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WHAT KIND OF LINEARLY DISTRIBUTIVE CATEGORY DO POLYNOMIAL FUNCTORS FORM?

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Abstract.

This paper has two purposes. The first is to extend the theory of linearly distributive categories by considering the structures that emerge in a special case: the normal duoidal category $(\mathsf{Poly}, y, \otimes, \triangleleft)$ of polynomial functors under Dirichlet and substitution product. This is an isomix LDC which is neither $*$-autonomous nor fully symmetric. The additional structures of interest here are a closure for $\otimes$ and a co-closure for $\triangleleft$, making Poly a bi-closed LDC, which is a notion we introduce in this paper.

The second purpose is to use Poly as a source of examples and intuition about various structures that can occur in the setting of LDCs, including duals, cores, linear monoids, and others, as well as how these generalize to the non-symmetric setting. To that end, we characterize the linearly dual objects in Poly: every linear polynomial has a right dual which is a representable. It turns out that the linear and representable polynomials also form the left and right cores of Poly. Finally, we provide examples of linear monoids, linear comonoids, and linear bialgebras in Poly.

Contents

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1. Introduction

Cockett and Seely introduced weakly distributive categories, renamed to linear distributive categories (LDCs), as a categorical semantics for multiplicative linear logic [Cockett and Seely (1997b)]. Adding negation to the semantics produces $*$-autonomous categories. The primary contribution of this paper is to present a detailed study of a reasonably approachable yet a rich example of an LDC, one which is neither symmetric nor $*$-autonomous, namely the category Poly of polynomial functors and natural transformations. While Poly is a well-known category with a vast body of literature, to the best of our knowledge this paper is the first to explore Poly purely from the perspective of LDCs.

Indeed, though this project is focused on the category Poly as an LDC, our definitions and results are more general. We first prove a straightforward result, that normal duoidal categories are isomix LDCs—there is a faithful functor as we'll see in Lemma 3.3—thereby providing a family of examples for isomix LDCs, one of which is Poly. These results are presented in Section 3.

The category Poly is not $*$-autonomous: not every object in Poly has a dual. However, there are certain pairs of polynomials which are duals: for any set A, the linear polynomial Ay is left dual to the representable y^A, and these are the only polynomials with duals.[1] We prove these results in more generality by introducing the notion of *biclosed LDCs* which are LDCs with a closure for the $\otimes$ operation and a coclosure for the $\triangleleft$ operation, i.e. LDCs for which $(-\otimes a)$ has a right adjoint $[a,-]$, and $(-\triangleleft a)$ has a left adjoint $\left[\begin{smallmatrix}-\\a\end{smallmatrix}\right]$. In Poly, the former $[-,-]$ is the closure for the Dirichlet product and the latter $\left[\begin{smallmatrix}-\\-\end{smallmatrix}\right]$ is the left Kan extension. We characterize the dual objects in biclosed LDCs, generalizing the above results for Poly; all this is shown in Section 4.

Next we explore the notion of *core* in mix LDCs within our context. Mix LDCs are equipped with a mix map $\mathsf{m} : \bot \longrightarrow \top$, which induces a natural transformation $\mathsf{indep}_{a,b} : a \otimes b \longrightarrow a \triangleleft b$ for all objects a, b. An isomix LDC is a mix LDC in which the mix map $\mathsf{m} : \bot \cong \top$ is an isomorphism. The core of a symmetric (iso)mix LDC is the full subcategory spanned by those objects for which the $\mathsf{indep}_{a,-}$ is a natural isomorphism. That is, an object a is in the core if the map $\mathsf{indep}_{a,x} : a \otimes x \cong a \triangleleft x$ is an isomorphism, for all x; from this it follows from the symmetry that $\mathsf{indep}_{x,a} : x \otimes a \cong x \triangleleft a$ is also an isomorphism.

The notion of core in mix LDCs has been studied only for symmetric LDCs. Inspired by Poly, this paper explores the notion of core in non-symmetric settings. Without the assumption of symmetry, one may define the left core (resp. right core) to be the full subcategory spanned by the objects a for which $\mathsf{indep}_{a,-}$ (resp. $\mathsf{indep}_{-,a}$) is a natural isomorphism. We show that in Poly, the linear polynomials (Ay) comprise the left core and the representables (y^A) comprise the right core. The cores and the duals thus coincide: a polynomial is in the left core iff it is a left dual iff it is linear, and it is in the right core iff it is a right dual iff it is representable.

For a mix LDC $\mathbb{X}$, we say that the left and the right cores are *opposing* when $\mathsf{core}_\ell(\mathbb{X}) \cong$

[1] We will explain this notation in Section 2.1.

$\mathsf{core_r}(\mathbb{X})^{\mathsf{op}}$. We will show that Poly has opposing cores, which are equivalent to Set and $\mathsf{Set}^{\mathsf{op}}$, respectively. Another example of mix LDCs with opposing cores are compact closed categories. Section 5 provides the above definitions and proves relevant results.

Linear monoids and linear comonoids generalize Frobenius algebras from monoidal categories to LDCs. We show that Poly has non-trivial linear-monoid-*like* structures: every monoid in Set defines a *left* linear monoid and every set induces a *right* linear monoid in Poly; these definitions are new. Such linear monoids and linear comonoids in Poly interact to produce left and right linear bialgebras. Section 6 contains these results.

Contributions. The following are the main technical contributions of this paper:

(1) We prove that all normal duoidal categories are isomix LDCs. In particular, the category Poly is an isomix LDC.
(2) We define the notion of biclosed LDCs.
(3) We characterize the linear duals in isomix biclosed LDCs.
(4) We prove that a polynomial q is left dual of p, i.e. $q \dashv\!\!\dashv p$ iff $q = Ay$ and $p = y^A$ for some set A.
(5) We define the left and the right cores of non-symmetric LDCs. In Poly, we show that the left core consists of the linear polynomials (Ay) and the right core consists of the representable polynomials (y^A).
(6) We show that for every monoid in Set, we get a left linear monoid and a right linear comonoid in Poly. Moreover, every set induces a left linear comonoid and a right linear monoid in Poly. These structures interact to produce right and left linear bialgebras which are the notion we introduce in this paper.

In Section 2 we recollect the preliminary definitions and results used in this paper.

2. Preliminaries

For preliminaries on Poly see Section 2.1; for preliminaries on LDCs see Section 2.7.

2.1. The category Poly. While there are many equivalent definitions of polynomial functors [Niu and Spivak (2008)],[2] in this paper we view them as coproducts of representables.

Given an object $A : \mathbb{C}$, a functor $F : \mathbb{C} \longrightarrow \mathsf{Set}$ is **represented by** A if there is an isomorphism $F \cong \mathbb{C}(A, -)$; in this case we also say that F is **representable**. For functors $\mathsf{Set} \longrightarrow \mathsf{Set}$, we denote the functor $\mathsf{Set}(A, -)$ by y^A.

2.2. Definition. [Niu and Spivak (2008), Defintion 2.1] *A* **polynomial functor** *is a functor* $p : \mathsf{Set} \longrightarrow \mathsf{Set}$ *that is isomorphic to a coproduct of representables, i.e.*

$$p \cong \sum_{i:I} y^{A_i}$$

[2] For a friendly and very non-technical introduction to polynomial functors, see https://topos.site/blog/2022-01-19-poly-makes-me-happy/ or https://topos.site/blog/2023-11-13-solving-problem-solving/.

for some indexing set I : Set and sets A_i : Set. We refer to the elements of I as the **positions** *of p, and we refer to elements of A_i as the* **directions** *of p at position $i : I$.*

The following are a few examples of polynomial functors:

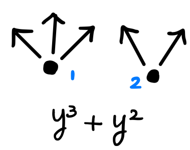
(i) y^A is a representable polynomial, for any $A : \mathsf{Set}$. The terminal object of Poly is y^0; we denote this by 1.

(ii) $\sum_{i:I} y^0$ is a constant polynomial; it sends $X \mapsto I$ for all $X : \mathsf{Set}$. We denote this by $I := \sum_{i:I} 1$.

(iii) $\sum_{i:I} y \cong Iy$ is a linear polynomial, for any $I : \mathsf{Set}$.

(iv) $y^3 + y^2$ where 2 and 3 are 2-element and 3-element sets respectively. For this polynomial, the index set $I = 2$, $A_1 = 3$, and $A_2 = 2$.

Corolla forests are an intuitive visualization tool for polynomials [Niu and Spivak (2008), Chapter 2]. For example, the polynomial $y^3 + y^2$, can be drawn as a corolla forest as follows:



For a polynomial $p \cong \sum_{i:I} y^{A_i}$, the elements of I are referred to as the **positions** of p. Thus, each corolla in a corolla forest corresponds to a position of its polynomial. For each $i : I$, the elements of A_i are referred to as the **directions at that position**. The directions at position i are denoted as the branches of the corolla corresponding to that position.

The positions of a polynomial p can be found by evaluating the polynomial functor at the single element set 1:

$$p(1) = \sum_{i:I} y^{A_i}(1) \cong \sum_{i:I} 1^{A_i} \cong \sum_{i:I} 1 \cong I$$

Hence, writing $p[P] := A_P$ for any position $P : p(1)$, we can denote a polynomial succinctly as follows:

$$p \cong \sum_{P:p(1)} y^{p[P]} \tag{2.1}$$

2.3. Definition. *We define* Poly *to be the category of polynomial functors* $\mathsf{Set} \longrightarrow \mathsf{Set}$ *and the natural transformations between them.*

We can give a fully set-theoretic account of the natural transformations between any two polynomial functors by using the universal property of coproducts and the Yoneda Lemma. Indeed, for any two polynomials $p = \sum_{P:p(1)} y^{p[P]}$ and $q = \sum_{Q:q(1)} y^{q[Q]}$,

$$\begin{aligned} \mathsf{Poly}(p, q) &= \mathsf{Poly}\left(\sum_{i:P} y^{p[P]}, q\right) \\ &\cong \prod_{P:p(1)} \mathsf{Poly}\left(y^{p[P]}, q\right) \qquad \text{(Universal property of coproduct)} \end{aligned}$$

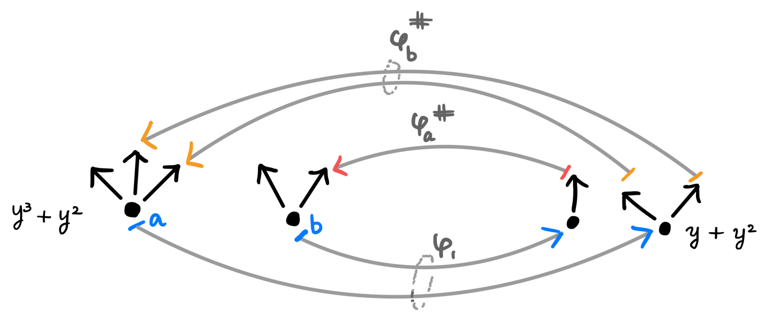
$$
\begin{aligned}
&\cong \prod_{P:p(1)} q(p[P]) \qquad \text{(Yoneda Lemma)} \\
&\cong \prod_{P:p(1)} \sum_{Q:q(1)} p[P]^{q[Q]} \\
&= \prod_{P:p(1)} \sum_{Q:q(1)} \prod_{e:q[Q]} \sum_{d:p[P]} 1 \qquad (2.2)
\end{aligned}
$$

Since products distribute over sums, the expression in (2.2) tells us that a natural transformation $\varphi : p \to q$ can be identified with:

- a function $\varphi_1 : p(1) \to q(1)$, from the positions of p to the the positions of q, and
- for each $P : p(1)$, a function $\varphi_P^\sharp : q[\varphi_1(P)] \to p[P]$, from the directions of q at $\varphi_1(P)$ to directions of p at P.

A map φ of polynomials is called **cartesian** if, for each $P : p(1)$, the map $\varphi_P^\sharp$ is a bijection.[3]

As an example of polynomial maps, let us consider the polynomials $p = y^3 + y^2$ and $q = y + y^2$. The following diagram depicts a natural transformation $\varphi : p \to q$:



One can count that there are $(3^1 + 3^2) \times (2^1 + 2^2) = 72$ natural transformations $p \to q$.

Here are two useful functors between Poly and Set, based on the positions and the directions of polynomials:

- the functor $\mathsf{Poly}(y, -) : \mathsf{Poly} \to \mathsf{Set}$, which we denote by $-(1)$. For any polynomial p, an element of $p(1)$ is a position of p.

- the functor $\mathsf{Poly}(-, y) : \mathsf{Poly} \to \mathsf{Set}^{\mathsf{op}}$, which we denote by Γ_-. For any polynomial p, an element of Γ_p is a *global section*: it picks a direction for each position of p:

$$\Gamma_p \cong \prod_{P:p(1)} p[P] \qquad (2.3)$$

While Poly has many monoidal structures, in this article we are concerned with only two: the tensor (Dirichlet) product and the substitution product [Spivak (2022)]. The

[3] In general, a natural transformation φ is called *cartesian* if its naturality squares are pullbacks. A map φ of polynomials is cartesian in this sense iff it satisfies the above condition, that $\varphi_P^\sharp$ is a bijection.

tensor product $\otimes$ is given by the Day convolution of Cartesian product $\times$ in Set, and the substitution product $\triangleleft$ is given by functor composition. They have straightforward formulas:

$$p \otimes q = \sum_{P:p(1)} y^{p[P]} \otimes \sum_{Q:q(1)} y^{q[Q]} := \sum_{(P,Q):p(1)\times q(1)} y^{p[P]\times q[Q]} \tag{2.4}$$

$$p \triangleleft q = \sum_{P:p(1)} y^{p[P]} \triangleleft \sum_{Q:q(1)} y^{q[Q]} := \sum_{P:p(1)} \Big(\sum_{Q:q(1)} y^{q[Q]} \Big)^{p[P]} = \sum_{P:p(1)} \prod_{d:p[P]} \sum_{Q:q(1)} \prod_{e:q[Q]} y$$

We leave the proofs of the following statements as exercises for the reader.

2.4. Lemma. *For any polynomial p, its set of positions $p(1)$ is isomorphic to $1 \otimes p$ (and $p \otimes 1$).*

2.5. Proposition.

a. *The functor $A \mapsto Ay$ is strong monoidal $(\mathsf{Set}, 1, \times) \longrightarrow (\mathsf{Poly}, y, \otimes)$.*
b. *The functor $p \mapsto p(1)$ is strong monoidal $(\mathsf{Poly}, y, \otimes) \longrightarrow (\mathsf{Set}, 1, \times)$.*
c. *The functor $A \mapsto y^A$ is strong monoidal $(\mathsf{Set}^{\mathsf{op}}, 1, \times) \longrightarrow (\mathsf{Poly}, y, \otimes)$.*
d. *The functor $p \mapsto \Gamma_p$ is colax monoidal $(\mathsf{Poly}, y, \triangleleft) \longrightarrow (\mathsf{Set}^{\mathsf{op}}, 1, \times)$.*

2.6. Remark. Proposition 2.5-(a) is a special case of [Garner and Franco (2016), Example 24] once Poly is proven to be a normal duoidal category, see Lemma 3.4.

2.7. Linearly distributive categories (LDCs). Here we discuss various flavors of LDCs and the appropriate sorts of functors between them.

2.7.1. Flavors of LDCs. A **linearly distributive category (LDC)** is a category $\mathbb{X}$, equipped with two monoidal structures[4]

$$(\mathbb{X}, \otimes, \top) \quad \text{and} \quad (\mathbb{X}, \triangleleft, \bot)$$

linked by natural transformations

$$\begin{aligned} \partial_L^L &: a \otimes (b \triangleleft c) \to (a \otimes b) \triangleleft c \\ \partial_R^R &: (b \triangleleft c) \otimes a \to b \triangleleft (c \otimes a) \end{aligned} \tag{2.5}$$

called *linear distributors*,[5] such that the the associators and unitors of the $\otimes$ and the $\triangleleft$ products interact coherently with the linear distributors [Blute et.al. (1996), Cockett and Seely (1997b)].

[4]The operation $a \triangleleft b$ is usually written as $a \oplus b$ in the literature.

[5]The maps ∂^L and ∂^R are called *linear* distributors because they do not copy a as in distributive categories:

$$a \times (b + c) \cong (a \times b) + (a \times c)$$

In this article, we use the word *linear* for two different but firmly-established notions. First, a polynomial is *linear* if it has the form Ay for some set A. Second, the terminology of linear distributive categories often appends the term “linear” to refer to various notions in that context, e.g. *linear duals*. Hopefully, this name-space collision will not cause confusion.

2.8. Example. *Every monoidal category is an LDC in which the two tensor products coincide.*

For more examples, see [Srinivasan (2021), Chapter 2].

A **symmetric LDC** is an LDC in which both monoidal structures are symmetric, i.e. having symmetry maps $\sigma_\otimes$ and $\sigma_\triangleleft$ such that the following diagram commutes:

$$\begin{tikzcd}
(a \triangleleft b) \otimes c \arrow[r, "\sigma_\otimes"] \arrow[d, "\partial_R^R"'] & c \otimes (a \triangleleft b) \arrow[r, "\sigma_\triangleleft"] & c \otimes (b \triangleleft a) \arrow[d, "\partial_L^L"] \\
a \triangleleft (b \otimes c) & a \triangleleft (c \otimes b) \arrow[l, "\sigma_\triangleleft"] & a \triangleleft (b \otimes c) \arrow[l, "\sigma_\otimes"]
\end{tikzcd}$$

In a symmetric LDC, the symmetry maps induce the following *permuting* distributivity maps:

$$\begin{aligned} \partial_R^L &: a \otimes (b \triangleleft c) \longrightarrow b \triangleleft (a \otimes c) \\ \partial_L^R &: (b \triangleleft c) \otimes a \longrightarrow (b \otimes a) \triangleleft c \end{aligned} \tag{2.6}$$

An LDC which is equipped with all the four linear distributivity maps is said to be **non-planar** [Cockett and Seely (1997b)].

Motivated by Poly, we consider LDCs in which only the tensor $\otimes$ is symmetric.

2.9. Definition. *A* **$\otimes$-symmetric LDC** $(\mathbb{X}, \otimes, \top, \triangleleft, \bot)$ *is an LDC such that:*

1. $(\mathbb{X}, \otimes, \top)$ *is a symmetric monoidal category (*parallel *structure), and*
2. $(\mathbb{X}, \triangleleft, \bot)$ *is a (not-necessarily-symmetric) monoidal category (*sequential *structure).*

2.10. Lemma. *All $\otimes$-symmetric LDCs are non-planar.*

Proof Sketch. In a $\otimes$-symmetric LDC, the symmetry of the tensor product $\otimes$ induces the two permuting distributivity maps as follows:

$$\partial_R^L := \left(a \otimes (b \triangleleft c) \xrightarrow{\sigma_\otimes} (b \triangleleft c) \otimes a \xrightarrow{\partial_R^R} b \triangleleft (c \otimes a) \xrightarrow{b \triangleleft \sigma_\otimes} b \triangleleft (a \otimes c) \right)$$

The other permuting map ∂_L^R, is constructed similarly. ■

An LDC satisfies the mix inference rule in linear logic if it has a map $\bot \longrightarrow \top$ satisfying certain coherences:

2.11. Definition. [Cockett and Seely (1997a)] *A* **mix LDC** *is an LDC with a map* $\mathsf{m} : \bot \longrightarrow \top$*, called the* **mix map**. *The mix map induces, for any $a, b : \mathbb{X}$, a natural map* $\mathsf{indep}_{a,b} : a \otimes b \longrightarrow a \triangleleft b$*, as shown as the dotted arrow in the following commutative diagram:*

$$\begin{tikzcd}
a \otimes b \arrow[r, "\mathsf{id} \otimes u_\triangleleft^{L^{-1}}"] \arrow[d, "(u_\triangleleft^R)^{-1} \otimes \mathsf{id}"'] \arrow[ddrr, dotted, "\mathsf{indep}_{a,b}"] & a \otimes (\bot \triangleleft b) \arrow[r, "\mathsf{id} \otimes (\mathsf{m} \triangleleft \mathsf{id})"] & a \otimes (\top \triangleleft b) \arrow[d, "\partial^L"] \\
(a \triangleleft \bot) \otimes b \arrow[d, "\partial^R"'] & & (a \otimes \top) \triangleleft b \arrow[d, "u_\otimes^R \triangleleft \mathsf{id}"] \\
a \triangleleft (\bot \otimes b) \arrow[r, "\mathsf{id} \triangleleft (\mathsf{m} \otimes \mathsf{id})"'] & a \triangleleft (\top \otimes b) \arrow[r, "\mathsf{id} \triangleleft u_\otimes^L"'] & a \triangleleft b
\end{tikzcd} \tag{2.7}$$

An **isomix LDC** *is a mix LDC in which the mix map* $\top \xrightarrow[\cong]{\mathsf{m}} \perp$ *is an isomorphism.*

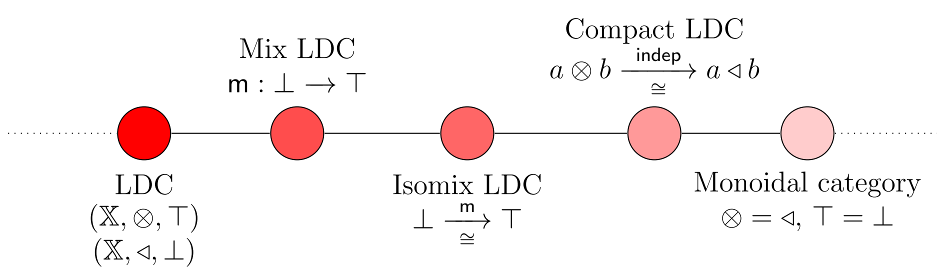
When $\mathsf{m} : \perp \longrightarrow \top$ is an isomorphism, the coherence requirement (2.7) is automatically satisfied (see [Cockett and Seely (1997a), Lemma 6.6]).

In the LDC literature, the indep map is usually referred to as the **mixor** and is written as mx. In this article, we use **indep**, because of its semantics in Poly; see below (3.3).

A **compact LDC** is an isomix LDC in which for all a, b, the map $\mathsf{indep}_{a,b} : a \otimes b \xrightarrow{\cong} a \triangleleft b$ is an isomorphism. In fact, any compact LDC is linearly equivalent to a monoidal category [Srinivasan (2021), Corollary 2.18]. The term 'compact' in the context of LDCs refers to the fact that the two monoidal products are isomorphic via the indep map.

A **monoidal category** is a compact LDC in which for all a, b, the map $\mathsf{indep}_{a,b} = \mathsf{id}_{a\otimes b}$ is the identity natural transformation.

The following schematic diagram summarizes some relevant flavors of LDCs:



We will be focused mainly on the center dot: we will see that Poly is an isomix LDC.

Notation. In this article, we write ∂^L and ∂^R for ∂^L_L and ∂^R_R respectively, and we will have very little use for the other two maps in (2.6). For the rest of the paper, the term LDC will refer to non-symmetric LDCs unless otherwise specified.

2.11.1. FUNCTORS BETWEEN LDCS. Linear functors [Cockett and Seely (1999)] are the most general notion of maps between LDCs. In this article we use a simpler notion: that of Frobenius functors [Cockett and Pastro (2009)].

2.12. DEFINITION. [Srinivasan (2021), Lemma 2.20] *Suppose* $\mathbb{X}$ *and* $\mathbb{Y}$ *are LDCs. A* **Frobenius functor** *consists of a functor* $F : \mathbb{X} \longrightarrow \mathbb{Y}$ *equipped with:*

(i) a lax monoidal structure $(F, m_\otimes, m_I) : (\mathbb{X}, \otimes, \top) \longrightarrow (\mathbb{Y}, \otimes, \top)$ *and*

(ii) a colax monoidal structure $(F, n_\triangleleft, n_I) : (\mathbb{X}, \triangleleft, \perp) \longrightarrow (\mathbb{Y}, \triangleleft, \perp)$

such that the laxors $m_\otimes$ *and* $n_\triangleleft$ *and the distributivity maps interact as follows:*

$$\textbf{[F.1]} \qquad \begin{tikzcd} F(a) \otimes F(b \triangleleft c) \arrow[r, "\mathsf{id} \otimes n_\triangleleft"] \arrow[d, "m_\otimes"'] & F(a) \otimes (F(b) \triangleleft F(c)) \arrow[d, "\partial^L"] \\ F(a \otimes (b \triangleleft c)) \arrow[d, "F(\partial^L)"'] & (F(a) \otimes F(b)) \triangleleft F(c) \arrow[d, "m_\otimes \triangleleft \mathsf{id}"] \\ F((a \otimes b) \triangleleft c) \arrow[r, "n_\triangleleft"'] & F(a \otimes b) \triangleleft F(c) \end{tikzcd}$$

$$[\mathbf{F.2}]\qquad \begin{tikzcd} F(a \triangleleft b) \otimes F(c) \arrow[r, "n_\triangleleft \otimes \mathsf{id}"] \arrow[d, "m_\otimes"'] & (F(a) \triangleleft F(b)) \otimes F(c) \arrow[d, "\partial^R"] \\ F((a \triangleleft b) \otimes c) \arrow[d, "F(\partial^R)"'] & F(a) \triangleleft (F(b) \otimes F(c)) \arrow[d, "\mathsf{id} \triangleleft m_\otimes"] \\ F(a \triangleleft (b \otimes c)) \arrow[r, "n_\triangleleft"'] & F(a) \triangleleft F(b \otimes c) \end{tikzcd}$$

2.13. Definition. [Srinivasan (2021), Definition 2.22] *Suppose $\mathbb{X}$ and $\mathbb{Y}$ are mix LDCs. A* **mix functor** *$F : \mathbb{X} \longrightarrow \mathbb{Y}$ is a Frobenius functor making the following diagram commute:*

$$\begin{tikzcd} F(\bot) \arrow[r, "n_I"] \arrow[d, "F(\mathsf{m})"'] & \bot \arrow[d, "\mathsf{m}"] \\ F(\top) & \top \arrow[l, "m_I"] \end{tikzcd}$$

A mix functor F is **isomix** *if $m_I : \top \longrightarrow F(\top)$ and $n_I : F(\bot) \longrightarrow \bot$ are isomorphisms.*

3. Poly is an isomix LDC

In this section, we prove that Poly is an isomix LDC. This result follows from the general observation that there is a faithful functor from the category of normal duoidal categories and functors to that of (non-planar) isomix LDCs and isomix Frobenius functors.

3.1. Definition. [Aguiar and Mahajan (2010), Definition 6.1.] *A* **duoidal category** *is a category $\mathbb{X}$ with two monoidal structures $(\mathbb{X}, \otimes, \top)$ and $(\mathbb{X}, \triangleleft, \bot)$ along with a natural transformation:*

$$\mathsf{duo} : \; (a \triangleleft b) \otimes (c \triangleleft d) \longrightarrow (a \otimes c) \triangleleft (b \otimes d)$$

called the **interchange law**, *and morphisms:*

$$e_\top : \top \longrightarrow \top \triangleleft \top$$
$$e_\bot : \bot \otimes \bot \longrightarrow \bot$$
$$k : \top \longrightarrow \bot$$

such that the functors $\triangleleft : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}$ and $\bot : \mathbb{1} \longrightarrow \mathbb{X}$ are $\otimes$-lax monoidal, and the associator and unitor natural isomorphisms of $(\triangleleft, \bot)$ are $\otimes$-monoidal natural transformations. From the above data, it can be proven that the following squares commute [Aguiar and Mahajan (2010), Proposition 6.9.]:

$$\begin{tikzcd} (\top \triangleleft \bot) \otimes (\bot \triangleleft \top) \arrow[d, "\mathsf{duo}"'] & \top \otimes \top \arrow[l, "\cong"'] \arrow[d, phantom, "="] & \top \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "k"] & \top \otimes \top \arrow[r, "\cong"] \arrow[d, phantom, "="] & (\bot \triangleleft \top) \otimes (\top \triangleleft \bot) \arrow[d, "\mathsf{duo}"] \\ (\top \otimes \bot) \triangleleft (\bot \otimes \top) \arrow[r, "\cong"'] & \bot \triangleleft \bot \arrow[r, "\cong"'] & \bot & \bot \triangleleft \bot \arrow[l, "\cong"] & (\bot \otimes \top) \triangleleft (\top \otimes \bot) \arrow[l, "\cong"] \end{tikzcd}$$

We say that a duoidal category is **normal** *if the above composite k is an isomorphism.*

3.2. Definition. [Aguiar and Mahajan (2010), Definition 6.50.] *A* **bilax duoidal functor** $F : (\mathbb{X}, \otimes, \top, \triangleleft, \bot) \to (\mathbb{Y}, \otimes, \top, \triangleleft, \bot)$ *consists of a functor* $F : \mathbb{X} \to \mathbb{Y}$ *that is* $\otimes$*-lax monoidal,* $\triangleleft$*-colax monoidal, and such that the following diagrams commute:*

[**D.1**]
$$\begin{tikzcd}
F((a \triangleleft b) \otimes (c \triangleleft d)) \arrow[d, "F(\mathsf{duo})"'] & F(a \triangleleft b) \otimes F(c \triangleleft d) \arrow[l, "m_\otimes"'] \arrow[r, "n_\triangleleft \otimes n_\triangleleft"] & (F(a) \triangleleft F(b)) \otimes (F(c) \triangleleft F(d)) \arrow[d, "\mathsf{duo}"] \\
F((a \otimes c) \triangleleft (b \otimes d)) \arrow[r, "n_\triangleleft"'] & F(a \otimes c) \triangleleft F(b \otimes d) & (F(a) \otimes F(c)) \triangleleft (F(b) \otimes F(d)) \arrow[l, "m_\otimes \triangleleft m_\otimes"]
\end{tikzcd}$$

[**D.2**]
$$\begin{tikzcd}
\top \arrow[r, "m_I"] \arrow[d, "e_\top"'] & F(\top) \arrow[r, "F(e_\top)"] & F(\top \triangleleft \top) \arrow[d, "n_\triangleleft"] \\
\top \triangleleft \top \arrow[rr, "m_I \triangleleft m_I"'] & & F(\top) \triangleleft F(\top)
\end{tikzcd}$$

[**D.3**]
$$\begin{tikzcd}
F(\bot) \otimes F(\bot) \arrow[rr, "n_I \otimes n_I"] \arrow[d, "m_\otimes"'] & & \bot \otimes \bot \arrow[d, "e_\bot"] \\
F(\bot \otimes \bot) \arrow[r, "F(e_\bot)"'] & F(\bot) \arrow[r, "n_I"'] & \bot
\end{tikzcd}$$

[**D.4**]
$$\begin{tikzcd}
\top \arrow[r, "m_I"] \arrow[d, "k"'] & F(\top) \arrow[d, "F(k)"] \\
\bot & F(\bot) \arrow[l, "n_I"]
\end{tikzcd}$$

A bilax duoidal functor F *between normal duoidal categories is* **normal** *if the unit laxors are isomorphisms:*

$$m_I : \top \xrightarrow{\cong} F(\top) \qquad \textit{and} \qquad n_I : F(\bot) \xrightarrow{\cong} \bot$$

Let $\mathsf{normalDuo}$ be the category of normal duoidal categories and bilax normal duoidal functors. Let Isomix be the category of isomix categories and isomix functors.

3.3. Lemma. *There is a faithful functor* $H : \mathsf{normalDuo} \to \mathsf{Isomix}$.

Proof Sketch. Suppose $(\mathbb{X}, \otimes, \top, \triangleleft, \bot)$ is a normal duoidal category. We show that it can be given an isomix LDC structure by defining the linear distributivity and the mix maps. The linear distributivity maps are constructed from the duoidal map and the isomorphism $\top \xrightarrow[\cong]{k} \bot$ by introducing and eliminating monoidal units. The construction of the left distributor ∂^L is shown in (3.1).

$$\begin{tikzcd}
a \otimes (b \triangleleft c) \arrow[rr, "u_\triangleleft^{-1} \otimes \mathsf{id}", "\cong"'] \arrow[d, "\partial^L"'] \arrow[drr, phantom, "\partial^L :="] & & (a \triangleleft \bot) \otimes (b \triangleleft c) \arrow[d, "\mathsf{duo}"] \\
(a \otimes b) \triangleleft c & (\top \otimes b) \triangleleft (a \otimes c) \arrow[l, "\mathsf{id} \triangleleft u_\otimes", "\cong"'] & (a \otimes b) \triangleleft (\bot \otimes c) \arrow[l, "k^{-1}", "\cong"']
\end{tikzcd} \tag{3.1}$$

The **mix map** $\mathsf{m} : \bot \to \top$ is defined to be k^{-1}. The right distributivity map ∂^R and the two permuting distributivity maps ∂^L_R, ∂^R_L are constructed similarly.

Suppose $\mathbb{X}$ and $\mathbb{Y}$ are normal duoidal categories, and $F : \mathbb{X} \to \mathbb{Y}$ is a normal duoidal functor. Then, $H(F) := F$ is a Frobenius functor, see Fig. 1 for the proof that [F.1] holds.

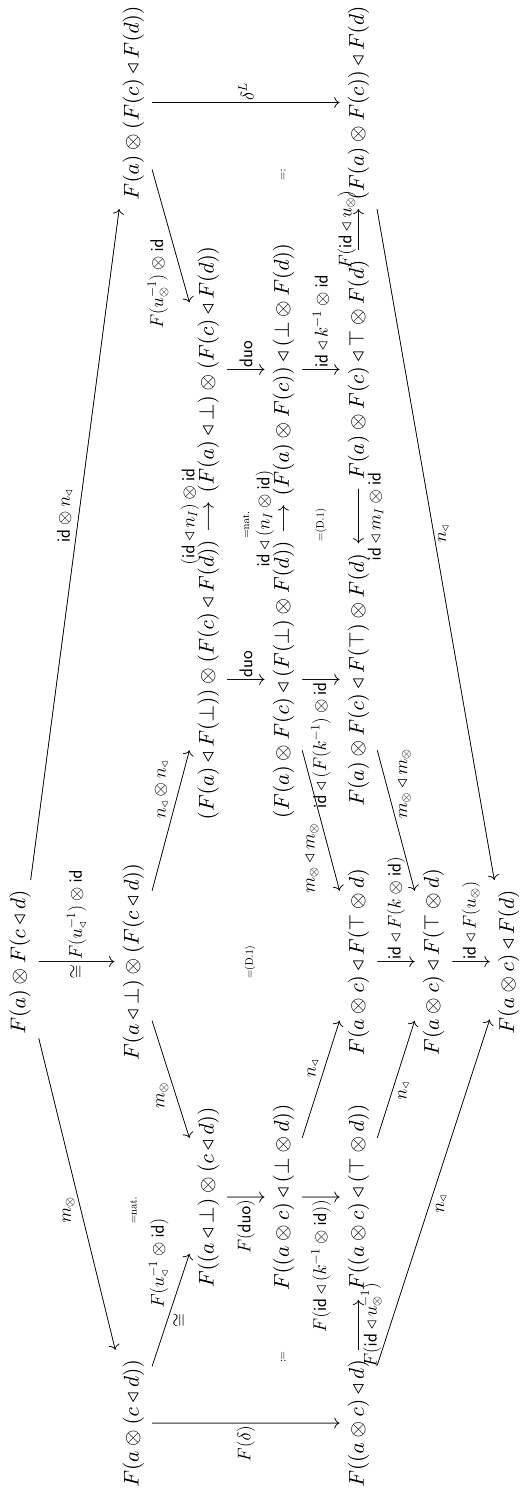


Figure 1: Normal duoidal functor is also isomix Frobenius

In Fig. 1, the inner squares shown without label commute because F is a lax monoidal functor or a co(lax)monoidal functor. Similary, it can be proven that [F.2] holds.

The normal duoidal functor F is also an isomix functor because $\top \xrightarrow{k} \bot$ is an isomorphism, and **[D.4]** holds. ■

The category of polynomial functors and natural transformations is normal duoidal:

3.4. Lemma. [Spivak (2022)] *The category* $(\mathsf{Poly}, \otimes, \triangleleft, y)$ *is normal duoidal.*

Proof Sketch of proof. For any symmetric monoidal structure $\cdot$ on Set, there is a symmetric monoidal structure $\odot$ on Poly which is the Day convolution of $\cdot$. The Cartesian product on Set induces the $\otimes$ structure on Poly given in Eq. (2.4). Thus, for $p, q : \mathsf{Poly}$, and $A, B : \mathsf{Set}$:

$$p(A) \times q(B) \xrightarrow{\mathsf{Day}} (p \otimes q)(A \times B)$$

natural in $A, B : \mathsf{Set}$, where $p(A) := p \triangleleft A$ denotes functor application. The above map satisfies the following universal property: for all functors $r \colon \mathsf{Set} \to \mathsf{Set}$ and functions $\phi : p(A) \times q(B) \to r(A \times B)$, there exists a unique (dotted) map which makes the following diagram commute:

$$\begin{tikzcd} p(A) \times q(B) \arrow[r, "\mathsf{Day}"] \arrow[dr, "\phi"'] & (p \otimes q)(A \times B) \arrow[d, dotted] \\ & r(A \times B) \end{tikzcd} \tag{3.2}$$

With this data, one can prove that $(\mathsf{Poly}, \otimes, y, \triangleleft, y)$ is normal duoidal. Indeed, to give the duoidal map d for $p_1, p_2, q_1, q_2 : \mathsf{Poly}$,

$$d : (p_1 \triangleleft p_2) \otimes (q_1 \triangleleft q_2) \longrightarrow (p_1 \otimes q_1) \triangleleft (p_2 \otimes q_2)$$

it suffices to give, naturally for all $A, B : \mathsf{Set}$, a map of the form

$$d_{A,B} : \big((p_1 \triangleleft p_2) \otimes (q_1 \triangleleft q_2)\big)(A \times B) \longrightarrow (p_1 \otimes q_1) \triangleleft (p_2 \otimes q_2)(A \times B)$$

And by the universal property of the Day convolution product, in order to give $d_{A,B}$, it is enough to provide a map of the form

$$\phi : (p_1 \triangleleft p_2)(A) \times (q_1 \triangleleft q_2)(B) \longrightarrow (p_1 \otimes q_1) \triangleleft (p_2 \otimes q_2)(A \times B)$$

which can be constructed as follows:

$$\begin{aligned} (p_1 \triangleleft p_2)(A) \times (q_1 \triangleleft q_2)(B) &= (p_1 \triangleleft (p_2(A))) \times (q_1 \triangleleft (q_2(B))) \\ &\xrightarrow{\mathsf{Day}} (p_1 \otimes q_1)(p_2(A) \times q_2(B)) \\ &\xrightarrow{\mathsf{id} \triangleleft \mathsf{Day}} (p_1 \otimes q_1)\big((p_2 \otimes q_2)(A \times B)\big) \\ &= (p_1 \otimes q_1) \triangleleft (p_2 \otimes q_2)(A \times B) \end{aligned}$$

Then, one can define the duoidal map as the unique map given by the universal property of the Day convolution product as in (3.2). ■

It has been brought to our attention by one of our reviewers that the normal duoidal structure of Poly given by Dirichlet and functor composition is likely to be closely related[6] to the normal duoidal structure of generalized species given by arithmetic and substitution products [Maia and Méndez (2008), Garner and Franco (2016), Gambino et. al. (2024)]. It might be an interesting future direction to explore this connection further.

3.5. Corollary. *The category* $(\mathsf{Poly}, \otimes, \triangleleft, y)$ *is an isomix non-planar linearly distributive category.*

Proof. By Lemmas 3.3 and 3.4, the category Poly is isomix LDC. By Lemma 2.10, it is non-planar because it is $\otimes$-symmetric. ∎

Now that we know that Poly is an isomix LDC, let us examine its mix and indep maps. In Poly, the units of the $\otimes$- and the $\triangleleft$-products coincide: $\perp = \top = y$. Hence, the mix map m can be taken to be the identity map. For any two polynomials p and q, the indep map $p \otimes q \xrightarrow{\mathsf{indep}} p \triangleleft q$ is defined as follows:

$$\sum_{P:p(1)} \sum_{Q:q(1)} \prod_{d:p[P]} \prod_{b:q[Q]} y \xrightarrow{\mathsf{indep}} \sum_{P:p(1)} \prod_{d:p[P]} \sum_{Q:q(1)} \prod_{b:q[Q]} y$$

$$\mathsf{indep}_1 : (P, Q) \mapsto (P, constQ : p[P] \to q(1); d \mapsto Q)$$

$$\mathsf{indep}^\sharp_{P,Q} : (d, b) \mapsto (d, b) : p[P] \times q[Q] \tag{3.3}$$

The fact that, under this map, Q arrives in the codomain as a constant, i.e. that it is *independent* of d, is the reason for the name indep. In passing, we note that for each $P : p(1)$ and $Q : q(1)$, the function $\mathsf{indep}^\sharp_{P,Q}$ is a bijection, and hence we record the following.

3.6. Corollary. *For any* $p, q : \mathsf{Poly}$*, the map* $\mathsf{indep}_{p,q}$ *is cartesian.*

4. Linear duals

In this section, we introduce biclosed LDCs and characterize the dual objects in these categories. The category Poly is a biclosed LDC. We apply the general results on biclosed LDCs to identify the dual objects in Poly.

[6] It seems likely that the normal duoidal structure on Poly described in Lemma 3.4 may be seen to arise from more general two-dimensional structure. For comparison, the category of species is also equipped with a normal duoidal structure [Garner and Franco (2016)], which was shown in [Gambino et. al. (2024)] to arise from a normal oplax monoidal structure on the bicategory of coloured symmetric sequences. Abstractly, the bicategory of coloured symmetric sequences is the Kleisli bicategory for a pseudo distributive law between the symmetric monoidal category 2-monad and the presheaf construction relative pseudo monad [Fiore et.al. (2018)]. The category of species arises as the presheaf construction on the free symmetric monoidal category on the terminal category 1, and thus as the endo-hom of the bicategory of coloured symmetric sequences on 1. Similarly, the category of polynomial functors arises as the free cocartesian category on the free cartesian category on 1. It is therefore natural to expect that the normal duoidal structure on Poly arises from normal oplax monoidal structure on the Kleisli bicategory for a pseudo distributive law between the cartesian category 2-monad and the cocartesian category 2-monad.

4.1. Linear duals in LDCs. Let us recall the definition of dual objects in LDCs.

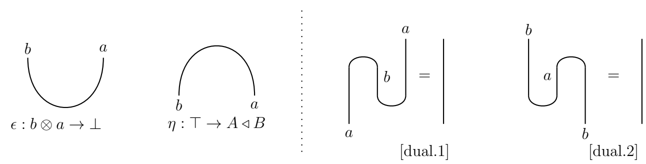
4.2. Definition. [Srinivasan (2021), Def. 2.8.] *Suppose* $\mathbb{X}$ *is a LDC and* $a, b : \mathbb{X}$ *are objects. Then* b *is* **left dual** *to* a*—and* a *is* **right dual** *to* b*—written as*

$$(\eta, \epsilon) : b \dashv\!\!\mid a$$

if there exists a unit map $\eta : \top \longrightarrow a \triangleleft b$*, and a counit map* $\epsilon : b \otimes a \longrightarrow \bot$ *such that the following diagrams commute:*

$$\textbf{[dual.1]}\quad \begin{tikzcd} a \arrow[r, "u_\otimes^{-1}"] \arrow[d, equal] & \top \otimes a \arrow[r, "\eta \otimes \mathsf{id}"] & (a \triangleleft b) \otimes a \arrow[d, "\partial^R"] \\ a & a \triangleleft \bot \arrow[l, "u_\triangleleft"'] & a \triangleleft (b \otimes a) \arrow[l, "1 \triangleleft \epsilon"'] \end{tikzcd} \qquad \textbf{[dual.2]}\quad \begin{tikzcd} b \arrow[r, "u_\otimes^{-1}"] \arrow[d, equal] & b \otimes \top \arrow[r, "\mathsf{id} \otimes \eta"] & b \otimes (a \triangleleft b) \arrow[d, "\partial^L"] \\ b & a \triangleleft \bot \arrow[l, "u_\triangleleft"'] & (b \otimes a) \triangleleft b \arrow[l, "\epsilon \triangleleft 1"'] \end{tikzcd} \tag{4.1}$$

Note that the above coherence conditions involve mainly η, ϵ and the linear distributivity maps. For easy understanding of the coherences, the η and the ϵ can be drawn as a cap and a cup respectively (by hiding the units), even though they refer to different monoidal products. The two coherence diagrams **[dual.1]** and **[dual.2]** can then be drawn as snake diagrams, as shown below.[7]

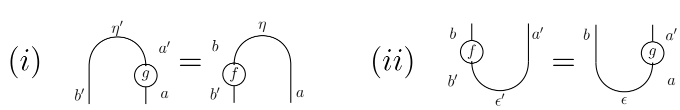


We record the following definition and lemma for completeness, but they will not be used again in the paper.

4.3. Definition. [Srinivasan (2021), Definition 2.10] *A morphism of linear duals,* $(f, g) : (\eta, \epsilon) \longrightarrow (\eta', \epsilon')$*, is given by a pair of maps*

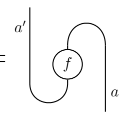
$$\begin{tikzcd} b \arrow[r, "(\eta,\epsilon)", "\dashv\!\mid"'] \arrow[d, "f"'] & a \\ b' \arrow[r, "(\eta',\epsilon')"', "\dashv\!\mid"] & a' \arrow[u, "g"'] \end{tikzcd}$$

such that the following equations hold:



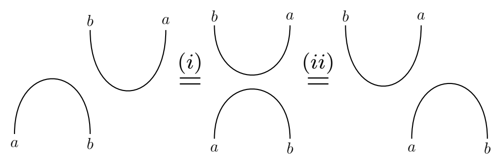
[7]All string diagrams in this article are to be read top-to-bottom, i.e. in the direction of gravity.

Notice that a morphism of duals is determined completely by either of the pair of maps (f completely determines g, and vice versa), and are referred to as Australian mates [Cockett et. al. (2000)].

$g :=$ 

The map f is an isomorphism if and only if g is an isomorphism. Also, notice that if f or g is an isomorphism, equations (i) and (ii) imply one another in the definition of morphism of duals.

4.4. Lemma. *In a mix LDC, we have the following equations:*



Equivalently, the following squares commute:

$$(i)\qquad \begin{tikzcd}
b \otimes a \arrow[r, "\epsilon"] \arrow[d, "u_\otimes^{-1}"'] & \bot \arrow[r, "\mathsf{m}"] & \top \arrow[r, "\eta"] & a \triangleleft b \arrow[dd, equal] \\
\top \otimes (b \otimes a) \arrow[d, "\eta \otimes \mathsf{id}"'] & & & \\
(a \triangleleft b) \otimes (b \otimes a) \arrow[r, "\mathsf{id} \otimes \epsilon"'] & (a \triangleleft b) \otimes \bot \arrow[r, "\mathsf{id} \otimes \mathsf{m}"'] & (a \triangleleft b) \otimes \top \arrow[r, "u_\otimes"'] & a \triangleleft b
\end{tikzcd}$$

$$(ii)\qquad \begin{tikzcd}
b \otimes a \arrow[r, "\epsilon"] \arrow[d, "u_\otimes^{-1}"'] & \bot \arrow[r, "\mathsf{m}"] & \top \arrow[r, "\eta"] & a \triangleleft b \arrow[dd, equal] \\
(b \otimes a) \otimes \top \arrow[d, "\mathsf{id} \otimes \eta"'] & & & \\
(b \otimes a) \otimes (a \triangleleft b) \arrow[r, "\epsilon \otimes \mathsf{id}"'] & \bot \otimes (a \triangleleft b) \arrow[r, "\mathsf{m} \otimes \mathsf{id}"'] & \top \otimes (a \triangleleft b) \arrow[r, "u_\otimes"'] & a \triangleleft b
\end{tikzcd}$$

Proof. We prove below that (i) commutes. The proof that (ii) commutes follows similarly.

$$\begin{tikzcd}
b \otimes a \arrow[r, "\epsilon"] \arrow[d, "u_\otimes^{-1}"'] \arrow[dr, phantom, "=\text{nat.}"] & \bot \arrow[r, "\mathsf{m}"] \arrow[d, "u_\otimes^{-1}"] \arrow[dr, phantom, "=\text{nat.}"] & \top \arrow[r, "\eta"] \arrow[d, "u_\otimes^{-1}"] \arrow[dr, phantom, "=\text{inv.}"] & a \triangleleft b \arrow[dd, equal, bend left] \\
\top \otimes (b \otimes a) \arrow[r, "\mathsf{id} \otimes \epsilon"] \arrow[d, "\eta \otimes \mathsf{id}"'] \arrow[dr, phantom, "= \otimes\text{-bifunctor}"] & \top \otimes \bot \arrow[r, "\mathsf{id} \otimes \mathsf{m}"] \arrow[d, "\eta \otimes \mathsf{id}"] \arrow[dr, phantom, "= \otimes\text{-bifunctor}"] & \top \otimes \top \arrow[r, "u_\otimes"] \arrow[d, "\eta \otimes \mathsf{id}"] \arrow[dr, phantom, "=\text{nat.}"] & \top \arrow[d, "\eta"] \\
(a \triangleleft b) \otimes (b \otimes a) \arrow[r, "\mathsf{id} \otimes \epsilon"'] & (a \triangleleft b) \otimes \bot \arrow[r, "\mathsf{id} \otimes \mathsf{m}"'] & (a \triangleleft b) \otimes \top \arrow[r, "u_\otimes"'] & (a \triangleleft b)
\end{tikzcd}$$

∎

4.5. Duals in biclosed LDCs. In this section, we discuss what happens when the parallel structure ($\otimes$) has a left closure and the sequential structure ($\triangleleft$) has a right coclosure. These are related to dual objects in LDCs.

4.6. Definition. *We define a linearly distributive category* $(\mathbb{X}, \otimes, \top, \triangleleft, \bot)$ *to be:*

- $\otimes$**-closed** *if for all* $a : \mathbb{X}$*, the functor* $- \otimes a : \mathbb{X} \longrightarrow \mathbb{X}$ *has a right adjoint, denoted* $a \multimap -$*,*
- $\triangleleft$**-coclosed** *if for all* $a : \mathbb{X}$*, the functor* $- \triangleleft a : \mathbb{X} \longrightarrow \mathbb{X}$ *has a left adjoint, denoted* $a/-$*,*
- **biclosed** *if* $\mathbb{X}$ *is both* $\otimes$*-closed and* $\triangleleft$*-coclosed.*

Let a be any object in a $\otimes$-closed LDC. Then, by the adjunction, for all b, and c, we have an isomorphism:

$$\mathbb{X}(a \otimes b, c) \cong \mathbb{X}(b, a \multimap c) \tag{4.2}$$

We refer to the counit of the $a \otimes - \dashv a \multimap -$ adjunction as the *eval* map:

$$\mathsf{ev} : a \otimes (a \multimap b) \longrightarrow b \tag{4.3}$$

Let a be any object in a $\triangleleft$-coclosed LDC. Then, by the adjunction, for all b, and c, we have an isomorphism:

$$\mathbb{X}\,(a/c, b) \cong \mathbb{X}(a, b \triangleleft c) \tag{4.4}$$

We refer to the counit of the $- \triangleleft b \dashv -/b$ adjunction as the *coeval* map:

$$\mathsf{coev} : a \longrightarrow (a/b) \triangleleft b \tag{4.5}$$

4.7. Example. [Spivak (2022), Niu and Spivak (2008)] *The category* Poly *is a biclosed LDC where, for any two polynomials* p, q*:*

$$p \multimap q := [p, q] = \prod_{P:p(1)} \sum_{Q:q(1)} \prod_{d:q[Q]} \sum_{d':p[P]} y \tag{4.6}$$

$$p/q := \left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right] = \sum_{P:p(1)} y^{q \triangleleft p[P]} = \sum_{P:p(1)} \prod_{Q:q(1)} \prod_{d:q[Q] \longrightarrow p[P]} y \tag{4.7}$$

Although we will not need it, the coclosure $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$ *is the left Kan extension of* p *along* q*:*

$$\begin{tikzcd}
\mathsf{Set} \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\Downarrow \mathsf{coev}"] & \mathsf{Set} \\
\mathsf{Set} \arrow[ur, bend right, "{\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]}"'] &
\end{tikzcd}$$

In the rest of the paper, without loss of generality, we will assume $\top = \bot$ for isomix LDCs. For an isomix LDC, we will denote the monoidal unit by y. Since we are mainly concerned with Poly category, in the rest of the paper, we will use the notation $[a, b]$ for $a \multimap b$, and $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]$ for a/b for readability and uniformity.

For any object $a : \mathbb{X}$ in a biclosed isomix LDC, there are two sorts of "double dual" maps, Φ_a and Ψ_a, each of which uses both the $\otimes$-closure and the $\triangleleft$-coclosure. They are defined as the following composites:

(For brevity, we suppress identities and the unitors $u_\otimes$ and $u_\triangleleft$ in (4.8) and (4.9). We follow this convention in the rest of the document when writing proofs and composites. For full version of the equations, see Appendix A.)

$$\begin{tikzcd} a \arrow[r, "\mathsf{coev}"] \arrow[rrr, bend right, "\Phi_a"'] & \left(\left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right] \triangleleft [a,y]\right) \otimes a \arrow[r, "\partial^R"] & \left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right] \triangleleft (([a,y]) \otimes a) \arrow[r, "\mathsf{ev}"] & \left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right] \end{tikzcd} \tag{4.8}$$

$$\begin{tikzcd} \left[\left[\begin{smallmatrix}a\\ y\end{smallmatrix}\right], y\right] \arrow[r, "\mathsf{coev}"] \arrow[rrr, bend right, "\Psi_a"'] & \left[\left[\begin{smallmatrix}a\\ y\end{smallmatrix}\right], y\right] \otimes \left(\left[\begin{smallmatrix}a\\ y\end{smallmatrix}\right] \triangleleft a\right) \arrow[r, "\partial^L"] & \left(\left[\left[\begin{smallmatrix}a\\ y\end{smallmatrix}\right], y\right] \otimes \left[\begin{smallmatrix}a\\ y\end{smallmatrix}\right]\right) \triangleleft a \arrow[r, "\mathsf{ev}"] & a \end{tikzcd} \tag{4.9}$$

4.8. THEOREM. *For an object a in a biclosed isomix LDC, the following are equivalent:*

(i) Φ_a has a retraction,

$$a \xrightarrow{\Phi_a} \left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right] \xrightarrow{\chi_a} a$$

and the following composite is the identity on $[a, y]$,

$$[a,y] \xrightarrow{\mathsf{coev}} [a,y] \otimes \left(\left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right] \triangleleft [a,y]\right) \xrightarrow{\partial^R} \left([a,y] \otimes \left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right]\right) \triangleleft [a,y]$$
$$\xrightarrow{\chi_a} ([a,y] \otimes a) \triangleleft [a,y] \xrightarrow{\mathsf{ev}} [a,y]$$

(ii) $[a, y] \dashv\!\vert\; a$, with $\epsilon = (\mathsf{ev} : [a, y] \otimes a \longrightarrow y)$.

PROOF.

(i) $\Rightarrow$ (ii): Given that (i) holds, we must show that $[a, y] \dashv\!\vert\; a$ with $\epsilon = \mathsf{ev}$. Define η as follows:

$$\eta := \left(y \xrightarrow{\mathsf{coev}} \left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right] \triangleleft [a,y] \xrightarrow{\chi_a} a \triangleleft [a,y]\right) \tag{4.10}$$

The following diagram proves that **[dual.1]** from Definition 4.2 holds:

$$
\begin{tikzcd}
a \arrow[r, "\mathsf{coev}"] \arrow[rr, bend left, "\eta"] \arrow[dd, equal] \arrow[dr, phantom, "=\text{assumption}"] & \left(\left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right] \triangleleft [a,y]\right) \otimes a \arrow[r, "\chi_a"] \arrow[d, "\partial^R"] \arrow[dr, phantom, "=\text{nat.}"] & (a \triangleleft [a,y]) \otimes a \arrow[ddl, bend left, "\partial^R"] \\
& \left[\begin{smallmatrix}[a,y]\\ a\end{smallmatrix}\right] \triangleleft ([a,y] \otimes a) \arrow[d, "\chi_a"] & \\
a & a \triangleleft ([a,y] \otimes a) \arrow[l, "\epsilon = \mathsf{ev}"'] &
\end{tikzcd}
$$

The proof that **[dual.2]** holds is similar.

(ii) $\Rightarrow$ *(i)*: Given that $[a,y] \dashv\!\!| \, a$ with $\epsilon = \mathsf{ev}$, we must prove that *(i)* holds. Since the LDC is $\triangleleft$-coclosed, we know that that $- \triangleleft [a,y]$ has a left adjoint as shown in Eq. (4.4). Define $\chi_a \colon \left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right] \longrightarrow a$ to be the universal map induced by η:

$$
\begin{tikzcd}
y \arrow[d, "\mathsf{coev}"'] \arrow[dr, "\eta"] & \\
\left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right] \triangleleft [a,y] \arrow[r, dotted, "\chi_a \triangleleft \mathsf{id}"'] & a \triangleleft [a,y]
\end{tikzcd}
$$

We first prove that the specified composite is indeed the identity on $[a,y]$:

$$
\begin{tikzcd}
{[a,y]} \arrow[r, "\mathsf{coev}"] \arrow[d, equal] \arrow[dr, phantom, "=\text{univ. prop.}"] & {[a,y]} \otimes \left(\left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right] \triangleleft [a,y]\right) \arrow[r, "\partial^L"] \arrow[d, "\chi_a"] \arrow[dr, phantom, "=\text{nat.}"] & \left([a,y] \otimes \left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right]\right) \arrow[d, "\chi_a"] \\
{[a,y]} \arrow[r, "\eta"] \arrow[drr, bend right, equal] & {[a,y]} \otimes ([a,y] \triangleleft a) \arrow[r, "\partial^L"] \arrow[dr, phantom, "=\text{[dual.2]}"] & ([a,y] \otimes a) \triangleleft a \arrow[d, "\mathsf{ev}"] \\
& & {[a,y]}
\end{tikzcd}
$$

Next we show that $\Phi_a \mathbin{⨾} \chi_a$ is a retract, completing the proof:

$$
\begin{tikzcd}
a \arrow[r, "\mathsf{coev}"] \arrow[rrr, bend left, "\Phi_a"] \arrow[d, equal] \arrow[dr, phantom, "=\text{couniv. prop.}"] & \left(\left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right] \triangleleft [a,y]\right) \otimes a \arrow[r, "\partial^L"] \arrow[d, "\chi_a"] \arrow[dr, phantom, "=\text{nat.}"] & \left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right] \triangleleft ([a,y] \otimes a) \arrow[r, "\mathsf{ev}"] \arrow[d, "\chi_a"] \arrow[dr, phantom, "=\text{nat.}"] & \left[\begin{smallmatrix}[a,y]\\ y\end{smallmatrix}\right] \arrow[d, "\chi_a"] \\
a \arrow[r, "\eta"'] \arrow[rrr, bend right, equal] & (a \triangleleft [a,y]) \otimes a \arrow[r, "\partial^L"'] & a \triangleleft ([a,y] \otimes a) \arrow[r, "\mathsf{ev}"'] & a
\end{tikzcd}
$$

$\blacksquare$

4.9. Theorem. *For an object b in a biclosed isomix LDC, the following are equivalent:*

(i) Ψ_b *has a section,*

$$b \xrightarrow{\Omega_b} \left[\begin{bmatrix} b \\ y \end{bmatrix}, y\right] \xrightarrow{\Psi_b} b$$

and the following composite is the identity on $\begin{bmatrix} b \\ y \end{bmatrix}$,

$$\begin{bmatrix} b \\ y \end{bmatrix} \xrightarrow{\mathsf{coev}} \left(\begin{bmatrix} b \\ y \end{bmatrix} \triangleleft b\right) \otimes \begin{bmatrix} b \\ y \end{bmatrix} \xrightarrow{\partial} \begin{bmatrix} b \\ y \end{bmatrix} \triangleleft \left(b \otimes \begin{bmatrix} b \\ y \end{bmatrix}\right) \xrightarrow{\Omega_b} \begin{bmatrix} b \\ y \end{bmatrix} \triangleleft \left(\left[\begin{bmatrix} b \\ y \end{bmatrix}, y\right] \otimes \begin{bmatrix} b \\ y \end{bmatrix}\right) \xrightarrow{\mathsf{ev}} \begin{bmatrix} b \\ y \end{bmatrix}$$

(ii) $b \dashv\!\!\mid \begin{bmatrix} b \\ y \end{bmatrix}$ *with* $\eta = \mathsf{coev}$ *maps* $y \longrightarrow \begin{bmatrix} b \\ y \end{bmatrix} \triangleleft b$

Proof Sketch of proof.

(i) $\Rightarrow$ *(ii)*: Given that (i) is true, we must show that $b \dashv\!\!\mid \begin{bmatrix} b \\ y \end{bmatrix}$ with $\eta = \mathsf{coev}$. Define ϵ as follows:

$$\epsilon := \left(b \otimes \begin{bmatrix} b \\ y \end{bmatrix} \xrightarrow{\Omega_b} \left[\begin{bmatrix} b \\ y \end{bmatrix}, y\right] \otimes \begin{bmatrix} b \\ y \end{bmatrix} \xrightarrow{\mathsf{ev}} y\right) \tag{4.11}$$

(ii) $\Rightarrow$ *(i)*: $b \dashv\!\!\mid \begin{bmatrix} b \\ y \end{bmatrix}$ with $\eta = \mathsf{coev}$, we must show that (i) holds. Since the LDC is $\otimes$-closed, we know that that $- \otimes \begin{bmatrix} b \\ y \end{bmatrix}$ has a right adjoint, see Eq. (4.2). Define $\Omega_b \colon b \to \left[\begin{bmatrix} b \\ y \end{bmatrix}, y\right]$ to be the universal map induced by ϵ:

$$\begin{tikzcd} b \otimes \begin{bmatrix} b \\ y \end{bmatrix} \arrow[r, dotted, "\Omega_b \otimes \mathsf{id}"] \arrow[dr, "\epsilon"'] & \left[\begin{bmatrix} b \\ y \end{bmatrix}, y\right] \otimes \begin{bmatrix} b \\ y \end{bmatrix} \arrow[d, "\mathsf{ev}"] \\ & y \end{tikzcd}$$

The rest follows similarly to the proof of Theorem 4.8. ∎

4.10. Theorem. *In a biclosed isomix LDC, suppose there are maps* $y \xrightarrow{\eta} a \triangleleft b$ *and* $b \otimes a \xrightarrow{\epsilon} y$ *defining a linear duality* $(\eta, \epsilon) : b \dashv\!\!\mid a$. *Then adjoint of* η, *that is* $\eta' \colon \begin{bmatrix} b \\ y \end{bmatrix} \to a$, *is a retraction and the adjoint of* ϵ, *that is* $\epsilon' \colon b \longrightarrow [a, y]$, *is a section.*

Proof. Given that $(\eta, \epsilon) : b \dashv\!\!\mid a$, we first prove that $\begin{bmatrix} b \\ y \end{bmatrix} \xrightarrow{\eta'} a$ is a retraction. Define the map $a \xrightarrow{\varphi} \begin{bmatrix} b \\ y \end{bmatrix}$ as the following composite:

$$\varphi := \left(a \xrightarrow{\mathsf{coev}} \left(\begin{bmatrix} b \\ y \end{bmatrix} \triangleleft b\right) \otimes a \xrightarrow{\partial^R} \begin{bmatrix} b \\ y \end{bmatrix} \triangleleft (b \otimes a) \xrightarrow{\epsilon} \begin{bmatrix} b \\ y \end{bmatrix}\right) \tag{4.12}$$

The following commuting diagram shows that $\varphi \fatsemi \eta' = \mathsf{id}_a$ making η' a retraction.

$$
\begin{tikzcd}
a \arrow[r, "\mathsf{coev}"] \arrow[dd, equal] \arrow[rr, bend left, "\eta"] & \left(\begin{bmatrix} b \\ y \end{bmatrix} \triangleleft b\right) \otimes a \arrow[r, "\eta'"] \arrow[d, "\partial^R"] & (a \triangleleft b) \otimes a \arrow[d, "\partial^R"] \\
{\scriptstyle =\text{Defn. }\varphi} & \begin{bmatrix} b \\ y \end{bmatrix} \triangleleft (b \otimes a) \arrow[r, "\eta'"'] \arrow[d, "\epsilon"] \arrow[ur, phantom, "{\scriptstyle =\text{nat.}}"] & a \triangleleft (b \otimes a) \arrow[d, "\epsilon"] \\
a \arrow[r, "\varphi"'] \arrow[rr, equal, bend right] & \begin{bmatrix} b \\ y \end{bmatrix} \arrow[r, "\eta'"'] \arrow[ur, phantom, "{\scriptstyle = \triangleleft-\text{bifunctor and nat. of } u_{\triangleleft}}"] & a
\end{tikzcd}
$$

Indeed, the large left square commutes by the definition of φ. The outer diagram commutes by **[dual.1]**. Thus, bottom triangle commutes, making η' a retraction.

Next we prove that $b \xrightarrow{\epsilon'} [a, y]$ is a section. Define the map $[a, y] \xrightarrow{\psi} b$ to be the following composite:

$$
\psi := \left([a, y] \xrightarrow{\eta} [a, y] \otimes (a \triangleleft b) \xrightarrow{\partial^R} ([a, y] \otimes a) \triangleleft b \xrightarrow{\mathsf{ev}} b\right)
$$

The following commuting diagram shows that $\epsilon' \fatsemi \psi = \mathsf{id}_b$, making ϵ' a section. The outer diagram commutes because of **[dual.2]**. The large right square is the definition of the ψ map.

$$
\begin{tikzcd}
b \arrow[r, "\epsilon'"] \arrow[d, "\eta"'] \arrow[rr, equal, bend left] \arrow[dr, phantom, "{\scriptstyle = \otimes\text{-bifunctor and nat. of } u_{\otimes}^{-1}}"] & {[a, y]} \arrow[r, "\psi"] \arrow[d, "\eta"'] & b \arrow[dd, equal] \\
b \otimes (a \triangleleft b) \arrow[r, "\epsilon'"] \arrow[d, "\partial^L"'] \arrow[dr, phantom, "{\scriptstyle =\text{nat.}}"] & {[a, y] \otimes (a \triangleleft b)} \arrow[d, "\partial^L"] \arrow[r, phantom, "{\scriptstyle \psi :=}"] & \\
(b \otimes a) \triangleleft b \arrow[r, "\epsilon'"] \arrow[rr, bend right, "\epsilon"'] & ([a, y] \otimes a) \triangleleft b \arrow[r, "\mathsf{ev}"] & b
\end{tikzcd}
$$

■

4.11. Duals in Poly. In this section, we characterize the dual objects of Poly using its left closure and right coclosure properties. We begin with some simple lemmas.

4.12. Lemma. *For any polynomial p, there is a natural isomorphism* $\begin{bmatrix} p \\ y \end{bmatrix} \cong y^{p(1)}$

Proof. By Eq. (4.7)

$$
\begin{bmatrix} p \\ y \end{bmatrix} \cong \begin{bmatrix} p \\ y^1 \end{bmatrix} \cong y^{p \triangleleft 1} \cong y^{p(1)}
$$

■

4.13. Lemma. *For any set A, there are natural isomorphisms in* Poly*:*

(i) $[Ay, y] \cong y^A$

(ii) $[y^A, y] \cong Ay$

Proof. By Eq. (4.6),

$$\begin{aligned} \textit{(i):} \quad [Ay, y] &\cong \prod_{a:A}\sum_{1}\prod_{1}\sum_{1} y \cong \prod_{a:A} y \cong y^A \\ \textit{(ii):} \quad [y^A, y] &\cong \prod_{1}\sum_{1}\prod_{1}\sum_{a:A} y \cong \sum_{a:A} y \cong Ay \end{aligned}$$

■

Let us examine the coeval map (4.5) in Poly,

$$\sum_{P:p(1)} y^{p[P]} = p \xrightarrow{\mathsf{coev}} \left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right] \triangleleft q = \sum_{P:p(1)} y^{q \triangleleft p[P]} \triangleleft q$$

We can write this out using only $\sum$ and $\prod$:

$$\sum_{P:p(1)}\prod_{d:p[P]} y \to \sum_{P:p(1)}\prod_{Q:q(1)}\prod_{d:q[Q]\to p[P]}\sum_{Q':q(1)}\prod_{b:q[Q']} y$$

The coeval map is defined in Poly as follows:

$$\mathsf{coev}_1 : P \mapsto (P, (Q, d) \mapsto Q) \qquad \text{and} \qquad \mathsf{coev}^\sharp_P : (Q, d, b) \mapsto d(b). \tag{4.13}$$

4.14. Theorem. *For any polynomial p :* Poly*, the following are equivalent:*

(i) $p = y^A$ *for some* $A : \mathsf{Set}$*,*

(ii) The map $\Phi_p : p \longrightarrow \begin{bmatrix} [p,y] \\ y \end{bmatrix}$ *from (4.8) is the identity,*

(iii) The map Φ_p *has a retraction,*

$$p \xrightarrow{\Phi_p} \begin{bmatrix} [p,y] \\ y \end{bmatrix} \xrightarrow{\chi_p} p$$

If one (and hence all) of (i), (ii), and (iii) holds, then it is also the case that the following composite is the identity on $[p, y]$*,*

$$\begin{aligned} [p, y] \xrightarrow{\mathsf{coev}} [p, y] \otimes \left(\begin{bmatrix} [p,y] \\ y \end{bmatrix} \triangleleft [p, y]\right) &\xrightarrow{\partial^L} \left([p, y] \otimes \begin{bmatrix} [p,y] \\ y \end{bmatrix}\right) \triangleleft [p, y] \\ &\xrightarrow{\chi_p} ([p, y] \otimes p) \triangleleft [p, y] \xrightarrow{\mathsf{ev}} [p, y] \end{aligned} \tag{4.14}$$

Proof.

(i)⇒ (ii): Given that $p = y^A$ for some A : Set, using Lemmas 4.12 and 4.13 we have that Φ_p is the composite of natural maps:

$$y^A \to (y^A \triangleleft Ay) \otimes y^A \to y^A \triangleleft (Ay \otimes y^A) \to y^A$$

and one can check directly that this is the identity. We rewrite it as

$$y^A \to A^A y^{A\times A} \to A^A y^{A\times A} \to y^A.$$

The first map sends the unique position to id_A and on directions sends the pair $(a_1, a_2) \mapsto a_2$. The second map is the identity. The third map sends a position $f\colon A \to A$ to the unique position and a direction $a : A$ to $(a, f(a))$. So the composite sends a direction a to $\mathsf{id}_A(a) = a$.

(ii) $\Rightarrow$ *(iii)* **:** Φ_p is the identity, hence is also a retract.

(iii) $\Rightarrow$ *(i)***:** Given that $\Phi_p \mathbin{⨟} \chi_p$ is a retract, since functors preserve retracts, we know that $\Phi_p(1) \mathbin{⨟} \chi_p(1)$ is a retract:

$$\begin{tikzcd} p(1) \arrow[r, "\Phi_p(1)"] \arrow[rr, bend right, equal] & y^{[p,y](1)}(1) \arrow[r, "\chi_p(1)"] & p(1) \end{tikzcd}$$

However $y^{[p,y](1)}(1) = 1$. Hence, we have that $p(1) = 1$. Thus $p = y^A$ for some $A : \mathsf{Set}$.

Finally, assuming that (i) holds, use (4.13) to prove that the composite $\Phi_p \mathbin{⨟} \chi_p$ is the identity map. ■

Using the above result, we can deduce that in Poly, each linear polynomial is left dual to its corresponding representable, as shown in the following corollary.

4.15. Corollary. *For any set A, there is a linear dual $Ay \dashv\!\mid y^A$ in* Poly*, with $\eta = \mathsf{coev}$ and $\epsilon = \mathsf{ev}$.*

Proof. Using Theorem 4.14 we have that the map Φ_{y^A} and therefore the map χ_{y^A} is the identity. Then, from Theorem 4.8, it follows that $[y^A, y] \dashv\!\mid y^A$ with $\epsilon = \mathsf{ev}$ and $\eta = \mathsf{coev} \mathbin{⨟} \chi_{y^A} = \mathsf{coev}$. Finally, from Lemma 4.13-(ii), we know that $[y^A, y] = Ay$. ■

4.16. Theorem. *For any polynomial p :* Poly*, the following are equivalent:*

(i) $q = Ay$ for some A : Set*,*

(ii) The map $\Psi_q : q \to \left[\begin{bmatrix} q \\ y \end{bmatrix}, y\right]$ in (4.9) is the identity,

(iii) The map Ψ_q from has a section,

$$q \xrightarrow{\Omega_q} \left[\begin{bmatrix} q \\ y \end{bmatrix}, y\right] \xrightarrow{\Psi_q} q$$

If one (and hence all) of (i), (ii), and (iii) holds, it is also the case that the following composite is the identity on $\begin{bmatrix} q \\ y \end{bmatrix}$.

$$\begin{bmatrix} q \\ y \end{bmatrix} \xrightarrow{\mathsf{coev}} \left(\begin{bmatrix} q \\ y \end{bmatrix} \triangleleft q\right) \otimes \begin{bmatrix} q \\ y \end{bmatrix} \xrightarrow{\partial^L} \begin{bmatrix} q \\ y \end{bmatrix} \triangleleft \left(q \otimes \begin{bmatrix} q \\ y \end{bmatrix}\right) \xrightarrow{\Omega_q} \begin{bmatrix} q \\ y \end{bmatrix} \triangleleft \left(\left[\begin{bmatrix} q \\ y \end{bmatrix}, y\right] \otimes \begin{bmatrix} q \\ y \end{bmatrix}\right) \xrightarrow{\mathsf{ev}} \begin{bmatrix} q \\ y \end{bmatrix} \tag{4.15}$$

Proof. The proof for *(i)* ⇒ *(ii)* and *(ii)* ⇒ *(iii)* are similar to that of Theorem 4.14. For *(iii)* ⇒ *(i)*, we have by Lemmas 4.12 and 4.13 that

$$\left[\begin{bmatrix} q \\ y \end{bmatrix}, y\right] = \left[y^{q(1)}, y\right] = q(1)y$$

Since functors preserve retracts, applying $\Gamma_{(-)} : \mathsf{Poly} \longrightarrow \mathsf{Set}^{\mathsf{op}}$ from Eq. (2.3) to the retract in (iii), we get the following composite to be a retract in Set:

$$\Gamma_q \xrightarrow{\Gamma_{\Psi_q}} \Gamma_{q(1)y} \xrightarrow{\Gamma_{\Omega_q}} \Gamma_q$$

Since $\Gamma_{q(1)y} = 1$, it follows that $\Gamma_q = 1$ and hence that $q[Q] = 1$ for all $Q : q(1)$. So $q = q(1)y$. ∎

Next we show that the linear polynomials and the representables are the only objects with duals in Poly:

4.17. Theorem. *In the category* Poly*, if* $(\eta, \epsilon) : q \dashv\!\vdash p$*, then* $p = y^A$ *and* $q = Ay$ *for some* $A : \mathsf{Set}$*.*

Proof. Given that $q \dashv\!\vdash p$ is a linear dual, we first show that for $A, B : \mathsf{Set}$, $p = y^A$ and $q = By$, and then that $A \cong B$.

We know from Theorem 4.10 the following composite is a retract:

$$p \xrightarrow{\varphi} \begin{bmatrix} q \\ y \end{bmatrix} \xrightarrow{\eta'} p \tag{4.16}$$

where φ is as in (4.12). By Lemma 4.12, we have $\begin{bmatrix} q \\ y \end{bmatrix} \cong y^{q(1)}$, and hence $\begin{bmatrix} q \\ y \end{bmatrix}(1) = 1$. Applying the functor $-(1) : \mathsf{Poly} \longrightarrow \mathsf{Set}$ to the retract in (4.16), we have the following retract:

$$p(1) \xrightarrow{\varphi_1} 1 \xrightarrow{\eta'_1} p(1)$$

so $p(1) = 1$, and hence $p = y^A$ for some $A : \mathsf{Set}$.

From Theorem 4.10, we have the other retract:

$$q \xrightarrow{\epsilon'} [p, y] \xrightarrow{\psi} q \tag{4.17}$$

From the previous step we know that $p = y^A$. Substituting, we get $[p, y] = [y^A, y] = Ay$. Hence, we have the following retracts:

$$q \xrightarrow{\epsilon'} Ay \xrightarrow{\psi} q$$

Again using Γ (as in the proof of Theorem 4.16), we find that $q = By$ for some $B : \mathsf{Set}$.

So we get $\begin{bmatrix} q \\ y \end{bmatrix} = \begin{bmatrix} By \\ y \end{bmatrix} = y^B$. Substituting in (4.16), we get the retracts:

$$\frac{y^A \xrightarrow{\varphi} y^B \xrightarrow{\eta'} y^A}{A \xrightarrow{\eta'^\sharp} B \xrightarrow{\varphi^\sharp} A}$$

Similarly, using the fact that $p = y^A$ for some $A : \mathsf{Set}$, we have that $[p, y] = [y^A, y] = Ay$. Substituting in (4.17), we get the following retracts:

$$\frac{By \xrightarrow{\epsilon'} Ay \xrightarrow{\psi} By}{B \xrightarrow{\epsilon'_1} A \xrightarrow{\psi_1} B}$$

These two retracts together form the isomorphism $A \cong B$. ■

4.18. Lemma. *In* Poly*, if $p \dashv\!\!\mid q$ and $q \dashv\!\!\mid p$ then $p = q = y$.*

Proof. This follows directly from Theorem 4.17. ■

5. The core of Poly

Central to mix LDCs is the notion of *core*. An important way in which compact LDCs (for all objects a, b, an isomorphism $a \otimes b \xrightarrow[\cong]{\mathsf{indep}} a \triangleleft b$) arise is as the *core* of a mix category. The core of a mix LDC is defined as follows.

5.1. Definition. [Blute et.al. (2000)] *An object u is in the* **core** *of a mix category if and only if the following natural transformations are isomorphisms:*

$$u \otimes - \xrightarrow{\mathsf{indep}_{u,-}} u \triangleleft - \qquad \textit{and} \qquad - \otimes u \xrightarrow{\mathsf{indep}_{-,u}} - \triangleleft u$$

To the best of our knowledge, the notion of core has been studied only within symmetric LDCs. In this article, motivated by Poly, we consider a more general notion of core, suitable in (possibly non-symmetric) LDCs.

5.2. Definition. *For a* **mix ($\otimes$-symmetric) LDC** $(\mathbb{X}, \otimes, \top, \triangleleft, \bot)$,

- *its* **left core** *is defined to be the full sub-category of objects $a : \mathbb{X}$ such that, for all $x : \mathbb{X}$, the map*
$$[\mathsf{core_l}] \qquad a \otimes x \xrightarrow[\cong]{\mathsf{indep}_{a,x}} a \triangleleft x$$
is an isomorphism.
- *its* **right core** *of $\mathbb{X}$ is the full sub-category of objects $a : \mathbb{X}$ such that, for all $x : \mathbb{X}$, the map*
$$[\mathsf{core_r}] \qquad x \otimes a \xrightarrow[\cong]{\mathsf{indep}_{x,a}} x \triangleleft a$$
is an isomorphism.

One may have empty left and right cores in a mix LDC. However, in an isomix category, the monoidal unit belongs to both the left and the right core.

5.3. Lemma. *If* $\mathbb{X}$ *is an isomix LDC with unit* y*, then* $y \in \mathsf{core}_\ell(\mathbb{X})$ *and* $y \in \mathsf{core}_\mathsf{r}(\mathbb{X})$*.*

In an isomix LDC, both the left and the right cores are compact LDCs: that is, for all objects $a, b : \mathsf{core}_\ell$, the map $a \otimes b \xrightarrow[\cong]{\mathsf{indep}} a \triangleleft b$ is an isomorphism, and same for core_r. Hence, by Corollary [Srinivasan (2021), Corollary 2.18] both the left and right cores are linearly equivalent to monoidal categories.

In a symmetric mix LDC $\mathbb{X}$, the left and the right cores coincide. The left and right cores of any isomix LDC may be related to each other as follows.

5.4. Definition. *A mix LDC* $(\mathbb{X}, \otimes, \top, \triangleleft, \bot)$ *is said to have* **opposing cores** *if there exists an isomorphism,*

$$\star : \mathsf{core}_\mathsf{r}(\mathbb{X})^{\mathsf{op}} \xrightarrow{\cong} \mathsf{core}_\ell(\mathbb{X})$$

5.5. Example. *Compact closed categories have opposing cores which coincide (the entire category is the core) and the* $\star$ *is the same as the contravariant involution.*

5.6. Example. *We will show that* Poly *has opposing cores in Corollary 5.9.*

The following theorem generalizes the result [Blute et.al. (2000)] that for any linear dual in a symmetric mix LDC, if the left dual is in the core then the right dual is also in the core, and vice versa. We prove an analogous result in non-symmetric isomix LDCs.

5.7. Lemma. *Let* $(\eta, \epsilon) : b \dashv\!\!| \, a$ *be a linear dual in an isomix LDC* $\mathbb{X}$*. Then* $b : \mathsf{core}_\mathsf{r}(\mathbb{X})$ *iff* $a : \mathsf{core}_\ell(\mathbb{X})$*.*

Proof. We have $\eta \colon y \longrightarrow a \triangleleft b$ and $\epsilon \colon (b \otimes a) \longrightarrow y$ satisfying the diagrams in (4.1).

(⇐): Suppose $b : \mathsf{core}_\mathsf{r}(\mathbb{X})$. We want to prove that $a : \mathsf{core}_\ell(\mathbb{X})$, i.e. that the map $\mathsf{indep}_{a,x} : a \otimes x \longrightarrow a \triangleleft x$ is an isomorphism for all $x : \mathbb{X}$.

Define a tentative inverse map $a \triangleleft x \longrightarrow a \otimes x$ as follows:

$$\begin{aligned} a \triangleleft x \xrightarrow{\eta} (a \triangleleft b) \otimes (a \triangleleft x) &\xrightarrow{\mathsf{indep}^{-1}_{a,b}} (a \otimes b) \otimes (a \triangleleft x) \\ &\xrightarrow[\cong]{a_\otimes} a \otimes (b \otimes (a \triangleleft x)) \xrightarrow{\partial^L} a \otimes ((b \otimes a) \triangleleft x) \xrightarrow{\epsilon} a \otimes x \end{aligned}$$

The second step uses the assumption that $b : \mathsf{core}_\mathsf{r}(\mathbb{X})$. The proof that the tentative inverse map defined as above is indeed the inverse of $\mathsf{indep}_{a,x}$ follows similarly to [Blute et.al. (2000), Proposition 5] and uses the inverse of the mix map m.

(⇒): [Sketch of proof] The proof of the converse is similar. Suppose $a : \mathsf{core}_\ell(\mathbb{X})$. We want to prove that $b : \mathsf{core}_\mathsf{r}(\mathbb{X})$, that is, the map $\mathsf{indep}_{x,b} : x \otimes b \longrightarrow x \triangleleft b$ is an isomorphism for all $x : \mathbb{X}$.

One checks that a map $x \triangleleft b \longrightarrow x \otimes b$ inverse to $\mathsf{indep}_{x,b}$ can be defined as follows:

$$\begin{aligned} x \triangleleft b \xrightarrow{\eta} (x \triangleleft b) \otimes (a \triangleleft b) &\xrightarrow{\mathsf{indep}^{-1}_{a,b}} (x \triangleleft b) \otimes (a \otimes b) \\ &\xrightarrow[\cong]{a_\otimes} ((x \triangleleft b) \otimes a) \otimes b \xrightarrow{\partial^L} (x \triangleleft (b \otimes a)) \otimes b \xrightarrow{\epsilon} x \otimes b \end{aligned}$$

■

The next result shows that in Poly only the linear polynomials reside in the left core, and only the representables reside in the right core.

5.8. LEMMA. *In the category* Poly*, the following statements hold for any polynomial* p*:*
(i) $p \in \mathsf{core}_\ell(\mathsf{Poly})$ *iff* $p \cong Ay$ *for some* $A : \mathsf{Set}$*, and*
(ii) $q \in \mathsf{core_r}(\mathsf{Poly})$ *iff* $q \cong y^A$ *for some* $A : \mathsf{Set}$*.*

PROOF.

(i) **(⇐):** Suppose $p \cong Ay$ and r is any polynomial. Then,

$$Ay \otimes r \cong Ar \cong Ay \triangleleft r$$

Thus, $Ay \in \mathsf{core}_\ell(\mathsf{Poly})$.

(⇒): Suppose $p : \mathsf{core}_\ell(\mathsf{Poly})$. So, for all $r : \mathsf{Poly}$, we have that $\mathsf{indep} : p \otimes r \longrightarrow p \triangleleft r$ is an isomorphism. Let $r \coloneqq 2$ (any set of cardinality > 1 will work). We have

$$p \otimes r = p \otimes 2 = \sum_{P:p(1)} 2$$

$$p \triangleleft r = p \triangleleft 2 = \sum_{P:p(1)} 2^{p[P]}$$

and in this case indep is the $p(1)$-indexed sum of diagonals, i.e. the function

$$\sum_{P:p(1)} 2 \xrightarrow{\sum_{P:p(1)} \Delta} \sum_{P:p(1)} 2^{p[P]}.$$

This is a bijection iff $p[P] = 1$ for all $P : p(1)$. Thus $p = p(1)y$ is of the required form.

(ii) **(⇐):** Suppose $q \cong y^A$, we must prove that for all $p : \mathsf{Poly}$, the map $p \otimes y^A \xrightarrow{\mathsf{indep}_{p,y^A}} p \triangleleft y^A$ is an isomorphism. Using (3.3), the map indep_{q,y^A} is defined as follows:

$$p \otimes y^A = \sum_{P:p(1)} \prod_{d:p[P]} \prod_{a:A} y \xrightarrow{\mathsf{indep}_{p,y^A}} \sum_{P:p(1)} \prod_{d:p[P]} \prod_{a:A} y = p \triangleleft y^A$$

$$\mathsf{indep}_1 : (P, !) \mapsto (P, \mathit{const}\,! : p[P] \to \{!\}; d \mapsto !)$$

$$\mathsf{indep}^\sharp_{P,Q} : (d, a) \mapsto (d, a) : p[P] \times q[Q]$$

Thus, the map indep_1 is an isomorphism, and $\mathsf{indep}^\sharp$ is always an isomorphism by Corollary 3.6. Thus, we have that $q : \mathsf{core_r}(\mathsf{Poly})$.

(⇒): Suppose $q : \mathsf{core_r}(\mathsf{Poly})$, so for all $x : \mathsf{Poly}$, the following map is an isomorphism:

$$\mathsf{indep}_{x,q} : x \otimes q \xrightarrow{\cong} x \triangleleft q$$

Then taking $x = 1$ we have $1 \otimes q \cong 1 \triangleleft q \cong 1$. But from Lemma 2.4 we also have $1 \otimes q \cong q(1)$. Hence $q(1) \cong 1$, making q a representable.

■

5.9. Corollary. *The category* Poly *has opposing cores.*

Proof. For any $A : \mathsf{Set}$, the object $\left(y^A\right)^\star := Ay$. By Lemma 5.8 this defines a mapping on objects (up to isomorphism, but any choice is sufficient).

For any $A, B : \mathsf{Set}$ and a map $\varphi : y^B \to y^A$, the map $\varphi^\star : Ay \to By$ is given by φ, where $(\varphi^\star)_1 := \varphi^\sharp$. ∎

It also follows from Lemma 5.8 that the left duals in Poly are precisely the left core of Poly and the right duals are the right core.

5.10. Corollary. *In the category* Poly*, for any two polynomials p and q, we have $q \dashv\!\!\!\mid p$ iff $q : \mathsf{core}_\ell(\mathsf{Poly})$ and $p : \mathsf{core}_\mathsf{r}(\mathsf{Poly})$.*

Proof. Suppose that $q \dashv\!\!\!\mid p$, then by Theorem 4.17, we have that $q = Ay$ and $p = y^A$ for some set A. The rest of the proof is the direct consequence of Lemma 5.8. For the converse, apply Corollary 4.15. ∎

6. Linear monoids, linear comonoids, linear bialgebras in Poly

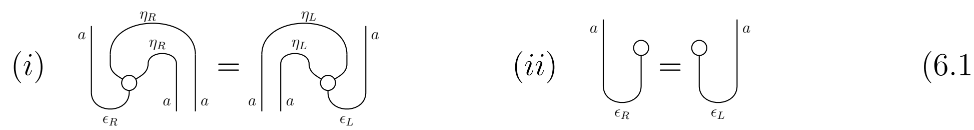
When an object in a cyclic linear dual ($b \dashv\!\!\!\mid a$ and $a \dashv\!\!\!\mid b$) is equipped with a $\otimes$-monoid structure, it is called a linear monoid. Similarly, if such an object is equipped with a $\otimes$-comonoid structure, it is called a linear comonoid. Linear monoids and linear comonoids in LDCs are analogous to Frobenius algebras in monoidal categories. Linear monoids and linear comonoids were studied in the context of quantum observables in [Srinivasan (2021), Chapter 9].

In Poly, the only cyclic linear dual is $a = b = y$ by Lemma 4.18. However, we can consider the notions of left and right linear monoids and comonoids on duals that are not cyclic. This section, focuses on these structures in the category Poly.

In what follows, we are aiming to define a notion of right and left linear monoids and comonoids: four cases. Each is about a linear dual $b \dashv\!\!\!\mid a$ such that both sides carry additional structure. The word left and right will refer to the side that carries the $\otimes$-structure, and in this case the other side will (implicitly) carry a complementary $\triangleleft$-structure. For example, a left linear comonoid is a dual $b \dashv\!\!\!\mid a$ such that the left side, b carries the $\otimes$-comonoid structure, and the terminology leaves implicit the fact that a carries a $\triangleleft$-monoid structure.

6.1. Linear monoids. Before defining left and right linear monoids, let's consider the standard definition that we are generalizing. In an LDC, a linear monoid consists of linear duals $a \dashv\!\!\!\mid b$ and $b \dashv\!\!\!\mid a$, with either a or b carrying a $\otimes$-monoid structure and satisfying certain coherences [Srinivasan (2021), Definition 8.7]. Thus, a linear monoid requires cyclic duals, that is, each object is both the left and the right dual of the other.

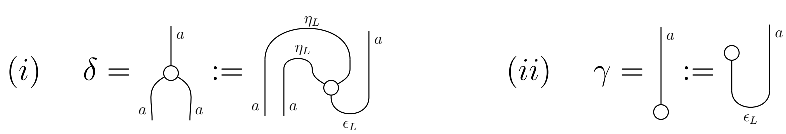
6.2. Definition. *In any LDC, a* **linear monoid**, $b \overset{\circ}{\dashv\!\!\!\mid} a$*, consists of linear duals $(\eta_L, \epsilon_L) : b \dashv\!\!\!\mid a$ and $(\eta_R, \epsilon_R) : a \dashv\!\!\!\mid b$, and a $\otimes$-monoid (b, μ, ν) such that the $\triangleleft$-comonoid* *structures induced on a by the dualities coincide:*



We now fine-grain this notion of linear monoid by considering linear duals which are non-cyclic. We separate the left and right notions as Definitions 6.3 and 6.4 so that we can include diagrams that explain the implicit $\triangleleft$-comonoid structures.

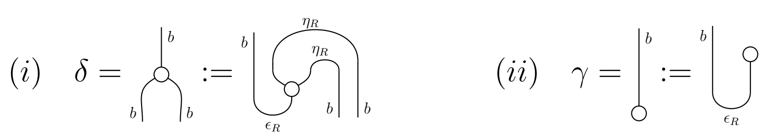
6.3. Definition. *In an LDC, a* **left linear monoid** $(\eta_L, \epsilon_L) : b \overset{\otimes\triangleleft}{\dashv\!\vert} a$, *consists of a linear dual* $(\eta_L, \epsilon_L) : b \dashv\!\vert\, a$ *equipped with a* $\otimes$*-monoid structure* $(b, \mu : b \otimes b \longrightarrow b, \nu : \top \longrightarrow b)$ *on the left dual.*

The reason for the "$\triangleleft$" near the a in our notation for a left linear monoid $b \overset{\otimes\triangleleft}{\dashv\!\vert} a$ is that the $\otimes$-monoid structure on b induces a $\triangleleft$-comonoid structure (a, δ, γ) on a, as can be seen from the following diagram:



6.4. Definition. *In an LDC, a* **right linear monoid** $(\eta_R, \epsilon_R) : b \overset{\triangleleft\otimes}{\dashv\!\vert} a$ *consists of a linear dual* $(\eta_R, \epsilon_R) : b \dashv\!\vert\, a$, *equipped with a* $\otimes$*-monoid structure* $(a, \mu : a \otimes a \longrightarrow a, \nu : \top \longrightarrow a)$ *on the right dual.*

The $\otimes$-monoid structure on b induces a $\triangleleft$-comonoid structure on (a, δ, γ) via the duality as follows:



A linear monoid $b \overset{\circ}{\dashv\!\vert} a$ in an LDC can be equivalently defined to consist of a left linear monoid $(\eta_L, \epsilon_L) : b \overset{\otimes\triangleleft}{\dashv\!\vert} a$, and a right linear monoid $(\eta_R, \epsilon_R) : a \overset{\triangleleft\otimes}{\dashv\!\vert} b$ such that the monoid structures on b coincide, i.e. the equations in (6.1) hold.

The monoids in Set produce left linear monoids in Poly as follows.

6.5. Lemma. *If* $(M, *, u) : \mathsf{Set}$ *is a monoid, then* $My \overset{\otimes\triangleleft}{\dashv\!\vert} y^M$ *has the structure of a left linear monoid where the linear duality* $(\mathsf{coev}, \mathsf{ev})\colon My \dashv\!\vert\, y^M$ *is given as in Corollary 4.15.*

Proof. The functor $M \mapsto My$ is strong monoidal by Proposition 2.5, so it preserves monoids. ∎

Explicitly, the $\otimes$-monoid structure on My induced by M is as follows.

$$\mu_1 := (a_1, a_2) \mapsto a_1 * a_2 : M \times M \longrightarrow M \tag{6.2}$$
$$\nu_1 := * \mapsto e : \{*\} \longrightarrow M \tag{6.3}$$

The directional components $\mu_M^\sharp$ and $\nu_M^\sharp$ are trivial.

The following Lemma shows that in Poly, every representable y^A is a right linear monoid.

6.6. Lemma. *In* Poly*, for any set* A*, there is a right linear monoid* $Ay \overset{\triangleleft\,\otimes}{\dashv\!\vdash} y^A$ *where the linear duality* $(\mathsf{coev}, \mathsf{ev})\colon Ay \dashv\!\!\vdash y^A$ *is given as in Corollary 4.15.*

Proof. The functor $A \mapsto y^A$ is strong monoidal by Proposition 2.5, so it preserves comonoids, and every set is a comonoid in a unique way, making y^A a $\otimes$-monoid. ∎

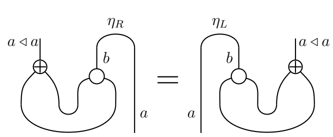
Explicitly, the $\otimes$-monoid structure on y^A induced by the unique comonoid structure on A is as follows.

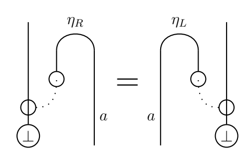
$$\mu_!^\sharp := a \mapsto (a, a) : A \longrightarrow A \times A \tag{6.4}$$
$$\nu_!^\sharp := a \mapsto * : A \longrightarrow \{*\} \tag{6.5}$$

The positional components μ_1 and ν_1 are trivial.

6.7. Linear comonoids. In this section, we simply run the same story except with $\otimes$-comonoids in place of $\otimes$-monoids, and implicitly also $\triangleleft$-monoids in place of $\triangleleft$-comonoids. If the reader does not need this detail, they can safely skip to Lemma 6.11.

6.8. Definition. [Srinivasan (2021), Definition 8.22] *In an LDC, a* **linear comonoid** $b \underset{\circ}{\dashv\!\vdash} a$ *consists of linear duals* $(\eta_L, \epsilon_L) : b \dashv\!\!\vdash a$ *and* $(\eta_R, \epsilon_R) : a \dashv\!\!\vdash b$*, and a* $\otimes$*-comonoid* (b, δ, ϵ)*, such that the* $\triangleleft$*-comonoid structures induced on* a *by the two dualities coincide:*

(i) 

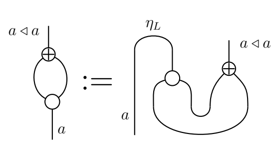
(ii) 

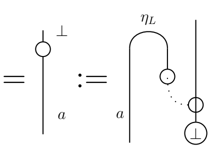
(6.6)

Similar to the definition of linear monoids, we can fine-grain the definition of linear comonoids to non-cyclic duals.

6.9. Definition. *In an LDC, a* **left linear comonoid**, $(\eta_L, \epsilon_L) : b \underset{\otimes\,\triangleleft}{\dashv\!\vdash} a$*, consists of a linear dual* $(\eta_L, \epsilon_L) : b \dashv\!\!\vdash a$ *equipped with a* $\otimes$*-comonoid structure* $(b, \delta : b \otimes b \longrightarrow b, \gamma : b \longrightarrow y)$ *on the left dual.*

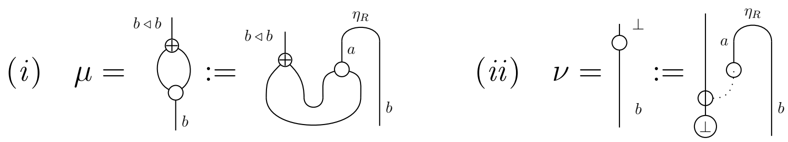
The $\otimes$-comonoid structure on b induces a $\triangleleft$-monoid structure (a, μ, ν) as follows:

(i) $\mu =$ 

(ii) $\nu =$ 

6.10. Definition. *In an LDC, a* **right linear comonoid** $(\eta_R, \epsilon_R) : b \underset{\triangleleft \otimes}{\dashv\!\!\mid} a$ *consists of a linear dual* $(\eta_R, \epsilon_R) : b \dashv\!\!\mid a$ *equipped with a* $\otimes$*-comonoid structure* $(a, \delta : a \longrightarrow a \otimes a, \gamma : a \longrightarrow y)$ *on the right dual.*

The $\otimes$-comonoid structure on a induces a $\triangleleft$-monoid structure (b, μ, ν) as follows:



A **linear comonoid** $b \underset{\circ}{\dashv\!\!\mid} a$ in any LDC can be equivalently defined to consist of a left linear comonoid $(\eta_L, \epsilon_L) : b \underset{\otimes \triangleleft}{\dashv\!\!\mid} a$ and a right linear comonoid $(\eta_R, \epsilon_R) : a \underset{\triangleleft \otimes}{\dashv\!\!\mid} b$ such that the $\otimes$-comonoid structures on b coincide and the equations in (6.6) hold.

The following Lemma shows that in Poly, every linear polynomial My is a left linear comonoid:

6.11. Lemma. *In* Poly*, for any set* A*, there is a left linear comonoid* $Ay \underset{\otimes \triangleleft}{\dashv\!\!\mid} y^A$ *where the linear duality* $(\mathsf{coev}, \mathsf{ev})\colon Ay \dashv\!\!\mid y^A$ *is given as in Corollary 4.15.*

Proof. The functor $A \mapsto Ay$ is strong monoidal by Proposition 2.5, so it preserves comonoids, and every set is a comonoid in a unique way. ■

Explicitly, the $\otimes$-comonoid structure on Ay is as follows:

$$\delta_1 := a \mapsto (a, a) : A \longrightarrow A \times A \tag{6.7}$$
$$\gamma_1 := a \mapsto * : A \longrightarrow \{*\} \tag{6.8}$$

The monoids in Set produce right linear comonoids in Poly as follows.

6.12. Lemma. *If* $(M, *, u) : \mathsf{Set}$ *is a monoid, then* $My \underset{\otimes \triangleleft}{\dashv\!\!\mid} y^M$ *has the structure of a right linear comonoid where the linear duality* $(\mathsf{coev}, \mathsf{ev})\colon My \dashv\!\!\mid y^M$ *is given as in Corollary 4.15.*

Proof. The functor $M \mapsto My$ is strong monoidal by Proposition 2.5, so it preserves monoids. ■

6.13. Linear bialgebras. In LDCs, a linear bialgebra on $b \dashv\!\!\mid a$ arises from a linear monoid structure interacting bialgebraically with a linear comonoid structure. In this section, we again define a left and right versions of linear bialgebra in $\otimes$-symmetric LDCs, and then we proceed to identify them in Poly. We will see that they can be identified with monoids in Set.

Let us first recall the definition of a bialgebra in a symmetric monoidal category (SMC).

6.14. Definition. [Porst (2008)] *In a SMC, a* **bialgebra** $(a, \ , \ , \ , \ )$ *consists of a monoid* $(a, \ : a \otimes a \to a, \ : y \to a)$ *and a comonoid* $(a, \ : a \to a \otimes a, \ : a \to y)$ *satisfying the following equations:*

(i) *(ii)* *(iii)* *(iv)*

(6.9)

The final equation is often referred to as the *bialgebra rule*. Eq. (6.9) $(i)-(iv)$ can equivalently be seen as the multiplication map acting as a comonoid morphism for the $\otimes$-comonoid $A \otimes A$ or the comultiplication acts as a monoid morphism for the $\otimes$-monoid $A \otimes A$ [Srinivasan (2021), Lemma 6.19].

Linear bialgebras were introduced as structures suitable to study complementary quantum observables in isomix LDCs.

6.15. Definition. [Srinivasan (2021), Defn. 8.29] *In a symmetric LDC, a* **linear bialgebra**, $\frac{(\eta,\epsilon)}{(\eta',\epsilon')} : b \overset{\circ}{\underset{\bullet}{\dashv\!\vdash}} a$, *consists of:*

(i) *a linear monoid,* $(\eta, \epsilon) : b \overset{\circ}{\dashv\!\vdash} a$, *and*
(ii) *a linear comonoid,* $(\eta', \epsilon') : b \underset{\bullet}{\dashv\!\vdash} a$,

such that:

(i) $(b, \ , \ , \ , \ )$ *is a* $\otimes$*-bialgebra, and*
(ii) $(a, \ , \ , \ , \ )$ *is a* $\triangleleft$*-bialgebra.*

As before we fine-grain the definition of linear bialgebra to non-cyclic duals.

6.16. Definition. *A* **left linear bialgebra**, $\frac{(\eta_L,\epsilon_L)}{(\eta'_L,\epsilon'_L)} : b \overset{\otimes \triangleleft}{\underset{\otimes \triangleleft}{\dashv\!\vdash}} a$ *in a* $\otimes$*-symmetric LDC consists of:*

(i) *a left linear monoid,* $(\eta_L, \epsilon_L) : b \overset{\otimes \triangleleft}{\dashv\!\vdash} a$, *denoted below by white* $\circ$, *and*
(ii) *a left linear comonoid,* $(\eta'_L, \epsilon'_L) : b \underset{\otimes \triangleleft}{\dashv\!\vdash} a$, *denoted below by black* $\bullet$,

such that $(b, \ , \ , \ , \ )$ *is a* $\otimes$*-bialgebra, that is, the left dual is a bialgebra via the left linear monoid and comonoid.*

6.17. Definition. *A* **right linear bialgebra**, $\frac{(\eta_R,\epsilon_R)}{(\eta'_R,\epsilon'_R)} : b \overset{\triangleleft \otimes}{\underset{\triangleleft \otimes}{\dashv\!\vdash}} a$ *in a* $\otimes$*-symmetric LDC consists of:*

(i) *a right linear monoid,* $(\eta_R, \epsilon_R) : b \overset{\triangleleft \otimes}{\dashv\!\vdash} a$, *denoted below by white* $\circ$, *and*
(ii) *a right linear comonoid,* $(\eta'_R, \epsilon'_R) : b \underset{\triangleleft \otimes}{\dashv\!\vdash} a$, *denoted below by black* $\bullet$,

such that $(a, \ , \ , \ , \ )$ *is a* $\otimes$*-bialgebra, that is, the right dual is a bialgebra via the right linear monoid and comonoid.*

Poly is a $\otimes$-symmetric LDC, and it has a left linear bialgebra and a right linear bialgebra for every monoid $M : \mathsf{Set}$.

6.18. Lemma. *Suppose* $(M, *, e)$ *is a monoid in* Set.

i. The left linear monoid $(\mathsf{coev}, \mathsf{ev}) : My \overset{\otimes \triangleleft}{\dashv} y^M$ *as defined in Lemma 6.5, and left linear comonoid* $(\mathsf{coev}, \mathsf{ev}) : My \underset{\otimes \triangleleft}{\dashv} y^M$ *as defined in Lemma 6.11 together produce a left linear bialgebra.*

ii. The right linear monoid $(\mathsf{coev}, \mathsf{ev}) : My \overset{\triangleleft \otimes}{\dashv} y^M$ *as defined in Lemma 6.6, and right linear comonoid* $(\mathsf{coev}, \mathsf{ev}) : My \underset{\triangleleft \otimes}{\dashv} y^M$ *as defined in Lemma 6.12 together produce a right linear bialgebra.*

Proof Sketch of proof. Using the $\otimes$-monoid structure specified in (6.2) and (6.3) and the $\otimes$-comonoid structure specified in (6.7) and (6.8) to prove that My is a $\otimes$-bialgebra. The same idea applies to proving that the right linear monoid and comonoid gives a $\otimes$-bialgebra on y^M. ■

6.19. Corollary. *In* Poly, *we have that* $My \overset{\otimes \triangleleft}{\underset{\otimes \triangleleft}{\dashv}} y^M$ *is a left linear bialgebra and* $My \overset{\triangleleft \otimes}{\underset{\triangleleft \otimes}{\dashv}} y^M$ *is a right linear bialgebra iff* M *is a monoid in* Set.

Proof. Direct consequence of Proposition 2.5, Lemma 6.18, and Theorem 4.17. ■

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Appendix A

Unabridged version of the composite map in (4.8):

$$\begin{tikzcd}
a \arrow[r, "u_\otimes^{-1}", "\simeq"'] \arrow[d, "\Phi_a"'] & y \otimes a \arrow[r, "\mathsf{coev} \otimes \mathsf{id}"] & \left(\begin{bmatrix} [a,y] \\ y \end{bmatrix} \triangleleft [a,y]\right) \otimes a \arrow[d, "\partial^R"] \\
\begin{bmatrix} [a,y] \\ y \end{bmatrix} & \begin{bmatrix} [a,y] \\ y \end{bmatrix} \triangleleft y \arrow[l, "\simeq", "u_\triangleleft"'] & \begin{bmatrix} [a,y] \\ y \end{bmatrix} \triangleleft (([a,y]) \otimes a) \arrow[l, "\mathsf{ev}"']
\end{tikzcd}$$

Unabridged version of the composite map in (4.9):

$$\begin{tikzcd}
\left[\begin{bmatrix} a \\ y \end{bmatrix}, y\right] \otimes y \arrow[r, "u_\otimes^{-1}", "\simeq"'] \arrow[d, "\Psi_a"'] & \left[\begin{bmatrix} a \\ y \end{bmatrix}, y\right] \arrow[r, "\mathsf{id} \otimes \mathsf{coev}"] & \left[\begin{bmatrix} a \\ y \end{bmatrix}, y\right] \otimes \left(\begin{bmatrix} a \\ y \end{bmatrix} \triangleleft a\right) \arrow[d, "\partial^L"] \\
a & y \triangleleft a \arrow[l, "\simeq", "u_\triangleleft"'] & \left(\left[\begin{bmatrix} a \\ y \end{bmatrix}, y\right] \otimes \begin{bmatrix} a \\ y \end{bmatrix}\right) \triangleleft a \arrow[l, "\mathsf{ev}"']
\end{tikzcd}$$

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