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AN EXTENSION OF BATANIN'S APPROACH TO GLOBULAR ALGEBRAS

SIMON FOREST

Abstract. In an earlier work, Batanin has shown that an important class of definitions of higher categories could be apprehended together simply as monads over globular sets. This allowed him to generalize the notion of polygraph, initially introduced by Street and Burroni for strict categories, to all algebraic globular higher categories. In this work, we refine this perspective and introduce new constructions and properties for this class of higher categories. In particular, we define the notion of cellular extension and its associated free construction, from which we obtain another definition of polygraphs and the adjunction between globular algebras and polygraphs. We moreover introduce two criteria allowing one to use most of the constructions of this article without having to describe explicitly the underlying globular monad.

Introduction

Over the last years, higher categories have emerged as a convenient tool to study problems in mathematics, physics and computer science [3, 24]. The notion of "higher category" encompasses informally all the structures that have higher-dimensional cells which can be composed together with several operations. It admits a vast number of definitions, which can make it hard to apprehend.

In order to get a more global view and factor out several common constructions and properties across the different possible definitions higher categories, it is useful to consider a restriction of this general notion to a more formal class of theories. This was done by Batanin [4], who introduced a unified formalism for algebraic globular higher categories. The latter are very common, since they include all the globular higher categories defined by a set of operations and equations between them. Moreover, the instances of such higher categories form *locally finitely presentable categories* and, as such, have very good properties, like being complete and cocomplete [1]. The setting of Batanin then enables one to derive several common constructions for such higher categories. In particular, one can generalize to those the notion of *polygraph*, which can be thought as higher-dimensional signatures, originally defined by Street [35] for strict 2-categories and Burroni [11] for

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strict ω-categories. Batanin moreover defined an adjunction between n-polygraphs and n-categories, and stated that it was monadic (as proved by Street for 2-polygraphs of strict categories), allowing for presenting higher categories of this framework using polygraphs.

Although this is already valuable, it seems that the work of Batanin can be refined in several aspects. First, several constructions of interest for concrete applications are too implicit, if not absent of Batanin's treatment. For example, in the context of higher dimensional rewriting, one is usually interested in a notion intermediate to higher category and polygraph, called *cellular extension*, which is simply a higher category with a set of generators in the next dimension. One can then consider the free n-category obtained from an n-cellular extension by freely generating the n-cells from the set of generators. Thus, cellular extensions can be considered as a more flexible construction than the one on polygraphs, where each dimension has to be freely generated.

Second, in order to use the results of Batanin, one has to work directly with a monad and its Eilenberg–Moore category. However, one usually does not have direct access to the underlying monad of the considered theory of higher category, but only the operations of the theory together with the equations satisfied by the theory, from which one can derive a monad, but the description of the latter can be very tedious [30]. Instead, one would prefer results that can work directly with the equational theory, the category of models of this theory, and the straight-forward functors that one can build from it. In particular, such a shortcut feels particularly desirable for showing the *truncatability* of the monad, which is required for the constructions and properties of Batanin to work.

Third, it seems that deeper understanding about globular algebras can be gained by seeing them through the lenses of the formal theory of monads of Street [34]. Indeed, several natural constructions on globular algebras, like truncations functors and their right adjoints, are in fact images, through the Eilenberg–Moore functor, of constructions happening in the category **MND** of monads and monad functors, whose compositions involve vertical composition of square-shaped natural transformation. Then, several computations on these functors can be nicely described using string diagrams, facilitating their readability and making them more intuitive.

In this work, we attempt to address these three points.

Outline In Section 1, we recall some elements of Street's formal theory of monads [34]. In particular, we define the category **MND** of monads over categories together with the Eilenberg–Moore construction, which produces a category of algebras from a monad. We rely on the convenient language of string diagrams to ease the presentation. In Section 2, we recall Batanin's framework for globular higher categories, where theories of higher categories are simply studied as monads on globular sets, and extend it with new results and constructions. Among other, we prove two criterions (Theorem 2.38 and Theorem 2.55) which help bridge the gap between that framework and the usual definitions of higher categories as structures with operations satisfying equations. Our use of string diagrams will notably make the proof of the second criterion particularly easy to grasp. In Section 3, we recover the notion of polygraph through a different path than the one of Batanin using the intermediate and intuitive notion of cellular extensions. We also describe explicitly the free constructions associated to both notions. The adjunctions shown by Batanin between higher categories and polygraphs can then be recovered from an adjunction between cellular extensions and polygraphs (Theorem 3.57). Finally, in Section 4, we illustrate the use of our properties and constructions on the case of strict categories, revisiting known results in our setting.

Acknowledgements I would like to thank all the people with whom I was able to discuss this work and who gave me useful feedback about it. In particular, I would like to thank Samuel Mimram, Tom Hirschowitz and Dominic Verity who took the time to read in depth an earlier version of this work under the form of the first chapter of my PhD thesis. The comments of Dominic were particularly valuable to this work, since he opened my eyes on some very naïve arguments I used in that earlier version, and indicated me some more natural ones. Most of the changes from the earlier version to this one can be attributed to him, so that I am grateful to him.

Also, some additional thanks to Samuel Mimram for having developed `satex`[1] that I have used to draw most of the string diagrams of this article.

1. Highlights of the formal theory of monads

In order to better manipulate globular algebras and constructions upon them, we use the formal theory of monads of Street [34], of which we recall briefly the salient points.

We start by recalling the notion of algebra for a monad and the associated Eilenberg–Moore category in Section 1.1. Then, in Section 1.4, we recall that the construction of the Eilenberg–Moore category is part of an adjunction between the 2-categories **MND**, describing monads over categories, and the 2-category **CAT** of large categories, functors and natural transformations. Finally, in Section 1.11, we give several useful properties to build morphisms in **MND** from adjunctions in **CAT**.

In the following, we introduce monads either as (T, η, μ), where T is the endofunctor of the monad and (η, μ) is the unit-multiplication pair; or we introduce them as $(\mathcal{C}, T)$, where $\mathcal{C}$ is the base category for which T is the endofunctor. When the latter notation is used, the unit and counit, when required, are explicitly introduced.

1.1. Algebras. We recall here the standard notions of an algebra over a monad, and of the Eilenberg–Moore category of a monad.

1.2. Definition. *Given a monad (T, η, μ) on a category $\mathcal{C}$, a T-*algebra *is the data of an object $X \in \mathcal{C}$ together with a morphism $h\colon TX \to X$ such that*

$$h \circ \eta_X = \mathrm{id}_X \quad \textit{and} \quad h \circ \mu_X = h \circ T(h).$$

A morphism *between two algebras (X, h) and (X', h') is given by a morphism $f\colon X \to X'$ of $\mathcal{C}$ satisfying*

$$f \circ h = h' \circ T(f).$$

[1] https://github.com/smimram/satex.

1.3. Definition. *The* Eilenberg–Moore category *of T, denoted $\mathcal{C}^T$, is the category of T-algebras and T-algebra morphisms. There is a canonical forgetful functor*

$$\mathcal{U}^T : \mathcal{C}^T \to \mathcal{C}$$

which maps the T-algebra (X, h) to X. This functor has a canonical left adjoint

$$\mathcal{F}^T : \mathcal{C} \to \mathcal{C}^T$$

which maps $X \in \mathcal{C}$ to the T-algebra (TX, μ_X), such that the unit of $\mathcal{F}^T \dashv \mathcal{U}^T$ is $\eta^T = \eta$, and the associated counit, denoted ϵ^T, is such that $\epsilon^T_{(X,h)} = h$ for a given T-algebra (X, h).

Note that, in the above definition, the monad induced by $\mathcal{F}^T \dashv \mathcal{U}^T$ is then exactly (T, η, μ).

1.4. The category **MND** and the **EM** functor. We now recall the definition of the 2-category **MND** of monads on (large) categories, together with the Eilenberg–Moore functor $\mathbf{EM} : \mathbf{MND} \to \mathbf{CAT}$, which extends the construction seen in Definition 1.3 of the Eilenberg–Moore category.

1.5. Definition. *Given two monads (S, γ, ν) and (T, η, μ) on two categories $\mathcal{C}$ and $\mathcal{D}$ respectively, a* monad functor *betweem (S, γ, ν) and (T, η, μ) is the data of a functor $F : \mathcal{C} \to \mathcal{D}$ together with a natural transformation $\alpha : TF \Rightarrow FS$ such that*

$$\alpha \circ (\eta F) = F\gamma \qquad \text{and} \qquad \alpha \circ (\mu F) = (F\nu) \circ (\alpha S) \circ (T\alpha). \tag{MF}$$

1.6. Definition. *A* monad functor transformation *(often abbreviated* monad transformation*) between two monad functors $(F, \alpha), (G, \beta) : (\mathcal{C}, S) \to (\mathcal{D}, T)$ is a natural transformation $m : F \Rightarrow G$ such that*

$$(mS) \circ \alpha = \beta \circ (Tm).$$

1.7. Definition. *The (very large) 2-category* **MND** *is defined as the 2-category of monads, monad functors and monad transformations.*

1.8. Definition. *There is a functor*

$$\mathbf{Inc} : \mathbf{CAT} \to \mathbf{MND}$$

which maps a large category $\mathcal{C}$ to the identity monad $(\mathcal{C}, 1_{\mathcal{C}})$. This functor admits a right adjoint

$$\mathbf{EM} : \mathbf{MND} \to \mathbf{CAT}$$

which maps a monad $(\mathcal{C}, T)$ to the Eilenberg–Moore category $\mathcal{C}^T$, and a monad functor

$$(F, \alpha) : (\mathcal{C}, S) \to (\mathcal{D}, T)$$

to a functor

$$F^\alpha : \mathcal{C}^S \to \mathcal{D}^T$$

mapping an algebra $(X,h) \in \mathcal{C}^S$ *to the algebra* $(FX, Fh \circ \alpha_X) \in \mathcal{D}^T$*, and mapping an algebra morphism* f *to* $F(f)$*. Finally, a monad transformation*

$$m\colon (F,\alpha) \Rightarrow (G,\beta)\colon (\mathcal{C},S) \to (\mathcal{D},T)$$

is mapped by **EM** *to the natural transformation* $F^\alpha \Rightarrow G^\beta$ *whose component at an* S*-algebra* (X,h) *is* m_X*.*

The component of the counit of the above adjunction at a monad $(\mathcal{C},T)$ is given by $(\mathcal{U}^T, \mathcal{U}^T\epsilon^T)$. Moreover, the monad functor $(T,\mu)\colon (\mathcal{C},1_{\mathcal{C}}) \to (\mathcal{C},T)$ corresponds to the functor $\mathcal{F}^T\colon \mathcal{C} \to \mathcal{C}^T$ by the adjunction $\mathbf{Inc} \dashv \mathbf{EM}$.

1.9. Definition. *We write*

$$\mathbf{Und}\colon \mathbf{MND} \to \mathbf{CAT}$$

for the functor which maps a monad $(\mathcal{C},T)$ *to* $\mathcal{C}$ *and whose action on monad functors and monad transformations is the expected one.*

1.10. Remark. We mention that this functor is left adjoint to **Inc**, even though this is not really useful for our purposes.

1.11. Morphisms of **MND** from adjunctions. In the following, we give several results that allows building morphisms of **MND** from adjunctions. For this purpose, we first recall one of the several definitions of adjunctions, which we will take as our working definition and will be particularly amenable to graphical proofs:

1.12. Definition. *An* adjunction *between two categories* $\mathcal{C}$ *and* $\mathcal{D}$ *is the data of two functors*

$$L\colon \mathcal{C} \to \mathcal{D} \qquad R\colon \mathcal{D} \to \mathcal{C}$$

together with two natural transformations

$$\eta\colon 1_{\mathcal{C}} \Rightarrow RL \qquad \epsilon\colon LR \Rightarrow 1_{\mathcal{D}}$$

such that the zigzag *equations hold:*

$$(\epsilon L) \circ (L\eta) = 1_{\mathcal{C}} \qquad \text{and} \qquad (R\epsilon) \circ (\eta R) = 1_{\mathcal{D}}\,.$$

The data of such an adjunction is denoted $L \dashv R\colon \mathcal{D} \to \mathcal{C}$*.*

As is well-known, an adjunction as above induces a monad $(RL, \eta, R\epsilon L)$. Drawing  for η and  for ϵ, the equations can be depicted using string diagrams:

(ZZ)

In the following, we will often prefer string diagrams for the computations, and in particular, make use of the above (ZZ) equations.

In order to lift adjunctions to **MND**, the notion of cocartesian morphism will be useful:

1.13. Definition. *Given a functor $F\colon C \to D$ between two categories C and D, a morphism $f\colon x_1 \to x_2 \in C$ is said* cocartesian *when, for all $y \in C$ and $h\colon x_1 \to y$ and $\tilde{g}\colon F(x_2) \to F(y)$ such that $F(h) = \tilde{g} \circ F(f)$, there is a unique $g\colon x_2 \to y$ such that $\tilde{g} = F(g)$ and $h = g \circ f$. Graphically,*

$$\begin{tikzcd} x_2 \arrow[dr, "\exists! g"] & \\ x_1 \arrow[u, "f"] \arrow[r, "\forall h"'] & x' \end{tikzcd} \qquad \Big| \qquad \begin{tikzcd} F(x_2) \arrow[dr, "\forall \tilde{g}"] & \\ F(x_1) \arrow[u, "F(f)"] \arrow[r, "F(h)"'] & F(x'). \end{tikzcd}$$

$$C \qquad\qquad\qquad D$$

We now prove that every functor which is a right adjoint can be lifted canonically to **MND**:

1.14. Proposition. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, (S, η^S, μ^S) be a monad on $\mathcal{C}$, and*

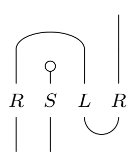
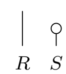
$$L \dashv R\colon \mathcal{C} \to \mathcal{D}$$

be an adjunction for some $R\colon \mathcal{C} \to \mathcal{D}$ and $L\colon \mathcal{D} \to \mathcal{C}$. The functor R lifts through **Und** *to a cocartesian map*

$$(R, RS\epsilon)\colon (\mathcal{C}, S) \to (\mathcal{D}, T)$$

where $T = (T, \eta^T, \mu^T)$ is the canonical monad with $T = RSL$ and ϵ denotes the counit of $L \dashv R$.

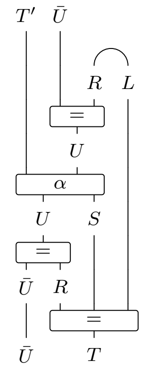
Proof. The fact that we obtain a monad functor can readily be verified using string diagrams. For example, the equation $(RS\epsilon) \circ (\eta^T R) = R\eta^S$ asserts that the two diagrams

 and 

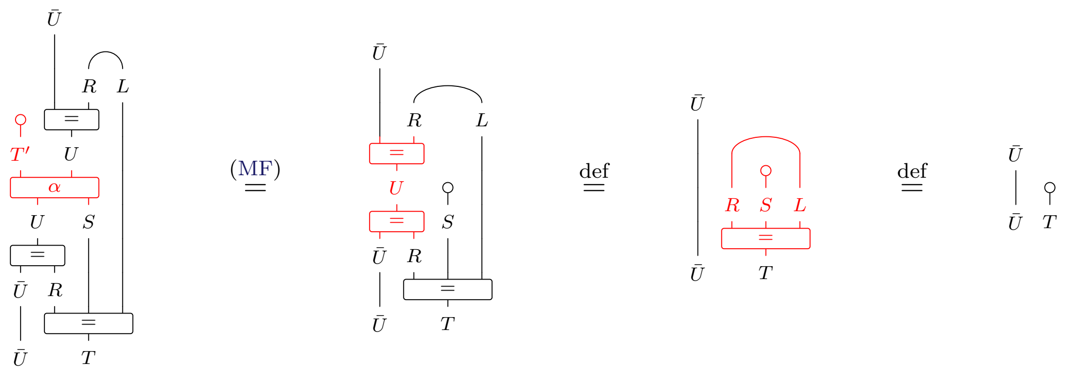
represents the same natural transformation, which is true by the zigzag equations (ZZ) for $L \dashv R$. The other equation $(RS\epsilon) \circ (\mu^T R) = (R\mu^S) \circ (RS\epsilon S) \circ (TRS\epsilon)$ is verified similarly.

We are now left to prove the cocartesianness of $(R, RS\epsilon)$. So let $(\mathcal{D}', T')$ be a monad, $U\colon \mathcal{C} \to \mathcal{D}'$ and $\bar{U}\colon \mathcal{D} \to \mathcal{D}'$ such that $U = \bar{U}R$, and $(U, \alpha)\colon (\mathcal{C}, S) \to (\mathcal{D}', T')$ be a monad functor. We define $\bar{\alpha}\colon T'\bar{U} \Rightarrow \bar{U}T$ by

$\bar{\alpha} \quad =$



and verify that it defines a monad functor $(\bar{U}, \bar{\alpha})\colon (\mathcal{D}, T) \to (\mathcal{D}', T')$. We start with the unit equation, *i.e.*, $\bar{\alpha} \circ (\eta^{T'}\bar{U}) = \bar{U}\eta^{T}$:



In the above computation, we draw in red the region on which a transformation is applied in the following equality. Also, we annotated the equals signs in order to specify when an axiom was used (like for the first equality with (MF)) or when an equality holds by definition (like the last ones). We will keep using these conventions in the future computations on string diagrams.

The second equation, *i.e.*, $\bar{\alpha} \circ (\mu^{T}\bar{U}) = (\bar{U}\mu^{S}) \circ (\bar{\alpha}T) \circ (T'\bar{\alpha})$, is proved in Figure 1. Thus, we conclude that $(\bar{U}, \bar{\alpha})$ is a monad functor. We verify that it is a factorisation

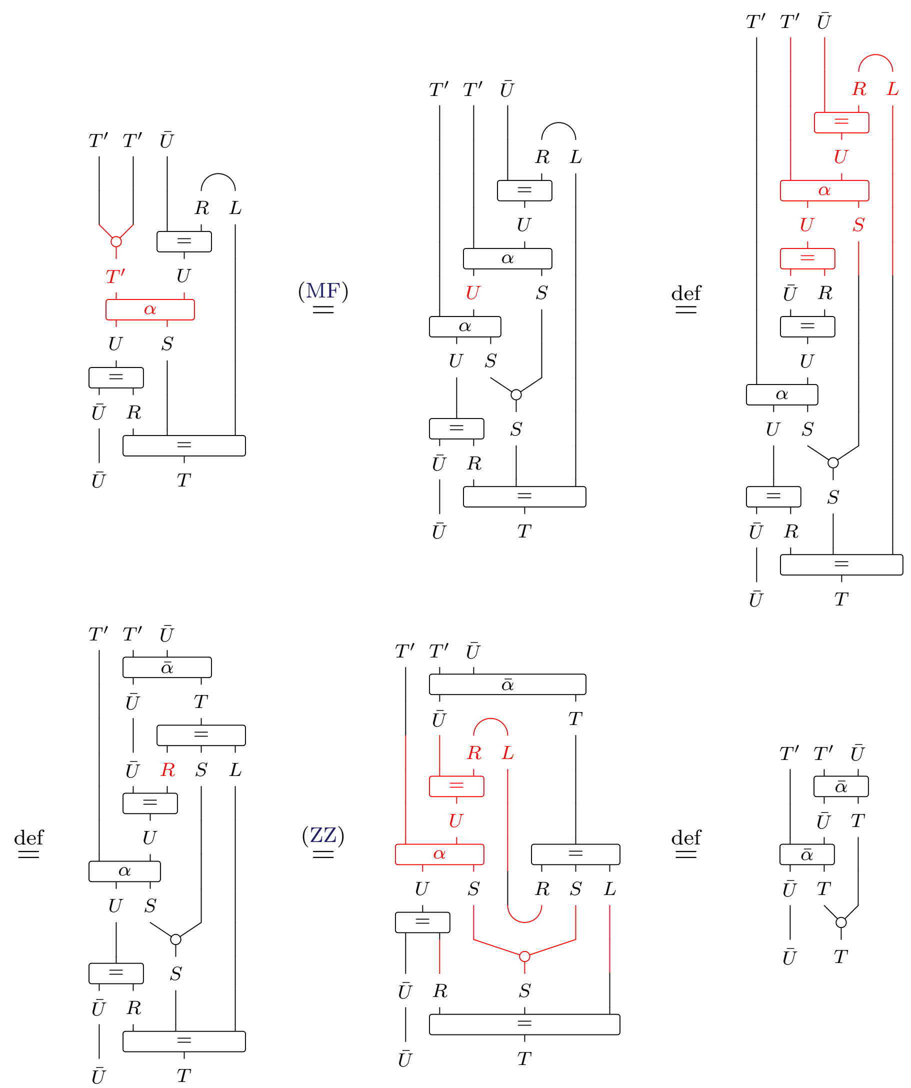
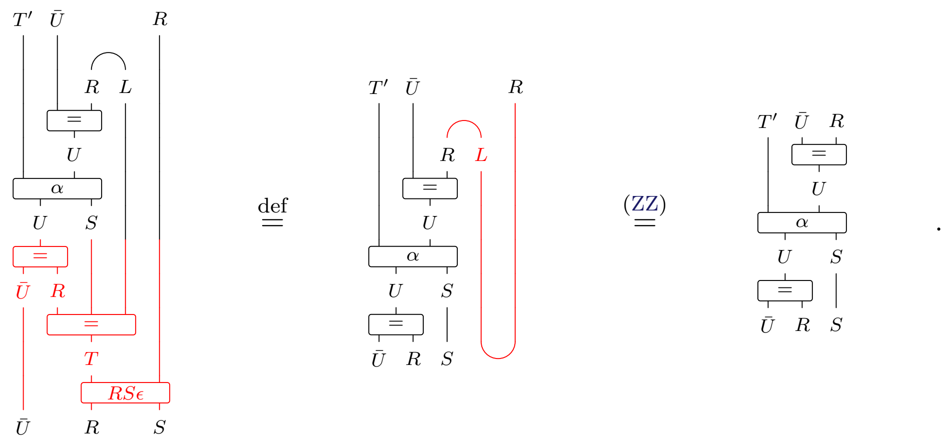


Figure 1: Proof of $\bar{\alpha} \circ (\mu^T \bar{U}) = (\bar{U}\mu^S) \circ (\bar{\alpha}T) \circ (T'\bar{\alpha})$.

of (U,α) through $(R, RS\epsilon)$:



Moreover, by the zigzag equations (ZZ), we observe that the vertical pasting operation

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, "\bar{U}"] \arrow[d, "T"'] \arrow[dr, phantom, "\Downarrow \beta"] & \mathcal{D}' \arrow[d, "T'"] \\
\mathcal{D} \arrow[r, "\bar{U}"'] & \mathcal{D}'
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, "R"] \arrow[d, "S"'] \arrow[dr, phantom, "\Downarrow RS\epsilon"] & \mathcal{D} \arrow[r, "\bar{U}"] \arrow[d, "T"] \arrow[dr, phantom, "\Downarrow \beta"] & \mathcal{D}' \arrow[d, "T'"] \\
\mathcal{C} \arrow[r, "R"'] & \mathcal{D} \arrow[r, "\bar{U}"'] & \mathcal{D}'
\end{tikzcd}
$$

is a bijective operation between natural transformations of adequate types, whose inverse is $\beta' \mapsto (\beta' L) \circ (T'\bar{U}\eta)$ where η is the unit of $L \dashv R$. Thus, we conclude that $(\bar{U}, \bar{\alpha})$ is the unique monad functor above $\bar{U}$ which factors (U, α) through $(R, RS\epsilon)$. Thus, the latter is a cocartesian morphism for **Und**. ∎

1.15. Remark. The morphism of Proposition 1.14 is even 2-*cocartesian*, meaning that the cocartesian lifting property extends to monad transformations: given a monad transformation $\theta\colon (U,\alpha) \Rightarrow (U',\alpha')\colon (\mathcal{D},T) \to (\mathcal{D}',T')$ and $\bar{\theta}\colon \bar{U} \Rightarrow \bar{U}'$ such that $\bar{\theta}R = \theta$, then $\bar{\theta}$ uniquely lifts through **Und** to a monad modification $\bar{\theta}\colon (\bar{U},\bar{\alpha}) \Rightarrow (\bar{U}',\bar{\alpha}')$ which factors θ seen as a monad modification through $(R, RS\epsilon)$. The proof that $\bar{\theta}$ induces a monad transformation is done by observing that the bijection introduced in the end of the proof of Proposition 1.14 generalises to a bijection

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, "\bar{U}"] \arrow[d, "T"'] \arrow[dr, phantom, "\Downarrow \beta"] & \mathcal{D}' \arrow[d, "T'"] \\
\mathcal{D} \arrow[r, "\bar{U}'"'] & \mathcal{D}'
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, "U"] \arrow[d, "S"'] \arrow[dr, phantom, "\Downarrow \beta'"] & \mathcal{D}' \arrow[d, "T'"] \\
\mathcal{C} \arrow[r, "U'"'] & \mathcal{D}'
\end{tikzcd}
.
$$

Then, the equation required for $\bar{\theta}$ to be a monad transformation is given by the one of θ, through this bijection. For a more complete view of 2-cartesianness (of which 2-cocartesianness can be derived as dual), we refer the reader to the existing literature [18, 10].

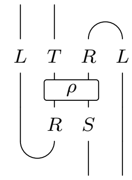
Given a situation as in the statement of Proposition 1.14, we might wonder whether the adjunction $L \dashv R$ lifts to an adjunction in $\mathbf{MND}$. In fact, we can ask this question in a more general setting: given $(\mathcal{C}, S)$ and $(\mathcal{D}, T)$ two objects of $\mathbf{MND}$ and

$$(R, \rho)\colon (\mathcal{C}, S) \to (\mathcal{D}, T) \in \mathbf{MND}$$

such that $R = \mathbf{Und}(R, \rho)$ is part of an adjunction $L \dashv R$, when does this adjunction lift to an adjunction in $\mathbf{MND}$, *i.e.*, there exists $\lambda\colon SL \Rightarrow LT$ such that $(L, \lambda) \dashv (R, \rho)$ in $\mathbf{MND}$? This question can be answered by considering the mate of ρ.

1.16. Definition. *Given the situation just described, the* mate *[22] of ρ is the natural transformation $\rho^*\colon LT \Rightarrow SL$ defined by*

$\rho^* \quad =$



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We then have the following property:

1.17. Proposition. *Given an adjunction $L \dashv R\colon \mathcal{C} \to \mathcal{D}$ and a monad functor*

$$(R, \rho)\colon (\mathcal{C}, S) \to (\mathcal{D}, T),$$

the following are equivalent:

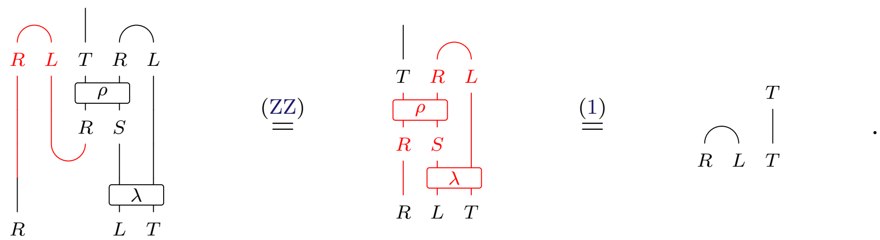
(i) *there exists $\lambda\colon SL \Rightarrow LT$ such that $(L, \lambda) \dashv (R, \rho)$ is an adjunction in* $\mathbf{MND}$*, whose image by* $\mathbf{Und}$ *is the adjunction $L \dashv R$;*

(ii) *the natural transformation ρ^* is an isomorphism.*

Proof. We first show that (i) implies (ii). So let λ be as in (i). We prove that λ is an inverse for ρ^*. We start by showing that $\lambda \circ \rho^* = \mathbb{1}_{LT}$. By the correspondence given by the adjunction $L \dashv R$, it is equivalent to show that $(R(\lambda \circ \rho^*)) \circ (\eta T) = \eta T$. From the fact that $(L, \lambda) \dashv (R, \rho)$ is an adjunction, we get in particular the following equality

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "L"] \arrow[d, "T"'] \arrow[rr, bend left=60, "1_{\mathcal{D}}"] \arrow[rr, bend left=25, phantom, "\Downarrow\eta"] \arrow[dr, phantom, "\Downarrow\lambda"] & \mathcal{C} \arrow[r, "R"] \arrow[d, dashed, "S"] \arrow[dr, phantom, "\Downarrow\rho"] & \mathcal{D} \arrow[d, "T"] \\
\mathcal{D} \arrow[r, "L"'] & \mathcal{C} \arrow[r, "R"'] & \mathcal{D}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{D} \arrow[rr, "1_{\mathcal{D}}"] \arrow[d, "T"'] \arrow[drr, phantom, "="] & & \mathcal{D} \arrow[d, "T"] \\
\mathcal{D} \arrow[rr, "1_{\mathcal{D}}"] \arrow[dr, bend right, "L"'] \arrow[rr, phantom, bend right=40, "\Downarrow\eta"'] & & \mathcal{D} \\
& \mathcal{C} \arrow[ur, bend right, "R"'] &
\end{tikzcd}
\tag{1}$$

from which we can prove the wanted equality as follows:



Symmetrically, in order to prove that $\rho^* \circ \lambda = \mathbb{1}_{SL}$, it is equivalent to prove that

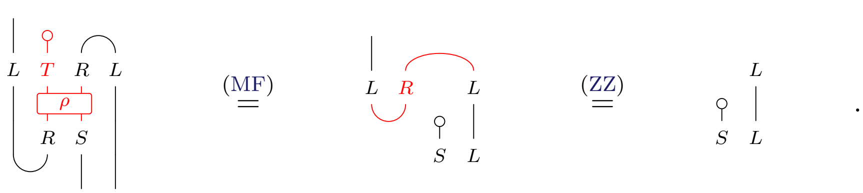
$$(S\epsilon) \circ ((\rho^* \circ \lambda)R) = S\epsilon.$$

In order to prove the latter, we use the equality satisfied by ϵ as a monad transformation:

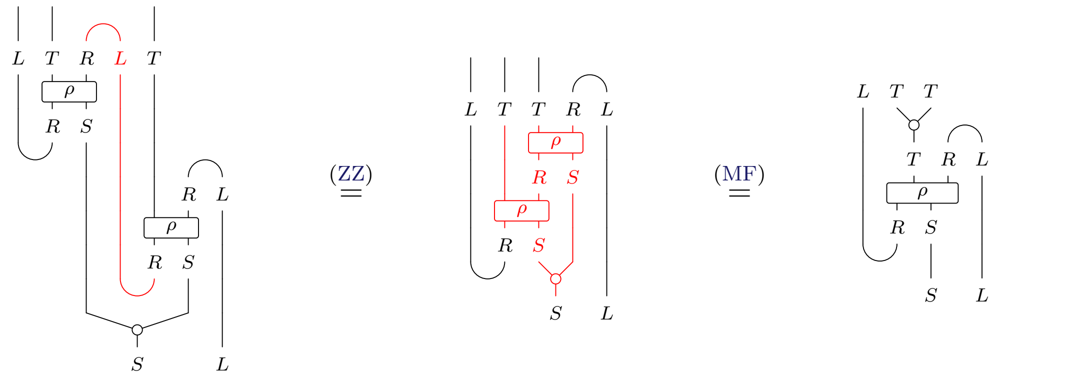
$$\begin{tikzcd} \mathcal{C} \arrow[r, "R"] \arrow[d, "S"'] \arrow[dr, phantom, "\Downarrow\rho"] & \mathcal{D} \arrow[r, "L"] \arrow[d, "T"] \arrow[dr, phantom, "\Downarrow\lambda"] & \mathcal{C} \arrow[d, "S"] \\ \mathcal{C} \arrow[r, "R"] \arrow[rr, bend right=60, "1_{\mathcal{C}}"'] \arrow[rr, bend right=30, phantom, "\Downarrow\epsilon"] & \mathcal{D} \arrow[r, "L"] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} & \mathcal{D} \arrow[dr, bend left, "L"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\ \mathcal{C} \arrow[ur, bend left, "R"] \arrow[rr, "1_{\mathcal{C}}"] \arrow[d, "S"'] \arrow[drr, phantom, "="] & & \mathcal{C} \arrow[d, "S"] \\ \mathcal{C} \arrow[rr, "1_{\mathcal{C}}"'] & & \mathcal{C} \end{tikzcd} . \tag{2}$$

Then, the wanted equality can be proved using string diagrams just as before. Thus, (ii) is proved.

Conversely, assume that (ii) holds. We put $\lambda = (\rho^*)^{-1}$ and show that (L, λ) is a monad functor. First, we need to show that $\lambda \circ (\eta^S L) = (L\eta^T)$, or equivalently, that $(\eta^S L) = \rho^* \circ (L\eta^T)$. We show the latter using string diagrams:



Second, we need to show that $\lambda \circ (\mu^S L) = (L\mu^T) \circ (\lambda T) \circ (S\lambda)$, or equivalently, that the equation $(\mu^S L) \circ (S\rho^*) \circ (\rho^* T) = \rho^* \circ (L\mu^T)$ holds. We show the latter using string diagrams again:



Thus, (L, λ) is a monad functor. We must now prove that η and ϵ induce monad transformations. First, we show that (1) holds, or equivalently, that

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "L"] \arrow[rr, bend left, "1_{\mathcal{D}}"] & \mathcal{C} \arrow[r, "R"] \arrow[d, "S"'] \arrow[u, phantom, "\Downarrow\eta"] \arrow[dr, phantom, "\Downarrow\rho"] & \mathcal{D} \arrow[d, "T"] \\
 & \mathcal{C} \arrow[r, "R"'] & \mathcal{D}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{D} \arrow[r, "T"] \arrow[d, "L"'] \arrow[dr, phantom, "\Downarrow\rho^*"] & \mathcal{D} \arrow[r, "1_{\mathcal{D}}"] \arrow[d, "L"] \arrow[dr, phantom, "\Downarrow\eta"] & \mathcal{D} \arrow[d, "1_{\mathcal{D}}"] \\
\mathcal{C} \arrow[r, "S"'] & \mathcal{C} \arrow[r, "R"'] & \mathcal{D}
\end{tikzcd}\ .$$

But the latter equation follows directly from the string diagram definition of ρ^* and the zigzag equations (ZZ) of the adjunction $L \dashv R$.

The other required equation (2), or equivalently,

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "R"] \arrow[d, "S"'] \arrow[dr, phantom, "\Downarrow\rho"] & \mathcal{D} \arrow[d, "T"] & \\
\mathcal{C} \arrow[r, "R"] \arrow[rr, bend right, "1_{\mathcal{C}}"'] & \mathcal{D} \arrow[r, "L"] \arrow[d, phantom, "\Downarrow\epsilon"] & \mathcal{C} \\
 & {} &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, "R"] \arrow[d, "1_{\mathcal{C}}"'] \arrow[dr, phantom, "\Downarrow\epsilon"] & \mathcal{D} \arrow[r, "T"] \arrow[d, "L"] \arrow[dr, phantom, "\Downarrow\rho^*"] & \mathcal{D} \arrow[d, "L"] \\
\mathcal{C} \arrow[r, "1_{\mathcal{C}}"'] & \mathcal{C} \arrow[r, "S"'] & \mathcal{C}
\end{tikzcd}$$

holds by a similar argument. Finally, the zigzag equations (ZZ) for η and ϵ seen as monad transformations follows from the zigzag equations satisfied by them as natural transformation in **CAT**. Hence, (i) holds. ∎

We have the same kind of property for left adjoints (first suggested by Dominic Verity to the author):

1.18. Proposition. *Given an adjunction $L \dashv R \colon \mathcal{C} \to \mathcal{D}$ and a monad functor*

$$(L, \lambda) \colon (\mathcal{D}, T) \to (\mathcal{C}, S),$$

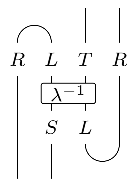
the following are equivalent:

(i) there exists $\rho\colon SR \Rightarrow RT$ *such that* $(L,\lambda) \dashv (R,\rho)$ *is an adjunction in* **MND**, *whose image by* **Und** *is the adjunction* $L \dashv R$*;*

(ii) the natural transformation λ *is an isomorphism.*

PROOF. The proof is essentially the same as the one for Proposition 1.17. For (ii) implies (i), ρ is now defined by the string diagram

$\rho \quad =$



and we prove just as before that it induces a monad functor (R,ρ) such that $(L,\lambda) \dashv (R,\rho)$ is an adjunction in **MND**. ∎

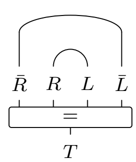
1.19. PROPOSITION. *Given two adjunctions* $L \dashv R$ *and* $\bar{L} \dashv \bar{R}$*, in the configuration*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift right, "R"'] & \mathcal{D} \arrow[l, shift right, "L"'] \arrow[r, shift right, "\bar{R}"'] & \mathcal{E} \arrow[l, shift right, "\bar{L}"'] \end{tikzcd} ,$$

if we write S *for the monad associated with* $\bar{L} \dashv \bar{R}$ *and* T *for the monad associated with* $L\bar{L} \dashv \bar{R}R$*, and* η *for the unit of* $L \dashv R$*, we have a monad functor*

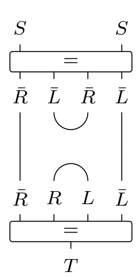
$$(1_{\mathcal{E}}, \bar{R}\eta\bar{L})\colon (\mathcal{E},T) \to (\mathcal{E},S).$$

PROOF. The fact that it is a monad functor can be checked using string diagrams. On the one hand, the compatibility of $\bar{R}\eta\bar{L}$ with the units of S and T asserts that the two diagrams

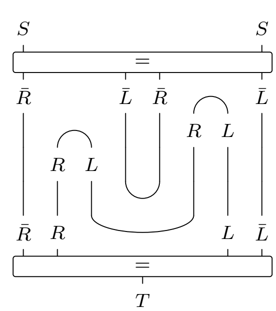


and

represents the same cell, which is true by definition of the unit of T. On the other hand, the compatibility of $\bar{R}\eta\bar{L}$ with the multiplications of S and T asserts that the two diagrams



and



represents the same cell, which is true by the zigzag equations (ZZ) of $L \dashv R$. Thus, $(1_{\mathcal{E}}, \bar{R}\eta\bar{L})$ is indeed a monad functor. ∎

2. Higher categories as globular algebras

The notion of "higher category" encompasses informally all the structures that have higher-dimensional cells which can be composed together with several operations. Such structures can differ on many points. First, there are several possible shapes for the cells of higher categories. For example, *globular higher categories* have 0-cells, 1-cells, 2-cells, 3-cells, *etc.* of the form

$$
x, \qquad x \xrightarrow{f} y, \qquad
\begin{tikzcd} x \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & y \end{tikzcd}, \qquad
\begin{tikzcd} x \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, "\phi\Downarrow \overset{F}{\Rrightarrow} \Downarrow\psi"] & y \end{tikzcd}, \qquad \textit{etc.}
$$

But one can consider higher categories with other shapes than the globular ones. Common variants include *cubical* [2] and *simplicial* [20] higher categories, whose 2-cells for example are respectively of the form

$$
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\phi"] & y \arrow[d, "h"] \\
x' \arrow[r, "f'"'] & y'
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
& y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\phi"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z
\end{tikzcd}.
$$

Moreover, higher categories have several operations which satisfy axioms that can take different forms, according to their position in the strict/weak spectrum. For example, a *strict* 2*-category* is a globular 2-dimensional category that have, among others, an operation $*_0$ to compose 1-cells in dimension 0, as in

$$
x \xrightarrow{f} y \;*_0\; y \xrightarrow{g} z \quad = \quad x \xrightarrow{f *_0 g} z,
$$

and operations $*_0$ and $*_1$ to compose 2-cells in dimensions 0 and 1 respectively, as in

$$
\begin{tikzcd} x \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & y \end{tikzcd}
\;*_0\;
\begin{tikzcd} y \arrow[r, bend left=60, "f'"] \arrow[r, bend right=60, "g'"'] \arrow[r, phantom, "\Downarrow\phi'"] & z \end{tikzcd}
\quad = \quad
\begin{tikzcd} x \arrow[r, bend left=60, "f *_0 f'"] \arrow[r, bend right=60, "g *_0 g'"'] \arrow[r, phantom, "\Downarrow\phi *_0 \phi'"] & z \end{tikzcd}
$$

and

$$
\begin{tikzcd} x \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & y \end{tikzcd}
\;*_1\;
\begin{tikzcd} x \arrow[r, bend left=60, "g"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, "\Downarrow\psi"] & y \end{tikzcd}
\quad = \quad
\begin{tikzcd} x \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, "\Downarrow\phi *_1 \psi"] & y \end{tikzcd}.
$$

These operations are required to satisfy several axioms consisting in equalities, like the associativity axiom: given 0-composable 1- or 2-cells u, v, w,

$$(u *_0 v) *_0 w = u *_0 (v *_0 w)$$

and, given 1-composable 2-cells ϕ, ψ, χ,

$$(\phi *_1 \psi) *_1 \chi = \phi *_1 (\psi *_1 \chi).$$

An example of a weak higher category is given by a *bicategory*, which is a globular 2-dimensional category that has operations similar to a strict 2-category but which satisfy axioms in the form of "weak equalities". For example, the 0-composition of 1-cells is only required to be weakly associative, in the sense that, given 0-composable 1-cells

$$w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z,$$

the equality $(f *_0 g) *_0 h = f *_0 (g *_0 h)$ does not hold necessarily, but there should exist a *coherence cell* between the two sides, *i.e.*, an invertible 2-cell $\alpha_{f,g,h}$ as in

$$\begin{tikzcd} w \arrow[r, bend left=60, "(f *_0 g) *_0 h"] \arrow[r, bend right=60, "f *_0 (g *_0 h)"'] \arrow[r, phantom, "\Downarrow \alpha_{f,g,h}"] & z. \end{tikzcd}$$

Finally, a subtle difference between the different kinds higher categories is the *algebraicity* of their definition [24, 17]. This notion essentially pertains to weak higher categories. Informally, a definition of some sort of higher categories is algebraic when it can be equivalently described by means of a monad. Concretely, algebraic definitions of weak higher categories involve coherence cells that are *distinguished* (like the definition of bicategories, which requires that "there exists an invertible 2-cell $\alpha_{f,g,h}$ between $(f *_0 g) *_0 h$ and $f *_0 (g *_0 h)$"), whereas non-algebraic definitions of weak higher categories involve coherence cells that are not (a non-algebraic definition of bicategories would only require that "there exists *some* invertible 2-cell between $(f *_0 g) *_0 h$ and $f *_0 (g *_0 h)$").

In the remainder of this article, we will restrict our attention to globular algebraic higher categories. A particular theory of k-categories can be seen as a monad on the category of k-globular sets, and an instance of this theory is an algebra for this monad. A lot of globular higher categories that one usually encounters fit in this setting: strict k-categories, bicategories, Gray categories, *etc.* Several constructions can then be defined at this level of abstractions, like the truncation of globular algebras and its left adjoint.

In Section 2.1, we recall the definition of globular sets and elementary operations on them. We then introduce globular algebras as algebras of a monad on globular sets in Section 2.14, and then define the truncation and inclusion functors which relate globular algebras from different dimensions in Section 2.23. Then, in the case of a monad on $\mathbf{Glob}_\omega$, we use the truncation functors to formally express $\mathbf{Alg}_\omega$ as a (bi)limit over the $\mathbf{Alg}_k$'s in Section 2.30. In Section 2.36, we introduce a criterion to recognize a tower of categories as equivalent to the categories of globular algebras over a monad, simplifying the concrete use of the theory developped in this article. In Section 2.40, we introduce the notion of truncatable monads, which correspond more to the notions of higher categories that we are accustomed to than general globular monads, and, in Section 2.51, we prove a criterion to easily recognize such monads in the wild. Hopefully, our use of the formal theory of monads and of string diagrams for the computations will provide a better understanding of the notions and properties of this section, in accordance to the last of our three goals listed in the introduction.

2.1. Globular sets and operations. Here, we recall the classical notion of *globular set*. It is the underlying structure of a globular higher category which describes *globes* of different dimensions together with their sources and targets. We moreover define the truncation and inclusion functors between globular sets of different dimensions.

2.2. Definition. *Given $n \in \mathbb{N} \cup \{\omega\}$, the n-*globe category *is defined as the category $\mathbf{O}^n$ whose objects are the natural numbers $k \leq n$ (or $k \in \mathbb{N}$ when $n = \omega$), and whose arrows are generated by morphisms $\mathrm{s}_k^-, \mathrm{s}_k^+ \colon k \to k+1$ for $k < n$, subject to the equations*

$$\begin{aligned} \mathrm{s}_{k+1}^- \circ \mathrm{s}_k^- &= \mathrm{s}_{k+1}^+ \circ \mathrm{s}_k^- \\ \mathrm{s}_{k+1}^- \circ \mathrm{s}_k^+ &= \mathrm{s}_{k+1}^+ \circ \mathrm{s}_k^+ \end{aligned}$$

for $k < n-1$.

2.3. Definition. *Given $n \in \mathbb{N} \cup \{\omega\}$, an n-*globular set *is a presheaf on $\mathbf{O}^n$. Concretely, such presheaves $X \in \mathbf{O}^n$ can be seen as diagrams*

$$X_0 \underset{\partial_0^+}{\overset{\partial_0^-}{\leftleftarrows}} X_1 \underset{\partial_1^+}{\overset{\partial_1^-}{\leftleftarrows}} X_2 \underset{\partial_2^+}{\overset{\partial_2^-}{\leftleftarrows}} \cdots \underset{\partial_{k-1}^+}{\overset{\partial_{k-1}^-}{\leftleftarrows}} X_k \underset{\partial_k^+}{\overset{\partial_k^-}{\leftleftarrows}} X_{k+1} \underset{\partial_{k+1}^+}{\overset{\partial_{k+1}^-}{\leftleftarrows}} \cdots$$

such that

$$\partial_i^- \circ \partial_{i+1}^- = \partial_i^- \circ \partial_{i+1}^+ \quad \text{and} \quad \partial_i^+ \circ \partial_{i+1}^- = \partial_i^+ \circ \partial_{i+1}^+ \quad \text{for } i < n.$$

*When there is no ambiguity on i, we often write ∂^- and ∂^+ for ∂_i^- and ∂_i^+. An element u of X_i is called an i-*globe *of X and, for $i > 0$, the globes $\partial_{i-1}^-(u)$ and $\partial_{i-1}^+(u)$ are respectively called the* source *and* target *and u. Given n-globular sets X and Y, a* morphism of n-globular set *between X and Y is a family of functions $F = (F_k \colon X_k \to Y_k)_{k<n+1}$, such that*

$$\partial_i^\epsilon \circ F_{i+1} = F_i \circ \partial_i^\epsilon \qquad \text{for } i < n \text{ and } \epsilon \in \{-,+\}.$$

We write $\mathbf{Glob}_n$ for the category of n-globular sets.

2.4. Remark. As a presheaf category, $\mathbf{Glob}_n$ is complete and cocomplete. More generally, since globular sets can be seen as models of an evident algebraic theory, $\mathbf{Glob}_n$ is a *locally finitely presentable category*, which is a class of categories with good properties (see Section A for details).

2.5. Definition. *Given $n \in \mathbb{N} \cup \{\omega\}$, $X \in \mathbf{Glob}_n$ and $i, j \in \mathbb{N}$ with $i + j < n + 1$, for $\epsilon \in \{-, +\}$, we write*

$$\partial^\epsilon_{i,j} = \partial^\epsilon_i \circ \partial^\epsilon_{i+1} \circ \cdots \circ \partial^\epsilon_{i+j-1} \colon X_{i+j} \to X_i$$

for the iterated source *(when $\epsilon = -$) and* target *(when $\epsilon = +$) operations. We generally omit the index j when there is no ambiguity and simply write $\partial^\epsilon_i(u)$ for $\partial^\epsilon_{i,j}(u)$ for $u \in X_{i+j}$.*

2.6. Definition. *Given $n \in \mathbb{N} \cup \{\omega\}$, $X \in \mathbf{Glob}_n$ and $i, k, l < n + 1$ with $i < \min(k, l)$, we write $X_k \times_i X_l$ for the pullback*

$$\begin{tikzcd}
X_k \times_i X_l \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & X_l \arrow[d, "\partial^-_i"] \\
X_k \arrow[r, "\partial^+_i"'] & X_i
\end{tikzcd}.$$

2.7. Definition. *Given $n \in \mathbb{N} \cup \{\omega\}$, $X \in \mathbf{Glob}_n$, $p \geq 2$ and $k_1, \ldots, k_p < n + 1$, a sequence of globes $u_1 \in X_{k_1}, \ldots, u_p \in X_{k_p}$ is said i-*composable *for some $i < \min(k_1, \ldots, k_p)$, when $\partial^+_i(u_j) = \partial^-_i(u_{j+1})$ for $j < p$. Given $k < n + 1$ and $u, v \in X_k$, u and v are said* parallel *when $k = 0$ or $\partial^\epsilon_{k-1}(u) = \partial^\epsilon_{k-1}(v)$ for $\epsilon \in \{-, +\}$. To remove the side condition $k = 0$, we use the convention that X_{-1} is the set $\{*\}$ and that $\partial^-_{-1}, \partial^+_{-1}$ are the unique function $X_0 \to X_{-1}$.*

2.8. Notation. *For $u \in X_{i+1}$, we sometimes write $u \colon v \to w$ to indicate that $\partial^-_i(u) = v$ and $\partial^+_i(u) = w$. In low dimension, we use n-arrows such as $\Rightarrow$, $\Rrightarrow$, etc. to indicate the sources and the targets of n-globes in several dimensions. For example, given a 2-globular set X and $\phi \in X$, we sometimes write $\phi \colon f \Rightarrow g \colon x \to y$ to indicate that*

$$\phi \in X_2, \quad \partial^-_1(\phi) = f, \quad \partial^+_1(\phi) = g, \quad \partial^-_0(\phi) = x \quad \text{and} \quad \partial^+_0(\phi) = y.$$

We also use these arrows in graphical representations to picture the elements of a globular set X. For example, given an n-globular set X with $n \geq 2$, the drawing

$$\begin{tikzcd}
x \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g"{description, name=G}] \arrow[r, bend right=60, "h"'{name=H}] & y \arrow[r, "k"] & z
\arrow[from=F, to=G, phantom, "\Downarrow\phi"] \arrow[from=G, to=H, phantom, "\Downarrow\psi"]
\end{tikzcd} \tag{3}$$

figures two 2-cells $\phi, \psi \in X_2$, four 1-cells $f, g, h, k \in X_1$ and three 0-cells $x, y, z \in X_0$ such that

$$\partial^-_1(\phi) = f, \qquad \partial^+_1(\phi) = \partial^-_1(\psi) = g, \qquad \partial^+_1(\psi) = h,$$
$$\partial^-_0(f) = \partial^-_0(g) = \partial^-_0(h) = x, \qquad \partial^+_0(f) = \partial^+_0(g) = \partial^+_0(h) = \partial^-_0(k) = y, \qquad \partial^+_0(k) = z.$$

2.9. Definition. *Given $n \in \mathbb{N} \cup \{\omega\}$, $m < n+1$, the canonical embedding $\mathbf{O}^m \to \mathbf{O}^n$ induces a* truncation *functor*

$$\mathcal{T}^{\mathrm{G}}_{n,m} \colon \mathbf{Glob}_n \to \mathbf{Glob}_m$$

*by precomposition and, given $X \in \mathbf{Glob}_n$, we denote by $X_{\leq m}$ the resulting m-*truncation *of X, i.e., the m-globular set obtained from X by removing the sets of i-globes for $i < n+1$ with $i > m$. Similarly, given $f \colon X \to Y \in \mathbf{Glob}_n$, we write $f_{\leq m}$ for the m-*truncation *of f. The operation $\mathcal{T}^{\mathrm{G}}_{n,m}$ is often denoted $\mathcal{T}^{\mathrm{G}}_m$ when there is no ambiguity on n.*

As is well-known (see [19, Example A4.1.4]), a functor between presheaf categories, like the above truncation functor, induced by a functor between the underlying small categories, like the embedding $\mathbf{O}^m \to \mathbf{O}^n$, has both a left and a right adjoint, that we now introduce and describe for the truncation functor.

2.10. Definition. *The truncation functor of Definition 2.9 admits a left adjoint*

$$\mathcal{I}^{\mathrm{G}}_{m,n} \colon \mathbf{Glob}_m \to \mathbf{Glob}_n$$

*often denoted $\mathcal{I}^{\mathrm{G}}_n$ when there is no ambiguity, and which maps an m-globular set X to the n-globular set $X_{\uparrow n}$, called n-*inclusion *of X, and which is defined by $(X_{\uparrow n})_{\leq m} = X$ and $(X_{\uparrow n})_i = \emptyset$ for $i < n+1$ with $i > m$. Similarly, given $f \colon X \to Y \in \mathbf{Glob}_m$, we write $f_{\uparrow n}$ for the n-*inclusion *of f. The unit of the adjunction $\mathcal{I}^{\mathrm{G}}_n \dashv \mathcal{T}^{\mathrm{G}}_m$ is the identity and the counit is the natural transformation denoted $\mathrm{i}^{m,n}$, or simply i^m when there is no ambiguity, which is given by the family of canonical morphisms*

$$\mathrm{i}^m_X \colon (X_{\leq m})_{\uparrow n} \to X$$

for $X \in \mathbf{Glob}_n$.

2.11. Definition. *Given $k \in \mathbb{N}$, we write*

$$\mathcal{B}_k \colon \mathbf{Glob}_k \to \mathbf{Set}$$

for the functor mapping $X \in \mathbf{Glob}_k$ to the set

$$\mathcal{B}_k(X) = \{(u,v) \in X_k \mid u \textit{ and } v \textit{ are parallel}\}$$

and with obvious action on morphisms.

2.12. Definition. *The truncation functor of Definition 2.9 also admits a right adjoint*

$$\mathcal{J}^{\mathrm{G}}_{m,n} \colon \mathbf{Glob}_m \to \mathbf{Glob}_n$$

denoted $\mathcal{J}^{\mathrm{G}}_n$ when there is no ambiguity. It maps an m-globular set to the n-globular set $X_{\Uparrow n}$ defined by $(X_{\Uparrow n})_{\leq m} = X$, and, for $i < n+1$ with $i > m$, $(X_{\Uparrow n})_i = \mathcal{B}_m(X)$ such that, for $(u,v) \in (X_{\Uparrow n})_i$,

$$\partial^-_{i-1}((u,v)) = \begin{cases} u & \textit{when } i-1 = m, \\ (u,v) & \textit{when } i-1 > m, \end{cases} \qquad \partial^+_{i-1}((u,v)) = \begin{cases} v & \textit{when } i-1 = m, \\ (u,v) & \textit{when } i-1 > m. \end{cases}$$

The action of $\mathcal{J}^{\mathrm{G}}_n$ *on morphisms is then the obvious one. We write* $f_{\Uparrow n}$ *for the image by* $\mathcal{J}^{\mathrm{G}}_{m,n}$ *of a morphism* $f\colon X \to Y \in \mathbf{Glob}_m$*. The counit of the adjunction is the identity, and we write* $\mathrm{j}^{m,n}\colon 1_{\mathbf{Glob}_n} \Rightarrow \mathcal{J}^{\mathrm{G}}_{m,n}\mathcal{T}^{\mathrm{G}}_{n,m}$*, or more simply* j^m*, for the unit of the adjunction: given* $X \in \mathbf{Glob}_n$*,* j^m_X *is such that* $(\mathrm{j}^m_X)_k = \mathrm{id}_{X_k}$ *for* $k \le m$*, and* $(\mathrm{j}^m_X)_k(u) = (\partial^-_m(u), \partial^+_m(u))$ *for* $k \in \mathbb{N}$ *with* $m < k \le n$ *and* $u \in X_k$*.*

2.13. Remark. Note that, since they are left adjoints, the functors $\mathcal{I}^{\mathrm{G}}_{m,n}$ and $\mathcal{T}^{\mathrm{G}}_{n,m}$ preserves colimits.

2.14. Globular algebras. We now introduce categories of *globular algebras*, *i.e.*, the Eilenberg–Moore categories induced by monads on globular sets, as were first introduced by Batanin in [4]. We moreover give several additional constructions and properties on these objects. For the remainder of this section, we fix $n \in \mathbb{N} \cup \{\omega\}$ and (T, η, μ) a finitary monad on $\mathbf{Glob}_n$.

2.15. Definition. *We write* $\mathbf{Alg}_n$ *for the category of* T*-algebras* $\mathbf{Glob}^T_n$ *and*

$$\mathcal{U}^{\mathrm{G}}_n\colon \mathbf{Alg}_n \to \mathbf{Glob}_n \qquad\qquad \mathcal{F}^{\mathrm{G}}_n\colon \mathbf{Glob}_n \to \mathbf{Alg}_n$$

for the induced left and right adjoints, that were denoted $\mathcal{U}^T$ *and* $\mathcal{F}^T$ *in Section 1: given an algebra* $(X,h) \in \mathbf{Alg}_n$*, the image of* (X,h) *by* $\mathcal{U}^{\mathrm{G}}_n$ *is* X *and, given* $Y \in \mathbf{Glob}_n$*,* $\mathcal{F}^{\mathrm{G}}_n Y$ *is the free* T*-algebra*

$$(TY, \mu_Y\colon TTY \to TY).$$

2.16. Definition. *Given* $k < n$*, using Proposition 1.14 with the adjunction* $\mathcal{T}^{\mathrm{G}}_k \dashv \mathcal{I}^{\mathrm{G}}_n$*, we derive a monad* (T^k, η^k, μ^k) *on* $\mathbf{Glob}_k$ *where*

$$T^k = \mathcal{T}^{\mathrm{G}}_k T \mathcal{I}^{\mathrm{G}}_n$$

and such that $\eta^k\colon 1_{\mathbf{Glob}_k} \Rightarrow T^k$ *is the composite*

$$1_{\mathbf{Glob}_k} =\!=\!=\!= \mathcal{T}^{\mathrm{G}}_k\mathcal{I}^{\mathrm{G}}_n \xrightarrow{\mathcal{T}^{\mathrm{G}}_k \eta \mathcal{I}^{\mathrm{G}}_n} T^k$$

i.e., $\eta^k_X = (\eta_{X_{\uparrow n}})_{\le k}$ *for* $X \in \mathbf{Glob}_k$*, and such that* $\mu^k\colon T^kT^k \to T^k$ *is the composite*

$$T^kT^k \xrightarrow{\mathcal{T}^{\mathrm{G}}_k T\, \mathrm{i}^k\, T\mathcal{I}^{\mathrm{G}}_n} \mathcal{T}^{\mathrm{G}}_k TT\mathcal{I}^{\mathrm{G}}_n \xrightarrow{\mathcal{T}^{\mathrm{G}}_k \mu \mathcal{I}^{\mathrm{G}}_n} T^k.$$

2.17. Definition. *Given* $k < n+1$*, we write* $\mathbf{Alg}_k$ *for the category* $\mathbf{Glob}^{T^k}_k$*, and*

$$\mathcal{U}^{\mathrm{G}}_k\colon \mathbf{Alg}_k \to \mathbf{Glob}_k \qquad\qquad \mathcal{F}^{\mathrm{G}}_k\colon \mathbf{Glob}_k \to \mathbf{Alg}_k$$

for the canonical functors defined like $\mathcal{U}^{\mathrm{G}}_n$ *and* $\mathcal{F}^{\mathrm{G}}_n$ *above, forming an adjunction* $\mathcal{F}^{\mathrm{G}}_k \dashv \mathcal{U}^{\mathrm{G}}_k$ *whose counit is denoted* ϵ^k*. The objects of* $\mathbf{Alg}_k$ *are called* k-globular algebras*, or even* k-categories*. Moreover, given a* k*-category* $C = (X,h)$*, the elements of* X_i *are called the* i-cells of C *for* $i < k+1$*.*

2.18. Remark. In the above definition, we require that the monad (T, η, μ) is finitary in order to prove later the existence of several free constructions on the k-categories. This is not too restrictive, since it includes all the monads of algebraic globular higher categories that have operations with finite arities, *i.e.*, most theories of algebraic globular higher categories.

We can already derive several properties of the categories $\mathbf{Alg}_k$:

2.19. Proposition. *For $k \in \mathbb{N} \cup \{\omega\}$ with $k \leq n$, the category $\mathbf{Alg}_k$ is locally finitely presentable. In particular, it is complete and cocomplete. Moreover, the functor $\mathcal{U}_k^{\mathrm{G}}$ preserves and creates directed colimits, and creates limits.*

Proof. The category $\mathbf{Alg}_k$ is locally finitely presentable as a consequence of Proposition A.13 since $\mathbf{Glob}_k$ is locally finitely presentable by Remark 2.4. The functor $\mathcal{U}_k^{\mathrm{G}}$ preserves directed colimits by Proposition A.13. Moreover, since $\mathcal{U}_k^{\mathrm{G}}$ reflects isomorphisms and $\mathbf{Alg}_k$ is cocomplete, $\mathcal{U}_k^{\mathrm{G}}$ creates directed colimits. Finally, it is well-known that the forgetful functor associated to an Eilenberg–Moore category creates limits (see [9, Proposition 4.3.1] for example). ∎

We can usually derive monads from equational definitions of higher categories as illustrated by the following examples.

2.20. Example. The canonical forgetful functor $\mathbf{Cat} \to \mathbf{Glob}_1$ is a finitary right adjoint (see Example A.36 for a detailed argument) which induces a finitary monad (T, η, μ) on $\mathbf{Glob}_1$. This monad maps a 1-globular set G to the underlying 1-globular set of the category of paths on G seen as a graph. Using Beck's monadicity theorem (recalled later as Theorem 4.12), one can verify that the functor $\mathbf{Cat} \to \mathbf{Glob}_1$ is monadic, so that $\mathbf{Alg}_1 \simeq \mathbf{Cat}$. Moreover, the monad (T^0, η^0, μ^0) is essentially the identity monad on $\mathbf{Glob}_0$, and thus $\mathbf{Alg}_0 \simeq \mathbf{Set}$. More generally, we will see in Section 4 that the monads of strict k-categories for $k \in \mathbb{N}$ are derived from the monad of strict ω-categories.

2.21. Example. We define a notion of *weird 2-category* as follows: a weird 2-category is a 2-globular set C equipped with an operation

$$* \colon C_2 \times C_2 \to C_0.$$

Note that we do not require the composability of the arguments of $*$, and we do not enforce any axiom on $*$. A morphism between two weird 2-categories is then a morphism between the underlying 2-globular sets that is compatible with $*$. The category **Weird** of weird 2-categories and their morphisms is essentially algebraic, and the functor which maps a weird 2-category to its underlying 2-globular set is induced by an essentially algebraic theory morphism, so that it is a right adjoint and finitary by Proposition A.33. From the adjunction, we derive a finitary monad (T, η, μ) on $\mathbf{Glob}_2$, and, given $X \in \mathbf{Glob}_2$, we have that

$$(TX)_0 \cong X_0 \sqcup (X_2 \times X_2) \qquad (TX)_1 \cong X_1 \qquad (TX)_2 \cong X_2$$

so that, for $\mathbf{Alg}_2$ derived from the monad T, $\mathbf{Alg}_2 \cong \mathbf{Weird}$. Moreover, the monads (T^0, η^0, μ^0) and (T^1, η^1, μ^1) are essentially the identity monads on $\mathbf{Glob}_0$ and $\mathbf{Glob}_1$ respectively, so that the associated notions of weird 0- and 1-categories are simply 0- and 1-globular sets.

The previous example moreover illustrates the unusual operations that notions of higher categories defined in the setting of Batanin can have. It is also an example of a monad on globular sets which is not truncatable (*c.f.* Example 2.43).

2.22. Example. Monoids can be considered in any category with a monoidal structure. Given a theory of globular algebraic higher category, one can define an associated notion of strict monoidal higher category by considering monoids in the category of algebras, choosing the monoidal structure to be the cartesian one. Monadically, we have an operation mapping a monad T on n-globular sets to a monad T' on n-globular sets representing the theory of strictly monoidal higher categories which are instances of T. This operation can be seen to preserve a finitary hypothesis on T.

2.23. Truncation and inclusion functors. We now introduce truncation and inclusion functors between the categories $\mathbf{Alg}_k$ together with some of their properties.

Let $n \in \mathbb{N} \cup \{\omega\}$ and (T, η, μ) be a finitary monad on $\mathbf{Glob}_n$. Given $k < n$, using Proposition 1.14, we get that $\mathcal{T}^{\mathrm{G}}_{n,k}$ lifts to a cocartesian morphism

$$(\mathbf{Glob}_n, T) \xrightarrow{(\mathcal{T}^{\mathrm{G}}_k, \mathcal{T}^{\mathrm{G}}_k T\,\mathrm{i}^k)} (\mathbf{Glob}_k, T^k)$$

with respect to the functor $\mathbf{Und}$.

2.24. Definition. *We write*

$$\mathcal{T}^{\mathrm{A}}_{n,k} \colon \mathbf{Alg}_n \to \mathbf{Alg}_k$$

for the image by $\mathbf{EM}$ *of the above functor. We might also denote it by* $\mathcal{T}^{\mathrm{A}}_k$ *when there is no ambiguity on* n*.*

Now, given $k, l \in \mathbb{N}$ with $k < l < n$, since $\mathcal{T}^{\mathrm{G}}_{n,k}$ can be factored as $\mathcal{T}^{\mathrm{G}}_{l,k}\mathcal{T}^{\mathrm{G}}_{n,l}$, the cocartesianness of $(\mathcal{T}^{\mathrm{G}}_l, \mathcal{T}^{\mathrm{G}}_l T\,\mathrm{i}^l)$ ensures that there is a unique morphism (F, α) which factorises $(\mathcal{T}^{\mathrm{G}}_k, \mathcal{T}^{\mathrm{G}}_k T\,\mathrm{i}^k)$ through $(\mathcal{T}^{\mathrm{G}}_l, \mathcal{T}^{\mathrm{G}}_l T\,\mathrm{i}^l)$ in $\mathbf{MND}$.

2.25. Lemma. *Given* $k, l \in \mathbb{N}$ *with* $k < l < n$*, the morphism*

$$(\mathcal{T}^{\mathrm{G}}_{l,k}, \mathcal{T}^{\mathrm{G}}_k T^l\,\mathrm{i}^{k,l}) \colon (\mathbf{Glob}_l, T^l) \to (\mathbf{Glob}_k, T^k)$$

is the factorisation of $(\mathcal{T}^{\mathrm{G}}_k, \mathcal{T}^{\mathrm{G}}_k T\,\mathrm{i}^k)$ *through the cocartesian morphism*

$$(\mathcal{T}^{\mathrm{G}}_l, \mathcal{T}^{\mathrm{G}}_l T\,\mathrm{i}^l).$$

PROOF. By Proposition 1.14, there is a morphism

$$(\mathcal{T}^{\mathrm{G}}_{l,k}, \mathcal{T}^{\mathrm{G}}_{k} T^l\, \mathrm{i}^{k,l}) \colon (\mathcal{T}^{\mathrm{G}}_{l}, T^l) \to (\mathcal{T}^{\mathrm{G}}_{k}, T')$$

where (T', η', μ') is the monad on $\mathcal{T}^{\mathrm{G}}_{k}$ defined by

$$\begin{aligned} T' &= \mathcal{T}^{\mathrm{G}}_{k} T^l \mathcal{I}^{\mathrm{G}}_{l} \\ \eta' &= \mathcal{T}^{\mathrm{G}}_{k} \eta^l \mathcal{I}^{\mathrm{G}}_{l} \\ \mu' &= (\mathcal{T}^{\mathrm{G}}_{k} \mu^l \mathcal{I}^{\mathrm{G}}_{k}) \circ (\mathcal{T}^{\mathrm{G}}_{k} T^l\, \mathrm{i}^{k,l}\, T^l \mathcal{I}^{\mathrm{G}}_{l}). \end{aligned}$$

By a straight-forward computation, we get $T' = T^k$, $\eta' = \eta^k$ and $\mu' = \mu^k$. To verify that the above morphism is the wanted factorisation, we are left to check that

$$(\mathcal{T}^{\mathrm{G}}_{l,k}, \mathcal{T}^{\mathrm{G}}_{k} T^l\, \mathrm{i}^{k,l}) \circ (\mathcal{T}^{\mathrm{G}}_{l}, \mathcal{T}^{\mathrm{G}}_{l} T\, \mathrm{i}^{l}) = (\mathcal{T}^{\mathrm{G}}_{k}, \mathcal{T}^{\mathrm{G}}_{k} T\, \mathrm{i}^{k})$$

in **MND**. But it is straight-forward too, since

$$\mathrm{i}^{l,n} \circ (\mathcal{I}^{\mathrm{G}}_{n,l}\, \mathrm{i}^{k,l}\, \mathcal{T}^{\mathrm{G}}_{n,l}) = \mathrm{i}^{k,n}.$$

∎

2.26. DEFINITION. *We write*

$$\mathcal{T}^{\mathrm{A}}_{l,k} \colon \mathbf{Alg}_l \to \mathbf{Alg}_k$$

for the image by the **EM** *functor of the factorisation morphism given by Lemma 2.25. We might also denote it by $\mathcal{T}^{\mathrm{A}}_{k}$ when there is no ambiguity. Concretely, given a T^l-algebra*

$$(X, h \colon T^l X \to X),$$

its image by $\mathcal{T}^{\mathrm{A}}_{k}$ is the T^k-algebra $(X_{\le k}, h')$, where h' is defined as the composite

$$T^k(X_{\le k}) \xrightarrow{(\mathcal{T}^{\mathrm{G}}_{l,k} T^l\, \mathrm{i}^{k,l})_X} (T^l X)_{\le k} \xrightarrow{\quad h_{\le k} \quad} X_{\le k}.$$

*The image of (X, h) in $\mathbf{Alg}_l$ by $\mathcal{T}^{\mathrm{A}}_{k}$ is called the k-*truncation *of (X, h) and we denote it $(X, h)_{\le k}$. Note that the image of a morphism $f \colon (X, h) \to (X', h')$ by $\mathcal{T}^{\mathrm{A}}_{k}$ is $f_{\le k}$ (the globular k-truncation of f). The same concrete description holds for the functors $\mathcal{T}^{\mathrm{A}}_{n,k}$.*

The definition of the truncation functors allows for the following compatibility property:

2.27. PROPOSITION. *Given $j, k, l \le n$ with $j < k < l$, we have*

$$\mathcal{T}^{\mathrm{A}}_{k,j} \mathcal{T}^{\mathrm{A}}_{l,k} = \mathcal{T}^{\mathrm{A}}_{l,j}.$$

PROOF. The two morphisms

$$(\mathcal{T}^{\mathrm{G}}_{k,j}, \mathcal{T}^{\mathrm{G}}_{j} T^k\, \mathrm{i}^{j,k}) \circ (\mathcal{T}^{\mathrm{G}}_{l,k}, \mathcal{T}^{\mathrm{G}}_{l,k} T^l\, \mathrm{i}^{k,l}) \quad \text{and} \quad (\mathcal{T}^{\mathrm{G}}_{l,j}, \mathcal{T}^{\mathrm{G}}_{l,j} T^l\, \mathrm{i}^{j,l})$$

provide a factorisation of $(\mathcal{T}^{\mathrm{G}}_{n,j}, \mathcal{T}^{\mathrm{G}}_{n,j} T\, \mathrm{i}^{j,n})$ through the cocartesian morphism

$$(\mathcal{T}^{\mathrm{G}}_{n,l}, \mathcal{T}^{\mathrm{G}}_{n,l} T\, \mathrm{i}^{l,n}).$$

Thus, they are equal. The conclusion follows from the functoriality of **EM**. ∎

The finitary assumption on T enables the existence of a left adjoint to truncation functors:

2.28. Proposition. *Given $k, l \leq n$ with $k < l$, the functor $\mathcal{T}^{\mathrm{A}}_{l,k}$ is finitary and admits a left adjoint.*

Proof. By the adjunction $\mathbf{Inc} \dashv \mathbf{EM}$, we have a commutative diagram

$$\begin{tikzcd} \mathbf{Alg}_l \arrow[r, "\mathcal{T}^{\mathrm{A}}_k"] \arrow[d, "\mathcal{U}^{\mathrm{G}}_l"'] & \mathbf{Alg}_k \arrow[d, "\mathcal{U}^{\mathrm{G}}_k"] \\ \mathbf{Glob}_l \arrow[r, "\mathcal{T}^{\mathrm{G}}_k"'] & \mathbf{Glob}_k \end{tikzcd} .$$

The functor $\mathcal{T}^{\mathrm{A}}_{l,k}$ is finitary since, by Proposition 2.19, $\mathcal{U}^{\mathrm{G}}_k$ creates directed colimits and the functor

$$\mathcal{U}^{\mathrm{G}}_k \mathcal{T}^{\mathrm{A}}_{l,k} = \mathcal{T}^{\mathrm{G}}_k \mathcal{U}^{\mathrm{G}}_l$$

preserves directed colimits. Moreover, $\mathcal{T}^{\mathrm{A}}_{l,k}$ preserves limits since $\mathcal{U}^{\mathrm{G}}_k$ creates limits and the functor $\mathcal{U}^{\mathrm{G}}_k \mathcal{T}^{\mathrm{A}}_{l,k} = \mathcal{T}^{\mathrm{G}}_k \mathcal{U}^{\mathrm{G}}_l$ preserves limits (both $\mathcal{T}^{\mathrm{G}}_{l,k}$ and $\mathcal{U}^{\mathrm{G}}_l$ are right adjoints). Then, by Proposition A.12, the functor $\mathcal{T}^{\mathrm{A}}_{l,k}$ admits a left adjoint. ∎

2.29. Definition. *Given $k, l \leq n$ with $k < l$, we write*

$$\mathcal{I}^{\mathrm{A}}_{k,l} \colon \mathbf{Alg}_k \to \mathbf{Alg}_l$$

*for the left adjoint to $\mathcal{T}^{\mathrm{A}}_{l,k}$ given by Proposition 2.28, or even $\mathcal{I}^{\mathrm{A}}_l$ when there is no ambiguity on k. The image of (X, h) in $\mathbf{Alg}_k$ by $\mathcal{I}^{\mathrm{A}}_l$ is called the l-*inclusion *of (X, h) and we denote it $(X, h)_{\uparrow l}$.*

We moreover write $\eta^{\mathrm{A},k,l}$ for the unit of the adjunction $\mathcal{I}^{\mathrm{A}}_{k,l} \dashv \mathcal{T}^{\mathrm{A}}_{l,k}$. When $l = k+1$, we might write $\eta^{\mathrm{A},k}$, or even η^{A} when is no ambiguity on k, for this unit.

2.30. $\mathbf{Alg}_\omega$ as a limit. Let (T, η, μ) be a finitary monad on $\mathbf{Glob}_\omega$. The purpose of this section is to characterise $\mathbf{Alg}_\omega$ as a limit on the categories $\mathbf{Alg}_k$ for $k \in \mathbb{N}$ using the truncation functors $\mathcal{T}^{\mathrm{A}}_k$.

Recall from Definition 2.10 the definition of the counit $\mathrm{i}^{m,n}$ (also denoted i^m) of the adjunction $\mathcal{I}^{\mathrm{G}}_n \dashv \mathcal{T}^{\mathrm{G}}_m$.

2.31. Proposition. *The cone*

$$((\mathbf{Glob}_\omega, T), (\mathcal{T}^{\mathrm{G}}_k, \mathcal{T}^{\mathrm{G}}_k T\, \mathrm{i}^k)_{k \in \mathbb{N}})$$

is a limit cone in $\mathbf{MND}$ on the diagram

$$(\mathbf{Glob}_0, T^0) \xleftarrow{\mathcal{T}^{\mathrm{G}}_0} (\mathbf{Glob}_1, T^1) \xleftarrow{\mathcal{T}^{\mathrm{G}}_1} (\mathbf{Glob}_2, T^2) \xleftarrow{\mathcal{T}^{\mathrm{G}}_2} (\mathbf{Glob}_3, T^3) \xleftarrow{\mathcal{T}^{\mathrm{G}}_3} \cdots$$

where we abbreviated by $\mathcal{T}^{\mathrm{G}}_k$ the monad functor $(\mathcal{T}^{\mathrm{G}}_k, \mathcal{T}^{\mathrm{G}}_k T^{k+1}\, \mathrm{i}^k)$ for $k \in \mathbb{N}$ for readability.

Proof. We already know that it is a cone by Lemma 2.25. Let us prove that it is a limit cone in **MND** (seen here as a 1-category).

Let $(\mathcal{C}, S)$ be a monad and $((\Gamma^k, \gamma^k) \colon (\mathcal{C}, S) \to (\mathbf{Glob}_k, T^k))_{k\in\mathbb{N}}$ be a cone on the diagram of the statement. By applying **Und**, we get a cone $(\Gamma^k \colon \mathcal{C} \to \mathbf{Glob}_k)_k$ and thus a factorising functor $\Gamma \colon \mathcal{C} \to \mathbf{Glob}_\omega$. In order to get a monad functor, we still need to build a 2-cell $\gamma \colon T\Gamma \Rightarrow \Gamma S$. Such a 2-cell is the data of morphisms $\gamma_X \colon T\Gamma X \to \Gamma S X$ for $X \in \mathcal{C}$. Write R^k for

$$R^k = \mathcal{I}^{\mathrm{G}}_\omega \mathcal{T}^{\mathrm{G}}_k \colon \mathbf{Glob}_\omega \to \mathbf{Glob}_\omega$$

and ι^k for the natural transformation

$$\iota^k = \mathcal{I}^{\mathrm{G}}_\omega \, \mathrm{i}^{k,k+1} \, \mathcal{T}^{\mathrm{G}}_{k+1} \colon R^k \Rightarrow R^{k+1}$$

which is just applying $\mathrm{i}^{k,k+1}$ in the middle of a truncation from and an inclusion to ω dimension. Note that, for all $Y \in \mathbf{Glob}_\omega$, $(\mathrm{i}^{k,\omega}_Y \colon R^k Y \to Y)_{k\in\mathbb{N}}$ is a colimit cocone in $\mathbf{Glob}_\omega$ on the diagram

$$R^0 Y \xrightarrow{\iota^0_Y} R^1 Y \xrightarrow{\iota^1_Y} \cdots \xrightarrow{\iota^{k-1}_Y} R^k Y \xrightarrow{\iota^k_Y} R^{k+1} Y \xrightarrow{\iota^{k+1}_Y} \cdots$$

Indeed, the above is a directed diagram and, for $l \in \mathbb{N}$, every l-globe of Y is the image of "essentially one" l-globe of the $R^k Y$'s. The criterion of Proposition A.4 to recognise a colimit cocone on a directed diagram in **Set** then is readily adapted to the presheaf case, so that $(\mathrm{i}^{k,\omega}_Y \colon R^k Y \to Y)_{k\in\mathbb{N}}$ is a colimit cocone.

Now, since T is finitary, $((T\,\mathrm{i}^{k,\omega})_Y \colon TR^k Y \to TY)_{k\in\mathbb{N}}$ is a colimit cocone on the diagram

$$TR^0 Y \xrightarrow{(T\,\iota^0)_Y} TR^1 Y \xrightarrow{(T\,\iota^1)_Y} \cdots \xrightarrow{(T\,\iota^{k-1})_Y} TR^k Y \xrightarrow{(T\,\iota^k)_Y} TR^{k+1} Y \xrightarrow{(T\,\iota^{k+1})_Y} \cdots$$

In particular, given $X \in \mathcal{C}$, a morphism $f \colon T\Gamma X \to \Gamma S X$ (like for instance the morphism γ_X we are looking for) can be defined through a cocone

$$(f^k \colon TR^k \Gamma X \to \Gamma S X)_{k\in\mathbb{N}}$$

on the above diagram, instantiated for $Y = \Gamma X$, so that f^k could be recovered from f through coprojection, that is, $f^k = f \circ (T\,\mathrm{i}^{k,\omega})_{\Gamma X} = f \circ (T\,\mathrm{i}^{k,\omega}\,\Gamma)_X$ for every $k \in \mathbb{N}$.

Generalising this fact from morphisms to natural transformations, we have a limit cone

$$((T\,\mathrm{i}^{k,\omega}\,\Gamma)^* \colon \operatorname{Hom}(T\Gamma, \Gamma S) \to \operatorname{Hom}(TR^k\Gamma, \Gamma S))_{k\in\mathbb{N}}$$

on the diagram

$$\operatorname{Hom}(TR^0\Gamma, \Gamma S) \xleftarrow{(T\,\iota^0\,\Gamma)^*} \operatorname{Hom}(TR^1\Gamma, \Gamma S) \xleftarrow{(T\,\iota^1\,\Gamma)^*} \operatorname{Hom}(TR^2\Gamma, \Gamma S) \xleftarrow{(T\,\iota^2\,\Gamma)^*} \cdots$$

On the other hand, given a pair of functors $G, G' \colon \mathcal{C} \to \mathbf{Glob}_\omega$, every natural transformation $\alpha \colon G \Rightarrow G'$ is uniquely characterised by the projections $\mathcal{T}^{\mathrm{G}}_k \alpha$ for $k \in \mathbb{N}$. In other words, we have a limit cone

$$(\mathcal{T}^{\mathrm{G}}_l \colon \operatorname{Hom}(G, G') \to \operatorname{Hom}(\mathcal{T}^{\mathrm{G}}_l G, \mathcal{T}^{\mathrm{G}}_l G'))_{l\in\mathbb{N}}$$

on the diagram

$$\mathrm{Hom}(\mathcal{T}_0^{\mathrm{G}}G,\mathcal{T}_0^{\mathrm{G}}G') \xleftarrow{\mathcal{T}_0^{\mathrm{G}}} \mathrm{Hom}(\mathcal{T}_1^{\mathrm{G}}G,\mathcal{T}_1^{\mathrm{G}}G') \xleftarrow{\mathcal{T}_1^{\mathrm{G}}} \mathrm{Hom}(\mathcal{T}_2^{\mathrm{G}}G,\mathcal{T}_2^{\mathrm{G}}G') \xleftarrow{\mathcal{T}_3^{\mathrm{G}}} \cdots$$

By combination of the two limit diagrams, we have that $\mathrm{Hom}(T\Gamma,\Gamma S)$ is limit cone on the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "\mathcal{T}_{l+1}^{\mathrm{G}}"] & \vdots \arrow[d, "\mathcal{T}_{l+1}^{\mathrm{G}}"] & \\
\cdots & \mathrm{Hom}(\mathcal{T}_{l+1}^{\mathrm{G}}TR^k\Gamma,\mathcal{T}_{l+1}^{\mathrm{G}}\Gamma S) \arrow[l, "(\iota^{k-1})^*"'] \arrow[d, "\mathcal{T}_l^{\mathrm{G}}"] & \mathrm{Hom}(\mathcal{T}_{l+1}^{\mathrm{G}}TR^{k+1}\Gamma,\mathcal{T}_{l+1}^{\mathrm{G}}\Gamma S) \arrow[l, "(\iota^{k})^*"'] \arrow[d, "\mathcal{T}_l^{\mathrm{G}}"] & \cdots \arrow[l, "(\iota^{k+1})^*"'] \\
\cdots & \mathrm{Hom}(\mathcal{T}_{l}^{\mathrm{G}}TR^k\Gamma,\mathcal{T}_{l}^{\mathrm{G}}\Gamma S) \arrow[l, "(\iota^{k-1})^*"'] \arrow[d, "\mathcal{T}_{l-1}^{\mathrm{G}}"] & \mathrm{Hom}(\mathcal{T}_{l}^{\mathrm{G}}TR^{k+1}\Gamma,\mathcal{T}_{l}^{\mathrm{G}}\Gamma S) \arrow[l, "(\iota^{k})^*"'] \arrow[d, "\mathcal{T}_{l-1}^{\mathrm{G}}"] & \cdots \arrow[l, "(\iota^{k+1})^*"'] \\
 & \vdots & \vdots &
\end{tikzcd}$$

expressed on the category $(\mathbb{N}\times\mathbb{N},\leq\times\leq)^{\mathrm{op}}$. Note that the diagonal functor

$$\Delta^{\mathrm{op}}\colon(\mathbb{N},\leq)^{\mathrm{op}}\to(\mathbb{N}\times\mathbb{N},\leq\times\leq)^{\mathrm{op}},$$

is *final* (see Definition B.3 and Example B.7 for details). Thus, by Proposition B.5, $\mathrm{Hom}(T\Gamma,\Gamma S)$ is in fact a limit cone on the diagram

$$\mathrm{Hom}(\mathcal{T}_0^{\mathrm{G}}TR^0\Gamma,\mathcal{T}_0^{\mathrm{G}}\Gamma S) \longleftarrow \mathrm{Hom}(\mathcal{T}_1^{\mathrm{G}}TR^1\Gamma,\mathcal{T}_1^{\mathrm{G}}\Gamma S) \longleftarrow \mathrm{Hom}(\mathcal{T}_2^{\mathrm{G}}TR^2\Gamma,\mathcal{T}_2^{\mathrm{G}}\Gamma S) \longleftarrow \cdots$$

One can check that a compatible sequence for this diagram is precisely given by the γ^k's (the natural transformation components from the family we started from at the beginning of the proof). Indeed, it is a consequence of the fact that $((\Gamma^k,\gamma^k))_{k\in\mathbb{N}}$ is a cone. Thus, we get $\gamma\colon T\Gamma\Rightarrow\Gamma S$ such that $(\mathcal{T}_k^{\mathrm{G}}\gamma)\circ(\mathcal{T}_k^{\mathrm{G}}T\,\mathrm{i}^{k,\omega}\,\Gamma)=\gamma^k$ and γ is uniquely defined by this property. Thus, we have the uniqueness of a factorising (Γ,γ) in **MND** for the cone we started from. We are left to show that (Γ,γ) is indeed a monad functor for existence.

Let's prove the first required equality, *i.e.*, $\gamma\circ(\eta^T\Gamma)=\Gamma\eta^S$. By an argument similar to the one we used above, it is enough to prove that

$$(\mathcal{T}_k^{\mathrm{G}}\gamma)\circ(\mathcal{T}_k^{\mathrm{G}}\eta^T\Gamma)\circ(\mathcal{T}_k^{\mathrm{G}}\,\mathrm{i}^{k,\omega}\,\Gamma)=(\mathcal{T}_k^{\mathrm{G}}\Gamma\eta^S)\circ(\mathcal{T}_k^{\mathrm{G}}\,\mathrm{i}^{k,\omega}\,\Gamma)$$

for every $k \in \mathbb{N}$. We compute that

$$\begin{aligned} & (\mathcal{T}_k^{\mathrm{G}}\gamma) \circ (\mathcal{T}_k^{\mathrm{G}}\eta^T\Gamma) \circ (\mathcal{T}_k^{\mathrm{G}}\,\mathrm{i}^{k,\omega}\,\Gamma) \\ = {} & (\mathcal{T}_k^{\mathrm{G}}\gamma) \circ (\mathcal{T}_k^{\mathrm{G}}T\,\mathrm{i}^{k,\omega}\,\Gamma) \circ (\mathcal{T}_k^{\mathrm{G}}\eta^T R^k\Gamma) && \text{(by naturality)} \\ = {} & \gamma^k \circ (\mathcal{T}_k^{\mathrm{G}}\eta^T R^k\Gamma) \\ = {} & \gamma^k \circ (\eta^k\Gamma^k) && \text{(since the unit of } \mathcal{I}_\omega^{\mathrm{G}} \dashv \mathcal{T}_k^{\mathrm{G}} \text{ is the identity)} \\ = {} & \Gamma^k\eta^S \\ = {} & \mathcal{T}_k^{\mathrm{G}}\Gamma\eta^S \\ = {} & (\mathcal{T}_k^{\mathrm{G}}\Gamma\eta^S) \circ (\mathcal{T}_k^{\mathrm{G}}\,\mathrm{i}^{k,\omega}\,\Gamma) && \text{(since } \mathcal{T}_k^{\mathrm{G}}\,\mathrm{i}^{k,\omega} \text{ is an identity).} \end{aligned}$$

In order to show the other equality, that is,

$$\gamma \circ (\mu^T\Gamma) = (\Gamma\mu^S) \circ (\gamma S) \circ (T\gamma) \tag{4}$$

we need to use twice the argument we used earlier to get that

$$(\pi_k \colon \operatorname{Hom}(TT\Gamma, \Gamma S) \to \operatorname{Hom}(\mathcal{T}_k^{\mathrm{G}}TR^kTR^k\Gamma, \mathcal{T}_k^{\mathrm{G}}\Gamma S))_{k\in\mathbb{N}}$$

is a limit cone on the adequate diagram, where π_k is the composite

$$\operatorname{Hom}(TT\Gamma, \Gamma S) \xrightarrow{\mathcal{T}_k^{\mathrm{G}}} \operatorname{Hom}(\mathcal{T}_k^{\mathrm{G}}TT\Gamma, \mathcal{T}_k^{\mathrm{G}}\Gamma S) \xrightarrow{(\mathcal{T}_k^{\mathrm{G}}T\,\mathrm{i}^k\,T\,\mathrm{i}^k\,\Gamma)^*} \operatorname{Hom}(\mathcal{T}_k^{\mathrm{G}}TR^kTR^k\Gamma, \mathcal{T}_k^{\mathrm{G}}\Gamma S).$$

Then, using the existing equations and naturality, the reader can prove similarly as above that

$$(\mathcal{T}_k^{\mathrm{G}}(\gamma \circ (\mu^T\Gamma))) \circ (\mathcal{T}_k^{\mathrm{G}}T\,\mathrm{i}^k\,T\,\mathrm{i}^k\,\Gamma) = (\mathcal{T}_k^{\mathrm{G}}((\Gamma\mu^S) \circ (\gamma S) \circ (T\gamma))) \circ (\mathcal{T}_k^{\mathrm{G}}T\,\mathrm{i}^k\,T\,\mathrm{i}^k\,\Gamma)$$

so that (4) holds and (Γ, γ) is indeed a monad functor, which moreover factorises uniquely the cone $(\Gamma^k, \gamma^k)_{k\in\mathbb{N}}$. Thus, $(\mathbf{Glob}_\omega, T)$ is indeed a limit cone in **MND** as stated. ∎

2.32. Proposition. *$(\mathcal{T}_k^{\mathrm{A}} \colon \mathbf{Alg}_\omega \to \mathbf{Alg}_k)_{k\in\mathbb{N}}$ is a limit cone in* **CAT** *on the diagram*

$$\mathbf{Alg}_0 \xleftarrow{\mathcal{T}_0^{\mathrm{A}}} \mathbf{Alg}_1 \xleftarrow{\mathcal{T}_1^{\mathrm{A}}} \mathbf{Alg}_2 \xleftarrow{\mathcal{T}_2^{\mathrm{A}}} \mathbf{Alg}_3 \xleftarrow{\mathcal{T}_3^{\mathrm{A}}} \cdots \tag{5}$$

Proof. This is a consequence of Proposition 2.31 and that **EM** is a right adjoint. ∎

2.33. Remark. The above proof only show that $\mathbf{Alg}_\omega$ is a limit in the 1-categorical sense, *i.e.*, only factorises cones of functors by a functor (and not morphisms of cones as natural transformations, *a priori*). But, by a generic argument (stated for example as [31, Lemma 7.5.1]), any limit in the 1-categorical sense in **CAT** is also a strict 2-limit, *i.e.*, factorises morphisms between cones as well.

2.34. Remark. In the context of a 2-category and, in particular, diagrams of categories, one is usually more interested in the notion of *bilimit*, which also allows factorising *pseudocones*, which are more natural in a 2-categorical setting. The good news is that $\mathbf{Alg}_\omega$ is also a bilimit since the truncation functors $\mathcal{T}_i^{\mathrm{A}}$'s are isofibrations[2]. Indeed, given a pseudocone on (5), that is, a category $\mathcal{C}$, functors $\mathcal{H}^k\colon \mathcal{C} \to \mathbf{Alg}_k$ for $k \in \mathbb{N}$ and natural isomorphisms $\theta^k\colon \mathcal{T}_k^{\mathrm{A}}\mathcal{H}^{k+1} \Rightarrow \mathcal{H}^k$ for $k \in \mathbb{N}$, we are able to build a (strict) cone which is equivalent to this pseudocone, by proceeding incrementally on the dimension k. We describe one step of the procedure for some $k \in \mathbb{N}$: given a triangle

$$
\begin{tikzcd}
& \mathcal{C} \arrow[dl, "\tilde{\mathcal{H}}^k"'] \arrow[dr, "\mathcal{H}^{k+1}"] \arrow[d, phantom, "\overset{\tilde{\theta}^k}{\Longleftarrow}"] & \\
\mathbf{Alg}_k & & \mathbf{Alg}_{k+1} \arrow[ll, "\mathcal{T}_k^{\mathrm{A}}"]
\end{tikzcd}
$$

for some functor $\tilde{\mathcal{H}}^k$ and natural transformation $\tilde{\theta}^k$, a pointwise use of the fact that $\mathcal{T}_k^{\mathrm{A}}$ is an isofibration allows us to construct a functor $\tilde{\mathcal{H}}^{k+1}$ and a natural isomorphism $\beta^{k+1}\colon \mathcal{H}^{k+1} \Rightarrow \tilde{\mathcal{H}}^{k+1}$ such that

$$
\tilde{\theta}^k \qquad = \qquad
\begin{tikzcd}
& \mathcal{C} \arrow[dl, "\tilde{\mathcal{H}}^k"'] \arrow[dr, "\tilde{\mathcal{H}}^{k+1}"'] \arrow[dr, bend left=60, "\mathcal{H}^{k+1}"] \arrow[dr, phantom, bend left=30, "\Downarrow\beta^{k+1}"] \arrow[d, phantom, "="] & \\
\mathbf{Alg}_k & & \mathbf{Alg}_{k+1} \arrow[ll, "\mathcal{T}_k^{\mathrm{A}}"]
\end{tikzcd}
.
$$

Using this elementary step adequately for every $k \in \mathbb{N}$, we get a strict cone on (5) which is equivalent (for a suitable notion of “equivalence”) to the pseudocone we started from, and which can thus be factored through $\mathbf{Alg}_\omega$. Thus, $(\mathcal{T}_k^{\mathrm{A}}\colon \mathbf{Alg}_\omega \to \mathbf{Alg}_k)_{k\in\mathbb{N}}$ is also a bilimit on (5).

2.35. Remark. If we restrict our interest to *weakly truncatable monads* (defined in Section 2.40), another proof is possible which dispenses with the specific and technical argument of Proposition 2.31. Indeed, writing $\mathbf{MND}_{\mathrm{psd}}$ for the sub-2-category of $\mathbf{MND}$ consisting of monads, *pseudo* monad functors and monad transformations, where a monad functor (F,ϕ) is said *pseudo* when ϕ is an isomorphism, we have that the *underlying functor* $\mathbf{Und}\colon \mathbf{MND}_{\mathrm{psd}} \to \mathbf{CAT}$, mapping a monad $(\mathcal{C},T)$ to its underlying category $\mathcal{C}$, creates bilimits, and the latter are moreover preserved by the inclusion $\mathbf{MND}_{\mathrm{psd}} \hookrightarrow \mathbf{MND}$ (this is left as an exercise for now). Thus, a bilimit version of Proposition 2.31 can be deduced from the fact that $\mathbf{Glob}_\omega$ is a bilimit of the $\mathbf{Glob}_k$'s for $k \in \mathbb{N}$. This argument does not apply for $\mathbf{MND}$ as a whole, and it is really the finitary hypothesis on T which makes Proposition 2.31 work.

[2]Recall that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *isofibration* when for every $c \in \mathcal{C}$, $d \in \mathcal{D}$ and isomorphism $\theta\colon F(c) \cong d \in \mathcal{D}$, there exists $c' \in \mathcal{C}$ and an isomorphism $\bar{\theta}\colon c \cong c' \in \mathcal{C}$ such that $F(\bar{\theta}) = \theta$.

2.36. A criterion for globular algebras. Usually, a specific notion of higher category and the associated truncation and inclusion functors are not directly derived from a monad. Instead, we often manipulate higher categories that are defined, in each dimension $k \in \mathbb{N}$, as structures with operations satisfying some equations, and the truncation and inclusion functors are defined by hand. Such equational definitions surely induce monads on k-globular sets, but it is not immediate that the monad in dimension l_1 is obtained by truncating the monad in dimension l_2 for $l_1 < l_2$, as was done earlier, nor is the equivalence between the boilerplate definitions of truncation and inclusion functors and the ones defined earlier in this section. Verifying the equivalences of these definitions is required in order to use general constructions for globular algebras, like the ones of the next section. But, without a generic argument, the verification can be tedious since it involves, among others, an explicit description of the different monads. In this section, we give a criterion, in the form of Theorem 2.38, to recognize that categories and functors between them are equivalent to categories of globular algebras and truncation functors derived from some monad on globular sets, which is a first tool to contribute to the second of the three goals from the introduction.

We first prove a technical lemma:

2.37. Lemma. *Given a commutative diagram of functors*

$$\begin{tikzcd} C \arrow[r, "U"] \arrow[d, "\mathcal{T}"'] & D \arrow[d, "\mathcal{T}'"] \\ \bar{C} \arrow[r, "\bar{U}"'] & \bar{D} \end{tikzcd}$$

which are part of adjunctions $F \dashv U$, $\bar{F} \dashv \bar{U}$, $\mathcal{I} \dashv \mathcal{T}$, $\mathcal{I}' \dashv \mathcal{T}'$, we have a commutative diagram in **MND**

$$\begin{tikzcd} (C, 1_C) \arrow[rrr, "{(U, U\epsilon^{F,U})}"] \arrow[d, "\mathbf{Inc}(\mathcal{T})"'] & & & (D, R) \arrow[d, "{(\mathcal{T}', \mathcal{T}' R \epsilon^{\mathcal{I}', \mathcal{T}'})}"] \\ (\bar{C}, 1_{\bar{C}}) \arrow[r, "{(\bar{U}, \bar{U}\epsilon^{\bar{F},\bar{U}})}"'] & (\bar{D}, \bar{T}) & (\bar{D}, \bar{S}) \arrow[l, "{(1_{\bar{D}}, \bar{U}\eta^{\mathcal{I},\mathcal{T}}\bar{F})}"] & (\bar{D}, S) \arrow[l, "\sim"] \end{tikzcd}$$

where R, S, $\bar{S}$ and $\bar{T}$ are the monads respectively derived from the adjunctions $F \dashv U$, $F\mathcal{I}' \dashv \mathcal{T}'U$, $\mathcal{I}\bar{F} \dashv \bar{U}\mathcal{T}$ and $\bar{F} \dashv \bar{U}$, and $(\bar{D}, S) \to (\bar{D}, \bar{S})$ is the isomorphism of monads induced by $\mathcal{T}'U = \bar{U}\mathcal{T}$, and where we wrote $\eta^{L,R}$ and $\epsilon^{L,R}$ for the unit and counit of each adjunction $L \dashv R$. Moreover, if the unit of the adjunction $\mathcal{I} \dashv \mathcal{T}$ is an isomorphism (or, equivalently, $\mathcal{I}$ is fully faithful), then the morphism $(\bar{D}, \bar{S}) \to (\bar{D}, \bar{T})$ is an isomorphism.

Proof. By Propositions 1.14 and 1.19 already introduced, the morphisms shown on the diagram are indeed monad functors. We are left to check that the diagram commutes.

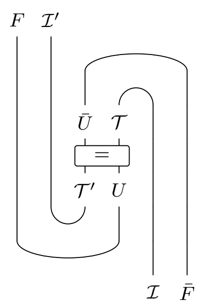
More precisely, we need to show that pasting of 2-cells

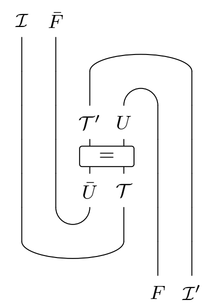
$$\begin{tikzcd}
C \arrow[r, "U"] \arrow[d, "1_C"'] \arrow[dr, phantom, "="] & D \arrow[r, "\mathcal{T}'"] \arrow[d, "R"] \arrow[dr, phantom, "\Downarrow \mathcal{T}'R\epsilon^{\mathcal{I}',\mathcal{T}'}"] & \bar{D} \arrow[r, "1_{\bar{D}}"] \arrow[d, "S"] \arrow[dr, phantom, "\Downarrow \bar{U}\mathcal{T}\theta^{-1}"] & \bar{D} \arrow[r, "1_{\bar{D}}"] \arrow[d, "\bar{S}"] \arrow[dr, phantom, "\Downarrow \bar{U}\eta^{\mathcal{I},\mathcal{T}}\bar{F}"] & \bar{D} \arrow[d, "\bar{T}"] \\
C \arrow[r, "U"'] & D \arrow[r, "\mathcal{T}'"'] & \bar{D} \arrow[r, "1_{\bar{D}}"'] & \bar{D} \arrow[r, "1_{\bar{D}}"'] & \bar{D}
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
C \arrow[r, "\mathcal{T}"] \arrow[d, "1_C"'] \arrow[dr, phantom, "="] & \bar{C} \arrow[r, "\bar{U}"] \arrow[d, "1_{\bar{C}}"] \arrow[dr, phantom, "\Downarrow \bar{U}\epsilon^{\bar{F},\bar{U}}"] & \bar{D} \arrow[d, "\bar{T}"] \\
C \arrow[r, "\mathcal{T}"'] & \bar{C} \arrow[r, "\bar{U}"'] & \bar{D}
\end{tikzcd}$$

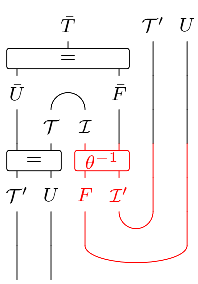
where $\theta\colon F\mathcal{I}' \Rightarrow \mathcal{I}\bar{F}$ is the canonical isomorphism between the two functors which are left adjoint to $\bar{U}\mathcal{T} = \mathcal{T}'U$. This morphism and its inverse are defined by

$\theta \quad = $ 

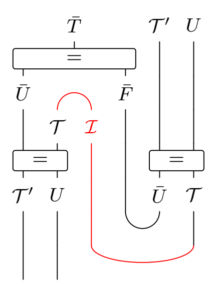
$\theta^{-1} \quad = $ 

.

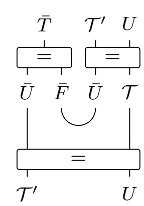
We check that the wanted equality holds using string diagrams:



$\overset{\text{def}+(\text{ZZ})}{=}$



$\overset{(\text{ZZ})}{=}$


.

■

The criterion for recognizing categories and functors as categories of globular algebras and truncation functors between them is the following:

2.38. Theorem. *Let (T, η, μ) be a finitary monad on $\mathbf{Glob}_\omega$, and*

$$(C_k)_{k\in\mathbb{N}} \quad \textit{and} \quad C_\omega$$

be categories, and

$$(U_k\colon C_k \to \mathbf{Glob}_k)_{k\in\mathbb{N}} \quad \textit{and} \quad U_\omega\colon C_\omega \to \mathbf{Glob}_\omega$$

be monadic functors, and

$$(\mathcal{T}_k^{k+1}\colon C_{k+1} \to C_k)_{k\in\mathbb{N}} \quad \textit{and} \quad (\mathcal{T}_k^{\omega}\colon C_\omega \to C_k)_{k\in\mathbb{N}}$$

be right adjoint functors with fully faithful left adjoints such that $\mathcal{T}_k^{k+1}\mathcal{T}_{k+1}^{\omega} = \mathcal{T}_k^{\omega}$ *and*

$$\begin{tikzcd} C_\omega \arrow[r, "U_\omega"] \arrow[d, "\mathcal{T}_k^\omega"'] & \mathbf{Glob}_\omega \arrow[d, "\mathcal{T}_{\omega,k}^{\mathrm{G}}"] \\ C_k \arrow[r, "U_k"'] & \mathbf{Glob}_k \end{tikzcd}$$

commute for every $k \in \mathbb{N}$. *Then, there exist equivalences of categories*

$$H_\omega\colon C_\omega \to \mathbf{Alg}_\omega \quad \textit{and} \quad (H_k\colon C_k \to \mathbf{Alg}_k)_{k\in\mathbb{N}}$$

such that

$$\begin{tikzcd} C_\omega \arrow[r, "H_\omega"] \arrow[d, "\mathcal{T}_k^\omega"'] & \mathbf{Alg}_\omega \arrow[d, "\mathcal{T}_{\omega,k}^{\mathrm{A}}"] \\ C_k \arrow[r, "H_k"'] & \mathbf{Alg}_k \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} C_{k+1} \arrow[r, "H_{k+1}"] \arrow[d, "\mathcal{T}_k^{k+1}"'] & \mathbf{Alg}_{k+1} \arrow[d, "\mathcal{T}_{k+1,k}^{\mathrm{A}}"] \\ C_k \arrow[r, "H_k"'] & \mathbf{Alg}_k \end{tikzcd} \tag{6}$$

commute for every $k \in \mathbb{N}$.

2.39. Remark. The statement of the above theorem can easily be adapted when working on a monad T on $\mathbf{Glob}_n$ for some $n \in \mathbb{N}$ by requiring instead that

$$\begin{tikzcd} C_{k+1} \arrow[r, "U_{k+1}"] \arrow[d, "\mathcal{T}_k^{k+1}"'] & \mathbf{Glob}_{k+1} \arrow[d, "\mathcal{T}_{k+1,k}^{\mathrm{G}}"] \\ C_k \arrow[r, "U_k"'] & \mathbf{Glob}_k \end{tikzcd}$$

commutes for every $k < n$, and then one only gets the right commuting square of (6).

Proof. Let $k \in \mathbb{N}$. Taking C_ω, C_k, $\mathbf{Glob}_\omega$ and $\mathbf{Glob}_k$ respectively for C, $\bar{C}$, D and $\bar{D}$, and U_ω, U_k, $\mathcal{T}_k^\omega$ and $\mathcal{T}_{\omega,k}^{\mathrm{G}}$ respectively for U, $\bar{U}$, $\mathcal{T}$ and $\mathcal{T}'$ in Lemma 2.37, and using the adjunction $\mathbf{Inc} \dashv \mathbf{EM}$, we get a commutative square

$$\begin{tikzcd} C_\omega \arrow[rrr, "H_\omega"] \arrow[d, "\mathcal{T}_k^\omega"'] & & & \mathbf{Alg}_\omega \arrow[d, "\mathcal{T}_{\omega,k}^{\mathrm{A}}"] \\ C_k \arrow[r, "\bar{H}_k"'] & \mathbf{Glob}_k^{\bar{T}} & \mathbf{Glob}_k^{\bar{S}} \arrow[l, "\sim"] & \mathbf{Alg}_k \arrow[l, "\sim"] \end{tikzcd}$$

where $\bar{S}$ and $\bar{T}$ are the monads defined in the lemma. Remember that both $\mathbf{Alg}_k$ and $\mathcal{T}^{\mathrm{A}}_{\omega,k}$ were defined using Proposition 1.14 so that the arrow on the right of the above diagram is indeed $\mathcal{T}^{\mathrm{A}}_{\omega,k}$. Also, H_ω and $\bar{H}_k$ are equivalences since U_ω and U_k were supposed monadic. Finally, by the final part of Lemma 2.37 and the hypothesis, the middle arrow in the bottom line is an isomorphism. Thus, by defining H_k to be the composition of the three morphisms on the bottom line (after inverting the second and third ones), we get the commutative diagram on the left of (6), and we are left to get the one on the right. Writing $\mathcal{I}^{k+1}_\omega$ for a left adjoint to $\mathcal{T}^\omega_{k+1}$, since the unit $1_{C_\omega} \Rightarrow \mathcal{T}^\omega_{k+1}\mathcal{I}^{k+1}_\omega$ is an isomorphism, it is enough to check that

$$\mathcal{T}^{\mathrm{A}}_k H_{k+1} \mathcal{T}^\omega_{k+1} \mathcal{I}^{k+1}_\omega = H_k \mathcal{T}^{k+1}_k \mathcal{T}^\omega_{k+1} \mathcal{I}^{k+1}_\omega .$$

But

$$\begin{aligned} \mathcal{T}^{\mathrm{A}}_k H_{k+1} \mathcal{T}^\omega_{k+1} \mathcal{I}^{k+1}_\omega &= \mathcal{T}^{\mathrm{A}}_k \mathcal{T}^{\mathrm{A}}_{\omega,k+1} H_\omega \mathcal{I}^{k+1}_\omega \\ &= \mathcal{T}^{\mathrm{A}}_{\omega,k} H_\omega \mathcal{I}^{k+1}_\omega \\ &= H_k \mathcal{T}^\omega_k \mathcal{I}^{k+1}_\omega \\ &= H_k \mathcal{T}^{k+1}_k \mathcal{T}^\omega_{k+1} \mathcal{I}^{k+1}_\omega \end{aligned}$$

which concludes the proof. ∎

2.40. Truncatable globular monads. The general setting of higher category theories as monads over globular sets allows defining theories with unusual operations, like compositions of l-cells that produce unrelated l'-cells for some $l' < l$ (*c.f.* Example 2.21). Anticipating the next section, such theories are badly behaved when it comes to freely adding new $(k{+}1)$-generators to k-categories, since the underlying k-categories will not be preserved in the process. In order not to allow such monads, we recall from [4] the notion of *truncatable monad* which forbids those problematic operations and still includes most usual theories for higher categories: those are the monads which "commute with truncation" in a suitable sense. As we will see in the nextsection, the k-categories of these theories are preserved when freely adding $(k{+}1)$-generators.

In this section, we fix $n \in \mathbb{N} \cup \{\omega\}$ and a finitary monad (T, η, μ) on $\mathbf{Glob}_n$.

For $k < n + 1$, remember that we used Proposition 1.14 to define both the monad (T^k, η^k, μ^k) and the cocartesian morphism

$$(\mathcal{T}^{\mathrm{G}}_{n,k}, \mathcal{T}^{\mathrm{G}}_k T\, \mathrm{i}^{k,n})\colon \quad (\mathbf{Glob}_n, T) \quad \to \quad (\mathbf{Glob}_k, T^k).$$

2.41. Definition. *We write* t^k *for the natural transformation*

$$\mathcal{T}^{\mathrm{G}}_k T\, \mathrm{i}^{k,n}\colon \quad \mathcal{T}^{\mathrm{G}}_k T \mathcal{I}^{\mathrm{G}}_{k,n} \mathcal{T}^{\mathrm{G}}_{n,k} \quad \Rightarrow \quad \mathcal{T}^{\mathrm{G}}_k T.$$

The monad T is said weakly truncatable *when* t^k *is an isomorphism for each $k < n$; it is* truncatable *when, for each $k < n$,* t^k *is the natural identity transformation (so that $T_k \mathcal{T}^{\mathrm{G}}_k = \mathcal{T}^{\mathrm{G}}_k T$).*

2.42. Example. The monad (T,η,μ) of categories on $\mathbf{Glob}_1$ defined in Example 2.20 is weakly truncatable. By adequately choosing the left adjoint $\mathbf{Glob}_1 \to \mathbf{Cat}$ that defines T, we can even suppose that T is truncatable. More generally, we will see in Section 4.19 that the monad of strict ω-categories is weakly truncatable, and even truncatable up to an isomorphism of monads.

2.43. Example. The monad (T,η,μ) of weird 2-categories on $\mathbf{Glob}_2$ defined in Example 2.21 is not truncatable since, for $X \in \mathbf{Glob}_2$, we have

$$(TX)_0 \cong X_0 \sqcup (X_2 \times X_2) \quad \text{and} \quad (T^0(X_{\le 0}))_0 \cong X_0.$$

The following property tells that we can always adapt a weakly truncatable monad to a truncatable monad:

2.44. Proposition. *If (T,η,μ) is weakly truncatable, then it is isomorphic to a monad which is truncatable.*

Proof. We define a truncatable monad $(\bar{T},\bar{\eta},\bar{\mu})$ on $\mathbf{Glob}_n$ and an isomorphism $\phi\colon T \to \bar{T}$ from their trunctations

$$\mathcal{T}^{\mathrm{G}}_k\bar{T} \quad \text{and} \quad \phi_{\le k}\colon \mathcal{T}^{\mathrm{G}}_k T \to \mathcal{T}^{\mathrm{G}}_k\bar{T}$$

and we define those using an induction on k for $k < n+1$. In dimension 0, we put

$$\mathcal{T}^{\mathrm{G}}_0\bar{T} = T^0\mathcal{T}^{\mathrm{G}}_0 \quad \text{and} \quad \phi_0 = (\mathrm{t}^0)^{-1}$$

Then, given $k < n+1$ and a $(k{+}1)$-globular set X, we define $(\bar{T}X)_{\le k+1}$ as the $(k{+}1)$-globular set Y where

$$Y_{\le k} = (\bar{T}X)_{\le k} \quad \text{and} \quad Y_{k+1} = (T^{k+1}(X_{\le k+1}))_{k+1}$$

and the operation $\partial^\epsilon_k\colon Y_{k+1} \to Y_k$ is defined as the composite

$$Y_{k+1} = (T^{k+1}(X_{\le k+1}))_{k+1} \xrightarrow{\partial^\epsilon_k} (T^{k+1}(X_{\le k+1}))_k \xrightarrow{(\mathrm{t}^{k+1})_k} (TX)_k \xrightarrow{(\phi_k)_k} (\bar{T}X)_k$$

for $\epsilon \in \{-,+\}$. Our definition extends canonically to a functor

$$\mathcal{T}^{\mathrm{G}}_{k+1}\bar{T}\colon \mathbf{Glob}_n \to \mathbf{Glob}_{k+1}.$$

We also extend $\phi_{\le k}$ on dimension $k+1$ by putting, for $X \in \mathbf{Glob}_n$,

$$(\phi_X)_{k+1} = ((\mathrm{t}^{k+1}_X)^{-1})_{k+1}\colon (TX)_{k+1} \to (\bar{T}X)_{k+1}$$

So we defined $\bar{T}\colon \mathbf{Glob}_n \to \mathbf{Glob}_n$ together with an isomorphism $\phi\colon T \to \bar{T}$. Finally, we put

$$\bar{\eta} = \phi \circ \eta \quad \text{and} \quad \bar{\mu} = \phi \circ \mu \circ (\phi^{-1}\phi^{-1})$$

so that $(\bar{T},\bar{\eta},\bar{\mu})$ is a monad and $(1_{\mathbf{Glob}_n},\phi^{-1})\colon (\mathbf{Glob}_n,T) \to (\mathbf{Glob}_n,\bar{T})$ a monad isomorphism. By the definition of $\bar{T}$, we easily verify that $(\bar{T},\bar{\eta},\bar{\mu})$ is truncatable. ∎

The truncatability with respect to dimension n implies the truncatability with respect to dimension l for any $l < n$:

2.45. Lemma. *If T is weakly truncatable (resp. truncatable), then, for every $k, l \in \mathbb{N}$ with $k < l < n$, $\mathcal{T}_k^{\mathrm{G}} T^l \,\mathrm{i}^{k,l}$ is an isomorphism (resp. an identity).*

Proof. We have $\mathrm{i}^{k,n} = \mathrm{i}^{k,l} \circ (\mathcal{I}_{k,l}^{\mathrm{G}} \,\mathrm{i}^{k,l}\, \mathcal{T}_{l,k}^{\mathrm{G}})$, so that

$$\mathcal{T}_{n,k}^{\mathrm{G}} T \,\mathrm{i}^{k,n} = (\mathcal{T}_{l,k}^{\mathrm{G}} \mathcal{T}_{n,l}^{\mathrm{G}} T \,\mathrm{i}^{l,n}) \circ (\mathcal{T}_{l,k}^{\mathrm{G}} T^l \,\mathrm{i}^{k,l}\, \mathcal{T}_{n,l}^{\mathrm{G}}).$$

By the 2-out-of-3 property for isomorphisms, we get that $\mathcal{T}_{l,k}^{\mathrm{G}} T^l \,\mathrm{i}^{k,l}\, \mathcal{T}_{n,l}^{\mathrm{G}}$ is an isomorphism. By precomposing with $\mathcal{I}_{l,n}^{\mathrm{G}}$, we get that $\mathcal{T}_{l,k}^{\mathrm{G}} T^l \,\mathrm{i}^{k,l}$ is an isomorphism. When T is truncatable, this isomorphism is an equality. ∎

When T is truncatable, the T^k, η^k and μ^k can be related through the equations given by the following lemma:

2.46. Lemma. *If T is truncatable, then, for $k, l \leq n$ with $k < l$, we have*

$$T^k \mathcal{T}_{l,k}^{\mathrm{G}} = \mathcal{T}_{l,k}^{\mathrm{G}} T^l \quad \textit{and} \quad \mathcal{T}_{l,k}^{\mathrm{G}} \eta^l = \eta^k \mathcal{T}_{l,k}^{\mathrm{G}} \quad \textit{and} \quad \mathcal{T}_{l,k}^{\mathrm{G}} \mu^l = \mu^k \mathcal{T}_{l,k}^{\mathrm{G}}$$

Proof. By Lemma 2.45, we have that $\mathcal{T}_k^{\mathrm{G}} T^l \,\mathrm{i}^{k,l}$ is an identity. In particular,

$$T^k \mathcal{T}_{l,k}^{\mathrm{G}} = \mathcal{T}_{l,k}^{\mathrm{G}} T^l.$$

Moreover, from the equations satisfied by the monad functor $(\mathcal{T}_{l,k}^{\mathrm{G}}, \mathcal{T}_k^{\mathrm{G}} T^l \,\mathrm{i}^{k,l})$, we deduce that

$$\mathcal{T}_{l,k}^{\mathrm{G}} \eta^l = \eta^k \mathcal{T}_{l,k}^{\mathrm{G}} \quad \text{and} \quad \mathcal{T}_{l,k}^{\mathrm{G}} \mu^l = \mu^k \mathcal{T}_{l,k}^{\mathrm{G}}.$$

∎

We now prove several properties of truncatable monads regarding truncation of algebras. First, the truncation of algebras has now a simpler definition:

2.47. Proposition. *If T is truncatable, then given $k, l \leq n$ such that $k < l$, and an l-algebra $(X, h) \in \mathbf{Alg}_l$, we have $(X, h)_{\leq k} = (X_{\leq k}, h_{\leq k})$.*

Proof. Indeed, since T is truncatable, we have

$$(\mathcal{T}_{l,k}^{\mathrm{G}} T^l \,\mathrm{i}^{k,l})_X = (T^k \mathcal{T}_{l,k}^{\mathrm{G}} \,\mathrm{i}^{k,l})_X = \mathrm{id}_{T^k X}$$

so that $(X, h)_{\leq k} = (X_{\leq k}, h_{\leq k})$. ∎

Moreover, the operation of truncation of algebras is now a left adjoint:

2.48. PROPOSITION. *If T is truncatable, then, given $k, l \leq n$ with $k < l$, the functor*

$$\mathcal{T}^{\mathrm{A}}_{l,k} \colon \mathbf{Alg}_l \to \mathbf{Alg}_k$$

is a left adjoint. In particular, it preserves colimits.

PROOF. By Lemma 2.45, $\mathcal{T}^{\mathrm{G}}_k T^l\, \mathrm{i}^{k,l}$ is an identity and in particular an isomorphism. Thus, by Proposition 1.18, since we have an adjunction $\mathcal{T}^{\mathrm{G}}_{l,k} \dashv \mathcal{J}^{\mathrm{G}}_{k,l}$, this adjunction lifts canonically through $\mathbf{Und}$ to an adjunction

$$(\mathcal{T}^{\mathrm{G}}_{l,k}, \mathcal{T}^{\mathrm{G}}_k T^l\, \mathrm{i}^{k,l}) \dashv (\mathcal{J}^{\mathrm{G}}_{k,l}, \rho) \tag{7}$$

in $\mathbf{MND}$ for some ρ given by Proposition 1.18. This adjunction is sent by $\mathbf{EM}$ to another adjunction, whose left adjoint is $\mathcal{T}^{\mathrm{A}}_{l,k}$. ∎

2.49. DEFINITION. *If T is truncatable, given $k, l \leq n$ with $k < l$, we write $\mathcal{J}^{\mathrm{A}}_{k,l}$ for the right adjoint to the functor $\mathcal{T}^{\mathrm{A}}_{l,k}$ built in the proof of Proposition 2.48.*

2.50. REMARK. In the proof of Proposition 2.48, since the counit of $\mathcal{T}^{\mathrm{G}}_{l,k} \dashv \mathcal{J}^{\mathrm{G}}_{k,l}$ is the identity (*c.f.* Definition 2.12), the counit of the lifted adjunction (7) is also the identity. Thus, the image by $\mathbf{EM}$ of the adjunction (7), which is $\mathcal{T}^{\mathrm{A}}_{l,k} \dashv \mathcal{J}^{\mathrm{A}}_{k,l}$, has identity as counit.

2.51. CHARACTERISATION OF TRUNCATABLE MONADS. Earlier, we introduced Theorem 2.38 that allows recognizing that some categories and functors between them are equivalent to the categories of globular algebras and the associated truncation functors derived from a monad T on globular sets, without having to explicitly describe this monad. But, by the current definition of truncatability, in order to show that the monad T is truncatable, a direct proof would require to show that the natural transformations $\mathcal{T}^{\mathrm{G}}_l T\, \mathrm{i}^{l,n}$ are isomorphisms, so that a description of T is still needed. Below, we introduce a characterisation of the truncatability of T that does not rely on such tedious description, which is another tool, in addition to Theorem 2.38, to contribute to the second of the three goals from the introduction.

We start by proving the following lemma, relating the functors $\mathcal{F}^{\mathrm{G}}_k$ (the free T^k-algebra functors from Definition 2.17):

2.52. LEMMA. *Given $k \in \mathbb{N}$ such that $k < n$, we have*

$$\mathcal{T}^{\mathrm{A}}_k \mathcal{F}^{\mathrm{G}}_n \mathcal{I}^{\mathrm{G}}_{k,n} = \mathcal{F}^{\mathrm{G}}_k.$$

PROOF. By the adjunction $\mathbf{Inc} \dashv \mathbf{EM}$, it amounts to prove that the two pastings of 2-cells

$$\begin{tikzcd}
\mathbf{Glob}_k \arrow[r, "\mathcal{I}^{\mathrm{G}}_n"] \arrow[d, "1"'] \arrow[dr, phantom, "="] & \mathbf{Glob}_n \arrow[r, "T"] \arrow[d, dashed, "1"] \arrow[dr, phantom, "\Downarrow \mu"] & \mathbf{Glob}_n \arrow[r, "\mathcal{T}^{\mathrm{G}}_k"] \arrow[d, dashed, "T"] \arrow[dr, phantom, "\Downarrow \mathcal{T}^{\mathrm{G}}_k T\, \mathrm{i}^k"] & \mathbf{Glob}_k \arrow[d, "T^k"] \\
\mathbf{Glob}_k \arrow[r, "\mathcal{I}^{\mathrm{G}}_n"'] & \mathbf{Glob}_n \arrow[r, "T"'] & \mathbf{Glob}_n \arrow[r, "\mathcal{T}^{\mathrm{G}}_k"'] & \mathbf{Glob}_k
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{Glob}_k \arrow[r, "T^k"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow \mu^k"] & \mathbf{Glob}_k \arrow[d, "T^k"] \\
\mathbf{Glob}_k \arrow[r, "T^k"'] & \mathbf{Glob}_k
\end{tikzcd}$$

are equal. But this directly follows from the definition of μ^k. So the wanted equality holds. ∎

Now, we prove that truncatable monads can be characterised through the associated globular algebras:

2.53. Proposition. *The monad (T, η, μ) is weakly truncatable (resp. truncatable) if and only if, for $k < n$, the natural transformation*

$$\mathcal{T}^{\mathrm{A}}_k \mathcal{F}^{\mathrm{G}}_n \, \mathrm{i}^{k,n} \colon \mathcal{F}^{\mathrm{G}}_k \mathcal{T}^{\mathrm{G}}_k \Rightarrow \mathcal{T}^{\mathrm{A}}_k \mathcal{F}^{\mathrm{G}}_n$$

is an isomorphism (resp. an identity).

Proof. Note that the domain of $\mathcal{T}^{\mathrm{A}}_k \mathcal{F}^{\mathrm{G}}_n \, \mathrm{i}^{k,n}$ is the claimed one by Lemma 2.52. Now, for $k < n$, we have that

$$\begin{aligned}
\mathcal{U}^{\mathrm{G}}_k \mathcal{T}^{\mathrm{A}}_k \mathcal{F}^{\mathrm{G}}_n \, \mathrm{i}^k &= \mathcal{T}^{\mathrm{G}}_k \mathcal{U}^{\mathrm{G}}_n \mathcal{F}^{\mathrm{G}}_n \, \mathrm{i}^k \\
&= \mathcal{T}^{\mathrm{G}}_k T \, \mathrm{i}^k .
\end{aligned}$$

The proposition follows from the fact that $\mathcal{U}^{\mathrm{G}}_k$ reflects isomorphisms (resp. identities). ∎

We will need the following folklore property about adjunctions:

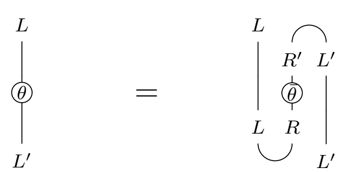
2.54. Proposition. *Let*

$$L \dashv R \colon C \to D \quad \text{and} \quad L' \dashv R' \colon C \to D$$

be two adjunctions with respective unit-counit pairs (γ, ϵ) and (γ', ϵ'), and

$$\theta \colon L \Rightarrow L' \quad \text{and} \quad \bar{\theta} \colon R' \Rightarrow R$$

be two natural transformations such that $\theta = (\epsilon L') \circ (L \bar{\theta} L') \circ (L \gamma')$, i.e., graphically:



.

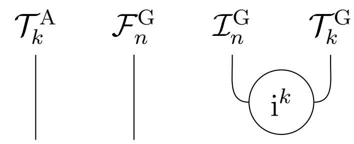
Then, θ is an isomorphism if and only if $\bar{\theta}$ is an isomorphism.

Proof. An inverse for θ can be straight-forwardly constructed from an inverse for $\bar{\theta}$ using string diagrams, and symmetrically. ∎

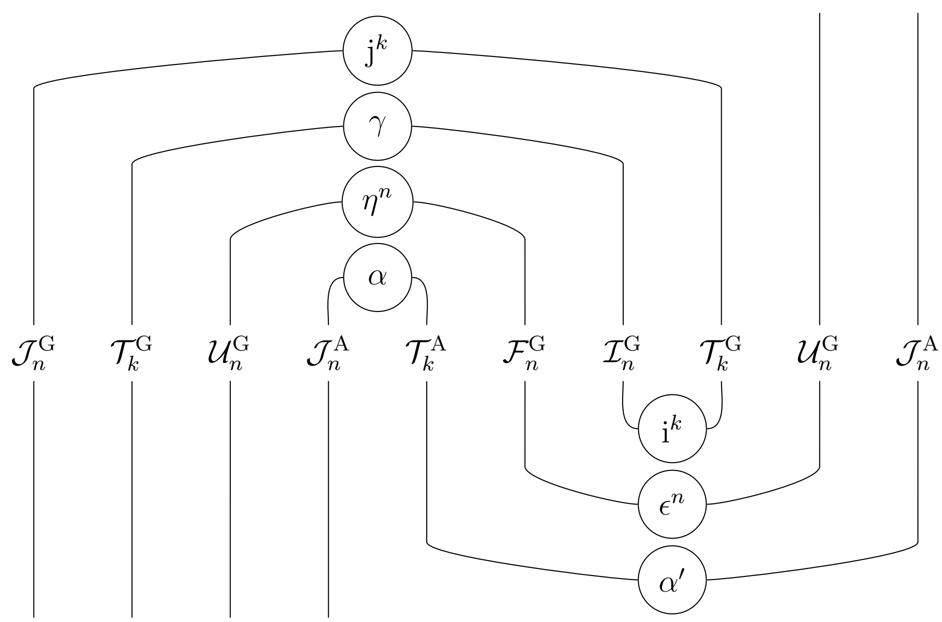
Recall the natural transformation j^k from Definition 2.12. We now introduce the criterion for showing the truncatability of monads through their globular algebras:

2.55. THEOREM. *Let $n \in \mathbb{N} \cup \{\omega\}$ and (T, η, μ) be a finitary monad on $\mathbf{Glob}_n$. The monad (T, η, μ) is weakly truncatable if and only if, for $k < n$, the functor $\mathcal{T}^{\mathrm{A}}_{n,k}$ has a right adjoint, which we write $\mathcal{J}^{\mathrm{A}}_{k,n}$ (as in Definition 2.49), such that $\mathrm{j}^k \mathcal{U}^{\mathrm{G}}_n \mathcal{J}^{\mathrm{A}}_{k,n}$ is an isomorphism.*

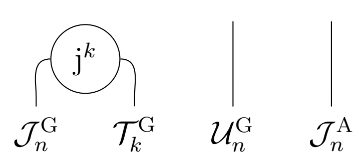
PROOF. By Proposition 2.48, if T is weakly truncatable, $\mathcal{T}^{\mathrm{A}}_{n,k}$ has a right adjoint, so we can assume that this adjoint exists and denote it by $\mathcal{J}^{\mathrm{A}}_{k,n}$. Then, the morphism $\mathcal{T}^{\mathrm{A}}_k \mathcal{F}^{\mathrm{G}}_n \mathrm{i}^k$, pictured by



is a natural transformation between two composites of left adjoints. Then, by deriving the units and counits of these composite adjunctions (as in [25, IV.§8 Theorem 1] for example), and using (the dual of) Proposition 2.54, the above natural transformation is an isomorphism if and only if the natural transformation depicted by the string diagram



is an isomorphism, where (α, α'), (η^n, ϵ^n), (γ, i^k) are the pairs of units and counits associated with the adjunctions $\mathcal{T}^{\mathrm{A}}_k \dashv \mathcal{J}^{\mathrm{A}}_n$, $\mathcal{F}^{\mathrm{G}}_n \dashv \mathcal{U}^{\mathrm{G}}_n$ and $\mathcal{I}^{\mathrm{G}}_n \dashv \mathcal{T}^{\mathrm{G}}_k$ respectively. Using the zigzag equations (ZZ) satisfied by adjunctions to reduce the above diagram, we obtain



which is the diagram associated to the morphism $\mathrm{j}^k \mathcal{U}_k^{\mathrm{G}} \mathcal{J}_k^{\mathrm{A}}$. Thus, $\mathcal{T}_k^{\mathrm{A}} \mathcal{F}_n^{\mathrm{G}}\, \mathrm{i}^k$ is an isomorphism if and only if $\mathrm{j}^k \mathcal{U}_k^{\mathrm{G}} \mathcal{J}_k^{\mathrm{A}}$ is an isomorphism. We conclude with Proposition 2.53. ■

We will use the above criterion to show that the monad associated to the theories of strict categories is weakly truncatable (*c.f.* Theorem 4.22).

3. Free higher categories on generators

Given some theory of higher categories, an important construction is the one that builds a k-category which is freely generated on a set of generators. Indeed, like for other algebraic theories, a k-category can be described by means of a *presentation*, *i.e.*, by quotienting a free k-category by a set of relations. For example, a formal adjunction can be described as the strict 2-category generated by two 0-cells x and y, two 1-cells $l\colon y \to x$ and $r\colon x \to y$, and two 2-cells $\gamma\colon \mathrm{id}_y \Rightarrow l *_0 r$ and $\epsilon\colon r *_0 l \Rightarrow \mathrm{id}_x$ satisfying the zigzag equations (ZZ). Given a theory of higher categories expressed in Batanin's setting, *i.e.*, as a monad (T, η, μ) on $\mathbf{Glob}_n$ for some $n \in \mathbb{N} \cup \{\omega\}$, there are several free constructions that one can consider. First, the functors $\mathcal{F}_k^{\mathrm{G}}\colon \mathbf{Glob}_k \to \mathbf{Alg}_k$ already enable to construct the free k-category on a k-globular set. Moreover, there is a construction which produces a k-category from a *k-cellular extension*, *i.e.*, a pair consisting of a $(k{-}1)$-category and a set of k-generators. Such construction was introduced for strict categories in [11]. Finally, one can consider the free k-category on a *k-polygraph*: the latter is a system of i-generators for $i < n+1$ which is organised inductively as cellular extensions. It differs from a mere k-globular set in the sense that a k-polygraph allows generators to have complex sources and targets that are composites of other generators, whereas the sources and targets of generators organised in a k-globular set can only be globes. Polygraphs (also called computads) were first introduced by Street [35] and Burroni [11] for strict categories, and then generalised to any finitary monad on globular sets by Batanin [4]. We should also mention the work of Garner [15], which provides a simplified presentation of polygraphs which does not require the finitary hypothesis.

In this section, we define free constructions for algebraic globular higher categories following the path of Burroni [11] by relying on cellular extensions, giving in the process another light on the results of Batanin [4], who did not use them, and addressing this way the first of our three points listed in the introduction. More precisely, we define the notion of *cellular extension* for any algebraic theory of globular higher categories, and then derive the notion *polygraphs* from it, together with the free construction associated to each notion. Since most of the definitions rely on pullbacks in **CAT**, we first recall some properties of these pullbacks (Section 3.1). Then, we introduce *cellular extensions* together with the associated free construction for any finitary monad on globular sets, and, in the case of a truncatable monad, we show that this construction preserves the lower dimensions, *i.e.*, that freely adding k-generators does not change the underlying $(k{-}1)$-category (Section 3.8); we show some specific properties in the case where the underlying monad is truncatable (Section 3.26). Then, we introduce *polygraphs*, in finite dimensions (Section 3.32) and infinite dimension (Section 3.47), together with the associated free construction for any finitary monad on globular sets. Finally, we recover in our setting the adjunction of Batanin between higher categories and polygraphs (Section 3.54). As before, we try to stick to formal arguments and computations through string diagrams in order to give a new light to these constructions, in accordance to the last of the three goals stated in the introduction.

3.1. Pullbacks in **CAT**. In the following sections, we define the categories of cellular extensions and polygraphs using pullbacks in **CAT**. We will be interested in showing that these categories are cocomplete and that several of the projection functors are left or right adjoints. Such properties are consequence of general properties of pullbacks that we recall below. In particular, a pullback of an isofibration (see Footnote 2) has good properties with regard to cocompleteness and preservation of colimits. This is convenient since, as we will see below, all the truncation functors introduced until now are isofibrations.

We begin with a property of compatibility of pullbacks in **CAT** with left and right adjoints:

3.2. Proposition. *Given a pullback in* **CAT**

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "F'"] \arrow[d, "G'"'] & \mathcal{C} \arrow[d, "G"] \\ \mathcal{D}' \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

and a functor $H\colon \mathcal{D} \to \mathcal{C}$ such that $GH = 1_{\mathcal{D}}$, then there exists a canonical $H'\colon \mathcal{D}' \to \mathcal{C}'$ such that $G'H' = 1_{\mathcal{D}'}$. Moreover, if there is an adjunction $H \dashv G$ (resp. $G \dashv H$) whose unit (resp. counit) is $\mathbb{1}_{1_{\mathcal{D}}}$, then there is an adjunction $H' \dashv G'$ (resp. $H \dashv G$) whose unit (resp. counit) is $\mathbb{1}_{1_{\mathcal{D}'}}$.

Proof. We define H' using the universal property of pullbacks by

$$\begin{tikzcd} \mathcal{D}' \arrow[dr, dotted, "H'"] \arrow[drr, bend left, "H \circ F"] \arrow[ddr, bend right, "1_{\mathcal{D}'}"'] & & \\ & \mathcal{C}' \arrow[r, "F'"] \arrow[d, "G'"'] & \mathcal{C} \arrow[d, "G"] \\ & \mathcal{D}' \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

which satisfies $G'H' = 1_{\mathcal{D}'}$ by definition. Moreover, suppose that there is an adjunction $H \dashv G$ whose unit is $\mathbb{1}_{1_{\mathcal{D}}}$. Then, considering a morphism $f\colon H'X \to Y \in \mathcal{C}'$, since $\mathcal{C}'$ is defined by a pullback, f is the data of morphisms $f_l\colon X \to G'Y$ and $f_r\colon HFX \to F'Y$ with $F(f_l) = G(f_r)$. But, since the unit of $H \dashv G$ is $\mathbb{1}_{1_{\mathcal{D}}}$, G induces a bijective correspondence between $\mathcal{C}(HFX, F'Y)$ and $\mathcal{D}(FX, GF'Y)$, so that f_r is uniquely defined by $F(f_l)$. Thus, G' induces a bijective natural correspondence between $\mathcal{C}'(H'X, Y)$ and $\mathcal{D}'(X, G'Y)$ for all $X \in \mathcal{D}'$ and $Y \in \mathcal{C}'$, so that there is an adjunction $H' \dashv G'$ with unit $\mathbb{1}_{1_{\mathcal{D}'}}$. The case where G is left adjoint is similar. ∎

Moreover, we prove that isofibrations are well-behaved regarding pullbacks in **CAT**. We recall that a functor $G\colon \mathcal{C} \to \mathcal{D} \in \mathbf{CAT}$ is an *isofibration* when it lifts isomorphisms, *i.e.*, for all $X \in \mathcal{C}$ and $\tilde{Y} \in \mathcal{D}$, given an isomorphism $\tilde{f}\colon GX \to \tilde{Y}$ in $\mathcal{D}$, there exists $Y \in \mathcal{C}$ and an isomorphism $f\colon X \to Y$ such that $GY = \tilde{Y}$ and $G(f) = \tilde{f}$. We then have:

3.3. Proposition. *Given a pullback in* **CAT**

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "F'"] \arrow[d, "G'"'] & \mathcal{C} \arrow[d, "G"] \\ \mathcal{D}' \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

such that G is an isofibration, the following hold:

(i) G' is an isofibration,

(ii) given a small category I, if $\mathcal{C}$ and $\mathcal{D}'$ have all I-colimits and F and G preserve them, then $\mathcal{C}'$ has all I-colimits and F' and G' preserve them.

Proof. *Proof of (i):* Let $X \in \mathcal{C}'$, $Y_L \in \mathcal{D}'$ and $\theta_L\colon G'X \to Y_L$ be an isomorphism. Then, since G is an isofibration, there is $Y_R \in \mathcal{C}$ and an isomorphism $\theta_R\colon F'X \to Y_R$ such that $F(\theta_L) = G(\theta_R)$. Moreover, $F(\theta_L^{-1}) = G(\theta_R^{-1})$ so that $(\theta_L, \theta_R)\colon X \to (Y_L, Y_R)$ is an isomorphism of $\mathcal{C}'$.

Proof of (ii): Let $d\colon I \to \mathcal{C}'$ be a functor, which is the data of $d_L\colon I \to \mathcal{D}'$ and $d_R\colon I \to \mathcal{C}$. Then, there are colimit cocones $(p_{L,i}\colon d_L(i) \to X_L)_{i\in I}$ and $(p_{R,i}\colon d_R(i) \to X_R)_{i\in I}$. Since both F and G preserve colimits, both

$$(F(p_{L,i})\colon F(d_L(i)) \to F(X_L))_{i\in I} \quad \text{and} \quad (G(p_{R,i})\colon F(d_L(i)) \to G(X_R))_{i\in I}$$

are colimit cocones for $F \circ d_L$. So there exists an isomorphism $\theta\colon F(X_L) \to G(X_R)$ between the two cocones. Since G is an isofibration, we can suppose that $F(X_L) = G(X_R)$ and $\theta = \mathrm{id}_{F(X_L)}$. Thus, we have a cocone $((p_{L,i}, p_{R,i})\colon d(i) \to (X_L, X_R))_i$ on d, and we easily verify that it is a colimit cocone. ■

3.4. Remark. Pullbacks in **CAT** should normally raise suspicion since strict limits are not well-behaved in **CAT** in general. Indeed, a limit cone in **CAT** on a diagram is not preserved when replacing some functors of the diagram by isomorphic functors. Moreover, the limit cone is defined up to isomorphism, and not up to equivalence of categories. To solve this problem, one usually considers a weaker notion of limits, where the triangles of cones commute only up to isomorphisms, as with weighted bilimits [27]. But the strict limit on a diagram is generally not equivalent to the associated weighted bilimit. However, introducing weighted bilimits here would be an unnecessary pain for what we want to do, since the pullbacks along isofibrations are equivalent to the weighted bipullbacks (see [27, Proposition 5.1.1]).

We say that a monad functor is an *isofibration* when the underlying functor is an isofibration. We then have:

3.5. Lemma. *The functor* **EM** *preserves isofibrations.*

Proof. Let $(F,\alpha)\colon (\mathcal{C},S) \to (\mathcal{D},T)$ be a monad functor isofibration, $(X, h\colon SX \to X)$ be an S-algebra, and $f\colon F^\alpha(X,h) \to (Y,k)$ be a T-algebra isomorphism. Since F is an isofibration, there is an isomorphism $\tilde{f}\colon X \to \tilde{Y}$ such that $F(\tilde{f}) = f$. One can equip $\tilde{Y}$ with a structure of S-algebra $(\tilde{Y},\tilde{k})$ by putting $\tilde{k} = \tilde{f} \circ h \circ S(\tilde{f}^{-1})$. Then, we have an S-algebra morphism $\tilde{f}\colon (X,h) \to (\tilde{Y},\tilde{k})$. Moreover, $(\tilde{Y},\tilde{k})$ is mapped to (Y,k) by F^α, so that $\tilde{f}$ is an isomorphism which lifts f. Hence, **EM** preserves isofibration. ∎

We now verify that several functors of interest to us are isofibrations:

3.6. Proposition. *Given $k,l \in \mathbb{N} \cup \{\omega\}$ with $k < l$, the functor $\mathcal{T}^{\mathrm{G}}_{l,k}$ is an isofibration.*

Proof. Straight-forward. ∎

3.7. Proposition. *Let $n \in \mathbb{N} \cup \{\omega\}$ and (T,η,μ) be a finitary monad on $\mathbf{Glob}_n$. Given $k,l \in \mathbb{N} \cup \{\omega\}$ with $k < l$, the functor $\mathcal{T}^{\mathrm{A}}_{l,k}$ is an isofibration.*

Proof. The functor $\mathcal{T}^{\mathrm{A}}_{l,k}$ is the image by **EM** of $(\mathcal{T}^{\mathrm{G}}_{l,k}, \mathcal{T}^{\mathrm{G}}_{l,k} T^l\, \mathrm{i}^{k,l})$, which is an isofibration. Thus, by Lemma 3.5, it is an isofibration. ∎

3.8. Cellular extensions. In this section, we introduce the notion of *k-cellular extension*, which describes a $(k-1)$-category (for some theory of higher categories) equipped with a set of *k-generators*. We moreover give the construction of the free k-category on a k-cellular extension together with more specific results when the theory we are considering is associated with a truncatable monad.

In the following, let $n \in \mathbb{N} \cup \{\omega\}$ and (T,η,μ) be a finitary monad on $\mathbf{Glob}_n$.

3.9. Definition. *Given $k \in \mathbb{N}$ with $0 < k \leq n$, we define the category $\mathbf{Alg}^+_k$ of* k-cellular extensions, *together with functors $\mathcal{T}^+_{k-1}\colon \mathbf{Alg}^+_k \to \mathbf{Alg}_{k-1}$ and $\mathcal{G}_k\colon \mathbf{Alg}^+_k \to \mathbf{Glob}_k$, as the pullback*

$$\begin{tikzcd}
\mathbf{Alg}^+_k \arrow[r, dotted, "\mathcal{G}_k"] \arrow[d, dotted, "\mathcal{T}^+_{k-1}"'] & \mathbf{Glob}_k \arrow[d, "\mathcal{T}^{\mathrm{G}}_{k-1}"] \\
\mathbf{Alg}_{k-1} \arrow[r, "\mathcal{U}^{\mathrm{G}}_{k-1}"'] & \mathbf{Glob}_{k-1}.
\end{tikzcd} \tag{8}$$

For better handling side cases, we moreover define $\mathbf{Alg}^+_0 = \mathbf{Glob}_0$ and $\mathcal{G}_0 = 1_{\mathbf{Glob}_0}$.

We verify that:

3.10. Proposition. *Given $k \in \mathbb{N}$ with $k \leq n$, the category $\mathbf{Alg}^+_k$ is locally finitely presentable. In particular, it is complete and cocomplete. Moreover, both $\mathcal{G}_k$ and $\mathcal{T}^+_{k-1}$ (when $k > 0$) are finitary right adjoints.*

Proof. We can restrict to the case where $k > 0$. The categoy $\mathbf{Alg}_k^+$ is defined as the pullback of the finitary right adjoint isofibration $\mathcal{T}_{k-1}^{\mathrm{G}}$ along the right adjoint $\mathcal{U}_{k-1}^{\mathrm{G}}$. Thus, it is a bipullback of right adjoints between locally finitely presentable categories. Since the 2-category of locally finitely presentable and finitary right adjoint functors is closed under bilimits (see [7, Theorem 2.17]), $\mathbf{Alg}_k^+$ is locally finitely presentable and the two projections $\mathcal{G}_k$ and $\mathcal{T}_{k-1}^+$ are finitary right adjoint functors. ∎

3.11. Remark. In fact, one can check by hand that the finitely presentable objects of $\mathbf{Alg}_k^+$ are exactly the compatible pairs (C, X), where C is a finitely presentable object of $\mathbf{Alg}_{k-1}$ and X is a compatible k-globular set such that X_k is finite.

Given a cellular extension (C, X), we observe that much of the data of X is already contained in C. We provide a more compact definition of $\mathbf{Alg}_k^+$ which can often be more convenient than the one given on the pullback (8).

3.12. Proposition. *Given $k \in \mathbb{N}$ with $0 < k \leq n$, the category $\mathbf{Alg}_k^+$ is isomorphic to the category*

– *whose objects are the pairs (C, S) where $C \in \mathbf{Alg}_{k-1}$ and S is a set, equipped with two functions*
$$\mathrm{d}_{k-1}^-, \mathrm{d}_{k-1}^+ \colon S \to C_{k-1}$$
such that $\partial_{k-2}^\epsilon \circ \mathrm{d}_{k-1}^- = \partial_{k-2}^\epsilon \circ \mathrm{d}_{k-1}^+$ for $\epsilon \in \{-, +\}$,

– *and whose morphisms between two such pairs (C, S) and (C', S') are the pairs (F, f) where*
$$F \colon C \to C' \in \mathbf{Alg}_{k-1} \quad \textit{and} \quad f \colon S \to S' \in \mathbf{Set}$$
and such that $\mathrm{d}_{k-1}^\epsilon \circ f = F_{k-1} \circ \mathrm{d}_{k-1}^\epsilon$ for $\epsilon \in \{-, +\}$.

Proof. Write $\overline{\mathbf{Alg}}_k^+$ for the category described in the statement. An isomorphism between $\mathbf{Alg}_k^+$ and $\overline{\mathbf{Alg}}_k^+$ can be described as follows. Given $(C, X) \in \mathbf{Alg}_k^+$, we map (C, X) to the pair (C, X_k) and, for $\epsilon \in \{-, +\}$ and $x \in X_k$, we put $\mathrm{d}_{k-1}^\epsilon(x) = \partial_{k-1}^\epsilon(x)$ (where ∂_{k-1}^ϵ is the operation of the globular structure on X), and we extend this mapping to morphisms of $\mathbf{Alg}_k^+$ as expected. Since $\mathcal{U}_{k-1}^{\mathrm{G}} C = X_{\leq k-1}$ for $(C, X) \in \mathbf{Alg}_k^+$, the resulting functor is an isomorphism of categories. ∎

3.13. Remark. This other description of $\mathbf{Alg}_k^+$ helps better understand how a colimit of a diagram (C^i, S^i) in $\mathbf{Alg}_k^+$ is computed: first, one computes a colimit C of the C^i's in $\mathbf{Alg}_{k-1}$, then one compute a colimit S of the sets S^i's and equips it with the canonical source and target operations $\mathrm{d}_{k-1}^-, \mathrm{d}_{k-1}^+ \colon S \to C_{k-1}$.

3.14. Definition. *Given $k \in \mathbb{N}$ with $0 < k \leq n$, following the description of $\mathbf{Alg}_k^+$ given by Proposition 3.12, a left adjoint to the functor $\mathcal{T}_{k-1}^+$ is given by the functor*
$$\mathcal{I}_k^+ \colon \mathbf{Alg}_{k-1} \to \mathbf{Alg}_k^+$$
which maps $C \in \mathbf{Alg}_{k-1}$ to $(C, \emptyset) \in \mathbf{Alg}_k^+$ and with the obvious action on morphisms of $\mathbf{Alg}_{k-1}$.

3.15. Proposition. *Given $k \in \mathbb{N}$ with $0 < k \leq n$, the functor $\mathcal{T}^+_{k-1}$ is an isofibration and has a right adjoint.*

Proof. This is a consequence of Propositions 3.2 and 3.6. ■

3.16. Definition. *Given $k \in \mathbb{N}$ with $0 < k \leq n$, we write*

$$\mathcal{J}^+_k \colon \mathbf{Alg}_{k-1} \to \mathbf{Alg}^+_k$$

for a right adjoint to $\mathcal{T}^+_{k-1}$. Concretely, one can define $\mathcal{J}^+_k$ as the functor mapping the $(k-1)$-algebra (X, h) to the pair $((X, h), X_{\Uparrow k}) \in \mathbf{Alg}^+_k$.

3.17. Remark. Using the concrete definition of Definition 3.16, we observe that the counit of the adjunction $\mathcal{T}^+_{k-1} \vdash \mathcal{J}^+_k$ is the identity.

3.18. Proposition. *Given $k \in \mathbb{N}$ with $0 < k \leq n$, the functor $\mathcal{T}^+_{k-1}$ preserves finitely presentable objects.*

Proof. By adjunction, it is equivalent to prove that its right adjoint is finitary. By the constructive content of Proposition 3.2, this right adjoint is built from a cone of finitary functors on the (bi)pullback defining $\mathbf{Alg}^+_k$, itself made of finitary functors. Thus, the factorising functor is itself finitary. ■

3.19. Definition. *Given $k \in \mathbb{N}$ with $0 < k \leq n$, we write $\mathcal{U}^{\mathrm{E}}_k \colon \mathbf{Alg}_k \to \mathbf{Alg}^+_k$ for the functor defined as the factorisation arrow*

$$\begin{tikzcd}
\mathbf{Alg}_k \arrow[dr, dotted, "\mathcal{U}^{\mathrm{E}}_k"] \arrow[drr, bend left, "\mathcal{U}^{\mathrm{G}}_k"] \arrow[ddr, bend right, "\mathcal{T}^{\mathrm{A}}_{k-1}"'] & & \\
& \mathbf{Alg}^+_k \arrow[r, "\mathcal{G}_k"] \arrow[d, "\mathcal{T}^+_{k-1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Glob}_k \arrow[d, "\mathcal{T}^{\mathrm{G}}_{k-1}"] \\
& \mathbf{Alg}_{k-1} \arrow[r, "\mathcal{U}^{\mathrm{G}}_{k-1}"'] & \mathbf{Glob}_{k-1}
\end{tikzcd} .$$

For convenience, we also write $\mathcal{U}^{\mathrm{E}}_0$ for $\mathcal{U}^{\mathrm{G}}_0$.

Then, there is an operation which produces a k-category from a k-cellular extension. It is the left adjoint to $\mathcal{U}^{\mathrm{E}}_k$, that exists by the following property:

3.20. Theorem. *Given $k \in \mathbb{N}$ with $k \leq n$, $\mathcal{U}^{\mathrm{E}}_k$ has a left adjoint.*

Proof. The case $k = 0$ is immediate. So assume $k > 0$ and let

$$\begin{array}{rccc}
\Phi^{\mathrm{L}}\colon & \mathbf{Alg}_k(\mathcal{I}^{\mathrm{A}}_{k-1,k}(-), -) & \Rightarrow & \mathbf{Alg}_{k-1}(-, \mathcal{T}^{\mathrm{A}}_{k,k-1}(-)) \\
\Psi^{\mathrm{L}}\colon & \mathbf{Alg}_k(\mathcal{I}^{\mathrm{A}}_{k-1,k}\mathcal{F}^{\mathrm{G}}_{k-1}(-), -) & \Rightarrow & \mathbf{Glob}_{k-1}(-, \mathcal{U}^{\mathrm{G}}_{k-1}\mathcal{T}^{\mathrm{A}}_{k,k-1}(-)) \\
\Phi^{\mathrm{R}}\colon & \mathbf{Alg}_k(\mathcal{F}^{\mathrm{G}}_k(-), -) & \Rightarrow & \mathbf{Glob}_k(-, \mathcal{U}^{\mathrm{G}}_k(-)) \\
\Psi^{\mathrm{R}}\colon & \mathbf{Glob}_k(\mathcal{F}^{\mathrm{G}}_k\mathcal{I}^{\mathrm{G}}_{k-1,k}(-), -) & \Rightarrow & \mathbf{Glob}_{k-1}(-, \mathcal{T}^{\mathrm{G}}_{k,k-1}\mathcal{U}^{\mathrm{G}}_k(-))
\end{array}$$

be the natural bijections derived from the associated adjunctions. Note that these bijections can be defined using the units of the adjunctions. For example, given $C \in \mathbf{Alg}_{k-1}$ and $D \in \mathbf{Alg}_k$, Φ^{L} maps a morphism $f\colon C_{\uparrow k} \to D \in \mathbf{Alg}_k$ to the morphism

$$f_{\leq k-1} \circ \eta^{\mathrm{A},k-1}_C \colon C \to D_{\leq k-1} \in \mathbf{Alg}_{k-1}$$

where $\eta^{\mathrm{A},k-1}$ is the unit of $\mathcal{T}^{\mathrm{A}}_{k-1} \dashv \mathcal{I}^{\mathrm{A}}_k$ defined in Definition 2.29. Since

$$\mathcal{F}^{\mathrm{G}}_k \mathcal{I}^{\mathrm{G}}_k \quad \text{and} \quad \mathcal{I}^{\mathrm{A}}_k \mathcal{F}^{\mathrm{G}}_{k-1}$$

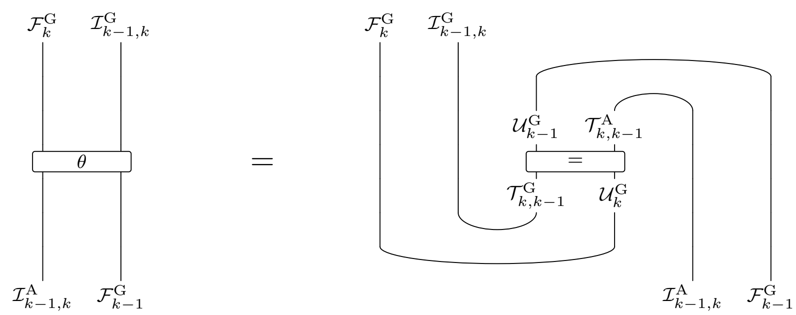
are both left adjoint to $\mathcal{U}^{\mathrm{G}}_{k-1} \mathcal{T}^{\mathrm{A}}_{k,k-1} = \mathcal{T}^{\mathrm{G}}_{k-1} \mathcal{U}^{\mathrm{G}}_k$, the natural morphism

$$\theta \colon \mathcal{F}^{\mathrm{G}}_k \mathcal{I}^{\mathrm{G}}_{k-1,k} \Rightarrow \mathcal{I}^{\mathrm{A}}_k \mathcal{F}^{\mathrm{G}}_{k-1}$$

defined as the composite

$$\theta = (\epsilon^k \mathcal{I}^{\mathrm{A}}_{k-1,k} \mathcal{F}^{\mathrm{G}}_{k-1}) \circ (\mathcal{F}^{\mathrm{G}}_k \, \mathrm{i}^{k-1} \, \mathcal{U}^{\mathrm{G}}_k \mathcal{I}^{\mathrm{A}}_{k-1,k} \mathcal{F}^{\mathrm{G}}_{k-1}) \circ (\mathcal{F}^{\mathrm{G}}_k \mathcal{I}^{\mathrm{G}}_{k-1,k} \mathcal{U}^{\mathrm{G}}_{k-1} \eta^{\mathrm{A},k-1} \mathcal{F}^{\mathrm{G}}_{k-1}) \circ (\mathcal{F}^{\mathrm{G}}_k \mathcal{I}^{\mathrm{G}}_{k-1,k} \eta^{k-1})$$

which can be represented by



is an isomorphism as a consequence of Proposition 2.54. In the following, given a morphism $f\colon X \to Y$ of a category $\mathcal{C}$, we write $f^*\colon \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$ for the function $g \mapsto g \circ f$ for all $Z \in \mathcal{C}$. One can verify using the zigzag equations (ZZ) that the natural transformation θ makes the diagram

$$\begin{tikzcd}
\mathbf{Alg}_k((\mathcal{F}^{\mathrm{G}}_{k-1} Z)_{\uparrow k}, A) \arrow[r, "\Psi^{\mathrm{L}}_{Z,A}"] \arrow[d, "(\theta_Z)^*"'] & \mathbf{Glob}_{k-1}(Z, \mathcal{U}^{\mathrm{G}}_{k-1}(A_{\leq k-1})) \arrow[d, equal] \\
\mathbf{Alg}_k(\mathcal{F}^{\mathrm{G}}_k(Z_{\uparrow k}), A) \arrow[r, "\Psi^{\mathrm{R}}_{Z,A}"'] & \mathbf{Glob}_{k-1}(Z, (\mathcal{U}^{\mathrm{G}}_k A)_{\leq k-1})
\end{tikzcd} \tag{9}$$

commutes for all $Z \in \mathbf{Glob}_{k-1}$ and $A \in \mathbf{Alg}_k$. Let $(C,X) \in \mathbf{Alg}^+_k$, $D \in \mathbf{Alg}_k$ and $(D_{\leq k-1}, Y)$ be $\mathcal{U}^{\mathrm{E}}_k D$. Since

$$\mathcal{U}^{\mathrm{G}}_{k-1} C = X_{\leq k-1} \quad \text{and} \quad \mathcal{U}^{\mathrm{G}}_{k-1}(D_{\leq k-1}) = Y_{\leq k-1}$$

and by the properties of adjunctions, we have a diagram

$$\begin{tikzcd}
\mathbf{Alg}_k(C_{\uparrow k}, D) \arrow[r, "\Phi^{\mathrm{L}}_{C,D}"] \arrow[d, "(e^{\mathrm{L}}_{(C,X)})^*"'] & \mathbf{Alg}_{k-1}(C, D_{\leq k-1}) \arrow[d, "\mathcal{U}^{\mathrm{G}}_{k-1}"] \\
\mathbf{Alg}_k((\mathcal{F}^{\mathrm{G}}_{k-1}\mathcal{U}^{\mathrm{G}}_{k-1}C)_{\uparrow k}, D) \arrow[r, "\Psi^{\mathrm{L}}_{\mathcal{U}^{\mathrm{G}}_{k-1}C,D}"] \arrow[d, "(\theta_{X_{\leq k-1}})^*"'] & \mathbf{Glob}_{k-1}(\mathcal{U}^{\mathrm{G}}_{k-1}C, \mathcal{U}^{\mathrm{G}}_{k-1}(D_{\leq k-1})) \arrow[d, equal] \\
\mathbf{Alg}_k(\mathcal{F}^{\mathrm{G}}_k((X_{\leq k-1})_{\uparrow k}), D) \arrow[r, "\Psi^{\mathrm{R}}_{X_{\leq k-1},D}"] & \mathbf{Glob}_{k-1}(X_{\leq k-1}, Y_{\leq k-1}) \\
\mathbf{Alg}_k(\mathcal{F}^{\mathrm{G}}_k X, D) \arrow[r, "\Phi^{\mathrm{R}}_{X,D}"] \arrow[u, "(e^{\mathrm{R}}_{(C,X)})^*"] & \mathbf{Glob}_k(X, Y) \arrow[u, "\mathcal{T}^{\mathrm{G}}_{k-1}"']
\end{tikzcd} \tag{10}$$

such that each square commutes and where e^{L} and e^{R} are the natural transformations

$$e^{\mathrm{L}} = \mathcal{I}^{\mathrm{A}}_k \epsilon^{k-1} \mathcal{T}^{+}_{k-1} \quad \text{and} \quad e^{\mathrm{R}} = \mathcal{F}^{\mathrm{G}}_k \, \mathrm{i}^{k-1} \, \mathcal{G}_k$$

respectively. Indeed, the middle square commutes by (9) and the top and bottom squares commute by the zigzag equations (ZZ). By definition of $\mathbf{Alg}^{+}_k$, the set $\mathbf{Alg}^{+}_k((C,X), \mathcal{U}^{\mathrm{E}}_k D)$ is the pullback

$$\begin{tikzcd}
\mathbf{Alg}^{+}_k((C,X), \mathcal{U}^{\mathrm{E}}_k D) \arrow[r, "\mathcal{G}_k"] \arrow[d, "\mathcal{T}^{+}_{k-1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Glob}_k(X, \mathcal{G}_k \mathcal{U}^{\mathrm{E}}_k D) \arrow[d, "\mathcal{T}^{\mathrm{G}}_{k-1}"] \\
\mathbf{Alg}_{k-1}(C, \mathcal{T}^{+}_{k-1}\mathcal{U}^{\mathrm{E}}_k D) \arrow[r, "\mathcal{U}^{\mathrm{G}}_{k-1}"'] & \mathbf{Glob}_{k-1}(X_{\leq k-1}, (\mathcal{G}_k \mathcal{U}^{\mathrm{E}}_k D)_{\leq k-1})
\end{tikzcd} .$$

Since

$$\mathcal{T}^{\mathrm{A}}_{k-1} = \mathcal{T}^{+}_{k-1}\mathcal{U}^{\mathrm{E}}_k \quad \text{and} \quad \mathcal{U}^{\mathrm{G}}_k = \mathcal{G}_k \mathcal{U}^{\mathrm{E}}_k$$

and by the commutative diagram (10), the following diagram is also a pullback:

$$\begin{tikzcd}
\mathbf{Alg}^{+}_k((C,X), \mathcal{U}^{\mathrm{E}}_k D) \arrow[r, "(\Phi^{\mathrm{R}}_{X,D})^{-1} \circ \mathcal{G}_k"] \arrow[d, "(\Phi^{\mathrm{L}}_{C,D})^{-1} \circ \mathcal{T}^{+}_{k-1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Alg}_k(\mathcal{F}^{\mathrm{G}}_k X, D) \arrow[d, "(e^{\mathrm{R}}_{(C,X)})^*"] \\
\mathbf{Alg}_k(C_{\uparrow k}, D) \arrow[r, "(e^{\mathrm{L}}_{(C,X)} \circ \theta_{X_{\leq k-1}})^*"'] & \mathbf{Alg}_k(\mathcal{F}^{\mathrm{G}}_k((X_{\leq k-1})_{\uparrow k}), D)
\end{tikzcd} .$$

Since $\mathbf{Alg}_k$ is cocomplete by Proposition 2.19, the diagram

$$\begin{tikzcd}
\mathbf{Alg}_k(C[X], D) \arrow[r, "(\mathrm{p}^{\mathrm{R}}_{(C,X)})^*"] \arrow[d, "(\mathrm{p}^{\mathrm{L}}_{(C,X)})^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Alg}_k(\mathcal{F}^{\mathrm{G}}_k X, D) \arrow[d, "(e^{\mathrm{R}}_{(C,X)})^*"] \\
\mathbf{Alg}_k(C_{\uparrow k}, D) \arrow[r, "(e^{\mathrm{L}}_{(C,X)} \circ \theta_{X_{\leq k-1}})^*"'] & \mathbf{Alg}_k(\mathcal{F}^{\mathrm{G}}_k((X_{\leq k-1})_{\uparrow k}), D)
\end{tikzcd}$$

is also a pullback, where $C[X]$, $\mathrm{p}^{\mathrm{L}}_{(C,X)}$ and $\mathrm{p}^{\mathrm{R}}_{(C,X)}$ are defined as the pushout

$$\begin{tikzcd} C[X] \arrow[dr, phantom, very near start, "\lrcorner"] & & \mathcal{F}^{\mathrm{G}}_k X \arrow[ll, dotted, "\mathrm{p}^{\mathrm{R}}_{(C,X)}"'] \\ C_{\uparrow k} \arrow[u, dotted, "\mathrm{p}^{\mathrm{L}}_{(C,X)}"] & & \mathcal{F}^{\mathrm{G}}_k((X_{\leq k-1})_{\uparrow k}) \arrow[ll, "e^{\mathrm{L}}_{(C,X)} \circ \theta_{X_{\leq k-1}}"] \arrow[u, "e^{\mathrm{R}}_{(C,X)}"'] \end{tikzcd} \quad . \tag{11}$$

Thus, there is an isomorphism

$$\mathbf{Alg}_k(C[X], D) \cong \mathbf{Alg}^+_k((C,X), \mathcal{U}^{\mathrm{E}}_k D)$$

which is natural in D. Hence, $\mathcal{U}^{\mathrm{E}}_k$ admits a left adjoint. ■

3.21. Definition. *Given $k \in \mathbb{N}$ with $0 < k \leq n$, the operation $(C,X) \mapsto C[X]$ defined in the proof of Theorem 3.20 extends to a functor*

$$\mathcal{F}^{\mathrm{E}}_k \colon \mathbf{Alg}^+_k \to \mathbf{Alg}_k$$

and which is left adjoint to $\mathcal{U}^{\mathrm{E}}_k$. The image $C[X]$ of some $(C,X) \in \mathbf{Alg}^+_k$ by $\mathcal{F}^{\mathrm{E}}_k$, is called the free extension on (C,X)*. The morphisms $\mathrm{p}^{\mathrm{L}}_{(C,X)}$ and $\mathrm{p}^{\mathrm{R}}_{(C,X)}$ defined in the proof of Theorem 3.20 extends to natural transformations*

$$\begin{array}{rcrcl} \mathrm{p}^{\mathrm{L},k} & : & \mathcal{I}^{\mathrm{A}}_k \mathcal{T}^+_{k-1} & \Rightarrow & \mathcal{F}^{\mathrm{E}}_k \\ \mathrm{p}^{\mathrm{R},k} & : & \mathcal{F}^{\mathrm{G}}_k \mathcal{G}_k & \Rightarrow & \mathcal{F}^{\mathrm{E}}_k . \end{array}$$

often denoted p^{L} and p^{R} when there is no ambiguity. For convenience, we also write $\mathcal{F}^{\mathrm{E}}_0$ for $\mathcal{F}^{\mathrm{G}}_0$.

3.22. Remark. By careful inspection of the proof of Theorem 3.20, we observe that a morphism $F \colon (C,X) \to \mathcal{U}^{\mathrm{E}}_k D$ is in correspondence with a morphism $\bar{F} \colon C[X] \to D$ which is the factorisation of two morphisms

$$\bar{F}_l \colon C_{\uparrow k} \to D \qquad \text{and} \qquad \bar{F}_r \colon \mathcal{F}^{\mathrm{G}}_k(X) \to D$$

in $\mathbf{Alg}_k$ through the pushout (11), where $\bar{F}_l$ is in correspondence with the morphism

$$\mathcal{T}^+_{k-1}(F) \colon C \to \mathcal{T}^+_{k-1}\mathcal{U}^{\mathrm{E}}_k(D) = D_{\leq k-1}$$

through the adjunction $\mathcal{I}^{\mathrm{A}}_k \dashv \mathcal{T}^{\mathrm{A}}_{k-1}$, and where $\bar{F}_r$ is in correspondence with the morphism

$$\mathcal{G}_k(F) \colon X \to \mathcal{G}_k \mathcal{U}^{\mathrm{E}}_k(D) = \mathcal{U}^{\mathrm{G}}_k(D)$$

through the adjunction $\mathcal{F}^{\mathrm{G}}_k \dashv \mathcal{U}^{\mathrm{G}}_k$.

3.23. Example. Consider the monad (T,η,μ) on $\mathbf{Glob}_1$ defined in Example 2.20. A 1-cellular extension (C,X) is then essentially the data of a set C_0 of 0-cells, a set X_1 of 1-generators, and functions $\mathrm{d}_0^-,\mathrm{d}_0^+\colon X_1 \to C_0$, *i.e.*, a graph. Moreover, the 1-category $C[X]$ is the image of (C,X) seen as a graph by the left adjoint to the functor $\mathbf{Cat} \to \mathbf{Gph}$ defined in Example A.36.

3.24. Remark. Theorem 3.20 is a particular case of the fact that the category of locally presentable categories and right adjoints are closed under weighted bilimits (see [7] and the end of [27, §5.1]). But the concrete pushout description given by the proof will be useful later to show properties of the functor $\mathcal{F}_k^{\mathrm{E}}$.

3.25. Proposition. *Given $k \in \mathbb{N}$ with $k \le n$, the functor $\mathcal{F}_k^{\mathrm{E}}$ preserves finitely presentable objects.*

Proof. By adjunction, it is equivalent to show that its right adjoint $\mathcal{U}_k^{\mathrm{E}}$ is finitary. Since T is finitary, $\mathcal{U}_k^{\mathrm{G}}$ is finitary, which concludes the proof when $k=0$. Otherwise, when $k>0$, $\mathcal{T}_k^{\mathrm{A}}$ is finitary by Proposition 2.28, so that the same argument as in the proof of Proposition 3.18 applies to conclude that $\mathcal{U}_k^{\mathrm{E}}$ is finitary. ■

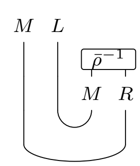
3.26. Cellular extensions and truncatability. In the following, let $n \in \mathbb{N} \cup \{\omega\}$ and (T,η,μ) be a finitary monad on $\mathbf{Glob}_n$.

We now consider the truncatable case in the context of cellular extensions, and, given $k \in \mathbb{N}$ with $0<k\le n$, show that the underlying $(k-1)$-category of a k-cellular extension is preserved by $\mathcal{F}_k^{\mathrm{E}}$. For this, we need to show that the natural transformation $\eta^{\mathrm{A},k-1}$, defined in Definition 2.29, is an isomorphism when the monad we consider is truncatable. Sadly, we cannot use a argument dual to the one of Remark 2.50 to conclude since, in general, the adjunction $\mathcal{I}_{k-1,k}^{\mathrm{G}} \dashv \mathcal{T}_{k,k-1}^{\mathrm{G}}$ does not lift through $\mathbf{Und}$ to provide a left adjoint to $(\mathcal{T}_{k,k-1}^{\mathrm{G}}, \mathcal{T}_{k-1}^{\mathrm{G}} T^k\, \mathrm{i}^{k-1,k})$ in $\mathbf{MND}$. Instead, we use the following property suggested to us by Dominic Verity, inspired by the principle of *unity and identity of opposites* [23]:

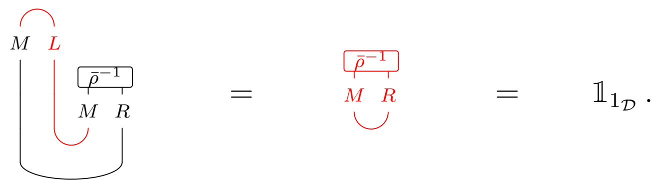
3.27. Proposition. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and $M\colon \mathcal{C} \to \mathcal{D}$ and $L,R\colon \mathcal{D} \to \mathcal{C}$ be three functors equipped with structures of adjunctions $L \dashv M$ and $M \dashv R$. If the counit of $M \dashv R$ is an isomorphism, then so is the unit of $L \dashv M$.*

Proof. Write $\lambda\colon 1_{\mathcal{D}} \Rightarrow ML$ for the unit of the adjunction $L \dashv M$, and $\rho\colon 1_{\mathcal{C}} \Rightarrow RM$ and $\bar\rho\colon MR \Rightarrow 1_{\mathcal{D}}$ respectively for the unit and the counit of the adjunction $M \dashv R$. Assume that $\bar\rho$ is an isomorphism. We define a natural transformation $\lambda'\colon ML \Rightarrow 1_{\mathcal{D}}$ using the following string diagram:

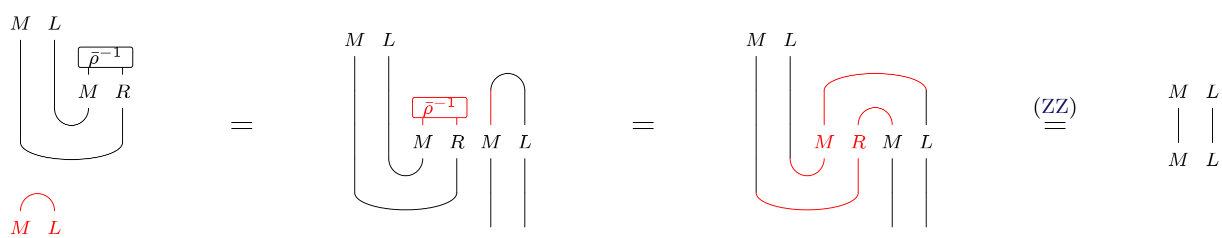
$\lambda' \quad =$



and show that it provides an inverse to λ. We first prove that $\lambda' \circ \lambda = \mathbb{1}_{1_{\mathcal{D}}}$ using string diagrams:



We now prove that $\lambda \circ \lambda' = \mathbb{1}_{ML}$ using string diagrams:



where we used for the first equation the exchange law of string diagrams and, for the middle equation the fact that $\bar{\rho}^{-1}M = M\rho$ since, by the zigzag equations (ZZ), we have $(\bar{\rho}M) \circ (M\rho) = \mathbb{1}_M$. Thus, λ' is an inverse to λ, which concludes the proof. ∎

3.28. Remark. By the equivalence between the units/counits of adjunctions being isomorphisms and left/right adjoints being fully faithful (see [8, Proposition 3.4.1] for example), an equivalent statement of Proposition 3.27 is: if R is fully faithful, then so is L.

3.29. Proposition. *Given $k \in \mathbb{N}$ with $k < n$, if T is truncatable, then $\eta^{\mathrm{A},k}$ is an isomorphism.*

Proof. By Remark 2.50, the counit of the adjunction $\mathcal{T}^{\mathrm{A}}_{k+1,k} \dashv \mathcal{J}^{\mathrm{A}}_{k,k+1}$ is an isomorphism, so that Proposition 3.27 applies, and the unit of the adjunction $\mathcal{I}^{\mathrm{A}}_{k,k+1} \dashv \mathcal{T}^{\mathrm{A}}_{k+1,k}$, which is $\eta^{\mathrm{A},k}$, is an isomorphism. ∎

We can conclude a conservation result for the underlying $(k-1)$-category of a k-category produced by $\mathcal{F}^{\mathrm{E}}_k$:

3.30. Proposition. *If T is truncatable, then, given $k < n$ and $(C, X) \in \mathbf{Alg}^+_{k+1}$, there is an isomorphism $C \cong C[X]_{\le k}$ which is natural in (C, X).*

Proof. Recall that $C[X]$ was defined in the proof of Theorem 3.20 as the pushout

$$\begin{tikzcd}
C[X] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathcal{F}^{\mathrm{G}}_{k+1}X \arrow[ll, "\mathrm{p}^{\mathrm{R}}_{(C,X)}"'] \\
C_{\uparrow k+1} \arrow[u, "\mathrm{p}^{\mathrm{L}}_{(C,X)}"] & & \mathcal{F}^{\mathrm{G}}_{k+1}((X_{\le k})_{\uparrow k+1}) \arrow[ll, "e^{\mathrm{L}}_{(C,X)} \circ \theta_{X_{\le k}}"] \arrow[u, "e^{\mathrm{R}}_{(C,X)}"']
\end{tikzcd}.$$

By Proposition 2.48, the following diagram is also a pushout

$$\begin{tikzcd}
C[X]_{\leq k} \arrow[dr, phantom, "\lrcorner", very near start] & & (\mathcal{F}^{\mathrm{G}}_{k+1} X)_{\leq k} \arrow[ll, "(\mathrm{p}^{\mathrm{R}}_{(C,X)})_{\leq k}"'] \\
(C_{\uparrow k+1})_{\leq k} \arrow[u, "(\mathrm{p}^{\mathrm{L}}_{(C,X)})_{\leq k}"] & & (\mathcal{F}^{\mathrm{G}}_{k+1}((X_{\leq k})_{\uparrow k+1}))_{\leq k} \arrow[ll, "(e^{\mathrm{L}}_{(C,X)} \circ \theta_{X_{\leq k}})_{\leq k}"] \arrow[u, "(e^{\mathrm{R}}_{(C,X)})_{\leq k}"']
\end{tikzcd}.$$

Since T is truncatable, we have $\mathcal{T}^{\mathrm{A}}_k \mathcal{F}^{\mathrm{G}}_{k+1} = \mathcal{F}^{\mathrm{G}}_k \mathcal{T}^{\mathrm{G}}_k$. Thus,

$$\begin{aligned}
(e^{\mathrm{R}})_{\leq k} &= \mathcal{T}^{\mathrm{A}}_k \mathcal{F}^{\mathrm{G}}_{k+1}\, \mathrm{i}^k\, \mathcal{G}_{k+1} \\
&= \mathcal{F}^{\mathrm{G}}_k \mathcal{T}^{\mathrm{G}}_k\, \mathrm{i}^k\, \mathcal{G}_{k+1} \\
&= \mathbb{1}_{\mathcal{F}^{\mathrm{G}}_k \mathcal{T}^{\mathrm{G}}_k \mathcal{G}_{k+1}} && (\text{since } (\mathrm{i}^k)_{\leq k} = 1_{\mathbf{Glob}_k})
\end{aligned}$$

so that $(e^{\mathrm{R}}_{(C,X)})_{\leq k} = \mathrm{id}_{\mathcal{F}^{\mathrm{G}}_k(X_{\leq k})}$. Hence, $(\mathrm{p}^{\mathrm{L}}_{(C,X)})_{\leq k}$ is an isomorphism, since the pushout of an isomorphism is an isomorphism. By Proposition 3.29, we conclude that the composite

$$C \xrightarrow{\eta^{\mathrm{A},k}_C} (C_{\uparrow k+1})_{\leq k} \xrightarrow{(\mathrm{p}^{\mathrm{L}}_{(C,X)})_{\leq k}} C[X]_{\leq k}$$

is an isomorphism. ∎

3.31. Remark. If T is truncatable, given $k < n$, by Proposition 3.7, we can suppose that we chose $\mathcal{F}^{\mathrm{E}}_{k+1}$ so that the isomorphism of Proposition 3.30 is the identity. When such a choice is made, we have $C[X]_{\leq k} = C$ for all $(k{+}1)$-cellular extension (C, X).

3.32. Finite-dimensional polygraphs. In this section, we recover the generalised notion of *polygraph* of Batanin using cellular extensions, focusing first on the finite-dimensional case. Intuitively, a polygraph is a system of generators which organises inductively as cellular extensions, so that a definition of polygraphs based on the latter structures seems relevant. In the process, we study some properties of the different functors involved, and consider the truncatable case.

Let $n \in \mathbb{N} \cup \{\omega\}$ and (T, η, μ) be a finitary monad on the category $\mathbf{Glob}_n$. We now give the definition of polygraphs through their category:

3.33. Definition. *For $k < n + 1$, we define the category $\mathbf{Pol}_k$ of* k-polygraphs *by induction on k, together with a functor*

$$\mathcal{F}^{\mathrm{P}}_k \colon \mathbf{Pol}_k \to \mathbf{Alg}_k$$

which maps a k-polygraph to the free k-category *on it. First, we put*

$$\mathbf{Pol}_0 = \mathbf{Glob}_0 \quad \textit{and} \quad \mathcal{F}^{\mathrm{P}}_0 = \mathcal{F}^{\mathrm{G}}_0.$$

Now suppose that $\mathbf{Pol}_k$ *and* $\mathcal{F}^{\mathrm{P}}_k$ *are defined for some* $k < n$*. We define* $\mathbf{Pol}_{k+1}$ *as the pullback*

$$\begin{tikzcd} \mathbf{Pol}_{k+1} \arrow[r, dotted, "\mathcal{E}_{k+1}"] \arrow[d, dotted, "\mathcal{T}^{\mathrm{P}}_{k+1,k}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Alg}^{+}_{k+1} \arrow[d, "\mathcal{T}^{+}_{k}"] \\ \mathbf{Pol}_k \arrow[r, "\mathcal{F}^{\mathrm{P}}_k"'] & \mathbf{Alg}_k \end{tikzcd} \tag{12}$$

and $\mathcal{F}^{\mathrm{P}}_{k+1}$ *as the composite*

$$\mathbf{Pol}_{k+1} \xrightarrow{\mathcal{E}_{k+1}} \mathbf{Alg}^{+}_{k+1} \xrightarrow{\mathcal{F}^{\mathrm{E}}_{k+1}} \mathbf{Alg}_{k+1}.$$

3.34. Notation. *For the sake of conciseness, we will often write* P^* *and* f^* *for the images by* $\mathcal{F}^{\mathrm{P}}_k$ *of a* k*-polygraph* P *and a* k*-polygraph morphism* $f\colon \mathsf{Q} \to \mathsf{Q}'$*. Also, as before, we write* $\mathsf{P}_{\le k}$ *and* $f_{\le k}$ *for the images of* P *and* f *by* $\mathcal{T}^{\mathrm{P}}_{k+1,k}$*, and we often simply write* $\mathcal{T}^{\mathrm{P}}_k$ *for the latter functor.*

3.35. Remark. Using the simpler definition of $\mathbf{Alg}^{+}_k$ from Proposition 3.12, we can give a more concrete description of $\mathbf{Pol}_k$ for $k < n+1$. A 0-polygraph P is the data of a set P_0 of 0-*generators*, and a morphism $\mathsf{P} \to \mathsf{P}'$ in $\mathbf{Pol}_0$ is the data of a function $F_0\colon \mathsf{P}_0 \to \mathsf{P}'_0$. Given $k < n$, a $(k{+}1)$-polygraph is the data of a pair

$$\mathsf{P} = (\mathsf{P}_{\le k}, \mathsf{P}_{k+1})$$

where $\mathsf{P}_{\le k}$ is a k-polygraph and P_{k+1} is a set of $(k{+}1)$-*generators*, together with functions

$$\mathrm{d}^-_k, \mathrm{d}^+_k \colon \mathsf{P}_{k+1} \to ((\mathsf{P}_{\le k})^*)_k$$

such that

$$\partial^{\epsilon}_{k-1} \circ \mathrm{d}^-_k = \partial^{\epsilon}_{k-1} \circ \mathrm{d}^+_k$$

for $\epsilon \in \{-,+\}$, where $\partial^-_{k-1}, \partial^+_{k-1}\colon ((\mathsf{P}_{\le k})^*)_k \to ((\mathsf{P}_{\le k})^*)_{k-1}$ are the source and target operations of the k-category $(\mathsf{P}_{\le k})^*$. Moreover, a morphism $\mathsf{P} \to \mathsf{P}'$ in $\mathbf{Pol}_{k+1}$ is the data of a pair $(F_{\le k}, F_{n+1})$ where $F_{\le k}\colon \mathsf{P}_{\le k} \to \mathsf{P}'_{\le k}$ is a morphism of $\mathbf{Pol}_k$ and $F_{n+1}\colon \mathsf{P}_{n+1} \to \mathsf{P}'_{n+1}$ is a function such that

$$\mathrm{d}^{\epsilon}_k \circ F_{n+1} = (F_{\le k})^* \circ \mathrm{d}^{\epsilon}_k$$

for $\epsilon \in \{-,+\}$, *i.e.*, a $(k{+}1)$-generator g is mapped by F_{n+1} to a generator g' whose k-source and k-target are exactly the images of the k-source and k-target of g by $(F_{\le k})^*$.

3.36. Remark. We recover this way a definition of the categories of polygraphs which is close to the one of Garner [15] using comma categories. Indeed, given $n \in \mathbb{N}$ the category of $(n{+}1)$-polygraphs is the comma category $\mathbf{Set} \downarrow (\mathcal{B}_n \mathcal{U}^{\mathrm{G}}_n \mathcal{F}^{\mathrm{P}}_n)$.

3.37. Remark. Note that the diagram

$$
\begin{tikzcd}
\mathbf{Pol}_{k+1} \arrow[r, "\mathcal{G}_{k+1}\mathcal{E}_{k+1}"] \arrow[d, "\mathcal{T}^{\mathrm{P}}_{k}"'] & \mathbf{Glob}_{k+1} \arrow[d, "\mathcal{T}^{\mathrm{G}}_{k}"] \\
\mathbf{Pol}_{k} \arrow[r, "\mathcal{U}^{\mathrm{G}}_{k}\mathcal{F}^{\mathrm{P}}_{k}"'] & \mathbf{Glob}_{k}
\end{tikzcd}
\tag{13}
$$

is a pullback, since $\mathbf{Alg}^{+}_{k+1}$ is defined as a pullback and the concatenation of two pullbacks is still a pullback.

In order to better handle side conditions, we use the convention that $\mathcal{E}_0 = 1_{\mathbf{Glob}_0}$, so that $\mathcal{F}^{\mathrm{P}}_0 = \mathcal{F}^{\mathrm{E}}_0 \circ \mathcal{E}_0$. We then have:

3.38. Proposition. *For $k < n+1$, the following hold:*

(i) the category $\mathbf{Pol}_k$ is locally ω_1-presentable (in particular, complete and cocomplete);

(ii) the functor $\mathcal{E}_k$ is a left adjoint which preserves ω_1-presentable objects;

(iii) when $k > 0$, the functor $\mathcal{T}^{\mathrm{P}}_{k-1}$ is a left adjoint which preserves ω_1-presentable objects;

(iv) the functor $\mathcal{F}^{\mathrm{P}}_k$ is a left adjoint which preserves ω_1-presentable objects.

Proof. We prove this property by induction on k. The category $\mathbf{Pol}_0$ is locally finitely presentable since equivalent to $\mathbf{Set}$. Moreover, since $\mathcal{U}^{\mathrm{G}}_0$ is finitary, it is also ω_1-accessible so that $\mathcal{F}^{\mathrm{G}}_0$ preserves finitely presentable objects. Thus, the property holds for $k = 0$.

Now assume that it holds in dimension k. By Proposition 3.25, the functor $\mathcal{F}^{\mathrm{E}}_k$ is a left adjoint which preserves finitely presentable objects. Thus, its right adjoint is finitary and in particular ω_1-accessible. Thus, $\mathcal{F}^{\mathrm{E}}_k$ also preserves ω_1-presentable objects. By induction hypothesis, $\mathcal{F}^{\mathrm{P}}_k$ is a left adjoint which preserves ω_1-presentable objects. By Proposition 3.18 and the same argument, $\mathcal{T}^{+}_k$ is also a left adjoint which preserves ω_1-presentable objects. By [7, Proposition 3.14], the pullback $\mathbf{Pol}_{k+1}$ of these two functors is a locally ω_1-presentable category, and the two projecting functors $\mathcal{T}^{\mathrm{P}}_k$ and $\mathcal{E}_{k+1}$ are left adjoints which preserve ω_1-presentable objects. ∎

3.39. Remark. Sadly, [7, Proposition 3.14] requires us to deal with uncountable cardinals, so that we can not use this general argument to prove that $\mathbf{Pol}_k$ is locally finitely presentable. But one can use an *ad hoc* argument to prove that it actually is. Moreover, the finitely presentable objects of $\mathbf{Pol}_k$ are the expected ones, that is, the k-polygraphs with a finite number of generators. However, the arguments are rather technical and not really illuminating, so that we keep those results in the appendix (see Section C).

3.40. Proposition. *For $k < n$, the functor $\mathcal{T}^{\mathrm{P}}_{k+1,k}$ is an isofibration which has both a left adjoint and a right adjoint.*

Proof. We know already that it has a right adjoint by Proposition 3.38. But it is also the consequence, as is the remainder of the statement, of Propositions 3.2 and 3.3. ∎

3.41. Definition. *Given $k < n$, we write*

$$\mathcal{I}^{\mathrm{P}}_{k,k+1} \colon \mathbf{Pol}_k \to \mathbf{Pol}_{k+1} \quad \text{and} \quad \mathcal{J}^{\mathrm{P}}_{k,k+1} \colon \mathbf{Pol}_k \to \mathbf{Pol}_{k+1}$$

or more concisely $\mathcal{I}^{\mathrm{P}}_{k+1}$ and $\mathcal{J}^{\mathrm{P}}_{k+1}$, for the left and right adjoints to the functor $\mathcal{T}^{\mathrm{P}}_{k+1,k}$.

We quickly describe these left and right adjoints, which are similar to the ones for the truncation of globular sets (see Definitions 2.10 and 2.12). The left adjoint $\mathcal{I}^{\mathrm{P}}_{k,k+1}$ maps a k-polygraph P to the $(k{+}1)$-polygraph $(\mathsf{P}, \emptyset)$, and maps morphisms of k-polygraphs as expected. The right adjoint $\mathcal{J}^{\mathrm{P}}_{k,k+1}$ maps a k-polygraph P to the $(k{+}1)$-polygraph $(\mathsf{P}, \mathcal{B}_k\mathcal{U}^{\mathrm{G}}_k(\mathsf{P}^*))$, where $\mathcal{B}_k\mathcal{U}^{\mathrm{G}}_k(\mathsf{P}^*)$ is the set of parallel k-globes of the globular k-algebra P^*, and maps morphisms as expected.

3.42. Definition. *Given $i, k \in \mathbb{N}$ such that $i \le k < n+1$ and $\mathsf{P} \in \mathbf{Pol}_k$, we write P_i for the set of i-generators of P. This operations extends to a functor*

$$(-)^k_i \colon \mathbf{Pol}_k \to \mathbf{Set}$$

or simply $(-)_i$ when there is no ambiguity on k.

We verify that colimits of polygraphs are computed dimensionwise:

3.43. Proposition. *Given $i, k < n+1$ such that $i \le k$, the functor $(-)^k_i$ preserves colimits.*

Proof. This functor can be described as the composite

$$(-)_i \circ \mathcal{E}_i \circ \mathcal{T}^{\mathrm{P}}_i \circ \cdots \circ \mathcal{T}^{\mathrm{P}}_{k-1}$$

where $(-)_i \colon \mathbf{Alg}^+_i \to \mathbf{Set}$ on the left is the analogue of the functor of Definition 3.42 for top-level generators of cellular extensions, mapping an i-cellular extension (X, S) to its set of i-generator S. By Remark 3.13, this functor $(-)_i$ preserves colimits. Moreover, all the other functors of the above composition preserve colimits by Proposition 3.38, so that the conclusion holds. ∎

In the case where T is truncatable, given $k, l \in \mathbb{N}$ with $k < l$, the underlying k-category of the free l-category on an l-polygraph is only determined by the underlying k-polygraph, as stated by the following proposition:

3.44. Proposition. *If T is truncatable, then, given $k \in \mathbb{N}$ such that $k < n$ and a $(k{+}1)$-polygraph, there exists an isomorphism $(\mathsf{P}^*)_{\le k} \cong (\mathsf{P}_{\le k})^*$.*

Proof. By definition of $\mathcal{F}^{\mathrm{P}}_k$, we have

$$\mathsf{P}^* = (\mathsf{P}_{\le k})^*[\mathsf{P}_{k+1}]$$

so that the wanted isomorphism comes from Proposition 3.30. ∎

3.45. REMARK. When T is truncatable, under the assumption of Remark 3.31, the isomorphism given by Proposition 3.44 is the identity. This enables us to simplify some notations: given $k, l < n+1$ with $k \le l$ and an l-polygraph P, we directly write $\mathsf{P}^*_{\le k}$ for both $(\mathsf{P}^*)_{\le k}$ and $(\mathsf{P}_{\le k})^*$, and P^*_k for both $(\mathsf{P}^*)_k$ and $((\mathsf{P}_{\le k})^*)_k$.

3.46. REMARK. When T is truncatable, given $k < n+1$, a k-polygraph P can be alternatively described as a diagram in **Set** of the form

$$
\begin{tikzcd}
\mathsf{P}_0 \arrow[d, "\mathrm{e}_0"] & \mathsf{P}_1 \arrow[dl, shift right, "\mathrm{d}_0^-"'] \arrow[dl, shift left, "\mathrm{d}_0^-"] \arrow[d, "\mathrm{e}_1"] & \mathsf{P}_2 \arrow[dl, shift right, "\mathrm{d}_1^-"'] \arrow[dl, shift left, "\mathrm{d}_1^-"] \arrow[d, "\mathrm{e}_2"] & \cdots & \mathsf{P}_{k-1} \arrow[dl, shift right, "\mathrm{d}_{k-2}^-"'] \arrow[dl, shift left, "\mathrm{d}_{k-2}^-"] \arrow[d, "\mathrm{e}_k"] & \mathsf{P}_k \arrow[dl, shift right, "\mathrm{d}_{k-1}^-"'] \arrow[dl, shift left, "\mathrm{d}_{k-1}^-"] \\
\mathsf{P}^*_0 & \mathsf{P}^*_1 \arrow[l, shift right, "\partial_0^-"'] \arrow[l, shift left, "\partial_0^+"] & \cdots \arrow[l, shift right, "\partial_1^-"'] \arrow[l, shift left, "\partial_1^+"] & \mathsf{P}^*_{k-2} & \mathsf{P}^*_{k-1} \arrow[l, shift right, "\partial_{k-2}^-"'] \arrow[l, shift left, "\partial_{k-2}^+"] &
\end{tikzcd}
$$

where, for $i < k$, e_i is the embedding of the i-generators in the i-cells induced by the unit of the adjunction $\mathcal{F}^{\mathrm{E}}_i \dashv \mathcal{U}^{\mathrm{E}}_{i-1}$ at $((\mathsf{P}_{\le i-1})^*, \mathsf{P}_i)$, such that

$$\partial_i^- \circ \mathrm{d}_{i+1}^- = \partial_i^- \circ \mathrm{d}_{i+1}^+ \qquad \text{and} \qquad \partial_i^+ \circ \mathrm{d}_{i+1}^- = \partial_i^+ \circ \mathrm{d}_{i+1}^+$$

for $i < k$. The above description of polygraphs can already be found in the original paper of Burroni [11] for polygraphs of strict categories.

3.47. ω-POLYGRAPHS. Let (T, η, μ) be a finitary monad on $\mathbf{Glob}_\omega$. In this section, we now give the definition of infinite-dimensional polygraphs, or ω-polygraphs:

3.48. DEFINITION. *The category of ω-polygraphs $\mathbf{Pol}_\omega$ is defined as the limit in* **CAT**

$$(\mathcal{T}^{\mathrm{P}}_{\omega,k} \colon \mathbf{Pol}_\omega \to \mathbf{Pol}_k)_{k\in\mathbb{N}}$$

on the diagram

$$\mathbf{Pol}_0 \xleftarrow{\mathcal{T}^{\mathrm{P}}_0} \mathbf{Pol}_1 \xleftarrow{\mathcal{T}^{\mathrm{P}}_1} \cdots \xleftarrow{\mathcal{T}^{\mathrm{P}}_{k-1}} \mathbf{Pol}_k \xleftarrow{\mathcal{T}^{\mathrm{P}}_k} \mathbf{Pol}_{k+1} \xleftarrow{\mathcal{T}^{\mathrm{P}}_{k+1}} \cdots$$

Concretely, an ω-polygraph P is the data of a sequence $(\mathsf{P}^k)_{k\in\mathbb{N}}$, where P^k is a k-polygraph, such that $(\mathsf{P}^{k+1})_{\le k} = \mathsf{P}^k$ for $k \in \mathbb{N}$. We start with a presentability result for $\mathbf{Pol}_\omega$:

3.49. PROPOSITION. *The category $\mathbf{Pol}_\omega$ is locally ω_1-presentable (in particular complete and cocomplete), and the functors $\mathcal{T}^{\mathrm{P}}_{\omega,k}$ are both left and right adjoints and preserve ω_1-presentable objects.*

PROOF. By Proposition 3.38, the functors $\mathcal{T}^{\mathrm{P}}_{k+1,k}$ are isofibrations, so that the limit defining $\mathbf{Pol}_\omega$ is in fact a bilimit (by a direct adaptation of Remark 2.34 from $\mathbf{Alg}_\omega$ to $\mathbf{Pol}_\omega$). Since $\mathcal{T}^{\mathrm{P}}_{k+1,k}$ preserves colimits as left adjoints, they are ω_1-accessible right adjoints. By [7, Theorem 2.17], $\mathbf{Pol}_\omega$ is locally ω_1-presentable and the functors $\mathcal{T}^{\mathrm{P}}_{\omega,k}$ are ω_1-accessible right adjoint functors.

The functors $\mathcal{T}^{\mathrm{P}}_{k+1,k}$ are also left adjoints which preserve ω_1-presentable objects by Proposition 3.38. Thus, [7, Proposition 3.14] also applies, so that we moreover get that $\mathcal{T}^{\mathrm{P}}_{\omega,k}$ are left adjoints which preserve ω_1-presentable objects. ∎

3.50. Remark. Assuming the finite presentability of the $\mathbf{Pol}_k$ given by Remark 3.39, we can apply [7, Theorem 2.17] to deduce that $\mathbf{Pol}_\omega$ is in fact locally finitely presentable. A small additional argument would then prove that the finitely presentable objects of $\mathbf{Pol}_k$ are the ω-polygraph with a finite number of generators. A proof of these facts without using Bird can be found in [12, Paragraph 1.3.3.16].

Now, in the truncatable case, we can easily define the free ω-category on an ω-polygraph, just like for finite-dimensional polygraphs:

3.51. Proposition. *If T is truncatable, there is a functor $\mathcal{F}^{\mathrm{P}}_\omega \colon \mathbf{Pol}_\omega \to \mathbf{Alg}_\omega$ which is uniquely defined by*

$$\mathcal{T}^{\mathrm{A}}_{\omega,k}\mathcal{F}^{\mathrm{P}}_\omega = \mathcal{F}^{\mathrm{P}}_k\mathcal{T}^{\mathrm{P}}_{\omega,k}$$

for $k \in \mathbb{N}$.

Proof. By Remarks 3.31 and 3.45, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{Pol}_0 \arrow[d, "\mathcal{F}^{\mathrm{P}}_0"] & \mathbf{Pol}_1 \arrow[l, "\mathcal{T}^{\mathrm{P}}_0"'] \arrow[d, "\mathcal{F}^{\mathrm{P}}_1"] & \cdots \arrow[l, "\mathcal{T}^{\mathrm{P}}_1"'] & \mathbf{Pol}_k \arrow[l, "\mathcal{T}^{\mathrm{P}}_{k-1}"'] \arrow[d, "\mathcal{F}^{\mathrm{P}}_k"] & \mathbf{Pol}_{k+1} \arrow[l, "\mathcal{T}^{\mathrm{P}}_k"'] \arrow[d, "\mathcal{F}^{\mathrm{P}}_{k+1}"] & \cdots \arrow[l, "\mathcal{T}^{\mathrm{P}}_{k+1}"'] \\
\mathbf{Alg}_0 & \mathbf{Alg}_1 \arrow[l, "\mathcal{T}^{\mathrm{A}}_0"] & \cdots \arrow[l, "\mathcal{T}^{\mathrm{A}}_1"] & \mathbf{Alg}_k \arrow[l, "\mathcal{T}^{\mathrm{A}}_{k-1}"] & \mathbf{Alg}_{k+1} \arrow[l, "\mathcal{T}^{\mathrm{A}}_k"] & \cdots \arrow[l, "\mathcal{T}^{\mathrm{A}}_{k+1}"]
\end{tikzcd}$$

which, by the definition of $\mathbf{Pol}_\omega$ and Proposition 2.32, induces a functor $\mathcal{F}^{\mathrm{P}}_\omega$ which satisfies the wanted properties. ∎

3.52. Remark. In the case where T is only weakly truncatable, the squares in the proof of Proposition 3.51 only commute up to isomorphism, so that we only get a *pseudocone* on the $\mathbf{Alg}_k$'s of vertex $\mathbf{Pol}_\omega$. In this case, we can use the fact that $\mathbf{Alg}_\omega$ is a bilimit on the $\mathbf{Alg}_k$'s (as discussed in Remark 2.34) to get a functor $\mathcal{F}^{\mathrm{P}}_\omega \colon \mathbf{Pol}_\omega \to \mathbf{Alg}_\omega$ which factorises up to isomorphism that pseudocone.

3.53. Remark. We can still define a functor $\mathcal{F}^{\mathrm{P}}_\omega \colon \mathbf{Pol}_\omega \to \mathbf{Alg}_\omega$ in the case where T is not weakly truncatable. Indeed, given $\mathsf{P} \in \mathbf{Pol}_\omega$, we can construct a canonical chain of embeddings

$$\mathcal{I}^{\mathrm{A}}_{0,\omega}(\mathsf{P}_{\leq 0})^* \longrightarrow \mathcal{I}^{\mathrm{A}}_{1,\omega}(\mathsf{P}_{\leq 1})^* \longrightarrow \mathcal{I}^{\mathrm{A}}_{2,\omega}(\mathsf{P}_{\leq 2})^* \longrightarrow \cdots$$

in $\mathbf{Alg}_\omega$, and call its colimit P^*. This construction then readily extends to a functor $\mathcal{F}^{\mathrm{P}}_\omega$, which is again to be considered the free ω-category construction on an ω-polygraph (see [15, Definition 5.4] for an earlier occurrence of this construction). But, this functor is not expected to be compatible with the functors $\mathcal{T}^{\mathrm{A}}_k$ as in Proposition 3.51. Indeed, in this case, the functor $\mathcal{F}^{\mathrm{E}}_k$ does not preserve the underlying $(k{-}1)$-category C of a k-cellular extension $(C, S) \in \mathbf{Alg}^+_k$. In the weakly truncatable case, this compatibility holds by definition.

3.54. The polygraphic adjunctions. We now translate into our setting the definition given by Batanin in [4, Section 3] of the adjunction between globular algebras and polygraphs:

$$\begin{tikzcd} \mathbf{Pol}_k \arrow[r, shift left, "\mathcal{F}^{\mathrm{P}}_k"] \arrow[r, phantom, "\perp"] & \mathbf{Alg}_k \arrow[l, shift left, dashed] \end{tikzcd}$$

In contrast to other approaches [4, 15], we will derive it from an adjunction with cellular extensions

$$\begin{tikzcd} \mathbf{Pol}_k \arrow[r, shift left, "\mathcal{E}_k"] \arrow[r, phantom, "\perp"] & \mathbf{Alg}^+_k \arrow[l, shift left, dashed] \end{tikzcd}.$$

Using as much as possible the already existing adjunctions with their units/counits will allow us to keep using formal arguments for the proofs. As before, our use of string diagrams will greatly simplify the definitions and the calculations of this section, and will hopefully make them more intuitive to the reader.

Our approach will require an inductive and mutual proof of the following properties:

3.55. Proposition. *For $k \in \mathbb{N}$, there is a canonical functor*

$$\mathcal{S}_k\colon \mathbf{Alg}^+_k \to \mathbf{Alg}^+_k$$

together with, when $k > 0$ natural transformations

$$\mathrm{q}^{\mathrm{L},k}\colon \mathcal{S}_k \Rightarrow \mathcal{J}^+_k \mathcal{F}^{\mathrm{P}}_{k-1} \mathcal{U}^{\mathrm{P}}_{k-1} \mathcal{T}^+_{k-1} \qquad \textit{and} \qquad \mathrm{q}^{\mathrm{R},k}\colon \mathcal{S}_k \Rightarrow 1_{\mathbf{Alg}^+_k}$$

often simply denoted q^{L} and q^{R}, and where $\mathcal{U}^{\mathrm{P}}_{k-1}$ is introduced by Theorem 3.57 below.

3.56. Proposition. *Given $k \in \mathbb{N}$, there is a canonical functor*

$$\mathcal{U}^{+,\mathrm{P}}_k \quad : \quad \mathbf{Alg}^+_k \;\to\; \mathbf{Pol}_k$$

and natural transformations

$$\begin{aligned} \xi^{+,k} \quad &: \quad 1_{\mathbf{Pol}_k} \;\Rightarrow\; \mathcal{U}^{+,\mathrm{P}}_k \mathcal{E}_k \\ \nu^{+,k} \quad &: \quad \mathcal{E}_k \mathcal{U}^{+,\mathrm{P}}_k \;\Rightarrow\; 1_{\mathbf{Alg}^+_k} \end{aligned}$$

which form a unit-counit pair witnessing that $\mathcal{U}^{+,\mathrm{P}}_k$ is right adjoint to $\mathcal{E}_k$.

3.57. Theorem. *Given $k \in \mathbb{N}$, there is a canonical functor*

$$\mathcal{U}^{\mathrm{P}}_k \quad : \quad \mathbf{Alg}_k \;\to\; \mathbf{Pol}_k$$

and natural transformations

$$\begin{aligned} \xi^{k} \quad &: \quad 1_{\mathbf{Pol}_k} \;\Rightarrow\; \mathcal{U}^{\mathrm{P}}_k \mathcal{F}^{\mathrm{P}}_k \\ \nu^{k} \quad &: \quad \mathcal{F}^{\mathrm{P}}_k \mathcal{U}^{\mathrm{P}}_k \;\Rightarrow\; 1_{\mathbf{Alg}_k} \end{aligned}$$

which form a unit-counit pair witnessing that $\mathcal{U}^{\mathrm{P}}_k$ is right adjoint to $\mathcal{F}^{\mathrm{P}}_k$.

Proof of the above propositions. We proceed by induction on $k \in \mathbb{N}$.

Base case. When $k = 0$, we take $\mathcal{S}_k = \mathcal{U}_k^{+,\mathrm{P}} = 1_{\mathbf{Glob}_0}$, $\mathcal{U}_0^{\mathrm{P}} = \mathcal{U}_0^{\mathrm{G}}$, $\xi^{+,k} = \nu^{+,k} = \mathbb{1}_{1_{\mathbf{Glob}_0}}$, $\xi^0 = \eta^0$ and $\nu^0 = \epsilon^0$ (defined in Definition 2.17).

Inductive case. We now assume that we have proved Propositions 3.55 and 3.56 and theorem 3.57 up to dimension $k \in \mathbb{N}$ and prove them in the dimension $k+1$.

The functor $\boldsymbol{\mathcal{S}_{k+1}}$: $\mathbf{Alg}^+_{k+1} \to \mathbf{Alg}^+_{k+1}$. Given $(C,S) \in \mathbf{Alg}^+_{k+1}$, we define the $(k{+}1)$-cellular extension $\mathcal{S}_{k+1}(C,S)$ as the pullback

$$\begin{tikzcd}
\mathcal{S}_{k+1}(C,S) \arrow[r, dashed, "\mathrm{q}^{\mathrm{R},k+1}_{(C,S)}"] \arrow[d, dashed, "\mathrm{q}^{\mathrm{L},k+1}_{(C,S)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (C,S) \arrow[d, "\mathrm{j}^k_{(C,S)}"] \\
\mathcal{J}^+_{k+1}((\mathcal{U}^{\mathrm{P}}_k C)^*) \arrow[r, "(\mathcal{J}^+_{k+1}\nu^k)_C"'] & \mathcal{J}^+_{k+1} C
\end{tikzcd} \tag{14}$$

where j^k denotes the unit of the adjunction $\mathcal{T}^+_k \dashv \mathcal{J}^+_{k+1}$, with $\mathcal{J}^+_{k+1}$ being the right adjoint given by Proposition 3.15 (whose concrete definition is close to the ones of $\mathcal{J}^{\mathrm{G}}_{k,k+1}$ and $\mathcal{J}^{\mathrm{A}}_{k,k+1}$), and where ν^k is the counit of the adjunction $\mathcal{F}^{\mathrm{P}}_k \dashv \mathcal{U}^{\mathrm{P}}_k$ defined by induction hypothesis. The idea behind this pullback is the following: we forget the k-category C as a k-polygraph and freely generate it back immediately to a k-category (the left leg), then we attach back the generators of S to this new k-category, taking into account that there now several choices of cells which evaluate to the old sources and targets of the elements of S (the right leg).

Since $\mathcal{T}^+_k$ preserves limits (as a right adjoint), the diagram

$$\begin{tikzcd}
\mathcal{T}^+_k(\mathcal{S}_{k+1}(C,S)) \arrow[r, dashed, "\mathcal{T}^+_k(\mathrm{q}^{\mathrm{R},k+1}_{(C,S)})"] \arrow[d, dashed, "\mathcal{T}^+_k(\mathrm{q}^{\mathrm{L},k+1}_{(C,S)})"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "1_C"] \\
(\mathcal{U}^{\mathrm{P}}_k C)^* \arrow[r, "\nu^k_C"'] & C
\end{tikzcd}$$

which is the image by $\mathcal{T}^+_k$ of the previous square, is a pullback (we picked $\mathcal{J}^+_{k+1}$ so that the counit of $\mathcal{T}^+_k \dashv \mathcal{J}^+_{k+1}$ is the identity, as stated in Remark 3.17). Moreover, since $\mathcal{T}^+_k$ is an isofibration, we can suppose that we defined $\mathcal{S}_{k+1}$ such that

$$\mathcal{T}^+_k(\mathcal{S}_{k+1}(C,S)) = (\mathcal{U}^{\mathrm{P}}_k C)^* \qquad \text{and} \qquad \mathcal{T}^+_k(\mathrm{q}^{\mathrm{L},k+1}_{(C,S)}) = 1_{(\mathcal{U}^{\mathrm{P}}_k C)^*} \tag{15}$$

holds, so that we moreover have

$$\mathcal{T}^+_k(\mathrm{q}^{\mathrm{R},k+1}_{(C,S)}) = \nu^k_C. \tag{16}$$

Note that the above pullback is functorial in (C,S), so that we get a functor

$$\mathcal{S}_{k+1} \colon \mathbf{Alg}^+_{k+1} \to \mathbf{Alg}^+_{k+1}$$

together with projecting natural transformations

$$\mathrm{q}^{\mathrm{L},k+1}\colon \mathcal{S}_{k+1} \Rightarrow \mathcal{J}^{+}_{k+1}\mathcal{F}^{\mathrm{P}}_{k}\mathcal{U}^{\mathrm{P}}_{k}\mathcal{T}^{+}_{k} \qquad \text{and} \qquad \mathrm{q}^{\mathrm{R},k+1}\colon \mathcal{S}_{k+1} \Rightarrow 1_{\mathbf{Alg}^{+}_{k+1}}$$

which concludes the proof of Proposition 3.55.

The functor $\mathcal{U}^{+,\mathrm{P}}_{k+1}\colon \mathbf{Alg}^{+}_{k+1} \to \mathbf{Pol}_{k+1}$. By the pullback definition (12) of $\mathbf{Pol}_{k+1}$, defining $\mathcal{U}^{+,\mathrm{P}}_{k+1}$ requires introducing functors

$$\mathcal{U}^{+,\mathrm{P}}_{k+1,l}\colon \mathbf{Alg}^{+}_{k+1} \to \mathbf{Pol}_{k} \qquad \text{and} \qquad \mathcal{U}^{+,\mathrm{P}}_{k+1,r}\colon \mathbf{Alg}^{+}_{k+1} \to \mathbf{Alg}^{+}_{k+1}$$

such that

$$\begin{tikzcd} \mathbf{Alg}^{+}_{k+1} \arrow[rr, dashed, bend left, "\mathcal{U}^{+,\mathrm{P}}_{k+1,r}"] \arrow[dr, dashed, bend right, "\mathcal{U}^{+,\mathrm{P}}_{k+1,l}"'] & & \mathbf{Alg}^{+}_{k+1} \arrow[d, "\mathcal{T}^{+}_{k}"] \\ & \mathbf{Pol}_{k} \arrow[r, "\mathcal{F}^{\mathrm{P}}_{k}"'] \arrow[ur, phantom, "="] & \mathbf{Alg}_{k}. \end{tikzcd} \tag{17}$$

The functor $\mathcal{U}^{+,\mathrm{P}}_{k+1,l}\colon \mathbf{Alg}^{+}_{k+1} \to \mathbf{Pol}_{k}$ is defined as the composite

$$\mathbf{Alg}^{+}_{k+1} \xrightarrow{\mathcal{T}^{+}_{k}} \mathbf{Alg}_{k} \xrightarrow{\mathcal{U}^{\mathrm{P}}_{k}} \mathbf{Pol}_{k}$$

and we take $\mathcal{U}^{+,\mathrm{P}}_{k+1,r}\colon \mathbf{Alg}^{+}_{k+1} \to \mathbf{Alg}^{+}_{k+1}$ to be the functor $\mathcal{S}_{k+1}$. By (15), the equality (17) holds, so that we get, as a factorisation through $\mathbf{Pol}_{k+1}$ of the cone made of the functors $\mathcal{U}^{+,\mathrm{P}}_{k+1,l}$ and $\mathcal{U}^{+,\mathrm{P}}_{k+1,r}$, the functor

$$\mathcal{U}^{+,\mathrm{P}}_{k+1}\colon \mathbf{Alg}^{+}_{k+1} \to \mathbf{Pol}_{k+1}$$

which is characterised by $\mathcal{T}^{\mathrm{P}}_{k}\mathcal{U}^{+,\mathrm{P}}_{k+1} = \mathcal{U}^{\mathrm{P}}_{k}\mathcal{T}^{+}_{k}$ and $\mathcal{E}_{k+1}\mathcal{U}^{+,\mathrm{P}}_{k+1} = \mathcal{S}_{k+1}$.

The adjunction between $\mathcal{E}_{k+1}$ and $\mathcal{U}^{+,\mathrm{P}}_{k+1}$. We now equip $(\mathcal{E}_{k+1},\mathcal{U}^{+,\mathrm{P}}_{k+1})$ with a structure of an adjunction. First, we build the unit

$$\xi^{+,k+1} = \langle \xi^{+,k+1}_{l}, \xi^{+,k+1}_{r} \rangle\colon\ 1_{\mathbf{Pol}_{k+1}} \Rightarrow \mathcal{U}^{+,\mathrm{P}}_{k+1}\mathcal{E}_{k+1}$$

through the pullback definition (12) of $\mathbf{Pol}_{k+1}$ (indeed, the universal property of the pullback of categories extend to natural transformations). The projection

$$\xi^{+,k+1}_{l}\colon \mathcal{T}^{\mathrm{P}}_{k} \Rightarrow \mathcal{T}^{\mathrm{P}}_{k}\mathcal{U}^{+,\mathrm{P}}_{k+1}\mathcal{E}_{k+1}\colon \mathbf{Pol}_{k+1} \to \mathbf{Pol}_{k}$$

of $\xi^{+,k+1}$ through $\mathcal{T}^{\mathrm{P}}_{k}$ is defined to be $\xi^{k}\,\mathcal{T}^{\mathrm{P}}_{k}$; this is well-typed, since

$$\mathcal{T}^{\mathrm{P}}_{k}\mathcal{U}^{+,\mathrm{P}}_{k+1}\mathcal{E}_{k+1} = \mathcal{U}^{\mathrm{P}}_{k}\mathcal{T}^{+}_{k}\mathcal{E}_{k+1} = \mathcal{U}^{\mathrm{P}}_{k}\mathcal{F}^{\mathrm{P}}_{k}\mathcal{T}^{\mathrm{P}}_{k}.$$

Moreover, the projection

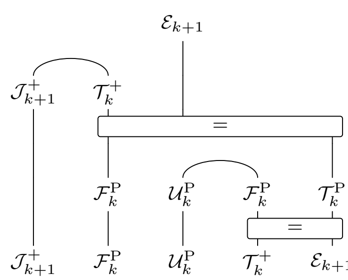
$$\xi_r^{+,k+1} \colon \mathcal{E}_{k+1} \Rightarrow \mathcal{E}_{k+1}\mathcal{U}^{+,\mathrm{P}}_{k+1}\mathcal{E}_{k+1} = \mathcal{S}_{k+1}\mathcal{E}_{k+1} \colon \mathbf{Pol}_{k+1} \to \mathbf{Alg}^+_{k+1}$$

of $\xi^{+,k+1}$ through $\mathcal{E}_{k+1}$ is defined using the pullback definition (14) of $\mathcal{S}_{k+1}$ as the natural transformation factorising the cone

$$\begin{tikzcd}
\mathcal{E}_{k+1} \arrow[dr, dashed, "\xi_r^{+,k+1}"] \arrow[drr, bend left, "\mathbb{1}_{\mathcal{E}_{k+1}}"] \arrow[ddr, bend right, "\lambda^{k+1}"'] & & \\
& \mathcal{S}_{k+1}\mathcal{E}_{k+1} \arrow[r, "\mathrm{q}^{\mathrm{R}}\,\mathcal{E}_{k+1}"] \arrow[d, "\mathrm{q}^{\mathrm{L}}\,\mathcal{E}_{k+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}_{k+1} \arrow[d, "\mathrm{j}^k\,\mathcal{E}_{k+1}"] \\
& \mathcal{J}^+_{k+1}\mathcal{F}^{\mathrm{P}}_k\mathcal{U}^{\mathrm{P}}_k\mathcal{T}^+_k\mathcal{E}_{k+1} \arrow[r, "\mathcal{J}^+_{k+1}\nu^k\mathcal{T}^+_k\mathcal{E}_{k+1}"'] & \mathcal{J}^+_{k+1}\mathcal{T}^+_k\mathcal{E}_{k+1}
\end{tikzcd} \tag{18}$$

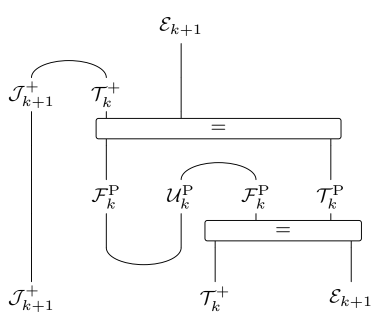
where λ^{k+1} is defined using a string diagram as

$$\lambda^{k+1} \quad = $$

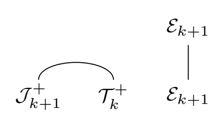


.

The pair $(\lambda^{k+1}, \mathbb{1}_{\mathcal{E}_{k+1}})$ is indeed a cone, as one can easily verify using the zigzag equations (ZZ) that the following two string diagrams



and

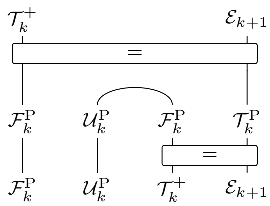


represent the same natural transformation. Thus, we get a factorising natural transformation $\xi_r^{+,k+1} \colon \mathcal{E}_{k+1} \Rightarrow \mathcal{E}_{k+1}\mathcal{U}^{+,\mathrm{P}}_{k+1}\mathcal{E}_{k+1}$.

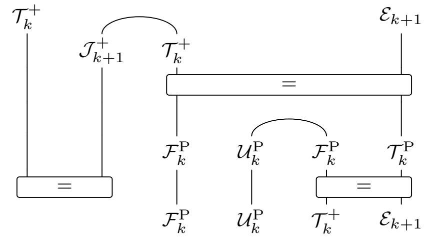
We still need to verify that the pair $(\xi_l^{+,k+1}, \xi_r^{+,k+1})$ satisfies

$$\mathcal{F}^{\mathrm{P}}_k\,\xi_l^{+,k+1} = \mathcal{T}^+_k\,\xi_r^{+,k+1} \tag{19}$$

for it to define a cone of natural transformations for the diagram (12). The string diagram representations of the two sides of the equation are



and



so that the equation holds by the zigzag equations of adjunctions (the equality on the right diagram is the counit of the adjunction $\mathcal{T}_k^+ \dashv \mathcal{J}_{k+1}^+$). Thus,

$$\xi^{+,k+1} = \langle \xi_l^{+,k+1}, \xi_r^{+,k+1} \rangle \colon 1_{\mathbf{Pol}_{k+1}} \Rightarrow \mathcal{U}_{k+1}^{+,\mathrm{P}} \mathcal{E}_{k+1}$$

is well-defined.

We are now required to give a counit

$$\nu^{+,k+1} \colon \mathcal{E}_{k+1} \mathcal{U}_{k+1}^{+,\mathrm{P}} \Rightarrow 1_{\mathbf{Alg}_{k+1}^+} .$$

We take $\nu^{+,k+1} = \mathrm{q}^{\mathrm{R}}$, which is well-typed since $\mathcal{E}_{k+1}\mathcal{U}_{k+1}^{+,\mathrm{P}} = \mathcal{S}_{k+1}$.

We now check that $(\xi^{+,k+1}, \nu^{+,k+1})$ is the unit-counit pair of an adjunction $\mathcal{E}_{k+1} \dashv \mathcal{U}_{k+1}^{+,\mathrm{P}}$. We first verify the right zigzag equation of (ZZ), by computing that

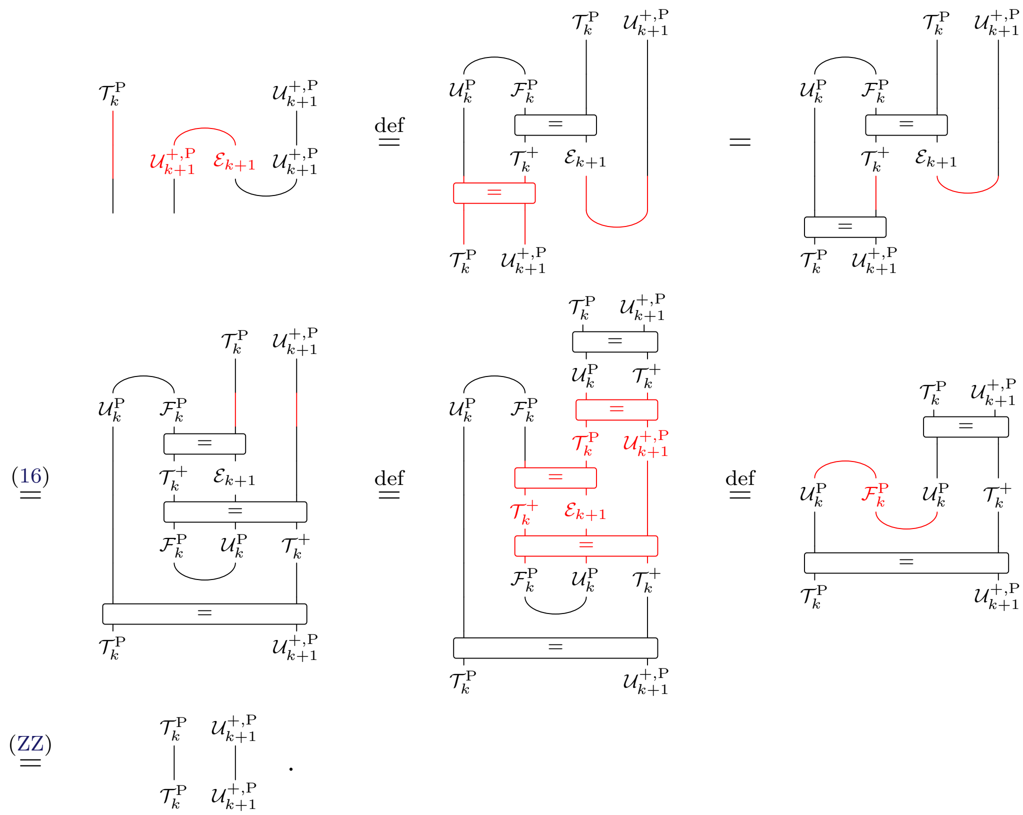
$$\begin{aligned}(\nu^{+,k+1}\mathcal{E}_{k+1}) \circ (\mathcal{E}_{k+1}\, \xi^{+,k+1}) &= (\nu^{+,k+1}\mathcal{E}_{k+1}) \circ (\xi_r^{+,k+1}) \\ &= \mathbb{1}_{\mathcal{E}_{k+1}}\end{aligned}$$

by the definition $\xi_r^{+,k+1}$ from (18), since $\nu^{+,k+1} = \mathrm{q}^{\mathrm{R}}$.

We now verify the left zigzag equation of (ZZ), namely

$$(\mathcal{U}_{k+1}^{+,\mathrm{P}} \nu^{+,k+1}) \circ (\xi^{+,k+1}\, \mathcal{U}_{k+1}^{+,\mathrm{P}}) = \mathbb{1}_{\mathcal{U}_{k+1}^{+,\mathrm{P}}} . \tag{20}$$

We first check that the projection along $\mathcal{T}_k^{\mathrm{P}}$ is an identity using string diagrams:



We now check that the projection along $\mathcal{E}_{k+1}$ is also an identity, *i.e.*,

$$(\mathcal{S}_{k+1}\,\mathrm{q}^{\mathrm{R}}) \circ (\xi_r^{+,k+1}\,\mathcal{U}_{k+1}^{+,\mathrm{P}}) = \mathbb{1}_{\mathcal{S}_{k+1}}\,. \tag{21}$$

As an equation between two natural transformation with codomain $\mathcal{S}_{k+1}$, a functor defined by a pullback of natural transformations, we check this equation by verifying that post-compositions with the projections q^{L} and q^{R} are the same. We start with q^{L}, and thus are required to show that

$$\mathrm{q}^{\mathrm{L}} \circ (\mathcal{S}_{k+1}\,\mathrm{q}^{\mathrm{R}}) \circ (\xi_r^{+,k+1}\,\mathcal{U}_{k+1}^{+,\mathrm{P}}) = \mathrm{q}^{\mathrm{L}} \tag{22}$$

as natural transformations of type $\mathcal{S}_{k+1} \Rightarrow \mathcal{J}_{k+1}^{+}\mathcal{F}_k^{\mathrm{P}}\mathcal{U}_k^{\mathrm{P}}\mathcal{T}_k^{+}$. While we could directly use string diagrams to prove the above equality, it is more convenient to simplify it first using the following property:

3.58. Lemma. *Given two natural transformations $\alpha, \beta \colon F \Rightarrow \mathcal{J}_{k+1}^{+}G$ for two functors $F \colon \mathcal{C} \to \mathbf{Alg}_{k+1}^{+}$ and $G \colon \mathcal{C} \to \mathbf{Alg}_k$, we have $\alpha = \beta$ if and only if $\mathcal{T}_k^{+}\alpha = \mathcal{T}_k^{+}\beta$.*

Proof. The adjunction $\mathcal{T}_k^{+} \dashv \mathcal{J}_{k+1}^{+}$ induces a correspondence between natural transformations of type $F \Rightarrow \mathcal{J}_{k+1}^{+}G$ and the ones of type $\mathcal{T}_k^{+}F \Rightarrow G$. Since the counit of this adjunction is an identity (Remark 3.17), the forward map of this correspondence is given by the application of $\mathcal{T}_k^{+}$. ∎

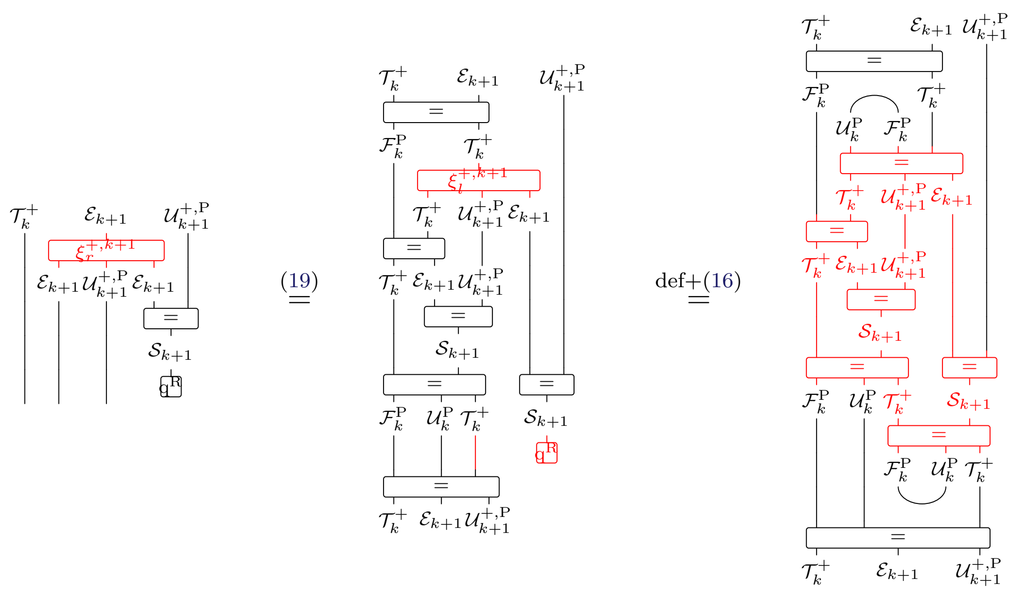
Thus, we just need prove that the equation (22) holds, up to application of $\mathcal{T}_k^+$. Since, by (15), we have $\mathcal{T}_k^+\,\mathrm{q}^{\mathrm{L}} = \mathbb{1}_{\mathcal{F}_k^{\mathrm{P}}\mathcal{U}_k^{\mathrm{P}}\mathcal{T}_k^+}$, equation (22) reduces to

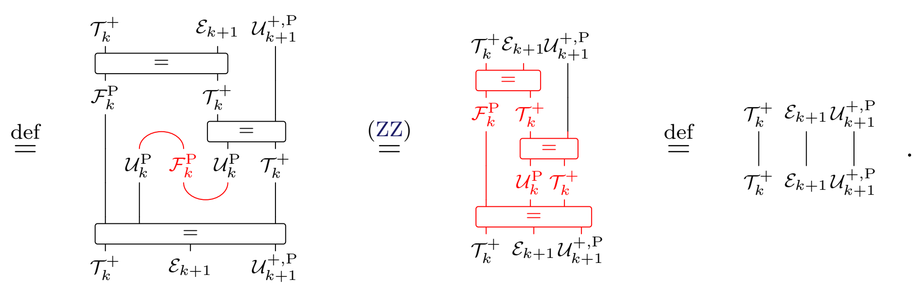
$$(\mathcal{T}_k^+\mathcal{S}_{k+1}\,\mathrm{q}^{\mathrm{R}}) \circ (\mathcal{T}_k^+\,\xi_r^{+,k+1}\,\mathcal{U}_{k+1}^{+,\mathrm{P}}) = \mathbb{1}_{\mathcal{T}_k^+\mathcal{S}_{k+1}}$$

Thus, by Lemma 3.58, q^{L} is a cofork[3] of the two sides of (21) if the equation

$$(\mathcal{T}_k^+\mathcal{S}_{k+1}\,\mathrm{q}^{\mathrm{R}}) \circ (\mathcal{T}_k^+\,\xi_r^{+,k+1}\,\mathcal{U}_{k+1}^{+,\mathrm{P}}) = \mathbb{1}_{\mathcal{T}_k^+\mathcal{S}_{k+1}}$$

holds. Using that $\mathcal{S}_{k+1} = \mathcal{E}_{k+1}\mathcal{U}_{k+1}^{+,\mathrm{P}}$, we prove the above equation using string diagrams:



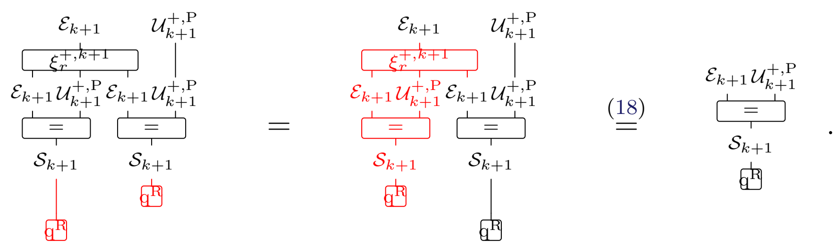


We proceed with verifying that q^{R} is a cofork of the two sides of equation (21), that is,

$$\mathrm{q}^{\mathrm{R}} \circ (\mathcal{S}_{k+1}\,\mathrm{q}^{\mathrm{R}}) \circ (\xi_r^{+,k+1}\,\mathcal{U}_{k+1}^{+,\mathrm{P}}) = \mathrm{q}^{\mathrm{R}}$$

[3]Given two morphisms $f, g\colon A \to B$ in some category, we say that a morphism $h\colon B \to C$ is a *cofork* of f and g when $h \circ f = h \circ g$.

and we show it again using string diagrams:

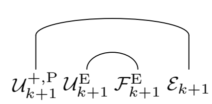


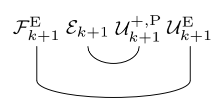
Thus, equation (21) holds up to post-composition by q^{L} and q^{R}, so that it plainly holds by the universal property of the pullback (14).

Hence, equation (20) holds up to post-composition by $\mathcal{T}^{\mathrm{P}}_k$ and $\mathcal{E}_{k+1}$, so that it plainly holds by the universal property of the pullback (12).

Thus, the pair $(\xi^{+,k+1}, \nu^{+,k+1})$ equips $(\mathcal{E}_{k+1}, \mathcal{U}^{+,\mathrm{P}}_{k+1})$ with the structure of an adjunction $\mathcal{E}_{k+1} \dashv \mathcal{U}^{+,\mathrm{P}}_{k+1}$.

The adjunction between $\mathcal{F}^{\mathbf{P}}_{k+1}$ and $\mathcal{U}^{\mathbf{P}}_{k+1}$. Writing $\mathcal{U}^{\mathrm{P}}_{k+1}$ for $\mathcal{U}^{+,\mathrm{P}}_{k+1}\mathcal{U}^{\mathrm{E}}_{k+1}$, we get an adjunction $\mathcal{F}^{\mathrm{P}}_{k+1} \dashv \mathcal{U}^{\mathrm{P}}_{k+1}$ by composition of the adjunctions $\mathcal{F}^{\mathrm{E}}_{k+1} \dashv \mathcal{U}^{\mathrm{E}}_{k+1}$ and $\mathcal{E}_{k+1} \dashv \mathcal{U}^{+,\mathrm{P}}_{k+1}$. The unit ξ^{k+1} and the counit ν^{k+1} of the new adjunction are then defined using string diagrams from the ones of the existing two adjunctions:

ξ^{k+1} = 

ν^{k+1} = 

The zigzag equations (ZZ) are then easily verified through string diagrams.

This concludes the inductive argument of this section. ∎

4. The full example of strict categories

In this section, we illustrate the previous constructions on the classical example of strict categories, which is a well-known theory of higher categories. Strict categories, as their name suggests, are a classical example of a theory for higher categories that lies on the strict side of the strict/weak spectrum of higher categories. As such, they do not represent faithfully the homotopical information of topological spaces (see [32] or [6]). Nevertheless, they admit a relatively simpler axiomatization than weak higher categories, and can be encountered in several situations of interest. As such, they will provide a simple concrete example to instantiate the abstract theory we have developed. While essentially no new results on strict categories will appear in this section, it will netherless provide a template one can easily adapt when considering another algebraic notion of higher categories.

A source of difficulties, which is common with other equational theories of higher categories, is the discrepancy between the presentation of higher categories as structures with operations and equations, and the one as globular algebras for some monad. A first problem is that it is not immediate that the operations that we defined for globular algebras, such as truncation and inclusion, are equivalent to the truncation and inclusion operations usually defined for strict categories (or other higher categories). A second problem is that several properties and simplications that we introduced earlier rely on the truncatability of the underlying monad, which can be difficult to prove since we only have an indirect definition of the monad through an equational theory. Our theoretical developments on globular algebras—more specifically, the two criterions that we introduced—will prove useful to deal with these problems, in the case of strict categories as we illustrate, but of other theories as well.

We start by recalling the equational definition of strict categories in Section 4.1. Then, we give the full calculation that the forgetful functor from strict categories to globular sets is monadic in Section 4.8. Next, we give the boilerplate definitions of the truncation and inclusion functors for strict categories in Section 4.14. Then, in Section 4.19, we show using the criteria of Theorems 2.38 and 2.55 that the categories of strict categories are coherently equivalent in each dimension to the categories of globular algebras derived from a monad on $\mathbf{Glob}_\omega$ (the monad of strict ω-categories). Finally, in Section 4.24, we instantiate the free constructions of cellular extensions and polygraphs introduced in Section 3 in the case of strict categories.

4.1. Equational definition. We recall here the usual equational definition of strict categories together with associated notions and elementary properties.

4.2. Definition. *Given $n \in \mathbb{N} \cup \{\omega\}$, a* strict n-category $(C, \partial^-, \partial^+, \mathrm{id}, *)$ *(often simply denoted C) is an n-globular set $(C, \partial^-, \partial^+)$ together with, for $k \in \mathbb{N}$ with $k < n$, identity operations*

$$\mathrm{id}^{k+1} \colon C_k \to C_{k+1}$$

often writen id *when there is no ambiguity on k, and, for $i, k < n+1$ with $i < k$, composition operations*

$$*_{i,k} \colon C_k \times_i C_k \to C_k$$

often denoted $*_i$ when there is no ambiguity on k, which satisfy the axioms (S-i) to (S-vi) below. Given $k, l < n+1$ such that $k \leq l$ and $u \in C_k$, we extend the notations for identity operations and write $\mathrm{id}^l(u)$ for

$$\mathrm{id}^l(u) = \mathrm{id}^l \circ \cdots \circ \mathrm{id}^{k+1}(u)$$

and, for the sake of conciseness, we often write id^l_u for $\mathrm{id}^l(u)$, or even id_u when $l = k+1$. The axioms are the following:

(S-i) for $k < n$ and $u \in C_k$,

$$\partial^-_k(\mathrm{id}^{k+1}_u) = \partial^+_k(\mathrm{id}^{k+1}_u) = u,$$

(S-ii) *for $i, k < n+1$ with $i < k$, $(u, v) \in C_k \times_i C_k$ and $\epsilon \in \{-,+\}$,*

$$\partial^{\epsilon}_{k-1}(u *_i v) = \begin{cases} \partial^{\epsilon}_{k-1}(u) *_i \partial^{\epsilon}_{k-1}(v) & \text{if } i < k-1, \\ \partial^{-}_{k-1}(u) & \text{if } i = k-1 \text{ and } \epsilon = -, \\ \partial^{+}_{k-1}(v) & \text{if } i = k-1 \text{ and } \epsilon = +, \end{cases}$$

(S-iii) *for $i, k < n+1$ such that $i < k$, and $u \in C_k$,*

$$\mathrm{id}^k(\partial^-_i(u)) *_i u = u = u *_i \mathrm{id}^k(\partial^+_i(u)),$$

(S-iv) *for $i, k < n+1$ such that $i < k$, and i-composable $u, v, w \in C_k$,*

$$(u *_i v) *_i w = u *_i (v *_i w),$$

(S-v) *for $i, k < n$ such that $i < k$, and $(u, v) \in C_k \times_i C_k$,*

$$\mathrm{id}^{k+1}(u *_i v) = \mathrm{id}^{k+1}(u) *_i \mathrm{id}^{k+1}(v),$$

(S-vi) *for $i, j, k < n+1$ such that $i < j < k$, and $u, u', v, v' \in C_k$ such that u, v are i-composable, and u, u' are j-composable, and v, v' are j-composable,*

$$(u *_i v) *_j (u' *_i v') = (u *_j u') *_i (v *_j v').$$

Note that the composition that appear in Axioms (S-iii), (S-iv), (S-v) and (S-vi) are well-defined as a consequence of Axioms (S-i) and (S-ii) and the equations satisfied by the source and target operations of a globular set. The Axiom (S-vi) is frequently called the *exchange law* of strict categories.

4.3. Example. Given a 2-category C and 0-, 1- and 2-globes as in the following configuration

$$\begin{tikzcd} x \arrow[r, bend left=60, "f_1"] \arrow[r, "f_2" description] \arrow[r, bend right=60, "f_3"'] \arrow[r, phantom, shift left=3, "\Downarrow u"] \arrow[r, phantom, shift right=3, "\Downarrow u'"] & y \arrow[r, bend left=60, "g_1"] \arrow[r, "g_2" description] \arrow[r, bend right=60, "g_3"'] \arrow[r, phantom, shift left=3, "\Downarrow v"] \arrow[r, phantom, shift right=3, "\Downarrow v'"] & z \end{tikzcd}$$

we have $(u *_0 v) *_1 (u' *_0 v') = (u *_1 u') *_0 (v *_1 v')$ by Axiom (S-vi).

Our definition of strict categories involves sets, but we could have written a similar definition using classes to define *large* strict categories. For such alternative definition, we have the following classical example:

4.4. Example. There is a large strict 2-category **Cat** whose 0-cells are the small categories, whose 1-cells are the functors between the 1-categories, and whose 2-cells are the natural transformations between functors, and where the operations $*_{0,1}$ is the composition of functors, and the operations $*_{0,2}$ and $*_{1,2}$ are respectively the horizontal and vertical compositions of natural transformations. Note that the exchange law Axiom (S-vi) in this setting corresponds to the usual *exchange law* for natural transformations.

4.5. Definition. *Let $n \in \mathbb{N} \cup \{\omega\}$. Given two strict n-categories C and D, a* morphism F *between C and D is the data of an n-globular morphism $F\colon C \to D$ which moreover satisfies that*

- $F(\mathrm{id}_u^{k+1}) = \mathrm{id}_{F(u)}^{k+1}$ *for every $k < n$ and $u \in C_k$,*
- $F(u *_i v) = F(u) *_i F(v)$ *for every $i, k < n+1$ with $i < k$ and i-composable $u, v \in C_k$.*

We often call such morphisms n-functors. *We write $\mathbf{Cat}_n$ for the category of strict n-categories and their morphisms.*

4.6. Definition. *We write*

$$\bar{\mathcal{U}}_n^{\mathrm{G}}\colon \mathbf{Cat}_n \to \mathbf{Glob}_n$$

for the functor which maps a strict n-category to its underlying n-globular set.

The definition of strict n-categories directly translates into an essentially algebraic theory (*c.f.* Section A.21), so that the functor $\bar{\mathcal{U}}_n^{\mathrm{G}}$ is induced by a morphism between the essentially algebraic theory of n-globular sets (*c.f.* Remark 2.4) and the one of strict n-categories. Thus, we get:

4.7. Proposition. *For every $n \in \mathbb{N} \cup \{\omega\}$, the category $\mathbf{Cat}_n$ is locally finitely presentable, complete and cocomplete. Moreover, the functor $\bar{\mathcal{U}}_n^{\mathrm{G}}$ is a right adjoint which preserves directed colimits.*

Proof. The category $\mathbf{Cat}_n$ is locally finitely presentable by Proposition A.28 and in particular cocomplete. It is moreover complete by Proposition A.11. The required properties on $\bar{\mathcal{U}}_n^{\mathrm{G}}$ are a consequence of Proposition A.33. ∎

4.8. Monadicity. While other proofs already exist in the literature (for example [24, Theorem F.2.2]), we give here a hopefully more direct version of the proof that the functors $\bar{\mathcal{U}}_n^{\mathrm{G}}$ are monadic. It uses Beck's monadicity theorem, that we first recall quickly. It relies on the notion of split coequaliser:

4.9. Definition. *Given a category $\mathcal{E}$ and morphisms $f, g\colon X \to Y$ and $h\colon Y \to Z$ in $\mathcal{E}$, we say that h is a* split coequaliser of (f, g) *when there exist $s\colon Z \to Y$ and $t\colon Y \to X$ as in*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[l, bend right=60, "t"'] \arrow[r, "h"] & Z \arrow[l, bend right=60, "s"'] \end{tikzcd}$$

such that $h \circ f = h \circ g$, $h \circ s = \mathrm{id}_Z$, $f \circ t = \mathrm{id}_Y$, and $g \circ t = s \circ h$.

4.10. Remark. In the above definition, it can be shown that h is indeed a coequaliser of f and g.

4.11. Definition. *Given a functor $\mathcal{R}\colon \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$, we say that a pair (f, g) of parallel morphisms $f, g\colon X \to Y$ is* $\mathcal{R}$-split *when there exists a split coequaliser of $(\mathcal{R}(f), \mathcal{R}(g))$.*

Beck's monadicity theorem is then:

4.12. Theorem. *(Beck's monadicity theorem) Given a functor $\mathcal{R}\colon \mathcal{C} \to \mathcal{D}$, the functor $\mathcal{R}$ is monadic if and only if the following conditions are satisfied:*

(i) $\mathcal{R}$ is a right adjoint,

(ii) $\mathcal{R}$ reflects isomorphisms,

(iii) for every $\mathcal{R}$-split pair (f, g) of morphisms $f, g\colon X \to Y$ in C, (f, g) has a coequaliser in $\mathcal{C}$ which is preserved by $\mathcal{R}$.

Proof. See [9, Theorem 4.4.4] or the original work of Beck [5]. ∎

We can then prove the following:

4.13. Proposition. *Given $n \in \mathbb{N} \cup \{\omega\}$, the functor $\bar{\mathcal{U}}^{\mathrm{G}}_n$ is monadic.*

Proof. By Proposition 4.7, $\bar{\mathcal{U}}^{\mathrm{G}}_n$ is a right adjoint. Moreover, given a morphism

$$F\colon C \to D \in \mathbf{Cat}_n,$$

if $F_k\colon C_k \to D_k$ is a bijection for $k < n+1$, then there is a morphism

$$F^{-1}\colon D \to C \in \mathbf{Cat}_n$$

defined by $(F^{-1})_k = (F_k)^{-1}$ for $k < n+1$, so that $\bar{\mathcal{U}}^{\mathrm{G}}_n$ reflects isomorphisms. Now, let $F, G\colon X \to Y$ be two morphisms of $\mathbf{Cat}_n$ such that there exist $Z \in \mathbf{Glob}_n$, and morphisms

$$H\colon \bar{\mathcal{U}}^{\mathrm{G}}_n Y \to Z, \quad S\colon Z \to \bar{\mathcal{U}}^{\mathrm{G}}_n Y \quad \text{and} \quad T\colon \bar{\mathcal{U}}^{\mathrm{G}}_n Y \to \bar{\mathcal{U}}^{\mathrm{G}}_n X$$

of $\mathbf{Glob}_n$, as in

$$\begin{tikzcd} \bar{\mathcal{U}}^{\mathrm{G}}_n X \arrow[r, shift left, "\bar{\mathcal{U}}^{\mathrm{G}}_n(F)"] \arrow[r, shift right, "\bar{\mathcal{U}}^{\mathrm{G}}_n(G)"'] & \bar{\mathcal{U}}^{\mathrm{G}}_n Y \arrow[r, "H"] \arrow[l, bend right, "T"'] & Z \arrow[l, bend right, "S"'] \end{tikzcd}$$

that witness that (F, G) is a $\bar{\mathcal{U}}^{\mathrm{G}}_n$-split coequaliser. We prove that F, G has a coequaliser which is preserved by $\bar{\mathcal{U}}^{\mathrm{G}}_n$. For this purpose, we shall equip Z with a structure of a strict n-category. For $i, k < n+1$ with $i < k$ and $(u, v) \in Z_k \times_i Z_k$, we put

$$u *_i v = H(S(u) *_i S(v))$$

and, given $k < n$ and $u \in C_k$, we put

$$\mathrm{id}^{k+1}_u = H(\mathrm{id}^{k+1}_{S(u)})$$

We verify that the axioms of strict n-categories are verified. Let $k < n$, $u \in Z_k$ and $\epsilon \in \{-,+\}$. We have

$$\begin{aligned}\partial_k^\epsilon(\mathrm{id}_u^{k+1}) &= \partial_k^\epsilon(H(\mathrm{id}_{S(u)}^{k+1})) \\ &= H(\partial_k^\epsilon(\mathrm{id}_{S(u)}^{k+1})) \\ &= H(S(u)) = u\end{aligned}$$

so that Axiom (S-i) is satisfied. Now, let $i, k < n+1$ such that $i < k$, $(u,v) \in Z_k \times_i Z_k$ and $\epsilon \in \{-,+\}$. We have

$$\begin{aligned}\partial_{k-1}^\epsilon(u *_i v) &= H(\partial_{k-1}^\epsilon(S(u) *_i S(v))) \\ &= \begin{cases} H(\partial_{k-1}^\epsilon(S(u)) *_i \partial_{k-1}^\epsilon(S(v))) & \text{if } i < k-1, \\ H(\partial_{k-1}^-(S(u))) & \text{if } i = k-1 \text{ and } \epsilon = -, \\ H(\partial_{k-1}^+(S(v))) & \text{if } i = k-1 \text{ and } \epsilon = +, \end{cases}\end{aligned}$$

so that, by reducing the last expressions, we see that Axiom (S-ii) is satisfied. Now, let $i, k < n+1$ such that $i < k$, and $u \in Z_k$. We have

$$\begin{aligned}\mathrm{id}^k(\partial_i^-(u)) *_i u &= H(S(H(\mathrm{id}^k_{S(\partial_i^-(u))})) *_i S(u)) \\ &= H(S(H(\mathrm{id}^k_{\partial_i^-(S(u))})) *_i SHS(u)) \\ &= H(GT(\mathrm{id}^k_{\partial_i^-(S(u))}) *_i GTS(u)) \\ &= HG(T(\mathrm{id}^k_{\partial_i^-(S(u))}) *_i TS(u)) \\ &= HF(T(\mathrm{id}^k_{\partial_i^-(S(u))}) *_i TS(u)) \\ &= H(FT(\mathrm{id}^k_{\partial_i^-(S(u))}) *_i FTS(u)) \\ &= H(\mathrm{id}^k_{\partial_i^-(S(u))} *_i S(u)) \\ &= H(S(u)) = u\end{aligned}$$

and, similarly, $u *_i \mathrm{id}^k(\partial_i^+(u)) = u$, so that Axiom (S-iii) holds. Now, let $i, k < n+1$ such that $i < k$, and i-composable $u, v, w \in C_k$. We have

$$\begin{aligned}(u *_i v) *_i w &= H(S(H(S(u) *_i S(v))) *_i S(w)) \\ &= H(SH(S(u) *_i S(v)) *_i SHS(w)) \\ &= H(GT(S(u) *_i S(v)) *_i GTS(w)) \\ &= HG(T(S(u) *_i S(v)) *_i TS(w)) \\ &= HF(T(S(u) *_i S(v)) *_i TS(w)) \\ &= H(FT(S(u) *_i S(v)) *_i FTS(w)) \\ &= H((S(u) *_i S(v)) *_i S(w)) \\ &= H(S(u) *_i S(v) *_i S(w))\end{aligned}$$

and, similarly, $u *_i (v *_i w) = H(S(u) *_i S(v) *_i S(w))$. So that Axiom (S-iv) is satisfied. Axioms (S-v) and (S-vi) are proved similarly, so Z is equipped with a structure of a strict n-category.

We now verify that H is a strict n-category morphism. Given $k < n$ and $u \in Y_k$, we have

$$\mathrm{id}^k_{H(u)} = H(\mathrm{id}^k_{SH(u)}) = H(\mathrm{id}^k_u)$$

and, given $i, k < n+1$ with $i < k$, and $(u, v) \in Y_k \times_i Y_k$, we have

$$\begin{aligned} H(u) *_i H(v) &= H(SH(u) *_i SH(v)) \\ &= H(GT(u) *_i GT(v)) \\ &= HG(T(u) *_i T(v)) \\ &= HF(T(u) *_i T(v)) \\ &= H(FT(u) *_i FT(v)) \\ &= H(u *_i v) \end{aligned}$$

so that H is a strict n-category morphism.

We now prove that H is the coequaliser of F and G in $\mathbf{Cat}_n$. Let $K\colon Y \to W$ be an n-functor such that $KF = KG$. Then, since H is the coequaliser of $\bar{\mathcal{U}}^{\mathrm{G}}_n(F)$ and $\bar{\mathcal{U}}^{\mathrm{G}}_n(G)$, there is a unique morphism

$$K'\colon \bar{\mathcal{U}}^{\mathrm{G}}_n Z \to \bar{\mathcal{U}}^{\mathrm{G}}_n W$$

of $\mathbf{Glob}_n$ such that $K'H = K$. We are only left to prove that K' is an n-functor. First, note that we have

$$K' = K'HS = KS \quad \text{and} \quad KSH = KGT = KFT = K.$$

Now, given $k < n$ and $u \in C_k$, we have

$$\begin{aligned} KS(\mathrm{id}^{k+1}_u) &= KSH(\mathrm{id}^{k+1}_{S(u)}) \\ &= K(\mathrm{id}^{k+1}_{S(u)}) \\ &= \mathrm{id}^{k+1}_{KS(u)}. \end{aligned}$$

Moreover, given $i, k < n+1$ with $i < k$, and $(u, v) \in C_k \times C_k$, we have

$$\begin{aligned} KS(u *_i v) &= KSH(S(u) *_i S(v)) \\ &= K(S(u) *_i S(v)) \\ &= KS(u) *_i KS(v), \end{aligned}$$

so that K' is an n-functor. Hence, H is the coequaliser in $\mathbf{Cat}_n$ of F and G. We can conclude with Theorem 4.12. ∎

4.14. TRUNCATION AND INCLUSION FUNCTORS. We now introduce the standard truncation and inclusion functors for strict categories. Contrary to the ones of Section 2.23, they do not refer to the underlying monad of strict categories.

Let $k, l \in \mathbb{N} \cup \{\omega\}$ such that $k < l$.

4.15. DEFINITION. *We write*

$$\mathcal{T}^{\mathrm{C}}_{l,k} \colon \mathbf{Cat}_l \to \mathbf{Cat}_k$$

for the truncation functor*, which maps a strict l-category C to its evident underlying strict k-category, denoted $C_{\leq k}$, and called the k-*truncation of C.

4.16. DEFINITION. *Conversely, there is an inclusion functor*

$$\mathcal{I}^{\mathrm{C}}_{k,l} \colon \mathbf{Cat}_k \to \mathbf{Cat}_l$$

*which maps a strict k-category C to the strict l-category $C_{\uparrow l}$, called the l-*inclusion of C, *and defined by*

$$(C_{\uparrow l})_{\leq k} = C \quad \textit{and} \quad (C_{\uparrow l})_m = C_k$$

for $m < l + 1$ with $k < m$, and such that

- *for $m < l$ with $k \leq m$ and $u \in (C_{\uparrow l})_{m+1}$, $\partial^-_m(u) = \partial^+_m(u) = u$,*
- *for $m < l$ with $k \leq m$ and $u \in (C_{\uparrow l})_m$, $\mathrm{id}^{m+1}_u = u$,*
- *for $i, m < l + 1$ with $i < k < m$ and $(u, v) \in (C_{\uparrow l})_m \times_i (C_{\uparrow l})_m$, $u *_{i,m} v = u *_{i,k} v$,*
- *for $i, m < l + 1$ with $k \leq i < m$ and $(u, v) \in (C_{\uparrow l})_m \times_i (C_{\uparrow l})_m$, $u *_{i,m} v = u = v$.*

4.17. DEFINITION. *We write*

$$\mathrm{i}^{k,l} \colon \mathcal{I}^{\mathrm{C}}_{k,l} \mathcal{T}^{\mathrm{C}}_{l,k} \Rightarrow 1_{\mathbf{Cat}_l}$$

for the natural transformation, given a strict l-category C, the l-functor $\mathrm{i}^{k,l}_C \colon (C_{\leq k})_{\uparrow l} \to C$ is defined by $(\mathrm{i}^{k,l}_C)_{\leq k} = \mathrm{id}_{C_{\leq k}}$ and, for $m < l+1$ with $m > k$, $\mathrm{i}^{k,l}_C$ maps $u \in ((C_{\leq k})_{\uparrow l})_m = C_k$ to id^m_u.

It is straight-forward to verify that:

4.18. PROPOSITION. *There is an adjunction $\mathcal{I}^{\mathrm{C}}_{k,l} \dashv \mathcal{T}^{\mathrm{C}}_{l,k}$ whose unit is the identity and whose counit is $\mathrm{i}^{k,l}$.*

4.19. STRICT CATEGORIES AS GLOBULAR ALGEBRAS. We now derive a monad on globular sets and relate, using the criterion of Theorem 2.38, the strict categories defined as in Definition 4.2 to the globular algebras induced by this monad. We also show that this monad is weakly truncatable using the criterion of Theorem 2.55.

4.20. Definition. *Given $n \in \mathbb{N} \cup \{\omega\}$ we write*

$$\bar{\mathcal{F}}^{\mathrm{G}}_n \colon \mathbf{Glob}_n \to \mathbf{Cat}_n$$

for the left adjoint of the functor $\bar{\mathcal{U}}^{\mathrm{G}}_n$ given by Proposition 4.7.

In particular, the adjunction $\bar{\mathcal{F}}^{\mathrm{G}}_\omega \dashv \bar{\mathcal{U}}^{\mathrm{G}}_\omega$ defines a monad (T, η, μ), which is finitary by Proposition 4.7, and it induces categories of algebras $\mathbf{Alg}_n$ for $n \in \mathbb{N} \cup \{\omega\}$ as explained in Section 2.14. By Proposition 4.13, the comparison functor $H_\omega \colon \mathbf{Cat}_\omega \to \mathbf{Alg}_\omega$ is an equivalence of categories, that moreover satisfies that $\mathcal{U}^{\mathrm{G}}_\omega H_\omega = \bar{\mathcal{U}}^{\mathrm{G}}_\omega$. Using the criterion introduced in Section 2.36, we prove that the other categories $\mathbf{Cat}_n$ are, up to equivalence, the categories of algebras $\mathbf{Alg}_n$:

4.21. Theorem. *There exists a family of equivalences*

$$(H_k \colon \mathbf{Cat}_k \to \mathbf{Alg}_k)_{k \in \mathbb{N}}$$

making the diagrams

$$\begin{tikzcd} \mathbf{Cat}_{k+1} \arrow[r, "H_{k+1}"] \arrow[d, "\mathcal{T}^{\mathrm{C}}_k"'] & \mathbf{Alg}_{k+1} \arrow[d, "\mathcal{T}^{\mathrm{A}}_k"] \\ \mathbf{Cat}_k \arrow[r, "H_k"'] & \mathbf{Alg}_k \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbf{Cat}_{\omega} \arrow[r, "H_{\omega}"] \arrow[d, "\mathcal{T}^{\mathrm{C}}_k"'] & \mathbf{Alg}_{\omega} \arrow[d, "\mathcal{T}^{\mathrm{A}}_k"] \\ \mathbf{Cat}_k \arrow[r, "H_k"'] & \mathbf{Alg}_k \end{tikzcd}$$

commute and such that $\mathcal{U}^{\mathrm{G}}_k H_k = \bar{\mathcal{U}}^{\mathrm{G}}_k$ for every $k \in \mathbb{N}$.

Proof. The unit of the adjunction $\mathcal{I}^{\mathrm{C}}_{n,\omega} \dashv \mathcal{T}^{\mathrm{C}}_n$ is the identity, so that $\mathcal{I}^{\mathrm{C}}_{n,\omega}$ is fully faithful and Theorem 2.38 applies. ∎

Finally, we prove the truncatability of the monad of strict ω-categories:

4.22. Theorem. *The monad (T, η, μ) on $\mathbf{Glob}_\omega$ derived from $\bar{\mathcal{F}}^{\mathrm{G}}_\omega \dashv \bar{\mathcal{U}}^{\mathrm{G}}_\omega$ is weakly truncatable.*

Proof. By Theorems 2.55 and 4.21, it is enough to show that, for every $k \in \mathbb{N}$, the functors $\mathcal{T}^{\mathrm{C}}_{\omega,k}$ have right adjoints $\mathcal{J}^{\mathrm{C}}_{k,\omega}$ such that $\mathrm{j}^k \mathcal{U}^{\mathrm{G}}_\omega \mathcal{J}^{\mathrm{C}}_{k,\omega}$ is an isomorphism, where j^k is the counit of $\mathcal{T}^{\mathrm{G}}_{\omega,k} \dashv \mathcal{J}^{\mathrm{G}}_{k,\omega}$ introduced in Definition 2.12. So let $k \in \mathbb{N}$. Given a strict k-category C, we define a strict ω-category C' whose underlying globular set is the image the underlying k-globular set of C by $\mathcal{J}^{\mathrm{G}}_{k,\omega}$, *i.e.*,

$$C'_{\leq k} = C \quad \text{and} \quad C'_l = \{(u,v) \in (C_k)^2 \mid u, v \text{ are parallel}\} \text{ for } l > k,$$

and we equip C' with a structure of a strict ω-category that extends the one on C by putting

$$\mathrm{id}^{k+1}_u = (u,u) \quad \text{for } u \in C_k, \qquad \mathrm{id}^{l+1}_{(u,v)} = (u,v) \quad \text{for } l \in \mathbb{N} \text{ with } l > k \text{ and } (u,v) \in C'_l,$$

and moreover, for $i, l \in \mathbb{N}$ with $\max(i,k) < l$ and i-composable $(u,v),(u',v') \in C'_l$,

$$(u,v) *_{i,l} (u',v') = \begin{cases} (u *_{i,k} u', v *_{i,k} v') & \text{if } i < k, \\ (u,v') & \text{if } i \geq k. \end{cases}$$

One can show that the axioms of strict ω-categories are verified by C'. Now, let D be a strict ω-category and $F\colon D_{\leq k} \to C$ be a k-functor. By the properties of the adjunction $\mathcal{T}^{\mathrm{G}}_{\omega,k} \dashv \mathcal{J}^{\mathrm{G}}_{k,\omega}$, there is a unique ω-globular morphism $F'\colon D \to C'$ such that $F'_{\leq k} = F$, which is defined by

$$F'(u) = (F(\partial^-_k(u)), F(\partial^+_k(u)))$$

for every $l \in \mathbb{N}$ with $k < l$ and $u \in D_l$. We verify that F' is an ω-functor by checking the compatibility with the id^l and $*_{i,l}$ operations. Given $l \in \mathbb{N}$ with $l \geq k$ and $u \in D_l$, we have

$$F'(\mathrm{id}^{l+1}_u) = (F(\partial^-_k(u)), F(\partial^+_k(u))) = \mathrm{id}^{k+1}_{F'(u)}.$$

Moreover, given $i, l \in \mathbb{N}$ with $\max(i,k) < l$ and i-composable $u, v \in D_l$, we have

$$\begin{aligned} F'(u *_i v) &= \begin{cases} (F(\partial^-_k(u)) *_i F(\partial^-_k(v)), F(\partial^+_k(u)) *_i F(\partial^+_k(v))) & \text{if } i < k \\ (F(\partial^-_k(u)), F(\partial^+_k(v))) & \text{if } i \geq k \end{cases} \\ &= F'(u) *_i F'(v). \end{aligned}$$

Thus, F' is an ω-functor. Hence, the natural bijective correspondence

$$\mathcal{T}^{\mathrm{G}}_{\omega,k}\colon \mathbf{Glob}_\omega(D, C') \to \mathbf{Glob}_k(D_{\leq k}, C)$$

induces a bijective correspondence

$$\mathcal{T}^{\mathrm{C}}_{\omega,k}\colon \mathbf{Cat}_\omega(D, C') \to \mathbf{Cat}_k(D_{\leq k}, C)$$

so that the operation $C \mapsto C'$ extends to a functor $\mathcal{J}^{\mathrm{C}}_{k,\omega}$ which is right adjoint to $\mathcal{T}^{\mathrm{C}}_{\omega,k}$. Moreover, by the definition of C' above, the natural morphism $\mathrm{j}^k \mathcal{U}^{\mathrm{G}}_\omega \mathcal{J}^{\mathrm{C}}_{k,\omega}$ is an isomorphism. Hence, Theorem 2.55 applies and (T, η, μ) is a weakly truncatable monad. ∎

4.23. Remark. We highlight that the criterions given by Theorems 2.38 and 2.55 enabled us to prove that the categories $\mathbf{Cat}_n$ are globular algebras derived from a truncatable monad on $\mathbf{Glob}_\omega$ without giving an explicit description of this monad, which could have been a tedious exercise [30].

4.24. Free constructions. Using Theorems 4.21 and 4.22, we can instantiate the definitions and properties developed in Section 3 to define free constructions on strict n-categories. In particular, for every $n \in \mathbb{N}$, there is a notion of n-cellular extension, with associated category $\mathbf{Cat}^+_n$ defined like $\mathbf{Alg}^+_n$. Moreover, there is a canonical forgetful functor $\mathbf{Cat}_n \to \mathbf{Cat}^+_n$ which has a left adjoint

$$\bar{\mathcal{F}}^{\mathrm{E}}_n\colon \mathbf{Cat}^+_n \to \mathbf{Cat}_n$$

which can be chosen such that $C[X]_{\leq n} = C$ for $(C, X) \in \mathbf{Cat}_n^+$. As was shown in [28], the n-cells of a free extension admit a syntactical description consisting of "well-typed" terms considered up to the axioms of strict categories (*c.f.* Section 4.1).

Using the functors $\bar{\mathcal{F}}_k^{\mathrm{E}}$, we can define, for every $n \in \mathbb{N} \cup \{\omega\}$, a notion of *$n$-polygraph* with associated category $\mathbf{Pol}_n$, and a functor

$$\bar{\mathcal{F}}_n^{\mathrm{P}} \colon \mathbf{Pol}_n \to \mathbf{Cat}_n$$

which maps an n-polygraph P to the free strict n-category P^* induced by the generators contained in P. Note that, when $n > 0$, as a consequence of the compatibility of $\bar{\mathcal{F}}_n^{\mathrm{E}}$ with truncation, the underlying strict $(n{-}1)$-category $(\mathsf{P}^*)_{\leq n-1}$ of P^* is exactly $(\mathsf{P}_{\leq n-1})^*$. As before, the cells of P^* admit a syntactic description as equivalence classes of well-typed terms (see [26] or [12, Proposition 1.4.1.16]).

The above free constructions provide convenient tools to introduce specific instances and define constructions on strict higher categories, as witnessed by the examples below.

4.25. Example. Given $n \in \mathbb{N}$ and a strict n-category C, we have an $(n{+}1)$-cellular extension $(C, C_{/\!/})$ where $C_{/\!/} = \mathcal{B}_n \bar{\mathcal{U}}_n^{\mathrm{G}}(C)$ and, for every $(u, v) \in C_{/\!/}$, we put $\mathrm{d}_n^-(u, v) = u$ and $\mathrm{d}_n^+(u, v) = v$. The image of C by $\mathcal{J}_{n+1}^{\mathrm{C}}$ can be described as the quotient of $C[C_{/\!/}]$ by the following equations:

(i) for all $u \in C_n$,
$$\mathrm{id}_u = (u, u),$$

(ii) for all $i < j \leq n$ and i-composable $u, u' \in C_j$ and $v, v' \in C_j$,
$$(u, v) *_i (u', v') = (u *_i u', v *_i v'),$$

(iii) for all two-by-two parallel $u, v, w \in C_n$,
$$(u, v) *_n (v, w) = (u, w).$$

Concretely, the above quotient takes the form of a coequaliser

$$C[A_i \sqcup A_{ii} \sqcup A_{iii}] \rightrightarrows C[C_{/\!/}] \dashrightarrow C_{\Uparrow n+1}$$

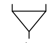
where A_i, A_{ii} and A_{iii} are the sets of instances of the respective axioms (i), (ii), (iii).

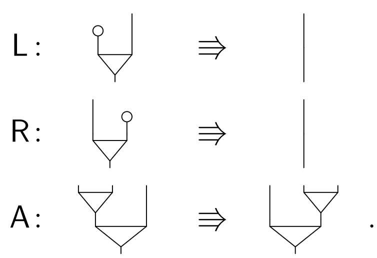
4.26. Example. We define a 3-*polygraph* P that aims at encoding the structure of a pseudomonoid in a 2-monoidal category as follows. We put

$$\mathsf{P}_0 = \{x\} \qquad \mathsf{P}_1 = \{\bar{1} \colon x \to x\} \qquad \mathsf{P}_2 = \{\mu \colon \bar{2} \Rightarrow \bar{1}, \eta \colon \bar{0} \Rightarrow \bar{1}\}$$

where, given $n \in \mathbb{N}$, we write $\bar{n}$ for the composite $\bar{1} *_0 \cdots *_0 \bar{1}$ of n copies of $\bar{1}$, and we define P_3 as the set with the following three elements

$$\begin{array}{rlcl} \mathsf{L}\colon & (\eta *_0 \mathrm{id}^2_{\bar{1}}) *_1 \mu & \Rrightarrow & \mathrm{id}^2_{\bar{1}} \\ \mathsf{R}\colon & (\mathrm{id}^2_{\bar{1}} *_0 \eta) *_1 \mu & \Rrightarrow & \mathrm{id}^2_{\bar{1}} \\ \mathsf{A}\colon & (\mu *_0 \mathrm{id}^2_{\bar{1}}) *_1 \mu & \Rrightarrow & (\mathrm{id}^2_{\bar{1}} *_0 \mu) *_1 \mu\ . \end{array}$$

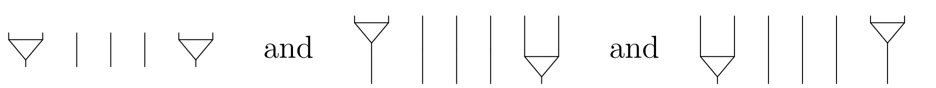
It is convenient to represent the 2-cells of P^* using string diagrams. In this representation, the 2-generators η and μ are represented by  and  respectively, and the 2-cells of the form $\mathrm{id}^2_{\bar{n}}$ are represented by sequences of n wires  for $n \in \mathbb{N}$. Moreover, given $u, v \in \mathsf{P}^*_2$, when u, v are 0-composable (resp. 1-composable), a representation of the 2-cell $u *_0 v$ (resp. $u *_1 v$) is obtained by concatenating horizontally (resp. vertically) representations of u and v. For example, using this representation, the 3-generators L, R and A, can be pictured by



Note that, by Axiom (S-vi), a 2-cell can admit several representations as string diagrams. For example, the 2-cell

$$\mu *_0 \mathrm{id}^2_{\bar{3}} *_0 \mu = (\mu *_0 \mathrm{id}^2_{\bar{5}}) *_1 (\mathrm{id}^2_{\bar{4}} *_0 \mu) = (\mathrm{id}^2_{\bar{5}} *_0 \mu) *_1 (\mu *_0 \mathrm{id}^2_{\bar{4}})$$

can be represented by the three string diagrams



4.27. Example. Given two strict 2-categories C and D, their *Gray tensor product* (more precisely, the *lax* variant, see [16, Section I, 4] for details) is a quotient of the free 2-category G^* on the polygraph G defined as follows:

- the elements of G_0 are the pairs (x, y) where $x \in C_0$ and $y \in D_0$,
- the elements of G_1 are the pairs

$$(f, y)\colon (x, y) \to (x', y) \quad \text{and} \quad (x, g)\colon (x, y) \to (x, y'),$$

  for $f\colon x \to x' \in C_1$ and $g\colon y \to y' \in C_2$,
- the elements of G_2 are the pairs

$$(\phi, y)\colon (f, y) \to (f', y) \quad \text{and} \quad (x, \psi)\colon (x, g) \to (x, g')$$

for $\phi\colon f \Rightarrow f' \in C_2$, $\psi\colon g \Rightarrow g' \in C_2$ and $x, y \in C_0$, and the pairs

$$(f, g)\colon (f, y) *_0 (x', g) \Rightarrow (x, g) *_0 (f, y')$$

which can be pictured as

$$\begin{tikzcd}
(x, y) \arrow[r, "{(f,y)}"] \arrow[d, "{(x,g)}"'] \arrow[dr, phantom, "\Downarrow (f,g)"] & (x', y) \arrow[d, "{(x',g)}"] \\
(x, y') \arrow[r, "{(f,y')}"'] & (x', y')
\end{tikzcd}$$

for $f\colon x \to x' \in C_1$ and $g\colon y \to y' \in C_1$.

The quotient on G^* is made by enforcing several equations on the generated cells. We refer the reader to [13, Section 2.1] for these equations.

More generally, Steiner [33] defined the Gray tensor product of n-categories for $n \geq 2$ as a quotient of free n-categories over particular polygraphs derived from so-called *augmented directed complexes* (see [33, Theorem 7.3] for details).

A. Local presentability

Locally presentable categories are a standard tool for deriving elementary properties on categories of algebraic structures (monoids, groups, but also categories, 2-categories, *etc.*). They are those categories where every object is a directed colimit of "finitely presentable" objects, which are a generalization of the notions of finitely presentable monoids or groups. Knowing that some categories are locally finitely presentable category is helpful since those categories are complete, cocomplete and satisfy other nice properties. For a more complete presentation, we refer to the existing literature [14, 1, 9].

We first recall the definition of locally finitely presentable categories (Section A.1) and then introduce *essentially algebraic theories*, which are a standard tool to show that some categories are locally finitely presentable (Section A.21).

A.1. Presentability. In this section, we define the notion of locally finitely presentable category, after recalling directed colimits and presentable objects of categories.

A.2. Definition. *A partial order* $(D, \leq)$ *is* directed *when* $D \neq \emptyset$ *and for all* $x, y \in D$, *there exists* $z \in D$ *such that* $x \leq z$ *and* $y \leq z$. *A small category* I *is called* directed *when it is isomorphic to a directed partial order* $(D, \leq)$.

Given a category $\mathcal{C} \in \mathbf{CAT}$, *a* diagram *in* $\mathcal{C}$ *is the data of a functor* $d\colon I \to \mathcal{C}$ *where* I *is a small category. We say that it is a* directed diagram *when* I *is moreover directed. A* directed colimit of C *is a colimit cocone* $(p_i\colon d(i) \to X)_{i \in I}$ *on a directed diagram* $d\colon I \to \mathcal{C}$.

A.3. Example. A set is a directed colimit of its finite subsets. A monoid is a directed colimit of its finitely generated submonoids.

In **Set**, we have the following characterisation of directed colimits:

A.4. Proposition. *Let* $d\colon I \to \mathbf{Set}$ *be a directed diagram in* **Set** *and* $(p_i\colon d(i) \to C)_{i \in I}$ *be a cocone on* d. *Then,* $(p_i\colon d(i) \to C)_{i \in I}$ *is a directed colimit on* d *if and only if*

(i) for all $x \in C$, *there is* $i \in I$ *and* $x' \in d(i)$ *such that* $p_i(x') = x$,

(ii) for all $i_1, i_2 \in I$, $x_1 \in d(i_1)$ *and* $x_2 \in d(i_2)$, *if* $p_{i_1}(x_1) = p_{i_2}(x_2)$, *then there exists* $i \in I$ *such that*

$$i_1 \to i \in I, \quad i_2 \to i \in I \quad \text{and} \quad d(i_1 \to i)(x_1) = d(i_2 \to i)(x_2).$$

Proof. See for example [8, Proposition 2.13.3]. ∎

A.5. Definition. *Let* $\mathcal{C} \in \mathbf{CAT}$. *An object* $P \in \mathcal{C}$ *is* finitely presentable *when its hom-functor*

$$\mathcal{C}(P, -)\colon \mathcal{C} \to \mathbf{Set}$$

commutes with directed colimits. By Proposition A.4, it means that, given a directed colimit

$$(p_i\colon d(i) \to X)_{i \in I}$$

on a directed diagram $d\colon I \to \mathcal{C}$, *we have*

(i) *for every* $X \in \mathcal{C}$ *and* $f\colon P \to X$*, there is a* factorisation of f through d*, i.e., there exists* $i \in I$ *and* $g\colon P \to d(i)$ *such that* $f = p_i \circ g$*;*

(ii) *this factorisation is* essentially unique*, i.e., given other* $i' \in I$ *and* $g'\colon P \to d(i)$ *such that* $f = p_{i'} \circ g'$*, then there exist* $j \in I$*,* $h\colon i \to j \in I$ *and* $h'\colon i' \to j \in I$ *such that*

$$d(h) \circ g = d(h') \circ g'.$$

A.6. Example. Given a set S, S is finitely presentable if and only if it is finite. See [1, Example 1.2(1)] for details.

A.7. Example. A monoid is *finitely presentable* when it admits a presentation consisting of a finite number of generators and equations. A similar description of finitely presentable objects holds for the other categories of algebraic structures (groups, rings, etc.). See [1, Theorem 3.12] for details.

A.8. Definition. *A locally small category* $\mathcal{C} \in \mathbf{CAT}$ *is* locally finitely presentable *when*

(i) *it has all small colimits,*

(ii) *every object of* $\mathcal{C}$ *is a directed colimit of locally finitely presentable objects,*

(iii) *the full subcategory of* $\mathcal{C}$ *whose objects are the finitely presentable objects is essentially small.*

A.9. Example. The category **Set** is locally finitely presentable. Indeed, it is cocomplete and every set is a directed colimit of its finite subsets, which are finitely presentable objects of **Set**.

A.10. Example. The category **Mon** of monoids is locally finitely presentable. More generally, the categories of algebraic structures (groups, rings, *etc.*) are locally finitely presentable. This is the consequence of the fact that such categories can be described by means of essentially algebraic theories, as we will see in the next section.

Identifying a category as locally finitely presentable enables to derive several elementary properties, like completeness:

A.11. Proposition. *A locally finitely presentable category is complete.*

Proof. See [1, Corollary 1.28, Remark 1.56(1), Theorem 1.58] for details. ∎

Moreover, showing that a functor between two locally finitely presentable categories is a left or right adjoint is easier than in the general case, since we do not need the existence of solution set like in Freyd's adjoint theorem ([8, Theorem 3.3.3]):

A.12. PROPOSITION. *Given a functor* $F\colon \mathcal{C} \to \mathcal{D}$ *between two locally finitely presentable categories* $\mathcal{C}$ *and* $\mathcal{D}$*, the following hold:*

(i) F *is left adjoint if and only if it preserves colimits,*

(ii) if F *preserves limits and directed colimits, then it is right adjoint.*

Finally, there is a simple criterion for a category of algebras on a monad to be locally finitely presentable. We recall that a functor F is *finitary* when F preserves directed colimits, and a monad (T, η, μ) on a category $\mathcal{C}$ is *finitary* when T is finitary. We then have:

A.13. PROPOSITION. *Given a locally finitely presentable category* $\mathcal{C}$ *and a finitary monad* (T, η, μ) *on* $\mathcal{C}$*, the category of algebras* $\mathcal{C}^T$ *is locally finitely presentable. Moreover, the canonical forgetful functor* $\mathcal{C}^T \to \mathcal{C}$ *preserves directed colimits.*

PROOF. The category $\mathcal{C}^T$ is finitely locally presentable by [1, Theorem 2.78 and the following remark]. Moreover, since T is finitary, the directed colimits of $\mathcal{C}^T$ are computed in $\mathcal{C}$, so that the mentioned forgetful functor preserves directed colimits. ∎

A.14. EXAMPLE. The category **Mon** is equivalent to the category of algebras $\mathbf{Set}^T$ where (T, η, μ) is the free monoid functor on **Set**. It can be shown that T is finitary, so that we obtain another proof that **Mon** is locally finitely presentable using Proposition A.13.

The above definitions and properties can be generalised to higher cardinals in order to define more general notions of presentable categories, in particular for the first non-countable cardinal $\aleph_1 = |\omega_1|$.

A.15. DEFINITION. *A partial order* $(D, \leq)$ *is* ω_1-directed *when every countable subset* $S \subseteq D$ *admits an upper bound in* D*. A small category* I *is called* ω_1-directed *when it is isomorphic to an* ω_1*-directed partial order* $(D, \leq)$*.*

A.16. DEFINITION. *Given a diagram* $d\colon I \to \mathcal{C}$ *in a category* $\mathcal{C}$*, we say that it is an* ω_1-directed diagram *when* I *is* ω_1*-directed. An* ω_1-directed colimit of C *is a colimit cocone* $(p_i\colon d(i) \to X)_{i\in I}$ *on a directed diagram* $d\colon I \to \mathcal{C}$*.*

A.17. DEFINITION. *Let* $\mathcal{C} \in \mathbf{CAT}$*. An object* $P \in \mathcal{C}$ *is* ω_1-presentable *when its hom-functor*

$$\mathcal{C}(P, -)\colon \mathcal{C} \to \mathbf{Set}$$

commutes with ω_1*-directed colimits.*

A.18. Definition. *A locally small category* $\mathcal{C} \in \mathbf{CAT}$ *is* locally ω_1-presentable *when*

(i) it has all small colimits,

(ii) every object of $\mathcal{C}$ *is an* ω_1*-directed colimit of locally* ω_1*-presentable objects,*

(iii) the full subcategory of $\mathcal{C}$ *whose objects are the* ω_1*-presentable objects is essentially small.*

A.19. Proposition. *A locally* ω_1*-presentable category is complete.*

Proof. See [1, Corollary 1.28]. ∎

A.20. Proposition. *Given a functor* $F \colon \mathcal{C} \to \mathcal{D}$ *between two locally* ω_1*-presentable categories* $\mathcal{C}$ *and* $\mathcal{D}$*, the following hold:*

(i) F *is left adjoint if and only if it preserves colimits,*

(ii) if F *preserves limits and* ω_1*-directed colimits, then it is right adjoint.*

We stop the copy-and-paste here and refer the reader to [1] for other properties shared by locally ω_1- and finitely presentable categories.

A.21. Essentially algebraic theories. The verification that some category is locally finitely presentable with the above definition can be tedious. A simpler way consists in describing it as the category of models of some *essentially algebraic theory*. The latter is similar to an algebraic theory (theory of monoids, theory of groups, *etc.*), except that operations with partial domains are allowed, as long as those domains are specified by equations. Another interesting property is that morphisms between such theories induce functors between the associated categories of model, and those functors are moreover right adjoints and preserve directed colimits. The main reference here is [1, Section 3.D].

In the following, let $(x_i)_{i \in \mathbb{N}}$ be a chosen sequence of distinct variable names.

A.22. Definition. *Given a set* S*, an* S-sorted signature *is the data of a set* Σ *of* symbols *such that each* $\sigma \in \Sigma$ *has an* arity *under the form of a finite sequence* $(s_i)_{i \in \{1,\ldots,n\}}$ *of elements of* S *for some* $n \in \mathbb{N}$*, and a* target *in the form of an element* $s \in S$ *and we write*

$$\sigma \colon s_1 \times \cdots \times s_n \to s$$

such a symbol σ *of* Σ *with such arity and target.*

A.23. Definition. *Given a set* S*, an* S-sorted context *is the data of a finite sequence* $\Gamma = (s_i)_{i \in \{1,\ldots,n\}}$ *of elements of* S *for some* $n \in \mathbb{N}$*. Under the context* Γ*, the variable* x_i *should be thought "of type* s_i*" for* $i \in \{1,\ldots,n\}$ *so that we often write*

$$x_1 \colon s_1, \ldots, x_n \colon s_n$$

for such a context Γ*.*

A.24. Definition. *Given a set S and S-sorted signature Σ and context Γ, we define the notions of* Σ-term *on* Γ *together with the one of judgement* $\Gamma \vdash t\colon s$ *where t is a Σ-term and $s \in S$, inductively as follows:*

- *if $\Gamma = (s_i)_{i\in\{1,\dots,n\}}$ for some $n \in \mathbb{N}$ and $s_1,\dots,s_n \in S$, then, for every $i \in \{1,\dots,n\}$, we have $\Gamma \vdash x_i\colon s_i$,*
- *given $\sigma\colon s_1 \times \cdots \times s_n \to s \in \Sigma$ and Σ-terms $t_1,\dots,t_n$ such that $\Gamma \vdash t_i\colon s_i$ for $i \in \{1,\dots,n\}$, then $\Gamma \vdash \sigma(t_1,\dots,t_n)\colon s$.*

Note that s is uniquely determined by t in a judgement $\Gamma \vdash t\colon s$.

A.25. Definition. *An* essentially algebraic theory *is a tuple*

$$\mathbf{T} = (S, \Sigma, E, \Sigma_t, \mathrm{Def})$$

where

- *S is a set,*
- *Σ is an S-sorted signature,*
- *E is a set of triples (Γ, t_1, t_2) where Γ is an S-sorted context, and t_1, t_2 are Σ-terms on Γ such that there exists $s \in S$ so that $\Gamma \vdash t_i\colon s$ for $i \in \{1,2\}$,*
- *Σ_t is a subset of Σ,*
- Def *is a function mapping a symbol $\sigma\colon s_1 \times \cdots \times s_n \to s \in \Sigma \setminus \Sigma_t$ to a set of pairs (t_1,t_2) of Σ_t-terms such that there exists $s \in S$ so that $(x_1\colon s_1,\dots,x_n\colon s_n) \vdash t_i\colon s$ for $i \in \{1,2\}$.*

The set S represents the different *sorts* of the theory, the set Σ the different operations that appear in the theory, the set E the global equations satisfied by the theory, the set Σ_t the operations whose domains are total, and the function Def the equations that define the domains of the partial operations.

A.26. Definition. *Given an essentially algebraic theory* $\mathbf{T}$*, a* model of $\mathbf{T}$*, or* $\mathbf{T}$-model*, is the data of*

- *for all $s \in S$, a set M_s,*
- *for all $\sigma\colon s_1 \times \cdots \times s_n \to s \in \Sigma_t$, a function*

$$M_\sigma\colon M_{s_1} \times \cdots \times M_{s_n} \to M_s,$$

- *for all $\sigma\colon s_1 \times \cdots \times s_n \to s \in \Sigma \setminus \Sigma_t$, a partial function*

$$M_\sigma\colon M_{s_1} \times \cdots \times M_{s_n} \to M_s,$$

such that

- *for all* $\sigma\colon s_1\times\cdots\times s_n \to s \in \Sigma\backslash\Sigma_t$, M_σ *is defined at* $\bar{y} = (y_1,\ldots,y_n) \in M_{s_1}\times\cdots\times M_{s_n}$ *if and only if, for all* $(t_1,t_2) \in \mathrm{Def}(\sigma)$, *we have* $[\![t_1]\!]_{\bar{y}} = [\![t_2]\!]_{\bar{y}}$,
- *for every triple* $(\Gamma, t_1, t_2) \in E$ *where* $\Gamma = (s_i)_{i\in\{1,\ldots,n\}}$ *for some* $n \in \mathbb{N}$ *and some sorts* $s_1,\ldots,s_n \in S$, *given a tuple* $\bar{y} = (y_1,\ldots,y_n) \in M_{s_1}\times\cdots\times M_{s_n}$, *if both* $[\![t_1]\!]_{\bar{y}}$ *and* $[\![t_2]\!]_{\bar{y}}$ *are defined, then* $[\![t_1]\!]_{\bar{y}} = [\![t_2]\!]_{\bar{y}}$,

where, given an S*-sorted context* $\Gamma = (s_i)_{i\in\{1,\ldots,n\}}$, *a sort* $s \in S$, *a* Σ*-term* t *such that* $\Gamma \vdash t\colon s$, *and a tuple* $\bar{y} = (y_1,\ldots,y_n) \in M_{s_1}\times\cdots\times M_{s_n}$, *the* evaluation of t at $\bar{y}$, *denoted* $[\![t]\!]_{\bar{y}}$, *is either undefined or an element of* M_s, *and is defined by induction on* t *by*

- *if* $t = x_i$ *for some* $i \in \{1,\ldots,n\}$, *then* $[\![t]\!]_{\bar{y}}$ *is defined and*
$$[\![t]\!]_{\bar{y}} = y_i,$$
- *if* $t = \sigma(t_1,\ldots,t_k)$ *for some* $k > 0$ *and* Σ_t*-terms* $t_1,\ldots,t_k$, *then* $[\![t]\!]_{\bar{y}}$ *is defined if and only if* $[\![t_1]\!]_{\bar{y}},\ldots,[\![t_k]\!]_{\bar{y}}$ *are defined and* M_σ *is defined at* $[\![t_1]\!]_{\bar{y}},\ldots,[\![t_k]\!]_{\bar{y}}$ *and, in this case,*
$$[\![t]\!]_{\bar{y}} = M_\sigma([\![t_1]\!]_{\bar{y}},\ldots,[\![t_k]\!]_{\bar{y}}).$$

A.27. Definition. *Given two models* M *and* M' *of* $\mathbf{T}$, *a* morphim of $\mathbf{T}$-model *between* M *and* M' *is a family of functions* $f = (f_s\colon M_s \to M'_s)_{s\in S}$ *such that*

- *for all* $\sigma\colon s_1\times\cdots\times s_n \to s \in \Sigma_t$, $f_s \circ M_\sigma = M'_\sigma \circ (f_{s_1}\times\cdots\times f_{s_n})$,
- *for all* $\sigma\colon s_1\times\cdots\times s_n \to s \in \Sigma\setminus\Sigma_t$ *and* $\bar{y} = (y_1,\ldots,y_n) \in M_{s_1}\times\cdots\times M_{s_n}$ *such that* M_σ *is defined on* $\bar{y}$, $f_s \circ M_s(\bar{y}) = M'_s(f_{s_1}(y_1),\ldots,f_{s_n}(y_n))$.

We then write $\mathrm{Mod}(\mathbf{T})$ *for the category of* $\mathbf{T}$*-models and their morphisms. We say that a (big) category* $\mathcal{C} \in \mathbf{CAT}$ *is* essentially algebraic *when it is equivalent to the category of models of some essentially algebraic theory.*

Identifying a category as essentially algebraic enables to deduce that it is locally finitely presentable, since the two notions are the same:

A.28. Theorem. *Given a category* $\mathcal{C} \in \mathbf{CAT}$, $\mathcal{C}$ *is essentially algebraic if and only if it is locally finitely presentable.*

Proof. See the proof of [1, Theorem 3.36]. ∎

A.29. Example. The category **Set** is essentially algebraic since it is the category of models of the essentially algebraic theory $(\{s\}, \emptyset, \emptyset, \emptyset, \perp)$.

A.30. Example. The category **Mon** is essentially algebraic since it is the category of models of the essentially algebraic theory

$$\mathbf{T}^{\mathrm{mon}} = (\{s\}, \{e\colon 1 \to s, m\colon s \times s \to s\}, E, \{e, m\}, \perp)$$

where E consists of three equations

- $m(e, x_1) = x_1$ in the context $(x_1\colon s)$,
- $m(x_1, e) = x_1$ in the context $(x_1\colon s)$,
- $m(m(x_1, x_2), x_3) = m(x_1, m(x_2, x_3))$ in the context $(x_1\colon s, x_2\colon s, x_3\colon s)$.

In particular, it gives a simple proof that **Mon** is locally finitely presentable.

A.31. Example. The category **Cat** of small categories is essentially algebraic since it is the category of models of the essentially algebraic theory $\mathbf{T}^{\mathrm{cat}} = (S, \Sigma, E, \Sigma_t, \mathrm{Def})$ defined as follows. The set S consists of two sorts c_0 and c_1 corresponding to 0-cells and 1-cells, and

$$\Sigma = \{\partial_0^-\colon c_1 \to c_0, \quad \partial_0^+\colon c_1 \to c_0, \quad \mathrm{id}^1\colon c_0 \to c_1, \quad *\colon c_1 \times c_1 \to c_1\}.$$

Moreover, E consists of the equations

- $\partial_0^-(\mathrm{id}^1(x_1)) = x_1$ and $\partial_0^+(\mathrm{id}^1(x_1)) = x_1$ in the context $(x_1\colon c_0)$,
- $\partial_0^-(*(x_1, x_2)) = \partial_0^-(x_1)$ and $\partial_0^+(*(x_1, x_2)) = \partial_0^+(x_2)$ in the context $(x_1\colon c_1, x_2\colon c_1)$,
- $*(\mathrm{id}^1(\partial_0^-(x_1)), x_1) = x_1$ and $*(x_1, \mathrm{id}^1(\partial_0^+(x_1))) = x_1$ in the context $(x_1\colon c_1)$,
- $*(*(x_1, x_2), x_3) = *(x_1, *(x_2, x_3))$ in the context $(x_1\colon c_1, x_2\colon c_1, x_3\colon c_1)$.

Finally, $\Sigma_t = \{\partial_0^-, \partial_0^+, \mathrm{id}^1\}$, and $\mathrm{Def}(*)$ is the singleton set containing the equation $\partial_0^+(x_1) = \partial_0^-(x_2)$. This shows that **Cat** is a locally finitely presentable category.

A.32. Definition. *Given two essentially algebraic theories*

$$\mathbf{T} = (S, \Sigma, E, \Sigma_t, \mathrm{Def}) \quad \textit{and} \quad \mathbf{T}' = (S', \Sigma', E', \Sigma'_t, \mathrm{Def}')$$

a morphism of essential algebraic theories *between* $\mathbf{T}$ *and* $\mathbf{T}'$ *is the data of*

- *a function* $f\colon S \to S'$,
- *a function* $g\colon \Sigma \to \Sigma'$,

such that

- *given* $\sigma\colon s_1 \times \cdots \times s_n \to s \in \Sigma$, *we have* $g(\sigma)\colon f(s_1) \times \cdots \times f(s_n) \to f(s) \in \Sigma'$,

– *given* $\sigma \in \Sigma$, $\sigma \in \Sigma_t$ *if and only if* $g(\sigma) \in \Sigma'_t$,

– *given* $(\Gamma, t_1, t_2) \in E$, *we have* $(f(\Gamma), g(t_1), g(t_2)) \in E'$,

– *given* $\sigma \in \Sigma \setminus \Sigma_t$ *and two* Σ_t*-terms* t_1 *and* t_2, *we have that* $(t_1, t_2) \in \mathrm{Def}(\sigma)$ *if and only if* $(g(t_1), g(t_2)) \in \mathrm{Def}'(g(\sigma))$,

where, given $\Gamma = (s_i)_{i\in\{1,\dots,n\}}$, *we write* $f(\Gamma)$ *for* $(f(s_i))_{i\in\{1,\dots,n\}}$ *and, given a* Σ*-term* t, *we write* $g(t)$ *for the* Σ'*-term defined by induction on* t *by*

– *for all variable* x_i,
$$g(x_i) = x_i,$$

– *for all* $\sigma\colon s_1 \times \cdots \times s_n \to s \in \Sigma$ *and* Σ*-terms* $t_1, \dots, t_n$,
$$g(\sigma(t_1, \dots, t_n)) = g(\sigma)(g(t_1), \dots, g(t_n)).$$

Such a morphism $(f, g)\colon \mathbf{T} \to \mathbf{T}'$ induces a functor
$$\mathrm{Mod}((f, g))\colon \mathrm{Mod}(\mathbf{T}') \to \mathrm{Mod}(\mathbf{T})$$
which maps a model $M' \in \mathrm{Mod}(\mathbf{T}')$ to a model $M \in \mathrm{Mod}(\mathbf{T})$ defined by

– for all $s \in S$, $M_s = M'_{f(s)}$,

– for all $\sigma \in \Sigma$, $M_\sigma = M'_{g(\sigma)}$,

and which maps morphisms of models as expected. The functors induced this way by morphisms between theories have good properties:

A.33. Theorem. *Given a morphism* $(f, g)\colon \mathbf{T} \to \mathbf{T}'$ *between two essentially algebraic theories* $\mathbf{T}$ *and* $\mathbf{T}'$, *the functor* $\mathrm{Mod}((f, g))$ *is a right adjoint which preserves directed colimits.*

Proof. The fact that it is a right adjoint is given by [29, Theorem 5.4]. Moreover, one easily verifies that the directed colimits are computed pointwise in both $\mathrm{Mod}(\mathbf{T})$ and $\mathrm{Mod}(\mathbf{T}')$, so that they are preserved by $\mathrm{Mod}((f, g))$. ∎

A.34. Remark. A more general definition of morphisms between essentially algebraic theories for which Proposition A.33 holds can be defined. However, it would require the introduction of formal deduction systems, which would be quite long and technical. This would be in vain since our definition of morphisms is enough for our purposes.

A.35. Example. One can define the essentially algebraic theory $\mathbf{T}^{\mathrm{grp}}$ of groups from the one of monoids given in Example A.30 by adding a symbol $i\colon s \to s$ representing a total function, and by adding the equations $m(i(x_1), x_1) = e$ and $m(x_1, i(x_1)) = e$ in the context $(x_1\colon s)$. The canonical embedding $\mathbf{T}^{\mathrm{mon}} \to \mathbf{T}^{\mathrm{grp}}$ induces a functor $\mathbf{Grp} \to \mathbf{Mon}$ between the categories of groups and monoids which is the expected forgetful functor. This functor is a right adjoint and preserves directed colimits by Proposition A.33.

A.36. Example. The essentially algebraic theory

$$\mathbf{T}^{\mathrm{gph}} = (\{c_0, c_1\}, \{\mathrm{d}_0^- : c_1 \to c_0, \mathrm{d}_0^+ : c_1 \to c_0\}, \emptyset, \{\mathrm{d}_0^-, \mathrm{d}_0^+\}, \bot)$$

exhibits the category **Gph** of graphs as an essentially algebraic category. Recalling from Example A.31 the definition of $\mathbf{T}^{\mathrm{cat}}$, the mappings $\mathrm{d}_0^- \mapsto \partial_0^-$ and $\mathrm{d}_0^+ \mapsto \partial_0^+$ define a morphism of essentially algebraic theories $\mathbf{T}^{\mathrm{gph}} \to \mathbf{T}^{\mathrm{cat}}$, which induces a functor $\mathbf{Cat} \to \mathbf{Gph}$ that is the expected forgetful functor. This functor is a right adjoint and preserves directed colimits by Proposition A.33.

B. Final and cofinal functors

In this section, we recall the notions of final and cofinal functors, together with elementary properties. We refer the reader to the standard references (like [25, Section IX.3], [8, Section 2.11] and [21, Section 2.5]) for details.

B.1. Definition. *Given a functor $F\colon I \to J$ between two categories I and J and an object $j \in J$, we write $F \downarrow j$ (resp. $j \downarrow F$) for the* slice category *of F above j (resp.* coslice category *of j above F), whose objects are the pairs $(i \in I, f\colon F(i) \to j \in J)$ (resp. the pairs $(i \in I, f\colon j \to F(i) \in J)$), and whose morphisms $(i, f) \to (i', f')$ are the morphisms $g\colon i \to i' \in I$ such that $f' \circ F(g) = f$ (resp. $F(g) \circ f = f'$), and whose identity and composition operations are derived from the ones on I.*

B.2. Definition. *Let I be a category. Write $\sim_I$ for the smallest equivalence relation on the objects of I satisfying $x \sim_I y$ when there exists $f\colon x \to y \in I$ for all $x, y \in I$. A category I is* connected *when there is exactly one equivalence class of objects of I under the equivalence $\sim_I$.*

B.3. Definition. *A functor $F\colon I \to J$ between two categories I and J is* final *(resp.* cofinal*) when, for each $j \in J$, the slice category $F \downarrow j$ (resp. the coslice category $j \downarrow F$) is connected.*

B.4. Remark. Alternatively, a functor $F\colon I \to J$ is cofinal when the opposite functor

$$F^{\mathrm{op}}\colon I^{\mathrm{op}} \to J^{\mathrm{op}}$$

is final.

Final and cofinal functors are useful to simplify the computation of limits and colimits respectively:

B.5. Proposition. *Let $F\colon I \to J$ be a final functor between two categories I and J, $d\colon J \to \mathcal{C}$ be a functor into a category $\mathcal{C}$ and $p\colon \Delta_l \Rightarrow d$ be a cone on d of vertex $l \in C$ (where Δ_l is the constant functor $J \to C$ of value l). Then, p is a limit cone on d if and only if pF is a limit cone on dF. A dual property holds for cofinal functors.*

Proof. See [25, Section IX.3] or [21, Section 2.5]. ■

The cofinal functors between two directed categories (see Definition A.2) are easily characterised:

B.6. Proposition. *Let I and J be two directed categories and $F\colon I \to J$ be a functor. F is cofinal if and only if, for each $j \in J$, there exists $i \in I$ and a morphism $j \to F(i) \in J$.*

Proof. The implication is clear. Conversely, suppose that for each $j \in J$, there exists $i \in I$ and a morphism $j \to F(i) \in J$. Then, given $j \in J$, the category $j \downarrow F$ is not empty. Moreover, given two morphisms $f_1\colon j \to F(i_1)$ and $f_2\colon j \to F(i_2)$ in J for some $i_1, i_2 \in I$, since I is directed, there exist morphisms $g_1\colon i_1 \to i$ and $g_2\colon i_2 \to i$ in I for some $i \in I$, that induce morphisms

$$g_1\colon (i_1, f_1) \to (i, F(g_1) \circ f_1) \quad \text{and} \quad g_2\colon (i_2, f_2) \to (i, F(g_2) \circ f_2)$$

in $j \downarrow F$. Since J is directed, we have $F(g_1) \circ f_1 = F(g_2) \circ f_2$. Thus, $j \downarrow F$ is connected. Hence, F is cofinal. ∎

B.7. Example. Both the categories $(\mathbb{N}, \leq)$ and $(\mathbb{N} \times \mathbb{N}, \leq \times \leq)$ are directed posets. The diagonal functor

$$\Delta\colon (\mathbb{N}, \leq) \to (\mathbb{N} \times \mathbb{N}, \leq \times \leq),$$

mapping $n \in \mathbb{N}$ to $(n, n) \in \mathbb{N}$, is cofinal. Indeed, given $(a, b) \in \mathbb{N} \times \mathbb{N}$, we have

$$(a, b) \leq (\max(a, b), \max(a, b)) = \Delta(\max(a, b))$$

so that Proposition B.6 applies. Now, by Remark B.4, the opposite diagonal functor

$$\Delta^{\mathrm{op}}\colon (\mathbb{N}, \leq)^{\mathrm{op}} \to (\mathbb{N} \times \mathbb{N}, \leq \times \leq)^{\mathrm{op}},$$

is final. In particular, given a complete category $\mathcal{C}$ and a functor $d\colon (\mathbb{N}\times\mathbb{N}, \leq \times \leq)^{\mathrm{op}} \to \mathcal{C}$, the limit of d is the same as the limit of $d\Delta^{\mathrm{op}}$, as a consequence of Proposition B.5.

C. Finite presentability of the categories of polygraphs

Let $n \in \mathbb{N} \cup \{\omega\}$ and (T, η, μ) be a finitary monad on $\mathbf{Glob}_n$. Following Remark 3.39, we now show a refinement of Proposition 3.38(i) stating that the different categories of polygraphs are locally finitely presentable.

In this section, we will use the following notations:

C.1. Notation. *Given $k \leq n$, for $l < k$, we write $\mathsf{P}_{\leq l}$ for $(\mathcal{T}^{\mathrm{P}}_{l+1,l} \cdots \mathcal{T}^{\mathrm{P}}_{k,k-1})(\mathsf{P}) \in \mathbf{Pol}_l$ and, for $l \in \mathbb{N}$ with $k < l \leq n$, we write $\mathsf{P}_{\uparrow l}$ for $\mathcal{I}^{\mathrm{P}}_{l-1,l} \cdots \mathcal{I}^{\mathrm{P}}_{k,k+1}(\mathsf{P}) \in \mathbf{Pol}_l$.*

As one can expect, the finitely presentable objects among polygraphs will be the finite polygraphs:

C.2. Definition. *Given $k \in \mathbb{N} \cup \{\omega\}$ with $k \leq n$, a k-polygraph P is* finite *when the set $\sqcup_{i \leq k} P_i$ is finite.*

Note that the full subcategory of $\mathbf{Pol}_k$ whose objects are the finite k-polygraph is *essentially small*, *i.e.*, there is a set $S \subseteq (\mathbf{Pol}_k)_0$ such that every finite k-polygraph P is isomorphic to an element of S. We prove that $\mathbf{Pol}_k$ is locally finitely presentable by showing that every k-polygraph is a directed colimit of finite k-polygraphs, and that those finite k-polygraphs are precisely the finitely presentable objects of $\mathbf{Pol}_k$. We start with the case $k \in \mathbb{N}$.

C.3. Proposition. *Given $k \in \mathbb{N}$ with $k \leq n$, every k-polygraph is a directed colimit of finite k-polygraphs.*

Proof. We prove this property by induction on k. If $k = 0$ the property holds, since every set is the directed colimit of its finite subsets. So suppose that $k > 0$. Let (P, S) be a k-polygraph and, by induction hypothesis, let $\mathsf{P}^{(-)} \colon I \to \mathbf{Pol}_{k-1}$ be a directed diagram on $\mathbf{Pol}_{k-1}$ together with colimit cocone

$$(F^i \colon \mathsf{P}^i \to \mathsf{P})_{i \in I}.$$

We write J for the small category whose objects are the tuples (i, U, g^-, g^+) where $i \in I$, U is a finite subset of S, and g^-, g^+ are functions of type $U \to (\mathsf{P}^i)^*_{k-1}$ such that

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "g^\epsilon"'] & S \arrow[d, "\mathrm{d}^\epsilon_{k-1}"] \\ (\mathsf{P}^i)^*_{k-1} \arrow[r, "(F^i)^*_{k-1}"'] & \mathsf{P}^*_{k-1} \end{tikzcd}$$

commutes for $\epsilon \in \{-, +\}$, and whose morphisms $(i_1, U_1, g_1^-, g_1^+) \to (i_2, U_2, g_2^-, g_2^+)$ are the morphisms $h \colon i_1 \to i_2 \in I$ such that $U_1 \subseteq U_2$ and

$$\begin{tikzcd} U_1 \arrow[r, hook] \arrow[d, "g_1^\epsilon"'] & U_2 \arrow[d, "g_2^\epsilon"] \\ (\mathsf{P}^{i_1})^*_{k-1} \arrow[r, "(\mathsf{P}^h)^*_{k-1}"'] & (\mathsf{P}^{i_2})^*_{k-1} \end{tikzcd}$$

commutes for $\epsilon \in \{-, +\}$.

We now prove that J is directed. There is a canonical functor $V \colon I \to J$ which maps $i \in I$ to $(i, \emptyset, \bot, \bot) \in J$. Thus, J is not empty since I is directed. Now, given

$$t_1 = (i_1, U_1, g_1^-, g_1^+) \quad \text{and} \quad t_2 = (i_2, U_2, g_2^-, g_2^+)$$

in J, we show that there are morphisms $h_1 \colon t_1 \to t$ and $h_2 \colon t_2 \to t$ in J for some $t \in J$. Since I is directed, there exist $i \in I$ and morphisms $h_1 \colon i_1 \to i$ and $h_2 \colon i_2 \to i$ in I, so we can suppose that $i_1 = i_2 = i$. We then have

$$(F^i)^*(g_1^\epsilon(x)) = \mathrm{d}^\epsilon_{k-1}(x) = (F^i)^*(g_2^\epsilon(x))$$

for all $x \in U_1 \cap U_2$. Note that, since T is finitary, $\mathcal{U}^{\mathrm{G}}_{k-1}$ preserves directed colimits. Thus, by Proposition 3.38(iv), $\mathcal{U}^{\mathrm{G}}_{k-1} \circ \mathcal{F}^{\mathrm{P}}_{k-1}$ preserves directed colimits, so that, since U_1 and U_2 are finite, there is $i' \in I$ and a morphism $h' \colon i \to i' \in I$ such that

$$(\mathsf{P}^{h'})^*(g_1^\epsilon(x)) = (\mathsf{P}^{h'})^*(g_2^\epsilon(x)).$$

Let $g^-, g^+ \colon U_1 \cup U_2 \to (\mathsf{P}^{i'})^*_{k-1}$ be the functions such that

$$g^\epsilon(x) = \begin{cases} g_1^\epsilon(x) & \text{if } x \notin U_2 \\ g_2^\epsilon(x) & \text{if } x \in U_2 \end{cases}$$

for $\epsilon \in \{-,+\}$ and $x \in U_1 \cup U_2$. We then have a tuple $t = (i', U_1 \cup U_2, g^-, g^+) \in J$, and h' induces morphisms of J between t_1 and t, and between t_2 and t. Thus, J is directed.

Now, we consider the functor

$$\mathsf{Q}^{(-)} \colon J \to \mathbf{Pol}_k$$

which maps $t = (i, U, g^-, g^+) \in J$ to the k-polygraph $\mathsf{Q}^t = (\mathsf{P}^i, S^t)$ defined by $S^t = U$ and such that $\mathrm{d}^-_{k-1}, \mathrm{d}^+_{k-1} \colon S^t \to (P^i)^*$ are the functions g^- and g^+ respectively. There is then a cocone

$$((F^i, \iota^t) \colon (\mathsf{P}^i, S^t) \to (\mathsf{P}, S))_{t=(i,U,g^-,g^+) \in J} \tag{23}$$

where $\iota^t \colon S^t \to S$ is the inclusion function $U \hookrightarrow S$ for $t = (i, U, g^-, g^+) \in J$.

We now prove that (23) is a colimit cocone. Given $x \in S$, since $\mathcal{U}^{\mathrm{G}}_{k-1}\mathcal{F}^{\mathrm{P}}_{k-1}$ preserves directed colimits, there are $i_-, i_+ \in I$, $u_- \in (\mathsf{P}^{i_-})^*_{k-1}$ and $u_+ \in (\mathsf{P}^{i_+})^*_{k-1}$ such that

$$(F^{i_\epsilon})^*(u_\epsilon) = \mathrm{d}^\epsilon_{k-1}(x) \quad \text{for } \epsilon \in \{-,+\}.$$

Since I is directed, we can suppose that $i_- = i_+ = i$ for some $i \in I$. Moreover, we have

$$(F^i)^*(\partial^\delta_{k-2}(u_-)) = \partial^\delta_{k-2} \circ \mathrm{d}^-_{k-1}(x) = \partial^\delta_{k-2} \circ \mathrm{d}^+_{k-1}(x) = (F^i)^*(\partial^\delta_{k-2}(u_+))$$

for $\delta \in \{-,+\}$, so that we can suppose that i is big enough such that $\partial^\delta_{k-2}(u_-) = \partial^\delta_{k-2}(u_+)$. Hence, there is $t = (i, \{x\}, g^-, g^+) \in J$ with g^-, g^+ defined by $g^\epsilon(x) = \mathrm{d}^\epsilon_{k-1}(x)$ for $\epsilon \in \{-,+\}$, so that $x = \iota^t(x)$. Moreover, if $x = \iota^{t_1}(x) = \iota^{t_2}(x)$ for some $t_1, t_2 \in J$, then, since J is directed, there exists $t' \in J$ and morphisms $h_1 \colon t_1 \to t'$ and $h_2 \colon t_2 \to t'$ in J so that both $x \in S^{t_1}$ and $x \in S^{t_2}$ are mapped to $x \in S^{t'}$. Thus, by Proposition A.4 and Proposition 3.43, we have

$$(\operatorname*{colim}_{t \in J} \mathsf{Q}^t)_k \cong S.$$

Now, write $W \colon J \to I$ for the functor which maps $(i, U, g^-, g^+) \in J$ to i. In particular, for every $i \in I$, we have $i = W(i, \emptyset, \bot, \bot)$, which is sufficient for W to be a *cofinal* functor (see Definition B.3) since both I and J are directed, as a consequence of Proposition B.6. Then, by Proposition B.5, since

$$\mathcal{T}^{\mathrm{P}}_{k-1}\mathsf{Q}^{(-)} = \mathsf{P}^{(-)}W$$

and since $\mathcal{T}^{\mathrm{P}}_{k-1}$ preserves colimits, we have $(\operatorname{colim}_{t \in J} \mathsf{Q}^t)_{\leq k-1} \cong \mathsf{P}$. Finally, since $\mathcal{T}^{\mathrm{P}}_{k-1}$ and $(-)_k$ are jointly conservative, we have $\operatorname{colim}_{t \in J} \mathsf{Q}^t \cong (\mathsf{P}, S)$. ∎

We now characterise the finitely presentable objects among finite-dimensional polygraphs:

C.4. Proposition. *Given $k \in \mathbb{N}$ with $k \leq n$ and $\mathsf{P} \in \mathbf{Pol}_k$, P is finitely presentable if and only if it is finite.*

Proof. Suppose first that P is finitely presentable. By Proposition C.3, there is a colimit cocone

$$(F^i \colon \mathsf{P}^i \to \mathsf{P})_{i \in I}$$

on some directed diagram $\mathsf{P}^{(-)} \colon I \to \mathbf{Pol}_k$ where P^i is finite for $i \in I$. Since P is finitely presentable, there is a factorisation of $\mathrm{id}_{\mathsf{P}} \colon \mathsf{P} \to \mathsf{P}$ through some P^i, so that we have $F^i \circ F = \mathrm{id}_{\mathsf{P}}$ for some $i \in I$ and $F \colon \mathsf{P} \to \mathsf{P}^i$. Then, for every $j \leq k$, we have $(F^i)_j \circ F_j = \mathrm{id}_{\mathsf{P}_j}$ in $\mathbf{Set}$. Since $(\mathsf{P}^i)_j$ finite, we deduce that P_j is finite for $j \leq k$.

We show converse implication by induction on k. If $k = 0$, the property holds since $\mathbf{Pol}_0 = \mathbf{Set}$ (see Example A.6). So suppose that $k > 0$. Let

$$(G^i \colon \mathsf{Q}^i \to \mathsf{Q})_{i \in I}$$

be a colimit cocone in $\mathbf{Pol}_k$ on a directed diagram

$$d \colon I \to \mathbf{Pol}_k$$

and $F \colon \mathsf{P} \to \mathsf{Q}$ be a morphism in $\mathbf{Pol}_k$. By Proposition 3.38(iii), $\mathcal{T}^{\mathrm{P}}_{k,k-1}$ preserves colimits. Thus, by induction hypothesis, there is $j_1 \in I$ and $\bar{F} \colon \mathsf{P}_{\leq k-1} \to \mathsf{Q}^{j_1}_{\leq k-1} \in \mathbf{Pol}_{k-1}$ such that

$$G^{j_1}_{\leq k-1} \circ \bar{F} = F_{\leq k-1}.$$

By Proposition 3.43 and the fact that P_k is finite, there exists $j_2 \in I$ and a function $f \colon \mathsf{P}_k \to \mathsf{Q}^{j_2}_k \in \mathbf{Set}$ such that

$$G^{j_2}_k \circ f = F_k.$$

Since I is directed, we can suppose $j_1 = j_2 = j$ for some $j \in I$. Moreover, since T is finitary, $\mathcal{U}^{\mathrm{G}}_{k-1}$ preserves directed colimits. Thus, by Proposition 3.38(iv), $\mathcal{U}^{\mathrm{G}}_{k-1} \circ \mathcal{F}^{\mathrm{P}}_{k-1}$ preserves directed colimits, so that we have a colimit cocone

$$((G^i_{\leq k-1})^*_{k-1} \colon (\mathsf{Q}^i_{\leq k-1})^*_{k-1} \to (\mathsf{Q}_{\leq k-1})^*_{k-1})_{i \in I}$$

on $(-)_{k-1}\mathcal{U}^{\mathrm{G}}_{k-1}\mathcal{F}^{\mathrm{P}}_{k-1}\mathsf{Q}^{(-)}$. Note that, for $\epsilon \in \{-,+\}$, the diagram

$$\begin{tikzcd}
\mathsf{P}_k \arrow[r, "f"] \arrow[d, "\mathrm{d}^{\epsilon}_{k-1}"'] & \mathsf{Q}^j_k \arrow[d, "\mathrm{d}^{\epsilon}_{k-1}"] \\
(\mathsf{P}_{\leq k-1})^*_{k-1} \arrow[r, "\bar{F}^*_{k-1}"'] & (\mathsf{Q}^j_{\leq k-1})^*_{k-1}
\end{tikzcd}$$

commutes when postcomposed with $(G^j_{\le k-1})^*_{k-1}$ since

$$\begin{aligned}(G^j_{\le k-1})^*_{k-1} \circ \mathrm{d}^\epsilon_{k-1} \circ f &= \mathrm{d}^\epsilon_{k-1} \circ G^j_k \circ f \\ &= \mathrm{d}^\epsilon_{k-1} \circ F_k \\ &= (F_{\le k-1})^*_{k-1} \circ \mathrm{d}^\epsilon_{k-1} \\ &= (G^j_{\le k-1})^*_{k-1} \circ \bar{F}^*_{k-1} \circ \mathrm{d}^\epsilon_{k-1}\,.\end{aligned}$$

Thus, by the properties of directed colimits and since P_k is finite, up to choosing a bigger $j \in I$, we can suppose that the above diagram commutes for $\epsilon \in \{-,+\}$. So, $(\bar{F}, f)$ is a morphism of $\mathbf{Pol}_k$ of type $\mathsf{P} \to \mathsf{Q}^j$ satisfying

$$F = G^j \circ (\bar{F}, f)$$

and this factorisation can be shown essentially unique using the fact that the factorisations

$$F_{\le k-1} = G^j_{\le k-1} \circ \bar{F} \quad \text{and} \quad F_k = G^j_k \circ f$$

are essentially unique. Hence, P is finitely presentable. ■

C.5. Theorem. *For every $k \in \mathbb{N}$ with $k \le n$, $\mathbf{Pol}_k$ is locally finitely presentable.*

Proof. The category $\mathbf{Pol}_k$ has all colimits by Proposition 3.38(i) and, by Proposition C.3, every k-polygraph is a directed colimit of finite k-polygraphs (that are finitely presentable by Proposition C.4), and the subcategory of finite k-polygraphs is essentially small. Thus, $\mathbf{Pol}_k$ is finitely presentable. ■

Until the end of the section, we suppose that $n = \omega$. We now verify that $\mathbf{Pol}_\omega$ is locally finitely presentable by showing properties similar to the ones of finite-dimensional polygraphs. In order to relate presentability in $\mathbf{Pol}_\omega$ to the one of the $\mathbf{Pol}_k$'s, we first make the following observation about the truncation functors $\mathcal{T}^{\mathrm{P}}_{\omega,k}$:

C.6. Proposition. *For $k \in \mathbb{N}$, the functor $\mathcal{T}^{\mathrm{P}}_{\omega,k} \colon \mathbf{Pol}_\omega \to \mathbf{Pol}_k$ has both a left and a right adjoint.*

Proof. Let $k \in \mathbb{N}$. Define $\mathcal{I}^{\mathrm{P}}_{k,\omega} \colon \mathbf{Pol}_k \to \mathbf{Pol}_\omega$ by the universal property of limit to be the unique functor such that

$$\mathcal{T}^{\mathrm{P}}_{\omega,l}\mathcal{I}^{\mathrm{P}}_{k,\omega} = \begin{cases} \mathcal{I}^{\mathrm{P}}_{k,l} & \text{if } k < l, \\ 1_{\mathbf{Pol}_k} & \text{if } k = l, \\ \mathcal{T}^{\mathrm{P}}_{k,l} & \text{if } k > l. \end{cases}$$

We have that $\mathcal{I}^{\mathrm{P}}_{k,\omega}$ is a left adjoint for $\mathcal{T}^{\mathrm{P}}_{\omega,k}$. Indeed, given $\mathsf{P} \in \mathbf{Pol}_k$, a morphism $\mathsf{P}_{\uparrow\omega} \to \mathsf{Q}$ is the data of a sequence of morphisms $F^l \colon (\mathsf{P}_{\uparrow\omega})_{\le l} \to \mathsf{Q}_{\le l}$ for $l \in \mathbb{N}$ with $l \ge k$ such that $(F^{l+1})_{\le l} = F^l$. But, for $l \in \mathbb{N}$ with $l > k$, we have $(\mathsf{P}_{\uparrow\omega})_{\le l} = \mathsf{P}_{\uparrow l}$ and, by the universal property of $\mathsf{P}_{\uparrow l}$, F^l is completely determined by $F^l_{\le k} = F^k$. So there is a natural correspondence between $\mathbf{Pol}_\omega(\mathsf{P}_{\uparrow\omega}, \mathsf{Q})$ and $\mathbf{Pol}_k(\mathsf{P}, \mathsf{Q}_{\le k})$. Thus, $\mathcal{I}^{\mathrm{P}}_{k,\omega}$ is a left adjoint for $\mathcal{T}^{\mathrm{P}}_{\omega,k}$.

Dually, let $\mathcal{J}^{\mathrm{P}}_{k,\omega} \colon \mathbf{Pol}_k \to \mathbf{Pol}_\omega$ be the functor such that

$$\mathcal{T}^{\mathrm{P}}_{\omega,l}\mathcal{J}^{\mathrm{P}}_{k,\omega} = \begin{cases} \mathcal{J}^{\mathrm{P}}_{k,l} & \text{if } k < l, \\ \mathrm{id}_{\mathbf{Pol}_k} & \text{if } k = l, \\ \mathcal{T}^{\mathrm{P}}_{k,l} & \text{if } k > l. \end{cases}$$

By a similar proof as above, we have that $\mathcal{J}^{\mathrm{P}}_{k,\omega}$ is a right adjoint for $\mathcal{T}^{\mathrm{P}}_{\omega,k}$. ■

C.7. Proposition. *Every ω-polygraph is a directed colimit of finite ω-polygraphs.*

Proof. Let P be an ω-polygraph. Using the proof of Proposition C.3, we can define, by induction on $k \in \mathbb{N}$, small directed categories I^k and diagrams $\mathsf{P}^{k,(-)} \colon I^k \to \mathbf{Pol}_k$ where $\mathsf{P}^{k,i}$ is finite for $i \in I^k$, with colimit cocones

$$(F^{k,i} \colon \mathsf{P}^{k,i} \to \mathsf{P}_{\le k})_{i \in I^k}.$$

In the following, given $k \in \mathbb{N}$, a category C and $x \in C_0$, we denote the constant functor $I^k \to C$ of value x by Δ^k_x . For $k \in \mathbb{N}$, the proof of Proposition C.3 moreover gives functors

$$V^k \colon I^k \to I^{k+1} \quad \text{and} \quad W^k \colon I^{k+1} \to I^k$$

such that $W^k V^k = 1_{I^k}$ (in particular, W^k is cofinal since both I^k and I^{k+1} are directed, as a consequence of Proposition B.6) and such that

$$\mathsf{P}^{k+1,(-)} \circ V^k = \mathcal{I}^{\mathrm{P}}_{k,k+1} \circ \mathsf{P}^{k,(-)}, \tag{24}$$

$$\mathsf{P}^{k,(-)} \circ W^k = \mathcal{T}^{\mathrm{P}}_{k+1,k} \circ \mathsf{P}^{k+1,(-)}, \tag{25}$$

$$F^k W^k = \mathcal{T}^{\mathrm{P}}_{k+1,k} F^{k+1} \tag{26}$$

where $F^k \mathrel{\hat{=}} (F^{k,i})_{i \in I^k}$ is interpreted as a natural transformation $\mathsf{P}^{k,(-)} \Rightarrow \Delta^k_{\mathsf{P}_{\le k}}$. Let

$$(\bar{V}^k \colon I^k \to I)_{k \in \mathbb{N}} \tag{27}$$

be a colimit on the diagram

$$I^0 \xrightarrow{V^0} I^1 \xrightarrow{V^1} I^2 \xrightarrow{V^2} I^3 \xrightarrow{V^3} \cdots$$

We prove that I is directed. First, I is not empty since every I^k is not empty. Now, given $x_1, x_2 \in I$, since the colimit (27) is directed, there is $k \in \mathbb{N}$, $x'_1, x'_2 \in I^k$ such that $\bar{V}^k(x'_i) = x_i$ for $i \in \{1, 2\}$. Thus, since I^k is directed, there exists $x' \in I^k$ and morphisms $h'_i \colon x'_i \to x'$ for $i \in \{1, 2\}$. Hence, we have morphisms $\bar{V}^k(h'_i) \colon x_i \to \bar{V}^k(x')$ for $i \in \{1, 2\}$. Finally, given $x, y \in I_0$ and $f_1, f_2 \colon x \to y \in I$, since (27) is a colimit cocone, there exist $k_i \in \mathbb{N}$, objects x'_i, y'_i and morphisms $f'_i \colon x'_i \to y'_i$ of I^{k_i} such that $\bar{V}^{k_i}(f'_i) = f_i$ for $i \in \{1,2\}$. Since (27) is directed, we can suppose that $k_1 = k_2$, $x'_1 = x'_2$ and $y'_1 = y'_2$. Thus, since I^{k_1} is directed, we have $f'_1 = f'_2$, so that $f_1 = f_2$. Hence, I is directed.

For $k \in \mathbb{N}$, we define a functor $\bar{W}^k \colon I \to I^k$ using the colimit (27) as the unique functor such that

$$\bar{W}^k \circ \bar{V}^l = W^k \circ \cdots \circ W^{l-1}$$

for $l \in \mathbb{N}$ with $l \geq k$. In particular, we have $\bar{W}^k \circ \bar{V}^k = 1_{I^k}$, so that $\bar{W}^k$ is cofinal by Proposition B.6, since both I and I^k are directed).

In the following, given $k \in \mathbb{N}$ and $l \in \{k+1, \omega\}$, we write $\epsilon^{k,l}$ for the counit of the adjunction

$$\mathcal{I}^{\mathrm{P}}_{k,l} \dashv \mathcal{T}^{\mathrm{P}}_{l,k}$$

given by Proposition 3.38(iii) and Proposition C.6. Then, given $k \in \mathbb{N}$, the cocone F^k induces another cocone

$$(\bar{F}^{k,i} \colon (\mathsf{P}^{k,i})_{\uparrow\omega} \to \mathsf{P})_{i \in I^k}$$

on the diagram $\mathcal{I}^{\mathrm{P}}_{k,\omega} \circ \mathsf{P}^{k,(-)}$, where

$$\bar{F}^k = (\epsilon^{k,\omega} \Delta^k_{\mathsf{P}}) \circ (\mathcal{I}^{\mathrm{P}}_{k,\omega} F^k).$$

By (24), we have

$$\mathcal{I}^{\mathrm{P}}_{k+1,\omega} \circ \mathsf{P}^{k+1,(-)} \circ V^k = \mathcal{I}^{\mathrm{P}}_{k,\omega} \circ \mathsf{P}^{k,(-)}$$

so that, by the colimit (27), there exists a unique functor $\mathsf{P}^{(-)} \colon I \to \mathbf{Pol}_\omega$ such that, for $k \in \mathbb{N}$,

$$\mathsf{P}^{(-)} \circ \bar{V}^k = \mathcal{I}^{\mathrm{P}}_{k,\omega} \circ \mathsf{P}^{k,(-)}.$$

In particular, since $\mathsf{P}^{k,i}$ is finite for $k \in \mathbb{N}$ and $i \in I^k$, we have that P^i is a finite ω-polygraph for $i \in I$. Given a category C and $x \in C_0$, we write Δ_x for the constant functor $I \to \mathbf{Pol}_\omega$ of value x. For $k \in \mathbb{N}$, we compute that

$$\begin{aligned}
\bar{F}^{k+1} V^k &= (\epsilon^{k+1,\omega} \Delta^{k+1}_{\mathsf{P}} V^k) \circ (\mathcal{I}^{\mathrm{P}}_{k+1,\omega} F^{k+1} V^k) \\
&= (\epsilon^{k+1,\omega} \Delta^k_{\mathsf{P}}) \circ (\mathcal{I}^{\mathrm{P}}_{k+1,\omega} F^{k+1} V^k) \circ (\mathcal{I}^{\mathrm{P}}_{k+1,\omega} \epsilon^{k,k+1} \mathcal{I}^{\mathrm{P}}_{k,k+1} \mathsf{P}^{k,(-)}) \\
&\qquad \text{(by (24) and since } \partial^-_1(F^{k+1}) = \mathsf{P}^{k+1,(-)} \text{ and } \epsilon^{k,k+1}\mathcal{I}^{\mathrm{P}}_{k,k+1} = \mathbb{1}_{\mathcal{I}^{\mathrm{P}}_{k,k+1}}) \\
&= (\epsilon^{k+1,\omega} \Delta^k_{\mathsf{P}}) \circ (\mathcal{I}^{\mathrm{P}}_{k+1,\omega} \epsilon^{k,k+1} \mathcal{T}^{\mathrm{P}}_{\omega,k+1} \Delta^k_{\mathsf{P}}) \circ (\mathcal{I}^{\mathrm{P}}_{k,\omega} \mathcal{T}^{\mathrm{P}}_{k+1,k} F^{k+1} V^k) \\
&\qquad \text{(by naturality)} \\
&= (\epsilon^{k,\omega} \Delta^k_{\mathsf{P}}) \circ (\mathcal{I}^{\mathrm{P}}_{k,\omega} F^k W^k V^k) \\
&\qquad \text{(since } \epsilon^{k+1,\omega} \circ (\mathcal{I}^{\mathrm{P}}_{k+1,\omega} \epsilon^{k,k+1} \mathcal{T}^{\mathrm{P}}_{\omega,k+1}) = \epsilon^{k,\omega}) \\
&= (\epsilon^{k,\omega} \Delta^k_{\mathsf{P}}) \circ (\mathcal{I}^{\mathrm{P}}_{k,\omega} F^k) \\
&= \bar{F}^k.
\end{aligned}$$

Thus, by the colimit (27), there exists a unique natural transformation $\bar{F}\colon \mathsf{P}^{(-)} \Rightarrow \Delta_{\mathsf{P}}$ such that, for $k \in \mathbb{N}$,

$$\bar{F}\bar{V}^k = \bar{F}^k.$$

We verify that it is a colimit cocone. For $k, l \in \mathbb{N}$ with $k \leq l$, we have

$$\begin{aligned}
\mathcal{T}^{\mathrm{P}}_{\omega,k}\bar{F}\bar{V}^l &= \mathcal{T}^{\mathrm{P}}_{\omega,k}\bar{F}^l \\
&= (\mathcal{T}^{\mathrm{P}}_{\omega,k}\epsilon^{l,\omega}\Delta^l_{\mathsf{P}}) \circ (\mathcal{T}^{\mathrm{P}}_{\omega,k}\mathcal{I}^{\mathrm{P}}_{l,\omega}F^l) \\
&= \mathcal{T}^{\mathrm{P}}_{l,k}F^l && \text{(since } \mathcal{T}^{\mathrm{P}}_{\omega,l}\epsilon^{l,\omega} = \mathbb{1}_{\mathcal{T}^{\mathrm{P}}_{\omega,l}}\text{)} \\
&= F^k W^k \cdots W^{l-1} && \text{(by (26))} \\
&= F^k \bar{W}^k \bar{V}^l
\end{aligned}$$

so that, by the colimit (27), we have $\mathcal{T}^{\mathrm{P}}_{\omega,k}\bar{F} = F^k\bar{W}^k$ for every $k \in \mathbb{N}$. By Proposition B.5, since F^k is a colimit cocone and $\bar{W}^k$ is cofinal,

$$\mathcal{T}^{\mathrm{P}}_{\omega,k}\bar{F}\colon \mathcal{T}^{\mathrm{P}}_{\omega,k}\mathsf{P}^{(-)} \Rightarrow \Delta_{\mathsf{P}_{\leq k}}$$

is a colimit cocone. Since $\mathbf{Pol}_\omega$ is cocomplete by Proposition 3.49, and $\mathcal{T}^{\mathrm{P}}_{\omega,k}$ preserves colimits by Proposition C.6 for every $k \in \mathbb{N}$, and the functors $\mathcal{T}^{\mathrm{P}}_{l,\omega}$ are jointly conservative for $l \in \mathbb{N}$, the latters jointly reflects colimits. Thus, $\bar{F}\colon \mathsf{P}^{(-)} \Rightarrow \Delta_{\mathsf{P}}$ is a colimit cocone. Hence, P is a directed colimit of finite ω-polygraphs. ∎

C.8. Proposition. *Given $\mathsf{P} \in \mathbf{Pol}_\omega$, P is finitely presentable if and only if it is finite.*

Proof. The proof of the first implication is similar to the one of Proposition C.4. So suppose that P is finite. Let

$$(F^i\colon \mathsf{Q}^i \to \mathsf{Q})_{i\in I}$$

be a directed colimit on a diagram $\mathsf{Q}^{(-)}\colon I \to \mathbf{Pol}_\omega$. Since P is finite, $\mathsf{P} = \bar{\mathsf{P}}_{\uparrow\omega}$ for some $k \in \mathbb{N}$ and k-polygraph $\bar{\mathsf{P}}$. Then, since there is an adjunction $\mathcal{I}^{\mathrm{P}}_{k,\omega} \dashv \mathcal{T}^{\mathrm{P}}_{\omega,k}$ by Proposition C.6, we have an isomorphism

$$\mathbf{Pol}_\omega(\mathsf{P},\mathsf{Q}) \cong \mathbf{Pol}_k(\bar{\mathsf{P}},\mathsf{Q}_{\leq k})$$

Since $\mathcal{T}^{\mathrm{P}}_{\omega,k}$ is a left adjoint by Proposition C.6,

$$(F^i_{\leq k}\colon \mathsf{Q}^i_{\leq k} \to \mathsf{Q}_{\leq k})_{i\in I}$$

is a directed colimit on $\mathcal{T}^{\mathrm{P}}_{\omega,k} \circ \mathsf{Q}^{(-)}$. By Proposition C.4, since $\bar{\mathsf{P}}$ is finite, we have an isomorphism

$$\mathbf{Pol}_k(\bar{\mathsf{P}},\mathsf{Q}_{\leq k}) \cong \operatorname*{colim}_{i\in I} \mathbf{Pol}_k(\bar{\mathsf{P}},\mathsf{Q}^i_{\leq k})$$

and, by the properties of the adjunction $\mathcal{I}^{\mathrm{P}}_{k,\omega} \dashv \mathcal{T}^{\mathrm{P}}_{\omega,k}$, we have

$$\operatorname*{colim}_{i\in I} \mathbf{Pol}_k(\bar{\mathsf{P}},\mathsf{Q}^i_{\leq k}) \cong \operatorname*{colim}_{i\in I} \mathbf{Pol}_\omega(\mathsf{P},\mathsf{Q}^i).$$

Hence, P is finitely presentable. ∎

We can conclude that:

C.9. Theorem. *The category* $\mathbf{Pol}_\omega$ *is locally finitely presentable.*

Proof. We already know that $\mathbf{Pol}_\omega$ is locally ω_1-presentable by Proposition 3.49. Moreover, every ω-polygraph can be written as a directed colimit of finite ω-polygraphs by Proposition C.7, that are finitely presentable by Proposition C.8. Finally, it is clear that the full subcategory of finite ω-polygraphs is essentially small. Hence, $\mathbf{Pol}_\omega$ is locally finitely presentable. ∎

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Loria, Campus Scientifique, BP 239, 54506 Vandœuvre-lès-Nancy, France
Email: `simon.forest@loria.fr`

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