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HYPERSUBTRACTIONS, SEMI-DIRECT PRODUCTS AND PROTOMODULAR CATEGORIES

DOMINIQUE BOURN

Abstract. In this article, we introduce an extrinsic approach to the notion of semi-direct product, an intrinsic one having been already done in [Bourn-Janelidze, 2003] and [Borceux-Janelidze-Kelly, 2005-1]. This led us to focus our attention on two algebraic structures, hyper-Słomínski setting and hypersubtraction which produce pointed protomodular categories, and will allow us to characterize this extrinsic approach.

1. Introduction

The notion of *semi-direct* product is well known for groups and associated with any group action of a group $(Y, *)$ on a group $(K, *)$. What is remarkable is that any split epimorphism $(f, s) : X \rightleftarrows Y$ in the category Gp of groups is obtained, up to isomorphism, by such a semi-direct product. The first attemp to understand this heavy structural fact intrinsically (namely inside the category Gp itself) was made in [Bourn-Janelidze, 2003]. Then, it was developed further in [Borceux-Janelidze-Kelly, 2005-1] and completed in [Borceux-Janelidze-Kelly, 2005-2] with the notion of *split extension classifier*, see further details in [Borceux-Bourn, 2007]. This also led, indirectly, to the remarkable observations on the category Gp produced in [JRA. Gray, 2012] with the notion of *algebraic exponentiation*. Here we try to understand this notion extrinsincally, namely through the inspection of the properties of the forgetful functor $V : \mathsf{Gp} \longrightarrow \mathbf{Set}_*$ from groups to pointed sets. We shall investigate the notion of *semi-direct index* which insures that any split epimorphim (f, s) in Gp is sent, up to isomorphism, on the canonical split epimorphism $(p_{V(Y)}, \iota_{V(Y)}) : V(Y) \times V(\mathrm{Ker} f) \rightleftarrows V(Y)$. Then we shall show that, for any left exact conservative functor $U : \mathbb{C} \longrightarrow \mathbb{D}$ between pointed categories and without any condition on $\mathbb{D}$, the existence of a semi-direct index implies that $\mathbb{C}$ is necessarily a protomodular category. For all that, we shall be led to focus our attention on two algebraic structures: hyper-Słomínski setting and hypersubtraction. Doing this, we shall be drawn into the exploration of rather unusual pointed protomodular contexts.

This article is organized along the following lines: Section 1 investigates these two structures; it contains also some considerations about internal quandles and introduces

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the notion of prequandle. Section 2 is devoted to a brief review about protomodular categories and partial Mal'tsev categories, and Section 3 to the characterization of the existence of the semi-direct indexes and hyperindexes.

In this article, we shall suppose that any category $\mathbb{E}$ is finitely complete and we shall denote by $\mathsf{Alex}\mathbb{E}$ the category whose objects are the pairs (X, f) of an object X in $\mathbb{E}$ and an isomorphism $f : X \longrightarrow X$ and whose morphisms are those maps $h : X \longrightarrow Y$ in $\mathbb{E}$ which commute with these isomorphisms. The symbol Alex will be justified in Section 2.25. Finally, recall that a finitely complete category $\mathbb{A}$ is additive if and only if any object X is endowed with a natural internal abelian group structure.

2. ILO settings, hyper-Słomínski settings and prequandles

In the section 4 concerning the notion of semi-direct product, we shall be interested in the isomorphisms ρ making commute the following diagram of split epimorphisms in a *pointed* category $\mathbb{D}$:

$$\begin{tikzcd}
X \times X \arrow[r, "\rho"] \arrow[d, shift right, "p_1^X"'] & X \times X \arrow[d, shift right, "p_0^X"'] \\
X \arrow[r, "="'] \arrow[u, shift right, "s_0^X"'] & X \arrow[u, shift right, "\iota_0^X"']
\end{tikzcd}$$

where $p_0^X, p_1^X, s_0^X, \iota_0^X$ are respectively the two projections, the diagonal and $(Id_X, 0_X)$.

2.1. ILO SETTINGS. Let us begin, in any category $\mathbb{D}$, with the isomorphisms ρ as on the left hand side:

$$\begin{tikzcd}
X \times X \arrow[r, "\rho"] \arrow[d, "p_1^X"'] & X \times X \arrow[d, "p_0^X"'] \\
X \arrow[r, "="'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times X \arrow[r, "d"] \arrow[d, "p_1^X"'] & X \arrow[d] \\
X \arrow[r] & 1
\end{tikzcd}$$

or, equivalently, with the data of a binary operation $d : X \times X \longrightarrow X$ making the right hand side diagram a pullback. The inverse ρ^{-1} produces another binary operation $\circ : X \times X \longrightarrow X$ such that: i) $d(x \circ z, x) = z$ and ii) $x \circ d(z, x) = z$. These equations mean that $x \circ -$ is the inverse of $d(-, x)$. We then get: i)+ii) $\iff$ iii): $x \circ y = t \iff y = d(t, x)$.

2.2. DEFINITION. *An internal* ILO *setting in a category* $\mathbb{D}$ *is a pair* (X, d) *of an object* X *and a binary operation* d *such that* $(p_1^X, d) : X \times X \longrightarrow X \times X$ *is an isomorphism. It is equivalent to a triple* $(X, d, \circ)$ *where* $(d, \circ)$ *is a pair of binary operations on* X *satisfying axioms* i) *and* ii)*.*

ILO is an acronyme for *invertible left hand sided operation.* Given any ILO setting (X, d), we shall call $\circ$ the *adjoint* binary operation associated with d. The map ρ being an isomorphism, a morphism $h : (X, d) \longrightarrow (Y, d)$ of ILO settings is, indifferently, a d-homomorphism or a $\circ$-homomorphism.

Examples

1) The ILO setting produced by the twisting isomorphism $\mathsf{tw}(x, y) = (y, x)$ is such that

$d(x, y) = x$ is the first projection, while $x \circ y = y$ is the second one.
2) Any group $(G, *)$ produces an ILO setting in **Set** with $d^*(x, y) = y^{-1} * x$ and the binary operation $*$.
3) The dual of an ILO setting defined by $(X, d, \circ)^{op} = (X, \circ^{op}, d^{op})$ is an ILO setting.
We shall denote by $\mathsf{ILO}\mathbb{E}$ the category of internal ILO settings in $\mathbb{E}$.

2.3. Definition. *An internal* ILO *setting* (X, d) *is called involutive when it is equal to its dual, namely when* $x \circ y = d(y, x)$*. It is called symmetric when* $d(x, y) = d(y, x)$*. It is called latin when* (X, d^{op}) *is an* ILO *setting as well, namely when* $(x, y) \mapsto (y, d(y, x))$ *produces an isomorphism* $X \times X \longrightarrow X \times X$*.*

The terminology *latin* comes from the fact that the table of the binary operation d of a latin ILO setting determines a latin square. Any symmetric ILO setting is a latin one. The ILO setting associated with a group $(X, \circ)$ is necessarily latin; it is involutive if and only if $x^2 = 1$; it is symmetric under the same characterization. We shall distinguish two main classes of ILO setting:
1) the class A with the further condition $d(x, x) = 1$ (which is equivalent to $x \circ 1 = x$) as expressed by the commutativity of our first diagram. This class A includes the ILO structure associated with a group;
2) the class B with the further condition $d(x, x) = x$ (which is equivalent to $x \circ x = x$) to which belongs the example 1.
Clearly an ILO setting $(X, d, 1)$ belonging to $A \cap B$ is reduced to the singleton 1. As we shall see in the next sections the structures of their respective categories are quite different.

2.4. Proposition. *Let* $(X, d, \circ)$ *be an* ILO *setting. The following conditions are equivalent:*

$$1)\ \text{the law } \circ \text{ is associative}\ ; \qquad 2)\ d(y, z) \circ d(x, y) = d(x, z)\ ;$$
$$3)\ d(d(x, z), d(y, z)) = d(x, y); \qquad 4)\ d(x, y) \circ t = d(x \circ t, y).$$

Proof. Suppose 1). Then $z \circ (d(y, z) \circ d(x, y)) = (z \circ d(y, z)) \circ d(x, y) = y \circ d(x, y) = x = z \circ d(x, z)$. Whence 2). Suppose 2). Then $d(y, z) \circ d(d(x, z) \circ d(y, z))) = d(x, z)$, while $d(y, z) \circ d(x, y) = d(x, z)$ whence 3). We get 4) if and only if:
$d(d(x \circ t, y), d(x, y)) = t$. From 3) we have $d(d(x \circ t, y), d(x, y)) = d(x \circ t, x) = t$.
We have 1) if and only if $y \circ z = d((x \circ y) \circ z, x)$. From 4) we get: $d((x \circ y) \circ z, x) = d(x \circ y, x) \circ z = y \circ z$. ■

2.5. Proposition. *Let* $(X, d, \circ)$ *be an* ILO *setting. The following conditions are equivalent:* 1) *the law* $\circ$ *is commutative;* 2) $x = d(x \circ y, y)$*;* 3) $d(y, d(y, x)) = x$*.*

Proof. By iii), we have 1), namely $y \circ x = x \circ y \iff$ 2). We have $d(y, d(y, x)) = x \iff d(y, x) \circ x = y = x \circ d(y, x)$. It is equivalent to commutativity of $\circ$, setting $t = d(y, x)$. ■

2.6. Autonomous ILO settings.

2.7. Definition. *An* ILO *setting* (X, d) *in* $\mathbb{E}$ *is said autonomous when:*
$d(d(x, x'), d(y, y')) = d(d(x, y), d(x', y'))$.

We shall denote by AILO the category of autonomous ILO settings. Let us now introduce the following relation $\mathbb{H}_X$ between pairs $(*, \circ)$ of binary operations on an object X: $*\mathbb{H}_X\circ \iff$ "$\circ$ is a $*$-homomorphism", namely $(x \circ y) * (x' \circ y') = (x * x') \circ (y * y')$. The relation $\mathbb{H}_X$ is symmetric and such that: $\circ\mathbb{H}_X* \iff \circ^{op}\mathbb{H}_X*$. Whence the following observation:

2.8. Proposition. *Let* (X, d) *be any autonomous* ILO *setting. From* $d\mathbb{H}_X d$*, we get* $d\mathbb{H}_X\circ$, $\circ\mathbb{H}_X d$ *and then* $\circ\mathbb{H}_X\circ$. *Accordingly, the full inclusion* $\mathsf{AILO}\mathbb{E} \hookrightarrow \mathsf{ILO}\mathbb{E}$*: is factorized through the left exact conservative forgetful functor* U*:*

$$\begin{tikzcd}
 & \mathsf{ILO}(\mathsf{ILO}\mathbb{E}) \arrow[rd, "U"] & \\
\mathsf{AILO}\mathbb{E} \arrow[ru, tail] \arrow[rr, tail] & & \mathsf{ILO}\mathbb{E}
\end{tikzcd}$$

So, $\mathsf{AILO}\mathbb{E}$ *is a full subcategory of* $\mathsf{ILO}(\mathsf{ILO}\mathbb{E})$ *which is stable under subobject and under pullback.*

Proof. The first point is a consequence of the fact that any morphism f of ILO settings is equivalently a d-homomorphism or a $\circ$-homomorphism. The second one follows immediately. ■

2.9. Class A: hyper-Słomínski settings. Now let us come back to our starting point with the pointed context:

$$\begin{tikzcd}
X \times X \arrow[r, "\rho"] \arrow[d, shift right, "p_1^X"'] & X \times X \arrow[d, shift right, "p_0^X"'] & X \times X \arrow[r, "d"] \arrow[d, shift right, "p_1^X"'] & X \arrow[d, shift right] \\
X \arrow[u, shift right, "s_0^X"'] \arrow[r, "="'] & X \arrow[u, shift right, "\iota_0^X"'] & X \arrow[u, shift right, "s_0^X"'] \arrow[r] & 1 \arrow[u, shift right]
\end{tikzcd}$$

namely with an isomorphism ρ of split epimorphisms or with a pullback of split epimorphims as on the right hand side. This adds to i) and ii) the axiom iv) $d(x, x) = 1$. As term identities of a variety, axioms iv) and ii) were considered in [Słomínski, 1960]. According to [Bourn-Janelidze, 2003], they are the characterization, in the case $n = 1$, of a pointed protomodular variety. Whence the following

2.10. Definition. *An internal Słomínski setting in a category* $\mathbb{E}$ *is a quadruple* $(X, d, \circ, 1)$ *of an object* X*, a constant* 1 *and two binary operations satisfying* iv) $d(x, x) = 1$ *and* ii) $x \circ d(z, x) = z$. *A hyper-Słomínski setting is the same data satisfying axioms* iv), ii) *and* i) $d(x \circ z, z) = x$ *or, equivalently, it is an* ILO *setting* $(X, d, 1)$ *satisfying* $d(x, x) = 1$.

Accordingly, the ILO settings belonging to the class A coincide with the hyper-Słomínski settings. The ILO setting associated with a group is a hyper-Słomínski setting.

When $\mathbb{E}$ is finitely complete, we shall denote by $\mathsf{Slom}\mathbb{E}$ and $\mathsf{\Sigma lom}\mathbb{E}$ the categories of internal Słomínski and hyper-Słomínski settings. They are both protomodular categories, see Section 3.2.

2.11. Proposition. *Given any Słomińsky setting* $(X, d, \circ, 1)$*, we get: 1)* $x \circ 1 = x$ *(the element* 1 *is a right unit); 2)* $1 \circ d(x, 1) = x$*; 3)* $x \circ d(1, x) = 1$ *(*x *has an inverse on the right hand side); and 4)* $x = y \iff d(x, y) = 1$*.*

Proof. Only the last point needs checking: $y = y \circ 1 = y \circ d(x, y) = x$. ■

We get examples of internal Słomínski and hyper-Słomínski settings with:

2.12. Proposition. *Let* $\mathbb{A}$ *be a finitely complete additive category. Then any internal Słomínski setting* $(X, d, \circ)$ *in* $\mathbb{A}$ *is determined by a triple* (X, f, g) *where* (f, g) *is a pair of* X*-endomorphims such that* $g.f = Id_X$*. With* $(X, d, \circ)$*, the triple is* $(X, d(-, 0), 0 \circ -)$*. Conversely, starting with* (X, f, g)*, we get* $(X, d, \circ)$ *with* $d(x, y) = f(x - y)$ *and* $x \circ y = x + g(y)$*.*

A Słomínski setting in $\mathbb{A}$ *is an hyper-Słomínski setting if and only if* f *is invertible and* $g = f^{-1}$*. The functor* $\Sigma_{\mathbb{A}} : \mathsf{Alex}\mathbb{A} \longrightarrow \mathsf{\Sigma lom}\mathbb{A}$ *defined by* $\Sigma_{\mathbb{A}}(X, f) = (X, d_f)$*, with* $d_f(a, b) = f(a - b)$*, is an isomorphism and makes* $\mathsf{\Sigma lom}\mathbb{A}$ *additive as well. Since* $d(0, x) = -f(x)$ *is an isomorphism, any internal hyper-Słomínski setting in* $\mathbb{A}$ *is a latin one.*

Proof. For any map $d : X \times X \longrightarrow X$ in $\mathbb{A}$ satisfying $d(x, x) = 0$, we get:
$d(x, y) = d((x - y, 0) + (y, y)) = d(x - y, 0) + d(y, y) = d(x - y, 0)$. Setting $d(x, 0) = f(x)$, we get $d(x, y) = f(x - y)$ and $d(0, y) = -f(y)$.

Setting $0 \circ x = g(x)$, from $x \circ y = (x + 0) \circ (0 + y) = (x \circ 0) + (0 \circ y)$, we get $x \circ y = x + g(y)$. Now, from $x \circ d(y, x) = x$, we get $x + gf(y - x) = y$; whence $g.f = Id_X$.

Since any additive category is protomodular, (X, d) is a hyper-Słomínski setting if and only if $d(-, 0) = f$ is an isomorphism, see Proposition 3.5. Accordingly we get an isomorphism: $\Sigma_{\mathbb{A}} : \mathsf{Alex}\mathbb{A} \longrightarrow \mathsf{\Sigma lom}\mathbb{A}$ being defined by $\Sigma_{\mathbb{A}}(X, f) = (X, d_f)$ with $d_f(x, y) = f(x - y)$ whose inverse $\Sigma^{-1} : \mathsf{\Sigma lom}\mathbb{A} \longrightarrow \mathsf{Alex}\mathbb{A}$ is defined by $\Sigma(X, d) = (X, d(-, 0))$. ■

2.13. Abelian Słomínski and hyper-Słomínski settings. We drew up a string of inclusions: $\mathsf{Gp}\mathbb{E} \subset \mathsf{\Sigma lom}\mathbb{E} \subset \mathsf{Slom}\mathbb{E}$ of protomodular categories. As such, each of them determines a subcategory of abelian objects, see Theorem 3.4, and consequently another string of inclusions: $\mathsf{Ab}\mathbb{E} \subset \mathsf{Ab}(\mathsf{\Sigma lom}\mathbb{E}) \subset \mathsf{Ab}(\mathsf{Slom}\mathbb{E})$ which we are now going to investigate.

2.14. Proposition. *An internal Słomínski setting* $(X, d, \circ, 1)$ *is abelian in* $\mathsf{Slom}\mathbb{E}$ *if and only if the internal magma object* $(X, *)$ *in* $\mathbb{E}$ *defined by* $x * y = x \circ d(y, 1)$ *is internal in* $\mathsf{Slom}\mathbb{E}$*. This characterizes the abelian objects in* $\mathsf{Slom}\mathbb{E}$ *and* $\mathsf{\Sigma lom}\mathbb{E}$*.*

Proof. By Theorem 3.4, there is at most one abelian group structure $((X, d), +)$ on (X, d) in $\mathsf{Slom}\mathbb{E}$. Take such an abelian object. Now, consider the binary operation $*$ in question. It is a unitary magma: $1 * y = 1 \circ d(y, 1) = y$ and $x * 1 = x \circ d(1, 1) = x \circ 1 = x$. Since $d\mathbb{H}_X+$ and $\circ\mathbb{H}_X+$, this binary operation is a $+$-homomorphism. Accordingly, we get

$*\mathbb{H}_X+$. The Ekman-Hilton argument [Eckmann-Hilton, 1962] applied to the pair $(+,*)$ in $\mathbb{E}$ implies $+=*$. Since $+$ is internal to $\mathsf{Slom}\mathbb{E}$, so has to be $*$. ∎

In addition, we have the following observation:

2.15. PROPOSITION. *We get the following isomorphisms:*
1) $\overrightarrow{\mathrm{Tw}}:\mathsf{Ab}(\mathsf{Slom}\mathbb{E})\simeq\mathsf{Slom}(\mathsf{Ab}\mathbb{E})$ *and 2)* $\overrightarrow{\mathrm{Tw}}:\mathsf{Ab}(\Sigma\mathsf{lom}\mathbb{E})\simeq\Sigma\mathsf{lom}(\mathsf{Ab}\mathbb{E})$
which make additive the two right hand side categories.

PROOF. According to the symmetry of the relation $\mathbb{H}_X$ (see Section 2.6), define: $\overrightarrow{\mathrm{Tw}}((X,d,\circ,1),+)=((X,+),d,\circ,1)$, a functor which preserves products. Symmetry being an involutive process, the formula $\overleftarrow{\mathrm{Tw}}((X,+),d,\circ,1)=((X,d,\circ,1),+)$ defines the inverse functor. ∎

2.16. AUTONOMOUS HYPER-SŁOMÍNSKI SETTINGS. We shall denote $\mathsf{A}\Sigma\mathsf{lom}$ the category of autonomous hyper-Słomínski settings.

2.17. PROPOSITION. *Suppose* $\mathbb{A}$ *additive. The isomorphism* $\Sigma_{\mathbb{A}}:\mathsf{Alex}\mathbb{A}\longrightarrow\Sigma\mathsf{lom}\mathbb{A}$ *actually takes its values in* $\mathsf{A}\Sigma\mathsf{lom}\mathbb{A}$*. Accordingly, we get* $\mathsf{A}\Sigma\mathsf{lom}\mathbb{A}=\Sigma\mathsf{lom}\mathbb{A}$*, and the isomorphism* $\Sigma_{\mathbb{A}}:\mathsf{Alex}\mathbb{A}\longrightarrow\mathsf{A}\Sigma\mathsf{lom}\mathbb{A}$.

PROOF. We have to check that d_f is autonomous:
$d_f(d_f(a,b),d_f(a',b'))=d_f(f(a)-f(b),f(a')-f(b'))$
$=f(f(a)-f(b))-f(f(a')-f(b'))=f^2(a)-f^2(b)-f^2(a')+f^2(b')$, while:
$d_f(d_f(a,a'),d_f(b,b'))=d_f(f(a)-f(a'),f(b)-f(b'))$
$=f(f(a)-f(a'))-f(f(b)-f(b')=f^2(a)-f^2(a')-f^2(b)+f^2(b')$. ∎

2.18. PROPOSITION. *The autonomous hyper-Słomińsky settings* $(X,d,1)$ *coincide with the Słomińsky settings* $(X,d,\circ,1)$ *satisfying* $d\mathbb{H}_X\circ$.

PROOF. By Section 2.6, for any hyper-Słomińsky setting, $d\mathbb{H}_X\circ$ and $d\mathbb{H}_Xd$ are equivalent. Conversely, let $(X,d,\circ,1)$ be a Słomińsky setting satisfying $d\mathbb{H}_X\circ$, namely $d(x\circ y,x'\circ y')=d(x,x')\circ d(y,y')$. We get $d(x\circ y,x)=d(x\circ y,x\circ 1)=d(x,x)\circ d(y,1)=1\circ d(y,1)=y$. So, $(X,d,1)$ is an hyper-Słomińsky setting. Now, from $d\mathbb{H}_X\circ$, we get $d\mathbb{H}_Xd$. ∎

2.19. PROPOSITION. *Any autonomous hyper-Słomińsky setting* $(X,d,1)$ *is abelian in* $\Sigma\mathsf{lom}\mathbb{E}$*. So, we get an inclusion* $\mathsf{A}\Sigma\mathsf{lom}\mathbb{E}\subset\mathsf{Ab}\Sigma\mathsf{lom}\mathbb{E}$ *which makes the category* $\mathsf{A}\Sigma\mathsf{lom}\mathbb{E}$ *additive. It is a strict inclusion since an abelian object* $(X,d,1)$ *belongs to* $\mathsf{A}\Sigma\mathsf{lom}\mathbb{E}$ *if and only if* $d(-,1)$ *is a* d*-homomorphism.*

PROOF. Let (X,d) be an autonomous hyper-Słomínski setting in $\mathbb{E}$. Again, consider the binary operation $x*y=x\circ d(y,1)$. Since d is autonomous, this unitary magma operation is internal in $\Sigma\mathsf{lom}\mathbb{E}$. Since this category is protomodular, it is underlying an abelain group structure. We check $x*1=x\circ d(1,1)=x\circ 1=x$, and $1*y=1\circ d(y,1)=y$. So, (X,d) is abelian in $\Sigma\mathsf{lom}\mathbb{E}$. And it is clear that $d(-,1)$ is a d-homomorphism.

Conversely, suppose (X, d) abelian in $\mathsf{\Sigma lom}\mathbb{E}$. We know that $x + y = x \circ d(y, 1)$ is a d-homomorphism or, equivalently a $\circ$-homomorphism, namely such that:

$$(x \circ d(y, 1)) \circ (x' \circ d(y', 1)) = (x \circ x') \circ d(y \circ y', 1)$$

Setting $d(y, 1) = t$ and $d(y', 1) = t'$, we get:

$$(x \circ t) \circ (x' \circ t') = (x \circ x') \circ d((1 \circ t) \circ (1 \circ t'), 1)$$

The operation d is autonomous, if and only if $\circ$ is autonomous, namely when:

$$(x \circ t) \circ (x' \circ t') = (x \circ x') \circ (t \circ t')$$

Now, d is autonomous if and only if

$$(x \circ x') \circ (t \circ t') = (x \circ x') \circ d((1 \circ t) \circ (1 \circ t'), 1)$$

namely if and only $d((1 \circ t) \circ (1 \circ t'), 1) = t \circ t'$ or, equivalently $1 \circ (y \circ y') = (1 \circ y) \circ (1 \circ y')$. Now $1 \circ -$ is a $\circ$-homomorphism if and only if its inverse $d(-, 1)$ is a $\circ$-homomorphism or, equivalently, a d-homomorphism. ■

2.20. Hypersubtractions. We shall be interested as well by the following more constraining commutations on the left hand side:

$$\begin{tikzcd}
X \arrow[r, shift left, "s_0^X"] \arrow[d, "="'] & X \times X \arrow[l, shift left, "p_1^X"] \arrow[r, shift right, "d"'] \arrow[d, "\rho"'] & X \arrow[l, shift right, "\iota_0^X"'] \arrow[d, "="'] \\
X \arrow[r, shift left, "\iota_0^X"] & X \times X \arrow[l, shift left, "p_0^X"] \arrow[r, shift right, "p_1^X"'] & X \arrow[l, shift right, "\iota_1^X"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times X \arrow[r, shift right, "d"'] \arrow[d, shift right, "p_1^X"'] & X \arrow[l, shift right, "\iota_0^X"'] \arrow[d, shift right] \\
X \arrow[u, shift right, "s_0^X"'] \arrow[r, shift right] & 1 \arrow[l, shift right] \arrow[u, shift right]
\end{tikzcd}$$

which is equivalent to the fact that the right hand side square is a pullback of split epimorphisms. The axiom added to a hyper-Słomínski setting is $\rho.\iota_0^X = \iota_1^X$, namely v) $d(x, 1) = x$. Recall that a *subtraction* in the sense of [Ursini, 1994] is binary operation d satisfying iv) $d(x, x) = 1$ and v) $d(x, 1) = x$. Whence the following

2.21. Definition. *A Słomínski subtraction on an object X in a category $\mathbb{E}$ is a Słomínski setting $(X, d, \circ, 1)$ such that $d(x, 1) = x$ (which implies $1 \circ x = x$). A hypersubtraction on an object X is a hyper-Słomínski setting $(X, d, 1)$ satisfying $d(x, 1) = x$, or equivalently $1 \circ x = x$.*

Given any abelian group $(A, +, 0)$, then $(A, d^+, 0)$ (with $d^+(a, b) = a - b$) is a hypersubtraction. When $\mathbb{E}$ is finitely complete, we shall denote by $\mathsf{HSbt}\mathbb{E} \subset \mathsf{SSbt} \subset \mathsf{Sbt}\mathbb{E}$ the string of the categories of internal subtractions, Słomínski subtraction and hypersubtractions. $\mathsf{Sbt}\mathbb{E}$ is a subtractive category in the sense of [Z.Janelidze, 2005], while obviously the subcategories $\mathsf{HSbt}\mathbb{E} \subset \mathsf{\Sigma lom}\mathbb{E}$ and $\mathsf{SSbt}\mathbb{E} \subset \mathsf{Slom}\mathbb{E}$ are protomodular ones.

2.22. Proposition. *Suppose $\mathbb{A}$ is an additive category. We get:*
$\mathbb{A} = \mathsf{HSbt}\mathbb{A} = \mathsf{SSbt}\mathbb{A} = \mathsf{Sbt}\mathbb{A}$.

Proof. About the first assertion, see Proposition 3.6. About the point $\mathbb{A} = \mathsf{Sbt}\mathbb{A}$, see [Bourn-ZJanelidze, 2009] as well. ■

2.23. Autonomous subtractions and hypersubtractions. Let us denote by $\mathsf{ASbt}\mathbb{E}$ and $\mathsf{AHSt}\mathbb{E}$ the categories of autonomous subtraction and hypersubtraction.

2.24. Proposition. *We get* $\mathsf{AHSt}\mathbb{E} = \mathsf{ASbt}\mathbb{E} \simeq \mathsf{Ab}\mathbb{E}$. *In other words any autonomous subtraction is the subtraction associated with an abelian group structure.*

Proof. Let $(X, d, 1)$ be an autonomous subtraction. Set $x \circ y = d(y, d(1, x))$. Then $x \circ d(y, x) = d(d(y, x), d(1, x)) = d(d(y, 1), d(x, x)) = d(y, 1) = y$. So $(X, d, \circ, 1)$ is a Słomínski subtraction with d autonomous. According to Proposition 2.18, it is an autonomous hypersubtraction if and only if $d\mathbb{H}_x\circ$. We check:
$d(x \circ y, x' \circ y') = d(d(y, d(1, x)), d(y', d(1, x'))) = d(d(y, y'), d(d(1, x), d(1, x'))$
$= d(d(y, y'), d(d(1, 1), d(x, x')) = d(d(y, y'), d(1, d(x, x')) = d(x, x') \circ d(y, y')$.

Now we get $d(d(x, z), d(y, z)) = d(d(x, y), d(z, z)) = d(d(x, y), 1) = d(x, y)$. So that $\circ$ is associative. On the other hand, we check: $d(y, d(y, x)) = d(d(y, 1), d(y, x))$
$= d(d(y, y), d(1, x)) = d(d(x, x), d(1, x)) = d(d(x, 1), d(x, x)) = d(x, 1) = x$. So, $\circ$ is commutative. ■

2.25. Class B: prequandles. It is worth saying a word about the class B which makes commutative the following diagram and idempotent the operation d:

$$\begin{tikzcd}
X \times X \arrow[r, "\rho"] \arrow[d, shift right, "p_1^X"'] & X \times X \arrow[d, shift right, "p_0^X"'] \\
X \arrow[r, "="'] \arrow[u, shift right, "s_0^X"'] & X \arrow[u, shift right, "s_0^X"']
\end{tikzcd}$$

First, recall the following definition, introduced independently [Joyce, 1982] and [Matseev, 1982], in relationship with the three Reidemeister moves in knot theory, see also [Bourn, 2015]:

2.26. Definition. *A quandle in a category* $\mathbb{E}$ *is an object* X *endowed with binary operation* $\triangleright : X \times X \longrightarrow X$ *which is idempotent and such that* $- \triangleright x$ *is* $\triangleright$*-automorphism, namely such that* $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

Examples

1) The example 1 of ILO setting is a quandle, namely a *trivial* quandle.
2) Any group $(G, *)$ produces a quandle with the law $x \triangleright_* y = y * x * y^{-1}$.
3) With any pair $((A, +, 0), f)$ of an abelian group and a $+$-automorphism f is associated the *Alexander* quandle on A defined by: $x \triangleright_f y = f(x) + y - f(y)$.

When $\mathbb{A}$ is additive, the category $\mathsf{Alex}\mathbb{A}$ is additive as well. The Alexander quandles defines a left exact conservative functor $\mathsf{Al} : \mathsf{AlexAb} \longrightarrow \mathsf{Qnd}$, which is consequently faithful.

So, it is meaningful to introduce the following:

2.27. Definition. *An* ILO *setting of class B (namely, an idempotent* ILO *setting) will be called a prequandle.*

We shall denote by $\circ_\triangleright$, the adjoint operation of $\triangleright$. The category of internal prequandles in $\mathbb{E}$ will be denoted by $\mathsf{PQd}\mathbb{E}$ and the category of internal quandles by $\mathsf{Qnd}\mathbb{E}$. In the set theorerical context, since the law $\rhd$ of a prequandle is idempotent, any element x of a prequandle $(X, \rhd)$ determines a map $x : 1 \longrightarrow (X, \rhd)$ in PQd, and given any map $h : (X, \rhd) \longrightarrow (Y, \rhd)$, any inverse image $h^{-1}(y)$ is a prequandle. Of course, the definition 2.3 applies to prequandles.

We have $\mathsf{Ab}(\mathsf{Qnd}) \simeq \mathsf{AlexAb}$, see [Bourn, 2015]. We get the same result for prequandles:

2.28. Proposition. *The functor* $\mathsf{Al} : \mathsf{AlexAb} \longrightarrow \mathsf{Ab}(\mathsf{PQd})$ *is an isomorphism.*

Proof. The functor $\mathsf{Al} : \mathsf{AlexAb} \longrightarrow \mathsf{PQd}$ being left exact and the category AlexAb being additive, actually takes its values in $\mathsf{Ab}(\mathsf{PQd})$.

Conversely, given any internal abelian group: $(X, \rhd) \times (X, \rhd) \longrightarrow (X, \rhd)$ in the category PQd, we get $(x + x') \rhd (y + y') = (x \rhd y) + (x' \rhd y')$. With $x' = 0 = y'$, we check that $f(x) = x \rhd 0$ defines a bijective group homomorphism. With $x = a - b$, $x' = b = y'$ and $y = 0$, we get $a \rhd b = ((a - b) \rhd 0) + (b \rhd b) = f(a - b) + b = a \rhd_f b$. So we get $\mathsf{Al}((X, +), f) = ((X, \rhd), +)$, with $f = - \rhd 0$. ■

More generally, let $\mathbb{A}$ be any left exact additive category and (X, f) any object of $\mathsf{Alex}\mathbb{A}$. Define the internal binary operation: $\rhd_f = X \times X \overset{f\times(Id_X - f)}{\longrightarrow} X \times X \overset{+}{\to} X$ which gives an internal prequandle structure on X in $\mathbb{A}$ whose adjoint operation $\circ$ is given by $x \circ y = y \rhd_{f^{-1}} x$. So, $\circ = \rhd^{op}_{f^{-1}}$. Whence a left exact conservative functor $\mathsf{Al}_{\mathbb{A}} : \mathsf{Alex}\mathbb{A} \longrightarrow \mathsf{PQd}\mathbb{A}$.

2.29. Proposition. *Given any finitely complete additive category* $\mathbb{A}$*, the functor* $\mathsf{Al}_{\mathbb{A}}$ *is an isomorphism which is factorized through the inclusion* $\mathsf{Qnd}\mathbb{A} \subset \mathsf{PQd}\mathbb{A}$ *and produces the isomorphisms* $\mathsf{PQd}\mathbb{A} = \mathsf{Qnd}\mathbb{A} \simeq \mathsf{Alex}\mathbb{A}$*. The quandle* $\mathsf{Al}_{\mathbb{A}}(X, f)$ *is a latin one if and only if* $Id_X - f$ *is an isomorphism.*

Proof. In the additive setting, the definition of $\rhd_f$ necessarily satifies the axiom of a quandle, whence the factorization $\mathsf{Al}_{\mathbb{A}} : \mathsf{Alex}_{\mathbb{A}} \longrightarrow \mathsf{Qnd}\mathbb{A} \subset \mathsf{PQd}\mathbb{A}$.

Now, let $\rhd$ be any internal binary idempotent operation on X in $\mathbb{A}$. Let us set $f(x) = x \rhd 0$. We get: $(x + x') \rhd (y + y') = (x \rhd y) + (x' \rhd y')$. From $(x, y) = (x - y, 0) + (y, y)$, we get $x \rhd y = (x - y \rhd 0) + (y \rhd y) = f(x - y) + y = f(x) + y - f(y)$. If, moreover, $\rhd$ is underlying a prequandle, the map $f = - \rhd 0$ is an isomorphism, and we get $\rhd = \rhd_f$. Since $0 \rhd_f (x) = (Id_X - f)(x)$, the last observation is straightforward. ■

Remark. Strangely enough, the additive context produces four isomorphic reduction: $\Sigma\mathsf{lom}\mathbb{A} \simeq \mathsf{Alex}\mathbb{A} \simeq \mathsf{PQd}\mathbb{A} \simeq \mathsf{Qnd}\mathbb{A}$.

3. A review of protomodular and Mal'tsev categories

On the one hand, we mentioned that any category $\mathsf{Slom}\mathbb{E}$ or $\Sigma\mathsf{lom}\mathbb{E}$ of internal Słomínśki or internal hyper-Słomínśki settings is protomodular. It is a strong structural property which requires a review.

On the other hand, the structural aspect of the category $\mathsf{PQd}\mathbb{E}$ of internal prequandles is weaker than protomodularity, but far from being insignificant. It is related to the notion of Mal'tsev category which will require a review as well. Let us recall the following

3.1. Definition. *A pair (u,v) of subobjects of X in $\mathbb{E}$ as on the left hand side is called jointly strongly epic*

$$\begin{tikzcd}
& W \arrow[dd, tail, "w"] & \\
U \arrow[ur, dotted] \arrow[dr, tail, "u"'] & & V \arrow[ul, dotted] \arrow[dl, tail, "v"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[dd, dotted] & \\
U \arrow[ur, tail, "u"] \arrow[dr, "f"'] & & V \arrow[ul, tail, "v"'] \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

when any subobject w of X containing u and v is necessarily an isomorphism.

When (u,v) is jointly strongly epic, there is at most one vertical factorization making the right hand side diagram commute: given any pair of such maps, take their equalizer.

3.2. Protomodular categories.

3.3. Definition. *[Bourn, 1991] A protomodular category $\mathbb{E}$ is a finitely category such that, given any pair of commutative squares of vertical split epimorphisms:*

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, shift right] & \bullet \arrow[r] \arrow[d, shift right] & \bullet \arrow[d, shift right] \\
\bullet \arrow[r] \arrow[u, shift right] & \bullet \arrow[r] \arrow[u, shift right] & \bullet \arrow[u, shift right]
\end{tikzcd}$$

the right hand side square is a pullback as soon as so are the left hand side one and the whole rectangle.

The major examples are the categories Gp of groups, Rng of rings, and any category of R-algebras when R is a ring. Any additive category is protomodular. The protomodular varieties are characterized in [Bourn-Janelidze, 2003]. Any protomodular category is a Mal'tsev one. The main point is that in any *pointed protomodular category* we get all the classical homological lemmas, see [Borceux-Bourn, 2004]. We shall produce further examples in the next section.

Given any finitely complete category, denote by $\mathrm{Pt}\mathbb{E}$ the category whose objects are the split epimorphisms (f,s), $X \rightleftarrows Y$ in $\mathbb{E}$ and whose morphisms are the commutative squares between them. Let $¶_{\mathbb{E}} : \mathrm{Pt}\mathbb{E} \longrightarrow \mathbb{E}$ be the functor associating with any split epimorphism (f,s) its codomain Y; it is a fibration (called *fibration of points*) whose cartesian maps are the pullbacks; we denote by $\mathrm{Pt}_Y\mathbb{E}$ the pointed fiber of split epimorphisms above Y. It is easy to check that $\mathbb{E}$ is protomodular if and only if any base-change functor of the fibration $¶_{\mathbb{E}}$ is conservative. When $\mathbb{E}$ is pointed, it is enough that any base-change functor $\alpha_Y^* : \mathrm{Pt}_Y\mathbb{E} \longrightarrow \mathbb{E}$ along the initial maps (namely any "kernel functor") is conservative. In the previous sections we systematically used the following result, see [Borceux-Bourn, 2004]:

3.4. Theorem. *Let $\mathbb{E}$ be a pointed protomodular category. On the one hand, for any pair (X,Y) of objects, the canonical pair (ι_X, ι_Y) of inclusions as on the left hand side diagram is jointly strongly epic:*

$$\begin{tikzcd}
X \arrow[r, "\iota_X"] \arrow[d, shift right] & X \times Y \arrow[d, shift right, "p_Y"'] \\
1 \arrow[u, shift right] \arrow[r] & Y \arrow[u, shift right, "\iota_Y"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\iota_0^X"] \arrow[d, shift right] & X \times X \arrow[d, shift right, "p_1^X"'] \\
1 \arrow[u, shift right] \arrow[r] & X \arrow[u, shift right, "s_0^X"']
\end{tikzcd}$$

Accordingly, on any object X there is atmost one structure of internal unitary magma which is necessarily an internal abelian group. When it is the case, we say that the object X is abelian in $\mathbb{E}$. In this way, we get a full inclusion $\mathsf{Ab}\mathbb{E} \subset \mathbb{E}$, where $\mathsf{Ab}\mathbb{E}$ is an additive category. So, an additive category $\mathbb{A}$ is nothing but a pointed protomodular category such that $\mathsf{Ab}\mathbb{A} = \mathbb{A}$.

On the other hand, the canonical pair (ι_0^X, s_0^X) of inclusions of the right hand side diagram is jointly strongly epic as well.

3.5. Proposition. *Let $\mathbb{E}$ be a pointed protomodular category. The object underlying a Słomínski setting $(X, d, \circ)$ in $\mathbb{E}$ is necessarily abelian. Whence $\mathsf{Ab}(\mathsf{Slom}\mathbb{E}) = \mathsf{Slom}\mathbb{E} = \mathsf{Slom}(\mathsf{Ab}\mathbb{E})$ and $\mathsf{Ab}(\mathsf{\Sigma lom}\mathbb{E}) = \mathsf{\Sigma lom}\mathbb{E} = \mathsf{\Sigma lom}(\mathsf{Ab}\mathbb{E})$. A hyper-Słomínski setting in $\mathbb{E}$ is necessarily a latin one. A Słomínski setting is a hyper-Słomínski setting if and only if only if $d(-,1)$ is an isomorphism.*

Proof. Let (X, d) be an internal Słomínski setting in $\mathbb{E}$. Then with $p(x, y, z) = x \circ d(z, y)$ we get an internal ternary operation in $\mathbb{E}$ which satisfies $p(x, x, z) = x \circ d(z, x) = z$ and $p(x, y, y) = x \circ d(y, y) = x \circ 1 = x$; namely, an internal Mal'tsev operation. Then the associated internal unitary magma on X defined by $x * y = x \circ d(y, 1)$ makes the object X abelian in the protomodular category $\mathbb{E}$. Whence $\mathsf{Slom}\mathbb{E} \subset \mathsf{Slom}(\mathsf{Ab}\mathbb{E})$, and consequently (since $\mathsf{Ab}\mathbb{E} \subset \mathbb{E}$) the string of inclusions $\mathsf{Slom}(\mathsf{Ab}\mathbb{E}) \subset \mathsf{Slom}\mathbb{E} \subset \mathsf{Slom}(\mathsf{Ab}\mathbb{E})$, and the same string for $\mathsf{\Sigma lom}\mathbb{E}$. On the other hand, we get $\mathsf{Ab}(\mathsf{Slom}\mathbb{E}) \subset \mathsf{Slom}\mathbb{E} \subset \mathsf{Slom}(\mathsf{Ab}\mathbb{E})$. By Proposition 2.15, this inclusion is an isomorphism, whence $\mathsf{Ab}(\mathsf{Slom}\mathbb{E}) = \mathsf{Slom}(\mathsf{Ab}\mathbb{E})$. The same considerations hold for $\mathsf{\Sigma lom}\mathbb{E}$, and we get $\mathsf{Ab}(\mathsf{\Sigma lom}\mathbb{E}) = \mathsf{\Sigma lom}\mathbb{E} = \mathsf{\Sigma lom}(\mathsf{Ab}\mathbb{E})$. By Proposition 2.12, the last equality shows that any hyper-Słomínski setting in $\mathbb{E}$ is necessarily a latin one.

For the last point, consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\iota_0^X"] \arrow[d, shift right] & X \times X \arrow[r, "d"] \arrow[d, shift right, "p_1^X"'] & X \arrow[d, shift right] \\
1 \arrow[r] \arrow[u, tail, shift right] & X \arrow[r] \arrow[u, tail, shift right, "s_0^X"'] & 1 \arrow[u, tail, shift right]
\end{tikzcd}$$

where the left hand side diagram is a kernel diagram. So, the right hand side square is a pullback, if and only if $d.\iota_0^X$ is an isomorphism. ■

3.6. Proposition. *Let $\mathbb{E}$ be a pointed protomodular category. On any object X there is at most one subtraction which is necessarily a hypersubtraction, whence $\mathsf{Sbt}\mathbb{E} \subset \mathbb{E}$ and $\mathsf{HSbt}\mathbb{E} = \mathsf{SSbt}\mathbb{E} = \mathsf{Sbt}\mathbb{E}$. This determines the identities $\mathsf{Ab}\mathbb{E} = \mathsf{HSbt}\mathbb{E} = \mathsf{SSbt}\mathbb{E} = \mathsf{Sbt}\mathbb{E}$, and, when $\mathbb{A}$ is additive, the identities $\mathbb{A} = \mathsf{HSbt}\mathbb{A} = \mathsf{SSbt}\mathbb{A} = \mathsf{Sbt}\mathbb{A}$.*

Proof. Let d be a subtraction on X. By the last assertion of Proposition 3.2, the equations $d.\iota_0^X = 1_X$ and $d.s_0^X$ guarantee the uniqueness of a subtraction d on X. Whence the inclusion $\mathsf{Sbt}\mathbb{E} \subset \mathbb{E}$. Since $d(-,1) = Id_X$ is an isomorphism, (X, d) is an hyper-Słomínski setting and d is a hypersubtraction. Whence $\mathsf{HSbt}\mathbb{E} = \mathsf{Sbt}\mathbb{E} \subset \mathbb{E}$. Any abelian object $(X, +)$ in $\mathbb{E}$ gives way to a hypersubtraction (X, d^+) in $\mathbb{E}$, whence $\mathsf{Ab}\mathbb{E} \subset \mathsf{HSbt}\mathbb{E}$. Finally any hypersubtraction (X, d) makes X abelian by Proposition 3.5, whence $\mathsf{HSbt}\mathbb{E} \subset \mathsf{Ab}\mathbb{E}$. ■

3.7. Mal'tsev categories and structural aspect of internal prequandles. The structural aspect of the category $\mathsf{PQd}\mathbb{E}$ of internal prequandles is partially related to the notion of Mal'tsev category. Recall from [Carboni-Lambek-Pedicchio, 1991]:

3.8. Definition. *A Mal'tsev category is a finitely complete category in which any internal reflexive relation is an equivalence relation.*

Any protomodular and, a fortiori, any additive category is a Mal'tsev one. There is a characterization of Mal'tsev categories through the fibration of points, see for instance [Borceux-Bourn, 2004]:

3.9. Proposition. *A finitely complete category $\mathbb{E}$ is a Mal'tsev one if and only if, given any pair $((f,s),(g,t))$ of split epimorphisms with common codomain Z, the canonical pair (ι_X^t, ι_Y^s) of inclusions toward the pullback $X \times_Z Y$*

$$\begin{tikzcd}
X \times_Z Y \arrow[r, shift right, "p_Y"'] \arrow[d, shift right, "p_X"'] \arrow[dr, phantom, "(M)"] & Y \arrow[l, shift right, "\iota_Y^s"'] \arrow[d, shift right, "g"'] \\
X \arrow[u, shift right, "\iota_X^t"'] \arrow[r, shift right, "f"'] & Z \arrow[l, shift right, "s"'] \arrow[u, shift right, "t"']
\end{tikzcd}$$

is jointly strongly epic.

3.10. Proposition. *Let $\mathbb{E}$ be a Mal'tsev (resp. protomodular) category. Then any internal prequandle $(X, \triangleright)$ in $\mathbb{E}$ is autonomous and consequently a quandle. Moreover, $\mathsf{PQd}\mathbb{E} = \mathsf{APq}\mathbb{E}$ is a Mal'tsev (resp. protomodular) category.*

Proof. First let us show that a prequandle is a quandle. We have to check $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$. From the previous characterization, it is enough to check it when $x = y$ and $y = z$. We indeed get: $(x \triangleright x) \triangleright z = x \triangleright z = (x \triangleright z) \triangleright (x \triangleright z)$; and: $(x \triangleright z) \triangleright z = (x \triangleright z) \triangleright (z \triangleright z)$. Now we have to check $(x \triangleright y) \triangleright (x' \triangleright y') = (x \triangleright x') \triangleright (y \triangleright y')$. Again it is enough to check it with $x' = y$ and $y = y'$. The first case is straightforward, while the second one is the quandle axiom. The last point comes from the fact that the left exact forgetful functor $U : \mathsf{PQd}\mathbb{E} \longrightarrow \mathbb{E}$ is conservative. ∎

We are now going to show that any category $\mathsf{PQd}\mathbb{E}$ satisfies the Mal'tsev property, but only relatively to a certain class Θ of split epimorphims of $\mathbb{E}$.

3.11. Definition. *[Bourn, 2015] Let $\mathbb{E}$ be a finitely complete category, and Θ a class of split epimorphisms of $\mathbb{E}$ containing the isomorphisms and stable under pullbacks along any map in $\mathbb{E}$. Then $\mathbb{E}$ is said to be a Θ-Mal'tsev category when the property of the diagram (M) is only demanded for the split epimorphism (g,t) belonging to Θ.*

Here is the class in question:

3.12. Definition. *A split epimorphism* $(f,s): X \rightleftarrows Y$ *in* $\mathsf{PQd}\mathbb{E}$ *is said to be acupuncturing when the upper horizontal map in the following diagram is an isomorphism:*

$$\begin{tikzcd}
X \arrow[r, "{(sf,1_X)}"] \arrow[d, shift right, "f"'] & X \times X \arrow[r, "\triangleright"] & X \arrow[d, shift right, "f"'] \\
Y \arrow[u, shift right, "s"'] \arrow[rr, equal] & & Y \arrow[u, shift right, "s"']
\end{tikzcd}$$

We shall denote by $\theta_{(f,s)}$ *its inverse and by* Θ *the class of acupuncturing split epimorphisms. We get in particular* $sf(x) \triangleright \theta_{(f,s)}(x) = x$.

In the set theoretical context, call acupunturing an element $x \in (X, \triangleright)$ such that the map $y \mapsto x \triangleright y$ is bijective. If $Y \neq \emptyset$, a split epimorphism $(f,s): (X,\triangleright) \longrightarrow (Y,\triangleright)$ is acupuncturing when for any $y \in Y$, the element $s(y)$ is acupuncturing in the prequandle $f^{-1}(y)$. A prequandle $(X,\triangleright)$ is a latin one if and only if any element is acupuncturing.

3.13. Proposition. *The class* Θ *in* $\mathsf{PQd}\mathbb{E}$ *is stable under pullback and contains the isomorphims.*

Proof. Consiser any pullback in $\mathsf{PQd}\mathbb{E}$ with $(f',s') \in \Theta$:

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\
Y \arrow[u, shift right, "s"'] \arrow[r, "h"'] & Y' \arrow[u, shift right, "s'"']
\end{tikzcd}$$

We get $f'.(\theta_{(f',s')}.k) = f'.k = h.f$, so that the pair $(\theta_{(f',s')}.k, f)$ produces a unique factorization: $\theta_{(f,s)}: X \longrightarrow X$ such that $f.\theta_{(f,s)} = f$ and $k.\theta_{(f,s)} = \theta(s',f').k$ which gives us the desired inverse to $\triangleright.(sf,1_X)$. The last point is straightforward. ∎

3.14. Proposition. *The category* $\mathsf{PQd}\mathbb{E}$ *is a* Θ*-Mal'tsev category.*

Proof. Consider any pullback in $\mathsf{PQd}\mathbb{E}$ with $(g,t) \in \Theta$:

$$\begin{tikzcd}
X \times_Z Y \arrow[r, shift right, "p_Y"'] \arrow[d, shift right, "p_X"'] & Y \arrow[l, shift right, "\iota^s_Y"'] \arrow[d, shift right, "g"'] \\
X \arrow[u, shift right, "\iota^t_X"'] \arrow[r, shift right, "f"'] & Z \arrow[l, shift right, "s"'] \arrow[u, shift right, "t"']
\end{tikzcd}$$

and $U \subset X \times_Z Y$ a subquandle such that for all $x \in X$, we have $\iota^t_X(x) = (x, tf(x)) \in U$ and for all $y \in Y$, we have $\iota^s_Y(y) = (sg(y), y) \in U$. For any $(x,y) \in X \times_Z Y$ with $f(x) = z = g(y)$, we get: $(x,y) = (((s(z) \circ_{\triangleright} x) \triangleright s(z), t(z) \triangleright \theta_{(g,t)}(y)))$
$= ((s(z) \circ_{\triangleright} x), t(z)) \triangleright (s(z), \theta_{(g,t)}(y)) = \iota^t_X(s(z) \circ_{\triangleright} x) \triangleright \iota^s_Y(\theta_{(g,t)}(y))$
where $\circ_{\triangleright}$ is the adjoint operation of $\triangleright$. So, any $(x,y) \in X \times_Z Y$ belongs to U. ∎

Let us briefly mention two meaningful consequences of this Θ-Mal'tsev property. Let $(d_0^R, d_1^R) : R \rightarrowtail X \times X$ be any internal reflexive relation in $\mathsf{PQd}\mathbb{E}$. Denote by $s_0^R : X \longrightarrow R$ the map characterizing the reflexivity, then define a *reflexive relation* R *as acupuncturing* when the split epimorphism (d_1^R, s_0^R) is acupuncturing.

3.15. Lemma. *A prequandle* $(X, \triangleright)$ *in* $\mathsf{PQd}\mathbb{E}$ *is a latin one if and only if the undiscrete relation* ∇_X *is acupuncturing.*

Proof. Saying that $(p_1^X, s_0^X) : X \times X \rightleftarrows X$ is acupuncturing is saying that $(x, y) \mapsto (y, y) \triangleright (x, y) = (y \triangleright x, y)$ is an isomorphism which is equivalent to $(x, y) \mapsto (y, y \triangleright x)$ is an isomorphism. ∎

3.16. Proposition. *Any internal acupuncturing reflexive relation* R *in* $\mathsf{PQd}\mathbb{E}$ *is transitive. When an internal equivalence relation* R *is acupuncturing, any class* $u : U \rightarrowtail X$ *of* R *is a latin prequandle.*

Proof. Consider the following pullback in $\mathsf{PQd}\mathbb{E}$:

$$\begin{tikzcd}
R \times_X R \arrow[r, shift right, "d_0^R"'] \arrow[d, shift right, "d_2^R"'] & R \arrow[l, shift right, "s_0^R"'] \arrow[d, shift right, "d_1^R"'] \\
R \arrow[u, shift right, "s_1^R"'] \arrow[r, shift right, "d_0^R"'] & X \arrow[l, shift right, "s_0^R"'] \arrow[u, shift right, "s_0^R"']
\end{tikzcd}$$

$R \times_X R$ is the prequandle of the pairs (uRv, vRw). Let us consider, the subquandle $U \subset R \times_X R$ whose objects satisfies uRw in addition. The map $s_0^R : R \longrightarrow R \times_X R$ is defined by $s_0^R(uRv) = (uRv, vRv)$ and the map $s_1^R : R \longrightarrow R \times_X R$ is defined by $s_1^R(uRv) = (uRu, uRv)$. The two maps factor through U. Since the reflexive relation R is acupuncturing, we get $U = R \times_X R$ and the reflexive relation R is transitive.

A subobject $u : U \rightarrowtail X$ is a class of R if and only if $u^{-1}(R) = \nabla_U$ and the induced diagram is such that any commutative square is a pullback.

$$\begin{tikzcd}
U \times_X U \arrow[r, "\tilde{u}"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & R \arrow[d, shift right=2, "d_0^R"'] \arrow[d, shift left=2, "d_1^R"] \\
U \arrow[u, "s_0"] \arrow[r, "u"'] & X \arrow[u, "s_0^R"]
\end{tikzcd}$$

Accordingly, if (d_1^R, s_0^R) is acupuncturing, so is $(p_1, s_0) : U \times U \rightleftarrows U$. ∎

3.17. Proposition. *The subcategory* $\mathsf{LPq}\mathbb{E} \hookrightarrow \mathsf{PQd}\mathbb{E}$ *of latin prequandles is stable under finite limits. The category* $\mathsf{LPq}\mathbb{E}$ *is a Mal'tsev one. In* $\mathsf{LPq}\mathbb{E}$*, any split epimorphism is acupuncturing.*

PROOF. It is clear that $\mathsf{PQd}\mathbb{E}$ is stable under product and equalizer, whence the first point. Let $(X, \triangleright)$ be a latin prequandle in $\mathbb{E}$ and let δ denote the binary law of which $\triangleright$ is adjoint. Set $\pi(x,y,z) = (y \circ x) \triangleright \delta(z,y)$. This ternary operation in $\mathbb{E}$ satisfies the Mal'tsev axioms: $\pi(x,y,y) = (y \circ x) \triangleright \delta(y,y) = (y \circ x) \triangleright y = x$, while
$\pi(x,x,z) = (x \circ x) \triangleright \delta(z,x) = x \triangleright \delta(z,x) = z$. Denote by $\mathsf{Mal}\mathbb{E}$ the category of internal Mal'tsev operations in $\mathbb{E}$; it is a Mal'tsev category. In this way, we get a left exact conservative functor $\pi : \mathsf{LPq}\mathbb{E} \longrightarrow \mathsf{Mal}\mathbb{E}$ with $\pi(X, \triangleright) = (X, \pi)$. So, $\mathsf{LPq}\mathbb{E}$ is a Mal'tsev category.

Let $(f,s) : X \rightleftarrows Y$ be any split epimorphism in $\mathsf{LPq}\mathbb{E}$. Then the map $\theta : X \longrightarrow X$ defined by $\theta(x) = \delta(x, sf(x))$ produces the desired inverse to $x \mapsto sf(x) \triangleright x$. ∎

3.18. PROPOSITION. *Let* $\mathbb{E}$ *be a pointed protomodular category. Then* $\mathsf{LPq}\mathbb{E}$ *is isomorphic to* $\mathsf{LPq}(\mathsf{Ab}\mathbb{E})$. *When* $\mathbb{A}$ *is additive,* $\mathsf{LPq}\mathbb{A}$ *is isomorphic to the full subcategory* $\mathsf{LAlex}\mathbb{A}$ *of* $\mathsf{Alex}\mathbb{A}$ *whose objects are the pairs* (X,f) *such that both* f *and* $Id_X - f$ *are isomorphisms.*

PROOF. Let $(X, \triangleright)$ be any latin prequandle in $\mathbb{E}$. Then $p(x,y,z) = (y \circ x) \triangleright \delta(z,y)$ becomes an internal Mal'tsev operation in the pointed protomodular category $\mathbb{E}$. So, the unitary magma structure on X given by $(x,y) \mapsto p(x,1,y)$ makes the object X an abelian object in $\mathbb{E}$ and $(X, \triangleright)$ a latin prequandle inside $\mathsf{Ab}\mathbb{E}$.

When $\mathbb{A}$ is additive, and f is an automorphism on X, the prequandle $(X, \triangleright_f)$ is a latin one if and only if $x \mapsto 0 \triangleright_f x = f(0-x) + x = (Id_X - f)(x)$ is an isomorphism. ∎

3.19. AUTONOMOUS LATIN PREQUANDLES. Denote by $\mathsf{APq}\mathbb{E} \subset \mathsf{PQd}\mathbb{E}$ the subcategory of internal autonomous prequandles. Any Alexander quandle $\mathsf{Al}(X,+)$ is autonomous. With $y = y'$ in the identity $(x \triangleright x') \triangleright (y \triangleright y') = (x \triangleright y) \triangleright (x' \triangleright y')$, we check that any autonomous prequandle is actually a quandle. So, autonomous prequandles and autonomous quandles coincide.

Consider now the subcategory $\mathsf{ALPq}\mathbb{E} \subset \mathsf{APq}\mathbb{E}$ of autonomous latin prequandles. By Proposition 2.8, from $\triangleright\mathbb{H}\triangleright$, we get $\triangleright\mathbb{H}d$. Accordingly, from $\triangleright\mathbb{H}\triangleright$, $\circ\mathbb{H}\triangleright$ and $d\mathbb{H}\triangleright$, the ternary term $\pi(x,y,z) = (y \circ x) \triangleright \delta(z,y)$ becomes internal to $\mathsf{ALPq}\mathbb{E}$. In this way, any object $(X, \triangleright) \in \mathsf{ALPq}\mathbb{E}$ is endowed with a natural ternary Mal'tsev operation . Whence, immediately, the following

3.20. PROPOSITION. *The category* $\mathsf{ALPq}\mathbb{E}$ *is a naturally Mal'tsev category.*

This notion was introduced in [Johnstone, 1989] as a category $\mathbb{A}$ in which any object X is endowed with a natural Mal'tsev operation and, strenghtening the notion of Mal'tsev category, characterized by the fact that on any internal graph there is at most one groupoid structure. A pointed category is a naturally Mal'tsev one if and only if it is additive.

4. Semi-direct index and hyperindex

Let $U : \mathbb{C} \longrightarrow \mathbb{D}$ be now a left exact functor between finitely complete *pointed* categories. Let $\mathrm{Pt}_U : \mathrm{Pt}\mathbb{C} \longrightarrow \mathrm{Pt}\mathbb{D}$ be its natural extension. Denote by $W_U : \mathrm{Pt}\mathbb{C} \longrightarrow \mathrm{Pt}\mathbb{D}$ the functor associating with any split epimorphism $(f, s) : X \rightleftarrows Y$ in $\mathbb{C}$ the split epimorphism $(p_{U(Y)}, \iota_{U(Y)}) : U(Y) \times U(\mathrm{Ker}f) \rightleftarrows U(Y)$ in $\mathbb{D}$.

4.1. Definition. *A semi-direct index for U is a natural isomorphism ρ between Pt_U and W_U given by an isomorphism $\rho_{(f,s)}$ making the following left hand side diagram commute in $\mathbb{D}$ for any split epimorphism $(f, s) : X \rightleftarrows Y$ in $\mathbb{C}$:*

$$\begin{tikzcd}
U(X) \arrow[r, "\rho_{(f,s)}"] \arrow[d, shift right, "U(f)"'] & U(Y) \times U(\mathrm{Ker}f) \arrow[d, shift right, "p_{U(Y)}"'] & & U(X) \arrow[r, "\gamma_{(f,s)}"] \arrow[d, shift right, "U(f)"'] & U(\mathrm{Ker}f) \arrow[d, shift right] \\
U(Y) \arrow[u, shift right, "U(s)"'] \arrow[r, "="'] & U(Y) \arrow[u, shift right, "\iota_{U(Y)}"'] & & U(Y) \arrow[u, shift right, "U(s)"'] \arrow[r] & 1 \arrow[u, shift right]
\end{tikzcd}$$

or, equivalently, it is given a natural map $\gamma_{(f,s)} : U(X) \longrightarrow U(\mathrm{Ker}f)$ in $\mathbb{D}$ making the right hand side square a pullback of split epimorphisms; namely, a pullback satisfying $(*)$: $\gamma_{(f,s)}.U(s) = 0$ in $\mathbb{D}$.

4.2. Definition. *A semi-direct hyperindex for U is a semi-direct index ρ making the following left hand side diagram commute in $\mathbb{D}$ for any split epimorphism $(f, s) : X \rightleftarrows Y$ in $\mathbb{C}$:*

$$\begin{tikzcd}
U(Y) \arrow[r, shift left, "U(s)"] \arrow[d, "="'] & U(X) \arrow[l, shift left, "U(f)"] \arrow[r, shift right, "\gamma_{(f,s)}"'] \arrow[d, "\rho_{(f,s)}"'] & U(\mathrm{Ker}f) \arrow[l, shift right, "U(k_f)"'] \arrow[d, "="] & & U(X) \arrow[r, shift right, "\gamma_{(f,s)}"'] \arrow[d, shift right, "U(f)"'] & U(\mathrm{Ker}f) \arrow[l, shift right, "U(k_f)"'] \arrow[d, shift right] \\
U(Y) \arrow[r, shift left, "\iota_{U(Y)}"] & U(Y) \times U(\mathrm{Ker}f) \arrow[l, shift left, "p_{U(Y)}"] \arrow[r, shift right, "p_{U(K)}"'] & U(\mathrm{Ker}f) \arrow[l, shift right, "\iota_{U(K)}"'] & & U(Y) \arrow[u, shift right, "U(s)"'] \arrow[r, shift right] & 1 \arrow[l, shift right] \arrow[u, shift right]
\end{tikzcd}$$

or, equivalently, making the following right hand side square a pullback of split epimorphisms; namely, adding to the index γ the axiom $\gamma_{(f,s)}.U(k_f) = Id_{U(\mathrm{Ker}f)}$.

Examples

1) Considering the forgetful functor $V : \mathsf{Gp} \longrightarrow \mathbf{Set}_*$ from the category of groups to the category of pointed sets, given any split epimorphism $(f, s) : X \rightleftarrows Y$ in Gp, the map $\gamma_{(f,s)} : V(X) \longrightarrow V(\mathrm{Ker}f)$ defined by $\gamma_{(f,s)}(x) = sf(x)^{-1}.x$ is a semi-direct hyperindex for U whose inverse is given by $(y, k) \mapsto s(y) \circ k$.

2) Let us start with any finitely complete category $\mathbb{E}$. Denoting the fiber $\mathrm{Pt}_1\mathbb{E}$ by $\mathbb{E}_*$ and the category of internal groups in $\mathbb{E}$ by $\mathsf{Gp}\mathbb{E}$, it is clear that the previous formula determines a semi-direct hyperindex for the forgetful functor $V_{\mathbb{E}} : \mathsf{Gp}\mathbb{E} \longrightarrow \mathbb{E}_*$.

3) Let R be a ring, and $\mathbb{C}$ be the category of any kind of R-algebras, the forgetful functor $U : \mathbb{C} \longrightarrow \mathsf{Ab}$ has a semi-direct hyperindex with the group homomorphism $\gamma_{(f,s)}$: $U(X) \longrightarrow U(\mathrm{Ker}f)$ defined by $\gamma_{(f,s)}(x) = x - sf(x)$.

4) Consider the category $\mathsf{\Sigma lom}\mathbb{D}$ of internal hyper-Słomínski settings in a finitely complete category $\mathbb{D}$. Then the forgetful functor $\Sigma_{\mathbb{D}} : \mathsf{\Sigma lom}\mathbb{D} \longrightarrow \mathbb{D}_*$ has, for any split epimorphism (f, s) in $\mathsf{\Sigma lom}\mathbb{D}$, a semi-direct index with the map γ^{Σ} defined by $\gamma^{\Sigma}_{(f,s)}(x) = d(x, sf(x))$, the inverse of $(f, \gamma^{\Sigma}_{(f,s)}) : X \longrightarrow Y \times \mathrm{Ker}f$ being produced by $(y, k) \mapsto s(y) \circ k$.

5) Consider the category $\mathsf{HSbt}\mathbb{D}$ of internal hypersubtractions in a finitely complete category $\mathbb{D}$. Then the forgetful functor $H_{\mathbb{D}} : \mathsf{HSbt}\mathbb{D} \longrightarrow \mathbb{D}_*$ has, for any split epimorphism (f, s)

in $\mathsf{HSbt}\mathbb{D}$, a semi-direct hyperindex with the map γ^{HS} defined by $\gamma^{HS}_{(f,s)}(x) = d(x, sf(x))$, the inverse of $(f, \gamma^{HS}_{(f,s)}) : X \longrightarrow Y \times \mathrm{Ker}f$ being produced in the same way by $(y, k) \mapsto s(y) \circ k$.
6) Let DiGp be the category of digroups, namely of quadruples $(X, *, \circ, 1)$ of a set X endowed with two group structures having same unit element, and SkB the subcategory of *left skew braces* [Guanieri-Vandramin, 2017] (see also [Rump, 2007]), where the two laws are related by the axiom: $a \circ (b * c) = (a \circ b) * a^{-*} * (a \circ c)$, where a^{-*} denotes the inverse of a in the group $(X, *)$. They are both protomodular, see [Bourn-Facchini-Pompili, 2023]. Clearly the two possible forgetful functors towards Gp composed with the forgetful functor $V : \mathsf{Gp} \longrightarrow \mathbf{Set}_*$ produce two absolutely independent (extrinsic) semi-direct hyperindexes for DiGp and SkB. So, the hyperindexes of a functor are far from being unique. It is shown in [Bourn, 2024] that the thorough description of the split epimorphisms in DiGp and SkB actually requires one more item than these only two hyperindexes.

4.3. Lemma. *Let $U : \mathbb{C} \longrightarrow \mathbb{D}$ be a left exact functor between finitely complete pointed categories which is endowed a semi-direct index ρ. If, in addition, the functor U is conservative, then the category $\mathbb{C}$ is protomodular, without any assumption on the category $\mathbb{D}$.*

Proof. Consider any morphism h of split epimorphisms as on the right-hand side:

$$\begin{tikzcd}
\mathrm{Ker}f \arrow[r, "k_f"] \arrow[d, shift right] & X \arrow[r, "h"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\
1 \arrow[r, tail, "\alpha_Y"'] \arrow[u, shift right] & Y \arrow[r, equal] \arrow[u, shift right, "s"'] & Y \arrow[u, shift right, "s'"']
\end{tikzcd}$$

Saying that $\alpha^*_Y(h)$ is an isomorphism is equivalent to saying that the whole rectangle is a pullback. Now since ρ is underlying a natural transformation, we have the following commutative diagram in $\mathbb{D}$:

$$\begin{tikzcd}
U(X) \arrow[r, "U(h)"'] \arrow[rr, bend left, "\gamma_{(f,s)}"] \arrow[d, shift right] & U(X') \arrow[r, "\gamma_{(f',s')}"'] \arrow[d, shift right] & \mathrm{Ker}f \arrow[d, shift right] \\
U(Y) \arrow[r, equal] \arrow[u, shift right] & U(Y) \arrow[r] \arrow[u, shift right] & 1 \arrow[u, shift right]
\end{tikzcd}$$

where the right-hand-side square and the whole rectangle are pullbacks. Accordingly the left-hand-side square is a pullback or, equivalently, $U(h)$ is an isomorphism. Since U is conservative, so is h; and the base-change α^*_Y is conservative as well. ∎

The fourth and the fith examples here above provide characterizations of the functors with semi-direct index and hyperindex.

4.4. Theorem. *Let $U : \mathbb{C} \longrightarrow \mathbb{D}$ be a left exact functor with semi-direct index ρ between finitely complete pointed categories. Then the functor U can be factorized through the forgetful functor $\Sigma_{\mathbb{D}} : \Sigma\mathsf{lom}\mathbb{D} \longrightarrow \mathbb{D}$ by a functor $\bar{U} : \mathbb{C} \longrightarrow \Sigma\mathsf{lom}\mathbb{D}$ in such a way that*

$\rho = \rho^{\Sigma}.\bar{U}$. When it has a semi-direct hyperindex, the functor U can be factorized through the forgetful functor $H_{\mathbb{D}} : \mathsf{HSbt}\mathbb{D} \longrightarrow \mathbb{D}$ by a functor $\bar{U} : \mathbb{C} \longrightarrow \mathsf{HSbt}\mathbb{D}$ in such a way that $\rho = \rho^{HS}.\bar{U}$.

Proof. Given any object X in $\mathbb{C}$, consider the split epimorphism $(p_1^X, s_0^X) : X \times X \rightleftarrows X$, where p_1^X is the second projection and $s_0^X : X \rightarrowtail X \times X$ is the diagonal. Then consider the following isomorphism given by the semi-direct index:

$$\begin{tikzcd}
U(X) \times U(X) \arrow[r, "\rho_{(p_1^X, s_0^X)}"] \arrow[d, shift right, "p_1^{U(X)}"'] & U(X) \times U(X) \arrow[d, shift right, "p_0^{U(X)}"'] \\
U(X) \arrow[u, shift right, "s_0^{U(X)}"'] \arrow[r, "="'] & U(X) \arrow[u, shift right, "\iota_0^{U(X)}"']
\end{tikzcd}$$

According to Definition 2.10, the map $\gamma_{(p_1^X, s_0^X)} : U(X) \times U(X) \longrightarrow U(X)$, which will be denoted by d_X for sake of simplicity, gives $U(X)$ a structure of hyper-Słomínski setting. Whence the factorization $\bar{U} : \mathbb{C} \longrightarrow \Sigma\mathsf{lom}\mathbb{D}$, defined by $\bar{U}(X) = (U(X), d_X, \alpha_X)$ through the forgetful functor $\Sigma\mathsf{lom}\mathbb{D} \longrightarrow \mathbb{D}$; it clearly satisfies $\rho^{\Sigma}_{(U(f),U(s))} = \rho_{(f,s)}$. When this index is a hyperindex, the map d_X gives $U(X)$ a structure of hypersubtraction and produces a factorization $\bar{U} : \mathbb{C} \longrightarrow \mathsf{HSbt}\mathbb{D}$, defined by $\bar{U}(X) = (U(X), d_X, \alpha_X)$ through the forgetful functor $\mathsf{HSbt}\mathbb{D} \longrightarrow \mathbb{D}$; it clearly satisfies $\rho^{HS}_{(U(f),U(s))} = \rho_{(f,s)}$. ∎

The examples 3) of hyperindex achieve their full meaning observing that:
$\mathsf{Ab} = \mathsf{UMag}(\mathsf{HSbt}) = \mathsf{HSbt}(\mathsf{UMag})$.

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Univ. Littoral Côte d'Opale, UR 2597, LMPA, Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, F-62100 Calais, France
Email: `bourn@univ-littoral.fr`

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