

UNIVERSAL CATEGORIES

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ABSTRACT. The category of models of any theory T in any first-order language L has the surprising property that any small category that is elementarily equivalent with it, already embeds in it. The proof uses an abstract argument via ultrapowers, leading one to wonder which concrete categorical axioms, depending on T and L , are responsible for this embedding result.

1. Introduction

The key notion of this paper is that of universality: let us call an (uncountable) L -structure *universal*, if any elementarily equivalent L -structure of lesser cardinality embeds in it. Uncountably categorical structures are examples of universal structures, as are saturated structures, but beyond these examples—which, incidentally, have the stronger property that the embedding can be taken to be elementary—, there seem to be no immediate examples. For instance, the real field is not universal, for there exist countable non-Archimedean real closed fields, which therefore are elementarily equivalent with the reals but can never be embedded in them.

In this paper, we will encounter nonetheless an abundance of examples, actually, one for each first-order theory T , to wit, the category of models of T , viewed as a first-order structure in the language of categories. Since categories make a distinction between objects and morphisms, a two-sorted language L_{cat} is therefore best suited for the present model-theoretic investigation. Apart from that, one needs a (partial) binary operation for composing morphisms, a unary function assigning to an object its identity morphism, and two unary functions assigning to a morphism its domain and range.¹ The axioms in this language are self-evident and the resulting L_{cat} -theory **CAT** will be called the *theory of categories*. To see how natural all this is: any **CAT**-model is in an obvious way a category, any substructure is a subcategory and the L_{cat} -homomorphisms between categories are just the functors. Moreover, this language is very expressive: the existence of any finite limit or co-limit is now first-order expressible. But the main observation is universality (see Theorem 6.5):

1.1. THEOREM. *Any category of the form $\text{Mod}_{\mathcal{U}}(T)$, for some theory T in some first-order*

Received by the editors 2026-01-13 and, in final form, 2026-10-07.

Transmitted by Tom Leinster. Published on 2026-10-27.

2020 Mathematics Subject Classification: 18A15, 03C52.

Key words and phrases: universal categories.

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¹In a future paper, [7], a one-sorted language instead is introduced, which, among other things, circumvents the issue that composition in L_{cat} is only partially defined.

language L , is universal.

Here, $\mathsf{Mod}_{\mathfrak{U}}(T)$ denotes the category whose objects are the models of T and whose morphisms are arbitrary L -homomorphisms. To not run into foundational issues, we restrict moreover to those models that belong to a once-and-for-all fixed Grothendieck universe $\mathfrak{U}$ (see §4 for more details).

In this introductory section, let me sketch a proof in one particular case, that of the category of Abelian groups $\mathsf{Mod}_{\mathfrak{U}}(T_{\text{Ab}})$, where T_{Ab} is the theory of Abelian groups in the language $L = (+, -, 0)$. This argument is how one would like to prove the main theorem, by explicitly giving the axioms. Unfortunately, so far, this is basically the only case where I succeeded in doing so. The general case will be proven using ultrapowers: it turns out that any ultrapower $\mathsf{Mod}_{\mathfrak{U}}(T)_{\mathfrak{d}}$ (with respect to an ultrafilter belonging to the universe $\mathfrak{U}$) of any category of models $\mathsf{Mod}_{\mathfrak{U}}(T)$ is already a subcategory of the original $\mathsf{Mod}_{\mathfrak{U}}(T)$. Together with a characterization of elementarity using ultrapowers—Proposition 3.2, a corollary of the famous Keisler-Shelah criterion, but which can easily be proven directly—, this will yield the main theorem. But let me explain what I consider to be a better proof at the hand of the category of Abelian groups.

1.2. ABELIAN GROUPS. Let $\mathbb{C}$ be an arbitrary category in our Grothendieck universe $\mathfrak{U}$. Assume it admits a null object 0 , meaning that for any object G , there are unique morphisms $0 \rightarrow G$ and $G \rightarrow 0$; their composition $G \rightarrow 0 \rightarrow G$ will be denoted again by 0 . We will also assume that $\mathbb{C}$ admits (finite) products $G_1 \times G_2$ with projection maps π_1 and π_2 . There are therefore unique morphisms $\Delta_G: G \rightarrow G \times G$ and $\tau_G: G \times G \rightarrow G \times G$, the *diagonal* and the *twist* respectively, such that $\pi_k \circ \Delta_G = 1_G$ and $\pi_k \circ \tau_G \circ \Delta_G = \pi_l$, with $\{k, l\} = \{1, 2\}$.² Let us call a morphism $\mu: G \times G \rightarrow G$ a *group arrow*, if it satisfies the following four conditions

1. (Associativity) $\mu \circ (1_G \times \mu) = \mu \circ (\mu \times 1_G)$;
2. (Unit) $\mu \circ (0 \times 1_G) = \pi_2$;
3. (Inverse) there exists a morphism $\lambda: G \rightarrow G$, such that $\mu \circ (1_G \times \lambda) \circ \Delta_G = 0$;
4. (Commutativity) $\mu \circ \tau_G = \mu$;

Tacitly assumed in these axioms are the various isomorphisms between products, such as $(G \times G) \times G \cong G \times (G \times G)$, which we simply denote by $G \times G \times G$, etc. If G is an actual (additive) Abelian group, then we can take $\mu(a, b) := a + b$ and $\lambda(a) := -a$. Note that commutativity is needed for μ and λ to be actual group homomorphisms.

To define the L_{cat} -theory **AB**, we now add to the existence of finite products and a null-object, the two axioms stating (Ab1) that any object G admits a unique group arrow μ_G , and (Ab2), if $f: G \rightarrow H$ is an arbitrary arrow, with μ_G and μ_H the respective unique group arrows on G and H , then

²In the category of sets, this would mean that $\Delta_G(a) = (a, a)$ and $\tau_G(a, b) = (b, a)$.

5. (Linearity) $\mu_H \circ (f \times f) = f \circ \mu_G$.

To see that $\mathsf{Mod}_{\mathfrak{U}}(T_{\mathsf{Ab}})$ satisfies these axioms—using the group law to define the group arrow—, everything trivially holds except the uniqueness requirement in (Ab1). But this follows from the following simple observation:

1.3. LEMMA. *Let $(G, +, 0)$ be an Abelian group and suppose $\mu: G \times G \rightarrow G$ defines a monoidal operation on G with the same neutral element 0. If μ is additive (for the induced addition on $G \times G$), then μ is just the group law.*

PROOF. By assumption, $\mu(a + a', b + b') = \mu(a, a') + \mu(b, b')$, for all $a, a', b, b' \in G$. Take $a' = 0 = b$, so that $\mu(a, b') = \mu(a, 0) + \mu(0, b')$ and since 0 is also the neutral element for the μ -operation, the latter is just $a + b'$. ■

Objects in a category $\mathbb{C} \models \mathbf{AB}$ are actually not yet groups, as the group arrow is only formal. But any functor (preserving products and null objects) into the category of sets will turn them into actual groups and morphisms into group homomorphisms. This is best achieved by positing the existence of a generator Z and taking as actualization the (faithful) functor $\mathrm{Hom}_{\mathbb{C}}(Z, -)$ (for details, see §4). In this way, $\mathbb{C}$ becomes a subcategory of $\mathsf{Mod}_{\mathfrak{U}}(T_{\mathsf{Ab}})$. We will therefore also add the axiom asserting the existence of a generator to $\mathbf{AB}$, and so we proved:

1.4. COROLLARY. *The category $\mathsf{Mod}_{\mathfrak{U}}(T_{\mathsf{Ab}})$ is universal.*

PROOF. If $\mathbb{C}$ is a category in the universe $\mathfrak{U}$ which is elementarily equivalent to $\mathsf{Mod}_{\mathfrak{U}}(T_{\mathsf{Ab}})$, it satisfies $\mathbf{AB}$, and hence the faithful functor $\mathrm{Hom}_{\mathbb{C}}(Z, -): \mathbb{C} \rightarrow \mathsf{Mod}_{\mathfrak{U}}(T_{\mathsf{Ab}})$ yields the desired embedding. ■

1.5. REMARK. Instead of considering all homomorphisms, model-theorists sometimes just work within the subcategory where the morphisms are restricted to elementary embeddings. However, one can obtain this special case from the present treatment through Morleyization. I thank the anonymous referee for pointing this out to me.

The use of category theory to analyze model-theory is of course not new: Makkai's work led to what is now called FOCL (=first-order categorical logic; see [4, 5]); or Lawvere's perspective—from whom I also borrow the use of the language L_{cat} —leading to ETCC (=elementary theory of the category of categories; see [2, 3]) and its variants, just to mention a few. We use these perspectives in the present paper only in a very simple-minded way, by combining them in order to study categories of the type $\mathsf{Mod}_{\mathfrak{U}}(T)$.

Of course, in going from a first-order theory in an arbitrary language to the first-order theory of its category of models, something must be lost in translation, for there are only continuum many (complete) theories of (small) categories, while there are far more languages L and first-order L -theories. This raises the question how much is lost in translation? And how essential is this loss? Let me illustrate this with a simple example. Consider, let's say, the first-order theory of fields FLD in the language of rings L_{ring} and the corresponding category of its small models $\mathsf{Mod}_{\mathfrak{U}}(FLD)$ (see §4 for

notation). Let K be a field of particular interest (of possibly large cardinality) and take its atomic diagram in the language $L_{\text{ring}}(K)$. The category of models of this latter theory is actually the co-slice category $K \downarrow \mathbb{M}\text{od}_{\mathbb{U}}(FLD)$ and therefore definable (with parameter K) inside $\mathbb{M}\text{od}_{\mathbb{U}}(FLD)$. As there are more than continuum many choices for K , there are non-elementarily equivalent fields K_1 and K_2 whose co-slice categories are elementary equivalent. How are K_1 and K_2 then ‘algebraically related’? Since they are both definable in $\mathbb{M}\text{od}_{\mathbb{U}}(FLD)$, therein then may lay the answer?

2. The model-theory of categories

While many categories have class-many objects and/or morphisms, technically, they cannot be considered as first-order structures. We will introduce the proper framework to deal with this issue, namely, Grothendieck universes, in §4. For now, we simply assume that our categories are small (i.e., have set-many objects and morphisms). Let L_{cat} be a two-sorted language, with the $\mathbf{o}$ -sort for objects and the $\mathbf{m}$ -sort for morphisms, together with a binary function $\circ$ on the $\mathbf{m}$ -sort, two unary functions dom and rng from the $\mathbf{m}$ -sort to the $\mathbf{o}$ -sort, and a unary function Id from the $\mathbf{o}$ -sort to the $\mathbf{m}$ -sort subject. The interpretation of these symbols in an arbitrary category $\mathbb{C}$ are then respectively the composition $g \circ f$ of morphisms $g: B \rightarrow C$ and $f: A \rightarrow B$, the domain A and range B of a morphism $f: A \rightarrow B$, and the identity morphism 1_A on an object A . Note that we treat $\circ$ as a partial function and we do not care what it does on non-composable morphisms. We take the convention that variables ranging over the $\mathbf{o}$ -sort are capitalized and those over the $\mathbf{m}$ -sort are lower-case, so that we do not need to specify this each time.

2.1. DEFINITION. Let **CAT** be the theory given by the following three axioms

1. $(\forall x, y)$, if $\text{rng}(x) = \text{dom}(y)$, then $\text{dom}(x) = \text{dom}(y \circ x)$ and $\text{rng}(y) = \text{rng}(y \circ x)$;
2. $(\forall x, y, z)$, if $\text{rng}(x) = \text{dom}(y)$ and $\text{rng}(y) = \text{dom}(z)$, then $(z \circ y) \circ x = z \circ (y \circ x)$;
3. $(\forall X, y)$, if $\text{dom}(y) = X$ (respectively, $\text{rng}(y) = X$), then $y \circ \text{Id}(X) = y$ (respectively, $\text{Id}(X) \circ y = y$);

To conform more to the usual notation, we will adopt the following notational conventions for arbitrary L_{cat} -structures. If f is a morphism with $A = \text{dom}(f)$ and $B = \text{rng}(f)$, then we simply indicate this by writing $f: A \rightarrow B$. The identity morphism $\text{Id}(A)$ on an object A is denoted simply 1_A . This is justified by the following fact.

2.2. LEMMA. If $\mathcal{C}$ is a model of **CAT**, then one can associate a category $\mathbb{C}$ to it such that as L_{cat} -structures, they are isomorphic. Moreover, if $\mathcal{D}$ is a second model of **CAT** with associated category $\mathbb{D}$, then the L_{cat} -homomorphisms between $\mathcal{C}$ and $\mathcal{D}$ induce precisely the functors between $\mathbb{C}$ and $\mathbb{D}$.

PROOF. Consider the category $\mathbb{C}$ by taking for objects, the elements of the $\mathbf{o}$ -sort in $\mathcal{C}$, and for morphisms, the elements of the $\mathbf{m}$ -sort, with composition of morphisms given by $\circ$ and the identity morphism given by Id . The axioms of **CAT** then guarantee that $\mathbb{C}$ is a category and it is not hard to see that as an L_{cat} -structure, it is isomorphic to $\mathcal{C}$. By definition of L_{cat} -structures, the interpretation of the symbols imply that a homomorphism $\varphi: \mathcal{C} \rightarrow \mathcal{D}$, sends a morphism $f: A \rightarrow B$ to a morphism $\varphi(f): \varphi(A) \rightarrow \varphi(B)$ and preserves composition, showing that φ induces a functor $\mathbb{C} \rightarrow \mathbb{D}$, and conversely. ■

Therefore, we will henceforward identify $\mathcal{C}$ with its associated category $\mathbb{C}$. In line with common notation, we also let $\text{Hom}_{\mathbb{C}}(A, B)$ denote the collection of all $f: A \rightarrow B$, that is to say, of all elements y of the $\mathbf{m}$ -sort such that $\text{dom}(y) = A$ and $\text{rng}(y) = B$. We refer to $\text{Hom}_{\mathbb{C}}(A, B)$ as a *hom-set*. Note that it is L_{cat} -definable in $\mathbb{C}$ with parameters A and B .

3. Ultraproducts and elementarity

Recall that an ultrafilter $\mathcal{U}$ on a set X is a collection of non-empty subsets closed under finite intersection and superset, such that any subset of X or its complement belongs to $\mathcal{U}$. One thinks of subsets in $\mathcal{U}$ as big and the others are then small. Given any sequence of non-empty sets S_x indexed by $x \in X$, let S_{∞} be their Cartesian product. On S_{∞} , the ultrafilter now induces an equivalence relation: we say that two elements $(a_x)_x$ and $(b_x)_x$ in S_{∞} are equivalent, if almost all their entries are the same, by which we mean that the set of all $x \in X$ such that $a_x = b_x$ lies in $\mathcal{U}$ (i.e., is big). We call the resulting quotient space the *ultraproduct* of the S_x and will often denote it as $S_{\mathfrak{f}}$. If L is a first-order language and all S_x are L -structures, then $S_{\mathfrak{f}}$ inherits an L -structure from the Cartesian product S_{∞} . The key result, called Los' Theorem, is that an L -sentence σ holds in $S_{\mathfrak{f}}$ if and only if it holds in almost all S_x (see, for instance, [6, §4.2]).

3.1. REMARK. In case some of the sets S_x are empty (which would force S_{∞} to be empty), then either almost all of them are empty, in which case we let $S_{\mathfrak{f}}$ be the empty set. In the remaining case, replacing the empty ones with arbitrary ones will not change the ultraproduct. These minor modifications will tacitly be assumed in our usage of ultraproducts below.

If all the S_x are the same set S , so that $S_{\infty} = S^X$, then we call $S_{\mathfrak{f}}$ the *ultrapower* of S . It follows from Los' Theorem that $S_{\mathfrak{f}}$ is elementarily equivalent to S . In fact, there is a canonical map $S \rightarrow S_{\mathfrak{f}}$ sending an element $s \in S$ to the equivalence class of the constant map $x \mapsto s$, called the *diagonal embedding*, and one checks that this actually yields an elementary embedding. By a famous theorem of Keisler and Shelah, elementarity can be detected by the ultrapower construction: two L -structures are elementarily equivalent if and only if they have some isomorphic ultrapowers (possibly with respect to different ultrafilters). Its proof, however, seems to rely on the generalized continuum hypothesis, but for our purposes, the following immediate corollary of it can easily be proven directly.

In the sequel, we will denote L -structures, for some first-order language L , by script letters $\mathcal{A}, \mathcal{M}, \dots$, and we let $D(\mathcal{A}), D(\mathcal{M}) \dots$ denote their underlying subsets.

3.2. PROPOSITION. *Two first-order structures are elementarily equivalent if and only if there exists an elementary embedding from one into an ultrapower of the other.*

PROOF. One direction is clear, so assume $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent L -structures with underlying sets $A := D(\mathcal{A})$ and $B := D(\mathcal{B})$. We want to show that there is an ultrapower of $\mathcal{A}$ containing $\mathcal{B}$ as an elementary submodel. Let b be a tuple enumerating all elements of B and let $\mathbf{b}$ be a set of new constant symbols of the same size. Let $L(\mathbf{b})$ be the language obtained from L by adding these new constant symbols. Hence the structure $(\mathcal{B}, b)$ is an $L(\mathbf{b})$ -structure by interpreting $\mathbf{b}$ by b . Consider the theory $\text{Th}_{L(\mathbf{b})}(\mathcal{B}, b)$ (aka, the *elementary diagram* of $\mathcal{B}$) in this language $L(\mathbf{b})$ and let X be the collection of all non-empty finite subsets of this theory. Any $\Sigma \in X$ is logically equivalent with a sentence of the form $\varphi(\mathbf{b})$ such that $(\mathcal{B}, b) \models \varphi(b)$. Since $\mathcal{B}$ is then a model of $(\exists x)\varphi(x)$, so is $\mathcal{A}$ by elementary equivalence. Therefore we can find an $L(\mathbf{b})$ -expansion $\mathcal{A}_\Sigma^* := (\mathcal{A}, c_\Sigma)$ which models Σ , where c_Σ is the tuple of elements that are the interpretation in $\mathcal{A}_\Sigma^*$ of $\mathbf{b}$. For each $\Sigma \in X$, let $\langle \Sigma \rangle$ be the set of all $\Psi \in X$ with $\Sigma \subseteq \Psi$. Since the collection of all $\langle \Sigma \rangle$ for $\Sigma \in X$, satisfies the finite intersection property, it is contained in some ultrafilter $\mathcal{U}$ on X . Let $\mathcal{A}_\mathfrak{h}^*$ be the ultraproduct of the $\mathcal{A}_\Sigma^*$ with respect to this ultrafilter. Hence the L -reduct $\mathcal{A}_\mathfrak{h}$ of $\mathcal{A}_\mathfrak{h}^*$ is the ultrapower of $\mathcal{A}$ with respect to the same ultrafilter. Let σ be a sentence in $\text{Th}_{L(\mathbf{b})}(\mathcal{B}, b)$. For each $\Sigma \in X$ containing σ , we have $\mathcal{A}_\Sigma^* \models \sigma$, and by construction of $\mathcal{U}$, this therefore holds for almost all Σ , so that $\mathcal{A}_\mathfrak{h}^*$ satisfies σ . Hence $\mathcal{A}_\mathfrak{h}$ is the reduct of a model of $\text{Th}_{L(\mathbf{b})}(\mathcal{B}, b)$, so that by the elementary diagram lemma (see, for instance, [6, Lemma 8.2.1]), $\mathcal{B}$ is an elementary substructure of $\mathcal{A}_\mathfrak{h}$. In fact, the map $\mathcal{B} \rightarrow \mathcal{A}_\mathfrak{h}$ is given by sending b to the interpretation of $\mathbf{b}$ in the $L(\mathbf{b})$ -structure $\mathcal{A}_\mathfrak{h}^*$, and it follows easily that this is an elementary embedding. ■

3.3. REMARK. Note that X , the underlying set of the ultrafilter, has size at most that of the power set of the language $L(\mathbf{b})$.

4. Ultrapowers of categories

Recall that a Grothendieck *universe*³ is a set $\mathfrak{U}$ with the following properties

1. $\mathfrak{U}$ is transitive, meaning that if $y \in \mathfrak{U}$ and $x \in y$, then $x \in \mathfrak{U}$;
2. if y belongs to $\mathfrak{U}$, then so does its power-set $\mathcal{P}(y)$;
3. if $I \in \mathfrak{U}$ and $x_i \in \mathfrak{U}$ for all $i \in I$, then the union of all x_i is also in $\mathfrak{U}$;
4. $\omega \in \mathfrak{U}$.

Fix, once and for all, a universe $\mathfrak{U}$. We view $\mathfrak{U}$ as a subcategory of the category of sets. Let me briefly argue why a Grothendieck universe suffices to ‘do model-theory.’ Firstly, if $\mathcal{M}$ is an L -structure of cardinality at most that of $\mathfrak{U}$, then by transport of structure,

³The existence of universes does not follow from ZFC alone; it requires some large cardinal axiom like the existence of inaccessibles.

we can replace it by an isomorphic structure and assume from the start that $D(\mathcal{M}) \subseteq \mathfrak{U}$. Moreover, we have the following variant of the Löwenheim-Skolem Theorem:

4.1. PROPOSITION. *Let L be a first-order language in $\mathfrak{U}$, let $\mathcal{M}$ be an L -structure, and let $A \in \mathfrak{U}$ be a subset of $D(\mathcal{M})$. Then there exists an elementary embedding $\mathcal{N} \preceq \mathcal{M}$ such that $A \subseteq D(\mathcal{N}) \in \mathfrak{U}$.*

PROOF. Let κ be the cardinality of $\mathfrak{U}$, which therefore is bigger than those of L and A . By the usual Löwenheim-Skolem Theorem (see, for instance, [6, Theorem 8.4.1]), there exists an elementary substructure $\mathcal{M}' \subseteq \mathcal{M}$ of cardinality $\lambda < \kappa$ with $A \subseteq D(\mathcal{M}')$. Choose some set $X \in \mathfrak{U}$ containing A and having cardinality λ . By what we just said, there exists an L -structure $\mathcal{N} \cong \mathcal{M}'$ with $D(\mathcal{N}) = X$. ■

The model $\mathcal{N}$ thus constructed will be considered small:

4.2. DEFINITION. [Smallness] *Given a first-order language $L \in \mathfrak{U}$, we call an L -structure $\mathcal{N}$ small, if its underlying set $D(\mathcal{N})$ is an element (not just a subset) of $\mathfrak{U}$.*

This applies in particular to categories, so that in our terminology, a category is small if both its collection of objects and its collection of morphisms are members of $\mathfrak{U}$. We will also encounter non-small categories, but even then we will assume that the collection of objects as well as the collection of morphisms, form a subset of $\mathfrak{U}$ (hence they are still small in the classical sense). They arise as follows:

4.3. DEFINITION. *Let us say that a category $\mathbb{C}$ is a $\mathfrak{U}$ -category, if all its hom-sets belong to $\mathfrak{U}$.⁴ If $\mathbb{C}$ admits moreover a generator, then we will say that $\mathbb{C}$ is $\mathfrak{U}$ -concrete. We call a category sub- $\mathfrak{U}$ -concrete, if it is a subcategory of a $\mathfrak{U}$ -concrete category.*

Recall that an object I in a category $\mathbb{C}$ is a *generator* for that category if for each pair of distinct arrows $f \neq g: A \rightrightarrows B$, there is an arrow $a: I \rightarrow A$, such that $f \circ a \neq g \circ a$.

4.4. COROLLARY. *Any (sub-) $\mathfrak{U}$ -concrete category $\mathbb{C}$ embeds in $\mathfrak{U}$.*

PROOF. For I a generator, the functor $\text{Hom}_{\mathbb{C}}(I, -)$ is by definition faithful (injective on morphisms); it is also injective on objects as different hom-sets are disjoint.⁵ ■

A faithful representable functor can be thought of as some sort of forgetful functor, and that is precisely what we will do. Given a $\mathfrak{U}$ -concrete category $\mathbb{C}$, we fix a generator I and we set

$$|A| := \text{Hom}_{\mathbb{C}}(I, A). \quad (1)$$

Note that generators are not necessarily unique (e.g., if I is a generator, then so is $I \times I$), and therefore, this forgetful functor depends on the choice of I , but we will not always make this explicit. Given an arrow $f: A \rightarrow B$, the map $|f|: |A| \rightarrow |B|$ is given by composition, that is to say, given $a \in |A|$, so that $a: I \rightarrow A$, the image of a under $|f|$ is then the morphism $f \circ a \in |B|$.

⁴This then corresponds to the classical notion of locally small.

⁵The common convention in category theory.

Let X be a set in $\mathfrak{U}$ and $\mathcal{U}$ an ultrafilter on X (so that also $\mathcal{U} \in \mathfrak{U}$); we can therefore express this by simply saying that $\mathcal{U}$ is an ultrafilter in $\mathfrak{U}$. Note that for a given category $\mathbb{C}$, viewing it as an L_{cat} -structure, we can then take its ultrapower $\mathbb{C}_{\mathfrak{h}}$ and we have an embedding of categories $\mathbb{C} \hookrightarrow \mathbb{C}_{\mathfrak{h}}$ via the diagonal embedding. Note that $\mathbb{C}_{\mathfrak{h}}$ is the quotient of the product category $\mathbb{C}^X$ modulo the equivalence relation given by the ultrafilter $\mathcal{U}$ on X . Hence any object B in the ultrapower is the equivalence class of a map $x \mapsto B_x$, for $x \in X$ and B_x an object in $\mathbb{C}$; we express this by saying that the map $x \mapsto B_x$ *represents* B .

4.5. LEMMA. *Let $\mathbb{C}$ be a $\mathfrak{U}$ -concrete category with generator I . The image $I_{\mathfrak{h}}$ of I under the diagonal embedding is a generator for $\mathbb{C}_{\mathfrak{h}}$, and so the latter is in particular again $\mathfrak{U}$ -concrete, yielding a forgetful functor $|-| := \text{Hom}_{\mathbb{C}_{\mathfrak{h}}}(I_{\mathfrak{h}}, -)$.*

Moreover, for each object B in $\mathbb{C}_{\mathfrak{h}}$, the set $|B|$ is the ultraproduct of the sets $|B_x|$, where $x \mapsto B_x$ is a map representing the object B .

PROOF. The first statement follows from the second because $|-|$ is faithful on $\mathbb{C}$, whence remains so on $\mathbb{C}_{\mathfrak{h}}$. An element in $|B| = \text{Hom}_{\mathbb{C}_{\mathfrak{h}}}(I_{\mathfrak{h}}, B)$ is the equivalence class of a map $x \mapsto f_x$, where each f_x is a morphism $I \rightarrow B_x$, that is to say, an element in $|B_x|$, and hence the map $x \mapsto |f_x|$ yields an element in the ultraproduct of the $|B_x|$, and conversely. It is now easy to see that this does not depend on any choices made. ■

4.6. REMARK. Nowhere in this proof did we use that the components in the ultrapower were all the same category $\mathbb{C}$, and so, the more general result is that if $\mathbb{C}_x$ is a family of $\mathfrak{U}$ -concrete categories with respective generator I_x , indexed by $x \in X$, and $\mathbb{C}_{\mathfrak{h}}$ is their ultraproduct, then the object $I_{\mathfrak{h}}$ of $\mathbb{C}_{\mathfrak{h}}$ given by the map $x \mapsto I_x$ is a generator for $\mathbb{C}_{\mathfrak{h}}$ and for each object B in $\mathbb{C}_{\mathfrak{h}}$, the set $|B| := \text{Hom}_{\mathbb{C}_{\mathfrak{h}}}(I_{\mathfrak{h}}, B)$ is the ultraproduct of the sets $|B_x|$, where $x \mapsto B_x$ is a map representing B .

Note that $\mathfrak{U}$ is itself $\mathfrak{U}$ -concrete, with generator the one-element set $\{\emptyset\}$. It follows from Lemma 4.5 that $\mathfrak{U}_{\mathfrak{h}}$ is in fact a subcategory of $\mathfrak{U}$ since the ultraproducts also belong to $\mathfrak{U}$. We will see in the next section that this ‘universality’ is not an isolated phenomenon.

4.7. EXAMPLE. If a category $\mathbb{C}$ admits a forgetful functor $\mathbb{C} \rightarrow \mathfrak{U}$ which has a left adjoint $U: \mathfrak{U} \rightarrow \mathbb{C}$, then $U(\{\emptyset\})$ is a generator for $\mathbb{C}$.

4.8. EXAMPLE. Given a group (or, more generally, a monoid) G , we can associate a category $\mathbb{G}$ to it, with a single object $\bullet$ the morphisms of which are given by the group G , and their composition by the group law. The sole object is trivially a generator and the “underlying set” is $|\bullet| = \text{Hom}(\bullet, \bullet) = G$, showing that the notion of forgetful functor is not just about objects. Clearly, $\mathbb{G}_{\mathfrak{h}}$ is then representing the group (monoid) $G_{\mathfrak{h}}$.

FAMILY OF GENERATORS. Given a subset of objects Σ of a category $\mathbb{C}$, let us define, for any object A , the set (given by the co-product/disjoint union in $\mathfrak{U}$)

$$\text{Hom}_{\mathbb{C}}(\Sigma, A) := \coprod_{I \in \Sigma} \text{Hom}_{\mathbb{C}}(I, A). \quad (2)$$

Note that Σ is a family of generators (meaning that given distinct $f, g: A \rightrightarrows B$, there is at least one $I \in \Sigma$ and a morphism $a: I \rightarrow A$ such that $f \circ a \neq g \circ a$) if and only if the functor $\text{Hom}_{\mathbb{C}}(\Sigma, -)$ is faithful. We can therefore again use this to define a forgetful functor $|-| := \text{Hom}_{\mathbb{C}}(\Sigma, -)$. Let us call therefore a $\mathfrak{U}$ -category $\mathbb{C}$ *weakly $\mathfrak{U}$ -concrete*, if it admits a family of generators.

We can now also prove a version of Lemma 4.5 for a weakly $\mathfrak{U}$ -concrete category $\mathbb{C}$; we leave the proof to the reader.

4.9. LEMMA. *Given a family Σ of generators in a weakly $\mathfrak{U}$ -concrete category $\mathbb{C}$ and putting $|-| := \text{Hom}_{\mathbb{C}}(\Sigma, -)$, let $\Sigma_{\mathfrak{h}}$ be all equivalence classes in $\mathbb{C}_{\mathfrak{h}}$ given by maps $x \mapsto I_x$, with $I_x \in \Sigma$. Then $\Sigma_{\mathfrak{h}}$ is a family of generators for $\mathbb{C}_{\mathfrak{h}}$ and for each object B in $\mathbb{C}_{\mathfrak{h}}$ given by a map $x \mapsto B_x$, the set $|B| := \text{Hom}_{\mathbb{C}_{\mathfrak{h}}}(\Sigma_{\mathfrak{h}}, B)$ is the ultraproduct of the sets $|B_x|$. $\square$*

5. The category of small models of a theory

Let L be a first-order language and T an L -theory. We assume that L is a member of $\mathfrak{U}$ (meaning that the collection of its constant, function and relation symbols is in $\mathfrak{U}$). Let $\text{Mod}_{\mathfrak{U}}(T)$ be the category of small models of T (that is to say all L -structures $\mathcal{N} \models T$ such that the underlying set $D(\mathcal{N}) \in \mathfrak{U}$), and with arbitrary L -homomorphisms for morphisms. If T is the empty theory, we simply write $\text{Mod}_{\mathfrak{U}}(L)$, the *category of L -structures*.

5.1. PROPOSITION. *Let L be a first-order language. Then $\text{Mod}_{\mathfrak{U}}(L)$ is $\mathfrak{U}$ -concrete, and the term algebra $\mathcal{T}$ in a single variable x is a generator. Moreover, the functors D and $|-| = \text{Hom}_{\text{Mod}_{\mathfrak{U}}(L)}(\mathcal{T}, -)$ are naturally isomorphic.*

PROOF. Recall that the *term algebra* $\mathcal{T}$ consists precisely of all the L -terms in the variable x , where the constant and function symbols of the language are interpreted by themselves, while the predicate symbols all define empty sets. If $f, g: \mathcal{M} \rightrightarrows \mathcal{N}$ are two distinct L -homomorphism, then there is some $a \in D(\mathcal{M})$ such that $f(a) \neq g(a)$. There is a unique homomorphism $s_a: \mathcal{T} \rightarrow \mathcal{M}$ sending the variable x to a (and each term $t(x)$ to its evaluation $t(a)$), and it follows that $f \circ s_a \neq g \circ s_a$. By the same argument, $\text{Hom}_{\text{Mod}_{\mathfrak{U}}(L)}(\mathcal{T}, \mathcal{M})$ is naturally isomorphic to the underlying set $D(\mathcal{M})$, by sending a homomorphism $s: \mathcal{T} \rightarrow \mathcal{M}$ to $s(x) \in D(\mathcal{M})$.⁶ $\blacksquare$

5.2. COROLLARY. *Given an ultrafilter $\mathcal{U}$ on $X \in \mathfrak{U}$, the ultrapower $\text{Mod}_{\mathfrak{U}}(L)_{\mathfrak{h}}$ with respect to $\mathcal{U}$ is isomorphic to a subcategory of $\text{Mod}_{\mathfrak{U}}(L)$. The same is true upon replacing $\text{Mod}_{\mathfrak{U}}(L)$ by $\text{Mod}_{\mathfrak{U}}(T)$ where T is any L -theory.*

More precisely, there is I in $\text{Mod}_{\mathfrak{U}}(L)_{\mathfrak{h}}$ and an embedding of categories $i: \text{Mod}_{\mathfrak{U}}(L)_{\mathfrak{h}} \hookrightarrow$

⁶In fact, the general term-algebra construction is a left adjoint of the functor $|-|$.

$\mathbb{M}\text{od}_{\mathfrak{U}}(L)$, making the following diagram commute

$$\begin{array}{ccc} \mathbb{M}\text{od}_{\mathfrak{U}}(L)_{\mathfrak{h}} & \xrightarrow{i} & \mathbb{M}\text{od}_{\mathfrak{U}}(L) \\ & \searrow | - | & \downarrow D \\ & & \mathfrak{U} \end{array} \quad (3)$$

where $|M| := \text{Hom}_{\mathbb{M}\text{od}_{\mathfrak{U}}(L)_{\mathfrak{h}}}(I, M)$.

PROOF. Let M be an object of $\mathbb{M}\text{od}_{\mathfrak{U}}(L)_{\mathfrak{h}}$ given by a sequence $x \mapsto \mathcal{M}_x$, for $x \in X$, with each $\mathcal{M}_x$ an object in $\mathbb{M}\text{od}_{\mathfrak{U}}(L)$, that is to say, an L -structure. As in Proposition 5.1, let $\mathcal{T}$ be the generator of $\mathbb{M}\text{od}_{\mathfrak{U}}(L)$ given by the one-variable term algebra, and let I be the object in $\mathbb{M}\text{od}_{\mathfrak{U}}(L)_{\mathfrak{h}}$ given by the constant sequence $x \mapsto \mathcal{T}$.

By Lemma 4.5, the set $|M|$ is equal to the ultraproduct of the sets $|\mathcal{M}_x|$. Let $\mathcal{M}_{\mathfrak{h}}$ be the ultraproduct of the $\mathcal{M}_x$. By Proposition 5.1, there are natural isomorphisms $|\mathcal{M}_x| \cong D(\mathcal{M}_x)$, and hence, by Łos' Theorem, a natural isomorphism

$$|M| \cong D(\mathcal{M}_{\mathfrak{h}}). \quad (4)$$

Define an L -structure on the set $|M|$ by pulling back the L -structure of $\mathcal{M}_{\mathfrak{h}}$ and let $i(M)$ denote this L -structure. In particular, $i(M) \cong \mathcal{M}_{\mathfrak{h}}$. To see that i is functorial, let N be a second object in $\mathbb{M}\text{od}_{\mathfrak{U}}(L)_{\mathfrak{h}}$ coming from a sequence of L -structures $x \mapsto \mathcal{N}_x$, and let $i(N)$ be the canonical L -structure obtained on $|N|$ isomorphic to the ultraproduct $\mathcal{N}_{\mathfrak{h}}$ of the $\mathcal{N}_x$. A morphism $f: M \rightarrow N$ in $\mathbb{M}\text{od}_{\mathfrak{U}}(L)_{\mathfrak{h}}$, arises from a sequence of morphisms $f_x: \mathcal{M}_x \rightarrow \mathcal{N}_x$ in $\mathbb{M}\text{od}_{\mathfrak{U}}(L)$, which in the ultraproduct yields a morphism of L -structures $f_{\mathfrak{h}}: \mathcal{M}_{\mathfrak{h}} \rightarrow \mathcal{N}_{\mathfrak{h}}$. Naturality of (4) now yields a morphism $i(f): i(M) \rightarrow i(N)$. Since $| - |$ is an embedding, so is therefore i by (3).

To prove the statement for arbitrary theories, we cannot just copy the above proof since $\mathcal{T}$ might not be a model of T . However, if M is an object in $\mathbb{M}\text{od}_{\mathfrak{U}}(T)_{\mathfrak{h}}$, the $\mathcal{M}_x$ are now also models of T , whence so is their ultraproduct $\mathcal{M}_{\mathfrak{h}}$ by Łos' Theorem. As $i(M) \cong \mathcal{M}_{\mathfrak{h}}$, the embedding i then factors through $\mathbb{M}\text{od}_{\mathfrak{U}}(T)$, as desired. ■

5.3. REMARK. Note that embedding $i: \mathbb{M}\text{od}_{\mathfrak{U}}(L)_{\mathfrak{h}} \rightarrow \mathbb{M}\text{od}_{\mathfrak{U}}(L)$ is not full, as the morphisms in the former are coming from ultraproducts. Pre-composing with the diagonal embedding $\mathbb{M}\text{od}_{\mathfrak{U}}(L) \hookrightarrow \mathbb{M}\text{od}_{\mathfrak{U}}(L)_{\mathfrak{h}}$, the ensuing endofunctor on $\mathbb{M}\text{od}_{\mathfrak{U}}(L)$ sends, up to a natural isomorphism, an L -structure $\mathcal{M}$ to its ultrapower $\mathcal{M}_{\mathfrak{h}}$, and homomorphisms to their ultrapower. In general this is not an elementary map, whence neither is i . Indeed, even with $L = \emptyset$, i.e., for the category of sets, this does not hold: the sentence expressing that there is a bijection between ω and 2^{ω} is clearly false downstairs, but it is true upstairs since $(2^{\omega})_{\mathfrak{h}} \cong \omega_{\mathfrak{h}}$. In particular, the embedding i might even fail to reflect isomorphisms.

STRONG HOMOMORPHISMS. In certain branches of mathematics, like graph theory, the more natural concept of morphism is what model-theorists call a *strong homomorphism*, i.e., a homomorphism where for any given relation symbol R , a tuple satisfies R if and only if its image under the morphism satisfies R . A special case is that of a strong embedding,

where we have added/treated the equality sign as a relation symbol. Our method can also cover this case. Let $\mathbb{M}\text{od}_{\mathfrak{U}}^{\text{str}}(L)$ be the (non-full) subcategory of $\mathbb{M}\text{od}_{\mathfrak{U}}(L)$, having the same objects—the L -structures—but where the homomorphisms are now assumed to be strong. Of course, if the language has no relation symbols (like, for instance, L_{cat}), then $\mathbb{M}\text{od}_{\mathfrak{U}}(L) = \mathbb{M}\text{od}_{\mathfrak{U}}^{\text{str}}(L)$. In general, let us write $\mathbf{fct}(L)$ for the restriction of L obtained by omitting all predicate symbols.

5.4. LEMMA. *Let $\mathcal{N}$ be an $\mathbf{fct}(L)$ -structure and $\mathcal{M}$ an L -structure. If $f: \mathcal{N} \rightarrow \mathcal{M}|_{\mathbf{fct}(L)}$ is an $\mathbf{fct}(L)$ -homomorphism, then it induces an L -expansion of $\mathcal{N}$, denoted $f^*\mathcal{M}$, such that $f: f^*\mathcal{M} \rightarrow \mathcal{M}$ is a strong L -homomorphism. Moreover, $f^*\mathcal{M}$ is the only expansion of $\mathcal{N}$ for which f becomes strong.*

PROOF. Let R be an n -ary relation symbol and $P \subseteq D(\mathcal{M})^n$ the subset it defines in $\mathcal{M}$. Then we let the interpretation of R on $D(\mathcal{N})$ be the subset $f^{-1}(P)$ (where we continue to write f for the n -fold product $f: D(\mathcal{N})^n \rightarrow D(\mathcal{M})^n$). By construction, f is then a strong L -homomorphism. The last assertion is now clear from the definition of strong homomorphism. ■

As before, let $\mathcal{T}$ be the one-variable term algebra. We cannot expand it into an L -structure which would make it a generator with respect to all L -structures, so we instead consider the collection Θ of all L -structures whose reduct to $\mathbf{fct}(L)$ is $\mathcal{T}$ (clearly $\Theta \in \mathfrak{U}$).

5.5. LEMMA. *The set Θ forms a family of generators of $\mathbb{M}\text{od}_{\mathfrak{U}}^{\text{str}}(L)$.*

PROOF. Let $f, g: \mathcal{M} \rightrightarrows \mathcal{N}$ be two distinct (strong) homomorphisms of L -structures. Their reducts are $\mathbf{fct}(L)$ -structures and so by Proposition 5.1, there exists a morphism $a: \mathcal{T} \rightarrow \mathcal{M}|_{\mathbf{fct}(L)}$ such that $f \circ a \neq g \circ a$. With $\mathcal{J} := a^*\mathcal{M}$ —whence $\mathcal{J} \in \Theta$ —, we get the desired (strong) homomorphism $a: \mathcal{J} \rightarrow \mathcal{M}$ by Lemma 5.4. ■

5.6. PROPOSITION. *With Θ defined as above, there is a natural isomorphism of functors, such that for each small L -structure $\mathcal{M}$, we have*

$$D(\mathcal{M}) \cong \text{Hom}_{\mathbb{M}\text{od}_{\mathfrak{U}}^{\text{str}}(L)}(\Theta, \mathcal{M}). \quad (5)$$

PROOF. Let $u \in D(\mathcal{M})$. There is a unique $\mathbf{fct}(L)$ -homomorphism $a_u: \mathcal{T} \rightarrow \mathcal{M}|_{\mathbf{fct}(L)}$ sending the free variable to u . By the proof of Lemma 5.5, the object $a_u^*\mathcal{M}$ is therefore the unique structure in Θ such that $a_u: a_u^*\mathcal{M} \rightarrow \mathcal{M}$ is strong.

Conversely, suppose $\mathcal{J} \in \Theta$ and we have a strong homomorphism $a: \mathcal{J} \rightarrow \mathcal{M}$. The induced map a from $D(\mathcal{J}) = D(\mathcal{T})$ to $D(\mathcal{M})$ sends the free variable to an element $u \in D(\mathcal{M})$. By the uniqueness in Lemma 5.4, we have $\mathcal{J} = a_u^*\mathcal{M}$, and hence $a = a_u$, proving that this assignment is the inverse of the previous one. ■

5.7. COROLLARY. *Let L be a first-order language and $\mathcal{U}$ an ultrafilter, both in $\mathfrak{U}$. Given an L -theory T , the ultrapower $\mathbb{M}\text{od}_{\mathfrak{U}}^{\text{str}}(T)_{\mathfrak{U}}$ with respect to $\mathcal{U}$ is isomorphic to a subcategory of $\mathbb{M}\text{od}_{\mathfrak{U}}^{\text{str}}(T)$.*

PROOF. We can copy the proof of Corollary 5.2, once we prove the equivalent statement of Lemma 4.5 in this setup. To this end, we may take T to be the empty theory. Therefore, let M be an object in $\mathbb{M}\text{od}_{\mathfrak{U}}^{\text{str}}(L)_{\mathfrak{q}}$, given by a map $x \mapsto \mathcal{M}_x$, for $x \in X$, with each $\mathcal{M}_x$ an object in $\mathbb{M}\text{od}_{\mathfrak{U}}^{\text{str}}(L)$. Let $\Theta_{\mathfrak{q}}$ be the image of the set Θ in $\mathbb{M}\text{od}_{\mathfrak{U}}^{\text{str}}(L)_{\mathfrak{q}}$, that is to say, all equivalence classes of maps $X \rightarrow \Theta$. Define

$$|M| := \text{Hom}_{\mathbb{M}\text{od}_{\mathfrak{U}}^{\text{str}}(L)_{\mathfrak{q}}}(\Theta_{\mathfrak{q}}, M).$$

By Lemma 4.9, the set $|M|$, being an ultraproduct of L -structures, inherits a canonical L -structure $\mathcal{M}$, and the rest of the proof now goes through as in Corollary 5.2. ■

6. Universal categories

Let $\mathcal{M}$ be an L -structure. We call $\mathcal{M}$ *universal* (respectively, *strongly universal*), if every L -structure $\mathcal{N}$ which is elementarily equivalent with $\mathcal{M}$ and has cardinality less than $\mathcal{M}$, embeds (respectively, elementarily embeds) in $\mathcal{M}$.

Any countable structure is trivially universal according to this definition, so we exclude this case. Any saturated structure is easily seen to be strongly universal (see, for instance, [6, Thm. 12.1.1]).

6.1. PROPOSITION. *Let T be a complete, uncountable categorical L -theory and $\mathcal{M}$ a model of T of uncountable cardinality λ . If λ is at least the size of the language L , then $\mathcal{M}$ is strongly universal.*

PROOF. Let $\mathcal{N}$ be a model of cardinality $\beta < \lambda$. By upward Löwenheim-Skolem, there exists an elementary extension $\mathcal{N}'$ of $\mathcal{N}$ of size λ (see, for instance, [6, Theorem 8.4.3]), and by categoricity, the former must be isomorphic to $\mathcal{M}$. ■

Rather than working with cardinalities, the following definition in terms of Grothendieck universes is more suitable for our needs. Recall that $D(\mathcal{M})$ denotes the underlying set of an L -structure and *small* means that this set belongs to $\mathfrak{U}$.

6.2. DEFINITION. *An L -structure $\mathcal{M}$ is called $\mathfrak{U}$ -universal, if $D(\mathcal{M}) \subseteq \mathfrak{U}$ and for any small L -structure $\mathcal{N}$ elementarily equivalent with it, there is an embedding $\mathcal{N} \hookrightarrow \mathcal{M}$.*

Since we will only consider this type of universality, we will henceforth drop reference to the universe $\mathfrak{U}$. For cardinality reasons, the underlying subset of a universal structure can never be an element of $\mathfrak{U}$, only a subset. Universality implies that small substructures form an elementary class:

6.3. PROPOSITION. *Let $\mathcal{M}$ be a universal structure in a first-order language L , let T be its complete L -theory, and let $\mathcal{N}$ be a small L -structure. Then $\mathcal{N}$ is (isomorphic to) a substructure of $\mathcal{M}$ if and only if it is a model of $T_{\forall}$.*

PROOF. One direction is clear by universal preservation (see [6, Lemma 6.2.2]), so suppose $\mathcal{N}$ is a small model of $T_{\forall}$. By the the same cited lemma, there exists $\mathcal{M}' \models T$ such that $\mathcal{N} \hookrightarrow \mathcal{M}'$. By Proposition 4.1 there is then some small $\mathcal{M}_0 \models T$ with $\mathcal{N} \hookrightarrow \mathcal{M}_0$. By universality, we also have an embedding $\mathcal{M}_0 \hookrightarrow \mathcal{M}$, proving that $\mathcal{N} \hookrightarrow \mathcal{M}$. ■

6.4. **EXAMPLE.** Let $\emptyset$ denote the empty theory in the empty language, that is to say, the theory of sets, so that $\text{Mod}_{\mathfrak{U}}(\emptyset)$ is just $\mathfrak{U}$. Let SET^∞ be the theory of infinite sets. Then $\text{Mod}_{\mathfrak{U}}(SET^\infty)$ does not contain a terminal object while $\mathfrak{U}$ does, so their complete L_{cat} -theories are not the same. However, the universal theories are the same since there is an embedding $\mathfrak{U} \hookrightarrow \text{Mod}_{\mathfrak{U}}(T^\infty)$ by sending a set $X \in \mathfrak{U}$ to $X \sqcup \omega$.

6.5. **THEOREM.** *Given a language L and an L -theory T , the category $\text{Mod}_{\mathfrak{U}}(T)$ is universal (in the language L_{cat} of categories), as is $\text{Mod}_{\mathfrak{U}}^{\text{str}}(T)$.*

PROOF. Let $\mathbb{C}$ be a small category that is L_{cat} -elementarily equivalent with $\text{Mod}_{\mathfrak{U}}(T)$. By Remark 3.3, there is some ultrapower $\text{Mod}_{\mathfrak{U}}(T)_{\mathfrak{h}}$ with respect to an ultrafilter on a set $X \in \mathfrak{U}$, such that $\mathbb{C}$ embeds elementarily in $\text{Mod}_{\mathfrak{U}}(T)_{\mathfrak{h}}$ and by Corollary 5.2, the latter embeds in $\text{Mod}_{\mathfrak{U}}(T)$. The case of $\text{Mod}_{\mathfrak{U}}^{\text{str}}(T)$ follows by the same argument, using instead Corollary 5.7. ■

6.6. **COROLLARY.** *Let L be a first-order language and E a collection of L -structures in $\mathfrak{U}$ closed under taking ultraproducts and isomorphic copies. Let $\mathbb{E}$ be the full subcategory of $\text{Mod}_{\mathfrak{U}}(L)$ the objects of which are the structures in E . Then $\mathbb{E}$ is universal.*

PROOF. Let $\mathbb{C}$ be a small category that is elementarily equivalent with $\mathbb{E}$. By Remark 3.3 again, there is an ultrafilter in $\mathfrak{U}$ such that $\mathbb{C}$ admits an elementary embedding into the ultrapower $\mathbb{E}_{\mathfrak{h}}$. Let $i: \text{Mod}_{\mathfrak{U}}(L)_{\mathfrak{h}} \rightarrow \text{Mod}_{\mathfrak{U}}(L)$ be the embedding given by Corollary 5.2 and consider its restriction to the subcategory $\mathbb{E}_{\mathfrak{h}}$. Let B be an object in $\mathbb{E}_{\mathfrak{h}}$, given by a sequence $x \mapsto \mathcal{B}_x$ with $\mathcal{B}_x \in E$. By assumption, the ultraproduct $\mathcal{B}_{\mathfrak{h}}$ of the latter also lies in E . As $i(B) \cong \mathcal{B}_{\mathfrak{h}}$, also $i(B) \in E$, and therefore we get an induced embedding $i: \mathbb{E}_{\mathfrak{h}} \hookrightarrow \mathbb{E}$, which composed with the elementary embedding $\mathbb{C} \hookrightarrow \mathbb{E}_{\mathfrak{h}}$ yields the desired embedding. ■

6.7. **EXAMPLE.** This yields plenty of examples of universal categories that do not arise as the category of models of a first-order theory. For instance, let $\mathbb{P}$ be the set of prime numbers and let S be any subset of $\mathbb{P}$. Let Fld_S be the subcategory of $\text{Mod}_{\mathfrak{U}}(FLD)$ of all (small) fields whose characteristic belongs to $S \cup \{0\}$. As the objects in Fld_S are closed under ultraproducts, the category is universal by Corollary 6.6. However, if $S \subseteq \mathbb{P}$ is an infinite and co-infinite set, Fld_S is not induced by a first-order theory.

7. Universalizing theories

To any theory T in a first-order language, we can now associate a (complete) L_{cat} -theory $\mathbf{T}$, namely, the theory of $\text{Mod}_{\mathfrak{U}}(T)$. However, if S is an extension of T (e.g., a completion), there is no immediate connection between $\mathbf{T}$ and $\mathbf{S}$ (as both are complete theories). It is of course still possible that $\mathbf{S} = \mathbf{T}$, and although I do not know any examples, disproving equality is not obvious, and at present, I only know ad hoc arguments.

7.1. **EXAMPLE.** Let FLD and ACF stand respectively for the theory of fields and the theory of algebraically closed fields in the language of rings, and let **FLD** and **ACF** be the corresponding L_{cat} -theories of $\text{Mod}_{\mathfrak{U}}(FLD)$ and $\text{Mod}_{\mathfrak{U}}(ACF)$. To see that these are not the same, observe that the Cantor-Bernstein property holds for the latter but not for the former. More precisely, let **CB** be the sentence expressing that $K \hookrightarrow L$ (i.e., there is a monomorphism $K \rightarrow L$) and $L \hookrightarrow K$ implies $K \cong L$; then $\text{Mod}_{\mathfrak{U}}(ACF) \models \text{CB}$ since K and L must then have the same characteristic and the same transcendence degree, whence are isomorphic by Steinitz's theorem. On the other hand, K and $K(x)$, with K uncountable and algebraically closed and x a variable, are mutually embeddable but not isomorphic as the latter is not algebraically closed, and hence **CB** does not belong to **FLD**.

One way to tell $\text{Mod}_{\mathfrak{U}}(ACF)$ and $\text{Mod}_{\mathfrak{U}}(RCF)$ apart: only the latter has an object that embeds in any other object (the field of algebraic reals). Let ACF_p be all algebraically closed fields of characteristic p , which now also has this property, but still the theories **ACF** _{p} and **RCF** are distinguishable as only the former is totally ordered: for any two algebraically closed fields K and L of the same characteristic, either $K \hookrightarrow L$ or $L \hookrightarrow K$ by categoricity. This fails miserably in $\text{Mod}_{\mathfrak{U}}(RCF)$.

7.2. **REMARK.** Since L_{cat} is a countable language, there are at most continuum many complete L_{cat} -theories. On the other hand, there are many more first-order theories in all possible languages in $\mathfrak{U}$. In particular, there will be disjoint languages L and L' and theories T and T' in these respective languages such that $\mathbf{T} = \mathbf{T}'$. Universality now yields a weak form of definitional equivalence between models of T and models of T' in the following sense. Let $\mathbb{C}$ be a small, elementary substructure of $\text{Mod}_{\mathfrak{U}}(T)$ (by Proposition 4.1 we can even assure it to contain a given small collection of models). Since $\mathbb{C}$ is then also elementarily equivalent to $\text{Mod}_{\mathfrak{U}}(T')$, it embeds in it by universality, yielding for each model of T belonging to $\mathbb{C}$ a corresponding model of T' .

Proposition 6.3 shows that as far as universality is concerned, the universal theory suffices. But neither the complete nor the universal theory are easily described in concrete cases. The next definition generalizes this in the hope that we can find some concrete axiomatization, like the case of Abelian groups, to obtain universality.

7.3. **DEFINITION.** *Given a language $L \in \mathfrak{U}$, let us call an L -theory U universalizing, if it admits a model $\mathcal{M}$ with $D(\mathcal{M}) \subseteq \mathfrak{U}$, such that any small model $\mathcal{N} \models U$ embeds in $\mathcal{M}$.*

It follows that the above $\mathcal{M}$ is universal, for any small model elementarily equivalent to it, is also a model of U , whence embeds in $\mathcal{M}$. We express this by saying that U is *universalizing for $\mathcal{M}$* . Clearly the complete theory T of a universal structure $\mathcal{M}$ is universalizing for that structure, as is $T_{\forall}$, by Proposition 6.3.

Example 6.4 shows that the same theory can be universalizing for different structures. The following result shows how any two universal structures then must at least compare. Here, given a theory T in a language L , we will say that an L -sentence σ belongs to $\Delta_2(T)$, if there exists a $\forall\exists$ -sentence σ_1 and a $\exists\forall$ -sentence σ_2 , such that $\mathcal{M} \models \sigma \leftrightarrow \sigma_1 \leftrightarrow \sigma_2$ in each model $\mathcal{M}$ of T .

7.4. PROPOSITION. *Suppose an L -theory U is universalizing for two L -structures $\mathcal{M}$ and $\mathcal{N}$. Then $\mathcal{M}$ and $\mathcal{N}$ agree on all $\Delta_2(U)$ -sentences.*

PROOF. We will construct a sequence $\mathcal{M}_0 \hookrightarrow \mathcal{N}_0 \hookrightarrow \mathcal{M}_1 \hookrightarrow \mathcal{N}_1 \hookrightarrow \dots$, with all $\mathcal{M}_i$ elementary substructures of $\mathcal{M}$ and all $\mathcal{N}_i$ elementary substructures of $\mathcal{N}$. Assume we have such a chain, let $\mathcal{L}$ be their filtered co-limit (i.e., direct limit). Note that $\mathcal{L}$ is then also the filtered co-limit of the sub-sequences $\mathcal{M}_0 \hookrightarrow \mathcal{M}_1 \dots$ and $\mathcal{N}_0 \hookrightarrow \mathcal{N}_1 \hookrightarrow \dots$ (these chains, however, need not be elementary). Let σ be a $\Delta_2(U)$ -sentence in L which holds in $\mathcal{M}$. Since the $\mathcal{M}_i$ are elementary substructures, σ holds in each of them, whence also in $\mathcal{L}$ by [1, Theorem 2.4.6], since σ is U -equivalent to a $\forall\exists$ -sentence. Suppose $\neg\sigma$ holds in $\mathcal{N}$. Being also in $\Delta_2(U)$, the same argument shows, via the sub-sequence $\mathcal{N}_i$, that $\neg\sigma$ holds in $\mathcal{L}$, contradiction.

To construct the given sequence, let $\mathcal{M}_0$ be any small elementary substructure of $\mathcal{M}$. Since it is then a model of U and $\mathcal{N}$ is universalizing for U , there is an embedding $\mathcal{M}_0 \hookrightarrow \mathcal{N}$. By Proposition 4.1, we can find a small elementary substructure $\mathcal{N}_0$ of $\mathcal{N}$ and an embedding $\mathcal{M}_0 \hookrightarrow \mathcal{N}_0$. We can now repeat this argument, with $\mathcal{N}_0$ instead of $\mathcal{M}_0$, to obtain an embedding $\mathcal{N}_0 \hookrightarrow \mathcal{M}_1$ with $\mathcal{M}_1$ a small elementary substructure of $\mathcal{M}$, etc. ■

Let us revisit the theory **AB** introduced in the introduction: it consist of the sentences **prod**, **null** and **gen** expressing respectively the existence of (finite) products, of a null-object, and of a generator, and then the two sentences (Ab1) and (Ab2) asserting respectively the existence of a unique group arrow and that every morphism is compatible with these group laws.

7.5. PROPOSITION. *The theory **AB** is universalizing (for $\mathbb{M}\text{od}_{\mathfrak{U}}(T_{\text{Ab}})$).*

As a partial converse, if $\mathbb{C}$ is a small, full, continuous⁷ subcategory of $\mathbb{M}\text{od}_{\mathfrak{U}}(T_{\text{Ab}})$ admitting products and a null-object, then $\mathbb{C} \models \mathbf{AB}$.

PROOF. Let $\mathbb{C}$ be a small model of **AB**, so that in particular, it admits a generator I . Put $|G| := \text{Hom}_{\mathbb{C}}(I, G)$, for G any object in $\mathbb{C}$. Using the unique group arrow $\mu_G: G \times G \rightarrow G$ on G , we can define an addition on $|G|$ as follows: for $\alpha, \beta: I \rightarrow G$, let $\alpha + \beta := \mu_G \circ (\alpha, \beta)$, where $(\alpha, \beta): I \rightarrow G \times G$ is the unique morphism given by the product. It is now not hard to show that the properties of a group arrow make $|G|$ into a group and any morphism $G \rightarrow H$ in $\mathbb{C}$ yields a group homomorphism $|G| \rightarrow |H|$. Since I is a generator, the functor $|-|$ therefore yields an embedding into not just $\mathfrak{U}$ but already into $\mathbb{M}\text{od}_{\mathfrak{U}}(T_{\text{Ab}})$.

For the second statement, assume now that $i: \mathbb{C} \hookrightarrow \mathbb{M}\text{od}_{\mathfrak{U}}(T_{\text{Ab}})$ is a full, continuous embedding and $\mathbb{C}$ satisfies **prod** and **null**. Let G be an object in $\mathbb{C}$, so that $i(G)$ is by assumption an Abelian group with addition $+$. By assumption, the embedding preserves products, so that $i(G \times G) \cong i(G) \times i(G)$. Composition with $+: i(G) \times i(G) \rightarrow i(G)$ yields a morphism $i(G \times G) \rightarrow i(G)$. Since i is full, there must be a morphism $\mu: G \times G \rightarrow G$ in $\mathbb{C}$ such that $i(\mu)$ is equal to this composition. It is now not hard to see, using faithfulness, that μ is a group arrow on G and that axioms (Ab1) and (Ab2) hold in $\mathbb{C}$, showing that it is a model of **AB**. ■

⁷i.e., preserving all limits whenever they exist.

As an illustration of the benefit of having a more concrete universalizing theory than just the complete theory, we have

7.6. COROLLARY. *If $\mathbb{D}$ is small category that is elementarily equivalent with $\text{Mod}_{\mathfrak{U}}(T_{Ab})$, then it admits a continuous embedding $\mathbb{D} \hookrightarrow \text{Mod}_{\mathfrak{U}}(T_{Ab})$.*

PROOF. Since $\mathbb{D}$ is then a model of **AB**, the above proof gives an embedding via a representable functor $\text{Hom}_{\mathbb{D}}(I, -)$ and such a functor is always continuous. ■

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