

ON THE WITT VECTORS IN THE LAWVERE QUANTALE

Dedicated to Professor Lars Hesselholt on the occasion of his sixtieth birthday.

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ABSTRACT. Categories enriched over the Lawvere quantale $\mathbb{L}$ have been studied as generalized metric spaces. In this paper, we consider the monoidal category $\mathbb{W}(\mathbb{L})$ of Witt vectors in $\mathbb{L}$ and explore some basic properties of categories enriched over it. We also construct a monoidal functor from $\mathbb{L}$ to $\mathbb{W}(\mathbb{L})$, which induces a fully faithful functor from the category of $\mathbb{L}$ -enriched categories to that of $\mathbb{W}(\mathbb{L})$ -enriched categories, and investigate the relationship between them.

1. Introduction

In [12], Lawvere showed that metric spaces are categories enriched over a certain monoidal category denoted by $\mathbb{L}$ in the present paper. This monoidal category has $[0, \infty]$ as its set of objects, and its morphisms are given by the poset structure which is the opposite to the usual one, and the monoidal product is the usual (extended) addition. Since this is not merely a monoidal category but is a quantale as well, we refer to it as *the Lawvere quantale*. With Lawvere's work as one of its starting points, the theory of categories enriched over a quantale has developed significantly [11].

While Witt vectors are usually studied for commutative rings, the works [17, 7, 5] have shown that they can also be considered for more general algebras, in particular commutative rigs. Witt vectors have traditionally been treated mainly in and around number theory and studied in connection with formal power series, but these studies reveal an intrinsic relationship between Witt vectors and symmetric functions.

The Lawvere quantale $\mathbb{L}$ can be viewed as a min-plus algebra, which is a rig of characteristic 1. In this paper, we consider the rig $\mathbb{W}(\mathbb{L})$ of Witt vectors in $\mathbb{L}$, defined as the set of rig homomorphisms from the rig Λ of symmetric functions to $\mathbb{L}$, with its rig structure induced by the birig structure of Λ . Using this rig structure, we equip $\mathbb{W}(\mathbb{L})$ with a monoidal category structure, which allows us to consider categories enriched over it. Such enriched categories may be interpreted as metric spaces enhanced by symmetric functions.

We let $\otimes_{\mathbb{W}(\mathbb{L})}$ denote the monoidal product of $\mathbb{W}(\mathbb{L})$, $\bar{0} \in \mathbb{W}(\mathbb{L})$ the unit, and $\leq_{\mathbb{W}(\mathbb{L})}$ the

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poset structure defined below. Let X be a category enriched over $\mathbb{W}(\mathbb{L})$, and let $d(x, x') \in \mathbb{W}(\mathbb{L})$ denote “the hom-set” for $x, x' \in X$. Then we have the following inequalities

$$\bar{0} \leq_{\mathbb{W}(\mathbb{L})} d(x, x),$$

$$d(x, x') \otimes_{\mathbb{W}(\mathbb{L})} d(x', x'') \leq_{\mathbb{W}(\mathbb{L})} d(x, x''),$$

for $x, x', x'' \in X$. These inequalities may appear fairly simple. However the second one, in particular, involves the substitutions of symmetric functions, which are difficult to compute. In this paper, we provide a partial analysis of these inequalities.

We also show the following.

1.1. THEOREM. [Theorem 3.5] *There is a monoidal functor $\theta : \mathbb{L} \rightarrow \mathbb{W}(\mathbb{L})$.*

We will explicitly construct θ in §3. We remark that there is no rig homomorphism from $\mathbb{L}$ to $\mathbb{W}(\mathbb{L})$, since the rig $\mathbb{W}(\mathbb{L})$ is not of characteristic 1. The addition $\oplus_{\mathbb{L}}$ of $\mathbb{L}$ and the poset structure $\leq_{\mathbb{L}}$, namely the category structure of $\mathbb{L}$, are equivalent in the following sense:

$$x \leq_{\mathbb{L}} y \Leftrightarrow \text{there exists } z \in \mathbb{L} \text{ such that } x \oplus_{\mathbb{L}} z = y.$$

Therefore, although θ is not a rig homomorphism, it does not discard the addition of $\mathbb{L}$. The (pre)orders defined as above are called difference (pre)orders in [6] and play a role in the theory of rigs. It is clear but important that difference orders do not work well on (non-trivial) rings because the additive inverses must exist uniquely.

We also remark that, for a commutative ring R , there is a ring isomorphism $\mathbb{W}(R) \cong 1 + tR[[t]]$, where $1 + tR[[t]]$ has the usual commutative multiplication of formal power series as its addition and is endowed with an intricate multiplication, which is governed by the combinatorics of elementary symmetric functions. Since θ is a monoidal functor, it respects this important structure. This approach may work for other rigs of characteristic 1.

By construction, θ induces a fully faithful functor from the category of categories enriched over $\mathbb{L}$ to that of those enriched over $\mathbb{W}(\mathbb{L})$. We will also describe the image of the functor explicitly.

Finally we briefly discuss some homotopical background. Witt vectors are connected to topological Hochschild homology (THH) in a profound way [10, 9]. THH is one of the most fundamental invariants of ring spectra and this connection has been playing a pivotal role in the field over the past several decades. Recently, the notion of spectra has been extended to *categorical spectra* [16], and the study of algebras of them is beginning as well [14]. However, there is still very little known about the invariants of such algebras. Rigs are elementary examples of algebras that lie outside the scope of ring spectra, yet they are captured by the framework of algebras of categorical spectra. In order to gain insights into the invariants of such algebras, we focus on Witt vectors in the important rig $\mathbb{L}$, which is not a ring, as we suspect they may relate to THH even in the categorical spectral setting.

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2. Witt vectors and Lawvere metric spaces

In the following, $\mathbb{N}$ denotes the set $\{0, 1, 2, \dots\}$ and $\mathbb{N}_{>0}$ denotes the set $\{1, 2, \dots\}$.

2.1. WITT VECTORS. In this section, we briefly recall the basics of the theory of Witt vectors, following [5] and [7], as preparation for later sections.

2.2. DEFINITION. A sequence of natural numbers $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathbb{N}^{\mathbb{N}_{>0}}$ is a partition if

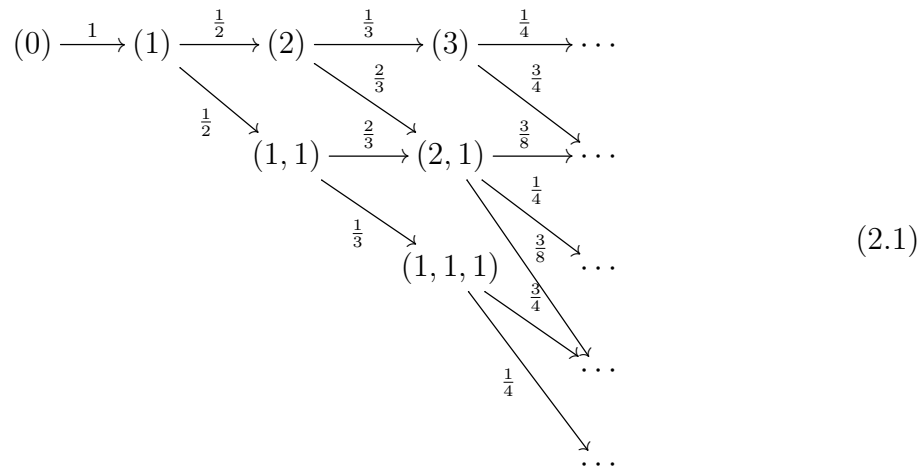
- $\lambda_1 \geq \lambda_2 \geq \dots$ and,
- there exists $i \in \mathbb{N}_{>0}$ such that $\lambda_i = 0$.

We let $\mathbb{P}$ denote the set of partitions.

For any partition $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathbb{P}$, we define its weight $|\lambda| \in \mathbb{N}$ to be the (finite) sum $\lambda_1 + \lambda_2 + \dots$. We say λ is a partition of $|\lambda|$. For any $n \in \mathbb{N}$, we let $\mathbb{P}_n \subset \mathbb{P}$ be the set of partitions of n .

2.3. REMARK. Note that $\mathbb{P} = \coprod_{n \in \mathbb{N}} \mathbb{P}_n$. We will usually write $(\lambda_1, \dots, \lambda_k)$ for the partition $(\lambda_1, \dots, \lambda_k, 0, 0, \dots)$.

Although the following well-known Markov process will not be used explicitly in this paper, it could be helpful to keep it in mind, and so we include it here.



2.4. DEFINITION. Let $x_1, x_2, \dots$ be commuting indeterminates and $\lambda \in \mathbb{P}$. We define the monomial symmetric function $m_\lambda(x_1, x_2, \dots)$ of λ to be the formal sum

$$\sum_{(\alpha_1, \alpha_2, \dots)} x_1^{\alpha_1} x_2^{\alpha_2} \cdots,$$

where $(\alpha_1, \alpha_2, \dots) \in \mathbb{N}^{\mathbb{N}_{>0}}$ runs over all permutations of λ . By convention, $m_{(0)} = 1$, where (0) is the unique partition of 0. We may write m_λ for short instead of $m_\lambda(x_1, x_2, \dots)$. We say a symmetric function with coefficients in $\mathbb{N}$ is a finite $\mathbb{N}$ -linear combination of monomial symmetric functions.

Let Λ denote the set of symmetric functions with coefficients in $\mathbb{N}$. As is shown in [5], Λ is a (commutative and co-commutative) birig, namely a commutative rig object in $\mathbf{CRig}^{op}$, where $\mathbf{CRig}$ denotes the category of commutative rigs and rig homomorphisms, and $\mathbf{CRig}^{op}$ its opposite category. More precisely, in the usual way Λ is a commutative rig, and with the following structures, it gives rise to a birig. See [5] for more detail:

$$\Delta^+ : \Lambda \rightarrow \Lambda \otimes_{\mathbb{N}} \Lambda,$$

$$\Delta^+(f(x_1, x_2, \dots)) = f(x_1 \otimes 1, 1 \otimes x_1, x_2 \otimes 1, 1 \otimes x_2, \dots, x_i \otimes 1, 1 \otimes x_i, \dots)$$

$$\varepsilon^+ : \Lambda \rightarrow \mathbb{N}, \quad \varepsilon^+(f(x_1, x_2, \dots)) = f(0, 0, 0, \dots)$$

$$\Delta^\times : \Lambda \rightarrow \Lambda \otimes_{\mathbb{N}} \Lambda,$$

$$\Delta^\times(f(x_1, x_2, \dots)) = f(x_1 \otimes x_1, x_1 \otimes x_2, \dots, x_i \otimes x_j, \dots)$$

$$\varepsilon^\times : \Lambda \rightarrow \mathbb{N}, \quad \varepsilon^\times(f(x_1, x_2, \dots)) = f(1, 0, 0, \dots),$$

where Δ^+ , ε^+ , $\Delta^\times$, and $\varepsilon^\times$ will be called co-addition, co-additive unit, co-multiplication, and co-multiplicative unit, respectively.

2.5. EXAMPLE. By definition

$$m_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + \cdots + x_1 x_2^2 + x_2^2 x_3 + \cdots.$$

Again by definition, we have

$$\Delta^+(m_{(2,1)}) = m_{(2,1)} \otimes 1 + m_{(2)} \otimes m_{(1)} + m_{(1)} \otimes m_{(2)} + 1 \otimes m_{(2,1)},$$

$$\Delta^\times(m_{(2,1)}) = m_{(2,1)} \otimes m_{(2,1)} + m_{(2,1)} \otimes m_{(3)} + m_{(3)} \otimes m_{(2,1)},$$

$$\varepsilon^+(m_{(2,1)}) = \varepsilon^\times(m_{(2,1)}) = 0.$$

Let $(S, +_S, \cdot_S)$ be a commutative rig and $\mathbb{W}(S)$ denote the set $\text{Hom}_{\mathbf{CRig}}(\Lambda, S)$ of rig homomorphisms. We give $\mathbb{W}(S)$ the following rig structure:

Let $f, g \in \mathbb{W}(S)$. The addition $f \oplus_{\mathbb{W}(S)} g \in \mathbb{W}(S)$ is given by

$$(f \oplus_{\mathbb{W}(S)} g)(\varphi) = \cdot_S \circ (f \otimes g) \circ \Delta^+(\varphi)$$

for any $\varphi \in \Lambda$. The additive unit is induced by ε^+ . Similarly, the multiplication $f \otimes_{\mathbb{W}(S)} g \in \mathbb{W}(S)$ is given by

$$(f \otimes_{\mathbb{W}(S)} g)(\varphi) = \cdot_S \circ (f \otimes g) \circ \Delta^\times(\varphi)$$

for any $\varphi \in \Lambda$ and the multiplicative unit is induced by $\varepsilon^\times$.

2.6. THEOREM. [5] *This gives rise to a functor $\mathbb{W} : \mathbf{CRig} \rightarrow \mathbf{CRig}$.*

This theorem has been proven by [17] for commutative rings.

The birig Λ also has an important structure called a composition $\mathbb{N}$ -algebra. As is shown in [5], the category of commutative and cocommutative birigs admits a monoidal structure where the monoidal product is denoted by $\odot$ and the unit object is the polynomial birig $\mathbb{N}[X]$.

2.7. DEFINITION. [5] *A monoid object with respect to this monoidal structure is called a composition $\mathbb{N}$ -algebra.*

2.8. THEOREM. [5] *The birig Λ is a composition $\mathbb{N}$ -algebra.*

The monoid operation $\circ : \Lambda \odot \Lambda \rightarrow \Lambda$ is the one known as plethysm or composition in [13]. The unit map $\mathbb{N}[X] \rightarrow \Lambda$ is induced by $X \mapsto m_{(1)}$.

By this structure, the functor $\mathbb{W} : \mathbf{CRig} \rightarrow \mathbf{CRig}$ gives rise to a comonad. Namely, the structure map, for any commutative rig S , $\mathbb{W}(S) \rightarrow \mathbb{W}(\mathbb{W}(S))$ is given by $f \mapsto \tilde{f}$, which is given by $\tilde{f}(\varphi) = f(- \circ \varphi) \in \mathbb{W}(S)$ for any $\varphi \in \Lambda$.

Restricting to commutative rings and abusing notation, $\mathbb{W} : \mathbf{CRing} \rightarrow \mathbf{CRing}$ is also a comonad, where $\mathbf{CRing}$ denotes the category of commutative rings and ring homomorphisms.

2.9. DEFINITION. [17] *A coalgebra over the comonad $\mathbb{W} : \mathbf{CRing} \rightarrow \mathbf{CRing}$ is called a λ -ring.*

In some literature, the λ -ring structure of a λ -ring R is defined in terms of endomaps on R indexed by $\mathbb{N}$. Such a definition and the one given above are connected via the elementary symmetric functions. See [17] for more detail. λ -rings have been playing a prominent role around the algebraic K -theory. They are also crucial in Borger's absolute geometry [3] and [4]. It would be fair to expect that coalgebras over the comonad $\mathbb{W} : \mathbf{CRig} \rightarrow \mathbf{CRig}$ could also play a role in mathematics. However we do not investigate such coalgebras in this paper, except for $\mathbb{W}(\mathbb{L})$ defined later.

2.10. LAWVERE METRIC SPACES. We let $[0, \infty]$ denote the set of non-negative real numbers together with ∞ , and view it as a linearly ordered set with the order $\leq_{\mathbb{L}}$ defined by

$$x \leq_{\mathbb{L}} y :\Leftrightarrow y \leq x,$$

where $\leq$ is the usual order on $[0, \infty]$. We also equip $[0, \infty]$ with a monoid structure by the usual addition, denoted here by $\otimes_{\mathbb{L}}$ with the unit 0. These two structures are compatible in such a way that they give rise to a monoidal poset, hence a monoidal category, which we denote by $\mathbb{L}$. It is well known that $\mathbb{L}$ is furthermore a quantale. We refer to it as the Lawvere quantale. While we do not discuss it in this paper, quantales are algebraic structures often studied in logic and related fields. See, for example, [1].

Note that the order structure $\leq_{\mathbb{L}}$ induces a binary operation $\oplus_{\mathbb{L}}$ on $[0, \infty]$ defined by

$$x \oplus_{\mathbb{L}} y := x \vee y,$$

where $\vee$ denotes the join with respect to $\leq_{\mathbb{L}}$. In other words, $\oplus_{\mathbb{L}}$ coincides with the minimum operation, $\min$, with respect to the usual order.

The Lawvere quantale $\mathbb{L}$ also determines the commutative rig whose underlying set is $[0, \infty]$, with addition given by $\min$ and multiplication given by $\otimes_{\mathbb{L}}$. Abusing both terminology and notation, we may also refer to this commutative rig as the Lawvere quantale and denote it by $\mathbb{L}$. Note that this rig is of characteristic 1, namely $0 \oplus_{\mathbb{L}} 0 = 0$, which is an object in idempotent analysis. Note also that $\leq_{\mathbb{L}}$ is recovered from $\oplus_{\mathbb{L}}$:

$$x \leq_{\mathbb{L}} y \Leftrightarrow x \oplus_{\mathbb{L}} z = y \text{ for some } z \in [0, \infty].$$

In this sense, the order and the addition of $\mathbb{L}$ are equivalent.

We remark that, for any ring R and $x, y \in R$, we have $y - x \in R$ and $x + (y - x) = y$. That is, the induced order on R is trivial.

Let $\mathbb{L}\text{-Cat}$ denote the category of (small) categories enriched over $\mathbb{L}$. Let $X \in \mathbb{L}\text{-Cat}$. Then, we have a hom-map $d : X \times X \rightarrow \mathbb{L}$, where, by abusing notation, we write X for the set of objects in X , and $\mathbb{L}$ for the set of objects in $\mathbb{L}$, which satisfies

$$0 \leq_{\mathbb{L}} d(x, x)$$

$$d(x, x') \otimes_{\mathbb{L}} d(x', x'') \leq_{\mathbb{L}} d(x, x''),$$

for any $x, x', x'' \in X$. The first one may be called the identity axiom, and the second one the composition axiom.

In the usual notation, we have

$$0 = d(x, x)$$

$$d(x, x') + d(x', x'') \geq d(x, x'').$$

So we can regard X as a kind of metric space. This brilliant idea is due to Lawvere [12]. Note that the distance is not necessarily symmetric and there may exist some points $x, x' \in X$ which are not the same but $d(x, x') = 0$. Viewing categories enriched over quantales as a kind of space has long been an active area of research. For example, in [2], probabilistic metric spaces are defined to be categories enriched over a quantale related to $\mathbb{L}$.¹ See [11] for more detail.

¹The author learned this from the anonymous referee.

3. $\mathbb{W}(\mathbb{L})$ -categories

In this section, viewing $\mathbb{L}$ as a rig, we consider the Witt vectors $\mathbb{W}(\mathbb{L})$. We give it a monoidal category structure induced by the structures of Λ and $\mathbb{L}$, and consider categories enriched over it. We would like to think of such categories as a generalization of Lawvere metric spaces.

3.1. RELATIONSHIP BETWEEN $\mathbb{W}(\mathbb{L})$ AND $\mathbb{L}$. By definition, $\mathbb{W}(\mathbb{L})$ is a rig. We equip it with a monoidal category structure as follows.

3.2. DEFINITION. Let $f, g \in \mathbb{W}(\mathbb{L})$. We define an order $\leq_{\mathbb{W}(\mathbb{L})}$ as follows:

$$f \leq_{\mathbb{W}(\mathbb{L})} g \Leftrightarrow \forall \lambda \in \mathbb{P}, f(m_\lambda) \leq_{\mathbb{L}} g(m_\lambda)$$

We denote the monoidal product on $\mathbb{W}(\mathbb{L})$ by $\otimes_{\mathbb{W}(\mathbb{L})}$, which is the multiplication of the rig $\mathbb{W}(\mathbb{L})$. Thus, by definition,

$$(f \otimes_{\mathbb{W}(\mathbb{L})} g)(m_\lambda) = \min_{\mu, \mu'} \{f(m_\mu) + g(m_{\mu'})\},$$

where μ and μ' are the partitions that appear in the unique expression $\Delta^\times(m_\lambda) = \sum_{\mu, \mu'} c_{\mu, \mu'}^\lambda m_\mu \otimes m_{\mu'}$.

Let $\bar{0} \in \mathbb{W}(\mathbb{L})$ be the element² defined by

$$\bar{0}(m_\lambda) = \begin{cases} 0 & \text{if } \lambda = (n) \text{ for some } n \\ \infty & \text{otherwise,} \end{cases}$$

which is the multiplicative unit of the rig $\mathbb{W}(\mathbb{L})$.

3.3. PROPOSITION. *With these structures, $\mathbb{W}(\mathbb{L})$ is a monoidal poset.*

PROOF. It is clear that $\mathbb{W}(\mathbb{L})$ is a poset with $\leq_{\mathbb{W}(\mathbb{L})}$. Since $\otimes_{\mathbb{W}(\mathbb{L})}$ and $\bar{0}$ are parts of the rig structure of $\mathbb{W}(\mathbb{L})$, the unit axiom and the associativity axiom hold. We only need to check the bifactoriality of $- \otimes_{\mathbb{W}(\mathbb{L})} - : \mathbb{W}(\mathbb{L}) \times \mathbb{W}(\mathbb{L}) \rightarrow \mathbb{W}(\mathbb{L})$. It is enough to show the functoriality on the first component.

Let $f, f' \in \mathbb{W}(\mathbb{L})$ with $f \leq_{\mathbb{W}(\mathbb{L})} f'$. Then by definition for any $\nu \in \mathbb{P}$ we have $f(m_\nu) \geq f'(m_\nu)$. For any $g \in \mathbb{W}(\mathbb{L})$ and $\lambda \in \mathbb{P}$, we have

$$\begin{aligned} (f \otimes_{\mathbb{W}(\mathbb{L})} g)(m_\lambda) &= \min_{\mu, \mu'} \{f(m_\mu) + g(m_{\mu'})\} \\ &\geq \min_{\mu, \mu'} \{f'(m_\mu) + g(m_{\mu'})\} \\ &= (f' \otimes_{\mathbb{W}(\mathbb{L})} g)(m_\lambda), \end{aligned}$$

where μ and μ' are the partitions that appear in the unique expression $\Delta^\times(m_\lambda) = \sum_{\mu, \mu'} c_{\mu, \mu'}^\lambda m_\mu \otimes m_{\mu'}$ with $c_{\mu, \mu'}^\lambda \in \mathbb{N}$.

Thus we have $f \otimes_{\mathbb{W}(\mathbb{L})} g \leq_{\mathbb{W}(\mathbb{L})} f' \otimes_{\mathbb{W}(\mathbb{L})} g$. ■

²Although this notation may seem awkward, we retain it because it is inherited from the min-plus algebra.

3.4. REMARK. Note that $\{m_\lambda\}_{\lambda \in \mathbb{P}}$ forms an $\mathbb{N}$ -module basis for Λ , and that elements of $\mathbb{W}(\mathbb{L})$ are rig homomorphisms. There might be ways to endow $\mathbb{W}(\mathbb{L})$ with a partial order that makes it a quantale.

Note also that, in [5], Borger studied the subrig $\Lambda^{Sch} \subset \Lambda$ generated by Schur symmetric functions and the associated notion of Witt vectors, Schur-Witt vectors, which we do not treat in this paper.

We relate the base monoidal categories as follows.

3.5. THEOREM. The map $\theta : \mathbb{L} \rightarrow \mathbb{W}(\mathbb{L})$ defined by

$$\theta(r)(m_\lambda) = \begin{cases} nr & \text{if } \lambda = (n) \text{ for some } n \\ \infty & \text{otherwise} \end{cases}$$

is a monoidal functor.

PROOF. We first show that $\theta(r) : \Lambda \rightarrow \mathbb{L}$ is a rig homomorphism for any $r \in \mathbb{L}$. Since $\{m_\lambda\}_{\lambda \in \mathbb{P}}$ is an $\mathbb{N}$ -module basis, $\theta(r)$ preserves addition. Let $n, n' \in \mathbb{N}$. Without loss of generality, we assume $n \geq n'$.

When $n > n'$, we have $m_{(n)}m_{(n')} = m_{(n+n')} + m_{(n,n')}$. So we have

$$\begin{aligned} \theta(r)(m_{(n)}m_{(n')}) &= \min\{\theta(r)(m_{(n+n')}), \theta(r)(m_{(n,n')})\} \\ &= \min\{(n+n')r, \infty\} \\ &= (n+n')r \\ &= \theta(r)(m_{(n)}) + \theta(r)(m_{(n')}) \\ &= \theta(r)(m_{(n)}) \otimes_{\mathbb{L}} \theta(r)(m_{(n')}). \end{aligned}$$

When $n = n'$, we have $m_{(n)}m_{(n')} = m_{(2n)} + 2m_{(n,n)}$. By the same argument above, $\theta(r)(m_{(n)}m_{(n')}) = \theta(r)(m_{(n)}) \otimes_{\mathbb{L}} \theta(r)(m_{(n')})$.

Let $\lambda \in \mathbb{P}$ be a partition not of the form (n') for any $n' \in \mathbb{N}$, then $m_{(n)}m_\lambda = \sum_{\mu} c_\mu m_\mu$, where any μ is not of the form (k) with $k \in \mathbb{N}$ and $c_\mu \in \mathbb{N}$. So we have

$$\begin{aligned} \theta(r)(m_{(n)}m_\lambda) &= \min_{\mu} \{\theta(r)(m_\mu)\} \\ &= \min_{\mu} \{\infty\} \\ &= \infty \\ &= \theta(r)(m_{(n)}) \otimes_{\mathbb{L}} \theta(r)(m_\lambda). \end{aligned}$$

Let $\lambda, \lambda' \in \mathbb{P}$ be partitions, neither of which is of the form (n) for any $n \in \mathbb{N}$. Then by the same argument above, we have $\theta(r)(m_\lambda m_{\lambda'}) = \theta(r)(m_\lambda) \otimes_{\mathbb{L}} \theta(r)(m_{\lambda'})$.

So we have shown that $\theta(r) \in \mathbb{W}(\mathbb{L})$ for all $r \in \mathbb{L}$.

Since $\Delta^\times(m_{(n)}) = m_{(n)} \otimes m_{(n)}$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} (\theta(r) \otimes_{\mathbb{W}(\mathbb{L})} \theta(r'))(m_{(n)}) &= \theta(r)(m_{(n)}) + \theta(r')(m_{(n)}) \\ &= n(r + r') \\ &= \theta(r \otimes_{\mathbb{L}} r')(m_{(n)}), \end{aligned}$$

for any $r, r' \in \mathbb{L}$.

For any $\lambda \in \mathbb{P}$ which is not of the form (n) , the term $m_{(k)} \otimes m_{(k)}$ with any $k \in \mathbb{N}$ does not appear in $\Delta^\times(m_\lambda)$. So $(\theta(r) \otimes_{\mathbb{W}(\mathbb{L})} \theta(r'))(m_\lambda) = \infty = \theta(r \otimes_{\mathbb{L}} r')(m_\lambda)$ for any $r, r' \in \mathbb{L}$.

By definition, $\theta(0) = \bar{0}$, and $\theta(r) \leq_{\mathbb{W}(\mathbb{L})} \theta(r')$ if $r \leq_{\mathbb{L}} r'$. ■

3.6. REMARK. Let $r, r' \in \mathbb{L}$, then by definition

$$\theta(r \oplus_{\mathbb{L}} r')(m_\lambda) = \begin{cases} n \min\{r, r'\} & \text{if } \lambda = (n) \text{ for some } n \\ \infty & \text{otherwise.} \end{cases}$$

On the other hand, for any $n \in \mathbb{N}$,

$$\begin{aligned} (\theta(r) \oplus_{\mathbb{W}(\mathbb{L})} \theta(r'))(m_{(n)}) &= \min\{\theta(r)(m_{(n)}), \theta(r')(m_{(n)})\} \\ &= \min\{nr, nr'\} \\ &= \theta(r \oplus_{\mathbb{L}} r')(m_{(n)}), \end{aligned}$$

since $\Delta^+(m_{(n)}) = m_{(n)} \otimes 1 + 1 \otimes m_{(n)}$.

However, by definition, since $\Delta^+(m_{(2,1)}) = m_{(2,1)} \otimes 1 + m_{(2)} \otimes m_{(1)} + m_{(1)} \otimes m_{(2)} + 1 \otimes m_{(2,1)}$, we have

$$(\theta(r) \oplus_{\mathbb{W}(\mathbb{L})} \theta(r'))(m_{(2,1)}) = \min\{\infty, 2r + r', r + 2r'\}.$$

So, in general, $\theta(r \oplus_{\mathbb{L}} r')(m_\lambda) \neq (\theta(r) \oplus_{\mathbb{W}(\mathbb{L})} \theta(r'))(m_\lambda)$, namely θ does not preserve addition and is not a rig homomorphism.

Note that, in contrast to the rig $\mathbb{L}$, the rig $\mathbb{W}(\mathbb{L})$ is not of characteristic one. Indeed, $\bar{0} \oplus_{\mathbb{W}(\mathbb{L})} \bar{0} \neq \bar{0}$, which can be checked directly by $(\bar{0} \oplus_{\mathbb{W}(\mathbb{L})} \bar{0})(m_{(2,1)}) = 0$ and $\bar{0}(m_{(2,1)}) = \infty$.

Thus there does not exist a rig homomorphism from the rig $\mathbb{L}$ to the rig $\mathbb{W}(\mathbb{L})$, but we have a monoidal functor from the monoidal poset $\mathbb{L}$ to the monoidal poset $\mathbb{W}(\mathbb{L})$. Note that the construction of the order $\leq_{\mathbb{L}}$ does not work well for rings as we have seen.

Note also that, for a commutative ring R , there is a famous ring isomorphism $\mathbb{W}(R) \cong 1 + tR[[t]]$. Under this ring isomorphism, the addition in $\mathbb{W}(R)$ corresponds to the usual multiplication in $1 + tR[[t]]$, and the multiplication in $\mathbb{W}(R)$ corresponds to an intricate multiplication in $1 + tR[[t]]$. In addition, we recall that categorical spectra are defined as the stabilisations of (∞, ∞) -categories [16] instead of $(\infty, 0)$ -categories, in which all morphisms are inverted. Therefore, a monoid object in categorical spectra may lack the underlying commutative group-like structure.

3.7. PROPOSITION. The map $\tau : \mathbb{W}(\mathbb{L}) \rightarrow \mathbb{L}$, $\tau(f) = f(m_{(1)})$ is a monoidal functor.

PROOF. Since $\Delta^\times(m_{(1)}) = m_{(1)} \otimes m_{(1)}$, for any $f, g \in \mathbb{W}(\mathbb{L})$, we have

$$\begin{aligned} \tau(f \otimes_{\mathbb{W}(\mathbb{L})} g) &= (f \otimes_{\mathbb{W}(\mathbb{L})} g)(m_{(1)}) \\ &= f(m_{(1)}) + g(m_{(1)}) \\ &= \tau(f) \otimes_{\mathbb{L}} \tau(g). \end{aligned}$$

Again by definition, $\tau(\bar{0}) = 0$, and $\tau(f) \leq_{\mathbb{L}} \tau(g)$ if $f \leq_{\mathbb{W}(\mathbb{L})} g$. ■

3.8. DEFINITION. We define a subset $\mathbb{W}(\mathbb{L})_l$ of $\mathbb{W}(\mathbb{L})$ to be

$$\{f \in \mathbb{W}(\mathbb{L}) \mid nf(m_{(1)}) = f(m_{(n)}), \forall n \in \mathbb{N}\}.$$

3.9. REMARK. ³ Note that, for any $f \in \mathbb{W}(\mathbb{L})$, $f(m_{(n)}) \leq_{\mathbb{L}} nf(m_{(1)})$ holds for any $n \in \mathbb{N}$. By direct calculation, $m_{(1)}^n$ is equal to the sum of $m_{(n)}$ and other terms, namely $m_{(1)}^n = m_{(n)} + \sum_{\lambda} a_{\lambda} m_{\lambda}$. Since f is a rig homomorphism,

$$\begin{aligned} nf(m_{(1)}) &= f(m_{(1)}^n) \\ &= \min\{f(m_{(n)}), f(\sum_{\lambda} a_{\lambda} m_{\lambda})\} \\ &\leq f(m_{(n)}). \end{aligned}$$

Thus, $f \in \mathbb{W}(\mathbb{L})_l$ if and only if $nf(m_{(1)}) \leq_{\mathbb{L}} f(m_{(n)}), \forall n \in \mathbb{N}$.

3.10. PROPOSITION. The monoidal category structure on $\mathbb{W}(\mathbb{L})$ restricts to $\mathbb{W}(\mathbb{L})_l$ and the monoidal functor $\tau : \mathbb{W}(\mathbb{L}) \rightarrow \mathbb{L}$ restricts to the monoidal functor $\tau : \mathbb{W}(\mathbb{L})_l \rightarrow \mathbb{L}$.

PROOF. We need to show the first claim. It is clear that the partial order structure, hence the category structure, on $\mathbb{W}(\mathbb{L})$ restricts to $\mathbb{W}(\mathbb{L})_l$. By definition, $\bar{0} \in \mathbb{W}(\mathbb{L})_l$. Let $f, g \in \mathbb{W}(\mathbb{L})_l$. Then by definition, for all $n \in \mathbb{N}$,

$$\begin{aligned} (f \otimes_{\mathbb{W}(\mathbb{L})} g)(m_{(n)}) &= f(m_{(n)}) + g(m_{(n)}) \\ &= n(f(m_{(1)}) + g(m_{(1)})) \\ &= n(f \otimes_{\mathbb{W}(\mathbb{L})} g)(m_{(1)}). \end{aligned}$$

■

Also by definition we have the following.

3.11. PROPOSITION. The monoidal functor $\theta : \mathbb{L} \rightarrow \mathbb{W}(\mathbb{L})$ factors through the monoidal functor $\theta : \mathbb{L} \rightarrow \mathbb{W}(\mathbb{L})_l$.

These give rise to the following adjunction.

3.12. PROPOSITION. The monoidal functor $\theta : \mathbb{L} \rightarrow \mathbb{W}(\mathbb{L})_l$ is left adjoint to $\tau : \mathbb{W}(\mathbb{L})_l \rightarrow \mathbb{L}$.

PROOF. Let $r \in \mathbb{L}$ and $f \in \mathbb{W}(\mathbb{L})_l$. Suppose $\theta(r) \leq_{\mathbb{W}(\mathbb{L})} f$. Then by definition for any $\lambda \in \mathbb{P}$, we have $\theta(r)(m_{\lambda}) \leq_{\mathbb{L}} f(m_{\lambda})$. Again by definition,

$$\theta(r)(m_{\lambda}) = \begin{cases} nr & \text{if } \lambda = (n) \text{ for some } n \\ \infty & \text{otherwise.} \end{cases}$$

So we have $r \leq_{\mathbb{L}} f(m_{(1)}) = \tau(f)$.

Conversely, suppose $r \leq_{\mathbb{L}} \tau(f)$. Since $\tau(f) = f(m_{(1)})$, for all $n \in \mathbb{N}$, we have $nr \leq_{\mathbb{L}} nf(m_{(1)})$. Since $f \in \mathbb{W}(\mathbb{L})_l$, we have $nr \leq_{\mathbb{L}} f(m_{(n)})$. By the definition of $\leq_{\mathbb{W}(\mathbb{L})}$ and that of $\theta(r)$, we have $\theta(r) \leq_{\mathbb{W}(\mathbb{L})} f$. ■

³This was pointed out to the author by the anonymous referee. In an earlier draft, the author defined $\mathbb{W}(\mathbb{L})_l$ to be $\{f \in \mathbb{W}(\mathbb{L}) \mid nf(m_{(1)}) \leq_{\mathbb{L}} f(m_{(n)}), \forall n \in \mathbb{N}\}$.

3.13. BASIC PROPERTIES OF $\mathbb{W}(\mathbb{L})$ -CATEGORIES. Let $\mathbb{W}(\mathbb{L})\text{-Cat}$ denote the category of (small) categories enriched over the monoidal poset $\mathbb{W}(\mathbb{L})$. Let $X \in \mathbb{W}(\mathbb{L})\text{-Cat}$. Then we have a hom-map $d : X \times X \rightarrow \mathbb{W}(\mathbb{L})$, where, by abusing notation, we write X for the set of objects in X , and $\mathbb{W}(\mathbb{L})$ for the set of objects in $\mathbb{W}(\mathbb{L})$, which satisfies

$$\bar{0} \leq_{\mathbb{W}(\mathbb{L})} d(x, x)$$

$$d(x, x') \otimes_{\mathbb{W}(\mathbb{L})} d(x', x'') \leq_{\mathbb{W}(\mathbb{L})} d(x, x''),$$

for any $x, x', x'' \in X$.

For any $\lambda \in \mathbb{P}$, we define for simplicity the map $d_\lambda : X \times X \rightarrow \mathbb{L}$ given by $d_\lambda(x, x') = d(x, x')(m_\lambda)$ for any $x, x' \in X$.

Then, from the first inequality above, we obtain by the definition of $\bar{0}$

$$0 = d_{(n)}(x, x)$$

for any $n \in \mathbb{N}$. Note that for any partition $\lambda \in \mathbb{P}$ not of the form (n) , we have only $\infty \geq d_\lambda(x, x)$, which is trivial. From the second one above, for any $\lambda \in \mathbb{P}$ we have

$$d_\lambda(x, x'') \leq \min_{\mu, \mu'} \{d_\mu(x, x') + d_{\mu'}(x', x'')\},$$

where μ and μ' run over the sum $\Delta^\times(m_\lambda) = \sum_{\mu, \mu'} c_{\mu, \mu'}^\lambda m_\mu \otimes m_{\mu'}$.

3.14. REMARK. Let X be a $\mathbb{W}(\mathbb{L})$ -category. For $x, x' \in X$, the distance $d(x, x')$ is not yet determined in some sense. Namely, we can evaluate it at $f \in \Lambda$ to obtain an observed value $d_f(x, x') := d(x, x')(f) \in [0, \infty]$. If f is expressed as $\sum a_\lambda m_\lambda$ with $a_\lambda \in \mathbb{N}$, the observed value is given by $d_f(x, x') = \min\{d_\lambda(x, x')\}$. The partition achieving the minimum depends on x, x' and f .

We explore some basic properties of $\mathbb{W}(\mathbb{L})$ -categories.

3.15. PROPOSITION. For any $\mathbb{W}(\mathbb{L})$ -category (X, d) and any $n \in \mathbb{N}$, $(X, d_{(n)})$ is an $\mathbb{L}$ -category.

PROOF. By the definition of $\mathbb{W}(\mathbb{L})$, for any $x \in X$ we have

$$\bar{0} \leq_{\mathbb{W}(\mathbb{L})} d(x, x).$$

Since $\bar{0}(m_{(n)}) = 0$, we have $0 = d_{(n)}(x, x)$ as we saw. Also for any $x, y, z \in X$, by the definition of $\mathbb{W}(\mathbb{L})$, we have

$$d(x, z) \geq_{\mathbb{W}(\mathbb{L})} d(x, y) \otimes_{\mathbb{W}(\mathbb{L})} d(y, z)$$

Since $\Delta^\times(m_{(n)}) = m_{(n)} \otimes m_{(n)}$, by the definition of $\leq_{\mathbb{W}(\mathbb{L})}$, we have

$$d_{(n)}(x, z) \leq d_{(n)}(x, y) + d_{(n)}(y, z).$$

■

So a $\mathbb{W}(\mathbb{L})$ -category (X, d) contains a sequence⁴ $\{(X, d_{(n)})\}_{n \in \mathbb{N}}$ of $\mathbb{L}$ -categories, with a common underlying set, indexed by $\mathbb{N}$, which may be viewed as discrete time. One may think of it as the shape of X changing moment by moment as time passes, while the particles remain the same. This sequence corresponds to the top line in the graph (2.1).

Note that there may be some particles $x, x' \in X$ with $x \neq x'$ that satisfy $d_{(n)}(x, x') = 0$ at some phase n and $d_{(n')}(x, x') \neq 0$ at another phase n' .

3.16. PROPOSITION. *For any $\mathbb{W}(\mathbb{L})$ -category (X, d) and any $\lambda \in \mathbb{P}$, d_λ satisfies the triangle inequality.*

PROOF. We prove, for any $x, y, z \in X$ and any $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{P}$, $d_\lambda(x, z) \leq d_\lambda(x, y) + d_\lambda(y, z)$.

By the definition of $\mathbb{W}(\mathbb{L})$ -category, we have

$$d_\lambda(x, z) \leq (d(x, y) \otimes_{\mathbb{W}(\mathbb{L})} d(y, z))(m_\lambda).$$

Again by the definition of $\otimes_{\mathbb{W}(\mathbb{L})}$, we have

$$(d(x, y) \otimes_{\mathbb{W}(\mathbb{L})} d(y, z))(m_\lambda) = \min_{\mu, \mu'} \{d_\mu(x, y) + d_{\mu'}(y, z)\},$$

where μ and μ' are the partitions which appear in the sum

$$\Delta^\times(m_\lambda) = \sum_{\mu, \mu'} c_{\mu, \mu'}^\lambda m_\mu \otimes m_{\mu'}.$$

By definition, the term $(x_1 \otimes x_1)^{\lambda_1} \cdots (x_n \otimes x_n)^{\lambda_n} = (x_1^{\lambda_1} \cdots x_n^{\lambda_n}) \otimes (x_1^{\lambda_1} \cdots x_n^{\lambda_n})$ appears in $\Delta^\times(m_\lambda) = m_\lambda(x_1 \otimes x_1, x_1 \otimes x_2, \dots, x_i \otimes x_j, \dots)$ at least once.

Therefore, we have

$$(d(x, y) \otimes_{\mathbb{W}(\mathbb{L})} d(y, z))(m_\lambda) \leq d_\lambda(x, y) + d_\lambda(y, z).$$

■

For a $\mathbb{W}(\mathbb{L})$ -category (X, d) and any $\lambda \in \mathbb{P}$ not of the form (n) for any $n \in \mathbb{N}$, (X, d_λ) is still a kind of metric space. However in them there may exist a particle $x \in X$ for which the self-distance $d_\lambda(x, x)$ is not 0. Note that, in a partial metric space, a point can have non-zero distance from itself [15].

Thus a $\mathbb{W}(\mathbb{L})$ -category (X, d) consists of a family $\{(X, d_\lambda)\}_{\lambda \in \mathbb{P}}$ of space-like structures indexed by $\mathbb{P}$ with a common underlying set, whose components are related by the following inequalities mentioned above, namely for any $x, x', x'' \in X$ and $\lambda \in \mathbb{P}$,

$$d_\lambda(x, x'') \leq \min_{\mu, \mu'} \{d_\mu(x, x') + d_{\mu'}(x', x'')\},$$

where μ and μ' are the partitions appearing in $\Delta^\times(m_\lambda)$. Note that, by the definition of $\Delta^\times$, we have $\mu, \mu' \in \mathbb{P}_{|\lambda|}$ for such μ and μ' .

Thus it may be natural to think about the complete symmetric functions $h_n = \sum_{\lambda \in \mathbb{P}_n} m_\lambda$ of degree n , for all n .

⁴The anonymous referee suggests that this sequence would be considered to be the “ghost components” of (X, d) , as $m_{(n)}$ ’s correspond to the ghost components of Witt vectors.

3.17. DEFINITION. For $(X, d) \in \mathbb{W}(\mathbb{L})$ and $n \in \mathbb{N}$, we define d_n as follows:

$$d_n : X \times X \rightarrow \mathbb{L}, (x, y) \mapsto d(x, y)(h_n).$$

3.18. PROPOSITION. For any $\mathbb{W}(\mathbb{L})$ -category (X, d) and any $n \in \mathbb{N}$, (X, d_n) is an $\mathbb{L}$ -category.

PROOF. For any $x \in X$, by the definition of d_n , we have

$$d_n(x, x) = \min_{|\lambda|=n} \{d_\lambda(x, x)\} \leq d_{(n)}(x, x) = 0.$$

Let $x, y, z \in X$. By definition, we have

$$\begin{aligned} d_n(x, z) &\leq (d(x, y) \otimes_{\mathbb{W}(\mathbb{L})} d(y, z))(h_n) \\ &= \min_{|\lambda|=n} \{(d(x, y) \otimes_{\mathbb{W}(\mathbb{L})} d(y, z))(m_\lambda)\} \\ &= \min_{|\lambda|=n} \{\min_{\mu, \mu'} \{d_\mu(x, y) + d_{\mu'}(y, z)\}\}. \end{aligned}$$

Here, μ and μ' are the partitions which appear in $\Delta^\times(m_\lambda)$. So in particular, $|\mu| = |\mu'| = n$. Moreover, in $\min_{|\lambda|=n} \{\min_{\mu, \mu'} \{d_\mu(x, y) + d_{\mu'}(y, z)\}\}$, λ runs over all partitions of size n . Therefore, we have

$$\begin{aligned} (d(x, y) \otimes_{\mathbb{W}(\mathbb{L})} d(y, z))(h_n) &= \min_{|\mu|=|\mu'|=n} \{d_\mu(x, y) + d_{\mu'}(y, z)\} \\ &= \min_{|\mu|=n} \{d_\mu(x, y)\} + \min_{|\mu'|=n} \{d_{\mu'}(y, z)\} \\ &= d_n(x, y) + d_n(y, z). \end{aligned}$$

■

3.19. REMARK. Consider, for example, the Plancherel measure on $\mathbb{P}_n$ for every $n \in \mathbb{N}$. These give rise to a Poissonized Plancherel measure on $\mathbb{P}$, shown in (2.1), which is a standard tool for the theory of asymptotic analysis of the Young lattice. See [8] for more detail.

Let $X \in \mathbb{W}(\mathbb{L})\text{-Cat}$. Then we obtain the family $\{(X, d_\lambda)\}_{\lambda \in \mathbb{P}}$. So it might be fine to say the enriched category X collects all possible ways in which a space may grow probabilistically. Note that the probability that a randomly chosen λ is of the form $(|\lambda|)$ tends to decrease as $|\lambda|$ becomes larger. One may also consider other stochastic processes on $\mathbb{P}$.

As Proposition 3.18 shows, $\{(X, d_n)\}_{n \in \mathbb{N}}$ is a sequence of $\mathbb{L}$ -categories, which does not depend on the measure. It might also be fine to say a $\mathbb{W}(\mathbb{L})$ -category X includes a discrete-time sequence of $\mathbb{L}$ -categories.

By replacing the base category $\mathbb{L}$ with $\mathbb{W}(\mathbb{L})$, we can endow spaces with a structure related to the composition operator on Λ in the following sense. Since $\mathbb{W}$ is a comonad, there is a map $c_{\mathbb{L}} : \mathbb{W}(\mathbb{L}) \rightarrow \mathbb{W}(\mathbb{W}(\mathbb{L}))$, which is induced by the composition operator on Λ .

Let $(X, d) \in \mathbb{W}(\mathbb{L})\text{-}\mathbf{Cat}$. Then we have a map

$$c_{\mathbb{L}} \circ d : X \times X \rightarrow \mathbb{W}(\mathbb{W}(\mathbb{L})),$$

defined for all $x, x' \in X$ by

$$(c_{\mathbb{L}} \circ d)(x, x') : \Lambda \rightarrow \mathbb{W}(\mathbb{L}),$$

where for each $f \in \Lambda$,

$$((c_{\mathbb{L}} \circ d)(x, x'))(f) : \Lambda \rightarrow \mathbb{L}, g \mapsto d_{g \circ f}(x, x').$$

Let $|X|$ denote the family $\{(X, d_f)\}_{f \in \Lambda}$, which carries a natural rig structure with addition $(X, d_f) \oplus (X, d_{f'}) := (X, d_{f+f'})$ and multiplication $(X, d_f) \otimes (X, d_{f'}) := (X, d_{f \cdot f'})$. We let $\text{End}(|X|)$ denote the set of maps from $|X|$ to itself, and equip it with the induced rig structure. By the definition of the composition operator on Λ , we have the following action, which allows us to leap from one phase of X to another.

3.20. PROPOSITION. *Let $(X, d) \in \mathbb{W}(\mathbb{L})\text{-}\mathbf{Cat}$. Then the action*

$$\Lambda \rightarrow \text{End}(|X|), \quad g \mapsto ((X, d_f) \mapsto (X, d_{g \circ f}))$$

is a rig homomorphism.

We investigate the relationship between $\mathbb{W}(\mathbb{L})$ -categories and $\mathbb{L}$ -categories. The functors θ and τ defined above induce functors again denoted by

$$\theta : \mathbb{L}\text{-}\mathbf{Cat} \rightarrow \mathbb{W}(\mathbb{L})\text{-}\mathbf{Cat}$$

and

$$\tau : \mathbb{W}(\mathbb{L})\text{-}\mathbf{Cat} \rightarrow \mathbb{L}\text{-}\mathbf{Cat}$$

respectively.

By construction, we have the following.

3.21. PROPOSITION. $\theta : \mathbb{L}\text{-}\mathbf{Cat} \rightarrow \mathbb{W}(\mathbb{L})\text{-}\mathbf{Cat}$ *is fully faithful.*

3.22. REMARK. *For $(X, d) \in \mathbb{L}\text{-}\mathbf{Cat}$ we denote $\theta(X, d) = (X, \tilde{d}) \in \mathbb{W}(\mathbb{L})\text{-}\mathbf{Cat}$. As we saw above, for any $n \in \mathbb{N}$ we have $(X, \tilde{d}_{(n)}) \in \mathbb{L}\text{-}\mathbf{Cat}$ such that for any $x, x' \in X$, $\tilde{d}_{(n)}(x, x') = nd(x, x')$. Thus we may obtain a sequence $\{(X, \tilde{d}_{(n)})\}_{n \in \mathbb{N}}$ of $\mathbb{L}$ -categories, linearly and uniformly expanding over discrete time, with $(X, \tilde{d}_{(1)}) = (X, d)$.*

Similarly, for any $\lambda \in \mathbb{P}$ which is not of the form (n) for any $n \in \mathbb{N}$, and for any $x, x' \in X$, we have $\tilde{d}_{\lambda}(x, x') = \infty$. Note that $(X, \tilde{d}_{\lambda})$ is no longer an $\mathbb{L}$ -category, but it is a kind of discrete space. If we consider the Plancherel growth process, the probability that a randomly chosen λ is of the form $(|\lambda|)$ will tend to decrease as $|\lambda|$ becomes larger, that is, as time passes. So it becomes increasingly difficult for $(X, \tilde{d}_{\lambda})$ to be an $\mathbb{L}$ -category as $|\lambda|$ becomes larger.

Note that, if a partition λ is not of the form $(|\lambda|)$, then $(X, \tilde{d}_\lambda)$ is not an $\mathbb{L}$ -category and the subsequent one also cannot be an $\mathbb{L}$ -category, since the transition probability from $(\lambda_1, \lambda_2, \dots, \lambda_k)$ with $\lambda_2 \neq 0$ to $(|\lambda| + 1)$ is 0.

Note also that we have $\tilde{d}_n = \tilde{d}_{(n)}$ for all $n \in \mathbb{N}$. So we cannot distinguish $\{(X, \tilde{d}_{(n)})\}_{n \in \mathbb{N}}$ from the non-stochastic sequence $\{(X, \tilde{d}_n)\}_{n \in \mathbb{N}}$.

Conversely, let $(Y, d) \in \mathbb{W}(\mathbb{L})\text{-Cat}$. By taking $\tau : \mathbb{W}(\mathbb{L})\text{-Cat} \rightarrow \mathbb{L}\text{-Cat}$, we have its initial-state $\mathbb{L}$ -category $(Y, d_{(1)})$.

The Lawvere quantale $\mathbb{L}$ is an example of (commutative) quantales. One might consider the Witt vectors in other quantales as well. Also, Λ is an example of composition algebras [17, 5, 7], so one might choose other composition algebras. It would be worth emphasizing that the combinatorial or probabilistic aspect of $\mathbb{P}$ has contributed to the theory of Witt vectors.

3.23. AN EXAMPLE OF $\mathbb{W}(\mathbb{L})$ -CATEGORIES. In this section, we describe a very explicit example of $\mathbb{W}(\mathbb{L})$ -categories to help the reader develop an intuition for $\mathbb{W}(\mathbb{L})$ -categories.

Let X be the three-point set $\{x, x', x''\}$. We are defining d_λ on X for any partition $\lambda \in \mathbb{P}$.

By convention $m_{(0)} = 1$, so the space $(X, d_{(0)})$ must be collapsed. Namely, for any $a, b \in X$,

$$d_{(0)}(a, b) = 0.$$

If the weight of a partition were regarded as a kind of discrete time, then any (non-empty) $\mathbb{W}(\mathbb{L})$ -category would be viewed as a one-point space at the very beginning.

Next, we define

$$\begin{aligned} d_{(1)}(x, x) &= d_{(1)}(x', x') = d_{(1)}(x'', x'') = 0, \\ d_{(1)}(x, x') &= d_{(1)}(x', x) = 3, \\ d_{(1)}(x', x'') &= d_{(1)}(x'', x') = 4, \\ d_{(1)}(x, x'') &= d_{(1)}(x'', x) = 5. \end{aligned}$$

So, one may view $(X, d_{(1)})$ as a right triangle. We could consider a non-symmetric metric space, but for simplicity, let us focus on a symmetric one here.

Then we consider when $|\lambda| = 2$. By direct calculation, we know the following inequalities must hold:

$$d_{(2)}(a, c) \leq d_{(2)}(a, b) + d_{(2)}(b, c)$$

$$\begin{aligned} d_{(1,1)}(a, c) &\leq \min\{d_{(2)}(a, b) + d_{(1,1)}(b, c), d_{(1,1)}(a, b) + d_{(2)}(b, c), \\ &\quad d_{(1,1)}(a, b) + d_{(1,1)}(b, c)\} \end{aligned}$$

for any $a, b, c \in X$.

We set $d_{(2)} = d_{(1)}$. So $(X, d_{(2)})$ may still be viewed as the right triangle.

We also define

$$d_{(1,1)}(x, x) = d_{(1,1)}(x', x') = d_{(1,1)}(x'', x'') = 0,$$

$$\begin{aligned} d_{(1,1)}(x, x') &= d_{(1,1)}(x', x) = d_{(1,1)}(x, x'') = d_{(1,1)}(x'', x) = 2, \\ d_{(1,1)}(x', x'') &= d_{(1,1)}(x'', x') = 4. \end{aligned}$$

So $(X, d_{(1,1)})$ may be viewed as a segment with length 4. One can check that the inequalities above indeed hold.

Recalling the Markov process of partitions, shown in (2.1), one might think of the right triangle $(X, d_{(1)})$ as staying the same right triangle $(X, d_{(2)})$ one unit of time later with probability $1/2$, and as becoming a line segment $(X, d_{(1,1)})$ with probability $1/2$.

We next consider when $|\lambda| = 3$. By direct calculation again, we know the following inequalities must hold:

$$d_{(3)}(a, c) \leq d_{(3)}(a, b) + d_{(3)}(b, c)$$

$$\begin{aligned} d_{(2,1)}(a, c) &\leq \min\{d_{(2,1)}(a, b) + d_{(2,1)}(b, c), d_{(2,1)}(a, b) + d_{(3)}(b, c), \\ &\quad d_{(3)}(a, b) + d_{(2,1)}(b, c)\} \end{aligned}$$

$$\begin{aligned} d_{(1,1,1)}(a, c) &\leq \min\{d_{(3)}(a, b) + d_{(1,1,1)}(b, c), d_{(1,1,1)}(a, b) + d_{(3)}(b, c), \\ &\quad d_{(2,1)}(a, b) + d_{(2,1)}(b, c), d_{(2,1)}(a, b) + d_{(1,1,1)}(b, c), \\ &\quad d_{(1,1,1)}(a, b) + d_{(2,1)}(b, c), d_{(1,1,1)}(a, b) + d_{(1,1,1)}(b, c)\} \end{aligned}$$

for any $a, b, c \in X$.

Note that $d_{(1,1,1)}$ is constrained by both $d_{(2,1)}$ and $d_{(3)}$, and $d_{(2,1)}$ is constrained by $d_{(3)}$. In other words, partitions of greater length are constrained by partitions of smaller length, but not conversely. In particular, the ghost component is not constrained by any other partition.

We set $d_{(3)} = d_{(2)}$.

We define

$$\begin{aligned} d_{(2,1)}(x, x) &= d_{(2,1)}(x', x') = d_{(2,1)}(x'', x'') = d_{(2,1)}(x, x') = d_{(2,1)}(x', x) = 0, \\ d_{(2,1)}(x', x'') &= d_{(2,1)}(x'', x') = d_{(2,1)}(x, x'') = d_{(2,1)}(x'', x) = 4. \end{aligned}$$

So $(X, d_{(2,1)})$ may be viewed as a segment with length 4 and one of the endpoints is doubled.

We define

$$\begin{aligned} d_{(1,1,1)}(x, x) &= d_{(1,1,1)}(x', x') = d_{(1,1,1)}(x, x') = d_{(1,1,1)}(x', x) = 0, \\ d_{(1,1,1)}(x'', x'') &= 1, \\ d_{(1,1,1)}(x', x'') &= d_{(1,1,1)}(x'', x') = d_{(1,1,1)}(x, x'') = d_{(1,1,1)}(x'', x) = 1. \end{aligned}$$

So $(X, d_{(1,1,1)})$ may be viewed as a segment with length 1, with one of the endpoints doubled and the other one having nontrivial self distance. Since $d_{(1,1,1)}(x'', x'') \neq 0$, $(X, d_{(1,1,1)})$ is not an $\mathbb{L}$ -category.

One can check that the inequalities above indeed hold.

When $|\lambda| = 3$, the probability that X keeps the same shape as the right triangle is considered to be $1/6$. Moreover, if X has already become the line segment when $|\lambda| = 2$, then it can never return to the right triangle at the next time step.

For $\lambda \in \mathbb{P}$ with $|\lambda| \geq 4$, we define $d_\lambda(a, b) = \infty$ for any $a, b \in X$. Note that if $|\lambda| \neq |\mu|$, then d_λ and d_μ do not constrain each other. Thus, from $|\lambda| = 4$ onward, the space X may be viewed as having completely fallen apart.

If we observe $d(x, x')$ via a symmetric function $f \in \Lambda$ and obtain a finite value $d_f(x, x')$, then we know that f contains no m_λ with $|\lambda| \geq 4$.

As we have shown, (X, d_3) is an $\mathbb{L}$ -category. Explicitly we have

$$d_3(x, x) = d_3(x', x') = d_3(x'', x'') = d_3(x, x') = d_3(x', x) = 0,$$

$$d_3(x', x'') = d_3(x'', x') = d_3(x, x'') = d_3(x'', x) = 1$$

By the definition of plethysm, we have

$$m_{(1,1)} \circ m_{(2)} = m_{(1,1)}(x_1^2, x_2^2, \dots) = x_1^2 x_2^2 + x_1^2 x_3^2 + \dots = m_{(2,2)}.$$

By Proposition 3.20, the action of $m_{(1,1)}$ allows us to leap from $(X, d_{(2)})$ to $(X, d_{(2,2)})$, namely, from the right triangle to the completely fallen-apart state.

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