11]). If a category $\mathcal{C}$ is such that two equivalence relations centralize in the sense of Smith if and only if their normalizations commute in the sense of Huq, then $\mathcal{C}$ is said to satisfy the so-called condition (SH) [11]. Note that any action representable category satisfies the condition (SH), since this is more generally true for any action accessible category [14]. Moreover, a reflexive graph in a Mal'tsev category $\mathcal{C}$

$$\begin{tikzcd} A_1 \arrow[r, shift left=2, "\delta"] \arrow[r, shift right=2, "\gamma"'] & A_0 \arrow[l, "i" description] \end{tikzcd} \tag{3}$$

has an internal structure of groupoid if and only if $\mathrm{Eq}(\delta)$ and $\mathrm{Eq}(\gamma)$, the kernel pairs of δ and γ, centralize with each other (in the sense of Smith) [15]. The results mentioned above are summarized in the following proposition:

2.2. PROPOSITION. *In a pointed, Mal'tsev category $\mathcal{C}$ which satisfies the condition (SH), the following statements are equivalent for the reflexive graph* (3)*:*

1. *it is a reflexive-multiplicative graph;*
2. *it is an internal category;*
3. *it is an internal groupoid;*
4. $\mathrm{Eq}(\delta)$ *and* $\mathrm{Eq}(\gamma)$ *centralize to each other (in the sense of Smith);*
5. *it satisfies the equality* $[\mathrm{Ker}(\delta), \mathrm{Ker}(\gamma)]_{\mathrm{Huq}} = \mathbf{0}$.

As mentioned above, the equivalence of conditions (1)–(4) was proved in [15]. We point out that their equivalence to (5) was introduced as a condition in [34] and then proven to be equivalent to (SH) for a semi-abelian category in [35].

The Zassenhaus Lemma. Recall from [31] that a category is *normal* if it is pointed regular and every regular epimorphism is a normal epimorphism. A normal category is *ideal determined* if it has binary coproducts and the normal image of a normal monomorphism is again a normal monomorphism [30]. In particular, any semi-abelian category is normal and ideal determined (see [29]). In [42, Proposition 5.2], O. Wyler stated the Zassenhaus Lemma in an axiomatic context that includes ideal determined categories, using the notion of *asymmetric join*: given a kernel $k : K \longrightarrow A$ in $\mathcal{C}$ with cokernel $f : A \longrightarrow A/K$ and a monomorphism $m : M \longrightarrow A$, the asymmetric join is the subobject $k \vee_A m : K \vee_A M \longrightarrow A$ of A such that $(K \vee_A M, k \vee_A m, \pi_2)$ is the pullback of the pair $(f, f \circ m)$. In [38, Lemma 2.8] it is shown that, if $\mathcal{C}$ is semi-abelian, the asymmetric join $K \vee_A M$ coincides with the supremum of K and M (as subobjects of A), i.e., the smallest subobject of A containing K and M, which always exists in a category with binary coproducts and a factorization of every morphism as a strong epimorphism followed by a monomorphsim. Thus, from [38, Theorem 3.3] and [38, Corollary 3.5], we recall the Zassenhaus Lemma for a semi-abelian category:

2.3. Theorem. *Let $\mathcal{C}$ be a semi-abelian category, $K \longrightarrow U$ and $L \longrightarrow V$ be two kernels in $\mathcal{C}$, and $U \longrightarrow A$ and $V \longrightarrow A$ be two monomorphisms in $\mathcal{C}$. Then we have isomorphisms*

$$\frac{K \vee_U (U \cap V)}{K \vee_U (U \cap L)} \cong \frac{U \cap V}{(K \cap V) \vee_{U \cap V} (L \cap U)} \cong \frac{L \vee_V (U \cap V)}{L \vee_V (K \cap V)}.$$

Simplicial objects and Moore complexes. First, recall that, given a pointed category $\mathcal{C}$, a *chain complex* $(X_\bullet, \partial_\bullet)$ in $\mathcal{C}$ is given by a collection of objects $(X_n)_{n\in\mathbb{Z}}$ in $\mathcal{C}$ and a collection of morphisms $(\partial_n : X_n \longrightarrow X_{n-1})_{n\in\mathbb{Z}}$ in $\mathcal{C}$, called differentials, such that $\partial_n \circ \partial_{n+1} = 0$, for all $n \in \mathbb{Z}$. A morphism in a pointed, regular, and protomodular category $\mathcal{C}$ is called *proper* if its image is a kernel [9]. A chain complex is proper whenever all its differentials are so [22]. We denote the category of chain complexes in $\mathcal{C}$ by $\mathrm{Ch}(\mathcal{C})$.

The *simplicial category* Δ has objects given by finite ordinals $[n] = \{0, \dots, n\}$, with $n \in \mathbb{N}$, and morphisms $[n] \longrightarrow [m]$ given by order preserving maps from $\{0, \dots, n\}$ to $\{0, \dots, m\}$. Given a category $\mathcal{C}$, the category $\mathrm{Simp}(\mathcal{C})$ of simplicial objects and simplicial morphisms of $\mathcal{C}$ is the functor category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ [36]. Thus, a *simplicial object* $X : \Delta^{\mathrm{op}} \longrightarrow \mathcal{C}$ in a category $\mathcal{C}$ is given by the following data: a collection of objects $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$, morphisms in $\mathcal{C}$

$$d_i^n : X_n \longrightarrow X_{n-1}, \text{ for } 0 \leq i \leq n \quad \text{and} \quad s_j^{n+1} : X_n \longrightarrow X_{n+1}, \text{ for } 0 \leq j \leq n$$

called the *face operators* and the *degeneracy operators*, respectively, subject to the following *simplicial identities*:

1) $d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n$ if $i < j$;

2) $s_i^{n+1} \circ s_j^n = s_{j+1}^{n+1} \circ s_i^n$ if $i \leq j$;

3) $d_i^n \circ s_j^n = s_{j-1}^{n-1} \circ d_i^{n-1}$ if $i < j$, $d_j^n \circ s_j^n = d_{j+1}^n \circ s_j^n = \mathrm{Id}$, $d_i^n \circ s_j^n = s_j^{n-1} \circ d_{i-1}^{n-1}$ if $i > j+1$

for all $n \in \mathbb{N}$.

Given $n \in \mathbb{N}$, a *n-truncated simplicial object* in a category $\mathcal{C}$ is a functor $X : \Delta_n^{\mathrm{op}} \longrightarrow \mathcal{C}$, where Δ_n is the full subcategory of Δ whose objects are natural numbers $\leq n$. We denote the category of n-truncated simplicial objects in $\mathcal{C}$ by $\mathrm{Simp}_n(\mathcal{C})$. For each $n \in \mathbb{N}$ there is a *truncation functor* $\mathrm{tr}_n : \mathrm{Simp}(\mathcal{C}) \longrightarrow \mathrm{Simp}_n(\mathcal{C})$ which simply forgets the objects X_i of a simplicial object X for dimensions higher than n and the corresponding face and degeneracy operators. If $\mathcal{C}$ has finite limits, then tr_n admits a right adjoint cosk_n called the *n-coskeleton functor*, while if $\mathcal{C}$ has finite colimits, then tr_n admits a left adjoint sk_n, called the *n-skeleton functor*, see [19].

Let us quickly recall the construction of the n-coskeleton functor for a finitely complete category $\mathcal{C}$. Given a n-truncated simplicial object X, consider the face operators $d_0^n, \dots, d_n^n : X_n \longrightarrow X_{n-1}$, then X_{n+1} is defined as an object in $\mathcal{C}$ with morphisms $f_0, \dots, f_{n+1} : X_{n+1} \longrightarrow X_n$ such that $d_i^n \circ f_j = d_{j-1}^n \circ f_i$ for all $i < j$ which is universal with this property: given morphisms $\pi_0, \dots, \pi_{n+1} : A \longrightarrow X_n$ in $\mathcal{C}$ such that $d_i^n \circ \pi_j = d_{j-1}^n \circ \pi_i$ for all $i < j$ then there is a unique morphism $\alpha : A \longrightarrow X_{n+1}$ such that $f_i \circ \alpha = \pi_i$:

$$\begin{tikzcd}
A \arrow[r, shift left=2, "\pi_{n+1}"] \arrow[r, phantom, "\cdots"] \arrow[r, shift right=2, "\pi_0"'] \arrow[d, "\alpha"'] & X_n \arrow[d, "\mathrm{Id}"] & & \\
X_{n+1} \arrow[r, shift left=2, "f_{n+1}"] \arrow[r, phantom, "\cdots"] \arrow[r, shift right=2, "f_0"'] & X_n \arrow[r, shift left=2, "d_n^n"] \arrow[r, phantom, "\cdots"] \arrow[r, shift right=2, "d_0^n"'] & X_{n-1} & \cdots\cdots
\end{tikzcd}$$

Furthermore, the universal property of X_{n+1} allows one to define degeneracy operators $s_i^{n+1} : X_n \longrightarrow X_{n+1}$ for $0 \leq i \leq n$. This yields an $(n+1)$-truncated simplicial objects in $\mathcal{C}$ and, proceeding inductively, one defines the n-coskeleton functor.

Given a simplicial object X in a pointed category $\mathcal{C}$ with pullbacks, the *Moore chain complex* $(M(X)_\bullet, \partial_\bullet)$ is the chain complex defined by:

1. $M(X)_n = \mathbf{0}$ for $n < 0$ and $M(X)_0 = X_0$;
2. $M(X)_n = \bigcap_{i=0}^{n-1} \mathrm{Ker}(d_i^n)$ for $n \geq 1$;
3. $\partial_n = d_n^n \circ \cap_{i=0}^{n-1} \mathrm{ker}(d_i^n) : M(X)_n \longrightarrow M(X)_{n-1}$ for $n \geq 1$ (and the zero morphism for $n \leq 0$),

see [22, Definition 3.1]. Hence there is a functor $M : \mathrm{Simp}(\mathcal{C}) \longrightarrow \mathrm{Ch}(\mathcal{C})$ and, if $\mathcal{C}$ is pointed and protomodular, then M is conservative, i.e., it reflects isomorphisms [10]. In [22, Theorem 3.6] it is shown that, given a pointed, exact and protomodular category $\mathcal{C}$ and a simplicial object X in $\mathcal{C}$, then $(M(X)_\bullet, \partial_\bullet)$ is a proper chain complex of $\mathcal{C}$. Moreover, we recall the following result:

2.4. THEOREM. [cf. [32, Theorem 5.3]] *Let $\mathcal{C}$ be a pointed category with finite limits. For a simplicial object X with corresponding Moore complex $M(X)$, then the Moore complex of* $\mathrm{cosk}_n\mathrm{tr}_n(X)$ *satisfies:*

1. $M(\mathrm{cosk}_n\mathrm{tr}_n(X))_i = M(X)_i$ *for* $i \leq n$*;*
2. $M(\mathrm{cosk}_n\mathrm{tr}_n(X))_{n+1} = \mathrm{Ker}(\partial_n : M(X)_n \longrightarrow M(X)_{n-1})$*;*
3. $M(\mathrm{cosk}_n\mathrm{tr}_n(X))_i = \mathbf{0}$ *for* $i > n+1$*.*

As a consequence, given a n-truncated simplicial object $X : \Delta_n^{\mathrm{op}} \longrightarrow \mathcal{C}$ in a pointed category with finite limits $\mathcal{C}$ and the simplicial object $\mathrm{cosk}_n(X)$, since $\mathrm{tr}_n\mathrm{cosk}_n(X) = X$ and then $\mathrm{cosk}_n\mathrm{tr}_n(\mathrm{cosk}_n(X)) = \mathrm{cosk}_n(X)$, by applying the previous theorem one obtains the following result:

2.5. Corollary. *Let $\mathcal{C}$ be a pointed category with finite limits and X an object in* $\mathrm{Simp}_n(\mathcal{C})$*. Then*

$$M(\mathrm{cosk}_n(X))_i = \mathbf{0} \textit{ if } i > n+1,\ M(\mathrm{cosk}_n(X))_{n+1} = \mathrm{Ker}(\partial_n : M(\mathrm{cosk}_n(X))_n \longrightarrow M(\mathrm{cosk}_n(X))_{n-1}).$$

The category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. We denote by $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ the category of cocommutative color Hopf algebras, i.e., cocommutative Hopf monoids in the category of G-graded vector spaces Vec_G, where G is an abelian group (see e.g. [1, Definition 1.15] for the definition of Hopf monoid in a braided monoidal category). We will use the same notations employed in [39] for the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. We refer the reader to [1] for many details about monoidal categories and to [41] for basic results in Hopf algebra theory.

We recall that, given a bicharacter on G, i.e., a map $\phi : G \times G \longrightarrow \Bbbk - \{0\}$ which satisfies

$$\phi(gh, l) = \phi(g, l)\phi(h, l) \text{ and } \phi(g, hl) = \phi(g, h)\phi(g, l) \text{ for every } g, h, l \in G,$$

one obtains a braiding c for the category Vec_G which is defined in the following way. For any $X = \bigoplus_{g\in G} X_g$ and $Y = \bigoplus_{g\in G} Y_g$ in Vec_G, the morphism $c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$ is defined on the components of the grading and extended by linearity: given $g, h \in G$ and $x \in X_g$, $y \in Y_h$, we have $c_{X,Y}(x \otimes y) := \phi(g, h)y \otimes x$. The braided monoidal category Vec_G is symmetric if and only if ϕ satisfies further $\phi(g, h)\phi(h, g) = 1_\Bbbk$, for all $g, h \in G$ (ϕ is said to be a skew-symmetric bicharacter). In the following, we fix a skew-symmetric bicharacter ϕ on G, hence a symmetric monoidal structure on Vec_G.

We recall that the objects of the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ are tuple $(H, m, u, \Delta, \epsilon, S)$, where:

i) (H, m, u) is a G-graded algebra, i.e., it is an object in $\mathrm{Mon}(\mathrm{Vec}_G)$;

ii) (H, Δ, ϵ) is a G-graded coalgebra which is cocommutative (so $a_1 \otimes a_2 = \phi(|a_1|, |a_2|)a_2 \otimes a_1$ for all $a \in H$), i.e., it is an object in $\mathrm{Comon}_{\mathrm{coc}}(\mathrm{Vec}_G)$;

iii) $(H, m, u, \Delta, \epsilon)$ is an object in $\mathrm{Bimon}_{\mathrm{coc}}(\mathrm{Vec}_G)$, so Δ and ϵ are morphisms in $\mathrm{Mon}(\mathrm{Vec}_G)$, i.e., the following compatibility conditions hold, for all $a, b \in H$:

$$\Delta(ab) = \phi(|a_2|, |b_1|)a_1b_1 \otimes a_2b_2, \quad \Delta(1_H) = 1_H \otimes 1_H, \quad \epsilon(ab) = \epsilon(a)\epsilon(b), \quad \epsilon(1_H) = 1_\Bbbk;$$

iv) $S : H \longrightarrow H$ is a morphism in Vec_G such that $a_1S(a_2) = \epsilon(a)1 = S(a_1)a_2$ for all $a \in H$, called antipode.

The morphisms in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ are simply algebra maps which are also coalgebra maps preserving gradings.

We recall that the cocommutativity assumption implies $\Delta(S(a)) = S(a_1) \otimes S(a_2)$ and $S^2(a) = a$ for all $a \in H$, while $S(ab) = \phi(|a|, |b|)S(b)S(a)$ for all $a, b \in H$. We also recall that the following commutativity condition $\phi(|a|, |b|)\epsilon(b)a = a\epsilon(b)$ holds for all $a, b \in H$ since this will be used frequently throughout the paper.

In [39, Theorem 6.1] we showed that the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is semi-abelian if the abelian group G is finitely generated and $\mathrm{char}\Bbbk \neq 2$ (not needed if G is finite of odd cardinality). Note that, in case $G = \{1\}$ is the trivial group, one recovers the semi-abelianness of the category $\mathrm{Hopf}_{\Bbbk,\mathrm{coc}}$ of cocommutative Hopf algebras [24, Theorem 2.10]. From now on we suppose that G is a finitely generated abelian group and $\mathrm{char}\Bbbk \neq 2$ and we study the categorical properties discussed above for $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$.

3. Huq commutators in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$

In this section, we give an explicit description of commutators in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$.

3.1. Remark. In $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ monomorphisms are exactly the injective morphisms as it is shown in [39, Lemma 5.22] and, since every injective map f in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ can be decomposed as an isomorphism followed by an inclusion, a subobject of an object A in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is an inclusion $i : X \longrightarrow A$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, i.e., X is a color Hopf subalgebra of A. Given X and Y color Hopf subalgebras of A, recall that $X \times Y = X \otimes Y$ is the binary product of X and Y in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and $\langle \mathrm{Id}_X, 0_{X,Y}\rangle = (\mathrm{Id}_X \otimes u_Y\epsilon_X)\circ\Delta_X : X \longrightarrow X \otimes Y$, $x \mapsto x \otimes 1_Y$, $\langle 0_{Y,X}, \mathrm{Id}_Y\rangle = (u_X\epsilon_Y \otimes \mathrm{Id}_Y)\circ\Delta_Y : Y \longrightarrow X \otimes Y$, $y \mapsto 1_X \otimes y$.

Given $i : X \longrightarrow A$ and $j : Y \longrightarrow A$ two color Hopf subalgebras of A in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, these commute in the sense of Huq if there exists a morphism $p : X \otimes Y \longrightarrow A$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ such that $p(x \otimes 1_Y) = x$ and $p(1_X \otimes y) = y$ for every $x \in X$ and $y \in Y$. But then p is uniquely determined if it exists, since it has to be a morphism of algebras and so

$$p(x \otimes y) = p((x \otimes 1_Y)(1_X \otimes y)) = p(x \otimes 1_Y)p(1_X \otimes y) = xy \text{ for all } x \in X,\ y \in Y.$$

Note that the uniqueness of p, in case of existence, was already known since $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is protomodular, hence unital. Hence we must have $p = m_A \circ (i \otimes j)$ which is clearly a morphism of coalgebras, since this is true for m_A with A color Hopf algebra. Thus, we only need that p is a morphism of algebras.

3.2. Lemma. *Given two color Hopf subalgebras* $i : X \longrightarrow A$ *and* $j : Y \longrightarrow A$ *of* A *in* $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, *the following conditions are equivalent:*

1) *there exists a unique morphism of color Hopf algebras* $p : X \otimes Y \longrightarrow A$ *that makes* i *and* j *commute in the sense of Huq;*
2) $xy = \phi(|x|, |y|)yx$, *for all* $x \in X$ *and* $y \in Y$;
3) $\phi(|x_2|, |y_1|)x_1y_1S(x_2)S(y_2) = \epsilon(x)\epsilon(y)1_A$, *for all* $x \in X$ *and* $y \in Y$;
4) $\phi(|x_2|, |y|)x_1yS(x_2) = \epsilon(x)y$, *for all* $x \in X$ *and* $y \in Y$.

Proof. If 1) holds, then p is a morphism of algebras and so, since Vec_G is a symmetric monoidal category, we obtain

$$\begin{aligned} xy &= p(\phi(|x|,|y|)\phi(|y|,|x|)x\otimes y) = \phi(|x|,|y|)p((1_X\otimes y)(x\otimes 1_Y)) \\ &= \phi(|x|,|y|)p(1_X\otimes y)p(x\otimes 1_Y) = \phi(|x|,|y|)yx \end{aligned}$$

for all $x\in X$ and $y\in Y$ and then 2) is satisfied, while if 2) holds then

$$pm_{X\otimes Y}(x\otimes y\otimes x'\otimes y') = \phi(|y|,|x'|)xx'yy' \overset{2)}{=} xyx'y' = m_A(p\otimes p)(x\otimes y\otimes x'\otimes y')$$

for every $x,x'\in X$ and $y,y'\in Y$ and then 1) is satisfied, since p is also a morphism of coalgebras. Thus, 1) and 2) are equivalent conditions and now we show the equivalence between 2) and 3). If 2) is satisfied then we immediately obtain 3) as

$$\phi(|x_2|,|y_1|)x_1y_1S(x_2)S(y_2) \overset{2)}{=} x_1S(x_2)y_1S(y_2) = \epsilon(x)\epsilon(y)1_A,$$

while if 3) holds then

$$\begin{aligned} xy &= x_1\epsilon(x_2)y = \phi(|x_2|,|y|)x_1y\epsilon(x_2) = \phi(|x_2\otimes x_3|,|y|)x_1y_1\epsilon(y_2)S(x_2)x_3 \\ &= \phi(|x_2|,|y_1|)\phi(|x_3|,|y|)x_1y_1S(x_2)\epsilon(y_2)x_3 = \phi(|x_2|,|y_1|)\phi(|x_3|,|y|)x_1y_1S(x_2)S(y_2)y_3x_3 \\ &\overset{3)}{=} \phi(|x_2|,|y|)\epsilon(x_1)\epsilon(y_1)y_2x_2 = \phi(|x_2|,|y|)\epsilon(x_1)yx_2 = \phi(|x_1|,|y|)\phi(|x_2|,|y|)y\epsilon(x_1)x_2 \\ &= \phi(|x|,|y|)yx \end{aligned}$$

so 2) is satisfied. Finally, we show that 3) and 4) are equivalent. Clearly, if 4) holds true then

$$\phi(|x_2|,|y_1|)x_1y_1S(x_2)S(y_2) \overset{4)}{=} \epsilon(x)y_1S(y_2) = \epsilon(x)\epsilon(y)1_A,$$

i.e., 3) is satisfied, while, by assuming 3), we can compute

$$\begin{aligned} \phi(|x_2|,|y|)x_1yS(x_2) &= \phi(|x_2|,|y_1\otimes y_2|)x_1y_1\epsilon(y_2)S(x_2) = \phi(|x_2|,|y_1|)x_1y_1S(x_2)\epsilon(y_2) \\ &= \phi(|x_2|,|y_1|)x_1y_1S(x_2)S(y_2)y_3 \overset{3)}{=} \epsilon(x)\epsilon(y_1)y_2 = \epsilon(x)y \end{aligned}$$

and then 4) holds. ■

Since the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is semi-abelian, a normal subobject $i : X \longrightarrow A$ of A in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is an inclusion which is a kernel of a morphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and this is equivalent, by [39, Corollary 5.21], to X being a *normal* color Hopf subalgebra of A, i.e., such that $\xi_A(a\otimes x) := \phi(|a_2|,|x|)a_1xS(a_2)\in X$ for every $a\in A$ and $x\in X$. Recall from [39, Lemma 5.4 1) and 2)] that $\xi_A : A\otimes A \longrightarrow A$ is a morphism of graded coalgebras and, given $f : A\longrightarrow B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, then $\xi_B\circ(f\otimes f) = f\circ\xi_A$. First, we can show the following result:

3.3. Lemma. *Given A in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, then*

$$\xi_A(a\otimes\xi_A(b\otimes c)) = \xi_A(ab\otimes c) \text{ and } \xi_A(1_A\otimes a) = a \text{ for all } a,b,c\in A, \tag{4}$$

i.e., (A,ξ_A) is in ${}_A\mathrm{Vec}_G$.

PROOF. Clearly ξ_A preserves the gradings. Given $a,b,c\in A$, we can compute

$$\begin{aligned}&\xi_A(a\otimes\xi_A(b\otimes c))\\&=\phi(|b_2|,|c|)\xi_A(a\otimes b_1cS(b_2))=\phi(|b_2|,|c|)\phi(|a_2|,|b_1cS(b_2)|)a_1b_1cS(b_2)S(a_2)\\&=\phi(|b_2|,|c|)\phi(|a_2|,|b_1c|)a_1b_1cS(a_2b_2)=\phi(|a_2b_2|,|c|)\phi(|a_2|,|b_1|)a_1b_1cS(a_2b_2)\\&=\phi(|(ab)_2|,|c|)(ab)_1cS((ab)_2)=\xi_A(ab\otimes c)\end{aligned}$$

and $\xi_A(1_A\otimes a)=\phi(1_G,|a|)1_AaS(1_A)=a$, hence (4) is satisfied. ■

3.4. REMARK. Observe that (4) can also be deduced from [26, Proposition 3.7.1] seeing A in ${}_A(\mathrm{Vec}_G)_A$ with A-actions given by multiplication m_A, in which case ξ_A becomes ad_A. Also note that 4) of Lemma 3.2 tells us that, when $i:X\longrightarrow A$ and $j:Y\longrightarrow A$ commute in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, the restriction of the action ξ_A to X and Y is trivial. From now on, we set $\xi_A(a\otimes b):=a\triangleright b$, for all $a,b\in A$. In case there will be different A and B in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ concurrently, the action $\triangleright$ will be clear from the context and we will omit putting indexes.

The Huq commutator of two normal color Hopf subalgebras X,Y of A in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, denoted by $[X,Y]_{\mathrm{Huq}}$, is then defined as the smallest normal color Hopf subalgebra of A such that, considered the cokernel of the inclusion $[X,Y]_{\mathrm{Huq}}\longrightarrow A$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ which is given by the canonical projection $q:A\longrightarrow A/A[X,Y]^+_{\mathrm{Huq}}$ [39, Corollary 5.21], the color Hopf subalgebras $q(X)$ and $q(Y)$ of $A/A[X,Y]^+_{\mathrm{Huq}}$, which are normal by [39, Lemma 5.4, 2)], commute in the sense of Huq. The latter means that there exists $p:q(X)\otimes q(Y)\longrightarrow A/A[X,Y]^+_{\mathrm{Huq}}$, $\overline{x}\otimes\overline{y}\mapsto\overline{xy}$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ which is equivalent, by Lemma 3.2, to the condition $\overline{xy}=\phi(|x|,|y|)\overline{yx}$ for all $x\in X$ and $y\in Y$ and then to $xy-\phi(|x|,|y|)yx\in A[X,Y]^+_{\mathrm{Huq}}$ for all $x\in X$ and $y\in Y$.

Now we give an explicit description of the Huq commutator of two normal color Hopf subalgebras X and Y of A in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. By analogy to the notation used in [24], we write $[X,Y]$ for the graded subalgebra of A generated by the elements of the form $[x,y]:=\phi(|x_2|,|y_1|)x_1y_1S(x_2)S(y_2)$ for any $x\in X$ and any $y\in Y$. Clearly, we have $[x,y]=(x\triangleright y_1)S(y_2)$ for all $x\in X$ and $y\in Y$. Note also that

$$[x,y]=\phi(|x_2|,|y_1|)x_1y_1S(x_2)S(y_2)=\phi(|x_2|,|y|)x_1(y\triangleright S(x_2))\text{ for all }x\in X,y\in Y.\quad(5)$$

The action $\triangleright$ satisfies some compatibility conditions with the antipode and the comultiplication of A, as shown in the following result.

3.5. LEMMA. *Given A in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, the following properties are satisfied:*

$$S(a\triangleright b)=a\triangleright S(b)\quad\text{for all }a,b\in A,\qquad(6)$$

$$a\triangleright bc=\phi(|a_2|,|b|)(a_1\triangleright b)(a_2\triangleright c)\quad\text{for all }a,b,c\in A.\qquad(7)$$

Proof. Given $a, b, c \in A$, we can compute

$$\begin{aligned} S(a \triangleright b) &= S(\phi(|a_2|, |b|)a_1 b S(a_2)) = \phi(|a_2|, |b|)\phi(|a_1 b|, |a_2|)S(S(a_2))S(a_1 b) \\ &= \phi(|a_1|, |a_2|)a_2 S(a_1 b) = a_1 S(a_2 b) = \phi(|a_2|, |b|)a_1 S(b)S(a_2) = a \triangleright S(b), \end{aligned}$$

so that (6) holds and

$$\begin{aligned} a \triangleright bc &= \phi(|a_2|, |bc|)a_1 bc S(a_2) = \phi(|a_3|, |bc|)a_1 \epsilon(a_2) bc S(a_3) \\ &= \phi(|a_2|, |b|)\phi(|a_3|, |bc|)a_1 b \epsilon(a_2) c S(a_3) = \phi(|a_2 \otimes a_3|, |b|)\phi(|a_4|, |bc|)a_1 b S(a_2) a_3 c S(a_4) \\ &= \phi(|a_2|, |b|)\phi(|a_3|, |bc|)(a_1 \triangleright b)a_2 c S(a_3) = \phi(|a_2 \otimes a_3|, |b|)\phi(|a_3|, |c|)(a_1 \triangleright b)a_2 c S(a_3) \\ &= \phi(|a_2|, |b|)(a_1 \triangleright b)(a_2 \triangleright c) \end{aligned}$$

and then also (7) is satisfied. ■

Recall that we already know that ξ_A is a morphism of graded coalgebras for all A in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, i.e., we have

$$(a \triangleright b)_1 \otimes (a \triangleright b)_2 = \phi(|a_2|, |b_1|)(a_1 \triangleright b_1) \otimes (a_2 \triangleright b_2) \text{ and } \epsilon(a \triangleright b) = \epsilon(a)\epsilon(b) \qquad (8)$$

for all $a, b \in A$. Moreover, we also know that, given $f : A \longrightarrow B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, we have $f(a \triangleright b) = f(a) \triangleright f(b)$ for all $a, b \in A$.

3.6. Proposition. *The graded algebra* $[X, Y]$ *is a normal color Hopf subalgebra of* A*, for any normal color Hopf subalgebras* X, Y *of* A.

Proof. By definition, $[X, Y]$ is a graded subalgebra of A. Thus, if we show that it is a graded subcoalgebra of A and that it is closed under the antipode of A, we obtain that it is a color Hopf subalgebra of A. From the fact that ξ_A is a morphism of graded coalgebras, we obtain that $[X, Y]$ is a subcoalgebra of A. Indeed, we can compute

$$\begin{aligned} \Delta_A([x, y]) &= \Delta_A((x \triangleright y_1)S(y_2)) = \phi(|(x \triangleright y_1)_2|, |S(y_2)_1|)(x \triangleright y_1)_1 S(y_2)_1 \otimes (x \triangleright y_1)_2 S(y_2)_2 \\ &\overset{(8)}{=} \phi(|x_2|, |y_1|)\phi(|x_2 \triangleright y_2|, |S(y_3)|)(x_1 \triangleright y_1)S(y_3) \otimes (x_2 \triangleright y_2)S(y_4) \\ &= \phi(|x_2|, |y_1|)\phi(|x_2|, |y_2|)(x_1 \triangleright y_1)S(y_2) \otimes (x_2 \triangleright y_3)S(y_4) \\ &= \phi(|x_2|, |y_1|)[x_1, y_1] \otimes [x_2, y_2] \in [X, Y] \otimes [X, Y]. \end{aligned}$$

Moreover, we have

$$\begin{aligned} S_A([x, y]) &= S_A((x \triangleright y_1)S_A(y_2)) = \phi(|x \otimes y_1|, |y_2|)y_2 S_A(x \triangleright y_1) = \phi(|x|, |y_1|)y_1 S_A(x \triangleright y_2) \\ &\overset{(6)}{=} \phi(|x|, |y_1|)y_1(x \triangleright S(y_2)) \overset{(5)}{=} \phi(|x|, |y|)[y, x] \in [X, Y], \end{aligned}$$

hence $[X, Y]$ is a color Hopf subalgebra of A. Finally, we show that $[X, Y]$ is normal:

$$\begin{aligned}
a \triangleright [x,y] &= a \triangleright ((x \triangleright y_1)S(y_2)) \overset{(7)}{=} \phi(|a_2|, |x \triangleright y_1|)(a_1 \triangleright (x \triangleright y_1))(a_2 \triangleright S(y_2)) \\
&\overset{(4)}{=} \phi(|a_2|, |x \otimes y_1|)(a_1 x \triangleright y_1)(a_2 \triangleright S(y_2)) \\
&= \phi(|a_2 \otimes a_3|, |x \otimes y_1|)(a_1 x \triangleright y_1)(\epsilon(a_2)a_3 \triangleright S(y_2)) \\
&= \phi(|a_2|, |x|)\phi(|a_3|, |x \otimes y_1|)(a_1 x \epsilon(a_2) \triangleright y_1)(a_3 \triangleright S(y_2)) \\
&= \phi(|a_2 \otimes a_3|, |x|)\phi(|a_4|, |x \otimes y_1|)(a_1 x S(a_2) a_3 \triangleright y_1)(a_4 \triangleright S(y_2)) \\
&\overset{(4)}{=} \phi(|a_2 \otimes a_3|, |x|)\phi(|a_4|, |x \otimes y_1|)((a_1 x S(a_2)) \triangleright (a_3 \triangleright y_1))(a_4 \triangleright S(y_2)) \\
&\overset{(6)}{=} \phi(|a_3|, |x|)\phi(|a_4|, |x \otimes y_1|)((\phi(|a_2|, |x|)a_1 x S(a_2)) \triangleright (a_3 \triangleright y_1))S(a_4 \triangleright y_2) \\
&= \phi(|a_2|, |x|)\phi(|a_3|, |x \otimes y_1|)((a_1 \triangleright x) \triangleright (a_2 \triangleright y_1))S(a_3 \triangleright y_2) \\
&= \phi(|a_2 \otimes a_3|, |x|)\phi(|a_3|, |y_1|)((a_1 \triangleright x) \triangleright (a_2 \triangleright y_1))S(a_3 \triangleright y_2) \\
&= \phi(|a_2|, |x|)((a_1 \triangleright x) \triangleright (a_2 \triangleright y)_1)S((a_2 \triangleright y)_2) \\
&= \phi(|a_2|, |x|)[a_1 \triangleright x, a_2 \triangleright y] \in [X, Y]
\end{aligned}$$

and then the thesis follows. ■

3.7. Proposition. *Given X and Y normal color Hopf subalgebras of A, then $[X, Y] = [X, Y]_{\mathrm{Huq}}$.*

Proof. Recall that $[X, Y]_{\mathrm{Huq}}$ is defined as the smallest normal color Hopf subalgebra of A such that

$$xy - \phi(|x|, |y|)yx \in A[X, Y]^+_{\mathrm{Huq}} = [X, Y]^+_{\mathrm{Huq}}A \tag{9}$$

for all $x \in X$ and $y \in Y$. By Proposition 3.6 we already know that $[X, Y]$ is a normal color Hopf subalgebra of A, hence we only have to show that it is the smallest which satisfies (9). Observe that

$$\begin{aligned}
xy &= x_1\epsilon(x_2)y = \phi(|x_2|, |y|)x_1 y \epsilon(x_2) = \phi(|x_2 \otimes x_3|, |y|)x_1 y_1 \epsilon(y_2) S(x_2) x_3 \\
&= \phi(|x_2|, |y_1|)\phi(|x_3|, |y|)x_1 y_1 S(x_2)\epsilon(y_2)x_3 = \phi(|x_2|, |y_1|)\phi(|x_3|, |y|)x_1 y_1 S(x_2) S(y_2) y_3 x_3 \\
&= \phi(|x_2|, |y|)[x_1, y_1]y_2 x_2
\end{aligned}$$

and

$$\begin{aligned}
\phi(|x|, |y|)yx &= \phi(|x_2|, |y|)\epsilon(x_1)\epsilon(y_1)y_2 x_2 = \phi(|x_3|, |y|)\epsilon(x_1)\epsilon(x_2)\epsilon(y_1)\epsilon(y_2)y_3 x_3 \\
&= \phi(|x_3|, |y|)\phi(|x_2|, |y_1|)\epsilon(x_1)\epsilon(y_1)\epsilon(S(x_2))\epsilon(S(y_2))y_3 x_3 \\
&= \phi(|x_3|, |y|)\epsilon(\phi(|x_2|, |y_1|)x_1 y_1 S(x_2) S(y_2))y_3 x_3 \\
&= \phi(|x_2|, |y|)\epsilon([x_1, y_1])y_2 x_2,
\end{aligned}$$

so that we obtain

$$xy - \phi(|x|, |y|)yx = \big([x_1, y_1] - \epsilon([x_1, y_1])\big)\phi(|x_2|, |y|)y_2 x_2 \in [X, Y]^+ A,$$

hence $[X,Y]$ satisfies (9). Finally, we prove that $[X,Y]$ is the smallest normal color Hopf subalgebra which satisfies (9) showing that $[X,Y] \subseteq \mathrm{Hker}(f)$ for every morphism $f : A \longrightarrow B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ such that $f(X)$ and $f(Y)$ commute in B, where $\mathrm{Hker}(f) = \{x \in A \mid x_1 \otimes f(x_2) = x \otimes 1_B\}$ is the categorical kernel of f in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. Hence, given $[x,y] = (x \triangleright y_1)S(y_2)$ in $[X,Y]$ for $x \in X$ and $y \in Y$, we can compute

$$\begin{aligned}
[x,y]_1 \otimes f([x,y]_2) &= ((x \triangleright y_1)S(y_2))_1 \otimes f(((x \triangleright y_1)S(y_2))_2) \\
&= \phi(|(x \triangleright y_1)_2|, |S(y_2)_1|)(x \triangleright y_1)_1 S(y_2)_1 \otimes f((x \triangleright y_1)_2 S(y_2)_2) \\
&\overset{(8)}{=} \phi(|x_2 \otimes y_2|, |y_3|)\phi(|x_2|, |y_1|)(x_1 \triangleright y_1)S(y_3) \otimes f((x_2 \triangleright y_2)S(y_4)) \\
&= \phi(|x_2|, |y_1|)\phi(|x_2|, |y_2|)(x_1 \triangleright y_1)S(y_2) \otimes f(x_2 \triangleright y_3)f(S(y_4)) \\
&= \phi(|x_2|, |y_1|)\phi(|x_2|, |y_2|)(x_1 \triangleright y_1)S(y_2) \otimes (f(x_2) \triangleright f(y_3))f(S(y_4)) \\
&\overset{(*)}{=} \phi(|x_2|, |y_1|)\phi(|x_2|, |y_2|)(x_1 \triangleright y_1)S(y_2) \otimes \epsilon(f(x_2))f(y_3)f(S(y_4)) \\
&= \phi(|x_2|, |y_1|)\phi(|x_2|, |y_2|)(x_1 \triangleright y_1)S(y_2) \otimes \epsilon(x_2)f(y_3 S(y_4)) \\
&= \phi(|x_2|, |y_1|)\phi(|x_2|, |y_2|)(x_1 \triangleright y_1)S(y_2) \otimes \epsilon(x_2)\epsilon(y_3)1_B \\
&= (x \triangleright y_1)S(y_2) \otimes \epsilon(y_3)1_B = [x,y] \otimes 1_B
\end{aligned}$$

where $(*)$ holds since $f(X)$ and $f(Y)$ commute in B, using 4) of Lemma 3.2. Hence $[X,Y]$ is the Huq commutator of X and Y in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. ■

The previous result generalizes [24, Proposition 4.3] given for the category of cocommutative Hopf algebras.

3.8. Remark. We recall that the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is locally algebraically cartesian closed [39, Proposition 6.3], then algebraically coherent by [18, Theorem 4.5]. Therefore, by [18, Theorem 6.18], $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ satisfies the condition (SH) (also recalled in the preliminaries) and the condition (NH) of normality of Higgins commutators of normal subobjects [16, 17]. Thus, for any normal color Hopf subalgebras X and Y of an object A in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, the object $[X,Y]$ is also the Higgins commutator of X and Y in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$.

4. Hall's criterion and Zassenhaus Lemma in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$

A cocommutative color Hopf algebra A is called *nilpotent* if $\gamma_A^n(A) = 0$ for a certain $n \in \mathbb{N}$ where, denoted by $\mathrm{Norm}(A)$ the set of all normal color Hopf subalgebras of A, the map $\gamma_A : \mathrm{Norm}(A) \longrightarrow \mathrm{Norm}(A)$ sends a normal color Hopf subalgebra X of A to $[A,X]_{\mathrm{Huq}}$, which is $[A,X]$ by Proposition 3.7. The least of such n is the nilpotency class of A. Recall that, by [39, Lemma 5.22], regular epimorphisms in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ are exactly the surjective maps. As recalled above, $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is algebraically coherent by [39, Proposition 6.3] and [18, Theorem 4.5]. Hence, in view of Theorem 2.1, we have the Hall's criterion for nilpotence in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, given by the following result:

4.1. PROPOSITION. *Let $p : E \longrightarrow B$ be a surjective map in* $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$*, where B is nilpotent in* $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$*. If* $\mathrm{Hker}(p) \subseteq [N, N]$ *for a nilpotent normal color Hopf subalgebra N of E, then E is nilpotent. Furthermore, if N is of nilpotency class c and B is of nilpotency class d, then E is of nilpotency class at most* $\frac{c(c+1)}{2}(d-1)+c$.

Moreover, we can also describe the Zassenhaus Lemma in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$.

4.2. LEMMA. *Let M and K be two color Hopf subalgebras of a cocommutative color Hopf algebra A, with K normal in A. Then, the graded vector spaces KM and MK, generated by linear combinations of elements of the form km and mk with $k \in K$ and $m \in M$, respectively, are the same object in* Vec_G.

PROOF. Since MK and KM are graded subspaces of A, in order to conclude it is sufficient to show that they are the same vector space. Given $k \in K$ and $m \in M$, we have

$$\begin{aligned} km &= k\epsilon(m_1)m_2 = \phi(|k|, |m_1|)\epsilon(m_1)km_2 \\ &= \phi(|k|, |m_1 \otimes m_2|)m_1S(m_2)kS(S(m_3)) \\ &= \phi(|k|, |m|)\phi(|m_3|, |k|)m_1S(m_2)kS(S(m_3)) \\ &= \phi(|k|, |m|)\phi(|S(m_2)_2|, |k|)m_1S(m_2)_1kS(S(m_2)_2) \in MK, \end{aligned}$$

since K is normal, hence $KM \subseteq MK$. Moreover, we also have

$$mk = m_1\epsilon(m_2)k = \phi(|m_2|, |k|)m_1k\epsilon(m_2) = \phi(|m_2|, |k|)\phi(|m_3|, |k|)m_1kS(m_2)m_3 \in KM,$$

using again that K is normal, thus the thesis follows. ■

4.3. LEMMA. *Let M and K be two color Hopf subalgebras of a cocommutative color Hopf algebra A, with K normal in A. Then the graded vector space KM is a color Hopf subalgebra of A.*

PROOF. Clearly KM contains 1_A, thus we only have to show that it is closed under m_A, Δ_A and S_A. Given $k, k' \in K$ and $m, m' \in M$, since $MK = KM$ in Vec_G by Lemma 4.2, we obtain

$$kmk'm' \in KMKM = KKMM = KM, \;\; \Delta(km) = \phi(|k_2|, |m_1|)k_1m_1 \otimes k_2m_2 \in KM \otimes KM$$

and $S(km) = \phi(|k|, |m|)S(m)S(k) \in MK = KM$. Hence KM is a color Hopf subalgebra of A, which is automatically cocommutative. ■

4.4. PROPOSITION. *Let M and K be two color Hopf subalgebras of a cocommutative color Hopf algebra A with K normal in A. Then KM is the supremum of K and M as color Hopf subalgebras of A, i.e.,* $K \vee_A M = KM$.

PROOF. If there exists a color Hopf subalgebra L of A containing K and M, then it clearly also contains KM, because a color Hopf subalgebra is closed under products and sums. ■

Hence, by applying the categorical results in [38], we obtain the analogs of [38, Propositions 4.4 and 4.5] and [38, Theorem 4.6]. Let us make explicit the Zassenhaus Lemma in the setting of cocommutative color Hopf algebras. Recall that, given a cocommutative color Hopf algebra A and a normal color Hopf subalgebra B of A, the cokernel object of the inclusion $i : B \longrightarrow A$ is given by A/AB^+. Hence, in view of Theorem 2.3, we obtain the following result:

4.5. Proposition. *Let U and V be two color Hopf subalgebras of a cocommutative color Hopf algebra A, K a normal color Hopf subalgebra of U and L a normal color Hopf subalgebra of V. Then the following*

$$\frac{K(U\cap V)}{K(U\cap V)(K(L\cap U))^+} \cong \frac{U\cap V}{(U\cap V)((K\cap V)(L\cap U))^+} \cong \frac{L(U\cap V)}{L(U\cap V)(L(K\cap V))^+}$$

are isomorphisms in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$.

5. Color Hopf crossed modules as internal crossed modules

In this section, we give a characterization of the category $\mathrm{XMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ of internal crossed modules in the semi-abelian category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ by providing an equivalence with another category, apparently different. To do this, we first have to understand the category $\mathrm{Pt}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ better. Recall that, according to what we said in the preliminary section, the category $\mathrm{XMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ is equivalent to $\mathrm{Grpd}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ which is isomorphic to $\mathrm{RMG}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$.

5.1. Remark. Recall that, given A in $\mathrm{Bimon}(\mathrm{Vec}_G)$, the category ${}_A\mathrm{Vec}_G$ is monoidal. The unit object is $\Bbbk$ with left A-action defined by $a \cdot k := \epsilon(a)k$ for all $a \in A$ and $k \in \Bbbk$ and, given V and W in ${}_A\mathrm{Vec}_G$, the tensor product $V \otimes W$ has left A-action given by $a\cdot(v\otimes w) := \phi(|a_2|,|v|)(a_1\cdot v)\otimes(a_2\cdot w)$, for all $a \in A$, $v \in V$ and $w \in W$. Furthermore, if we consider A in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ then the category ${}_A\mathrm{Vec}_G$ is symmetric monoidal since Vec_G is symmetric (see e.g. [1, Proposition 6.40]). Thus, we can consider the category $\mathrm{Hopf}({}_A\mathrm{Vec}_G)$, whose objects are color Hopf algebras $(H, m, u, \Delta, \epsilon, S)$ which are also graded left A-modules and such that $m, u, \Delta, \epsilon, S$ are morphisms of left A-modules. Hence, denoting by $\cdot : A \otimes H \longrightarrow H$ the left A-action, the following relations are satisfied

$$\begin{aligned} a\cdot 1_H = \epsilon(a)1_H, \qquad \epsilon(a\cdot h) = \epsilon(a)\epsilon(h),\\ \Delta(a\cdot h) = \phi(|a_2|,|h_1|)(a_1\cdot h_1)\otimes(a_2\cdot h_2),\\ a\cdot(hh') = \phi(|a_2|,|h|)(a_1\cdot h)(a_2\cdot h'), \end{aligned}$$

for every $a \in A$ and $h, h' \in H$. Note that we are considering the trivial braiding on the category ${}_A\mathrm{Vec}_G$, given by the braiding of Vec_G, since A is cocommutative. We call an object in $\mathrm{Hopf}({}_A\mathrm{Vec}_G)$ an *A-module color Hopf algebra.* Note that we can consider A as an A-module color Hopf algebra with left A-action given by $\triangleright$. In fact, A is in ${}_A\mathrm{Vec}_G$ by

Lemma 3.3, clearly $a \triangleright 1_A = \epsilon(a)1_A$ for all $a \in A$ and $\triangleright$ satisfies the second and third equations as said in (8). Moreover, also the last equation holds since $\triangleright$ satisfies (7).

The fact that $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is a semi-abelian category directly implies an equivalence between *internal actions* and *split extensions* of cocommutative color Hopf algebras [7]. As in the case of cocommutative Hopf algebras, one can define the category of (cocommutative) module color Hopf algebras.

5.2. DEFINITION. *The category of (cocommutative) module color Hopf algebras is defined as follows. Objects are (cocommutative) A-module color Hopf algebras, where A is an arbitrary cocommutative color Hopf algebra and, given a (cocommutative) A-module color Hopf algebra H and a (cocommutative) B-module color Hopf algebra K, with A, B in* $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$*, a morphism between them is given by a pair of morphisms of color Hopf algebras $\alpha : A \longrightarrow B$ and $\beta : H \longrightarrow K$ such that $\beta(a \cdot h) = \alpha(a) \cdot \beta(h)$, for $a \in A$ and $h \in H$. We denote the category of cocommutative module color Hopf algebras by* $\mathrm{Act}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$.

The category $\mathrm{Act}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ will turn out to be equivalent to $\mathrm{Pt}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$, so it will coincide with the category of internal actions in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, justifying the notation adopted.

Moreover, one can consider semi-direct products (also called smash products) starting from a general braided monoidal category and then, in case of Vec_G, one recovers the following definition:

5.3. DEFINITION. *Given A in* $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ *and a (cocommutative) A-module color Hopf algebra H, the semi-direct product of H and A, denoted by $H \rtimes A$, is the (cocommutative) color Hopf algebra which is $H \otimes A$ as a G-graded vector space, an algebra with unit $1_{H\rtimes A} := 1_H \otimes 1_A$ and multiplication defined by*

$$(h \otimes a)(h' \otimes a') := \phi(|a_2|, |h'|)h(a_1 \cdot h') \otimes a_2a'$$

for all $h, h' \in H$ and $a, a' \in A$ and a coalgebra with counit and comultiplication defined by

$$\epsilon(h \otimes a) := \epsilon(h)\epsilon(a) \text{ and } \Delta(h \otimes a) := \phi(|h_2|, |a_1|)h_1 \otimes a_1 \otimes h_2 \otimes a_2.$$

The antipode is given by $S(h \otimes a) := \phi(|h|, |a_1|)(S(a_1) \cdot S(h)) \otimes S(a_2)$.

The following result is a special case of [26, Theorem 3.10.4], see also [2] and [3, Theorem 5.1.5] for the original sources. Note also that it is a straightforward generalization of the result proved in [37] for the category $\mathrm{Hopf}_{\Bbbk,\mathrm{coc}}$.

5.4. PROPOSITION. *Let H and A be color Hopf algebras with A cocommutative and $\begin{tikzcd} H \arrow[r, shift left, "p"] & A \arrow[l, shift left, "i"] \end{tikzcd}$ a splitting of morphisms of color Hopf algebras, i.e., $p \circ i = \mathrm{Id}_A$. Then,* $\mathrm{Hker}(p)$ *is an A-module color Hopf algebra through*

$$\cdot : A \otimes \mathrm{Hker}(p) \longrightarrow \mathrm{Hker}(p), \ a \otimes k \mapsto \phi(|a_2|, |k|)i(a_1)ki(S(a_2)).$$

Furthermore, there exists an isomorphism of color Hopf algebras

$$f : \mathrm{Hker}(p) \rtimes A \longrightarrow H, \;\; k \otimes a \mapsto k i(a)$$

with inverse $g : H \longrightarrow \mathrm{Hker}(p) \rtimes A$, $h \mapsto h_1 i(p(S(h_2))) \otimes p(h_3)$.

Note that $a \cdot k = i(a) \triangleright k$, for all $a \in A$ and $k \in \mathrm{Hker}(p)$.

The category $\mathrm{Act}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ turns out to be equivalent to $\mathrm{Pt}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$. This result can be deduced from [4, Proposition 1.5]. However, we include here a direct proof for the reader's sake.

5.5. Proposition. *The categories* $\mathrm{Act}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ *and* $\mathrm{Pt}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ *are equivalent.*

Proof. Define a functor

$$F : \mathrm{Pt}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)) \longrightarrow \mathrm{Act}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$$

as follows

$$\begin{tikzcd}
H \arrow[r, shift left, "p"] \arrow[d, "f"'] & A \arrow[l, shift left, "i"] \arrow[d, "g"] & \mapsto & A \otimes \mathrm{Hker}(p) \arrow[r, "\cdot"] \arrow[d, "g \otimes \bar{f}"'] & \mathrm{Hker}(p) \arrow[d, "\bar{f}"] \\
H' \arrow[r, shift left, "p'"] & A' \arrow[l, shift left, "i'"] & & A' \otimes \mathrm{Hker}(p') \arrow[r, "\cdot"] & \mathrm{Hker}(p')
\end{tikzcd}$$

where the morphism $\cdot$ is defined as in Proposition 5.4 and $\bar{f}$ is induced by the universal property of the kernel. Indeed, calling $j : \mathrm{Hker}(p) \longrightarrow H$ the inclusion given by the kernel of p in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, we can compute

$$p' \circ f \circ j = g \circ p \circ j = g \circ u_A \circ \epsilon_H \circ j = u_{A'} \circ \epsilon_H \circ j = u_{A'} \circ \epsilon_{H'} \circ f \circ j,$$

so that there exists a unique $\bar{f} : \mathrm{Hker}(p) \longrightarrow \mathrm{Hker}(p')$ such that $f \circ j = j' \circ \bar{f}$, where $j' : \mathrm{Hker}(p') \longrightarrow H'$ is the inclusion. Clearly, $\bar{f}$ is simply given by the restriction of f to $\mathrm{Hker}(p)$. The diagram on the right actually commutes. Indeed, for all $a \in A$ and $x \in \mathrm{Hker}(p)$, we have

$$\begin{aligned} g(a) \cdot f(x) &= \phi(|g(a_2)|, |f(x)|) i'(g(a_1)) f(x) i'(S(g(a_2))) = \phi(|a_2|, |x|) f(i(a_1)) f(x) f(i(S(a_2))) \\ &= f(\phi(|a_2|, |x|) i(a_1) x i(S(a_2))) = f(a \cdot x), \end{aligned}$$

hence F is well-defined. Moreover, define a functor

$$G : \mathrm{Act}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)) \longrightarrow \mathrm{Pt}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$$

as follows

$$\begin{tikzcd}
A \otimes H \arrow[r, "\cdot"] \arrow[d, "\alpha \otimes \beta"'] & H \arrow[d, "\beta"] & \mapsto & H \rtimes A \arrow[r, shift left, "p_2"] \arrow[d, "\beta \otimes \alpha"'] & A \arrow[l, shift left, "\iota_2"] \arrow[d, "\alpha"] \\
B \otimes K \arrow[r, "\cdot"] & K & & K \rtimes B \arrow[r, shift left, "p_2"] & B \arrow[l, shift left, "\iota_2"]
\end{tikzcd}$$

where $p_2(h\otimes a)=\epsilon_H(h)a$ and $\iota_2(a)=1_H\otimes a$ for all $h\in H$ and $a\in A$. Hence $p_2\circ\iota_2=\mathrm{Id}_A$ and the following relations

$$p_2(\beta\otimes\alpha)(h\otimes a)=\epsilon_K(\beta(h))\alpha(a)=\alpha(\epsilon_H(h)a),\quad 1_K\otimes\alpha(a)=\beta(1_H)\otimes\alpha(a)$$

are satisfied, so that G is well-defined. Hence, we have

$$(H \underset{i}{\overset{p}{\rightleftarrows}} A) \overset{F}{\longmapsto} (A\otimes \mathrm{Hker}(p)\overset{\cdot}{\longrightarrow}\mathrm{Hker}(p)) \overset{G}{\longmapsto} (\mathrm{Hker}(p)\rtimes A \underset{\iota_2}{\overset{p_2}{\rightleftarrows}} A)$$

and we know, by Proposition 5.4, that $f:\mathrm{Hker}(p)\rtimes A\longrightarrow H$, $k\otimes a\mapsto ki(a)$ is an isomorphism of color Hopf algebras, so that (f,Id_A) is an isomorphism of split epimorphisms, since clearly $f\circ\iota_2=i$ and $p\circ f=p_2$ as $p(k)=\epsilon(k)1_A$ for $k\in\mathrm{Hker}(p)$. Furthermore, we have

$$(A\otimes H\overset{\cdot}{\longrightarrow}H)\overset{G}{\longmapsto}(H\rtimes A \underset{\iota_2}{\overset{p_2}{\rightleftarrows}} A) \overset{F}{\longmapsto}(A\otimes\mathrm{Hker}(p_2)\longrightarrow\mathrm{Hker}(p_2))$$

and

$$\begin{aligned}(\mathrm{Id}\otimes p_2)\Delta(h\otimes a)&=(\mathrm{Id}\otimes p_2)(\phi(|h_2|,|a_1|)h_1\otimes a_1\otimes h_2\otimes a_2)\\&=\phi(|h_2|,|a_1|)h_1\otimes a_1\otimes\epsilon(h_2)a_2=h_1\epsilon(h_2)\otimes a_1\otimes a_2=h\otimes a_1\otimes a_2,\end{aligned}$$

so that $\mathrm{Hker}(p_2)$ is given by elements in $H\otimes A^{\mathrm{co}A}=H\otimes\Bbbk 1_A\cong H$. Therefore, F and G form an equivalence of categories. ■

5.6. Remark. Since the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is semi-abelian, Proposition 5.5 implies that $\mathrm{Act}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ provides an alternative description of internal actions in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. Moreover, given a cocommutative color Hopf algebra A, the category $\mathrm{Pt}_A(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ results to be equivalent to the category $\mathrm{Hopf}_{\mathrm{coc}}({}_A\mathrm{Vec}_G)$ of cocommutative A-module color Hopf algebras, which coincides with the Eilenberg–Moore category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)^{\mathbf{T}_A}$ of the monad $\mathbf{T}_A$ corresponding to the monadic functor $u_A^*:\mathrm{Pt}_A(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))\longrightarrow\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, where $u_A:\Bbbk\longrightarrow A$ is the unit of A. Therefore, given H in $\mathrm{Hopf}_{\mathrm{coc}}({}_A\mathrm{Vec}_G)$, the semi-direct product $H\rtimes A$ introduced in Definition 5.3 is actually the categorical one defined in [13] for the semi-abelian category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$.

Generalizing [33, Definition 2.1] which is given for (cocommutative) Hopf algebras, we give the following definition for cocommutative color Hopf algebras.

5.7. Definition. *A color Hopf crossed module is a triple (A,H,d) where A is a cocommutative color Hopf algebra, H is a cocommutative A-module color Hopf algebra and $d:H\longrightarrow A$ is a morphism of color Hopf algebras such that*

$$d(a\cdot h)=a\triangleright d(h)=\phi(|a_2|,|h|)a_1d(h)S(a_2),\qquad \text{for all } h\in H \text{ and } a\in A, \tag{10}$$

$$d(g)\cdot h=g\triangleright h=\phi(|g_2|,|h|)g_1hS(g_2),\qquad \text{for all } h,g\in H. \tag{11}$$

A morphism $(\alpha,\beta):(A,H,d)\longrightarrow(A',H',d')$ of color Hopf crossed modules is given by a pair of color Hopf algebra morphisms $\alpha:H\longrightarrow H'$ and $\beta:A\longrightarrow A'$ such that $d'\circ\alpha=\beta\circ d$

and $\alpha(a \cdot h) = \beta(a) \cdot \alpha(h)$*, for all* $a \in A$ *and* $h \in H$*. We denote this category by* $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$*.*

Now we show that $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ is equivalent to $\mathrm{RMG}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$. In order to do this, we need to know how pullbacks in the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ are made.

Pullbacks in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. From the construction of binary products and equalizers in the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ given in [39] we can easily derive pullbacks in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. Let A, B, C be cocommutative color Hopf algebras and $f : A \longrightarrow C$, $g : B \longrightarrow C$ be morphisms of color Hopf algebras. The pullback object $A \times_C B$ of A and B over C is given by $\mathrm{Eq}(f \circ \pi_A, g \circ \pi_B)$, where $\pi_A := r_A \circ (\mathrm{Id}_A \otimes \epsilon_B)$ and $\pi_B := l_B \circ (\epsilon_A \otimes \mathrm{Id}_B)$ are the projections of $A \otimes B$, which is the binary product of A and B in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. So $A \times_C B$ is given by elements $a \otimes b \in A \otimes B$ such that

$$(\mathrm{Id}_{A\otimes B} \otimes f r_A(\mathrm{Id}_A \otimes \epsilon_B))\Delta_{A\otimes B}(a \otimes b) = (\mathrm{Id}_{A\otimes B} \otimes g l_B(\epsilon_A \otimes \mathrm{Id}_B))\Delta_{A\otimes B}(a \otimes b).$$

The first member is

$$\begin{aligned}(\mathrm{Id}_{A\otimes B} \otimes f r_A(\mathrm{Id}_A \otimes \epsilon_B))(\phi(|a_2|, |b_1|)a_1 \otimes b_1 \otimes a_2 \otimes b_2) &= \phi(|a_2|, |b_1|)a_1 \otimes b_1 \otimes f(a_2\epsilon(b_2)) \\ &= \phi(|a_2|, |b|)a_1 \otimes b \otimes f(a_2),\end{aligned}$$

while the second one is

$$\begin{aligned}(\mathrm{Id}_{A\otimes B} \otimes g l_B(\epsilon_A \otimes \mathrm{Id}_B))(\phi(|a_2|, |b_1|)a_1 \otimes b_1 \otimes a_2 \otimes b_2) &= \phi(|a_2|, |b_1|)a_1 \otimes b_1 \otimes g(\epsilon(a_2)b_2) \\ &= a \otimes b_1 \otimes g(b_2).\end{aligned}$$

Hence $A\times_C B$ is given by elements $a\otimes b \in A\otimes B$ such that $\phi(|a_2|, |b|)a_1\otimes b\otimes f(a_2) = a\otimes b_1\otimes g(b_2)$ or, equivalently by applying $\mathrm{Id}_A \otimes c_{B,C}$, such that $a_1 \otimes f(a_2) \otimes b = \phi(|b_1|, |b_2|)a \otimes g(b_2) \otimes b_1 = a \otimes g(b_1) \otimes b_2$, where the last equality follows by cocommutativity of B. Therefore, we obtain

$$A \times_C B = \{a \otimes b \in A \otimes B \mid a_1 \otimes f(a_2) \otimes b = a \otimes g(b_1) \otimes b_2\}$$

and $(A \times_C B, \pi_A, \pi_B)$ is the pullback of the pair (f, g) in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$.

Let us also recall that, since $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is action representable by [39, Proposition 6.3] and then it satisfies the condition (SH), Proposition 2.2 gives us the following result for a reflexive graph in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$.

5.8. Lemma. *Given a reflexive graph*

$$\begin{tikzcd} A_1 \arrow[r, shift left=2, "p"] \arrow[r, shift right=2, "\gamma"'] & A_0 \arrow[l, "i"'] \end{tikzcd} \tag{12}$$

in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$*, the following conditions are equivalent for* (12)*:*

1. *it is a reflexive-multiplicative graph;*

2. *it is an internal groupoid;*
3. *it satisfies* $[\mathrm{Hker}(p), \mathrm{Hker}(\gamma)]_{\mathrm{Huq}} = 0$*, i.e.,* $xy = \phi(|x|, |y|)yx$ *for all* $x \in \mathrm{Hker}(p)$ *and* $y \in \mathrm{Hker}(\gamma)$*.*

Finally, we can show the aforementioned equivalence between $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ and $\mathrm{RMG}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$. This result will allow us to obtain the equivalence between $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ and $\mathrm{XMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$, giving an explicit description of the latter.

5.9. Proposition. *There is an equivalence of categories between* $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ *and* $\mathrm{RMG}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$*. Hence also* $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ *and* $\mathrm{Grpd}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ *are equivalent categories.*

Proof. Given a reflexive graph in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ as in (12) which is multiplicative, one can consider the morphism of color Hopf algebras $d := \gamma \circ j : \mathrm{Hker}(p) \longrightarrow A_0$, where $j : \mathrm{Hker}(p) \longrightarrow A_1$ is the canonical inclusion. We know that $\mathrm{Hker}(p)$ is a cocommutative A_0-module color Hopf algebra by Proposition 5.4 with left A_0-action given by $a \cdot k = \phi(|a_2|, |k|)i(a_1)ki(S(a_2))$ for every $k \in \mathrm{Hker}(p)$ and $a \in A_0$. We only have to verify that d satisfies (10) and (11) of Definition 5.7. Thus, for all $a \in A_0$ and $k \in \mathrm{Hker}(p)$, we compute

$$\begin{aligned} d(a \cdot k) &= \gamma(\phi(|a_2|, |k|)i(a_1)ki(S(a_2))) = \phi(|a_2|, |k|)\gamma(i(a_1))\gamma(k)\gamma(i(S(a_2))) \\ &= \phi(|a_2|, |k|)a_1 d(k)S(a_2). \end{aligned}$$

Since the reflexive graph is multiplicative, by Lemma 5.8 we have $[\mathrm{Hker}(p), \mathrm{Hker}(\gamma)]_{\mathrm{Huq}} = 0$, i.e., $xy = \phi(|x|, |y|)yx$ for all $x \in \mathrm{Hker}(p)$ and $y \in \mathrm{Hker}(\gamma)$. But now $S(k_1)i(\gamma(k_2)) \in \mathrm{Hker}(\gamma)$ for all $k \in \mathrm{Hker}(p)$, as

$$\begin{aligned} (\mathrm{Id} \otimes \gamma)\Delta(S(k_1)i(\gamma(k_2))) &= (\mathrm{Id} \otimes \gamma)(\phi(|k_2|, |k_3|)S(k_1)i(\gamma(k_3)) \otimes S(k_2)i(\gamma(k_4))) \\ &= S(k_1)i(\gamma(k_2)) \otimes \gamma(S(k_3))\gamma(i(\gamma(k_4))) \\ &= S(k_1)i(\gamma(k_2)) \otimes \gamma(S(k_3)k_4) \\ &= S(k_1)i(\gamma(k_2)) \otimes \epsilon(k_3)1_{A_0} \\ &= S(k_1)i(\gamma(k_2)) \otimes 1_{A_0} \end{aligned}$$

and then, for all $b, k \in \mathrm{Hker}(p)$, we obtain

$$S(k_1)i(\gamma(k_2))b = \phi(|S(k_1)i(\gamma(k_2))|, |b|)bS(k_1)i(\gamma(k_2)) = \phi(|k_1 \otimes k_2|, |b|)bS(k_1)i(\gamma(k_2)). \tag{13}$$

Thus, we can compute

$$\begin{aligned} d(g) \cdot h = \gamma(g) \cdot h &= \phi(|g_2|, |h|)i(\gamma(g_1))hi(S(\gamma(g_2))) \\ &= \phi(|g_4|, |h|)g_1 S(g_2)i(\gamma(g_3))hi(S(\gamma(g_4))) \\ &\overset{(13)}{=} \phi(|g_4|, |h|)\phi(|g_2 \otimes g_3|, |h|)g_1 hS(g_2)i(\gamma(g_3))i(\gamma(S(g_4))) \\ &= \phi(|g_2 \otimes g_3|, |h|)g_1 hS(g_2)\epsilon(g_3) = \phi(|g_2|, |h|)g_1 hS(g_2) \end{aligned}$$

and then $(A_0, \mathrm{Hker}(p), d)$ is a color Hopf crossed module. Hence we have obtained a functor $F : \mathrm{RMG}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)) \longrightarrow \mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ defined as follows

$$\begin{tikzcd}
A_1 \arrow[r, shift left=2, "p"] \arrow[r, shift right=2, "\gamma"'] \arrow[d, "f_1"'] & A_0 \arrow[l, "i" description] \arrow[d, "f_0"] & {} \arrow[r, phantom, "\longmapsto"] & {} & \mathrm{Hker}(p) \arrow[r, "d"] \arrow[d, "\bar{f}_1"'] & A_0 \arrow[d, "f_0"] \\
A_1' \arrow[r, shift left=2, "p'"] \arrow[r, shift right=2, "\gamma'"'] & A_0' \arrow[l, "i'" description] & & & \mathrm{Hker}(p') \arrow[r, "d'"'] & A_0'
\end{tikzcd}$$

where $\bar{f}_1$ is the morphism induced by f_1 using the universal property of the kernel.

Given a color Hopf crossed module $(B, X, d : X \longrightarrow B)$, we define the reflexive graph

$$\begin{tikzcd}
X \rtimes B \arrow[r, shift left=2, "p_2"] \arrow[r, shift right=2, "p_1"'] & B \arrow[l, "\iota_2" description]
\end{tikzcd} \tag{14}$$

where $p_2(x \otimes b) = \epsilon(x)b$, $p_1(x \otimes b) = d(x)b$ for all $x \in X$, $b \in B$ and $\iota_2(b) = 1_X \otimes b$. Note that $p_1 = m_B \circ (d \otimes \mathrm{Id}_B)$, which is clearly a morphism of graded coalgebras, is also a morphism of algebras. Indeed, given $x, x' \in X$ and $b, b' \in B$, we have

$$\begin{aligned}
p_1((x \otimes b)(x' \otimes b')) &= p_1(\phi(|b_2|, |x'|)x(b_1 \cdot x') \otimes b_2 b') = \phi(|b_2|, |x'|)d(x(b_1 \cdot x'))b_2 b' \\
&= \phi(|b_2|, |x'|)d(x)d(b_1 \cdot x')b_2 b' \overset{(10)}{=} \phi(|b_3|, |x'|)\phi(|b_2|, |x'|)d(x)b_1 d(x')S(b_2)b_3 b' \\
&= \phi(|b_2|, |x'|)d(x)b_1 d(x')\epsilon(b_2)b' = d(x)bd(x')b' = p_1(x \otimes b)p_1(x' \otimes b').
\end{aligned}$$

Clearly $p_2 \circ \iota_2 = p_1 \circ \iota_2 = \mathrm{Id}_B$. The pullback of the pair (p_2, p_1) in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is given by elements $x \otimes b \otimes x' \otimes b' \in (X \rtimes B) \otimes (X \rtimes B)$ such that

$$(x \otimes b)_1 \otimes p_2((x \otimes b)_2) \otimes x' \otimes b' = x \otimes b \otimes p_1((x' \otimes b')_1) \otimes (x' \otimes b')_2,$$

i.e., $\phi(|x_2|, |b_1|)x_1 \otimes b_1 \otimes \epsilon(x_2)b_2 \otimes x' \otimes b' = \phi(|x_2'|, |b_1'|)x \otimes b \otimes d(x_1')b_1' \otimes x_2' \otimes b_2'$. Then $(X \rtimes B) \times_B (X \rtimes B)$ is given by

$$\{x \otimes b \otimes x' \otimes b' \in (X \rtimes B) \otimes (X \rtimes B) \mid x \otimes b_1 \otimes b_2 \otimes x' \otimes b' = \phi(|x_2'|, |b_1'|)x \otimes b \otimes d(x_1')b_1' \otimes x_2' \otimes b_2'\}.$$

By applying $\mathrm{Id}_X \otimes \epsilon_B \otimes \mathrm{Id}_B \otimes \mathrm{Id}_X \otimes \Delta_B$, it follows that

$$x \otimes b \otimes x' \otimes b_1' \otimes b_2' = \phi(|x_2'|, |b_1'|)x \otimes \epsilon(b)d(x_1')b_1' \otimes x_2' \otimes b_2' \otimes b_3'. \tag{15}$$

The reflexive graph (14) is multiplicative with $m : (X \rtimes B) \times_B (X \rtimes B) \longrightarrow X \rtimes B$ defined by $m := (m_X \otimes \mathrm{Id}_B) \circ (\mathrm{Id}_X \otimes \epsilon_B \otimes \mathrm{Id}_X \otimes \mathrm{Id}_B)$, i.e.,

$$m(x \otimes b \otimes x' \otimes b') = x\epsilon(b)x' \otimes b' \text{ for all } x, x' \in X \text{ and } b, b' \in B.$$

This is actually a morphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. Indeed, m is automatically a morphism of graded coalgebras, so we only have to show that it is a morphism of algebras. Given

$x \otimes b \otimes x' \otimes b' \otimes y \otimes c \otimes y' \otimes c' \in ((X \rtimes B) \times_B (X \rtimes B)) \otimes ((X \rtimes B) \times_B (X \rtimes B))$ we compute

$$\begin{aligned}
&m(x \otimes b \otimes x' \otimes b') \cdot m(y \otimes c \otimes y' \otimes c') = (x\epsilon(b)x' \otimes b')(y\epsilon(c)y' \otimes c') \\
&= \phi(|b'_2|, |y\epsilon(c)y'|)x\epsilon(b)x'(b'_1 \cdot y\epsilon(c)y') \otimes b'_2c' \\
&= \phi(|c|, |y'|)\phi(|b'_2|, |yy'|)x\epsilon(b)x'(b'_1 \cdot yy') \otimes b'_2\epsilon(c)c' \\
&= \phi(|c|, |y'|)\phi(|b'_3|, |yy'|)\phi(|b'_2|, |y|)x\epsilon(b)x'(b'_1 \cdot y)(b'_2 \cdot y') \otimes b'_3\epsilon(c)c' \\
&= \phi(|c|, |y'|)\phi(|b'_3|, |yy'|)\phi(|b'_2|, |y|)\phi(|x'_2|, |b'_1 \cdot y|)x\epsilon(b)x'_1(b'_1 \cdot y)\epsilon(x'_2)(b'_2 \cdot y') \otimes b'_3\epsilon(c)c' \\
&= \phi(|c|, |y'|)\phi(|b'_3|, |yy'|)\phi(|b'_2|, |y|)\phi(|x'_2 \otimes x'_3|, |b'_1 \cdot y|)x\epsilon(b)x'_1(b'_1 \cdot y)S(x'_2)x'_3(b'_2 \cdot y') \otimes b'_3\epsilon(c)c' \\
&\overset{(11)}{=} \phi(|c|, |y'|)\phi(|b'_3|, |yy'|)\phi(|b'_2|, |y|)\phi(|x'_2|, |b'_1 \cdot y|)x\epsilon(b)(d(x'_1) \cdot (b'_1 \cdot y))x'_2(b'_2 \cdot y') \otimes b'_3\epsilon(c)c'
\end{aligned}$$

and

$$\begin{aligned}
&m((x \otimes b \otimes x' \otimes b')(y \otimes c \otimes y' \otimes c')) \\
&= m(\phi(|x' \otimes b'|, |y \otimes c|)(x \otimes b)(y \otimes c) \otimes (x' \otimes b')(y' \otimes c')) \\
&= m(\phi(|x' \otimes b'|, |y \otimes c|)\phi(|b_2|, |y|)\phi(|b'_2|, |y'|)x(b_1 \cdot y) \otimes b_2c \otimes x'(b'_1 \cdot y') \otimes b'_2c') \\
&= \phi(|x' \otimes b'|, |y \otimes c|)\phi(|b_2|, |y|)\phi(|b'_2|, |y'|)x(b_1 \cdot y)\epsilon(b_2c)x'(b'_1 \cdot y') \otimes b'_2c' \\
&= \phi(|x' \otimes b'|, |y|)\phi(|b'_2|, |y'|)\phi(|c|, |y'|)x(b \cdot y)x'(b'_1 \cdot y') \otimes b'_2\epsilon(c)c' \\
&= \phi(|c|, |y'|)\phi(|b'_2|, |yy'|)\phi(|b'_1|, |y|)\phi(|x'|, |y|)x(b \cdot y)x'(b'_1 \cdot y') \otimes b'_2\epsilon(c)c' \\
&\overset{(15)}{=} \phi(|c|, |y'|)\phi(|b'_3|, |yy'|)\phi(|b'_2|, |y|)\phi(|x'_2|, |b'_1 \cdot y|)x(\epsilon(b)d(x'_1)b'_1 \cdot y)x'_2(b'_2 \cdot y') \otimes b'_3\epsilon(c)c',
\end{aligned}$$

so m is a morphism of algebras.

Moreover, for all $x \in X$ and $b \in B$, we have

$$\begin{aligned}
m(\mathrm{Id} \otimes \iota_2p_2)\Delta_{X \rtimes B}(x \otimes b) &= m(\phi(|x_2|, |b_1|)x_1 \otimes b_1 \otimes \iota_2p_2(x_2 \otimes b_2)) \\
&= m(\phi(|x_2|, |b_1|)x_1 \otimes b_1 \otimes \iota_2(\epsilon(x_2)b_2)) \\
&= m(x \otimes b_1 \otimes \iota_2(b_2)) = m(x \otimes b_1 \otimes 1_X \otimes b_2) \\
&= x\epsilon(b_1)1_X \otimes b_2 = x \otimes b
\end{aligned}$$

and also

$$\begin{aligned}
m(\iota_2p_1 \otimes \mathrm{Id})\Delta_{X \rtimes B}(x \otimes b) &= m(\phi(|x_2|, |b_1|)\iota_2p_1(x_1 \otimes b_1) \otimes x_2 \otimes b_2) \\
&= m(\phi(|x_2|, |b_1|)\iota_2(d(x_1)b_1) \otimes x_2 \otimes b_2) \\
&= m(\phi(|x_2|, |b_1|)1_X \otimes d(x_1)b_1 \otimes x_2 \otimes b_2) \\
&= \phi(|x_2|, |b_1|)1_X\epsilon(d(x_1)b_1)x_2 \otimes b_2 \\
&= \phi(|x_2|, |b_1|)\epsilon(x_1)\epsilon(b_1)x_2 \otimes b_2 = x \otimes b,
\end{aligned}$$

so (14) is a reflexive-multiplicative graph. Therefore, we have obtained a functor G :

$\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)) \longrightarrow \mathrm{RMG}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ defined as follows

$$\begin{tikzcd}
X \arrow[r, "d"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] & & X \rtimes B \arrow[r, shift left=2, "p_2"] \arrow[r, leftarrow, "\iota_2" description] \arrow[r, shift right=2, "p_1"'] \arrow[d, "\alpha\otimes\beta"'] & B \arrow[d, "\beta"] \\
X' \arrow[r, "d'"'] & B' & & X' \rtimes B' \arrow[r, shift left=2, "p_2"] \arrow[r, leftarrow, "\iota_2" description] \arrow[r, shift right=2, "p_1"'] & B'
\arrow[from=1-2, to=2-4, phantom, "\longmapsto"]
\end{tikzcd}$$

and the two functors F and G give rise to an equivalence of categories. Indeed, we obtain

$$\begin{tikzcd}
(X \arrow[r, "d"] & B) \arrow[r, mapsto, "G"] & (X \rtimes B \arrow[r, shift left=2, "p_2"] \arrow[r, leftarrow, "\iota_2" description] \arrow[r, shift right=2, "p_1"'] & B) \arrow[r, mapsto, "F"] & (\mathrm{Hker}(p_2) \arrow[r, "d'"] & B)
\end{tikzcd}$$

where $d' := p_1 \circ j$ and $j : \mathrm{Hker}(p_2) \longrightarrow X \rtimes B$ is the inclusion. We know that $\mathrm{Hker}(p_2) = X \otimes \Bbbk 1_B \cong X$ and so clearly we have an isomorphism of color Hopf crossed modules. Furthermore, we have

$$\begin{tikzcd}
(A_1 \arrow[r, shift left=2, "p"] \arrow[r, leftarrow, "i" description] \arrow[r, shift right=2, "\gamma"'] & A_0) \arrow[r, mapsto, "F"] & (\mathrm{Hker}(p) \arrow[r, mapsto, "\gamma\circ j'"] & A_0) \arrow[r, mapsto, "G"] & (\mathrm{Hker}(p) \rtimes A_0 \arrow[r, shift left=2, "p_2"] \arrow[r, leftarrow, "\iota_2" description] \arrow[r, shift right=2, "p_1"'] & A_0)
\end{tikzcd}$$

where $j' : \mathrm{Hker}(p) \longrightarrow A_1$ is the inclusion and $p_1(k \otimes a) = \gamma(k)a$ for all $k \in \mathrm{Hker}(p)$ and $a \in A_0$. The last reflexive-multiplicative graph is isomorphic to the starting one through the isomorphism $\phi : \mathrm{Hker}(p) \rtimes A_0 \longrightarrow A_1$, $k \otimes a \mapsto ki(a)$ of Proposition 5.4. Indeed, for all $k \in \mathrm{Hker}(p)$ and $a \in A_0$, we obtain

$$p\phi(k \otimes a) = p(ki(a)) = p(k)p(i(a)) = p(k)a = \epsilon(k)a = p_2(k \otimes a),$$

i.e., $p \circ \phi = p_2$. In addition $\phi \circ \iota_2 = i$ and

$$\gamma(\phi(k \otimes a)) = \gamma(ki(a)) = \gamma(k)\gamma(i(a)) = \gamma(k)a = p_1(k \otimes a),$$

i.e., $\gamma \circ \phi = p_1$. Then $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ and $\mathrm{RMG}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ are equivalent categories. Moreover, we know that $\mathrm{RMG}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ and $\mathrm{Grpd}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ are isomorphic since $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is Mal'tsev [15] and then also $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ and $\mathrm{Grpd}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ are equivalent. ∎

The previous result generalizes [24, Proposition 5.5]. Note also that it can be deduced from [4, Proposition 3.13], since a color Hopf algebra is precisely a Hopf monoid in the category Vec_G. Finally, we obtain the equivalence between $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ and $\mathrm{XMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$.

5.10. COROLLARY. *There is an equivalence of categories between* $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ *and* $\mathrm{XMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$.

PROOF. For any semi-abelian category $\mathcal{C}$, there is an equivalence between $\mathrm{Grpd}(\mathcal{C})$ and $\mathrm{XMod}(\mathcal{C})$, see [28], thus $\mathrm{XMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ is equivalent to $\mathrm{Grpd}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$. By Proposition 5.9, $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ is equivalent to $\mathrm{Grpd}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$, so the thesis follows. ∎

As a consequence of the previous results, we also obtain:

5.11. COROLLARY. *The categories* $\mathrm{XMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ *and* $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ *are semi-abelian.*

PROOF. By Proposition 5.9 and Corollary 5.10, the categories $\mathrm{XMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$, $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ and $\mathrm{Grpd}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ are all equivalent. Since $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is semi-abelian, $\mathrm{Grpd}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ is also semi-abelian (see [23] and [12, Lemma 4.1]) and so we are done. ■

5.12. REMARK. Since the category $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ is semi-abelian, one could study the category $\mathrm{XMod}(\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)))$ of internal crossed modules in the category of color Hopf crossed modules. Note that we have the following equivalences of categories

$$\mathrm{XMod}(\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))) \cong \mathrm{Grpd}(\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))) \cong \mathrm{Grpd}(\mathrm{Grpd}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)))$$

where the latter is the category of double internal groupoids in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. In [40, Definitions 2.3 and 2.4] the category of Hopf crossed squares $X^2(\mathrm{Hopf}_{\Bbbk,\mathrm{coc}})$ is introduced and in [40, Corollary 3.3] it is shown that this is equivalent to $\mathrm{XMod}(\mathrm{HXMod}(\mathrm{Hopf}_{\Bbbk,\mathrm{coc}}))$. It would be interesting to extend the previous result to $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ introducing the notion of color Hopf crossed squares. We leave this as a possible future project.

6. Simplicial color Hopf algebras and color Hopf crossed modules

In [20, Theorem 5.5] it is shown that the category of crossed modules of cocommutative Hopf algebras is equivalent to the category of simplicial cocommutative Hopf algebras with Moore complex of length one. We want to extend this result to the case of cocommutative color Hopf algebras. We start by giving the following two definitions which employ the notions given in the preliminary section.

6.1. DEFINITION. *A simplicial cocommutative color Hopf algebra* $\mathcal{H}$ *is a simplicial object in the category* $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$*. In other words, it is given by a collection of cocommutative color Hopf algebras* H_n*, with* $n \in \mathbb{N}$*, together with color Hopf algebra maps*

$$d_i^n : H_n \longrightarrow H_{n-1}, \text{ for } 0 \le i \le n \quad \text{and} \quad s_j^{n+1} : H_n \longrightarrow H_{n+1}, \text{ for } 0 \le j \le n$$

that satisfy the simplicial identities:

1) $d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n$ *if* $i < j$;

2) $s_i^{n+1} \circ s_j^n = s_{j+1}^{n+1} \circ s_i^n$ *if* $i \le j$;

3) $d_i^n \circ s_j^n = s_{j-1}^{n-1} \circ d_i^{n-1}$ *if* $i < j$, $d_j^n \circ s_j^n = d_{j+1}^n \circ s_j^n = \mathrm{Id}$, $d_i^n \circ s_j^n = s_j^{n-1} \circ d_{i-1}^{n-1}$ *if* $i > j+1$.

A simplicial cocommutative color Hopf algebra can be represented by the following diagram

$$\begin{tikzcd}
\mathcal{H}: \arrow[r, dotted, no head] & H_3 \arrow[r, shift left=6] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift right=6] & H_2 \arrow[l, shift left=8] \arrow[l, shift left=12] \arrow[l, shift left=16] \arrow[r, shift left=6, "d_2"] \arrow[r, shift left=2, "d_1" description] \arrow[r, shift right=2, "d_0" description] & H_1 \arrow[l, shift left=6, "s_0" description] \arrow[l, shift left=10, "s_1"] \arrow[r, shift left=4, "d_1"] \arrow[r, "d_0" description] & H_0 \arrow[l, shift left=4, "s_0"]
\end{tikzcd} \tag{16}$$

where we omit the upper indexes for the maps d_i and s_i.

6.2. Definition. *Given a simplicial cocommutative color Hopf algebra $\mathcal{H}$, the chain complex $(M(\mathcal{H})_\bullet, \partial_\bullet)$ is given by:*

1. *$M(\mathcal{H})_n = \{0\}$ for $n < 0$ and $M(\mathcal{H})_0 = H_0$,*
2. *$M(\mathcal{H})_n = \bigcap_{i=0}^{n-1} \mathrm{Hker}(d_i^n)$ for $n \geq 1$,*
3. *$\partial_n : M(\mathcal{H})_n \longrightarrow M(\mathcal{H})_{n-1}$ is the restriction of d_n^n to $M(\mathcal{H})_n$ for $n \geq 1$ (and the zero morphism for $n \leq 0$).*

This is called the Moore complex of $\mathcal{H}$. We say that it has length m if $M(\mathcal{H})_i$ is the zero object $\Bbbk$ for all $i > m$.

Note that given a simplicial cocommutative color Hopf algebra $\mathcal{H}$ as in (16), we have a splitting of morphisms $H_n \underset{s_i^n}{\overset{d_i^n}{\rightleftarrows}} H_{n-1}$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, for all $n \geq 1$, by 3) of the simplicial identities. Hence, from Proposition 5.4, we immediately obtain the following result:

6.3. Proposition. *In a simplicial cocommutative color Hopf algebra $\mathcal{H}$, there exists an action*

$$\cdot_i : H_{n-1} \otimes \mathrm{Hker}(d_i^n) \longrightarrow \mathrm{Hker}(d_i^n),\ a \otimes k \mapsto \phi(|a_2|, |k|) s_i^n(a_1) k s_i^n(S(a_2))$$

for all $0 \leq i \leq n-1$ and $n \geq 1$ which makes $\mathrm{Hker}(d_i^n)$ a cocommutative H_{n-1}-module color Hopf algebra. Moreover, we have an isomorphism of color Hopf algebras

$$H_n \cong \mathrm{Hker}(d_i^n) \rtimes_i H_{n-1}$$

for all $1 \leq n \in \mathbb{N}$ and $0 \leq i \leq n-1$.

6.4. Remark. Explicitly, in Proposition 6.3, the isomorphism is given by $f : \mathrm{Hker}(d_i^n) \rtimes_i H_{n-1} \longrightarrow H_n$, $k \otimes a \mapsto k s_i^n(a)$ with inverse given by $g : H_n \longrightarrow \mathrm{Hker}(d_i^n) \rtimes_i H_{n-1}$, $h \mapsto h_1 s_i^n d_i^n(S(h_2)) \otimes d_i^n(h_3)$, i.e., $g = (f_i^n \otimes d_i^n) \circ \Delta_{H_n}$ with $f_i^n : H_n \longrightarrow \mathrm{Hker}(d_i^n) \subseteq H_n$, $x \mapsto x_1 s_i^n d_i^n(S(x_2))$. Recall that $a \cdot_i k = \xi_{H_n}(s_i^n(a) \otimes k) = s_i^n(a) \triangleright k$. Note that, given $x \in H_n$ and

$$f_{i+1}^n f_i^n(x) = f_{i+1}^n(x_1 s_i^n d_i^n(S(x_2))) \overset{(*)}{=} x_1 s_i^n d_i^n(S(x_2)) s_{i+1}^n d_{i+1}^n(s_i^n d_i^n(x_3) S(x_4)),$$

where $(*)$ follows by cocommutativity, then

$$\begin{aligned} d_i^n(f_{i+1}^n f_i^n(x)) &= d_i^n(x_1) d_i^n s_i^n d_i^n(S(x_2)) d_i^n s_{i+1}^n d_{i+1}^n s_i^n d_i^n(x_3) d_i^n s_{i+1}^n d_{i+1}^n(S(x_4)) \\ &= d_i^n(x_1) d_i^n(S(x_2)) d_i^n s_{i+1}^n d_i^n(x_3) s_i^{n-1} d_i^{n-1} d_{i+1}^n(S(x_4)) \\ &= d_i^n(x_1 S(x_2)) s_i^{n-1} d_i^{n-1} d_i^n(x_3) s_i^{n-1} d_i^{n-1} d_i^n(S(x_4)) \\ &= \epsilon(x_1) s_i^{n-1} d_i^{n-1} d_i^n(x_2 S(x_3)) = \epsilon(x_1)\epsilon(x_2) 1_{H_{n-1}} = \epsilon(x) 1_{H_{n-1}}, \end{aligned}$$

and so $f_{i+1}^n f_i^n(x) \in \mathrm{Hker}(d_{i+1}^n) \cap \mathrm{Hker}(d_i^n)$. Hence $f^n := f_{n-1}^n \circ f_{n-2}^n \circ \cdots \circ f_1^n \circ f_0^n : H_n \longrightarrow \bigcap_{i=0}^{n-1} \mathrm{Hker}(d_i^n) = M(\mathcal{H})_n$, where we identify f_i^n with $j_i^n \circ f_i^n$ and $j_i^n : \mathrm{Hker}(d_i^n) \longrightarrow H_n$ is the canonical inclusion. In particular, if $n = 2$, $f^2 := f_1^2 \circ f_0^2 : H_2 \longrightarrow M(\mathcal{H})_2$, where $f_0^2 : H_2 \longrightarrow \mathrm{Hker}(d_0^2)$, $x \mapsto x_1 s_0^2 d_0^2(S(x_2))$ and $f_1^2 : H_2 \longrightarrow \mathrm{Hker}(d_1^2)$, $x \mapsto x_1 s_1^2 d_1^2(S(x_2))$. We will use the morphism f^2 in the following.

Note that, given a simplicial cocommutative color Hopf algebra $\mathcal{H}$ as in (16), we can obtain a new simplicial cocommutative color Hopf algebra by considering kernels, where the first three components are given by

$$\begin{tikzcd} \mathrm{Hker}(d_0^3) \subseteq H_3 \arrow[r, shift left=4, "d_3"] \arrow[r, shift left=2, "d_2" description] \arrow[r, "d_1" description] & \mathrm{Hker}(d_0^2) \subseteq H_2 \arrow[l, shift left=2, "s_1" description] \arrow[l, shift left=4, "s_2"'] \arrow[r, shift left=2, "d_2"] \arrow[r, "d_1" description] & \mathrm{Hker}(d_0^1) \subseteq H_1 \arrow[l, shift left=2, "s_1"] \end{tikzcd} \tag{17}$$

and one can repeat the same process taking kernels in (17) and obtaining a new simplicial cocommutative color Hopf algebra with first three components

$$\begin{tikzcd} \mathrm{Hker}(d_0^4) \cap \mathrm{Hker}(d_1^4) \subseteq H_4 \arrow[r, shift left=4, "d_4"] \arrow[r, shift left=2, "d_3" description] \arrow[r, "d_2" description] & \mathrm{Hker}(d_0^3) \cap \mathrm{Hker}(d_1^3) \subseteq H_3 \arrow[l, shift left=2, "s_2" description] \arrow[l, shift left=4, "s_3"'] \arrow[r, shift left=2, "d_3"] \arrow[r, "d_2" description] & \mathrm{Hker}(d_0^2) \cap \mathrm{Hker}(d_1^2) \subseteq H_2. \arrow[l, shift left=2, "s_2"] \end{tikzcd} \tag{18}$$

By applying Proposition 6.3 to (16) we obtain

$$H_1 \cong \mathrm{Hker}(d_0^1) \rtimes_0 H_0 = M(\mathcal{H})_1 \rtimes_0 M(\mathcal{H})_0$$

and

$$H_2 \cong \mathrm{Hker}(d_0^2) \rtimes_0 H_1 \cong \mathrm{Hker}(d_0^2) \rtimes_0 (M(\mathcal{H})_1 \rtimes_0 M(\mathcal{H})_0).$$

Moreover, if we apply Proposition 6.3 to (17) we further obtain

$$\begin{aligned} \mathrm{Hker}(d_0^2) &\cong \mathrm{Hker}(d_1^2)|_{\mathrm{Hker}(d_0^2)} \rtimes_1 \mathrm{Hker}(d_0^1) = (\mathrm{Hker}(d_1^2) \cap \mathrm{Hker}(d_0^2)) \rtimes_1 \mathrm{Hker}(d_0^1) \\ &= (M(\mathcal{H})_2 \rtimes_1 M(\mathcal{H})_1) \end{aligned}$$

and then

$$H_2 \cong (M(\mathcal{H})_2 \rtimes_1 M(\mathcal{H})_1) \rtimes_0 (M(\mathcal{H})_1 \rtimes_0 M(\mathcal{H})_0).$$

Let us also analyze the case of H_3, using (17) and (18). Since

$$\begin{aligned}\mathrm{Hker}(d_0^3)\cap\mathrm{Hker}(d_1^3) &\cong (\mathrm{Hker}(d_2^3)|_{(\mathrm{Hker}(d_0^3)\cap\mathrm{Hker}(d_1^3))})\rtimes_2(\mathrm{Hker}(d_0^2)\cap\mathrm{Hker}(d_1^2))\\ &=(\mathrm{Hker}(d_0^3)\cap\mathrm{Hker}(d_1^3)\cap\mathrm{Hker}(d_2^3))\rtimes_2(\mathrm{Hker}(d_0^2)\cap\mathrm{Hker}(d_1^2))\\ &=M(\mathcal{H})_3\rtimes_2 M(\mathcal{H})_2,\end{aligned}$$

we obtain

$$\begin{aligned}\mathrm{Hker}(d_0^3)&\cong(\mathrm{Hker}(d_1^3)|_{\mathrm{Hker}(d_0^3)})\rtimes_1\mathrm{Hker}(d_0^2)=(\mathrm{Hker}(d_0^3)\cap\mathrm{Hker}(d_1^3))\rtimes_1\mathrm{Hker}(d_0^2)\\ &\cong(M(\mathcal{H})_3\rtimes_2 M(\mathcal{H})_2)\rtimes_1(M(\mathcal{H})_2\rtimes_1 M(\mathcal{H})_1)\end{aligned}$$

and then $H_3\cong\mathrm{Hker}(d_0^3)\rtimes_0 H_2$ is isomorphic to

$$((M(\mathcal{H})_3\rtimes_2 M(\mathcal{H})_2)\rtimes_1(M(\mathcal{H})_2\rtimes_1 M(\mathcal{H})_1))\rtimes_0((M(\mathcal{H})_2\rtimes_1 M(\mathcal{H})_1)\rtimes_0(M(\mathcal{H})_1\rtimes_0 M(\mathcal{H})_0)).$$

Clearly, we can iterate this process arriving at a decomposition of H_n for all $n\geq 0$, analogously to [20, Theorem 3.18].

In order to show the equivalence between color Hopf crossed modules and simplicial cocommutative color Hopf algebras with Moore complex of length one, we prove some intermediate results.

6.5. LEMMA. *Given $x\in H_1$ and $y\in M(\mathcal{H})_1$ we obtain*

$$M(\mathcal{H})_2\ni f^2(s_0^2(x)\triangleright s_1^2(y))=\phi(|x_2|,|y_1|)(s_0^2(x_1)\triangleright s_1^2(y_1))S(s_1^2(x_2)\triangleright s_1^2(y_2))$$

PROOF. Given $x\in H_1$ and $y\in M(\mathcal{H})_1$, we compute

$$\begin{aligned}f_0^2(s_0^2(x)\triangleright s_1^2(y))&=(s_0^2(x)\triangleright s_1^2(y))_1 s_0^2d_0^2(S((s_0^2(x)\triangleright s_1^2(y))_2))\\ &\overset{(8)}{=}\phi(|x_2|,|y_1|)(s_0^2(x_1)\triangleright s_1^2(y_1))s_0^2d_0^2(S(s_0^2(x_2)\triangleright s_1^2(y_2)))\\ &=\phi(|x_2|,|y_1|)(s_0^2(x_1)\triangleright s_1^2(y_1))S(s_0^2d_0^2(s_0^2(x_2)\triangleright s_1^2(y_2)))\\ &=\phi(|x_2|,|y_1|)(s_0^2(x_1)\triangleright s_1^2(y_1))S(s_0^2d_0^2s_0^2(x_2)\triangleright s_0^2d_0^2s_1^2(y_2))\\ &=\phi(|x_2|,|y_1|)(s_0^2(x_1)\triangleright s_1^2(y_1))S(s_0^2(x_2)\triangleright s_0^2s_0^1d_0^1(y_2))\\ &\overset{(*)}{=}\phi(|x_2|,|y_1|)(s_0^2(x_1)\triangleright s_1^2(y_1))S(s_0^2(x_2)\triangleright\epsilon(y_2)1_{H_2})\\ &\overset{(6)}{=}\phi(|x_2|,|y_1|)(s_0^2(x_1)\triangleright s_1^2(y_1))(s_0^2(x_2)\triangleright\epsilon(y_2)1_{H_2})\\ &\overset{(7)}{=}s_0^2(x)\triangleright s_1^2(y)\end{aligned}$$

where $(*)$ follows since $y\in\mathrm{Hker}(d_0^1)$, i.e., $y_1\otimes d_0^1(y_2)=y_1\otimes\epsilon(y_2)1_{H_0}$ and then

$$s_1^2(y_1)\otimes s_0^2s_0^1d_0^1(y_2)=s_1^2(y_1)\otimes s_0^2s_0^1(\epsilon(y_2)1_{H_0})=s_1^2(y_1)\otimes\epsilon(y_2)1_{H_2}.$$

Hence we obtain

$$\begin{aligned} f^2(s_0^2(x) \triangleright s_1^2(y)) &= f_1^2 f_0^2(s_0^2(x) \triangleright s_1^2(y)) = f_1^2(s_0^2(x) \triangleright s_1^2(y)) \\ &= (s_0^2(x) \triangleright s_1^2(y))_1 s_1^2 d_1^2(S((s_0^2(x) \triangleright s_1^2(y))_2)) \\ &\overset{(8)}{=} \phi(|x_2|,|y_1|)(s_0^2(x_1) \triangleright s_1^2(y_1)) s_1^2 d_1^2(S(s_0^2(x_2) \triangleright s_1^2(y_2))) \\ &= \phi(|x_2|,|y_1|)(s_0^2(x_1) \triangleright s_1^2(y_1)) S(s_1^2 d_1^2 s_0^2(x_2) \triangleright s_1^2 d_1^2 s_1^2(y_2)) \\ &= \phi(|x_2|,|y_1|)(s_0^2(x_1) \triangleright s_1^2(y_1)) S(s_1^2(x_2) \triangleright s_1^2(y_2)) \end{aligned}$$

and then the thesis follows. ■

6.6. Proposition. *Let $\mathcal{H}$ be a simplicial cocommutative color Hopf algebra with Moore complex of length one. We obtain the color Hopf crossed module $(H_0, M(\mathcal{H})_1 = \mathrm{Hker}(d_0^1), \partial_1 : M(\mathcal{H})_1 \longrightarrow H_0)$, where $M(\mathcal{H})_1$ is a cocommutative H_0-module color Hopf algebra through*

$$\cdot : H_0 \otimes M(\mathcal{H})_1 \longrightarrow M(\mathcal{H})_1, \ k \otimes x \mapsto \phi(|k_2|,|x|) s_0^1(k_1) x s_0^1(S(k_2))$$

and $\partial_1 := d_1^1|_{M(\mathcal{H})_1} : M(\mathcal{H})_1 \longrightarrow H_0$.

Proof. By Proposition 6.3 we already know that $M(\mathcal{H})_1$ is a cocommutative H_0-module color Hopf algebra through the map $\cdot$ defined, so we only have to show that the morphism of color Hopf algebras ∂_1 satisfies (10) and (11). For all $k \in H_0$ and $x \in M(\mathcal{H})_1$ we compute

$$\begin{aligned} \partial_1(k \cdot x) &= d_1^1(\phi(|k_2|,|x|) s_0^1(k_1) x s_0^1(S(k_2))) = \phi(|k_2|,|x|) d_1^1 s_0^1(k_1) d_1^1(x) d_1^1 s_0^1(S(k_2)) \\ &= \phi(|k_2|,|x|) k_1 \partial_1(x) S(k_2) \end{aligned}$$

and then (10) is satisfied. Moreover, for all $x, y \in M(\mathcal{H})_1$, we have

$$\begin{aligned} \partial_1(x) \cdot y &= d_1^1(x) \cdot y = \phi(|x_2|,|y|) s_0^1 d_1^1(x_1) y s_0^1(S(d_1^1(x_2))) = \phi(|x_2|,|y|) d_2^2 s_0^2(x_1) y S(d_2^2 s_0^2(x_2)) \\ &= d_2^2 s_0^2(x) \triangleright y \end{aligned}$$

so that, in order to prove that also (11) holds, we have to show $d_2^2 s_0^2(x) \triangleright y = x \triangleright y$ for all $x, y \in M(\mathcal{H})_1$. Hence, using Lemma 6.5, we obtain

$$\begin{aligned} d_2^2 f^2(s_0^2(x) \triangleright s_1^2(y)) &= \phi(|x_2|,|y_1|) d_2^2(s_0^2(x_1) \triangleright s_1^2(y_1)) d_2^2 S(s_1^2(x_2) \triangleright s_1^2(y_2)) \\ &= \phi(|x_2|,|y_1|)(d_2^2 s_0^2(x_1) \triangleright d_2^2 s_1^2(y_1)) S(d_2^2 s_1^2(x_2) \triangleright d_2^2 s_1^2(y_2)) \\ &= \phi(|x_2|,|y_1|)(d_2^2 s_0^2(x_1) \triangleright y_1) S(x_2 \triangleright y_2) \\ &\overset{(6)}{=} \phi(|x_2|,|y_1|)(d_2^2 s_0^2(x_1) \triangleright y_1)(x_2 \triangleright S(y_2)). \end{aligned}$$

But we are assuming that the Moore complex of $\mathcal{H}$ has length one and then $M(\mathcal{H})_2 = \Bbbk 1_{H_2}$, so that $\partial_2 = d_2^2|_{M(\mathcal{H})_2} : M(\mathcal{H})_2 \longrightarrow M(\mathcal{H})_1$ is the zero morphism. Therefore, we obtain

$$\phi(|x_2|,|y_1|)(d_2^2 s_0^2(x_1) \triangleright y_1)(x_2 \triangleright S(y_2)) = \epsilon(x)\epsilon(y) 1_{H_1}.$$

Finally, we compute

$$\begin{aligned}
x \triangleright y &= \epsilon(x_1)x_2 \triangleright \epsilon(y_1)y_2 = \phi(|x_2|,|y_1|)\epsilon(x_1)\epsilon(y_1)(x_2 \triangleright y_2)\\
&= \phi(|x_3|,|y_1 \otimes y_2|)\phi(|x_2|,|y_1|)(d_2^2 s_0^2(x_1) \triangleright y_1)(x_2 \triangleright S(y_2))(x_3 \triangleright y_3)\\
&\overset{(7)}{=} \phi(|x_2|,|y_1|)(d_2^2 s_0^2(x_1) \triangleright y_1)(x_2 \triangleright \epsilon(y_2)1_{H_1})\\
&= \phi(|x_2|,|y|)(d_2^2 s_0^2(x_1) \triangleright y)(x_2 \triangleright 1_{H_1}) = \phi(|x_2|,|y|)(d_2^2 s_0^2(x_1) \triangleright y)\epsilon(x_2)\\
&= d_2^2 s_0^2(x) \triangleright y
\end{aligned}$$

and so also (11) is satisfied. ■

We can also obtain a simplicial cocommutative color Hopf algebra with Moore complex of length one starting from a color Hopf crossed module.

6.7. Remark. A 2-truncated simplicial cocommutative color Hopf algebra is given by

$$\begin{tikzcd}
H_2 \arrow[r, shift left=6, "d_2"] \arrow[r, shift left=3, "d_1" description] \arrow[r, "d_0" description] & H_1 \arrow[l, shift left=3, "s_0" description] \arrow[l, shift left=6, "s_1"'] \arrow[r, shift left=3, "d_1"] \arrow[r, "d_0" description] & H_0 \arrow[l, shift left=3, "s_0"]
\end{tikzcd}$$

where H_0, H_1 and H_2 are cocommutative color Hopf algebras and the maps d_i, s_i satisfy the simplicial identities. Recall that from it we can obtain a simplicial cocommutative color Hopf algebra by using the 2-coskeleton functor, as explained in the preliminary section.

6.8. Proposition. *Given a color Hopf crossed module* $(A, H, d : H \longrightarrow A)$ *we obtain a 2-truncated simplicial cocommutative color Hopf algebra*

$$\begin{tikzcd}
H \rtimes_* (H \rtimes A) \arrow[r, shift left=6, "d_2"] \arrow[r, shift left=3, "d_1" description] \arrow[r, "d_0" description] & H \rtimes A \arrow[l, shift left=3, "s_0" description] \arrow[l, shift left=6, "s_1"'] \arrow[r, shift left=3, "d_1"] \arrow[r, "d_0" description] & A \arrow[l, shift left=3, "s_0"]
\end{tikzcd} \tag{19}$$

and then, as said in Remark 6.7, a simplicial cocommutative color Hopf algebra. The latter has Moore complex of length one.

Proof. Since H is a cocommutative A-module color Hopf algebra, we can make the semidirect product $H \rtimes A$. Denote the A-action of H by $\cdot : A \otimes H \longrightarrow H$ and define $H_0 := A$ and $H_1 := H \rtimes A$. Moreover, define morphisms $d_0^1, d_1^1 : H_1 \longrightarrow H_0$ and $s_0^1 : H_0 \longrightarrow H_1$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ as

$$d_0^1(h \otimes a) := \epsilon(h)a,\ d_1^1(h \otimes a) := d(h)a,\ s_0^1(a) := 1_H \otimes a,$$

so that, clearly, $d_0^1 \circ s_0^1 = d_1^1 \circ s_0^1 = \mathrm{Id}_A$. Note that these morphisms are exactly those defined in (14). We can define a $(H \rtimes A)$-action on H as

$$* : (H \rtimes A) \otimes H \longrightarrow H,\ (h \otimes a) \otimes h' \mapsto (d(h)a) \cdot h',$$

which makes H a cocommutative $(H \rtimes A)$-module color Hopf algebra. Indeed, the morphism $*$ is an action since $(1_H \otimes 1_A) * h' = (d(1_H)1_A) \cdot h' = 1_A \cdot h' = h'$ and, since d is a morphism of algebras, we also have

$$\begin{aligned}
&((h' \otimes a')(h \otimes a)) * h'' \\
&= (\phi(|a'_2|, |h|)h'(a'_1 \cdot h) \otimes a'_2 a) * h'' = (d(\phi(|a'_2|, |h|)h'(a'_1 \cdot h))a'_2 a) \cdot h'' \\
&= (\phi(|a'_2|, |h|)d(h')d(a'_1 \cdot h)a'_2 a) \cdot h'' \overset{(10)}{=} (\phi(|a'_3|, |h|)\phi(|a'_2|, |h|)d(h')a'_1 d(h)S(a'_2)a'_3 a) \cdot h'' \\
&= (\phi(|a'_2|, |h|)d(h')a'_1 d(h)\epsilon(a'_2)a) \cdot h'' = (d(h')a'd(h)a) \cdot h'' = (d(h')a') \cdot ((d(h)a) \cdot h'') \\
&= (h' \otimes a') * ((h \otimes a) * h'').
\end{aligned}$$

Moreover, clearly $*$ makes H a cocommutative $(H \rtimes A)$-module color Hopf algebra since the compatibility conditions are satisfied by the action $\cdot : A \otimes H \longrightarrow H$.

Hence we can define $H_2 := H \rtimes_* (H \rtimes A)$ and morphisms $d^2_0, d^2_1, d^2_2 : H_2 \longrightarrow H_1$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ as

$$d^2_0(h \otimes h' \otimes a) := \epsilon(h)h' \otimes a,\ d^2_1(h \otimes h' \otimes a) := hh' \otimes a,\ d^2_2(h \otimes h' \otimes a) := h \otimes d(h')a$$

and morphisms $s^2_0, s^2_1 : H_1 \longrightarrow H_2$ as

$$s^2_0(h \otimes a) := 1_H \otimes h \otimes a,\ s^2_1(h \otimes a) := h \otimes 1_H \otimes a.$$

Observe that $d^2_1 = m_H \otimes \mathrm{Id}_A$, which is clearly a morphism of graded coalgebras since $H \rtimes_* (H \rtimes A)$ has the tensor product coalgebra structure, is also a morphism of algebras. Indeed, given $h \otimes h' \otimes a$ and $k \otimes k' \otimes a'$ in $H \rtimes_* (H \rtimes A)$, we have

$$\begin{aligned}
d^2_1((h \otimes h' \otimes a)(k \otimes k' \otimes a')) &= d^2_1(\phi(|(h' \otimes a)_2|, |k|)h((h' \otimes a)_1 * k) \otimes (h' \otimes a)_2(k' \otimes a')) \\
&= d^2_1(\phi(|h'_2 \otimes a_2|, |k|)\phi(|h'_2|, |a_1|)h((h'_1 \otimes a_1) * k) \otimes (h'_2 \otimes a_2)(k' \otimes a')) \\
&= d^2_1(\phi(|h'_2|, |a_1 \cdot k|)\phi(|a_2 \otimes a_3|, |k|)\phi(|a_3|, |k'|)h((d(h'_1)a_1) \cdot k) \otimes h'_2(a_2 \cdot k') \otimes a_3 a') \\
&= \phi(|h'_2|, |a_1 \cdot k|)\phi(|a_2 \otimes a_3|, |k|)\phi(|a_3|, |k'|)h(d(h'_1) \cdot (a_1 \cdot k))h'_2(a_2 \cdot k') \otimes a_3 a' \\
&\overset{(11)}{=} \phi(|h'_2 \otimes h'_3|, |a_1 \cdot k|)\phi(|a_2 \otimes a_3|, |k|)\phi(|a_3|, |k'|)hh'_1(a_1 \cdot k)S(h'_2)h'_3(a_2 \cdot k') \otimes a_3 a' \\
&= \phi(|h'_2|, |a_1 \cdot k|)\phi(|a_2 \otimes a_3|, |k|)\phi(|a_3|, |k'|)hh'_1(a_1 \cdot k)\epsilon(h'_2)(a_2 \cdot k') \otimes a_3 a' \\
&= \phi(|a_2|, |k|)\phi(|a_3|, |kk'|)hh'(a_1 \cdot k)(a_2 \cdot k') \otimes a_3 a' = \phi(|a_2|, |kk'|)hh'(a_1 \cdot kk') \otimes a_2 a' \\
&= (hh' \otimes a)(kk' \otimes a') = d^2_1(h \otimes h' \otimes a)d^2_1(k \otimes k' \otimes a')
\end{aligned}$$

and $d^2_2 = \mathrm{Id}_H \otimes d^1_1$ is a morphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. Moreover, the following relations

$$d^2_0 \circ s^2_0 = d^2_1 \circ s^2_0 = d^2_1 \circ s^2_1 = d^2_2 \circ s^2_1 = \mathrm{Id}_{H_1}$$

are trivially satisfied as well as $s^2_1 \circ s^1_0 = s^2_0 \circ s^1_0$. Furthermore, we have

$$d^2_0 s^2_1(h \otimes a) = \epsilon(h)1_H \otimes a = s^1_0 d^1_0(h \otimes a) \text{ and } d^2_2 s^2_0(h \otimes a) = 1_H \otimes d(h)a = s^1_0 d^1_1(h \otimes a)$$

for all $h \in H$ and $a \in A$. Finally, we have

$$d_0^1 d_1^2(h \otimes h' \otimes a) = \epsilon(hh')a = d_0^1 d_0^2(h \otimes h' \otimes a),\ d_0^1 d_2^2(h \otimes h' \otimes a) = \epsilon(h)d(h')a = d_1^1 d_0^2(h \otimes h' \otimes a)$$

and $d_1^1 d_2^2(h \otimes h' \otimes a) = d(hh')a = d_1^1 d_1^2(h \otimes h' \otimes a)$ for all $h, h' \in H$ and $a \in A$.

Therefore, we have obtained a 2-truncated simplicial cocommutative color Hopf algebra and then, as said in Remark 6.7, we obtain a simplicial cocommutative color Hopf algebra $\mathcal{H}$ with first three components H_0, H_1 and H_2 defined as before.

By definition $M(\mathcal{H})_0 := H_0 = A$, $M(\mathcal{H})_1 := \mathrm{Hker}(d_0^1)$ and $M(\mathcal{H})_2 := \mathrm{Hker}(d_0^2) \cap \mathrm{Hker}(d_1^2)$. Clearly, $\mathrm{Hker}(d_0^1) = H \otimes \Bbbk 1_A \cong H$ and we can show that $M(\mathcal{H})_2 = \Bbbk(1_H \otimes 1_H \otimes 1_A)$. As for $\mathrm{Hker}(d_0^1)$, it is easy to show that $\mathrm{Hker}(d_0^2) = H \otimes \Bbbk(1_H \otimes 1_A)$.

But now an element $h \otimes 1_H \otimes 1_A$ is in $\mathrm{Hker}(d_1^2)$ if

$$h_1 \otimes 1_H \otimes 1_A \otimes d_1^2(h_2 \otimes 1_H \otimes 1_A) = h \otimes 1_H \otimes 1_A \otimes 1_H \otimes 1_A,$$

i.e., $h_1 \otimes 1_H \otimes 1_A \otimes h_2 \otimes 1_A = h \otimes 1_H \otimes 1_A \otimes 1_H \otimes 1_A$ and then $h = \epsilon(h)1_H$. Thus $M(\mathcal{H})_2 = \Bbbk(1_H \otimes 1_H \otimes 1_A)$. Moreover, by Corollary 2.5, we know that $M(\mathcal{H})_i = \Bbbk 1$ for $i > 3$ and $M(\mathcal{H})_3 = \mathrm{Hker}(\partial_2)$ with $\partial_3 : \mathrm{Hker}(\partial_2) \longrightarrow M(\mathcal{H})_2$ given by the inclusion. Then $M(\mathcal{H})_3 \cong \Bbbk 1$ and so $\mathcal{H}$ has Moore complex with length one. ■

Note that, in the previous proof, we need a 2-truncated simplicial cocommutative color Hopf algebra because, if we stop at the 1-truncation level and we apply the 1-coskeleton functor, we obtain a simplicial cocommutative color Hopf algebra with Moore complex of length two and there is no cancellation.

Finally, denoting by $\mathrm{Simp}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))|_{l=1}$ the category of simplicial cocommutative color Hopf algebras with Moore complex of length one, we can show the aforementioned equivalence of categories:

6.9. Proposition. *There is an equivalence of categories between* $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ *and* $\mathrm{Simp}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))|_{l=1}$.

Proof. We get a functor $F : \mathrm{Simp}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))|_{l=1} \longrightarrow \mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ by Proposition 6.6, which is defined as follows

$$\begin{tikzcd}[column sep=large]
(\mathcal{H} : \cdots\cdots & H_3 \arrow[r, shift left=9] \arrow[r, shift left=6] \arrow[r, shift left=3] \arrow[r] & H_2 \arrow[l, shift left=3] \arrow[l, shift left=6] \arrow[l, shift left=9] \arrow[r, shift left=6, "d_2"] \arrow[r, shift left=3, "d_1" description] \arrow[r, "d_0" description] & H_1 \arrow[l, shift left=3, "s_0" description] \arrow[l, shift left=6, "s_1"] \arrow[r, shift left=3, "d_1"] \arrow[r, "d_0" description] & H_0) \arrow[l, shift left=3, "s_0"] & \mapsto (H_0, M(\mathcal{H})_1 = \mathrm{Hker}(d_0^1), \partial_1 : M(\mathcal{H})_1 \longrightarrow H_0)
\end{tikzcd}$$

and a functor $G : \mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)) \longrightarrow \mathrm{Simp}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))|_{l=1}$ by Proposition 6.8, which is defined as

$$\begin{tikzcd}[column sep=large]
(A, H, d : H \longrightarrow A) \mapsto (\mathcal{H} : \cdots\cdots & H \rtimes_* (H \rtimes A) \arrow[r, shift left=6, "d_2"] \arrow[r, shift left=3, "d_1" description] \arrow[r, "d_0" description] & H \rtimes A \arrow[l, shift left=3, "s_0" description] \arrow[l, shift left=6, "s_1"] \arrow[r, shift left=3, "d_1"] \arrow[r, "d_0" description] & A). \arrow[l, shift left=3, "s_0"]
\end{tikzcd}$$

The functors F and G form an equivalence of categories. Indeed, we have

$$\begin{tikzcd}[column sep=large]
(A,H,d:H\longrightarrow A) \overset{G}{\mapsto} (\mathcal{H}: \cdots\cdots & H \rtimes_* (H \rtimes A) \arrow[r, shift left=4, "d_2"] \arrow[r, shift left=2, "d_1" description] \arrow[r, "d_0" description] & H \rtimes A \arrow[l, shift left=2, "s_0" description] \arrow[l, shift left=4, "s_1"] \arrow[r, shift left=2, "d_1"] \arrow[r, "d_0" description] & A) \arrow[l, shift left=2, "s_0"]
\end{tikzcd}$$

$$\overset{F}{\mapsto} (A, M(\mathcal{H})_1 = \mathrm{Hker}(d_0^1), \partial_1 : M(\mathcal{H})_1 \longrightarrow A)$$

and, as noted in the proof of Proposition 6.8, we have $M(\mathcal{H})_1 = H \otimes \Bbbk 1_A \cong H$ and $\partial_1 = d_1^1|_{H\otimes \Bbbk 1_A} = d$, hence clearly $FG \cong \mathrm{Id}$. Moreover, we have

$$\begin{tikzcd}
(\mathcal{H}: \cdots & H_3 \arrow[r, shift left=6] \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] & H_2 \arrow[l, shift left=2] \arrow[l, shift left=4] \arrow[l, shift left=6] \arrow[r, shift left=4, "d_2"] \arrow[r, shift left=2, "d_1" description] \arrow[r, "d_0" description] & H_1 \arrow[l, shift left=2, "s_0" description] \arrow[l, shift left=4, "s_1"] \arrow[r, shift left=2, "d_1"] \arrow[r, "d_0" description] & H_0) \arrow[l, shift left=2, "s_0"] & \overset{F}{\mapsto} (H_0, M(\mathcal{H})_1 = \mathrm{Hker}(d_0^1), \partial_1 : M(\mathcal{H})_1 \longrightarrow H_0)
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large]
\overset{G}{\mapsto} (\mathcal{H}' : \cdots\cdots & (\mathrm{Hker}(d_0^1) \rtimes_* (\mathrm{Hker}(d_0^1) \rtimes H_0)) \arrow[r, shift left=4, "d'_2"] \arrow[r, shift left=2, "d'_1" description] \arrow[r, "d'_0" description] & (\mathrm{Hker}(d_0^1) \rtimes H_0) \arrow[l, shift left=2, "s'_0" description] \arrow[l, shift left=4, "s'_1"] \arrow[r, shift left=2, "d'_1"] \arrow[r, "d'_0" description] & H_0) \arrow[l, shift left=2, "s'_0"]
\end{tikzcd}$$

and both $\mathcal{H}'$ and $\mathcal{H}$ have Moore complex of length one, i.e., $M(\mathcal{H}')_i = M(\mathcal{H})_i = \Bbbk 1$ for all $i \geq 2$. Furthermore, $M(\mathcal{H}')_0 = H_0 = M(\mathcal{H})$ and $M(\mathcal{H}')_1 = \mathrm{Hker}(d_0'^1) = \mathrm{Hker}(d_0^1) \otimes \Bbbk 1_{H_0} \cong \mathrm{Hker}(d_0^1) = M(\mathcal{H})_1$. Hence $\mathcal{H}'$ and $\mathcal{H}$ are simplicial cocommutative color Hopf algebras which have isomorphic Moore complex, so they are isomorphic since the Moore functor $M : \mathrm{Simp}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)) \longrightarrow \mathrm{Ch}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ reflects isomorphisms (see [10]). ■

Note that, using Corollary 5.10, we also obtain that the categories $\mathrm{XMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ and $\mathrm{Simp}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))|_{l=1}$ are equivalent. This result can also be deduced from [5, Proposition 4.4]. Finally, we obtain:

6.10. Corollary. *The category* $\mathrm{Simp}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))|_{l=1}$ *is semi-abelian.*

Proof. By Corollary 5.11 both $\mathrm{XMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ and $\mathrm{HXMod}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ are semi-abelian, hence also $\mathrm{Simp}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))|_{l=1}$ is semi-abelian. ■

6.11. Remark. In [20, Definitions 6.5 and 6.6] the category of 2-crossed modules of cocommutative Hopf algebras is introduced and in [20, Theorem 6.17] it is shown that this category is equivalent to the category of simplicial cocommutative Hopf algebras with Moore complex of length 2. It would be interesting to extend this result to the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ introducing the notion of 2-color Hopf crossed module. We leave this as a possible future project.

Moreover, as it is said in [40, page 301], the notions of 2-crossed module and Hopf crossed square for the category of cocommutative Hopf algebras are not equivalent, hence it is reasonable to think that 2-color Hopf crossed modules and color Hopf crossed squares could be two not equivalent generalizations.

Acknowledgements. This paper was written while the author was member of the "National Group for Algebraic and Geometric Structures and their Applications" (GNSAGA-INdAM). He was partially supported by the project funded by the European Union - NextGenerationEU under NRRP, Mission 4 Component 2 CUP D53D23005960006 - Call PRIN 2022 No. 104 of February 2, 2022 of Italian Ministry of University and Research; Project 2022S97PMY "Structures for Quivers, Algebras and Representations" (SQUARE). The author would like to thank A. Ardizzoni and A. Cigoli for their careful reading of this paper and meaningful comments, and the referee for valuable suggestions.

References

M. Aguiar, S. Mahajan, *Monoidal functors, species and Hopf algebras.* With forewords by Kenneth Brown and Stephen Chase and André Joyal. CRM Monograph Series, 29. American Mathematical Society, Providence, RI, 2010.

Y. Bespalov, *Crossed modules and quantum groups in braided categories*, Appl. Categ. Structures 5 (1997), no. 2, 155–204.

Y. Bespalov, B. Drabant, *Hopf (bi-)modules and crossed modules in braided monoidal categories*, J. Pure Appl. Algebra 123 (1998), no. 1-3, 105–129.

G. Böhm, *Crossed modules of monoids II: Relative crossed modules*, Appl. Categ. Structures 28 (2020), no. 4, 601–653.

G. Böhm, *Crossed modules of monoids III: simplicial monoids of Moore length 1*, Appl. Categ. Structures (2021), 29(1):1–29.

F. Borceux, D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Mathematics and its Applications, 566. Kluwer Academic Publishers, Dordrecht, 2004.

F. Borceux, G. Janelidze, G. M. Kelly, *Internal object actions*, Comment. Math. Univ. Carol. 46 (2) (2005) 235–255.

D. Bourn, *Mal'cev categories and fibration of pointed objects*, Appl. Categ. Structures 4 (1996), no. 2-3, 307–327.

D. Bourn, *3 × 3 lemma and protomodularity*, J. Algebra 236 (2001), 778–795.

D. Bourn, *Moore normalization and Dold-Kan theorem for semi-abelian categories.* In "Categories in algebra, geometry and mathematical physics", 105–124. Contemp. Math., 431 American Mathematical Society, Providence, RI, 2007.

D. Bourn, M. Gran, *Centrality and normality in protomodular categories*, Theory Appl. Categ. 9 (8) (2002) 151–165.

D. Bourn, M. Gran, *Central extensions in semi-abelian categories*, J. Pure Appl. Algebra 175 (2002), no. 1-3, 31–44.

D. Bourn, G. Janelidze, *Protomodularity, descent and semi-direct product*, Theory Appl. Categories 4 (1998) 37–46.

D. Bourn, G. Janelidze, *Centralizers in action accessible categories*, Cah. Topol. Géom. Différ. Catég. 50 (2009) 211–232.

A. Carboni, M. C. Pedicchio, N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, in: Proc. Conference, Montreal, 1991, 1992, pp. 97–109.

A.S. Cigoli, *Centrality via internal actions and action accessibility via centralizers.* Ph.D. thesis, Università degli Studi di Milano, 2009.

A.S. Cigoli, J.R.A. Gray, T. Van der Linden, *On the normality of Higgins commutators*, J. Pure Appl. Algebra 219 (2015), 897-912.

A.S. Cigoli, J.R.A. Gray, T. Van der Linden, *Algebraically coherent categories*, Theory Appl. Categ. 30 (54) (2015) 1864–1905.

J. Duskin, *Simplicial methods and the interpretation of "triple" cohomology*, Mem. Am. Math. Soc., 163:135 (1975).

K. Emir, *The Moore complex of a simplicial cocommutative Hopf algebra*, Theory Appl. Categ. 37 (2021), Paper No. 7, 189–226.

T. Everaert, M. Gran, T. Van der Linden, *Higher Hopf formulae for homology via Galois theory*, Adv. Math. 217 (2008) 2231–2267.

T. Everaert, T. Van der Linden, *Baer invariants in semi-abelian categories II: Homology*, Theory Appl. Categ., 12:195–224 (2004).

M. Gran, *Central extensions and internal groupoids in Maltsev categories*, J. Pure Appl. Algebra 155 (2001), no. 2-3, 139–166.

M. Gran, F. Sterck, J. Vercruysse, *A semi-abelian extension of a theorem by Takeuchi*, J. Pure Appl. Algebra 223 (2019), no. 10, 4171–4190.

J.R.A. Gray, *Hall's criterion for nilpotence in semi-abelian categories*, Adv. Math. 349 (2019), 911–919.

I. Heckenberger, H-J. Schneider, *Hopf algebras and root systems*, Math. Surveys Monogr., 247 American Mathematical Society, Providence, RI, 2020.

S.A. Huq, *Commutator, nilpotency, and solvability in categories*, Q. J. Math. Oxf. Ser. (2) 19 (1968) 363–389.

G. Janelidze, *Internal crossed modules*, Georgian Math. J. 10 (1) (2003) 99–114.

G. Janelidze, L. Marki, W. Tholen, *Semi-abelian categories*, J. Pure. Appl. Alg. 168, 367-386, 2002.

G. Janelidze, L. Marki, W. Tholen, A. Ursini, *Ideal determined categories*, Cah. Topol. Géom. Différ. Catég. 51 (2010), no. 2, 115–125.

Z. Janelidze, *The pointed subobject functor, 3×3 lemmas, and subtractivity of spans*, Theory Appl. Categ. 23 (2010), No. 11, 221–242.

G. López Cafaggi, *Torsion theories of simplicial groups with truncated Moore complex*, Cah. Topol. Géom. Différ. Catég. 63 (2022), no. 4, 343–386.

S. Majid, *Strict quantum 2-groups*, preprint arXiv:1208.6265.

S. Mantovani, G. Metere, *Internal crossed modules and Peiffer condition*, Theory Appl. Categ. 23 (2010), No. 6, 113–135.

N. Martins-Ferreira, T. Van der Linden, *A note on the "Smith is Huq" condition*, Appl. Categ. Structures 20 (2012), no. 2, 175–187.

J. May, *Simplicial Objects in Algebraic Topology.* University of Chicago Press (1967).

R. Molnar, *Semi-direct products of Hopf algebras*, J. Algebra 47 (1977) 29–51.

O. Ngaha Ngaha, F. Sterck, *The Zassenhaus lemma in star-regular categories*, Theory Appl. Categ. 34 (2019), Paper No. 38, 1196–1219.

A. Sciandra, *Semi-abelian condition for color Hopf algebras*, J. Pure Appl. Algebra 228 (2024), no. 9, Paper No. 107677, 34 pp.

F. Sterck, *Crossed squares of cocommutative Hopf algebras*, J. Algebra 584 (2021), 270–303.

M. E. Sweedler, *Hopf algebras.* Mathematics Lecture Note Series W. A. Benjamin, Inc., New York 1969.

O. Wyler, *The Zassenhaus lemma for categories*, Arch. Math. (Basel) 22 (1971), 561–569.

University of Turin, Department of Mathematics "G. Peano", via Carlo Alberto
10, I-10123 Torino, Italy

Email: `andrea.sciandra@unito.it`

This article may be accessed at `http://www.tac.mta.ca/tac/`

THEORY AND APPLICATIONS OF CATEGORIES will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

Subscription information Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. Full text of the journal is freely available at `http://www.tac.mta.ca/tac/`.

Information for authors LaTeX2e is required. Articles may be submitted in PDF by email directly to a Transmitting Editor following the author instructions at
`http://www.tac.mta.ca/tac/authinfo.html`.

Managing editor. Geoff Cruttwell, Mount Allison University: `gcruttwell@mta.ca`

TeXnical editor. Nathanael Arkor, Tallinn University of Technology.

Assistant TeX editor. Gavin Seal, Ecole Polytechnique Fédérale de Lausanne:
`gavin_seal@fastmail.fm`

TeX editor emeritus. Michael Barr, McGill University: `michael.barr@mcgill.ca`

Transmitting editors.
Clemens Berger, Université Côte d'Azur: `clemens.berger@univ-cotedazur.fr`
Julie Bergner, University of Virginia: `jeb2md (at) virginia.edu`
John Bourke, Masaryk University: `bourkej@math.muni.cz`
Maria Manuel Clementino, Universidade de Coimbra: `mmc@mat.uc.pt`
Valeria de Paiva, Topos Institute: `valeria.depaiva@gmail.com`
Richard Garner, Macquarie University: `richard.garner@mq.edu.au`
Ezra Getzler, Northwestern University: `getzler (at) northwestern(dot)edu`
Rune Haugseng, Norwegian University of Science and Technology: `rune.haugseng@ntnu.no`
Dirk Hofmann, Universidade de Aveiro: `dirk@ua.pt`
Joachim Kock, Universitat Autònoma de Barcelona: `Joachim.Kock (at) uab.cat`
Stephen Lack, Macquarie University: `steve.lack@mq.edu.au`
Tom Leinster, University of Edinburgh: `Tom.Leinster@ed.ac.uk`
Sandra Mantovani, Università degli Studi di Milano: `sandra.mantovani@unimi.it`
Matias Menni, Conicet and Universidad Nacional de La Plata, Argentina: `matias.menni@gmail.com`
Giuseppe Metere, Università degli Studi di Palermo: `giuseppe.metere (at) unipa.it`
Kate Ponto, University of Kentucky: `kate.ponto (at) uky.edu`
Robert Rosebrugh, Mount Allison University: `rrosebrugh@mta.ca`
Jiri Rosický, Masaryk University: `rosicky@math.muni.cz`
Giuseppe Rosolini, Università di Genova: `rosolini@unige.it`
Michael Shulman, University of San Diego: `shulman@sandiego.edu`
Alex Simpson, University of Ljubljana: `Alex.Simpson@fmf.uni-lj.si`
James Stasheff, University of North Carolina: `jds@math.upenn.edu`
Tim Van der Linden, Université catholique de Louvain: `tim.vanderlinden@uclouvain.be`
Christina Vasilakopoulou, National Technical University of Athens: `cvasilak@math.ntua.gr`