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DISTRIBUTIVE LAWS AND HOPF QUASIGROUPS

RAMÓN GONZÁLEZ RODRÍGUEZ

ABSTRACT. In this paper we introduce the notion of a-monoidal distributive law between two Hopf quasigroups A and H in a symmetric monoidal category. We prove that every a-monoidal distributive law induces a product on $A \otimes H$, called the wreath product, such that $A \otimes H$ becomes a Hopf quasigroup. Finally, we show that several well-known constructions fit naturally into this framework. In particular, double cross products of Hopf quasigroups, cross products of Hopf quasigroups with a skew pairing between them, smash products of Hopf quasigroups and twisted smash products of Hopf quasigroups are examples of wreath products associated to a-monoidal distributive laws.

1. Introduction

The theory of distributive laws between monads was initiated by Beck [Beck, 1969] and Barr [Barr, 1969] in the 1970s. A distributive law between two monoids A and H in a symmetric monoidal category $\mathcal{C}$ is a morphism $\Psi : H \otimes A \to A \otimes H$ which is compatible with the monoid structures. It is well-known that a distributive law $\Psi : H \otimes A \to A \otimes H$ induces a monoid structure on the tensor product $A \otimes H$ commuting with the action associated to the product of A on the left and with the action associated to the product of H on the right. This monoid, denoted by $A \otimes_\Psi H$, admits several names in the literature (wreath product, twisted product, smash product, etc.) and its unit and product are defined by

$$\eta_{A\otimes_\Psi H} = \eta_A \otimes \eta_H, \quad \mu_{A\otimes_\Psi H} = (\mu_A \otimes \mu_H) \circ (id_A \otimes \Psi \otimes id_H),$$

where η_A, η_H denote the units of A and H and μ_A, μ_H the corresponding products.

On the other hand, bimonoids are monoids in the category of comonoids. From this point of view, a distributive law in the category of comonoids is a distributive law between the underlying monoids satisfying that is a comonoid morphism. These distributive laws induce a wreath product bimonoid, where the product is the wreath product and the comonoid structure is the one associated to the tensor product comonoid. A relevant example of these wreath products is the double crossed product introduced by Majid (see [Majid, 1995]) which contains, as particular case, the Drinfel'd double of a Hopf algebra.

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Without leaving the associative setting, the theory of distributive laws was extended to the weak context by Street [Street, 2009] and in [Böhm, Gómez-Torrecillas, 2013(1)] and [Böhm, Gómez-Torrecillas, 2013(2)] we can find sufficient conditions under which the corresponding weak wreath product, i.e., the product associated to a weak distributive law, defined as in the first paragraph of this introduction, becomes a weak bimonoid, where the comonoid structure is the tensor product comonoid structure. Also, if the weak bimonoids are weak Hopf monoids, the wreath product is a weak Hopf monoid and, as in the Hopf algebra case, it is possible to describe the Drinfel'd double of a weak Hopf algebra using the wreath product associated to a particular weak distributive law.

Recently, the notion of Hopf quasigroup in a category of vector spaces was introduced by Klim and Majid in [Klim, Majid, 2010] in order to understand the structure and relevant properties of the algebraic 7-sphere, which is the loop of nonzero octonions. Hopf quasigroups are a non-associative generalization of Hopf algebras and they are a particular case of unital coassociative H-bialgebras (see [Pérez-Izquierdo, 2007]) and also of quantum quasigroups (see [Smith, 2016] and [Im, Novak, Smith, 2021]). Hopf quasigroups include as particular cases the loop algebra for a loop L with the inverse property (see [Klim, Majid, 2010], [Bruck, 1946]) and also the enveloping algebra $U(M)$ of a Malcev algebra living in a category of modules over a ring R satisfying suitable conditions (see [Klim, Majid, 2010] and [Pérez-Izquierdo, Shestakov, 2004]). In [Alonso, Fernández, González, 2021] the authors proved that the category of loops with the inverse property, I.P. loops for short, is equivalent to the one of pointed cosemisimple Hopf quasigroups over a given field F and, if F is algebraically closed, the categories of I.P. loops and of cocommutative cosemisimple Hopf quasigroups are equivalent. Also, in [Alonso, Fernández, González, Soneira, 2015(1)] and [Alonso, Fernández, González, Soneira, 2015(2)] was proved that relevant properties of Hopf algebras can be obtained for Hopf quasigroups. Finally, and in a simplified manner, we can say that Hopf quasigroups unify I.P. loops and Malcev algebras in the same way that Hopf algebras unify groups and Lie algebras.

In the literature about Hopf quasigroups we can find several papers devoted to study smash products of Hopf quasigroups (see [Brzeziński, Jiao, 2012(1)], [Brzeziński, Jiao, 2012(2)]), twisted smash products of Hopf quasigroups (see [Fang, Wang, 2011], [Fang, Torrecillas, 2014]), Hopf quasigroups obtained by the twist double method (see [Fang, Torrecillas, 2014]), Hopf quasigroups associated to skew pairings and double cross products of Hopf quasigroups (see [Alonso, Fernández, González, 2020]). It is a remarkable fact that in all these products the magma structure is determined by a morphism Ψ satisfying some conditions that are close to the conditions involved in the definition of distributive law. Motivated by these observations, the main motivation of this paper is to introduce a broad and natural notion of distributive law for Hopf quasigroups that allows us to understand the products cited in the first lines of this paragraph from a unified perspective. In order to do it, firstly, in the third section of this paper, for two Hopf quasigroups A and H, we introduce the notion of a-comonoidal distributive law of H over A and we prove that $A\otimes H$ with the corresponding wreath product, i.e., the product associated to an a-comonoidal distributive law, becomes a Hopf quasigroup, where the comonoid structure is the one of the tensor product comonoid. As examples, in the fourth section, we show that the double cross product of two Hopf quasigroups (see [Alonso, Fernández, González, 2020]) is an example of a wreath product Hopf quasigroup. The same is the cross product associated to a skew pairing (see [Alonso, Fernández, González, 2020]). Moreover, the smash products of Hopf quasigroups in the sense of [Brzeziński, Jiao, 2012(1)], [Brzeziński, Jiao, 2012(2)] are examples of wreath product Hopf quasigroups as well as the twisted smash products defined in [Fang, Wang, 2011] or the construction obtained in [Fang, Torrecillas, 2014, Theorem 3.9].

2. Preliminaries

2.1. Magmas and comonoids in symmetric monoidal categories.

From now on $\mathcal{C}$ denotes a strict symmetric monoidal category with tensor product $\otimes$, unit object K and natural isomorphism of symmetry c. Recall that a monoidal category is a category $\mathcal{C}$ equipped with a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object K of $\mathcal{C}$ and a family of natural isomorphisms

$$a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P),$$

$$r_M : M \otimes K \to M, \quad l_M : K \otimes M \to M,$$

in $\mathcal{C}$ (called associativity, right unit and left unit constraints, respectively) satisfying the Pentagon Axiom and the Triangle Axiom, i.e.,

$$a_{M,N,P\otimes Q} \circ a_{M\otimes N,P,Q} = (id_M \otimes a_{N,P,Q}) \circ a_{M,N\otimes P,Q} \circ (a_{M,N,P} \otimes id_Q),$$

$$(id_M \otimes l_N) \circ a_{M,K,N} = r_M \otimes id_N,$$

where id_X denotes the identity morphism for each object X in $\mathcal{C}$ (see [Mac Lane, 1971]). A monoidal category is called strict if the associativity, right unit and left unit constraints are identities. On the other hand, a strict monoidal category $\mathcal{C}$ is symmetric if it has a natural family of isomorphisms $c_{M,N} : M \otimes N \to N \otimes M$ such that the equalities

$$c_{M,N\otimes P} = (id_N \otimes c_{M,P}) \circ (c_{M,N} \otimes id_P), \quad c_{M\otimes N,P} = (c_{M,P} \otimes id_N) \circ (id_M \otimes c_{N,P}),$$

$$c_{N,M} \circ c_{M,N} = id_{M\otimes N},$$

hold for all M, N in $\mathcal{C}$.

Since every non-strict monoidal category is monoidally equivalent to a strict one (see [Kassel, 1995]), we may assume without loss of generality that the category is strict. Consequently, all results obtained in this paper remain valid for arbitrary symmetric monoidal categories, which includes, for example, the categories of vector spaces over a field F, or the category of left modules over a commutative ring R. In what follows, for simplicity, given objects M, N, P in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

A magma in $\mathcal{C}$ is a pair $A = (A, \mu_A)$, where A is an object in $\mathcal{C}$ and $\mu_A : A \otimes A \to A$ (product) is a morphism in $\mathcal{C}$. A unital magma in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$, where (A, μ_A) is a magma in $\mathcal{C}$ and $\eta_A : K \to A$ (unit) is a morphism in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$. A monoid in $\mathcal{C}$ is a unital magma $A = (A, \eta_A, \mu_A)$ in $\mathcal{C}$ satisfying $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$, i.e., the product μ_A is associative. Given two unital magmas (monoids) A and B, $f : A \to B$ is a morphism of unital magmas (monoids) if $f \circ \eta_A = \eta_B$ and $\mu_B \circ (f \otimes f) = f \circ \mu_A$.

Also, if A, B are unital magmas (monoids) in $\mathcal{C}$, the object $A \otimes B$ is a unital magma (monoid) in $\mathcal{C}$, where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$. If $A = (A, \eta_A, \mu_A)$ is a unital magma so is $A^{op} = (A, \eta_A, \mu_A \circ c_{A,A})$.

A comagma in $\mathcal{C}$ is a pair $D = (D, \delta_D)$, where D is an object in $\mathcal{C}$ and $\delta_D : D \to D \otimes D$ (coproduct) is a morphism in $\mathcal{C}$. A counital comagma in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$, where (D, δ_D) is a comagma in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit) is a morphism in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$. A comonoid in $\mathcal{C}$ is a counital comagma in $\mathcal{C}$ satisfying $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$, i.e., the coproduct δ_D is coassociative. If D and E are counital comagmas (comonoids) in $\mathcal{C}$, $f : D \to E$ is a morphism of counital comagmas (comonoids) if $\varepsilon_E \circ f = \varepsilon_D$, and $(f \otimes f) \circ \delta_D = \delta_E \circ f$.

Moreover, if D, E are counital comagmas (comonoids) in $\mathcal{C}$, the object $D \otimes E$ is a counital comagma (comonoid) in $\mathcal{C}$, where $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$. If $D = (D, \varepsilon_D, \delta_D)$ is a counital comagma so is $D^{cop} = (D, \varepsilon_D, c_{D,D} \circ \delta_D)$.

Let $f : B \to A$ and $g : B \to A$ be morphisms between a comagma B and a magma A. We define the convolution product by $f * g = \mu_A \circ (f \otimes g) \circ \delta_B$. If A is unital and B counital, we will say that f is convolution invertible if there exists $f^{-1} : B \to A$ such that $f * f^{-1} = f^{-1} * f = \varepsilon_B \otimes \eta_A$. Note that, if $B = K$, we have that $f * g = \mu_A \circ (f \otimes g)$ and f is convolution invertible if there exists $f^{-1} : K \to A$ such that $f * f^{-1} = f^{-1} * f = \eta_A$.

2.2. Hopf quasigroups.

In this subsection we recall the definition and the main properties of Hopf quasigroups.

Definition. 2.2.1. *A non-associative bimonoid in $\mathcal{C}$ is a unital magma (H, η_H, μ_H) and a comonoid $(H, \varepsilon_H, \delta_H)$ such that ε_H and δ_H are morphisms of unital magmas (equivalently, η_H and μ_H are morphisms of counital comagmas).*

Then, if H is a non-associative bimonoid, the following identities hold:

$$\varepsilon_H \circ \eta_H = id_K, \tag{1}$$

$$\varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H, \tag{2}$$

$$\delta_H \circ \eta_H = \eta_H \otimes \eta_H, \tag{3}$$

$$\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}. \tag{4}$$

Definition. 2.2.2. *A non-associative bimonoid is called cocommutative if $\delta_H = c_{H,H} \circ \delta_H$, i.e., $H = H^{cop}$ as comonoids.*

Now we recall the notion of Hopf quasigroup in a monoidal setting.

Definition. 2.2.3. *A Hopf quasigroup H in $\mathcal{C}$ is a non-associative bimonoid such that there exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) satisfying*

$$\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \varepsilon_H \otimes H = \mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H) \quad (5)$$

and

$$\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = H \otimes \varepsilon_H = \mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H). \quad (6)$$

Note that composing with the morphism $H \otimes \eta_H$ in (5) we obtain that

$$\lambda_H * id_H = \varepsilon_H \otimes \eta_H, \quad (7)$$

and composing with $\eta_H \otimes H$ in (6) we obtain

$$id_H * \lambda_H = \varepsilon_H \otimes \eta_H. \quad (8)$$

Therefore, if H is a Hopf quasigroup, λ_H is convolution invertible and $\lambda_H^{-1} = id_H$ as in the classical Hopf algebra setting.

The above definition is the monoidal version of the notion of Hopf quasigroup (also called non-associative Hopf algebra with the inverse property, or non-associative IP Hopf algebra) introduced in [Klim, Majid, 2010] (in this case $\mathcal{C}$ is the category of vector spaces over a field F). Note that a Hopf quasigroup is associative, i.e., H is a monoid, if and only if it is a Hopf algebra.

If H is a Hopf quasigroup in $\mathcal{C}$ we know that the antipode λ_H is unique because, if $s_H : H \to H$ is a new morphism satisfying (5) and (6), then

$$s_H = (s_H * id_H) * s_H = (\lambda_H * id_H) * s_H = \mu_H \circ (\mu_H \otimes s_H) \circ (\lambda_H \otimes \delta_H) \circ \delta_H = \lambda_H.$$

Also, by [Klim, Majid, 2010, Proposition 4.2], the antipode is antimultiplicative and anticomultiplicative, i.e.,

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad (9)$$

$$\delta_H \circ \lambda_H = (\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H \quad (10)$$

and preserves the unit and counit (see [López, Villanueva, 2013]), i.e.,

$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H. \quad (11)$$

A morphism between Hopf quasigroups H and B is a morphism $f : H \to B$ of unital magmas and comonoids. Then (see Lemma 1.4 of [Alonso, Fernández, González, Soneira, 2015(1)]) the equality

$$\lambda_B \circ f = f \circ \lambda_H \quad (12)$$

holds.

Example. 2.2.4. *A loop $(L, \cdot, \diagup, \diagdown)$ is a quadruple where L is a set, $\cdot$ (multiplication), $\diagup$ (right division) and $\diagdown$ (left division) are binary operations, satisfying the identities*

$$v \diagdown (v \cdot u) = u, \quad u = (u \cdot v) \diagup v, \quad v \cdot (v \diagdown u) = u, \quad u = (u \diagup v) \cdot v,$$

and such that it contains an identity element e_L (i.e., $e_L \cdot x = x = x \cdot e_L$ hold for all x in L). In what follows multiplications on L will be expressed by juxtaposition.

Let R be a commutative ring and L a loop. Then, the loop algebra

$$RL = \bigoplus_{u \in L} Ru$$

is a cocommutative non-associative bimonoid with unit $\eta_{RL} = 1_R e_L$, product defined by linear extension of the one defined in L, $\delta_{RL}(u) = u \otimes u$ and $\varepsilon_{RL}(u) = 1_R$ on the basis elements (see [Pérez-Izquierdo, 2007]).

Let L be a loop. If for every element $u \in L$ there exists an element $u^{-1} \in L$ (the inverse of u) such that the equalities $u^{-1}(uv) = v = (vu)u^{-1}$ hold for every $v \in L$, we will say that L is a loop with the inverse property (for brevity an IP loop). As a consequence, one easily checks that, if L is an IP loop, for all $u \in L$ the element u^{-1} is unique and $u^{-1}u = e_L = uu^{-1}$. Moreover, $(uv)^{-1} = v^{-1}u^{-1}$ holds for any pair of elements $u, v \in L$.

Now let R be a commutative ring and let L be an IP loop. Then, by [Klim, Majid, 2010, Proposition 4.7], the non-associative bimonoid RL is a cocommutative Hopf quasigroup where the antipode is defined by the linear extension of the map $\lambda_{RL}(u) = u^{-1}$.

Example. 2.2.5. *Let R be a commutative ring with $\frac{1}{2}$ and $\frac{1}{3}$ in R. A Malcev algebra $(M, [,])$ over R is a free R-module M equipped with a bilinear and anticommutative operation $[\ ,\ ]$ such that*

$$[J(a, b, c), a] = J(a, b, [a, c]),$$

where $J(a, b, c) = [[a, b], c] - [[a, c], b] - [a, [b, c]]$ denotes the Jacobian in a, b, c (see [Pérez-Izquierdo, Shestakov, 2004]). Then, every Lie algebra is a Malcev algebra with $J = 0$. The universal enveloping algebra $U(M)$ can be provided with a Hopf quasigroup structure [Klim, Majid, 2010, Proposition 4.8].

3. Wreath Hopf quasigroups

Let A and H be Hopf quasigroups. In this section, we introduce the notion of a-comonoidal distributive law $\Psi : H \otimes A \to A \otimes H$ of H over A, and we prove that the wreath product $A \otimes_\Psi H$ built on $A \otimes H$ with tensor coproduct and unit, product

$$\mu_{A \otimes_\Psi H} = (\mu_A \otimes \mu_H) \circ (A \otimes \Psi \otimes H), \tag{13}$$

and antipode

$$\lambda_{A \otimes_\Psi H} = \Psi \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H}, \tag{14}$$

becomes a Hopf quasigroup.

Definition. 3.1. *Let H, A be Hopf quasigroups. A morphism $\Psi : H\otimes A\to A\otimes H$ is said to be a distributive law of H over A if the following identities*

$$\Psi\circ(H\otimes\mu_A)\circ(\lambda_H\otimes\lambda_A\otimes A)=(\mu_A\otimes H)\circ(A\otimes\Psi)\circ(\Psi\otimes A)\circ(\lambda_H\otimes\lambda_A\otimes A), \quad (15)$$

$$\Psi\circ(\mu_H\otimes A)\circ(H\otimes\lambda_H\otimes\lambda_A)=(A\otimes\mu_H)\circ(\Psi\otimes H)\circ(H\otimes\Psi)\circ(H\otimes\lambda_H\otimes\lambda_A), \quad (16)$$

$$\Psi\circ(H\otimes\eta_A)=\eta_A\otimes H, \quad (17)$$

$$\Psi\circ(\eta_H\otimes A)=A\otimes\eta_H, \quad (18)$$

hold.

If the antipodes of H and A are isomorphisms, the identities (15) and (16) are equivalent to

$$\Psi\circ(H\otimes\mu_A)=(\mu_A\otimes H)\circ(A\otimes\Psi)\circ(\Psi\otimes A), \quad (19)$$

$$\Psi\circ(\mu_H\otimes A)=(A\otimes\mu_H)\circ(\Psi\otimes H)\circ(H\otimes\Psi), \quad (20)$$

respectively. Then, in this situation, the conditions of the definition of distributive law for Hopf quasigroups are the usual conditions appearing in the classical definition of distributive law between monoids, i.e., Ψ is compatible with the unit and the product of A and H.

Definition. 3.2. *Let H, A be Hopf quasigroups and let $\Psi : H\otimes A\to A\otimes H$ be a distributive law of H over A. The distributive law Ψ is said to be comonoidal if it is a comonoid morphism, i.e., the following identities*

$$\delta_{A\otimes H}\circ\Psi=(\Psi\otimes\Psi)\circ\delta_{H\otimes A}, \quad (21)$$

$$(\varepsilon_A\otimes\varepsilon_H)\circ\Psi=\varepsilon_H\otimes\varepsilon_A, \quad (22)$$

hold.

Definition. 3.3. *Let H, A be Hopf quasigroups and let $\Psi : H\otimes A\to A\otimes H$ be a comonoidal distributive law of H over A. We will say that Ψ is an a-comonoidal distributive law of H over A if the following identities*

$$(A\otimes\mu_H)\circ(\Psi\otimes\mu_H)\circ(H\otimes\Psi\otimes H)\circ(((\lambda_H\otimes H)\circ\delta_H)\otimes A\otimes H)=\varepsilon_H\otimes A\otimes H, \quad (23)$$

$$(A\otimes\mu_H)\circ(\Psi\otimes\mu_H)\circ(H\otimes\Psi\otimes H)\circ(((H\otimes\lambda_H)\circ\delta_H)\otimes A\otimes H)=\varepsilon_H\otimes A\otimes H, \quad (24)$$

$$(\mu_A\otimes H)\circ(\mu_A\otimes\Psi)\circ(A\otimes\Psi\otimes A)\circ(A\otimes H\otimes((\lambda_A\otimes A)\circ\delta_A))=A\otimes H\otimes\varepsilon_A, \quad (25)$$

$$(\mu_A\otimes H)\circ(\mu_A\otimes\Psi)\circ(A\otimes\Psi\otimes A)\circ(A\otimes H\otimes((A\otimes\lambda_A)\circ\delta_A))=A\otimes H\otimes\varepsilon_A, \quad (26)$$

hold.

Note that, if H and A are Hopf algebras and $\Psi : H\otimes A\to A\otimes H$ is a distributive law between the monoids H and A, the equalities (23), (24), (25) and (26) always hold.

Theorem. 3.4. *Let A and H be Hopf quasigroups. Let $\Psi : H\otimes A\to A\otimes H$ be an a-comonoidal distributive law of H over A. Then the wreath product $A\otimes_\Psi H$ built on $A\otimes H$ with tensor product unit, product defined by (13), tensor coproduct and antipode defined by (14), is a Hopf quasigroup.*

PROOF. First note that, for $\eta_{A\otimes_\Psi H}=\eta_A\otimes\eta_H$, the identity

$$\mu_{A\otimes_\Psi H}\circ(\eta_{A\otimes_\Psi H}\otimes A\otimes H)=id_{A\otimes H}=\mu_{A\otimes_\Psi H}\circ(A\otimes H\otimes\eta_{A\otimes_\Psi H})$$

follows from (17), (18) and the properties of the units. Then, the triple

$$(A\otimes H,\eta_{A\otimes_\Psi H},\mu_{A\otimes_\Psi H})$$

is a unital magma. On the other hand, $(A\otimes H,\varepsilon_{A\otimes_\Psi H}=\varepsilon_A\otimes\varepsilon_H,\delta_{A\otimes_\Psi H}=\delta_{A\otimes H})$ is a comonoid, $\varepsilon_{A\otimes_\Psi H}\circ\eta_{A\otimes_\Psi H}=id_{A\otimes H}$ follows by (1) for H and A, $\varepsilon_{A\otimes_\Psi H}\circ\mu_{A\otimes_\Psi H}=\varepsilon_{A\otimes_\Psi H}\otimes\varepsilon_{A\otimes_\Psi H}$ follows by (2) for A and H and by (22). Also, $\delta_{A\otimes_\Psi H}\circ\eta_{A\otimes_\Psi H}=\eta_{A\otimes_\Psi H}\otimes\eta_{A\otimes_\Psi H}$ follows by (3) for A and H and by the naturality of c.

On the other hand,

$$\begin{aligned}
&(\mu_{A\otimes_\Psi H}\otimes\mu_{A\otimes_\Psi H})\circ\delta_{A\otimes_\Psi H\otimes A\otimes_\Psi H}\\
&=(\mu_A\otimes\mu_{H\otimes A}\otimes\mu_H)\circ(A\otimes A\otimes c_{A,H}\otimes c_{A,H}\otimes H\otimes H)\circ(A\otimes c_{A,A}\otimes H\otimes A\otimes c_{H,H}\otimes H)\\
&\quad\circ(\delta_A\otimes((\Psi\otimes\Psi)\circ\delta_{H\otimes A})\otimes\delta_H)\ \textit{(by the naturality of c)}\\
&=(\mu_A\otimes\mu_{H\otimes A}\otimes\mu_H)\circ(A\otimes A\otimes c_{A,H}\otimes c_{A,H}\otimes H\otimes H)\circ(A\otimes c_{A,A}\otimes H\otimes A\otimes c_{H,H}\otimes H)\\
&\quad\circ(\delta_A\otimes(\delta_{H\otimes A}\circ\Psi)\otimes\delta_H)\ \textit{(by (21))}\\
&=(A\otimes c_{A,H}\otimes H)\circ(((\mu_A\otimes\mu_A)\circ\delta_{A\otimes A})\otimes((\mu_H\otimes\mu_H)\circ\delta_{H\otimes H}))\circ(A\otimes\Psi\otimes H)\ \textit{(by}\\
&\quad\textit{the naturality of c)}\\
&=\delta_{A\otimes_\Psi H}\circ\mu_{A\otimes_\Psi H}\ \textit{(by (4) for A and H).}
\end{aligned}$$

Moreover, if the antipode is defined as in (77), we have:

- $$\begin{aligned}
&\mu_{A\otimes_\Psi H}\circ(\lambda_{A\otimes_\Psi H}\otimes\mu_{A\otimes_\Psi H})\circ(\delta_{A\otimes_\Psi H}\otimes A\otimes H)\\
&=(A\otimes\mu_H)\circ(((\mu_A\otimes H)\circ(A\otimes\Psi)\circ(\Psi\otimes A)\circ(\lambda_H\otimes\lambda_A\otimes A))\otimes H)\circ(c_{A,H}\otimes\mu_A\otimes\mu_H)\\
&\quad\circ(A\otimes c_{A,H}\otimes\Psi\otimes H)\circ(\delta_A\otimes\delta_H\otimes A\otimes H)\ \textit{(by definition)}\\
&=(A\otimes\mu_H)\circ((\Psi\circ(H\otimes\mu_A)\circ(\lambda_H\otimes\lambda_A\otimes A))\otimes H)\circ(c_{A,H}\otimes\mu_A\otimes\mu_H)\circ(A\otimes c_{A,H}\otimes\Psi\otimes H)\\
&\quad\circ(\delta_A\otimes\delta_H\otimes A\otimes H)\ \textit{(by (15))}\\
&=(A\otimes\mu_H)\circ((\Psi\circ(\lambda_H\otimes(\mu_A\circ(\lambda_A\otimes\mu_A)\circ(\delta_A\otimes A))))\otimes\mu_H)\circ(c_{A,H}\otimes\Psi\otimes H)\\
&\quad\circ(A\otimes\delta_H\otimes A\otimes H)\ \textit{(by the naturality of c)}\\
&=\varepsilon_A\otimes((A\otimes\mu_H)\circ(\Psi\otimes\mu_H)\circ(H\otimes\Psi\otimes H)\circ(((\lambda_H\otimes H)\circ\delta_H)\otimes A\otimes H))\ \textit{(by (5) and}\\
&\quad\textit{the naturality of c)}
\end{aligned}$$

$= \varepsilon_{A\otimes_\Psi H} \otimes A \otimes H$ *(by (23))*,

- $\mu_{A\otimes_\Psi H} \circ (A \otimes H \otimes \mu_{A\otimes_\Psi H}) \circ (A \otimes H \otimes \lambda_{A\otimes_\Psi H} \otimes A \otimes H) \circ (\delta_{A\otimes_\Psi H} \otimes A \otimes H)$

$= (\mu_A\otimes\mu_H)\circ(A\otimes\Psi\otimes\mu_H)\circ(A\otimes H\otimes((\mu_A\otimes H)\circ(A\otimes\Psi)\circ(\Psi\otimes A)\circ(\lambda_H\otimes\lambda_A\otimes A))\otimes H)$

$\circ(A \otimes H \otimes c_{A,H} \otimes A \otimes H) \circ (\delta_{A\otimes H} \otimes A \otimes H)$ *(by definition)*

$= (\mu_A \otimes \mu_H) \circ (A \otimes \Psi \otimes \mu_H) \circ (A \otimes H \otimes (\Psi \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A)) \otimes H)$

$\circ(A \otimes H \otimes c_{A,H} \otimes A \otimes H) \circ (\delta_{A\otimes H} \otimes A \otimes H)$ *(by (15))*

$= (\mu_A \otimes H) \circ (A \otimes ((A \otimes \mu_H) \circ (\Psi \otimes \mu_H) \circ (H \otimes \Psi \otimes H) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes A \otimes H)))$

$\circ(A \otimes H \otimes (\mu_A \circ (\lambda_A \otimes A)) \otimes H) \circ (A \otimes c_{A,H} \otimes A \otimes H) \circ (\delta_A \otimes H \otimes A \otimes H)$ *(by the naturality of c)*

$= ((\mu_A \circ (A \otimes \mu_A) \circ (A \otimes \lambda_A \otimes A) \circ (\delta_A \otimes A)) \otimes H) \circ (A \otimes \varepsilon_H \otimes A \otimes H)$ *(by (24) and the naturality of c)*

$= \varepsilon_{A\otimes_\Psi H} \otimes A \otimes H$ *(by (5) for A)*,

- $\mu_{A\otimes_\Psi H} \circ (\mu_{A\otimes_\Psi H} \otimes A \otimes H) \circ (A \otimes H \otimes \lambda_{A\otimes_\Psi H} \otimes A \otimes H) \circ (A \otimes H \otimes \delta_{A\otimes_\Psi H})$

$= (\mu_A\otimes\mu_H)\circ(\mu_A\otimes\Psi\otimes H)\circ(A\otimes((A\otimes\mu_H)\circ(\Psi\otimes H)\circ(H\otimes\Psi)\circ(H\otimes\lambda_H\otimes\lambda_A))\otimes A\otimes H)$

$\circ(A \otimes H \otimes c_{A,H} \otimes A \otimes H) \circ (A \otimes H \otimes \delta_{A\otimes H})$ *(by definition)*

$= (\mu_A \otimes \mu_H) \circ (\mu_A \otimes \Psi \otimes H) \circ (A \otimes (\Psi \circ (\mu_H \otimes A) \circ (H \otimes \lambda_H \otimes \lambda_A)) \otimes A \otimes H)$

$\circ(A \otimes H \otimes c_{A,H} \otimes A \otimes H) \circ (A \otimes H \otimes \delta_{A\otimes H})$ *(by (16))*

$= (A \otimes \mu_H) \circ (((\mu_A \otimes H) \circ (\mu_A \otimes \Psi) \circ (A \otimes \Psi \otimes A) \circ (A \otimes H \otimes ((\lambda_A \otimes A) \circ \delta_A))) \otimes H)$

$\circ(A \otimes (\mu_H \circ (H \otimes \lambda_H)) \otimes A \otimes H) \circ (A \otimes H \otimes c_{A,H} \otimes H) \circ (A \otimes H \otimes A \otimes \delta_H)$ *(by the naturality of c)*

$= (A \otimes (\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H))) \circ (A \otimes H \otimes \varepsilon_A \otimes H)$ *(by (25) and naturality of c)*

$= A \otimes H \otimes \varepsilon_{A\otimes_\Psi H}$ *(by (6) for H)*,

- $\mu_{A\otimes_\Psi H} \circ (\mu_{A\otimes_\Psi H} \otimes \lambda_{A\otimes_\Psi H}) \circ (A \otimes H \otimes \delta_{A\otimes_\Psi H})$

$= (\mu_A \otimes H) \circ (A \otimes ((A \otimes \mu_H) \circ (\Psi \otimes H) \circ (H \otimes \Psi) \circ (H \otimes \lambda_H \otimes \lambda_A))) \circ (\mu_A \otimes \mu_H \otimes c_{A,H})$

$\circ(A\otimes\Psi\otimes c_{A,H}\otimes H)\circ(A\otimes H\otimes\delta_A\otimes\delta_H)$ *(by definition)*

$= (\mu_A\otimes H)\circ(A\otimes(\Psi\circ(\mu_H\otimes A)\circ(H\otimes\lambda_H\otimes\lambda_A))\circ(\mu_A\otimes\mu_H\otimes c_{A,H})\circ(A\otimes\Psi\otimes c_{A,H}\otimes H)$

$\circ(A\otimes H\otimes\delta_A\otimes\delta_H)$ *(by (16))*

$= (\mu_A\otimes H)\circ(A\otimes\Psi)\circ(\mu_A\otimes(\mu_H\circ(\mu_H\otimes\lambda_H)\circ(H\otimes\delta_H))\otimes\lambda_A)\circ(A\otimes\Psi\otimes c_{A,H})$

$\circ(A\otimes H\otimes\delta_A\otimes H)$ *(by the naturality of c)*

$= ((\mu_A\otimes H)\circ(\mu_A\otimes\Psi)\circ(A\otimes\Psi\otimes A)\circ(A\otimes H\otimes((A\otimes\lambda_A)\circ\delta_A)))\otimes\varepsilon_H$ *(by (6) for H and the naturality of c)*

$= A\otimes H\otimes\varepsilon_{A\otimes_\Psi H}$ *(by (26) for A)*.

Therefore, (5) and (6) hold for $A\otimes_\Psi H$ and the proof is complete. ■

REMARK. 3.5. *Let A and H be Hopf quasigroups. Let $\Psi: H\otimes A\to A\otimes H$ be an a-comonoidal distributive law of H over A and let $A\otimes_\Psi H$ be the wreath product Hopf quasigroup contructed in Theorem 3.4. The morphisms*

$$i_A = A\otimes\eta_H : A\to A\otimes H,\;\; i_H=\eta_A\otimes H: H\to A\otimes H$$

are morphisms of Hopf quasigroups between A and $A\otimes_\Psi H$ and between H and $A\otimes_\Psi H$, respectively. Also, by (17), (18), and the properties of the units of A and H, they satisfy that

$$\mu_{A\otimes_\Psi H}\circ(i_A\otimes i_H)\circ\Psi=\mu_{A\otimes_\Psi H}\circ(i_H\otimes i_A), \tag{27}$$

$$\mu_{A\otimes_\Psi H}\circ(\mu_{A\otimes_\Psi H}\otimes\mu_{A\otimes_\Psi H})\circ(i_A\otimes i_A\otimes i_H\otimes i_H) \tag{28}$$
$$=\mu_{A\otimes_\Psi H}\circ((\mu_{A\otimes_\Psi H}\circ(A\otimes H\otimes\mu_{A\otimes_\Psi H}))\otimes A\otimes H)\circ(i_A\otimes i_A\otimes i_H\otimes i_H),$$

and

$$\mu_{A\otimes_\Psi H}\circ(\mu_{A\otimes_\Psi H}\otimes\mu_{A\otimes_\Psi H})\circ(i_A\otimes i_H\otimes i_A\otimes i_H). \tag{29}$$
$$=\mu_{A\otimes_\Psi H}\circ((\mu_{A\otimes_\Psi H}\circ(A\otimes H\otimes\mu_{A\otimes_\Psi H}))\otimes A\otimes H)\circ(i_A\otimes i_H\otimes i_A\otimes i_H),$$

$$\mu_{A\otimes_\Psi H}\circ(i_A\otimes i_H)=id_{A\otimes H}. \tag{30}$$

The wreath product Hopf quasigroup has the following universal property.

THEOREM. 3.6. *Let A and H be Hopf quasigroups. Let $\Psi: H\otimes A\to A\otimes H$ be an a-comonoidal distributive law of H over A and let $A\otimes_\Psi H$ be the wreath product Hopf quasigroup contructed in Theorem 3.4. Let D be a Hopf quasigroup and $u_A: A\to D$, $u_H: H\to D$ be morphisms of Hopf quasigroups such that*

$$\mu_D\circ(u_A\otimes u_H)\circ\Psi=\mu_D\circ(u_H\otimes u_A), \tag{31}$$

$$\mu_D\circ(\mu_D\otimes\mu_D)\circ(u_A\otimes u_A\otimes u_H\otimes u_H)=\mu_D\circ((\mu_D\circ(D\otimes\mu_D))\otimes D)\circ(u_A\otimes u_A\otimes u_H\otimes u_H), \tag{32}$$

and

$$\mu_D\circ(\mu_D\otimes\mu_D)\circ(u_A\otimes u_H\otimes u_A\otimes u_H) = \mu_D\circ((\mu_D\circ(D\otimes\mu_D))\otimes D)\circ(u_A\otimes u_H\otimes u_A\otimes u_H). \quad (33)$$

Then, there exists an unique morphism $\omega_D : A\otimes H \to D$ of Hopf quasigroups between $A\otimes_\Psi H$ and D such that $\omega_D\circ i_A = u_A$ and $\omega_D\circ i_H = u_H$. The morphism ω_D is given explicitly by $\omega_D = \mu_D\circ(u_A\otimes u_H)$.

Proof. Observe first that the Hopf quasigroup morphisms i_A and and i_H in Remark 3.5 satisfy the conditions (31)-(33) in the statement.

Suppose that $\omega : A\otimes H \to D$ is a morphism of Hopf quasigroups between $A\otimes_\Psi H$ and D such that $\omega\circ i_A = u_A$ and $\omega\circ i_H = u_H$. Then,

$$\begin{aligned}
&\omega\\
&= \omega\circ\mu_{A\otimes_\Psi H}\circ(i_A\otimes i_H) \;\textit{(by (30))}\\
&= \mu_D\circ((\omega\circ i_A)\otimes(\omega\circ i_A)) \;\textit{(by the condition of morphism of Hopf quasigroups for } \omega\textit{)}\\
&= \mu_D\circ(u_A\otimes u_H)) \;\textit{(by the conditions that satisfy } \omega\textit{)}\\
&= \omega_D \;\textit{(by the definition of } \omega_D\textit{)},
\end{aligned}$$

and this shows the uniqueness of ω_D.

One easily checks that $\omega_D\circ\eta_{A\otimes_\Psi H} = \eta_D$ because u_A and u_H are morphisms of Hopf quasigroups. Moreover,

$$\begin{aligned}
&\omega_D\circ\mu_{A\otimes_\Psi H}\\
&= \mu_D\circ((u_A\circ\mu_A)\otimes(u_H\circ\mu_H))\circ(A\otimes\Psi\otimes H) \;\textit{(by the definition of } \omega_D\textit{)}\\
&= \mu_D\circ((\mu_D\circ(u_A\otimes u_A))\otimes(\mu_H\circ(u_H\otimes u_H)))\circ(A\otimes\Psi\otimes H) \;\textit{(by the condition of morphism}\\
&\quad\textit{of Hopf quasigroups for } u_A \textit{ and } u_H\textit{)}\\
&= \mu_D\circ(\mu_D\otimes D)\circ(u_A\otimes(\mu_D\circ(u_A\otimes u_H)\circ\Psi)\otimes u_H) \;\textit{(by (32))}\\
&= \mu_D\circ(\mu_D\otimes D)\circ(u_A\otimes(\mu_D\circ(u_H\otimes u_A))\otimes u_H) \;\textit{(by (31))}\\
&= \mu_D\circ(\omega_D\otimes\omega_D) \;\textit{(by (33))}
\end{aligned}$$

and this shows that ω_D is a morphism of unital magmas.

Finally, using that u_A and u_H are comonoid morphisms, one easily checks that ω_D is a comonoid morphism. Therefore, ω_D is a morphism of Hopf quasigroups. ∎

The universal property obtained in the previous theorem is the Hopf quasigroup version of [Bulacu, Caenepeel, Panaite, Van Oystaeyen, 2019, Proposition 2.9].

4. Examples

In this section, we present several significant examples of structures related to Hopf quasigroups that give rise to a-comonoidal distributive laws. Then, by Theorem 3.4 we obtain that the double cross product $A \bowtie H$ of two Hopf quasigroups A and H, introduced in [Alonso, Fernández, González, 2020], is an example of a wreath product Hopf quasigroup. As a consequence, the cross product $A \bowtie_{\tau} H$ associated to a skew pairing (see [Alonso, Fernández, González, 2020]) also is. Moreover, the R-smash products of Hopf quasigroups introduced in [Brzeziński, Jiao, 2012(1)], [Brzeziński, Jiao, 2012(2)] are examples of wreath product Hopf quasigroups. Finally, the twisted smash products defined in [Fang, Wang, 2011] as well as Hopf quasigroups defined by the twisted double method (twisted smash products for biquasimodule Hopf quasigroups) in [Fang, Torrecillas, 2014] are examples of wreath product Hopf quasigroups.

4.1. Double cross products of Hopf quasigroups.

In this example we will show that the theory of double cross products of Hopf quasigroups, introduced in [Alonso, Fernández, González, 2020], produces examples of a-comonoidal distributive laws.

Let H be a Hopf quasigroup. The pair (M, φ_M) is said to be a left H-quasimodule if M is an object in $\mathcal{C}$ and $\varphi_M : H \otimes M \to M$ is a morphism in $\mathcal{C}$ (called the action) satisfying

$$\varphi_M \circ (\eta_H \otimes M) = id_M \tag{34}$$

and

$$\varphi_M \circ (H \otimes \varphi_M) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes M) = \varepsilon_H \otimes M = \varphi_M \circ (\lambda_H \otimes \varphi_M) \circ (\delta_H \otimes M). \tag{35}$$

Given two left H-quasimodules (M, φ_M), (N, φ_N) and a morphism $f : M \to N$ in $\mathcal{C}$, we will say that f is a morphism of left H-quasimodules if

$$\varphi_N \circ (H \otimes f) = f \circ \varphi_M. \tag{36}$$

We denote the category of left H-quasimodules by ${}_H\mathcal{QC}$. It is easy to prove that, if (M, φ_M) and (N, φ_N) are left H-quasimodules, the tensor product $M \otimes N$ is a left H-quasimodule with the diagonal action

$$\varphi_{M\otimes N} = (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\delta_H \otimes M \otimes N).$$

Thus, the category the category of left H-quasimodules becomes a strict monoidal category (see [Brzeziński, Jiao, 2012(1), Remark 3.3]).

We will say that a unital magma A is a left H-quasimodule magma if it is a left H-quasimodule with action $\varphi_A : H \otimes A \to A$ and the following equalities

$$\varphi_A \circ (H \otimes \eta_A) = \varepsilon_H \otimes \eta_A, \tag{37}$$

$$\mu_A \circ \varphi_{A\otimes A} = \varphi_A \circ (H \otimes \mu_A), \tag{38}$$

hold, i.e., η_A and μ_A are quasimodule morphisms.

A comonoid A is a left H-quasimodule comonoid if it is a left H-quasimodule with action φ_A and

$$\varepsilon_A \circ \varphi_A = \varepsilon_H \otimes \varepsilon_A, \tag{39}$$

$$\delta_A \circ \varphi_A = \varphi_{A\otimes A} \circ (H \otimes \delta_A), \tag{40}$$

hold, i.e., ε_A and δ_A are quasimodule morphisms.

If condition (35) is replaced by the equality

$$\varphi_M \circ (H \otimes \varphi_M) = \varphi_M \circ (\mu_H \otimes M), \tag{41}$$

we have the definition of left H-module and the definitions of left H-module magma and comonoid. Note that the pair (H, μ_H) is not an H-module but it is an H-quasimodule. Morphisms between left H-modules are defined as for H-quasimodules and we denote the category of left H-modules by ${}_H\mathcal{C}$. Obviously we have similar definitions for the right side.

By [Alonso, Fernández, González, 2020, Proposition 5.3, Corollary 5.6], we know that, if A, H are Hopf quasigroups, (A, φ_A) is a left H-module comonoid, (H, ϕ_H) is a right A-module comonoid and

$$\Psi = (\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A},$$

the following assertions are equivalent:

(i) The double cross product $A \bowtie H$ built on the object $A \otimes H$ with product

$$\mu_{A\bowtie H} = (\mu_A \otimes \mu_H) \circ (A \otimes \Psi \otimes H)$$

and tensor product unit, counit and coproduct, is a Hopf quasigroup with antipode

$$\lambda_{A\bowtie H} = \Psi \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H}.$$

(ii) The equalities

$$\varphi_A \circ (H \otimes \eta_A) = \varepsilon_H \otimes \eta_A, \tag{42}$$

$$\phi_H \circ (\eta_H \otimes A) = \eta_H \otimes \varepsilon_A, \tag{43}$$

$$(\phi_H \otimes \varphi_A) \circ \delta_{H\otimes A} = c_{A,H} \circ \Psi, \tag{44}$$

$$\varphi_A \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A) = \mu_A \circ (A \otimes \varphi_A) \circ ((\Psi \circ (\lambda_H \otimes \lambda_A)) \otimes A), \tag{45}$$

$$\mu_H \circ (\phi_H \otimes \mu_H) \circ (\lambda_H \otimes \Psi \otimes H) \circ (\delta_H \otimes A \otimes H) = \varepsilon_H \otimes \varepsilon_A \otimes H, \quad (46)$$

$$\mu_H \circ (\phi_H \otimes \mu_H) \circ (H \otimes \Psi \otimes H) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes A \otimes H) = \varepsilon_H \otimes \varepsilon_A \otimes H, \quad (47)$$

$$\phi_H \circ (\mu_H \otimes A) \circ (H \otimes \lambda_H \otimes \lambda_A) = \mu_H \circ (\phi_H \otimes H) \circ (H \otimes (\Psi \circ (\lambda_H \otimes \lambda_A))), \quad (48)$$

$$\mu_A \circ (\mu_A \otimes \varphi_A) \circ (A \otimes \Psi \otimes \lambda_A) \circ (A \otimes H \otimes \delta_A) = A \otimes \varepsilon_H \otimes \varepsilon_A, \quad (49)$$

$$\mu_A \circ (\mu_A \otimes \varphi_A) \circ (A \otimes \Psi \otimes A) \circ (A \otimes H \otimes ((\lambda_A \otimes A) \circ \delta_A)) = A \otimes \varepsilon_H \otimes \varepsilon_A, \quad (50)$$

hold.

If the previous equalities hold, we can prove that Ψ is an example of a-comonoidal distributive law of H over A. Indeed: First,

$$(\mu_A \otimes H) \circ (A \otimes \Psi) \circ (\Psi \otimes A) \circ (\lambda_H \otimes \lambda_A \otimes A)$$

$$= (\mu_A \otimes H) \circ (A \otimes ((\varphi_A \otimes \phi_H) \circ (H \otimes c_{H,A} \otimes A))) \circ (\varphi_A \otimes (\delta_H \circ \phi_H) \otimes A \otimes A) \circ ((\delta_{A\otimes H} \circ (\lambda_A \otimes \lambda_H)) \otimes \delta_A) \text{ (by definition)}$$

$$= (\mu_A \otimes H) \circ (A \otimes ((\varphi_A \otimes \phi_H) \circ (H \otimes c_{H,A} \otimes A))) \circ (\varphi_A \otimes ((\phi_H \otimes \phi_H) \circ \delta_{H\otimes A}) \otimes A \otimes A) \circ ((\delta_{A\otimes H} \circ (\lambda_A \otimes \lambda_H)) \otimes \delta_A) \text{ (by the condition of right } A\text{-module comonoid for } (H, \phi_H))$$

$$= (\mu_A \otimes H) \circ (A \otimes ((\varphi_A \otimes \phi_H) \circ (H \otimes c_{H,A} \otimes A))) \circ (\Psi \otimes H \otimes A \otimes \mu_A) \circ (H \otimes A \otimes H \otimes c_{A,A} \otimes A) \circ ((\delta_{A\otimes H} \circ (\lambda_A \otimes \lambda_H)) \otimes \delta_A) \text{ (by the condition of right } A\text{-module for } (H, \phi_H)\text{, the naturality of } c\text{, and the coassociativity of } \delta_A \text{ and } \delta_H)$$

$$= ((\mu_A \circ (A \otimes \varphi_A) \circ ((\Psi \circ (\lambda_H \otimes \lambda_A)) \otimes A)) \otimes \phi_H) \circ (H \otimes A \otimes c_{H,A} \otimes \mu_A) \circ (H \otimes c_{H,A} \otimes c_{A,A} \otimes A) \circ ((c_{H,H} \circ (\lambda_H \otimes H) \circ \delta_H) \otimes (c_{A,A} \circ (\lambda_A \otimes A) \circ \delta_A) \otimes \delta_A) \text{ (by (10) for } \lambda_A \text{ and } \lambda_H \text{ and the naturality of } c)$$

$$= ((\varphi_A \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A)) \otimes A)) \otimes \phi_H) \circ (H \otimes A \otimes c_{H,A} \otimes \mu_A) \circ (H \otimes c_{H,A} \otimes c_{A,A} \otimes A) \circ ((c_{H,H} \circ (\lambda_H \otimes H) \circ \delta_H) \otimes (c_{A,A} \circ (\lambda_A \otimes A) \circ \delta_A) \otimes \delta_A) \text{ (by (45))}$$

$$= (\varphi_A \otimes H) \circ (H \otimes \mu_A \otimes \phi_H) \otimes (H \otimes A \otimes c_{H,A} \otimes \mu_A) \circ (H \otimes A \otimes H \otimes c_{A,A} \otimes A) \circ ((\delta_{A\otimes H} \circ (\lambda_A \otimes \lambda_H)) \otimes \delta_A) \text{ (by (10) for } \lambda_A \text{ and } \lambda_H \text{ and the naturality of } c)$$

$$= (\varphi_A \otimes \phi_H) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes ((\mu_A \otimes \mu_A) \circ \delta_{A\otimes A})) \circ (\lambda_H \otimes \lambda_A \otimes A) \text{ (by the naturality of } c)$$

$$= \Psi \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A) \text{ (by (4) for } A)$$

and then (15) holds. Second, (16) holds because

$$(A\otimes\mu_H)\circ(\Psi\otimes H)\circ(H\otimes\Psi)\circ(H\otimes\lambda_H\otimes\lambda_A)$$

$$=(A\otimes\mu_H)\circ(((\varphi_A\otimes\phi_H)\circ(H\otimes c_{H,A}\otimes A))\otimes H)\circ(H\otimes H\otimes(\delta_A\circ\varphi_A)\otimes\phi_H)\circ(\delta_H\otimes(\delta_{H\otimes A}\circ(\lambda_H\otimes\lambda_A)))\ \text{(by definition)}$$

$$=(A\otimes\mu_H)\circ(((\varphi_A\otimes\phi_H)\circ(H\otimes c_{H,A}\otimes A))\otimes H)\circ(H\otimes H\otimes((\varphi_A\otimes\varphi_A)\circ\delta_{H\otimes A})\otimes\phi_H)\circ(\delta_H\otimes(\delta_{H\otimes A}\circ(\lambda_H\otimes\lambda_A)))\ \text{(by (40))}$$

$$=(A\otimes\mu_H)\circ(((\varphi_A\otimes\phi_H)\circ(H\otimes c_{H,A}\otimes A))\otimes H)\circ(\mu_H\otimes c_{H,A}\otimes\Psi)\circ(H\otimes c_{H,H}\otimes A\otimes H\otimes A)\circ(\delta_H\otimes(\delta_{H\otimes A}\circ(\lambda_H\otimes\lambda_A)))$$
(by the condition of left H-module for (A,φ_A), the naturality of c, and coassociativity of δ_A and δ_H)

$$=(\varphi_A\otimes(\mu_H\circ(\phi_H\otimes H)\circ(H\otimes(\Psi\circ(\lambda_H\otimes\lambda_A)))))\circ(\mu_H\otimes c_{H,A}\otimes H\otimes A)\circ(H\otimes c_{H,H}\otimes c_{H,A}\otimes A)\circ(\delta_H\otimes(c_{H,H}\circ(H\otimes\lambda_H)\circ\delta_H)\otimes(c_{A,A}\circ(A\otimes\lambda_A)\circ\delta_A))$$
by (10) for λ_A and λ_H and the naturality of c)

$$=(\varphi_A\otimes(\phi_H\circ(\mu_H\otimes A)\circ(H\otimes\lambda_H\otimes\lambda_A)))\circ(\mu_H\otimes c_{H,A}\otimes H\otimes A)\circ(H\otimes c_{H,H}\otimes c_{H,A}\otimes A)\circ(\delta_H\otimes(c_{H,H}\circ(H\otimes\lambda_H)\circ\delta_H)\otimes(c_{A,A}\circ(A\otimes\lambda_A)\circ\delta_A))\ \text{(by (48))}$$

$$=(A\otimes\phi_H)\circ(\varphi_A\otimes\mu_H\otimes A)\otimes(\mu_H\otimes c_{H,A}\otimes H\otimes A)\circ(H\otimes c_{H,H}\otimes A\otimes H\otimes A)\circ(\delta_H\otimes(\delta_{H\otimes A}\circ(\lambda_H\otimes\lambda_A)))$$
(by (10) for λ_A and λ_H and the naturality of c)

$$=(\varphi_A\otimes\phi_H)\circ(H\otimes c_{H,A}\otimes A)\circ(((\mu_H\otimes\mu_H)\circ\delta_{H\otimes H})\otimes\delta_A)\circ(H\otimes\lambda_H\otimes\lambda_A)\ \text{(by naturality of } c)$$

$$=\Psi\circ(\mu_H\otimes A)\circ(H\otimes\lambda_H\otimes\lambda_A)\ \text{(by (4) for } H).$$

On the other hand, (17) follows by

$$\Psi\circ(H\otimes\eta_A)=((\varphi_A\circ(H\otimes\eta_A))\otimes(\phi_H\circ(H\otimes\eta_A)))\circ\delta_H=\eta_A\otimes((\varepsilon_H\otimes H)\circ\delta_H)=\eta_A\otimes H,$$

where the first identity is a consequence of (3) for A and the naturality of c, in the second one we used (42) and the right A-module condition for (H,ϕ_H) and, finally, the third one follows by the properties of the counit of H. Similarly, by (3) for H, the naturality of c, (43), the left A-module condition for (A,φ_H) and the properties of the counit of A we obtain (18).

Therefore Ψ is s a distributive law of H over A. Also, Ψ is comonoidal because, in one hand, (21) follows by

$$\delta_{A\otimes H}\circ\Psi$$

$$= (A \otimes c_{A,H} \otimes H) \circ ((\delta_A \circ \varphi_A) \otimes (\delta_H \circ \phi_H)) \circ \delta_{H\otimes A} \text{ (by definition)}$$

$$= (A \otimes c_{A,H} \otimes H) \circ (\varphi_A \otimes \varphi_A \otimes \phi_H \otimes \phi_H) \circ (\delta_{H\otimes A} \otimes \delta_{H\otimes A}) \circ \delta_{H\otimes A} \text{ (by (40) and the similar property for } \Phi_H)$$

$$= (\varphi_A \otimes (c_{A,H} \circ \Psi) \otimes \phi_H) \circ (H \otimes c_{H,A} \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes H \otimes \delta_A) \circ \delta_{H\otimes A} \text{ (by the naturality of } c \text{ and the coassociativity of } \delta_A \text{ and } \delta_H)$$

$$= (\varphi_A \otimes ((\phi_H \otimes \varphi_A) \circ \delta_{H\otimes A}) \otimes \phi_H) \circ (H \otimes c_{H,A} \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes H \otimes \delta_A) \circ \delta_{H\otimes A} \text{ (by (44))}$$

$$= (\Psi \otimes \Psi) \circ \delta_{H\otimes A} \text{ (by the naturality of } c \text{ and the coassociativity of } \delta_A \text{ and } \delta_H)$$

and, on the other hand, by (39) for φ_A, the similar identity for Φ_H and the properties of the counits ε_H, ε_A we obtain (22).

Finally, we prove that Ψ is a-comonoidal. First, to demonstrate the identities (23), (24), (25) and (25), we need to show that

$$(A \otimes \mu_H) \circ (\Psi \otimes \mu_H) \circ (H \otimes \Psi \otimes H) \tag{51}$$
$$= ((\varphi_A \circ (\mu_H \otimes A)) \otimes (\mu_H \circ (\phi_H \otimes \mu_H) \circ (H \otimes \Psi \otimes H))) \circ (\delta_{H\otimes H\otimes A} \otimes H),$$

$$(\mu_A \otimes H) \circ (\mu_A \otimes \Psi) \circ (H \otimes \Psi \otimes A) \tag{52}$$
$$= ((\mu_A \circ (\mu_A \otimes \varphi_A) \circ (H \otimes \Psi \otimes A)) \otimes (\phi_H \circ (H \otimes \mu_A))) \circ (A \otimes \delta_{H\otimes A\otimes A}),$$

hold. Indeed, the equality (51) follows by

$$(A \otimes \mu_H) \circ (\Psi \otimes \mu_H) \circ (H \otimes \Psi \otimes H)$$

$$= (\varphi_A \otimes (\mu_H \circ (\phi_H \otimes H))) \circ (H \otimes c_{H,A} \otimes A \otimes H) \circ (\delta_H \otimes ((\delta_A \circ \varphi_A) \otimes (\mu_H \circ (\phi_H \otimes H))) \circ (H \otimes \delta_{H\otimes A} \otimes H) \text{ (by definition)}$$

$$= (\varphi_A \otimes (\mu_H \circ (\phi_H \otimes H))) \circ (H \otimes c_{H,A} \otimes A \otimes H) \circ (\delta_H \otimes (((\varphi_A \otimes \varphi_A) \circ \delta_{H\otimes A}) \otimes (\mu_H \circ (\phi_H \otimes H))) \circ (H \otimes \delta_{H\otimes A} \otimes H) \text{ (by (40))}$$

$$= (\varphi_A \otimes (\mu_H \circ (\phi_H \otimes H))) \circ (H \otimes c_{H,A} \otimes A \otimes \mu_H) \circ (\delta_H \otimes ((\varphi_A \otimes \Psi) \circ \delta_{H\otimes A}) \otimes H) \text{(by the coassociativity of } \delta_{H\otimes A})$$

$$= ((\varphi_A \circ (\mu_H \otimes A)) \otimes (\mu_H \circ (\phi_H \otimes \mu_H) \circ (H \otimes \Psi \otimes H))) \circ (\delta_{H\otimes H\otimes A} \otimes H) \text{ (by the naturality of } c \text{ and the condition of left } H\text{-module for } (A, \varphi_A))$$

and similarly, using the condition of right A-module comonoid for (H, ϕ_H) we obtain (52).

The identity (23) follows by

$$(A\otimes\mu_H)\circ(\Psi\otimes\mu_H)\circ(H\otimes\Psi\otimes H)\circ(((\lambda_H\otimes H)\circ\delta_H)\otimes A\otimes H)$$

$$=((\varphi_A\circ(\mu_H\otimes A))\otimes(\mu_H\circ(\phi_H\otimes\mu_H)\circ(H\otimes\Psi\otimes H)))\circ(\delta_{H\otimes H\otimes A}\otimes H)\circ(((\lambda_H\otimes H)$$
$$\circ\delta_H)\otimes A\otimes H) \text{(by (51))}$$

$$=((\varphi_A\circ((\lambda_H*id_H)\otimes A))\otimes(\mu_H\circ(\phi_H\otimes\mu_H)\circ(H\otimes\Psi\otimes H)))\circ(H\otimes c_{H,A}\otimes H\otimes A\otimes H)$$
$$\circ(c_{H,H}\otimes c_{H,A}\otimes A\otimes H)\circ(((\lambda_H\otimes\delta_H)\circ\delta_H)\otimes\delta_A\otimes H)$$ *(by the coassociativity of δ_H, the naturality of c and (10) for λ_H)*

$$=(A\otimes(\mu_H\circ(\phi_H\otimes\mu_H)\circ(H\otimes\Psi\otimes H)))\circ(c_{H,A}\otimes H\otimes A\otimes H)\circ(H\otimes c_{H,A}\otimes H\otimes A)$$
$$\circ(((\lambda_H\otimes H)\circ\delta_H)\otimes\delta_A\otimes H)$$ *(by (7) for H, the counit properties, the naturality of c and the condition of left H-module for A)*

$$=(A\otimes(\mu_H\circ(\phi_H\otimes\mu_H)\circ(\lambda_H\otimes\Psi\otimes H)\circ(\delta_H\otimes A\otimes H)))\circ(c_{A,H}\otimes A\otimes H)\circ(H\otimes\delta_A\otimes H)$$ *(by the naturality of c)*

$$=(A\otimes\varepsilon_H\otimes\varepsilon_A\otimes H)\circ(c_{A,H}\otimes A\otimes H)\circ(H\otimes\delta_A\otimes H)$$ *(by (46))*

$$=\varepsilon_H\otimes A\otimes H$$ *(by the naturality of c and the counit properties).*

Also,

$$(A\otimes\mu_H)\circ(\Psi\otimes\mu_H)\circ(H\otimes\Psi\otimes H)\circ(((H\otimes\lambda_H)\circ\delta_H)\otimes A\otimes H)$$

$$=((\varphi_A\circ(\mu_H\otimes A))\otimes(\mu_H\circ(\phi_H\otimes\mu_H)\circ(H\otimes\Psi\otimes H)))\circ(\delta_{H\otimes H\otimes A}\otimes H)\circ(((H\otimes\lambda_H)$$
$$\circ\delta_H)\otimes A\otimes H)$$ *(by (51))*

$$=((\varphi_A\circ(\mu_H\otimes H)\circ(H\otimes\lambda_H\otimes H))\otimes(\mu_H\circ(\phi_H\otimes\mu_H)\circ(H\otimes\Psi\otimes H)\circ(((H\otimes\lambda_H)$$
$$\circ\delta_H)\otimes A\otimes H)))\circ(H\otimes H\otimes c_{H,A}\otimes A\otimes H)\circ(H\otimes c_{H,H}\otimes A\otimes A\otimes H)\circ(((\delta_H\otimes H)$$
$$\circ\delta_H)\otimes\delta_A\otimes H)$$ *(by the coassociativity of δ_H, the naturality of c and (10) for λ_H)*

$$=((\varphi_A\circ(\mu_H\otimes H)\circ(H\otimes\lambda_H\otimes H))\otimes\varepsilon_H\otimes\varepsilon_A\otimes H)\circ(H\otimes H\otimes c_{H,A}\otimes A\otimes H)$$
$$\circ(H\otimes c_{H,H}\otimes A\otimes A\otimes H)\circ(((\delta_H\otimes H)\circ\delta_H)\otimes\delta_A\otimes H)$$ *(by (47))*

$$=(\varphi_A\circ((id_H*\lambda_H)\otimes A))\otimes H$$ *(by the naturality of c and the counit properties)*

$$=\varepsilon_H\otimes A\otimes H$$ *(by (8) for H and the condition of left H-module for A),*

which proves (24). The proof for (25) is the following:

$$(\mu_A \otimes H) \circ (\mu_A \otimes \Psi) \circ (A \otimes \Psi \otimes A) \circ (A \otimes H \otimes ((\lambda_A \otimes A) \circ \delta_A))$$

$$= ((\mu_A \circ (\mu_A \otimes \varphi_A) \circ (H \otimes \Psi \otimes A)) \otimes (\phi_H \circ (H \otimes \mu_A))) \circ (A \otimes \delta_{H\otimes A\otimes A})$$

$$\circ (A \otimes H \otimes ((\lambda_A \otimes A) \circ \delta_A)) \text{ (by (52))}$$

$$= ((\mu_A \circ (\mu_A \otimes \varphi_A) \circ (A \otimes \Psi \otimes A) \circ (A \otimes H \otimes ((\lambda_A \otimes A) \circ \delta_A))) \otimes (\phi_H \circ (H \otimes \mu_A)))$$

$$\circ (A \otimes H \otimes c_{H,A} \otimes A \otimes A) \circ (A \otimes H \otimes H \otimes c_{A,A} \otimes A) \circ (A \otimes \delta_H \otimes (((\lambda_A \otimes A) \circ \delta_A) \circ \delta_A)$$

(by the coassociativity of δ_A, the naturality of c and (10) for λ_A)

$$= (A \otimes \varepsilon_H \otimes \varepsilon_A \otimes (\phi_H \circ (H \otimes \mu_A))) \circ (A \otimes H \otimes c_{H,A} \otimes A \otimes A) \circ (A \otimes H \otimes H \otimes c_{A,A} \otimes A)$$

$$\circ (A \otimes \delta_H \otimes (((\lambda_A \otimes A) \circ \delta_A) \circ \delta_A) \text{ (by (50))}$$

$$= A \otimes (\phi_H \circ (H \otimes (\lambda_A * id_A))) \text{ (by the naturality of } c \text{ and the counit properties)}$$

$$= A \otimes H \otimes \varepsilon_A \text{ (by (7) for } A \text{ and the condition of right } A\text{-module for } H\text{)},$$

Finally,

$$(\mu_A \otimes H) \circ (\mu_A \otimes \Psi) \circ (A \otimes \Psi \otimes A) \circ (A \otimes H \otimes ((A \otimes \lambda_A) \circ \delta_A))$$

$$= ((\mu_A \circ (\mu_A \otimes \varphi_A) \circ (H \otimes \Psi \otimes A)) \otimes (\phi_H \circ (H \otimes \mu_A))) \circ (A \otimes \delta_{H\otimes A\otimes A})$$

$$\circ (A \otimes H \otimes ((A \otimes \lambda_A) \circ \delta_A)) \text{ (by (52))}$$

$$= ((\mu_A \circ (\mu_A \otimes \varphi_A) \circ (H \otimes \Psi \otimes A)) \otimes (\phi_H \circ (H \otimes (id_A * \lambda_A)))) \circ (A \otimes H \otimes A \otimes c_{H,A} \otimes A)$$

$$\circ (A \otimes H \otimes c_{H,A} \otimes c_{A,A}) \circ (A \otimes \delta_H \otimes ((\delta_A \otimes \lambda_A) \circ \delta_A))$$ *(by the coassociativity of δ_A, the naturality of c and (10) for λ_A)*

$$= ((\mu_A \circ (\mu_A \otimes \varphi_A) \circ (H \otimes \Psi \otimes A)) \otimes H) \circ (A \otimes H \otimes A \otimes c_{H,A}) \circ (A \otimes H \otimes c_{H,A} \otimes A)$$

$$\circ (A \otimes \delta_H \otimes ((A \otimes \lambda_A) \circ \delta_A))$$ *(by (8) for A, the counit properties, the naturality of c and the condition of left A-module for H)*

$$= ((\mu_A \circ (\mu_A \otimes \varphi_A) \circ (A \otimes \Psi \otimes \lambda_A) \circ (A \otimes H \otimes \delta_A)) \otimes H) \circ (A \otimes H \otimes c_{H,A}) \circ (A \otimes \delta_H \otimes A)$$

(by the naturality of c)

$$= (A \otimes \varepsilon_H \otimes \varepsilon_A \otimes H) \circ (A \otimes H \otimes c_{H,A}) \circ (A \otimes \delta_H \otimes A) \text{ (by (49))}$$

$$= A \otimes H \otimes \varepsilon_A \text{ (by the naturality of } c \text{ and the counit properties)}$$

and we obtain that (26) holds. Therefore, Ψ is an a-comonoidal distributive law.

4.2. Skew parings between Hopf quasigroups.

In this example, following Sections 4 and 5 of [Alonso, Fernández, González, 2020], we provide examples of a-comonoidal distributive laws associated to skew parings between Hopf quasigroups. If A, H are Hopf quasigroups, a skew pairing between A and H over K is a morphism $\tau : A\otimes H \to K$ such that the equalities

$$\tau \circ (\mu_A \otimes H) = (\tau\otimes\tau)\circ(A\otimes c_{A,H}\otimes H)\circ(A\otimes A\otimes \delta_H), \tag{53}$$

$$\tau\circ(A\otimes \mu_H) = \tau\circ(A\otimes\tau\otimes H)\circ(\delta_A\otimes H\otimes H), \tag{54}$$

$$\tau\circ(A\otimes\eta_H) = \varepsilon_A, \tag{55}$$

$$\tau\circ(\eta_A\otimes H) = \varepsilon_H, \tag{56}$$

hold.

If $\tau : A\otimes H\to K$ is a skew pairing, by [Alonso, Fernández, González, 2020, Proposition 4.3], we have that τ is convolution invertible and, if τ^{-1} is the inverse of τ, $\tau^{-1} = \tau\circ(\lambda_A\otimes H)$, $\tau = \tau^{-1}\circ(A\otimes\lambda_H)$ and satisfies

$$\tau^{-1}\circ(A\otimes\mu_H) = (\tau^{-1}\otimes\tau^{-1})\circ(A\otimes c_{A,H}\otimes H)\circ(\delta_A\otimes H\otimes H), \tag{57}$$

$$\tau^{-1}\circ(\mu_A\otimes H) = (\tau^{-1}\otimes\tau^{-1})\circ(A\otimes c_{A,H}\otimes H)\circ(A\otimes A\otimes(c_{H,H}\circ\delta_H)), \tag{58}$$

(55) and (56). Therefore, $\tau : A\otimes H\to K$ is a skew pairing between A and H if and only if τ^{-1} is a Hopf pairing between A and H in the sense of [Fang, Torrecillas, 2014]. As a consequence of the properties of τ

$$A\bowtie_\tau H = (A\otimes H, \eta_{A\bowtie_\tau H}, \mu_{A\bowtie_\tau H}, \varepsilon_{A\bowtie_\tau H}, \delta_{A\bowtie_\tau H}, \lambda_{A\bowtie_\tau H})$$

is a Hopf quasigroup (see [Fang, Torrecillas, 2014, Proposition 2.2] and [Alonso, Fernández, González, 2020, Corollary 4.10]) with unit, product, counit, coproduct and antipode defined by

$$\eta_{A\bowtie_\tau H} = \eta_{A\otimes H},\quad \mu_{A\bowtie_\tau H} = (\mu_A\otimes\mu_H)\circ(A\otimes\Psi\otimes H),$$

$$\varepsilon_{A\bowtie_\tau H} = \varepsilon_{A\otimes H},\quad \delta_{A\bowtie_\tau H} = \delta_{A\otimes H},$$

$$\lambda_{A\bowtie_\tau H} = (\lambda_A\otimes\lambda_H)\circ\Psi\circ c_{A,H},$$

where

$$\Psi = (\tau\otimes A\otimes H\otimes\tau^{-1})\circ(A\otimes H\otimes\delta_{A\otimes H})\circ\delta_{A\otimes H}\circ c_{H,A}. \tag{59}$$

By [Alonso, Fernández, González, 2020, Proposition 5.2]) we know that, if the morphisms $\varphi_A : H\otimes A\to A$ and $\phi_H : H\otimes A\to H$ are defined as

$$\varphi_A = (\tau\otimes A\otimes\tau^{-1})\circ(A\otimes H\otimes\delta_A\otimes H)\circ\delta_{A\otimes H}\circ c_{H,A}$$

and

$$\phi_H = (\tau\otimes H\otimes\tau^{-1})\circ(A\otimes H\otimes c_{A,H}\otimes H)\circ(A\otimes H\otimes A\otimes\delta_H)\circ\delta_{A\otimes H}\circ c_{H,A},$$

the pair (A, φ_A) is a left H-module comonoid, (H, ϕ_H) is a right A-module comonoid and

$$\Psi = (\varphi_A \otimes \phi_H) \circ \delta_{H \otimes A}.$$

As a consequence, $A \bowtie_\tau H$ (or, following the notation of [Fang, Torrecillas, 2014], $D(A, H, \tau^{-1})$) is the double cross product induced by the actions φ_A and ϕ_H. Therefore, by the previous example, the morphism Ψ defined in (59) is an a-comonoidal distributive law of A over H.

In the final part of this example we present a concrete case of a-comonoidal distributive law induced by a skew pairing. Let F be a field such that Char$(F) \neq 2$ and denote the tensor product over F as $\otimes$. Consider the nonabelian group $S_3 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$, where σ_0 is the identity, $o(\sigma_1) = o(\sigma_2) = o(\sigma_3) = 2$ and $o(\sigma_4) = o(\sigma_5) = 3$. Let u be an additional element such that $u^2 = 1$. By [Chein, 1946, Theorem 1] the set

$$L = M(S_3, 2) = \{\sigma_i u^\alpha \ ; \ \alpha = 0, 1\}$$

is a Moufang loop where the product is defined by

$$\sigma_i u^\alpha . \ \sigma_j u^\beta = (\sigma_i^\nu \sigma_j^\mu)^\nu u^{\alpha+\beta}, \ \ \nu = (-1)^\beta, \ \mu = (-1)^{\alpha+\beta}.$$

Then, L is an I.P. loop and FL is a cocommutative Hopf quasigroup. On the other hand, let H_4 be the 4-dimensional Taft Hopf algebra. The basis of H_4 is $\{1, x, y, w = xy\}$ and the multiplication table is defined by

	x	y	w
x	1	w	y
y	$-w$	0	0
w	$-y$	0	0

The costructure of H_4 is given by

$$\delta_{H_4}(x) = x \otimes x, \ \delta_{H_4}(y) = y \otimes x + 1 \otimes y, \ \delta_{H_4}(w) = w \otimes 1 + x \otimes w,$$

$$\varepsilon_{H_4}(x) = 1_F, \ \varepsilon_{H_4}(y) = \varepsilon_{H_4}(w) = 0$$

and the antipode λ_{H_4} is described by

$$\lambda_{H_4}(x) = x, \ \lambda_{H_4}(y) = w, \ \lambda_{H_4}(w) = -y.$$

The morphism $\tau : FL \otimes H_4 \to F$ defined by

$$\tau(\sigma_i u^\alpha \otimes z) = \left\{ \begin{array}{ccc} 1 & \text{if} & z = 1 \\ (-1)^\alpha & \text{if} & z = x \\ 0 & \text{if} & z = y, w \end{array} \right.$$

is a skew pairing such that $\tau = \tau^{-1}$. Then, $FL \bowtie_\tau H_4$ is a Hopf quasigroup where the a-comonoidal distributive law Ψ of FL over H_4 is defined by:

$$\Psi(1 \otimes \sigma_i u^\alpha) = \sigma_i u^\alpha \otimes 1, \ \ \Psi(x \otimes \sigma_i u^\alpha) = \sigma_i u^\alpha \otimes x,$$

$$\Psi(y \otimes \sigma_i u^\alpha) = (-1)^\alpha \sigma_i u^\alpha \otimes y, \ \ \Psi(w \otimes \sigma_i u^\alpha) = (-1)^\alpha \sigma_i u^\alpha \otimes w.$$

4.3. R-smash products of Hopf quasigroups.

Let A, H be Hopf quasigroups and assume that there exists an action $\varphi_A : H\otimes A\to A$ such that (A,φ_A) is a left H-quasimodule magma, (A,φ_A) a left H-quasimodule comonoid and satisfies the identities

$$(H\otimes\varphi_A)\circ((c_{H,H}\circ\delta_H)\otimes A)=(H\otimes\varphi_A)\circ(\delta_H\otimes A), \tag{60}$$

$$\varphi_A\circ(H\otimes(\varphi_A\circ(\lambda_H\otimes A)))=\varphi_A\circ((\mu_H\circ(H\otimes\lambda_H))\otimes A). \tag{61}$$

Define $\Psi : H\otimes A\to A\otimes H$ by

$$\Psi=(\varphi_A\otimes H)\circ(H\otimes c_{H,A})\circ(\delta_H\otimes A). \tag{62}$$

Then, by the monoidal version of the proofs developed in [Brzeziński, Jiao, 2012(2)], we can assure that the morphism Ψ satisfies (19),

$$\Psi\circ(\mu_H\otimes A)\circ(H\otimes\lambda_H\otimes A)=(A\otimes\mu_H)\circ(\Psi\otimes H)\circ(H\otimes\Psi)\circ(H\otimes\lambda_H\otimes A), \tag{63}$$

(17) and (18). Therefore, is an example of distributive law of H over A. Moreover, Ψ is comonoidal and we have that

$$(\varepsilon_A\otimes H)\circ\Psi=H\otimes\varepsilon_A. \tag{64}$$

Also, the proof of [Brzeziński, Jiao, 2012(2), Lemma 2.6] remains valid in this general symmetric monoidal setting and, as a consequence of the properties of Ψ, the equality

$$\Psi\circ(A\otimes\lambda_A)=(\lambda_A\otimes H)\circ\Psi \tag{65}$$

holds.

Finally, Ψ is a a-comonoidal distributive law because:

- $(A\otimes\mu_H)\circ(\Psi\otimes\mu_H)\circ(H\otimes\Psi\otimes H)\circ(((\lambda_H\otimes H)\circ\delta_H)\otimes A\otimes H)$

  $=((\varphi_A\circ(\lambda_H\otimes\varphi_A)\circ(\delta_H\otimes A))\otimes(\mu_H\circ(\lambda_H\otimes\mu_H)))\circ(H\otimes c_{H,A}\otimes H\otimes H)\circ(c_{H,H}\otimes c_{H,A}\otimes H)$

  $\circ(((\delta_H\otimes H)\circ\delta_H)\otimes A\otimes H)$ *(by the naturality of c, the coassociativity of δ_H, and (10) for λ_H)*

  $=(A\otimes(\mu_H\circ(\lambda_H\otimes\mu_H)\circ(\delta_H\otimes H)))\circ(c_{H,A}\otimes H)$ *(by (35) for φ_A)*

  $=\varepsilon_H\otimes A\otimes H$ *(by (5) and the naturality of c).*

- $(A\otimes\mu_H)\circ(\Psi\otimes\mu_H)\circ(H\otimes\Psi\otimes H)\circ(((H\otimes\lambda_H)\circ\delta_H)\otimes A\otimes H)$

  $=((\varphi_A\circ(H\otimes(\varphi_A\circ(\lambda_H\otimes A))))\otimes(\mu_H\circ(H\otimes\mu_H)\circ(H\otimes\lambda_H\otimes H)\circ(\delta_H\otimes H)))$

  $\circ(H\otimes H\otimes c_{H,A}\otimes H)\circ(H\otimes c_{H,H}\otimes A\otimes H)\circ(((\delta_H\otimes H)\circ\delta_H)\otimes A\otimes H)$ *(by the naturality of c, the coassociativity of δ_H, and (10) for λ_H)*

$= (\varphi_A \circ (H \otimes \varphi_A) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes A) \otimes H$ *(by (5), the naturality of c, and the counit properties)*

$= \varepsilon_H \otimes A \otimes H$ *(by (35) for φ_A).*

- $(\mu_A \otimes H) \circ (\mu_A \otimes \Psi) \circ (A \otimes \Psi \otimes A) \circ (A \otimes H \otimes ((\lambda_A \otimes A) \circ \delta_A))$

$= (((\mu_A \circ ((\mu_A \circ (A \otimes \lambda_A)) \otimes A)) \circ (A \otimes ((\varphi_A \otimes \varphi_A) \circ \delta_{H \otimes A}))) \otimes H) \circ (A \otimes H \otimes c_{H,A})$

$\circ (A \otimes \delta_H \otimes A)$ *(by (65), the naturality of c, and the coassociativity of δ_H)*

$= (((\mu_A \circ ((\mu_A \circ (A \otimes \lambda_A)) \otimes A)) \circ (A \otimes (\delta_A \circ \varphi_A))) \otimes H) \circ (A \otimes H \otimes c_{H,A}) \circ (A \otimes \delta_H \otimes A)$

(by (40))

$= A \otimes (((\varepsilon_A \circ \varphi_A) \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A))$ *(by (6))*

$= A \otimes H \otimes \varepsilon_A$ *(by (39), the naturality of c, and the counit properties).*

- $(\mu_A \otimes H) \circ (\mu_A \otimes \Psi) \circ (A \otimes \Psi \otimes A) \circ (A \otimes H \otimes ((A \otimes \lambda_A) \circ \delta_A))$

$= (((\mu_A \circ (\mu_A \otimes \lambda_A)) \circ (A \otimes ((\varphi_A \otimes \varphi_A) \circ \delta_{H \otimes A}))) \otimes H) \circ (A \otimes H \otimes c_{H,A}) \circ (A \otimes \delta_H \otimes A)$

(by (65), the naturality of c, and the coassociativity of δ_H)

$= (((\mu_A \circ (\mu_A \otimes \lambda_A)) \circ (A \otimes (\delta_A \circ \varphi_A))) \otimes H) \circ (A \otimes H \otimes c_{H,A}) \circ (A \otimes \delta_H \otimes A)$ *(by (40))*

$= A \otimes (((\varepsilon_A \circ \varphi_A) \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A))$ *(by (6))*

$= A \otimes H \otimes \varepsilon_A$ *(by (39), the naturality of c, and the counit properties).*

Let F be a field and denote the tensor product over F as $\otimes$. Let A, H be Hopf quasigroups in F-Vect. The morphism Ψ, defined in (62), was used in [Brzeziński, Jiao, 2012(1)] to introduce the notion of smash product for the quasigroups H and A. This product is a particular instance of a more general construction that we can find in [Brzeziński, Jiao, 2012(2)]. In [Brzeziński, Jiao, 2012(2)], for a morphism $R : H \otimes A \to A \otimes H$, the authors introduce a product in $A \otimes H$ defined by

$$\mu_{A \otimes H} = (\mu_A \otimes \mu_H) \circ (A \otimes R \otimes H),$$

and an antipode $\lambda_{A \otimes H} = R \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H}$. With this new product and the antipode $\lambda_{A \otimes H}$, they proved that, with the tensor product coproduct and the tensor product counit, if R satisfies (19) and (64), $A \otimes H$ has a new structure of Hopf quasigroup (denoted by $A \rtimes_R H$ and called the R-smash product Hopf quasigroup for H and A) if and only if R is a comonoid morphism, satisfies (17), (18), (63) and

$$(A \otimes \varepsilon_H) \circ R \circ c_{A,H} \circ (A \otimes \lambda_H) \circ R = \varepsilon_H \otimes A.$$

Then, if $A \rtimes_R H$ is a Hopf quasigroup, one easily checks that R is an a-comonoidal distributive law.

Note that the condition (64) in the Example 4.1 implies that

$$\phi_H = H \otimes \varepsilon_A,$$

i.e., ϕ_H is trivial. Then, the theory of R-smash products presented in [Brzeziński, Jiao, 2012(2)] does not cover as a particular cases the double cross products $A \bowtie H$ defined in [Alonso, Fernández, González, 2020].

4.4. Twisted smash products.

Let A, H be Hopf quasigroups and assume that there exists two actions $\varphi_A : H \otimes A \to A$, $\hat{\varphi}_A : H \otimes A \to A$ such that (A, φ_A), $(A, \hat{\varphi}_A)$ are left H-quasimodule magmas and left H-quasimodule comonoids and satisfy the identities (60), (61) and

$$\hat{\varphi}_A \circ (H \otimes \varphi_A) = \varphi_A \circ (H \otimes \hat{\varphi}_A) \circ (c_{H,H} \otimes A). \tag{66}$$

Define $\Gamma : H \otimes A \to A \otimes H$ by

$$\Gamma = (\varphi_A \otimes H) \circ (H \otimes T) \circ (\delta_H \otimes A), \tag{67}$$

where

$$T = (\hat{\varphi}_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A). \tag{68}$$

By the previous example, we known that the morphisms Ψ (defined in (62)) and T are examples of a-comonoidal distributive laws. Moreover, they satisfy the identities (19), (63), (64) and (65). On the other hand, by the naturality of c and the coassociativity of δ_H, we obtain

$$(A \otimes \delta_H) \circ T = (T \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) \tag{69}$$

and, using (66), again the naturality of c and the coassociativity of δ_H, we can prove that

$$T \circ (H \otimes \varphi_A) = (\varphi_A \otimes H) \circ (H \otimes T) \circ (c_{H,H} \otimes A) \tag{70}$$

holds. Then,

$$(c_{H,A} \otimes H) \circ (H \otimes T) \circ (\delta_H \otimes A) = (T \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A). \tag{71}$$

Indeed:

$$\begin{aligned}
&(c_{H,A} \otimes H) \circ (H \otimes T) \circ (\delta_H \otimes A)\\
&= (c_{H,A} \otimes H) \circ (H \otimes \hat{\varphi}_A \otimes H) \circ (\delta_H \otimes c_{H,A}) \circ (\delta_H \otimes A) \text{ (by coassociativity of } \delta_H)\\
&= (c_{H,A} \otimes H) \circ (H \otimes \hat{\varphi}_A \otimes H) \circ ((c_{H,H} \circ \delta_H) \otimes c_{H,A}) \circ (\delta_H \otimes A) \text{ (by (60))}\\
&= (T \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) \text{ (by naturality of } c).
\end{aligned}$$

Also, T satisfies

$$(\delta_A \otimes H) \circ T = (A \otimes T) \circ (T \otimes A) \circ (H \otimes \delta_A) \tag{72}$$

because

$$\begin{aligned}
&(\delta_A \otimes H) \circ T \\
&= (((\hat{\varphi}_A \otimes \hat{\varphi}_A) \circ \delta_{H\otimes A}) \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) \text{ (by (40) for } \hat{\varphi}_A\text{))} \\
&= (A \otimes T) \circ (T \otimes A) \circ (H \otimes \delta_A) \text{ (by the naturality of } c \text{ and the coassociativity of } \delta_H\text{)}
\end{aligned}$$

and, as a consequence of this fact,

$$(\delta_A \otimes H) \circ \Gamma = (A \otimes \Gamma) \circ (\Gamma \otimes A) \circ (H \otimes \delta_A) \tag{73}$$

also holds. Indeed:

$$\begin{aligned}
&(\delta_A \otimes H) \circ \Gamma \\
&= (((\varphi_A \otimes \varphi_A) \circ \delta_{H\otimes A}) \otimes H) \circ (H \otimes T) \circ (\delta_H \otimes A) \text{ (by (40) for } \varphi_A\text{))} \\
&= (\varphi_A \otimes \varphi_A \otimes H) \circ (H \otimes c_{H,A} \otimes T) \circ (\delta_H \otimes T \otimes A) \circ (\delta_H \otimes \delta_A) \text{ (by (72))} \\
&= (\varphi_A \otimes \varphi_A \otimes H) \circ (H \otimes ((c_{H,A} \otimes H) \circ (H \otimes T) \circ (\delta_H \otimes A)) \otimes A) \circ (\delta_H \otimes \delta_A) \text{ (by} \\
&\quad \text{the coassociativity of } \delta_H\text{)} \\
&= (A \otimes \Gamma) \circ (\Gamma \otimes A) \circ (H \otimes \delta_A) \text{ (by (69) and (71)).}
\end{aligned}$$

The morphism Γ is a distributive law of H over H because satisfy the identities (19), (63), (17) and (18). Indeed, (19) holds because

$$\begin{aligned}
&(\mu_A \otimes H) \circ (A \otimes \Gamma) \circ (\Gamma \otimes A) \\
&= ((\mu_A \circ (\varphi_A \otimes \varphi_A)) \otimes H) \circ (H \otimes A \otimes H \otimes T) \circ (H \otimes ((A \otimes \delta_H) \circ T) \otimes A) \circ (\delta_H \otimes A \otimes A) \\
&\quad \text{(by definition)} \\
&= ((\mu_A \circ (\varphi_A \otimes \varphi_A)) \otimes H) \circ (H \otimes A \otimes H \otimes T) \circ (H \otimes ((T \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)) \otimes A) \\
&\quad \circ (\delta_H \otimes A \otimes A) \text{ (by (69))} \\
&= ((\mu_A \circ (\varphi_A \otimes \varphi_A)) \otimes H) \circ (H \otimes A \otimes H \otimes T) \circ (H \otimes ((c_{H,A} \otimes H) \circ (H \otimes T) \circ (\delta_H \otimes A)) \otimes A) \\
&\quad \circ (\delta_H \otimes A \otimes A) \text{ (by (71))} \\
&= ((\mu_A \circ \varphi_{A\otimes A}) \otimes H) \circ (H \otimes A \otimes T) \circ (H \otimes T \otimes A) \circ (\delta_H \otimes A \otimes A) \text{ (by the coassociativity} \\
&\quad \text{of } \delta_H\text{)}
\end{aligned}$$

$$= ((\varphi_A \circ (H \otimes \mu_A)) \otimes H) \circ (H \otimes A \otimes T) \circ (H \otimes T \otimes A) \circ (\delta_H \otimes A \otimes A) \text{ (by (38))}$$

$$= \Gamma \circ (H \otimes \mu_A) \text{ (by (19) for } T.)$$

On the other hand,

$$(A \otimes \mu_H) \circ (\Gamma \otimes H) \circ (H \otimes \Gamma) \circ (H \otimes \lambda_H \otimes A)$$

$$= (\varphi_A \otimes \mu_H) \circ (H \otimes (T \circ (H \otimes \varphi_A)) \otimes H) \circ (H \otimes H \otimes H \otimes T) \circ (\delta_H \otimes (\delta_H \circ \lambda_H) \otimes A)$$

(by definition)

$$= (\varphi_A \otimes \mu_H) \circ (H \otimes ((\varphi_A \otimes H) \circ (H \otimes T) \circ (c_{H,H} \otimes A)) \otimes H) \circ (H \otimes H \otimes H \otimes T)$$

$$\circ (\delta_H \otimes (\delta_H \circ \lambda_H) \otimes A) \text{ (by (70))}$$

$$= ((\varphi_A \circ (H \otimes (\varphi_A \circ (\lambda_H \otimes A)))) \otimes H) \circ (H \otimes H \otimes ((A \otimes \mu_H) \circ (T \otimes H) \circ (H \otimes T)$$

$$\circ (H \otimes \lambda_H \otimes A))) \circ (H \otimes c_{H,H} \otimes H \otimes A) \circ (\delta_H \otimes (c_{H,H} \circ \delta_H) \otimes A) \text{ (by the naturality of } c \text{ and}$$

((10) for λ_H)

$$= ((\varphi_A \circ ((\mu_H \circ (H \otimes \lambda_H)) \otimes A)) \otimes H) \circ (H \otimes H \otimes (T \circ ((\mu_H \circ (H \otimes \lambda_H)) \otimes A)))$$

$$\circ (H \otimes c_{H,H} \otimes H \otimes A) \circ (\delta_H \otimes (c_{H,H} \circ \delta_H) \otimes A) \text{ (by (61) for } \varphi_A \text{ and (63) for } T)$$

$$= (\varphi_A \otimes H) \circ (H \otimes T) \circ (((\mu_H \otimes \mu_H) \circ \delta_{H \otimes H} \circ (H \otimes \lambda_H)) \otimes A) \text{ (by the naturality of } c \text{ and}$$

(10) for λ_H)

$$= \Gamma \circ (\mu_H \otimes A) \circ (H \otimes \lambda_H \otimes A) \text{(by (4))}$$

and, as a consequence, (63) holds. The identities (17) and (18) for Γ follow from (17) and (18) for T, (37), (34), (3), the naturality of c and the counit properties.

One easily checks that $\varepsilon_{A \otimes H} \circ \Gamma = \varepsilon_{H \otimes A}$ follows by (39), (22) for T and the properties of the counit. Also,

$$\delta_{A \otimes H} \otimes \Gamma$$

$$= (A \otimes c_{A,H} \otimes H) \circ (((\varphi_A \otimes \varphi_A) \circ \delta_{H \otimes A}) \otimes \delta_H) \circ (H \otimes T) \circ (\delta_H \otimes A) \text{ (by (40))}$$

$$= (A \otimes c_{A,H} \otimes H) \circ (((\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A)) \otimes \delta_H) \circ (\delta_H \otimes A \otimes T)$$

$$\circ (H \otimes T \otimes A) \circ (\delta_H \otimes \delta_A) \text{ (by (72))}$$

$$= (A \otimes H \otimes \varphi_A \otimes H) \circ (\varphi_A \otimes c_{H,H} \otimes A \otimes H) \circ (H \otimes c_{H,A} \otimes H \otimes A \otimes H) \circ (\delta_H \otimes A \otimes H \otimes T)$$

$$\circ (H \otimes ((c_{H,A} \otimes H) \circ (H \otimes T) \circ (\delta_H \otimes A)) \otimes A) \circ (\delta_H \otimes \delta_A) \text{ (by (69) and (71))}$$

$$= (\varphi_A \otimes ((H \otimes \varphi_A) \circ ((c_{H,H} \circ \delta_H) \otimes A)) \otimes H) \circ (H \otimes ((c_{H,A} \otimes T) \circ (H \otimes T \otimes A) \circ (\delta_H \otimes \delta_A)))$$

$$\circ(\delta_H \otimes A) \text{ (by the naturality of } c)$$

$$= (\varphi_A \otimes ((H \otimes \varphi_A) \circ (\delta_H \otimes A)) \otimes H) \circ (H \otimes ((T \otimes T) \circ \delta_{H\otimes A}))) \circ (\delta_H \otimes A) \text{ ((60))}$$

$$= (\varphi_A \otimes H \otimes \varphi_A \otimes H) \circ (H \otimes T \otimes H \otimes T) \circ (\delta_H \otimes ((c_{H,A} \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)) \otimes A)$$

$$\circ(\delta_H \otimes \delta_A) \text{ (by (69) and the coassociativity of } \delta_H)$$

$$= (\Gamma \otimes \Gamma) \circ \delta_{H\otimes A} \text{ (by the naturality of } c),$$

and this implies that Γ is comonoidal.

The morphism Γ also satisfies (64) because (39) holds and T satisfies (64). Moreover, as in the previous example, we can assure that (65) holds for Γ. Then,

$$(A \otimes \mu_H) \circ (\Gamma \otimes \mu_H) \circ (H \otimes \Gamma \otimes H) \circ (((\lambda_H \otimes H) \circ \delta_H) \otimes A \otimes H)$$

$$= (\varphi_A \otimes \mu_H) \circ (H \otimes (T \circ (H \otimes \varphi_A)) \otimes \mu_H) \circ ((\delta_H \circ \lambda_H) \otimes H \otimes T \otimes H) \circ (((H \otimes \delta_H)$$

$$\circ\delta_H) \otimes A \otimes H) \text{(by definition)}$$

$$= (\varphi_A \otimes \mu_H) \circ (H \otimes ((\varphi_A \otimes H) \circ (H \otimes T) \circ (c_{H,H} \otimes A)) \otimes \mu_H) \circ ((\delta_H \circ \lambda_H) \otimes H \otimes T \otimes H)$$

$$\circ(((H \otimes \delta_H) \circ \delta_H) \otimes A \otimes H) \text{ (by (70))}$$

$$= ((\varphi_A \circ (H \otimes \varphi_A) \circ (((\lambda_H \otimes H) \circ \delta_H) \otimes A)) \otimes \mu_H) \circ (H \otimes T \otimes \mu_H) \circ (((H \otimes \lambda_H) \circ c_{H,H}$$

$$\circ\delta_H) \otimes T \otimes H) \circ (\delta_H \otimes A \otimes H) \text{ (by (10), the coassociativity of } \delta_H\text{, and the naturality of } c)$$

$$= (A \otimes \mu_H) \circ (T \otimes \mu_H) \circ (H \otimes T \otimes H) \circ (((\lambda_H \otimes H) \circ \delta_H) \otimes A \otimes H) \text{ (by (35) for } \varphi_A,$$

$$\text{the naturality of } c \text{ and the counit properties)}$$

$$= \varepsilon_H \otimes A \otimes H \text{ (by (23) for } T),$$

$$(A \otimes \mu_H) \circ (\Gamma \otimes \mu_H) \circ (H \otimes \Gamma \otimes H) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes A \otimes H)$$

$$= (\varphi_A \otimes \mu_H) \circ (H \otimes (T \circ (H \otimes \varphi_A)) \otimes \mu_H) \circ (\delta_H \otimes H \otimes T \otimes H) \circ (((H \otimes (\delta_H \circ \lambda_H))$$

$$\circ\delta_H) \otimes A \otimes H) \text{ (by definition)}$$

$$= (\varphi_A \otimes \mu_H) \circ (H \otimes ((\varphi_A \otimes H) \circ (H \otimes T) \circ (c_{H,H} \otimes A)) \otimes \mu_H) \circ (\delta_H \otimes H \otimes T \otimes H)$$

$$\circ(((H \otimes (\delta_H \circ \lambda_H)) \circ \delta_H) \otimes A \otimes H) \text{ (by (70))}$$

$$= ((\varphi_A \circ (H \otimes \varphi_A)) \otimes H) \circ (H \otimes H \otimes ((A \otimes \mu_H) \circ (T \otimes \mu_H) \circ (H \otimes T \otimes H) \circ (((H \otimes \lambda_H)$$

$$\circ\delta_H) \otimes A \otimes H))) \circ (((H \otimes c_{H,H}) \circ (\delta_H \otimes \lambda_H) \circ \delta_H) \otimes A \otimes H) \text{ (by (10), the coassociativity}$$

$$\text{of } \delta_H \text{ and the naturality of } c)$$

$= (\varphi_A \circ (H \otimes \varphi_A) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes A)) \otimes H$ *(by (24) for T, the naturality of c, and the counit properties)*

$= \varepsilon_H \otimes A \otimes H$ *(by (35) for φ_A)*,

$(\mu_A \otimes H) \circ (\mu_A \otimes \Gamma) \circ (A \otimes \Gamma \otimes A) \circ (A \otimes H \otimes ((\lambda_A \otimes A) \circ \delta_A))$

$= (\mu_A \otimes H) \circ ((\mu_A \circ (A \otimes \lambda_A)) \otimes \Gamma) \circ (A \otimes \Gamma \otimes A) \circ (A \otimes H \otimes \delta_A)$ *(by (65) for Γ)*

$= ((\mu_A \circ (\mu_A \otimes A) \circ (A \otimes \lambda_A \otimes A) \circ (A \otimes \delta_A)) \otimes H) \circ (A \otimes \Gamma)$*(by (73))*

$= A \otimes ((\varepsilon_A \otimes H) \circ \Gamma)$ *(by (6) for φ_A)*

$= A \otimes H \otimes \varepsilon_A$ *(by (64) for Γ)*,

$(\mu_A \otimes H) \circ (\mu_A \otimes \Gamma) \circ (A \otimes \Gamma \otimes A) \circ (A \otimes H \otimes ((A \otimes \lambda_A) \circ \delta_A))$

$= ((\mu_A \circ (A \otimes \lambda_A)) \otimes H) \circ (\mu_A \otimes \Gamma) \circ (A \otimes \Gamma \otimes A) \circ (A \otimes H \otimes \delta_A)$ *(by (65) for Γ)*

$= ((\mu_A \circ (\mu_A \otimes \lambda_A) \circ (A \otimes \delta_A)) \otimes H) \circ (A \otimes \Gamma)$*(by (73))*

$= A \otimes ((\varepsilon_A \otimes H) \circ \Gamma)$ *(by (6) for φ_A)*

$= A \otimes H \otimes \varepsilon_A$ *(by (64) for Γ)*.

Therefore, Γ is a an a-comonoidal distributive law of H over A.

4.5. Twisted smash products for biquasimodule Hopf quasigroups.

Assume that (A, φ_A) is a left H-quasimodule magma-comonoid and suppose that there exists a right action $\phi_A : A \otimes H \to A$ such that (A, ϕ_A) is a right H-quasimodule magma-comonoid. Following [Fang, Torrecillas, 2014], we will say that (A, φ_A, ϕ_A) is a H-biquasimodule Hopf quasigroup if

$$\varphi_A \circ (H \otimes \phi_A) = \phi_A \circ (\varphi_A \otimes H) \tag{74}$$

holds.

The main result of [Fang, Wang, 2011] asserts that, if (61) and

$$\phi_A \circ ((\phi_A \circ (A \otimes \lambda_H)) \otimes H) = \phi_A \circ (A \otimes (\mu_H \circ (\lambda_H \otimes H)) \tag{75}$$

hold, the twisted smash product $A \circledast H$ built on $A \otimes H$ with tensor coproduct and unit, where the product and antipode are defined by

$$\mu_{A \circledast H} = \tag{76}$$

$$(\mu_A \otimes \mu_H) \circ (A \otimes ((\phi_A \otimes H) \circ (A \otimes ((\lambda_H \otimes H) \circ c_{H,H} \circ \delta_H)) \circ (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)) \otimes H)$$

and

$$\lambda_{A\circledast H} = \tag{77}$$

$$(\phi_A \otimes H) \circ (A \otimes ((\lambda_H \otimes H) \circ c_{H,H} \circ \delta_H)) \circ (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H},$$

respectively, is a Hopf quasigroup if and only if (60) and

$$(\phi_A \otimes H) \circ (A \otimes ((\lambda_H \otimes H) \circ c_{H,H} \circ \delta_H)) = (\phi_A \otimes H) \circ (A \otimes ((\lambda_H \otimes H) \circ \delta_H)) \tag{78}$$

hold.

Note that the product defined in (76) is

$$\mu_{A\circledast H} = (\mu_A \otimes \mu_H) \circ (A \otimes \Gamma \otimes H),$$

where

$$\Gamma = (\phi_A \otimes H) \circ (A \otimes ((\lambda_H \otimes H) \circ c_{H,H} \circ \delta_H)) \circ (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A),$$

and

$$\lambda_{A\circledast H} = \Gamma \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H}.$$

It is easy to show that, if (A, φ_A, ϕ_A) satisfies (74), (61), (75), (60) and (78), we obtain that the pairs (A, φ_A) and $(A, \hat{\varphi}_A = \phi_A \circ (A \otimes \lambda_H) \circ c_{H,A})$ are left H-quasimodule magmas-comonoids and satisfy the identities (60), (61) and (66). Moreover, under this conditions,

$$\Gamma = (\varphi_A \otimes H) \circ (H \otimes T) \circ (\delta_H \otimes A),$$

where T is defined as in (68).

Therefore, by Example 4.4, the product defined in [Fang, Wang, 2011] (see also [Fang, Torrecillas, 2014, Theorem 3.9]) is induced by an a-comonoidal distributive law of H over A.

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CITMAga. 15782 Santiago de Compostela. Spain

Departamento de Matemática Aplicada II. Universidade de Vigo. E-36310 Vigo. Spain

Email: rgon@dma.uvigo.es

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