Theory and Applications of Categories, Vol. 45, No. 6, 2026, pp. 206–245.

A DIRECT-CATEGORICAL APPROACH TO OPETOPIC SETS AND OPETOPES

TAICHI UEMURA

Abstract. We propose elementary definitions of opetopic sets and opetopes. We define opetopic sets by a simple structure on a direct category and several axioms. Opetopes are then opetopic sets satisfying one more axiom. We show that our definition is equivalent to the polynomial monad definition given by Kock, Joyal, Batanin, and Mascari. We also show that our category of opetopes is equivalent to the one given by Ho Thanh.

1. Introduction

Opetopes and *opetopic sets* were introduced by Baez and Dolan [1998] as a combinatorial approach to weak ω-categories. An opetope is a geometric shape of a many-in-single-out operator in higher dimension. Examples of opetopes of low dimensions are drawn in Fig. 1. There is only one opetope of dimension 0, the point. There is only one opetope of dimension 1, the arrow with one target and one source. There are countably many opetopes of dimension 2. The sources of an opetope of dimension 2 are opetopes of dimension 1 that form a "pasting diagram". It can be the case that an opetope of dimension 2 has no source in which case it looks like an arrow filling a loop. An opetope of dimension 3 is determined by its pasting diagram of sources, and its target is the opetope of dimension 2 that has the same "boundary" as the pasting diagram of sources; see Fig. 2. The opetopes form a category, and opetopic sets are presheaves on the category of opetopes.

Several equivalent definitions of opetopes have been proposed: Baez and Dolan [1998] using operads; Leinster [2004] using cartesian monads; Hermida et al. [2002] (called multitopes there) using multicategories. Comparison of these definitions is made by Cheng [2003, 2004b,a]. More recent accounts are given by Kock et al. [2010] using polynomial monads and by Curien et al. [2022] using type theory. Ho Thanh [2021] gives an explicit presentation of the category of opetopes by generators and relations and shows that presheaves on it, that is, opetopic sets, are equivalent to many-to-one polygraphs. Those definitions of opetopes, however, require some amount of prerequisites.

The author would like to thank Soichiro Fujii and the anonymous referee for helpful feedback. This work was partially supported by JST (JPMJMS2033).

Received by the editors 2025-04-22 and, in final form, 2026-01-07.

Transmitted by Richard Garner. Published on 2026-01-27.

2020 Mathematics Subject Classification: 18N20, 18N30, 18C15.

Key words and phrases: opetopic sets, opetopes, substitution, grafting, polynomial, polynomial monad.

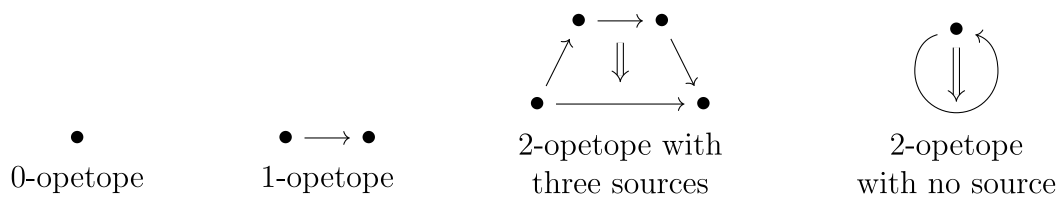


Figure 1: Examples of opetopes of dimension 0, 1, and 2

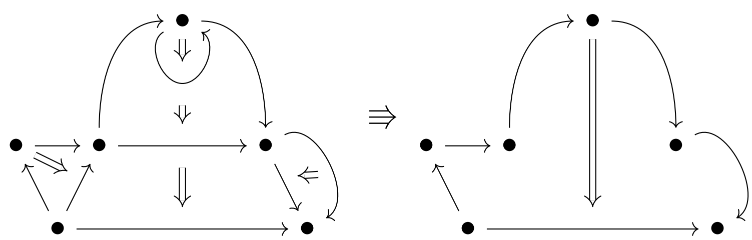


Figure 2: Example of opetope of dimension 3, the pasting diagram of sources on the left and the target on the right.

A more elementary approach to opetopes is taken by Leclerc [2024b]. There, opetopes are defined as posets of cells whose ordering expresses inclusion of cells. In addition, each subcell of codimension 1 is marked as a source or a target. That posetal approach works very well for "positive" opetopes [Zawadowski, 2023; Leclerc, 2024a] in which every subcell is either a source or a target but not both. However, a general opetope may contain a cell that is both the source and the target of a loop such as the point in the 2-opetope with no source (Fig. 1). Due to such loops, the posetal approach to general opetopes is more complicated than positive opetopes.

In this paper, we propose another elementary definition of opetopes. The idea is to store cells in a *category* rather than a poset to distinguish source and target inclusions of a cell into a loop. For example, the 2-opetope with no source is encoded by a poset and a category as in Fig. 3. The "source and target" arrow from the 0-cell to the 1-cell in the posetal encoding is split into a source arrow and a target arrow in the categorical encoding. This modification eliminates the need for special treatment on loops and simplifies the theory of opetopes.

We define, before opetopes, opetopic sets by a simple structure on a category and several axioms. Our formal definition of opetopic sets given in Section 2 takes less than two pages, and the only prerequisite is basic category theory. In Section 3, we show basic properties of opetopic sets and morphisms between them. One goal is that every slice of the category of opetopic sets is a presheaf category (Proposition 3.41).

Opetopes are defined in Section 4 as opetopic sets satisfying one more axiom. An inter-

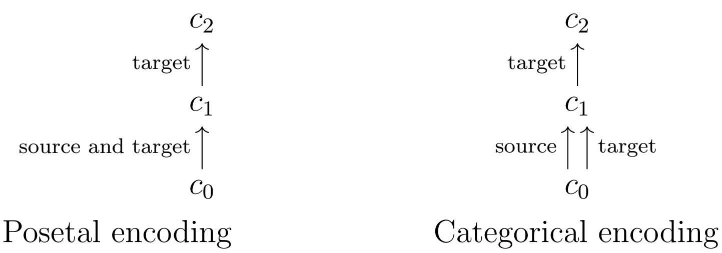


Figure 3: Posetal (left) and categorical (right) encodings of the 2-opetope with no source. c_0, c_1, and c_2 are the 0-cell, the 1-cell (loop), and the 2-cell, respectively, in the opetope. In the posetal encoding, there is only one arrow from c_0 to c_1 exhibiting c_0 as a source-and-target of c_1. In the categorical encoding, there are two arrows from c_0 to c_1. One exhibits c_0 as a source of c_1, and the other as a target.

esting phenomenon is that the category of opetopes is naturally turned into an opetopic set. Moreover, the opetopic set of opetopes is the terminal object in the category of opetopic sets. Consequently, the category of opetopic sets is a presheaf category (Theorem 4.8). In particular, colimits of opetopic sets exist. We provide in Section 5 some tools to compute colimits of opetopic sets.

In Section 6, we introduce *boundaries* and *pasting diagrams*. We show that an opetope is completely determined by its boundary (Proposition 6.20) or the pasting diagram formed by its sources (Corollary 6.24). We provide in Section 7 two operators on pasting diagrams, *substitution* and *grafting*. Since we already know the presheaf category of opetopic sets, these operators are simply defined by colimits.

Using the substitution and grafting operators, we show in Section 8 that our definition of opetopes is equivalent to the polynomial monad definition given by Kock et al. [2010] (Corollary 8.13). We also see that the category of opetopes is presented by the generators and relations described by Ho Thanh [2021].

Foundations The results in the present paper are valid in any constructive foundations of mathematics. For concreteness, we choose *Univalent Foundations* [The Univalent Foundations Program, 2013] because it seems to be a proper foundation especially for category theory in that isomorphic objects are identical [Ahrens et al., 2015]. Only one univalent universe $\mathcal{U}$ is needed. An object is said to be *small* if it is equivalent to an object in $\mathcal{U}$. We do not assume the law of excluded middle, the axiom of choice, or the propositional resizing axiom. We use notation $(x : A) \to B(x)$ for dependent function types (Π-types) and $(x : A) \times B(x)$ for dependent pair types (Σ-types). A category in Univalent Foundations [Ahrens et al., 2015] satisfies that the type of identifications $x = y$ between objects is equivalent to the type of equivalences $x \simeq y$, so equivalent objects satisfy the same properties. $\mathbf{Obj}(C)$ denotes the type of objects in a category C, and $\mathbf{Arr}_C(x, y)$ denotes the set of arrows from x to y in C.

All the types that appear in this paper are actually 1-truncated, so our language can be translated into set-theoretic foundations by interpreting types as groupoids [Hofmann and Streicher, 1998].

2. Definition of opetopic sets

We first introduce our formal definition of opetopic sets (Definition 2.7) and then explain intuition.

2.1. Definition. *We say a category C is* gaunt *if its type of objects* $\mathbf{Obj}(C)$ *is a set. The small gaunt categories and the functors between them form a category* $\mathbf{Gaunt}$.

2.2. Remark. In terms of set-theoretic foundations, a category is gaunt if its underlying groupoid is discrete. This is equivalent to that the identities are the only isomorphisms, which coincides with the definition given in [Barwick and Schommer-Pries, 2021, Definition 3.1].

2.3. Definition. *An* ω-direct category *is a gaunt category A equipped with a conservative functor* $\mathbf{deg}_A : A \to \omega$ *called the* degree functor*, where ω is the poset of natural numbers. Let* $\mathbf{DirCat}_\omega \subset \mathbf{Gaunt} \downarrow \omega$ *denote the full subcategory spanned by the ω-direct categories.*

2.4. Definition. *Let $k : \omega$ and let A be an ω-direct category. We write $f : x \to^k y$ to mean that $f : x \to y$ is an arrow in A between objects satisfying that $\mathbf{deg}(x) + k = \mathbf{deg}(y)$. Such an arrow is called a* k-step *arrow. Let* $\mathbf{Arr}^k(A)$ *denote the set of k-step arrows in A. We also define the* k-step slice $A \downarrow^k x$ *to be the full subcategory of $A \downarrow x$ spanned by the k-step arrows into x. Note that $A \downarrow^k x$ is discrete.*

2.5. Definition. *A* preopetopic set *is an ω-direct category A equipped with a subset* $\mathbf{S}(A) \subset \mathbf{Arr}^1(A)$ *with complement* $\mathbf{T}(A)$*. An arrow in* $\mathbf{S}(A)$ *is called a* source arrow *and written as $f : x \to^{\mathrm{s}} y$. An arrow in* $\mathbf{T}(A)$ *is called* target arrow *and written as $f : x \to^{\mathrm{t}} y$. A* morphism of preopetopic sets *is a morphism of ω-direct categories preserving source and target arrows. Let* $\mathbf{PreOSet}$ *denote the category of small preopetopic sets.*

2.6. Definition. *Let A be a preopetopic set and let $f : y \to^1 x$ and $g : z \to^1 y$ be 1-step arrows in A. We say (f, g) is* homogeneous *if either both f and g are source arrows or both f and g are target arrows. We say (f, g) is* heterogeneous *if either f is a source arrow and g is a target arrow or f is a target arrow and g is a source arrow. By a* homogeneous/heterogeneous factorization *of a 2-step arrow h we mean a factorization $h = f \circ g$ such that (f, g) is homogeneous/heterogeneous.*

2.7. Definition. *An* opetopic set *is a preopetopic set A satisfying the following axioms.*

O1. *$A \downarrow^1 x$ is finite for every $x : A$.*

O2. *For every object $x : A$ of degree ≥ 1, there exists a unique target arrow into x.*

O3. *For every object $x : A$ of degree 1, there exists a unique source arrow into x.*

O4. *Every 2-step arrow $y \to^2 x$ in A has a unique homogeneous factorization.*

O5. *Every 2-step arrow $y \to^2 x$ in A has a unique heterogeneous factorization.*

O6. *For every object $x : A$ of degree ≥ 2, there exists a 2-step arrow $r : A \downarrow^2 x$ such that, for every 2-step arrow $f : A \downarrow^2 x$, there exists a zigzag*

$$f = f_0 \xrightarrow{s_0} g_0 \xleftarrow{t_0} f_1 \xrightarrow{s_1} \dots \xrightarrow{s_{m-1}} g_{m-1} \xleftarrow{t_{m-1}} f_m = r, \tag{1}$$

where g_i's are source arrows into x, s_i's are source arrows in $A \downarrow x$, and t_i's are target arrows in $A \downarrow x$.

O7. *For every target arrow $f : y \to^{\mathrm{t}} x$ in A and object $z : A$ of degree $\leq \mathbf{deg}(y) - 2$, the postcomposition map $f_! : \mathbf{Arr}_A(z, y) \to \mathbf{Arr}_A(z, x)$ is injective.*

O8. *For every $k \geq 3$, every k-step arrow $y \to^k x$ in A factors as $f \circ g$ such that f is a $(k-1)$-step arrow and g is a 1-step arrow.*

Let $\mathbf{OSet} \subset \mathbf{PreOSet}$ *denote the full subcategory spanned by the opetopic sets.*

Let A be an opetopic set. We think of objects in A as *cells*, and arrows in A determine the configuration of the cells. A source arrow $y \to^{\mathrm{s}} x$ exhibits y as a source of x, and a target arrow $y \to^{\mathrm{t}} x$ exhibits y as a target of x. Axiom O1 asserts that every cell x has finitely many sources and targets. Recall that a set X is finite if there (merely) exist a natural number $n : \mathbb{N}$ and an equivalence $\{k : \mathbb{N} \mid k < n\} \simeq X$ [Rijke, 2022, Definition 16.3.1]. Axiom O2 asserts that every cell x of dimension ≥ 1 has a unique target, expressing the single-out nature of opetopes. An opetope may have many sources with the exception of the opetope of dimension 1 which has a unique source, so we introduce Axiom O3.

Axioms O4 and O5 assert that, for every 2-step arrow $y \to^2 x$, exactly one of the following holds.

1. y is a source of a source of x and a source of the target of x.
2. y is a source of a source of x and the target of a source of x.
3. y is the target of the target of x and a source of the target of x.
4. y is the target of the target of x and the target of a source of x.

Figure 4 illustrates each of these situations.

Axiom O6, combined with other axioms, expresses that the sources of a cell form a "tree". For example, consider the pasting diagram on the left of Fig. 5 which is the sources of a 3-dimensional cell. We see the zigzag

$$f_0 \to^{\mathrm{s}} g_0 \ {}^{\mathrm{t}}\!\!\leftarrow f_1 \to^{\mathrm{s}} g_1 \ {}^{\mathrm{t}}\!\!\leftarrow r,$$

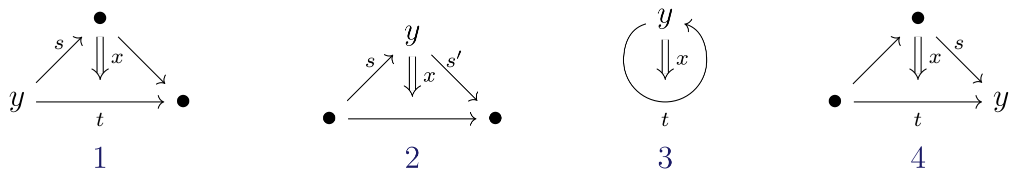


Figure 4: Illustration of Axioms O4 and O5. In Case 1, y is a source of the source s of x and a source of the target t of x. In Case 2, y is a source of the source s' of x and the target of the source s of x. In Case 3, y is the target of the target t of x and a source of the target t of x. In Case 4, y is the target of the target t of x and the target of the source s of x.

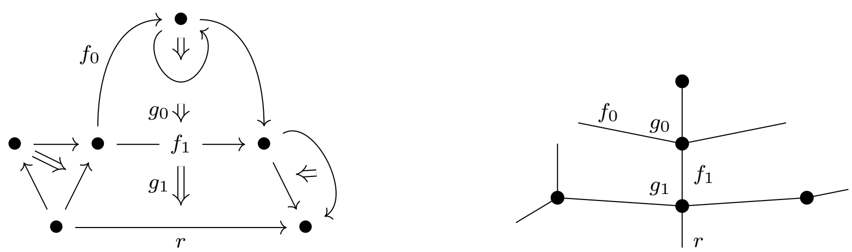


Figure 5: Illustration of Axiom O6. The pasting diagram on the left has the tree structure on the right. Dots and lines in the tree correspond to 2-dimensional cells and 1-dimensional cells, respectively, in the pasting diagram.

and one can find a similar zigzag from any 1-cell to r. Moreover, such a zigzag to r is unique; we will prove this in Lemma 3.7 later. In this way we see the tree structure of the pasting diagram displayed on the right of Fig. 5.

Axioms O1 to O6 are local conditions in that they mention only 1-step and 2-step arrows. Axioms O7 and O8 are global conditions for tying cells in all dimensions together. There can be several axiomatizations. We choose Axioms O7 and O8 as minimal assumptions to prove basic properties of opetopic sets in Section 3.

2.8. Example. We list in Fig. 6 the categorical encodings of the low-dimensional opetopes displayed in Fig. 1. One can verify Axioms O1 to O8 for each category.

2.9. Variants of the definition. There are some variants of Definition 2.7.

As already mentioned, Axioms O7 and O8 may be replaced by other axioms. Indeed, we will prove in Lemma 6.15 that, under Axioms O1 to O6, the conjunction of Axioms O7 and O8 is equivalent to the following stronger variant of Axiom O7.

O7'. For every target arrow $f : y \to^{\mathrm{t}} x$ in A and object $z : A$ of degree $\leq \mathbf{deg}(y) - 2$, the postcomposition map $f_! : \mathbf{Arr}_A(z, y) \to \mathbf{Arr}_A(z, x)$ is an equivalence.

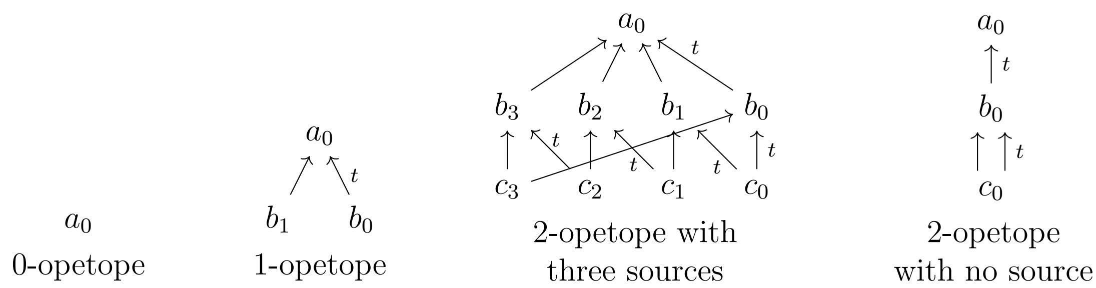


Figure 6: Examples of categorical encodings of opetopes of dimension 0, 1, and 2. Target arrows are marked with "t". All the other arrows are source arrows. In these examples, all diagrams into a_0 commute, and there is no other non-trivial commutative diagram. In particular, the parallel arrows from c_0 to b_0 in the 2-opetope with no source are distinct.

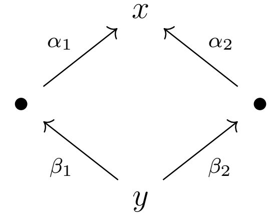
Axiom O3, which singles out degree 1, may be eliminated by augmentation. Specifically, the following modification yields an equivalent definition.

- Extend A by an initial object $\perp$ whose degree is -1.
- Drop Axiom O3.
- Amend Axiom O2 to state "For every object $x : A$ of degree ≥ 0".

Axiom O3 is recovered as follows. For every object $x : A$ of degree 1, the unique arrow $\perp \to x$ has a unique heterogeneous factorization $f \circ g$ by Axiom O5. By the amended version of Axiom O2, g must be a target arrow. Then f is the unique source arrow into x.

2.10. Comparison with posetal approaches. Our definition of opetopic sets shares some ideas with posetal approaches to combinatorics of higher categories.

Axioms O4 and O5 are categorified "oriented thinness" [Hadzihasanovic, 2020, Definition 5], which is also known as a "balanced coloring" [e.g. Chandler, 2019, Definition 4.6]. A similar property also appears in the study of parity structures [Nguyen, 2018, Theorem 1.36]. Its non-oriented variant is called the "diamond property" in the theory of abstract polytopes [McMullen and Schulte, 2002]. Axioms O4 and O5 asserts that every 2-step arrow $y \to^2 x$ factors in exactly two ways, homogeneously and heterogeneously, forming a diamond shape.



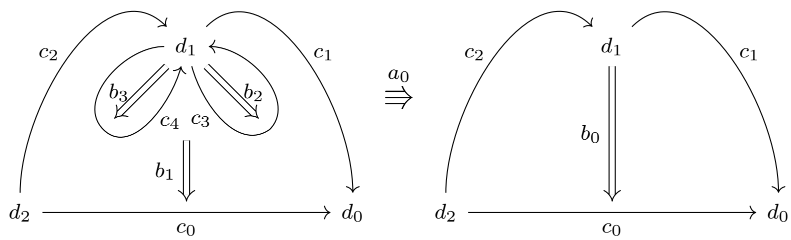


Figure 7: A 3-opetope with multiple loops on a cell.

Let us assign "−" sign to source arrows and "+" sign to target arrows. When signs α_1, β_1, α_2, and β_2 are assigned as in the above diagram, the homogeneity of one factorization and the heterogeneity of the other are expressed by the single "sign rule" $\alpha_1\beta_1 = -\alpha_2\beta_2$.

The posetal definition of opetopes given by Leclerc [2024b, Definition 2.9] also requires oriented thinness, but it is split into two cases, one for loops and the other for non-loops. In our categorical approach, there is no need for case splitting.

A posetal approach to opetopes faces the issue of multiple loops on a cell. For example, consider the 3-opetope displayed in Fig. 7. It contains loops c_4 and c_3. They are distinguished in the picture by the order of occurrence in the pasting diagram of sources of b_1

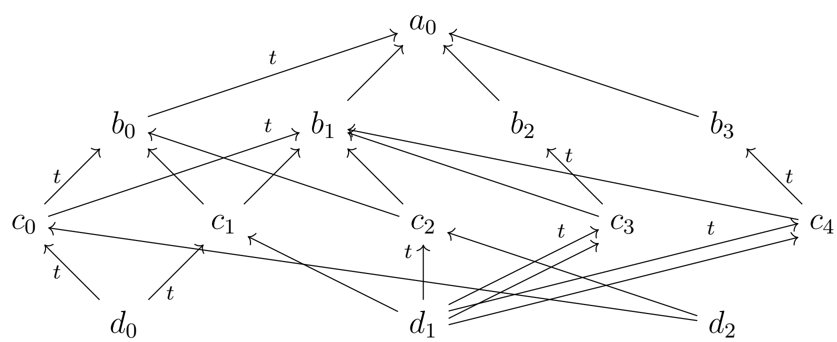
$$d_2 \xrightarrow{c_2} d_1 \xrightarrow{c_4} d_1 \xrightarrow{c_3} d_1 \xrightarrow{c_1} d_0.$$

In a posetal approach, however, there is no apparent way of distinguishing c_4 from c_3, because swapping c_4 with c_3 and b_3 with b_2 yields the same subcell relation. Leclerc [2024b] resolves this issue by further requiring a total order on loops on a cell as part of structure.

In our categorical approach, loops on a cell are distinguished by equality of arrows. The opetope in Fig. 7 is encoded by a category as in Fig. 8. Here, the first three diamonds in the lower half of Fig. 8 are the only non-trivial commutative diamonds from d_1 to b_1. Then the last diamond in Fig. 8, for example, does not commute. The commutativity of the first diamond and the non-commutativity of the last diamond distinguishes c_4 from c_3.

3. Properties of opetopic sets

We prove basic properties of opetopic sets and morphisms between them: the underlying category of an opetopic set has a canonical presentation (Section 3.1); every slice of an opetopic set is finite (Section 3.16); if two morphisms of opetopic sets $A \to A'$ agree at an object $x : A$, then they agree on the slice $A \downarrow x$ (Section 3.23); any morphism of opetopic sets induces an equivalence between slices (Section 3.31); every slice $\mathbf{OSet} \downarrow A$ is a presheaf category (Section 3.36).



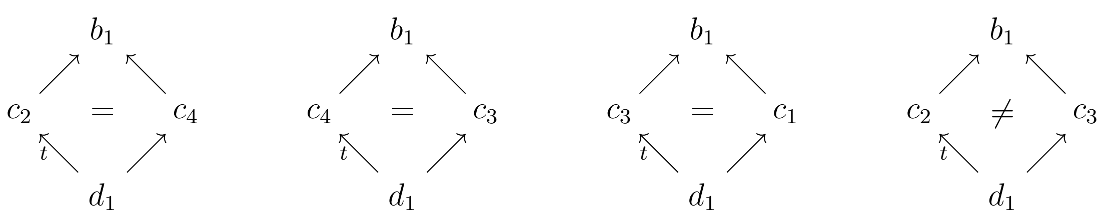


Figure 8: Categorical encoding of the opetope in Fig. 7 with selected commutative and non-commutative diamonds. All the diagrams into a_0, b_0, b_2, or b_3 commute, the diamond from d_2 to b_1 commutes, and the first three diamonds below are the only non-trivial commutative diamonds from d_1 to b_1. Target arrows are marked with "t", and all the other arrows are source arrows.

3.1. Generators and relations. We show that the underlying category of an opetopic set is canonically presented by generating 1-step arrows and equations between 2-step arrows (Proposition 3.15).

3.2. Construction. *Let A be an opetopic set. The set* $\mathbf{gen}(A)$ *of* canonical generators *for A is the set of 1-step arrows in A. The set* $\mathbf{rel}(A)$ *of* canonical relations *for A is the set of equations of the form $f_1 \circ g_1 = f_2 \circ g_2$ that hold in A such that (f_1, g_1) is heterogeneous and (f_2, g_2) is homogeneous. Let $\mathbf{C}(A)$ be the category with the same object as A and presented by generators $\mathbf{gen}(A)$ and relations $\mathbf{rel}(A)$. By definition, we have a canonical functor $\mathbf{C}(A) \to A$, which is conservative because there is no way to construct a non-trivial equivalence in $\mathbf{C}(A)$, by which we regard $\mathbf{C}(A)$ an ω-direct category.*

We show that the canonical functor $\mathbf{C}(A) \to A$ is an equivalence (Proposition 3.15). We prepare a lemma on a tree structure in an opetopic set (Lemma 3.7).

3.3. Definition. *A* graph *G consists of a set $\mathbf{V}(G)$ of* vertices *and a set $\mathbf{E}_G(x, y)$ of* edges from x to y *for every $x, y : \mathbf{V}(G)$. We may write $f : x \to y$ instead of $f : \mathbf{E}_G(x, y)$. For vertices $x, y : G$, a* path from x to y *is a chain of edges*

$$x = z_0 \xrightarrow{f_0} z_1 \xrightarrow{f_1} \dots \xrightarrow{f_{m-1}} z_m = y.$$

We say a graph G is a tree *if there exists a vertex $r : G$ such that, for every vertex $x : G$, there exists a unique path from x to r. In other words, r is the terminal object in the free category over G, from which it follows that such a vertex r is unique. We refer to r as the* root of G.

When G is a tree, the following induction principle is valid. Let P be a property on vertices in G. Suppose:

- the root of G satisfies P; and
- for every edge $f : x \to y$ in G, if y satisfies P, then x satisfies P.

Then every vertex $x : G$ satisfies P. This induction principle is justified by induction on the length of the unique path from x to the root.

3.4. Construction. *Let A be a preopetopic set and let $x : A$. We define the* source slice $A \downarrow^{\mathrm{s}} x \subset A \downarrow^{1} x$ *to be the subset spanned by the source arrows into x. We define the* target slice $A \downarrow^{\mathrm{t}} x \subset A \downarrow^{1} x$ *to be the subset spanned by the target arrows into x. By definition, $(A \downarrow^{\mathrm{s}} x) + (A \downarrow^{\mathrm{t}} x) \simeq (A \downarrow^{1} x)$.*

3.5. Construction. *Let A be an opetopic set and let $x : A$ be an object of degree ≥ 1. We refer to the unique target arrow into x, which exists by Axiom O2, as $\overline{\mathbf{t}}_A(x) : \mathbf{t}_A(x) \to^{\mathrm{t}} x$ or $\overline{\mathbf{t}}(x) : \mathbf{t}(x) \to^{\mathrm{t}} x$ if A is clear from the context. For $k \leq \mathbf{deg}(x)$, we define $\overline{\mathbf{t}}^k(x) : \mathbf{t}^k(x) \to^k x$ by $\overline{\mathbf{t}}^0(x) \equiv \mathbf{id}_x$ and $\overline{\mathbf{t}}^{k+1}(x) \equiv \overline{\mathbf{t}}^k(x) \circ \overline{\mathbf{t}}(\mathbf{t}^k(x))$.*

3.6. CONSTRUCTION. *Let A be an opetopic set and let $x : A$ be an object of degree ≥ 1. We define a graph $\mathbf{G}(A, x)$ as follows. The set of vertices in $\mathbf{G}(A, x)$ is $(A \downarrow^{\mathrm{s}} x)+(A \downarrow^2 x)$. There is no edge between vertices from $A \downarrow^{\mathrm{s}} x$. There is no edge between vertices from $A \downarrow^2 x$. An edge from $f : A \downarrow^{\mathrm{s}} x$ to $g : A \downarrow^2 x$ is a target arrow $t : g \to^{\mathrm{t}} f$ in $A \downarrow x$. An edge from $g : A \downarrow^2 x$ to $f : A \downarrow^{\mathrm{s}} x$ is a source arrow $s : g \to^{\mathrm{s}} f$ in $A \downarrow x$.*

3.7. LEMMA. *Let A be an opetopic set and let $x : A$ be an object of degree ≥ 1.*

1. *The graph $\mathbf{G}(A, x)$ is a tree.*
2. *If $\mathbf{deg}(x) = 1$, then $\mathbf{G}(A, x)$ is a singleton set with no edge.*
3. *If $\mathbf{deg}(x) \geq 2$, then the root of $\mathbf{G}(A, x)$ is $\bar{\mathbf{t}}^2(x) : A \downarrow^2 x$.*

PROOF. Suppose that $\mathbf{deg}(x) = 1$. Then the set of vertices in $\mathbf{G}(A, x)$ is a singleton set since $A \downarrow^2 x$ is empty and $A \downarrow^{\mathrm{s}} x$ is a singleton by Axiom O3. $\mathbf{G}(A, x)$ has no edge by definition. $\mathbf{G}(A, x)$ is a tree whose root is the unique vertex. Suppose that $\mathbf{deg}(x) \geq 2$. Take r as in Axiom O6. By the definition of $\mathbf{G}(A, x)$, Zigzag (1) is a path in $\mathbf{G}(A, x)$ from f to r. Thus, there is a path p from $\bar{\mathbf{t}}^2(x)$ to r. By Axiom O4, there is no edge from $\bar{\mathbf{t}}^2(x)$. Hence, the length of p is 0, and thus $\bar{\mathbf{t}}^2(x) = r$. For every $f : A \downarrow^2 x$, there is a path from f to $\bar{\mathbf{t}}^2(x)$. Such a path is unique by Axioms O4 and O2 and by the fact that there is no edge from $\bar{\mathbf{t}}^2(x)$. For every $f : A \downarrow^{\mathrm{s}} x$, we have the unique edge $t : f \to g$ by Axiom O2 and then a unique path from g to $\bar{\mathbf{t}}^2(x)$ as we have seen. ∎

We prove $\mathbf{C}(A) \simeq A$ by a *normalization* procedure.

3.8. DEFINITION. *Let A be an opetopic set and let $k \geq 2$. We say a composable tuple of 1-step arrows $(f_1, \ldots, f_k)$ in A is* in normal form *if $f_1, \ldots, f_{k-2}$ are target arrows and (f_{k-1}, f_k) is homogeneous. For $x, y : A$ such that $\mathbf{deg}(x) + k = \mathbf{deg}(y)$, let $\mathbf{NF}^k_A(x, y)$ denote the set of composable tuples of 1-step arrows $(f_1, \ldots, f_k)$ in normal form such that $f_1 \circ \ldots \circ f_k : x \to^k y$.*

We first consider normalization of three 1-step arrows.

3.9. CONSTRUCTION. *Let A be an opetopic set and let $f : y \to^3 x$ be a 3-step arrow. We define a graph $\mathbf{G}^3(A, f)$ as follows. A vertex in $\mathbf{G}^3(A, f)$ is a factorization (p, q, r) of f into three 1-step arrows $f = p \circ q \circ r$. We add an edge from (p_1, q_1, r_1) to (p_2, q_2, r_2) when one of the following holds.*

1. *$p_1 = p_2$ is a source arrow, $q_1 \circ r_1 = q_2 \circ r_2$, (q_1, r_1) is heterogeneous, and (q_2, r_2) is homogeneous.*
2. *$r_1 = r_2$ is a source arrow, $p_1 \circ q_1 = p_2 \circ q_2$, (p_1, q_1) is homogeneous, and (p_2, q_2) is heterogeneous.*
3. *$r_1 = r_2$ is a target arrow, $p_1 \circ q_1 = p_2 \circ q_2$, (p_1, q_1) is heterogeneous, and (p_2, q_2) is homogeneous.*

Condition	(p_1, q_1, r_1)	(p_2, q_2, r_2)
1	(s, s, t) or (s, t, s)	(s, s, s) or (s, t, t)
2	(s, s, s) or (t, t, s)	(s, t, s) or (t, s, s)
3	(s, t, t) or (t, s, t)	(s, s, t) or (t, t, t)

Table 1: Possible combinations of source (s) and target (t) arrows when there is an edge $(p_1, q_1, r_1) \to (p_2, q_2, r_2)$ in $\mathbf{G}^3(A, f)$.

These are all the edges in $\mathbf{G}^3(A, f)$. Table 1 lists all possible combinations of source and target arrows when there is an edge $(p_1, q_1, r_1) \to (p_2, q_2, r_2)$ in $\mathbf{G}^3(A, f)$.

3.10. Lemma. *Let A be an opetopic set, let f be a 3-step arrow in A, and let $v \equiv (p, q, r) : \mathbf{G}^3(A, f)$ be a vertex. Then either there is no edge from v or there is a unique edge from v. Moreover, the former holds if and only if (p, q, r) is in normal form.*

Proof. By Axioms O5 and O4. See also Table 1. ∎

3.11. Lemma. *Let A be an opetopic set and let $f : y \to^3 x$ be a 3-step arrow in A. Then there exists a natural number n such that every path in $\mathbf{G}^3(A, f)$ is of length at most n.*

Proof. By definition, a path π in $\mathbf{G}^3(A, f)$ is of the form

$$\ldots \to v_i \to v'_i \to v_{i+1} \to \ldots,$$

where Condition 1 holds for the edge $v_i \to v'_i$ and Condition 2 or 3 holds for the edge $v'_i \to v_{i+1}$. Let $v_i \equiv (p_i, q_i, r_i)$ and $v'_i \equiv (p_i, q'_i, r_{i+1})$, and let $g_i \equiv p_i \circ q'_i$ and $h_i = q_i \circ r_i$. Let z_i be the codomain of q_i (or the domain of p_i). Because a vertex (p, q, r) with p a target arrow appears only at the start or the end of a path by Table 1, we may assume that all the p_i's are source arrows. If r_{i+1} is a source arrow, then q'_i is a source arrow, q_{i+1} is a target arrow, and we have the edges $p_i \xleftarrow{q'_i} g_i \xleftarrow{q_{i+1}} p_{i+1}$ in $\mathbf{G}(A, x)$. If r_{i+1} is a target arrow, then q'_i is a target arrow, q_{i+1} is a source arrow, and we have the edges $p_i \xrightarrow{q'_i} g_i \xrightarrow{q_{i+1}} p_{i+1}$ in $\mathbf{G}(A, x)$. In particular, p_i's and g_i's form a zigzag ζ in $\mathbf{G}(A, x)$. Let $p : A \downarrow^{\mathrm{s}} x$ and let $i_1 < i_2 < \ldots$ be the indexes i such that $p_i = p$. Note that we can find these indexes because the identity type on $A \downarrow^{\mathrm{s}} x$ is decidable by Axiom O1.

Suppose that r_{i_j+1} is a source arrow and i_{j+1} exists. Since $\mathbf{G}(A, x)$ is a tree (Lemma 3.7), the zigzag ζ between p_{i_j} and $p_{i_{j+1}}$ begins with $p_{i_j} \xleftarrow{q'_{i_j}} g_{i_j} \xleftarrow{q_{i_j+1}} p_{i_j+1}$ and ends with $p_{i_j+1} \xrightarrow{q_{i_j+1}} g_{i_j} \xrightarrow{q'_{i_j}} p_{i_j} = p_{i_{j+1}}$. Hence, $p_{i_{j+1}-1} = p_{i_j+1}$, $q'_{i_{j+1}-1} = q_{i_j+1}$, and $q_{i_{j+1}} = q'_{i_j}$. Then $q_{i_{j+1}}$ is always a source arrow, and we have the edges $q_{i_j} \xrightarrow{r_{i_j}} h_{i_j} \xrightarrow{r_{i_j+1}} q'_{i_j} = q_{i_{j+1}}$ in $\mathbf{G}(A, z_{i_j})$ when q_{i_j} is also a source arrow.

Suppose that r_{i_j+1} is a target arrow and i_{j+1} exists. Let p' be the last p_i between p_{i_j} and $p_{i_{j+1}}$ in the path π with minimum depth (i.e. the length of the path to the root), and let $k_1, k_2 \ldots$ be the indices k between i_j and i_{j+1} such that $p_k = p'$. Then we have the edges

$p_{k_l-1} \xrightarrow{q'_{k_l-1}} g_{k_l-1} \xrightarrow{q_{k_l}} p_{k_l} \xleftarrow{q'_{k_l}} g_{k_l} \xleftarrow{q_{k_l+1}} p_{k_l+1}$ in $\mathbf{G}(A, x)$. We see that q_{k_l} and q'_{k_l} are source arrows, and we have the edges $q_{k_l} \xrightarrow{r_{k_l}} h_{k_l} \xrightarrow{r_{k_l+1}} q'_{k_l}$ in $\mathbf{G}(A, z_{k_l})$. From the observation in the previous paragraph, $q'_{k_l} = q_{k_{l+1}}$ is a source arrow. It then follows that p_{i_j} and $p_{i_{j+1}}$ lie in different branches of p' in $\mathbf{G}(A, x)$, which contradicts that $p_{i_j} = p_{i_{j+1}} = p$. Therefore, if r_{i_j+1} is a target arrow, then p_{i_j} is the end of occurrence of p in π.

These observations give a bound $2 \times \mathbf{card}(A \downarrow^1 x')$ for the number of occurrences of each $p : x' \to^{\mathrm{s}} x$ in a path in $\mathbf{G}^3(A, f)$, where $\mathbf{card}(A \downarrow^1 x')$ is the cardinality of the finite set $A \downarrow^1 x'$ (Axiom O1). Since $A \downarrow^{\mathrm{s}} x$ is also finite by Axiom O1, we obtain a bound $\sum_{x' : A \downarrow^{\mathrm{s}} x} 2 \times \mathbf{card}(A \downarrow^1 x')$ for the lengths of paths in $\mathbf{G}^3(A, f)$. ∎

3.12. Lemma. *Let A be an opetopic set, let f be a 3-step arrow in A, and let $v : \mathbf{G}^3(A, f)$. Then there exists a unique path in $\mathbf{G}^3(A, f)$ from v to a vertex in normal form.*

Proof. By Lemma 3.10, we can uniquely extend a path from v until it reaches a vertex in normal form. By Lemma 3.11, this procedure terminates at such a vertex. ∎

We now obtain the normalization procedure.

3.13. Lemma. *Let A be an opetopic set and $k \geq 2$. Then every k-step arrow f in $\mathbf{C}(A)$ factors into k 1-step arrows $g_1 \circ \ldots \circ g_k$ in normal form.*

Proof. We proceed by induction on k. The case when $k = 2$ is by Axiom O4. Suppose that $k \geq 3$. By definition, f factors into 1-step arrows $f_1 \circ \ldots \circ f_k$. By Lemma 3.12, $f_1 \circ f_2 \circ f_3 = g_1 \circ g_2 \circ g_3$ in $\mathbf{C}(A)$ with (g_1, g_2, g_3) in normal form, since $p_1 \circ q_1 \circ r_1 = p_2 \circ q_2 \circ r_2$ in $\mathbf{C}(A)$ whenever there is an edge $(p_1, q_1, r_1) \to (p_2, q_2, r_2)$ in $\mathbf{G}^3(A, f)$ by definition. Then apply the induction hypothesis for $g_2 \circ g_3 \circ f_4 \circ \ldots \circ f_k$. ∎

3.14. Lemma. *Let A be an opetopic set, let $k \geq 2$, and let $x, y : A$ be objects such that $\mathbf{deg}(x) + k = \mathbf{deg}(y)$. Then $\mathbf{NF}^k_A(x, y) \simeq \mathbf{Arr}_{\mathbf{C}(A)}(x, y) \simeq \mathbf{Arr}_A(x, y)$.*

Proof. Let $H : \mathbf{NF}^k_A(x, y) \to \mathbf{Arr}_{\mathbf{C}(A)}(x, y)$ and $K : \mathbf{Arr}_{\mathbf{C}(A)}(x, y) \to \mathbf{Arr}_A(x, y)$ denote the canonical maps. H is surjective by Lemma 3.13. K is surjective by Axioms O8 and O4. $K \circ H$ is injective by Axioms O2, O7 and O4. Thus, $K \circ H$ is an equivalence, and then H and K are also equivalences. ∎

3.15. Proposition. *Let A be an opetopic set. Then the canonical functor $\mathbf{C}(A) \to A$ is an equivalence.*

Proof. By construction, the canonical functor is an equivalence on objects and fully faithful on 1-step arrows. It is fully faithful on k-step arrows for $k \geq 2$ by Lemma 3.14. ∎

3.16. Local finiteness. We show that every slice of an opetopic set is finite (Proposition 3.22). We say a gaunt category C is finite if $\mathbf{Obj}(C)$ is finite and the set of arrows $\mathbf{Arr}_C(x, y)$ is finite for all $x, y : C$.

3.17. Lemma. *Let A be an opetopic set, let $k \geq 2$, and let $x : A$ be an object of degree $\geq k$. Then $A \downarrow^k x \simeq ((y : A \downarrow^{\mathrm{s}} \mathbf{t}^{k-2}(x)) \times (A \downarrow^{\mathrm{s}} y)) + ((y : A \downarrow^{\mathrm{t}} \mathbf{t}^{k-2}(x)) \times (A \downarrow^{\mathrm{t}} y))$.*

Proof. By Lemma 3.14. ■

3.18. Lemma. *Let A be an opetopic set, let $x : A$, and let $k \geq 0$. Then $A \downarrow^k x$ is finite.*

Proof. By case analysis on k. The case when $k = 0$ is because $A \downarrow^0 x \simeq \{\mathbf{id}_x\}$. The case when $k = 1$ is by Axiom O1. The case when $k \geq 2$ follows from Lemma 3.17 and Axiom O1. ■

3.19. Lemma. *Let C be a category and let $x, y : C$ be objects. Then $\mathbf{Arr}_C(x, y)$ is the fiber of $\mathbf{Obj}(C \downarrow y) \to \mathbf{Obj}(C)$ over $x : \mathbf{Obj}(C)$.*

Proof. By definition. ■

3.20. Lemma. *Let C be a category, $x : C$, and $y, z : C \downarrow x$. Then $\mathbf{Arr}_{C\downarrow x}(y, z)$ is the fiber of $\mathbf{Obj}(C \downarrow z) \to \mathbf{Obj}(C \downarrow x)$ over $y : \mathbf{Obj}(C \downarrow x)$.*

Proof. By Lemma 3.19. ■

3.21. Lemma. *Let C be a gaunt category. Suppose that $\mathbf{Obj}(C \downarrow x)$ is finite for every $x : C$. Then $C \downarrow x$ is finite for every $x : C$.*

Proof. By assumption, the object part of $C \downarrow x$ is finite. The arrow part of it is also finite by Lemma 3.20. ■

3.22. Proposition. *Let A be an opetopic set. Then $A \downarrow x$ is finite for every $x : A$.*

Proof. By Lemma 3.18, $\mathbf{Obj}(A \downarrow x)$ is finite. Then apply Lemma 3.21. ■

3.23. Local uniqueness of morphisms. The next goal is local uniqueness of morphisms of opetopic sets (Proposition 3.30), which asserts that, if two morphisms of opetopic sets $A \to A'$ agree at an object $x : A$, then they agree on the slice $A \downarrow x$.

3.24. Construction. *Let G be a graph and let $x : G$ be a vertex. We define $G \downarrow x \equiv (y : G) \times \mathbf{E}_G(y, x)$.*

3.25. Lemma. *Let A be an opetopic set and let $f : y \to^{\mathrm{s}} x$ be a source arrow in A. Then $\mathbf{G}(A, x) \downarrow f \simeq A \downarrow^{\mathrm{s}} y$.*

Proof. By definition. ■

3.26. Lemma. *Let A be an opetopic set and let $f : y \to^2 x$ be a 2-step arrow in A. Then $\mathbf{G}(A, x) \downarrow f \simeq \mathbf{0}$ or $\mathbf{G}(A, x) \downarrow f \simeq \mathbf{1}$. The former holds if and only if f factors as a source arrow followed by a target arrow. The latter holds if and only if f factors as a target arrow followed by a source arrow.*

Proof. By definition and Axiom O5. ■

3.27. Lemma. *Let $F_1, F_2 : A \to A'$ be morphisms of opetopic sets, let $x : A$, and let $x' : A'$ such that $F_1(x) = F_2(x) = x'$. Then the induced maps $F_1 \downarrow^1 x, F_2 \downarrow^1 x : A \downarrow^1 x \to A' \downarrow^1 x'$ are identical.*

Proof. We prove that the graph morphisms $\mathbf{G}(F_1, x), \mathbf{G}(F_2, x) : \mathbf{G}(A, x) \to \mathbf{G}(A', x')$ agree on vertices by induction on $\mathbf{deg}(x)$. This implies that $F_1 \downarrow^{\mathrm{s}} x, F_2 \downarrow^{\mathrm{s}} x : A \downarrow^{\mathrm{s}} x \to A' \downarrow^{\mathrm{s}} x'$ are identical. Then, since both F_1 and F_2 send $\bar{\mathbf{t}}(x)$ to $\bar{\mathbf{t}}(x')$, we see that $F_1 \downarrow^1 x = F_2 \downarrow^1 x$. The case when $\mathbf{deg}(x) = 0$ is trivial since $\mathbf{G}(A, x)$ is empty. Suppose that $\mathbf{deg}(x) \geq 1$. We show that $F_1(f) = F_2(f)$ by induction on $f : \mathbf{G}(A, x)$. Both F_1 and F_2 send the root of $\mathbf{G}(A, x)$ to the root of $\mathbf{G}(A', x')$ by Lemma 3.7. Let $h : g \to f$ be an edge in $\mathbf{G}(A, x)$ and suppose that $F_1(f) = F_2(f)$. If $f : A \downarrow^{\mathrm{s}} x$, then $F_1(g) = F_2(g)$ by Lemma 3.25 and by the induction hypothesis for the domain of f. If $f : A \downarrow^2 x$, then $F_1(g) = F_2(g)$ by Lemma 3.26. ∎

3.28. Lemma. *Let $F_1, F_2 : A \to A'$ be morphisms of opetopic sets, let $k \geq 0$, let $x : A$, and let $x' : A'$ such that $F_1(x) = F_2(x) = x'$. Then the induced maps $F_1 \downarrow^1 x, F_2 \downarrow^k x : A \downarrow^k x \to A' \downarrow^k x'$ are identical.*

Proof. By case analysis on k. The case when $k = 0$ is trivial since $A \downarrow^0 x$ is the singleton $\{\mathbf{id}_x\}$. The case when $k = 1$ is Lemma 3.27. The case when $k \geq 2$ follows from Lemmas 3.17 and 3.27. ∎

3.29. Lemma. *Let $F_1, F_2 : C \to C'$ be functors between gaunt categories. Suppose that, for every $x : C$ and $x' : C$ such that $F_1(x) = F_2(x) = x'$, the induced maps $\mathbf{Obj}(F_1 \downarrow x)$, $\mathbf{Obj}(F_2 \downarrow x) : \mathbf{Obj}(C \downarrow x) \to \mathbf{Obj}(C' \downarrow x')$ are identical. Then, for every $x : C$ and $x' : C'$ such that $F_1(x) = F_2(x) = x'$, the induced functors $F_1 \downarrow x, F_2 \downarrow x : C \downarrow x \to C' \downarrow x'$ are identical.*

Proof. The object parts of $F_1 \downarrow x$ and $F_2 \downarrow x$ are identical by assumption. The arrow parts of them are also identical by Lemma 3.20. ∎

3.30. Proposition. *Let $F_1, F_2 : A \to A'$ be morphisms of opetopic sets, let $x : A$, and let $x' : A'$ such that $F_1(x) = F_2(x) = x'$. Then the induced functors $F_1 \downarrow x, F_2 \downarrow x : A \downarrow x \to A' \downarrow x'$ are identical.*

Proof. By Lemma 3.28, $F_1 \downarrow x$ and $F_2 \downarrow x$ agree on objects. Then apply Lemma 3.29. ∎

3.31. Local equivalence. We show that any morphism of opetopic sets induces an equivalence between slices (Proposition 3.35).

3.32. Lemma. *Let $F : A \to A'$ be a morphism of opetopic sets and let $x : A$. Then the induced map $F \downarrow^1 x : A \downarrow^1 x \to A' \downarrow^1 F(x)$ is an equivalence.*

PROOF. Let $x' \equiv F(x)$. We prove that $\mathbf{G}(F, x) : \mathbf{G}(A, x) \to \mathbf{G}(A', x')$ is an equivalence on vertices by induction on $\mathbf{deg}(x)$. This implies that $F \downarrow^{\mathrm{s}} x : A \downarrow^{\mathrm{s}} x \to A' \downarrow^{\mathrm{s}} x'$ is an equivalence. Then, since the fiber of F over $\bar{\mathbf{t}}(x')$ is the singleton $\{\bar{\mathbf{t}}(x)\}$, we see that $F \downarrow^1 x$ is an equivalence. The case when $\mathbf{deg}(x) = 0$ is trivial since $\mathbf{G}(A, x)$ is empty. Suppose that $\mathbf{deg}(x) \geq 1$. We prove that the fiber of F over f' is contractible by induction on $f' : \mathbf{G}(A', x')$. The fiber of F over the root of $\mathbf{G}(A', x')$ consists of only the root of $\mathbf{G}(A', x')$ by Lemma 3.7. Let $h' : g' \to f'$ be an edge in $\mathbf{G}(A', x')$ and suppose that the fiber of F over f' is contractible with center f. If $f : A \downarrow^{\mathrm{s}} x$, then the fiber of F over g' is contractible by Lemma 3.25 and by the induction hypothesis for the domain of f. If $f : A \downarrow^2 x$, then the fiber of F over g' is contractible by Lemma 3.26. ∎

3.33. LEMMA. *Let $F : A \to A'$ be a morphism of opetopic sets, let $x : A$, and let $k \geq 0$. Then the induced map $F \downarrow^k x : A \downarrow^k x \to A' \downarrow^k F(x)$ is an equivalence.*

PROOF. By case analysis on k. The case when $k = 0$ is trivial since $A \downarrow^0 x$ is the singleton $\{\mathbf{id}_x\}$. The case when $k = 1$ is Lemma 3.27. The case when $k \geq 2$ follows from Lemmas 3.17 and 3.32. ∎

3.34. LEMMA. *Let $F : C \to C'$ be a functor. Suppose that the map $\mathbf{Obj}(F \downarrow x) : \mathbf{Obj}(C \downarrow x) \to \mathbf{Obj}(C' \downarrow F(x))$ is an equivalence for every $x : C$. Then the functor $F \downarrow x : C \downarrow x \to C' \downarrow F(x)$ is an equivalence for every $x : C$.*

PROOF. The object part of $F \downarrow x$ is an equivalence by assumption. The arrow part of it is also an equivalence by Lemma 3.20. ∎

3.35. PROPOSITION. *Let $F : A \to A'$ be a morphism of opetopic sets and let $x : A$. Then the induced functor $F \downarrow x : A \downarrow x \to A' \downarrow F(x)$ is an equivalence.*

PROOF. By Lemma 3.33, $\mathbf{Obj}(F \downarrow x)$ is an equivalence. Then apply Lemma 3.34. ∎

3.36. OPETOPIC SETS OVER A AS PRESHEAVES. We show that each slice $\mathbf{OSet} \downarrow A$ is a presheaf category (Proposition 3.41).

3.37. CONSTRUCTION. *Let A be a preopetopic set and let $p : B \to A$ be a morphism of ω-direct categories. We extend B to a preopetopic set by $\mathbf{S}(B) \equiv p^{-1}(\mathbf{S}(A))$ and $\mathbf{T}(B) \equiv p^{-1}(\mathbf{T}(A))$. As a special case, every slice $A \downarrow x$ is extended to a preopetopic set. This preopetopic set structure on B is characterized as the unique one that makes p a morphism of preopetopic sets. In other words, the forgetful functor $\mathbf{PreOSet} \downarrow A \to \mathbf{DirCat}_\omega \downarrow A$ is an equivalence. When A is an opetopic set, we regard $\mathbf{OSet} \downarrow A$ as a full subcategory of $\mathbf{DirCat}_\omega \downarrow A$ via the equivalence $\mathbf{PreOSet} \downarrow A \simeq \mathbf{DirCat}_\omega \downarrow A$.*

3.38. DEFINITION. *We say a property P on preopetopic sets is* local *if a preopetopic set A satisfies P if and only if every slice $A \downarrow x$ satisfies P.*

3.39. LEMMA. *Axioms O1 to O8 are local properties on preopetopic sets.*

PROOF. Straightforward. ∎

3.40. Definition. *Let $p : B \to A$ be a functor. We say p is a* right fibration *if the induced functor $p \downarrow y : B \downarrow y \to A \downarrow p(y)$ is an equivalence for every $y : B$. We say p is a* discrete fibration *if it is a right fibration and every fiber B_x is a set. For a small category A, the small discrete fibrations over A and the functors over A between them form a category $\mathbf{DFib}(A)$. Every discrete fibration is conservative. Thus, when A is an ω-direct category, $\mathbf{DFib}(A)$ is regarded as a full subcategory of $\mathbf{DirCat}_\omega \downarrow A$.*

It is a standard fact that $\mathbf{DFib}(A)$ is equivalent to the category of set-valued presheaves on A.

3.41. Proposition. *Let A be an opetopic set. Then $\mathbf{OSet} \downarrow A = \mathbf{DFib}(A)$ in the poset of full subcategories of $\mathbf{DirCat}_\omega \downarrow A$.*

Proof. Let $P : B \to A$ be a morphism of ω-direct categories. If P is a right fibration, then B is an opetopic set by Lemma 3.39 since $B \downarrow y \simeq A \downarrow P(y)$ for every $y : B$. If B is an opetopic set, then P is a right fibration by Proposition 3.35. ∎

4. Opetopes

Opetopes are defined as special opetopic sets.

4.1. Definition. *An* opetope *is an opetopic set in which a terminal object exists. We refer to the terminal object in an opetope A as $*_A$. Let $\mathbb{O} \subset \mathbf{OSet}$ denote the full subcategory spanned by the opetopes.*

An interesting phenomenon is that $\mathbb{O}$ is extended to a small opetopic set (Propositions 4.4 and 4.5), and thus $\mathbb{O}$ is regarded as an object in $\mathbf{OSet}$.

4.2. Construction. *We extend $\mathbb{O}$ to a preopetopic set as follows. The degree functor is $A \mapsto \mathbf{deg}(*_A)$. Clearly it is a functor to ω. It reflects equivalences by Proposition 3.35. It is then gaunt by Proposition 3.30. Thus, $\mathbb{O}$ is an ω-direct category. We say a 1-step morphism $F : A \to^1 A'$ of opetopes is a source/target arrow if the arrow $F(*_A) \to^1 *_{A'}$ is a source/target arrow in A'.*

4.3. Lemma. *Let A be an opetopic set. The morphism of preopetopic sets $A \to \mathbb{O} \downarrow A$ that sends $x : A$ to the forgetful functor $x_! : A \downarrow x \to A$ is an equivalence.*

Proof. The inverse is given by $(F : A' \to A) \mapsto F(*_{A'})$. For $x : A$, we have $x_!(\mathbf{id}_x) = x$. For $F : A' \to A$, we have $A' \simeq A \downarrow F(*_{A'})$ by Proposition 3.35. ∎

4.4. Proposition. *$\mathbb{O}$ is an opetopic set.*

Proof. By Lemmas 4.3 and 3.39. ∎

4.5. Proposition. *$\mathbb{O}$ is small.*

Proof. This is because every opetope is finite by Proposition 3.22. ∎

We show that $\mathbb{O} : \mathbf{OSet}$ is moreover the terminal object (Proposition 4.7). As a corollary, **OSet** is a presheaf category (Theorem 4.8).

4.6. Lemma. *Let C be a category and let $x : C$ be an object. Suppose that we have a natural transformation $t : \mathbf{id}_C \Rightarrow x$ such that $t_x = \mathbf{id}_x$. Then x is a terminal object.*

Proof. Let $x' : C$ be an object. We show that $\mathbf{Arr}_C(x', x)$ is contractible. We have the arrow $t_{x'} : x' \to x$. Let $f : x' \to x$ be an arrow. By the naturality of t, we have $t_{x'} = t_x \circ f$. Since $t_x = \mathbf{id}_x$, we have $t_{x'} = f$. ∎

4.7. Proposition. $\mathbb{O} : \mathbf{OSet}$ *is the terminal object.*

Proof. For $A : \mathbf{OSet}$, we define a morphism of opetopic sets $t_A : A \to \mathbb{O}$ by $t_A(x) \equiv A \downarrow x$. This is natural in A by Proposition 3.35. The component at $\mathbb{O} : \mathbf{OSet}$ is the morphism $(X \mapsto \mathbb{O} \downarrow X) : \mathbb{O} \to \mathbb{O}$, which is equivalent to the identity on $\mathbb{O}$ by Lemma 4.3. Then apply Lemma 4.6. ∎

4.8. Theorem. $\mathbf{OSet} \simeq \mathbf{DFib}(\mathbb{O})$.

Proof. By Propositions 3.41 and 4.7. ∎

Let us determine the opetopes of low degrees.

4.9. Notation. Let A be an ω-direct category and let $n : \omega$. The fiber of $\mathbf{deg} : A \to \omega$ over n is denoted by A_n.

4.10. Proposition. $\mathbb{O}_0 \simeq \mathbf{1}$.

Proof. The singleton $\{0\}$ with $\mathbf{deg}(0) \equiv 0$ is the only opetope of degree 0. ∎

4.11. Proposition. $\mathbb{O}_1 \simeq \mathbf{1}$. *Moreover, for the unique opetope A of degree 1, there exist a unique source morphism into A and a unique target morphism into A.*

Proof. Let A be an opetope of degree 1. By Axiom O3, there exists a unique source arrow $x \to^{\mathrm{s}} *_A$. Since $*_A$ is the terminal object and since the sets of source and target arrows are disjoint, we see that $x \neq \overline{\mathbf{t}}(*_A)$. Hence, A must look like

$$x \to^{\mathrm{s}} *_A \;{}^{\mathrm{t}}\!\!\leftarrow \mathbf{t}(*_A).$$

It is straightforward to check that this preopetopic set is indeed an opetopic set. ∎

5. The category of opetopic sets

We study the category **OSet** of opetopic sets in more detail. We first give a way to detect equivalences in **OSet**. Of course, equivalences in $\mathbf{OSet} \simeq \mathbf{DFib}(\mathbb{O})$ (Theorem 4.8) are detected fiberwise, but since we do not know yet much about opetopes, this is not so helpful. A more useful sufficient condition is degreewise equivalence (Proposition 5.3).

5.1. Lemma. *The forgetful functor* $\mathbf{OSet} \to \mathbf{Gaunt}$ *is conservative.*

Proof. By definition, a morphism $F : A \to B$ of opetopic sets is an equivalence if and only if its underlying functor is an equivalence and it reflects source and target arrows. The second condition automatically holds since the sets of source and target arrows are complement to each other. ■

5.2. Lemma. *Let $F : C \to D$ be a functor and suppose that $\mathbf{Obj}(F) : \mathbf{Obj}(C) \to \mathbf{Obj}(D)$ is an equivalence and that $F \downarrow x : C \downarrow x \to D \downarrow F(x)$ is an equivalence for every $x : C$. Then F is an equivalence.*

Proof. By assumption, F is an equivalence on objects. It is also fully faithful by Lemma 3.19. ■

5.3. Proposition. *The functors $(A \mapsto A_n) : \mathbf{OSet} \to \mathbf{Set}$ for all $n : \omega$ are jointly conservative.*

Proof. Let $F : A \to B$ be a morphism of opetopic sets and suppose that $F_n : A_n \to B_n$ is an equivalence for every $n : \omega$. To see that F is an equivalence, by Lemma 5.1, it suffices to see that the underlying functor of F is an equivalence, but this follows from Proposition 3.35 and Lemma 5.2 since F is an equivalence on objects by assumption. ■

We give some tools to compute colimits in $\mathbf{OSet}$. Fiberwise computation of colimits in $\mathbf{OSet} \simeq \mathbf{DFib}(\mathbb{O})$ is not helpful, and degreewise computation (Proposition 5.7) is what we want.

5.4. Construction. *Let A be an opetopic set and let $n : \omega$. We define an opetopic set $A_{<n}$ to be the category of elements for the proposition-valued presheaf $x \mapsto (\mathbf{deg}(x) < n)$ on A.*

5.5. Proposition. *Let $n : \omega$. Then the functor $(A \mapsto A_{<n}) : \mathbf{OSet} \to \mathbf{OSet}$ preserves small colimits and pullbacks.*

Proof. By construction and by Proposition 4.7, the functor $A \mapsto A_{<n}$ factors as the pullback functor $\mathbf{OSet} \simeq \mathbf{OSet} \downarrow \mathbb{O} \to \mathbf{OSet} \downarrow \mathbb{O}_{<n}$ followed by the forgetful functor $\mathbf{OSet} \downarrow \mathbb{O}_{<n} \to \mathbf{OSet}$. These functors preserve small colimits and pullbacks. ■

5.6. Proposition. *Let $n : \omega$. Then the functor $(A \mapsto A_n) : \mathbf{OSet} \to \mathbf{Set}$ preserves small colimits and pullbacks.*

Proof. The functor $A \mapsto A_n$ factors as the pullback functor $\mathbf{OSet} \simeq \mathbf{DFib}(\mathbb{O}) \to \mathbf{Set} \downarrow \mathbb{O}_n$ followed by the forgetful functor $\mathbf{Set} \downarrow \mathbb{O}_n \to \mathbf{Set}$. These functors preserve small colimits and pullbacks. ■

5.7. Proposition. *Let $A : I \to \mathbf{OSet}$ be a diagram. Then a cocone $(f(i) : A(i) \to B)_{i:I}$ under A is a colimit cocone if and only if $(f(i)_n : A(i)_n \to B_n)_{i:I}$ is a colimit cocone for every $n : \omega$.*

Proof. By Propositions 5.3 and 5.6. ■

5.8. Proposition. *Let $n \geq 0$ and let $A : I \to \mathbf{OSet} \downarrow \mathbb{O}_{<n+1}$. Then a cocone $(f(i) : A(i) \to B)_{i:I}$ under A is a colimit cocone if and only if $(f(i)_n : A(i)_n \to B_n)_{i:I}$ and $(f(i)_{<n} : A(i)_{<n} \to B_{<n})_{i:I}$ are colimit cocones.*

Proof. Note that $X_m \simeq \mathbf{0}$ for any $X : \mathbf{OSet} \downarrow \mathbb{O}_{<n+1}$ and $m \geq n+1$. Thus, the claim follows from Propositions 5.5 and 5.7. ∎

6. Boundaries and pasting diagrams

We introduce boundaries and pasting diagrams. We show that an opetope is completely determined by its pasting diagram of source objects (Corollary 6.24).

6.1. Definition. *Let $n : \omega$. An n-*opetopic set *is an opetopic set A whose degree functor factors through $n \subset \omega$. This is equivalent to that the morphism $A \to \mathbb{O}$ factors through $\mathbb{O}_{<n}$.*

6.2. Definition. *Let $n : \omega$. An n-*preboundary *is an n-opetopic set A equipped with a subset $\mathbf{S}^{\mathrm{Bd}}(A) \subset A_{n-1}$ with complement $\mathbf{T}^{\mathrm{Bd}}(A)$. Objects in $\mathbf{S}^{\mathrm{Bd}}(A)$ are called* source objects*. Objects in $\mathbf{T}^{\mathrm{Bd}}(A)$ are called* target objects*.*

6.3. Definition. *Let $n : \omega$. An n-*prepasting diagram *is an $(n+1)$-opetopic set A equipped with two families of sets $\mathbf{L}_A, \mathbf{R}_A : A_{n-1} \to \mathbf{Set}$. We say an object $x : A_{n-1}$ is a* leaf object *if $\mathbf{L}_A(x)$ is inhabited. We say an object $x : A_{n-1}$ is a* root object *if $\mathbf{R}_A(x)$ is inhabited. Let $\mathbf{L}(A) \equiv (x : A_{n-1}) \times \mathbf{L}_A(x)$ and $\mathbf{R}(A) \equiv (x : A_{n-1}) \times \mathbf{R}_A(x)$. When extending an $(n+1)$-opetopic set A to an n-prepasting diagram, we specify either $\mathbf{L}_A$ and $\mathbf{R}_A$ or $\mathbf{L}(A) \to A_{n-1}$ and $\mathbf{R}(A) \to A_{n-1}$.*

6.4. Construction. *Let $n \geq 1$ and let A be an n-preboundary. We define an n-opetopic set $\Lambda^{\mathrm{s}}(A)$ called the* source horn of A *to be the category of elements for the following proposition-valued presheaf on A.*

$$x \mapsto \begin{cases} \text{“}x \text{ is a source object”} & \text{if } \mathbf{deg}(x) = n-1 \\ \mathbf{1} & \text{if } \mathbf{deg}(x) < n-1 \end{cases}$$

We extend $\Lambda^{\mathrm{s}}(A)$ to an $(n-1)$-prepasting diagram by $\mathbf{L}_{\Lambda^{\mathrm{s}}(A)}(x) \equiv (y : \mathbf{T}^{\mathrm{Bd}}(A)) \times (x \to^{\mathrm{s}} y)$ and $\mathbf{R}_{\Lambda^{\mathrm{s}}(A)}(x) \equiv (y : \mathbf{T}^{\mathrm{Bd}}(A)) \times (x \to^{\mathrm{t}} y)$.

6.5. Construction. *Let $n \geq 0$ and let A be an n-prepasting diagram. We define an n-opetopic set $\partial(A)$ called the* boundary of *A to be the category of elements for the following set-valued presheaf on A.*

$$x \mapsto \begin{cases} \mathbf{0} & \text{if } \mathbf{deg}(x) = n \\ \mathbf{L}_A(x) + \mathbf{R}_A(x) & \text{if } \mathbf{deg}(x) = n-1 \\ \mathbf{1} & \text{if } \mathbf{deg}(x) < n-1 \end{cases}$$

We extend $\partial(A)$ to an n-preboundary by $\mathbf{S}^{\mathrm{Bd}}(\partial(A)) \equiv \mathbf{L}(A)$ and $\mathbf{T}^{\mathrm{Bd}}(\partial(A)) \equiv \mathbf{R}(A)$. When $n \geq 1$, $\Lambda^{\mathrm{s}}(\partial(A))$ is abbreviated to $\Lambda^{\mathrm{s}}(A)$.

6.6. Definition. *Let $n \geq 0$. We mutually define n-boundaries and n-pasting diagrams. An n-*boundary *is an n-preboundary A satisfying the following axioms.*

Bd1. *When $n \geq 1$, there exists a unique target object in A.*

Bd2. *When $n \geq 1$, $\Lambda^{\mathrm{s}}(A)$ is an $(n-1)$-pasting diagram.*

*An n-*pasting diagram *is an n-prepasting diagram A satisfying the following axioms.*

PD1. *A_n is finite.*

PD2. *For every $x : A_{n-1}$, the type $\mathbf{L}_A(x)$ is a proposition, and it holds if and only if there is no target arrow from x.*

PD3. *For every $x : A_{n-1}$, the type $\mathbf{R}_A(x)$ is a proposition, and it holds if and only if there is no source arrow from x.*

PD4. *When $n = 0$, A_0 is contractible.*

PD5. *For every object $x : A_{n-1}$, there is at most one target arrow from x.*

PD6. *For every object $x : A_{n-1}$, there is at most one source arrow from x.*

PD7. *When $n \geq 1$, there exists an object $r : A_{n-1}$ such that, for every object $x : A_{n-1}$, there exists a zigzag*

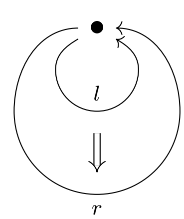
$$x = x_0 \xrightarrow{f_0} y_0 \xleftarrow{g_0} x_1 \xrightarrow{f_1} \dots \xrightarrow{f_{m-1}} y_{m-1} \xleftarrow{g_{m-1}} x_m = r, \tag{2}$$

where $x_i : A_{n-1}$, $y_i : A_n$, f_i's are source arrows, and g_i's are target arrows.

PD8. *$\partial(A)$ is an n-boundary.*

Let $\mathbf{Bd}_n$ denote the category of small n-boundaries whose morphisms are those morphisms of opetopic sets preserving source and target objects. Let $\mathbf{PD}_n$ denote the category of small n-pasting diagrams whose morphisms are those morphisms of opetopic sets preserving leaf and root objects. By definition, Λ^{s} is a functor $\mathbf{Bd}_n \to \mathbf{PD}_{n-1}$ when $n \geq 1$, and ∂ is a functor $\mathbf{PD}_n \to \mathbf{Bd}_n$.

We illustrate some examples of pasting diagrams and boundaries in Fig. 9. Axioms PD1 to PD7 express that a pasting diagram forms a tree; see Fig. 5. Axiom PD8 is necessary to exclude, for example, the following diagram.



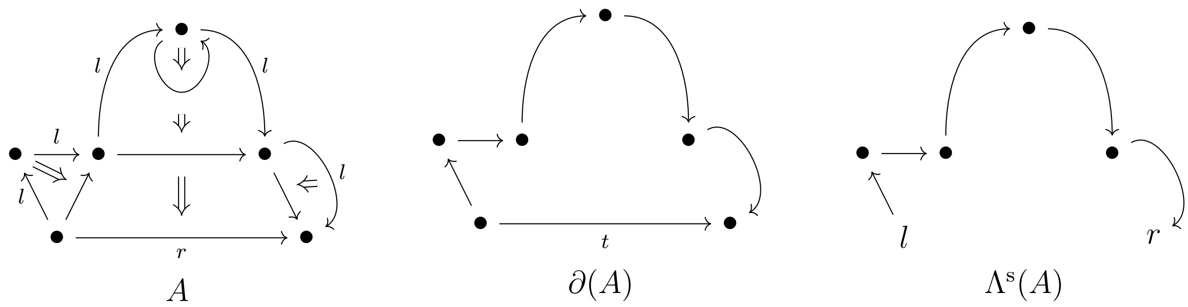
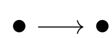
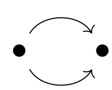


Figure 9: Examples of pasting diagrams and boundaries. A is a 2-pasting diagram, $\partial(A)$ is its boundary, and $\Lambda^{\mathrm{s}}(A)$ is its source horn. Leaves of pasting diagrams are marked with "l". The roots of pasting diagrams are marked with "r". The targets in boundaries are marked with "t".

It forms a tree with root r and leaf l, but it is not considered as a pasting diagram due to the hole surrounded by l. There are n-pasting diagrams A such that A_n is empty, in which case the root object in A is also a leaf object. For example, the following is a 2-pasting diagram.



Its boundary is the following, where the target object is marked with "t".



We construct the boundary of an opetope (Proposition 6.9).

6.7. Construction. *Let A be an opetope of degree $n \geq 0$. We define $\partial(A) \equiv A_{<n}$. We extend it to an n-preboundary where the source/target objects are those $x : A_{n-1}$ such that the unique arrow $x \to^1 *_A$ is a source/target arrow. We call $\partial(A)$ the* boundary *of A. When $n \geq 1$, $\Lambda^{\mathrm{s}}(\partial(A))$ is abbreviated to $\Lambda^{\mathrm{s}}(A)$ and called the* source horn *of A.*

6.8. Lemma. *Let A be an opetope of degree $n \geq 1$. Then $\partial(\Lambda^{\mathrm{s}}(A)) = \partial(\mathbf{t}_{\mathbb{O}}(A))$ in the type of $(n-1)$-preboundaries.*

Proof. $\partial(\Lambda^{\mathrm{s}}(A))$ is the category of elements for the following set-valued presheaf on A.

$$x \mapsto \begin{cases} \mathbf{0} & \text{if } \mathbf{deg}(x) \geq n-1 \\ (x \to^{\mathrm{s}} \mathbf{t}(*_A)) + (x \to^{\mathrm{t}} \mathbf{t}(*_A)) & \text{if } \mathbf{deg}(x) = n-2 \\ \mathbf{1} & \text{if } \mathbf{deg}(x) < n-2 \end{cases}$$

$\partial(\mathbf{t}_{\mathbb{O}}(A))$ is the category of elements for the following set-valued presheaf on A.

$$x \mapsto \begin{cases} \mathbf{0} & \text{if } \mathbf{deg}(x) \geq n-1 \\ (x \to^{\mathrm{s}} \mathbf{t}(*_A)) + (x \to^{\mathrm{t}} \mathbf{t}(*_A)) & \text{if } \mathbf{deg}(x) = n-2 \\ \mathbf{Arr}_A(x, \mathbf{t}(*_A)) & \text{if } \mathbf{deg}(x) < n-2 \end{cases}$$

They are equivalent because $\mathbf{Arr}_A(x, \mathbf{t}(*_A)) \simeq \mathbf{NF}_A^{k-1}(x, \mathbf{t}(*_A)) \simeq \mathbf{NF}_A^k(x, *_A) \simeq \mathbf{Arr}_A(x, *_A)$ (Lemma 3.14) is contractible when $\mathbf{deg}(x) = n-k$ for $k \geq 3$. ∎

6.9. Proposition. *$\partial(A)$ is an n-boundary for every opetope A of degree $n \geq 0$.*

Proof. We proceed by induction on n. Axiom Bd1 is by Axiom O2. Suppose that $n \geq 1$ and we verity the $(n-1)$-pasting diagram axioms for $\Lambda^{\mathrm{s}}(A)$. Axiom PD1 is by Axiom O1. For $x : A_{n-2}$, the type $\mathbf{L}_{\Lambda^{\mathrm{s}}(A)}(x)$ is the type of factorizations of $x \to^2 *_A$ into a source arrow followed by a target arrow. Thus, Axiom PD2 follows from Axiom O5. Similarly Axiom PD3 follows from Axiom O4. Axiom PD4 is by Axiom O3. Axiom PD5 follows from Axiom O5. Axiom PD6 follows from Axiom O4. Axiom PD7 is by Axiom O6. Axiom PD8 is by Lemma 6.8 and induction hypothesis. ∎

We then show that $\partial : \mathbb{O}_n \to \mathbf{Bd}_n$ is an equivalence (Proposition 6.20). We prepare basic lemmas for boundaries and pasting diagrams.

6.10. Construction. *Let $n \geq 0$ and let A be an n-prepasting diagram. We define a 0-graph $\mathbf{G}^{\mathrm{PD}}(A)$ as follows. The set of vertices in $\mathbf{G}^{\mathrm{PD}}(A)$ is $A_n + A_{n-1}$. There is no edge between vertices from A_n. There is no edge between vertices from A_{n-1}. An edge from $x : A_n$ to $y : A_{n-1}$ is a target arrow $f : y \to^{\mathrm{t}} x$. An edge from $y : A_{n-1}$ to $x : A_n$ is a source arrow $f : y \to^{\mathrm{s}} x$.*

6.11. Lemma. *Let $n \geq 0$ and let A be an n-pasting diagram.*

1. *$\mathbf{G}^{\mathrm{PD}}(A)$ is a tree.*
2. *When $n = 0$, $\mathbf{G}^{\mathrm{PD}}(A)$ is a singleton set with no edge.*
3. *When $n \geq 1$, the root of $\mathbf{G}^{\mathrm{PD}}(A)$ is the unique root object in A, which exists by Axiom Bd1 for $\partial(A)$.*

Proof. When $n = 0$, $\mathbf{G}^{\mathrm{PD}}(A)$ is a singleton set with no edge by Axiom PD4 and by definition. In particular, it is a tree. Suppose that $n \geq 1$. Take $r : A_{n-1}$ as in Axiom PD7. Then Zigzag (2) is a path from x to r in $\mathbf{G}^{\mathrm{PD}}(A)$. Let t be the unique root object in A. There is a path p from t to r. By Axiom PD3, the length of p must be 0, and thus $t = r$. For every $x : A_{n-1}$, there is a path from x to r. Such a path is unique by Axioms PD6, O2 and PD3. For every $x : A_n$, we have a unique edge $x \to y$ by Axiom O2 and then a unique path from y to r. ∎

6.12. Lemma. *Let $n \geq 1$, let A be an n-pasting diagram, and let $x : A_{n-1}$. Then either x is a root object or there exists a unique source arrow from x.*

Proof. Let d be the length of the unique path in $\mathbf{G}^{\mathrm{PD}}(A)$ from x to the root (Lemma 6.11). If $d = 0$, then there is no source arrow from x, and thus x is a root object by Axiom PD3. If $d \geq 1$, then there is a source arrow from x, and such a source arrow is unique by Axiom PD6. ∎

6.13. Lemma. *Let $n \geq 0$ and let A be an n-pasting diagram. Then A_{n-1} is finite.*

Proof. The case when $n = 0$ is trivial since A_{-1} is empty. Suppose that $n \geq 1$. We show that $A_{n-1} \simeq \mathbf{1}+((x : A_n)\times(A \downarrow^{\mathrm{s}} x))$, which is finite by Axioms PD1 and O1. Let $x : A_{n-1}$. We proceed by case analysis on x by Lemma 6.12. If x is a root object, we map x to $* : \mathbf{1}$. If there is a unique source arrow $f : x \to^{\mathrm{s}} y$, we map x to $(y, f) : (x' : A_n) \times (A \downarrow^{\mathrm{s}} x')$. This gives a one-to-one correspondence between A_{n-1} and $\mathbf{1} + ((x : A_n) \times (A \downarrow^{\mathrm{s}} x))$. ∎

6.14. Lemma. *Let $n \geq 1$, let A be an n-pasting diagram, and let $x : A_{n-1}$. Then either x is a leaf object or there exists a unique target arrow from x.*

Proof. By Axiom PD1 and Lemma 6.13, the proposition "there is a target arrow from x" is decidable: check if $\mathbf{t}(y) = x$ for all $y : A_n$. Thus, by Axiom PD2, either x is a leaf object or there is a target arrow from x. In the latter case, such a target arrow is unique by Axiom PD5. ∎

6.15. Lemma. *Let A be a preopetopic set satisfying Axiom O1 to O6. Then A satisfies Axioms O7 and O8 if and only if, for every $k \geq 3$ and $x, y : A$ such that $\mathbf{deg}(y) + k = \mathbf{deg}(x)$, the postcomposition map $\overline{\mathbf{t}}(x)_! : \mathbf{Arr}_A(y, \mathbf{t}(x)) \to \mathbf{Arr}_A(y, x)$ is an equivalence.*

Proof. The "only if" direction follows from Lemma 3.14 as $\mathbf{Arr}_A(y, \mathbf{t}(x)) \simeq \mathbf{NF}^{k-1}_A(y, \mathbf{t}(x)) \simeq \mathbf{NF}^k_A(y, x) \simeq \mathbf{Arr}_A(y, x)$. We show the "if" direction. Axiom O7 is immediate. By assumption and Axiom O4, we have $\mathbf{NF}^k_A(y, x) \simeq \mathbf{Arr}_A(y, x)$ for every $k \geq 2$ and x, $y : A$ such that $\mathbf{deg}(y) + k = \mathbf{deg}(x)$. Axiom O8 thus follows. ∎

6.16. Construction. *Let $n \geq 1$ and let A be an n-boundary. We refer to the unique target object in A, which exists by Axiom Bd1, as $\mathbf{t}^{\mathrm{Bd}}(A)$.*

6.17. Lemma. *Let $k \geq 2$, let $n \geq 1$, let A be an n-boundary, and let $x : A$ be an object of degree $n - 1 - k$. Then there exists a unique k-step arrow $x \to^k \mathbf{t}^{\mathrm{Bd}}(A)$.*

Proof. By induction on $k \geq 2$. Suppose that $k = 2$. Then x is an object in $\Lambda^{\mathrm{s}}(\Lambda^{\mathrm{s}}(A))$ of degree $n-3$. By Lemma 6.12, either there is a unique target arrow $f : x \to^{\mathrm{t}} \mathbf{t}^{\mathrm{Bd}}(\partial(\Lambda^{\mathrm{s}}(A)))$ or there is a unique source arrow $f : x \to^{\mathrm{s}} y$ to a source object y in $\partial(\Lambda^{\mathrm{s}}(A))$. In the former case, by construction, $\mathbf{t}^{\mathrm{Bd}}(\partial(\Lambda^{\mathrm{s}}(A))) = \mathbf{t}(\mathbf{t}^{\mathrm{Bd}}(A))$, and thus we have the unique target arrow $g \equiv \overline{\mathbf{t}}(\mathbf{t}^{\mathrm{Bd}}(A)) : \mathbf{t}^{\mathrm{Bd}}(\partial(\Lambda^{\mathrm{s}}(A))) \to^{\mathrm{t}} \mathbf{t}^{\mathrm{Bd}}(A)$. In the latter case, y is a leaf object in $\Lambda^{\mathrm{s}}(A)$ by the definition of $\partial(\Lambda^{\mathrm{s}}(A))$. Then we have a unique source arrow $g : y \to^{\mathrm{s}} \mathbf{t}^{\mathrm{Bd}}(A)$. In both cases, (g, f) is the unique homogeneous pair of 1-step arrows $f : y \to^1 \mathbf{t}^{\mathrm{Bd}}(A)$ and $g : x \to^1 y$. Then the composite $g \circ f$ is the unique arrow $x \to^2 \mathbf{t}^{\mathrm{Bd}}(A)$ by Axiom O4. Suppose that $k \geq 3$. As we have seen, $\mathbf{t}^{\mathrm{Bd}}(\partial(\Lambda^{\mathrm{s}}(A))) = \mathbf{t}(\mathbf{t}^{\mathrm{Bd}}(A))$. By induction hypothesis, we have a unique arrow $f : x \to^{k-1} \mathbf{t}^{\mathrm{Bd}}(\partial(\Lambda^{\mathrm{s}}(A)))$. Then the composite $\overline{\mathbf{t}}(\mathbf{t}^{\mathrm{Bd}}(A)) \circ f$ is the unique arrow $x \to^{k} \mathbf{t}^{\mathrm{Bd}}(A)$ by Lemma 6.15. ∎

We now construct an inverse of $\partial : \mathbb{O}_n \to \mathbf{Bd}_n$.

6.18. Construction. *Let $n \geq 0$ and let A be an n-preboundary. We construct a category $\mathbf{Fill}(A)$ from A by freely adjoining a terminal object $*$. We extend it to an ω-direct category by extending $\mathbf{deg}_A$ by $\mathbf{deg}_{\mathbf{Fill}(A)}(*) \equiv n$. We further extend it to a preopetopic set where an arrow is a source/target arrow if either it is a source/target arrow in A or it is the arrow $x \to^1 *$ from a source/target object x in A.*

6.19. Lemma. *Let $n \geq 0$ and let A be an n-boundary. Then $\mathbf{Fill}(A)$ is an opetope of degree n.*

Proof. Since $\mathbf{Fill}(A) \downarrow x \simeq A \downarrow x$ for $x : A$ by construction, it suffices to verify Axioms O1 to O8 for $x \equiv *$. Since $\mathbf{Fill}(A) \downarrow^1 * \simeq A_{n-1}$ by construction, Axiom O1 is trivial when $n = 0$ and follows from Axiom Bd1 and Axiom PD1 for $\Lambda^{\mathrm{s}}(A)$ when $n \geq 1$. Axiom O2 is by Axiom Bd1. Axiom O3 is by Axiom PD4 for $\Lambda^{\mathrm{s}}(A)$. Axiom O4 follows from Lemma 6.12 for $\Lambda^{\mathrm{s}}(A)$. Axiom O5 follows from Lemma 6.14 for $\Lambda^{\mathrm{s}}(A)$. Axiom O6 is by Axiom PD7 for $\Lambda^{\mathrm{s}}(A)$. Axioms O7 and O8 follow from Lemmas 6.15 and 6.17. ∎

6.20. Proposition. *Let $n \geq 0$. Then the functor $\partial : \mathbb{O}_n \to \mathbf{Bd}_n$ is an equivalence.*

Proof. The inverse is given by **Fill** (Lemma 6.19). ∎

6.21. Corollary. *Let $n \geq 0$. Then $\mathbf{Bd}_n$ is a discrete category over a set.*

Proof. By Proposition 6.20. ∎

We show that $\Lambda^{\mathrm{s}} : \mathbf{Bd}_{n+1} \to \mathbf{PD}_n$ is an equivalence (Proposition 6.23).

6.22. Lemma. *Let $n \geq 0$. Then the diagram*

$$\begin{array}{ccc} \mathbf{Bd}_{n+1} & \xrightarrow{\Lambda^{\mathrm{s}}} & \mathbf{PD}_n \\ {\scriptstyle \mathbf{t}^{\mathrm{Bd}}}\downarrow & & \downarrow{\scriptstyle \partial} \\ \mathbb{O}_n & \xrightarrow[\partial]{} & \mathbf{Bd}_n \end{array} \tag{3}$$

is a pullback of categories, where we identify $\mathbf{t}^{\mathrm{Bd}}(A) : A_n$ and the associated opetope $A \downarrow \mathbf{t}^{\mathrm{Bd}}(A)$ for $A : \mathbf{Bd}_{n+1}$.

Proof. We first note that Diagram (3) commutes, that is, $\partial(\Lambda^{\mathrm{s}}(A)) = \partial(\mathbf{t}^{\mathrm{Bd}}(A))$ for all $A : \mathbf{Bd}_{n+1}$. By Proposition 6.20, it suffices to show the case when A is of the form $\partial(A')$ for an opetope A' of degree $n+1$, but this is just Lemma 6.8. To see that Diagram (3) is a pullback, let $A : \mathbb{O}_n$, $B : \mathbf{PD}_n$, and $C : \mathbf{Bd}_n$ and suppose $\partial(A) = \partial(B) = C$. Let $D(A, B)$ be the following pushout in **OSet**.

$$\begin{array}{ccc} C & \longrightarrow & B \\ \downarrow & \ulcorner & \downarrow \\ A & \longrightarrow & D(A,B) \end{array}$$

By Proposition 5.6, we have $\mathbf{1} + B_n \simeq D(A,B)_n$. We extend $D(A,B)$ to an $(n+1)$-preboundary by $\mathbf{S}^{\mathrm{Bd}}(D(A,B)) \equiv B_n$ and $\mathbf{T}^{\mathrm{Bd}}(D(A,B)) \equiv \mathbf{1}$. By Proposition 5.5, we have $B_{<n} \simeq D(A,B)_{<n}$. It then follows that $B \simeq \Lambda^{\mathrm{s}}(D(A,B))$ by Proposition 5.3, and thus $D(A,B)$ is an $(n+1)$-boundary. By Proposition 3.35, $A \simeq \mathbf{t}^{\mathrm{Bd}}(D(A,B))$. We thus have a section D of $\mathbf{Bd}_{n+1} \to \mathbf{OSet}_n \times_{\mathbf{Bd}_n} \mathbf{PD}_n$. To see that D is moreover an inverse, let $X : \mathbf{Bd}_{n+1}$. We have a canonical morphism $D(\mathbf{t}^{\mathrm{Bd}}(X), \Lambda^{\mathrm{s}}(X)) \to X$ of $(n+1)$-boundaries, which is an equivalence by Corollary 6.21. ■

6.23. Proposition. *Let $n \geq 1$. Then the functor $\Lambda^{\mathrm{s}} : \mathbf{Bd}_n \to \mathbf{PD}_{n-1}$ is an equivalence.*

Proof. By Proposition 6.20 and Lemma 6.22. ■

6.24. Corollary. *Let $n \geq 1$. Then the map $\Lambda^{\mathrm{s}} : \mathbb{O}_n \to \mathbf{PD}_{n-1}$ is an equivalence.*

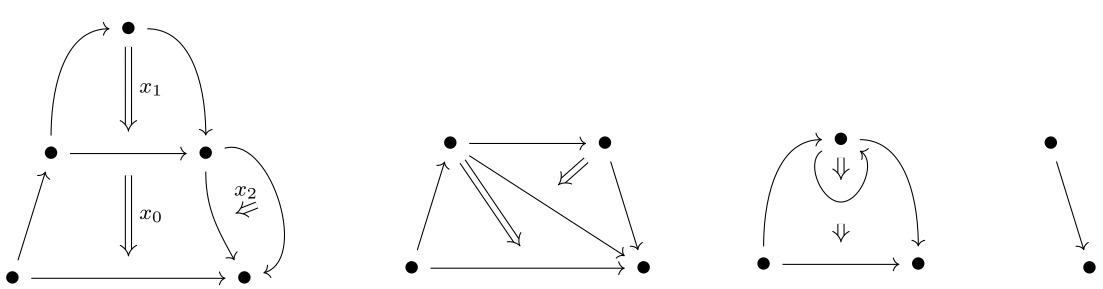
Proof. By Propositions 6.20 and 6.23. ■

6.25. Corollary. *Let $n \geq 0$. Then $\mathbf{PD}_n$ is a discrete category over a set.*

Proof. By Corollary 6.24. ■

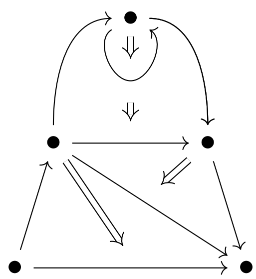
7. Substitution and grafting

We introduce two operators on pasting diagrams, substitution and grafting. Substitution is the operator on n-pasting diagrams replacing n-cells in an n-pasting diagram by n-pasting diagrams of the same boundaries. For example, let A, $B(x_0)$, $B(x_1)$, and $B(x_2)$ be the following 2-pasting diagrams.

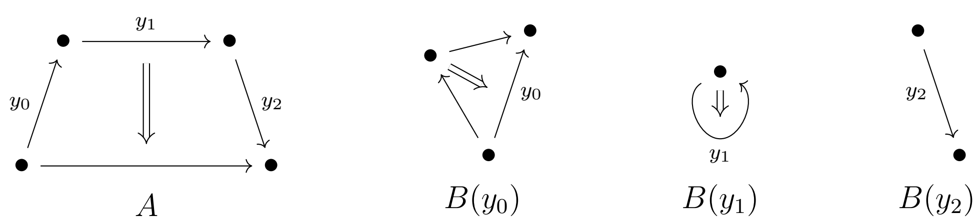


A $B(x_0)$ $B(x_1)$ $B(x_2)$

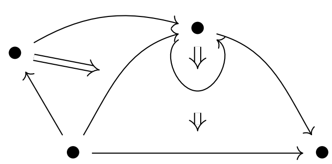
Then the result of substitution of B in A is the following 2-pasting diagram.



Grafting is the operator on n-pasting diagrams attaching an n-pasting diagram to each leaf of an n-pasting diagram. For example, let A, $B(y_0)$, $B(y_1)$, and $B(y_2)$ be the following 2-pasting diagrams.



Then the result of grafting of B to A is the following 2-pasting diagram.



We begin with the formal definition of substitution.

7.1. Notation. Let $f : B \to A$ be a map between types. We write the fiber of f over $x : A$ as $B[f = x]$.

7.2. Construction. *Let $n \geq 0$, let A be an n-pasting diagram, and let $B : (x : A_n) \to \mathbf{PD}_n[\partial = \partial(A \downarrow x)]$. We define an opetopic set $\mathbf{Subst}(A, B)$ called the* substitution of B in A *by the following pushout in* **OSet**.

$$\begin{array}{ccc} \coprod_{x:A_n} \partial(A \downarrow x) & \longrightarrow & A_{<n} \\ \downarrow & \ulcorner & \downarrow \\ \coprod_{x:A_n} B(x) & \longrightarrow & \mathbf{Subst}(A, B) \end{array}$$

We extend $\mathbf{Subst}(A, B)$ to an n-prepasting diagram by $\mathbf{L}(\mathbf{Subst}(A, B)) \equiv \mathbf{L}(A)$ and $\mathbf{R}(\mathbf{Subst}(A, B)) \equiv \mathbf{R}(A)$.

7.3. Lemma. *Let $n \geq 0$, let A be an n-pasting diagram, and let $B : (x : A_n) \to \mathbf{PD}_n[\partial = \partial(A \downarrow x)]$. Then $(x : A_n) \times B(x)_n \simeq \mathbf{Subst}(A, B)_n$.*

Proof. By Proposition 5.6. ∎

7.4. Lemma. *Let $n \geq 1$, let A be an n-pasting diagram, and let $B : (x : A_n) \to \mathbf{PD}_n[\partial = \partial(A \downarrow x)]$.*

1. $\mathbf{R}(A) + ((x : A_{n-1}) \times (B(x)_{n-1} \setminus \mathbf{R}(B(x)))) \simeq \mathbf{Subst}(A, B)_{n-1}$
2. $\mathbf{L}(A) + ((x : A_{n-1}) \times (B(x)_{n-1} \setminus \mathbf{L}(B(x)))) \simeq \mathbf{Subst}(A, B)_{n-1}$

Proof. We prove the first claim. The second one is similarly proved. Since $(x : A_n) \times (A \downarrow^{1} x) \simeq ((x : A_n) \times (A \downarrow^{\mathrm{s}} x)) + ((x : A_n) \times (A \downarrow^{\mathrm{t}} x))$, $(x : A_n) \times B(x)_{n-1} \simeq ((x : A_n) \times (B(x)_{n-1} \setminus \mathbf{R}(B(x)))) + ((x : A_n) \times \mathbf{R}(B(x)))$, and $(x : A_n) \times (A \downarrow^{\mathrm{t}} x) \simeq (x : A_n) \times \mathbf{R}(B(x))$ as $\partial(A \downarrow x) = \partial(B(x))$, we have the following pushout.

$$\begin{array}{ccc} (x : A_n) \times (A \downarrow^{\mathrm{s}} x) & \longrightarrow & (x : A_n) \times (A \downarrow^{1} x) \\ \downarrow & \ulcorner & \downarrow \\ (x : A_n) \times (B(x)_{n-1} \setminus \mathbf{R}(B(x))) & \longrightarrow & (x : A_n) \times B(x)_{n-1} \end{array}$$

By Lemma 6.12, $(x : A_n) \times (A \downarrow^{\mathrm{s}} x) \simeq A_{n-1} \setminus \mathbf{R}(A)$. We then have the following pushout by Proposition 5.6.

$$\begin{array}{ccc} A_{n-1} \setminus \mathbf{R}(A) & \hookrightarrow & A_{n-1} \\ \downarrow & \ulcorner & \downarrow \\ (x : A_n) \times (B(x)_{n-1} \setminus \mathbf{R}(B(x))) & \longrightarrow & \mathbf{Subst}(A, B)_{n-1} \end{array}$$

Since $\mathbf{R}(A) + (A_{n-1} \setminus \mathbf{R}(A)) \simeq A$, we have $\mathbf{R}(A) + ((x : A_n) \times (B(x)_{n-1} \setminus \mathbf{R}(B(x))))$. ∎

7.5. Lemma. *Let $n \geq 1$, let A be an n-pasting diagram, and let $B : (x : A_n) \to \mathbf{PD}_n[\partial = \partial(A \downarrow x)]$. Then $\partial(\mathbf{Subst}(A, B)) = \partial(A)$ in the type of n-preboundaries.*

Proof. By Proposition 5.5, $A_{<n-1} \simeq \mathbf{Subst}(A, B)_{<n-1}$. Thus, the claim is true by construction. ∎

7.6. Proposition. *Let $n \geq 0$, let A be an n-pasting diagram, and let $B : (x : A_n) \to \mathbf{PD}_n[\partial = \partial(A \downarrow x)]$. Then $\mathbf{Subst}(A, B)$ is an n-pasting diagram.*

Proof. When $n = 0$, A_0 is contractible by Axiom PD4 with center $*$, and then $B(*) \simeq \mathbf{Subst}(A, B)$. Suppose that $n \geq 1$. Axiom PD1 is by Lemma 7.3. Axioms PD2 and PD3 follow from Lemma 7.4. Axiom PD4 is vacuously true. Axioms PD5 to PD7 follow from Lemma 7.4. Axiom PD8 is by Lemma 7.5. ∎

The substitution operator is associative in the following sense.

7.7. Proposition. *Let $n \geq n$, let A be an n-pasting diagram, let $B : (x : A_n) \to \mathbf{PD}_n[\partial = \partial(A \downarrow x)]$, and let $C : (x : A_n) \to (y : B(x)_n) \to \mathbf{PD}_n[\partial = \partial(B(x) \downarrow y)]$. Then*

$$\mathbf{Subst}(A, (x \mapsto \mathbf{Subst}(B(x), C(x)))) \simeq \mathbf{Subst}(\mathbf{Subst}(A, B), ((x, y) \mapsto C(x, y))).$$

Proof. We first note that both sides of the stated equivalence are well-typed. The left side is well-typed by Lemma 7.5. For the right side, use Lemma 7.3 and the equivalence $B(x) \downarrow y \simeq \mathbf{Subst}(A, B) \downarrow (x, y)$ for every $x : A_n$ and $y : B(x)_n$ by Proposition 3.35. Let X be the following pushout

$$\begin{array}{ccccc}
 & & \coprod_{x:A_n} \partial(A \downarrow x) & \longrightarrow & A_{<n} \\
 & & \downarrow & \ulcorner & \downarrow \\
\coprod_{x:A_n} \coprod_{y:B(x)_n} \partial(B(x) \downarrow y) & \longrightarrow & \coprod_{x:A_n} B(x)_{<n} & \longrightarrow & \mathbf{Subst}(A, B)_{<n} \\
\downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\
\coprod_{x:A_n} \coprod_{y:B(x)_n} C(x, y) & \longrightarrow & \coprod_{x:A_n} \mathbf{Subst}(B(x), C(x)) & \longrightarrow & X,
\end{array}$$

where the upper right square is a pushout by Proposition 5.5. The composite of the upper right and lower right pushouts exhibits X as $\mathbf{Subst}(A, (x \mapsto \mathbf{Subst}(B(x), C(x))))$. The composite of the lower left and lower right pushouts exhibits X as $\mathbf{Subst}(\mathbf{Subst}(A, B), ((x, y) \mapsto C(x, y)))$. ■

Any opetope of degree n can be turned into an n-pasting diagram, which plays the role of the unit for substitution (Propositions 7.11 and 7.12).

7.8. Construction. *Let A be an opetope of degree $n \geq 0$. We define an n-prepasting diagram $\iota(A)$ as follows. The underlying opetopic set of $\iota(A)$ is A. An object $x : A_{n-1}$ is a leaf/root object if the arrow $x \to^1 *_A$ is a source/target arrow.*

7.9. Lemma. *Let A be an opetope of degree $n \geq 0$. Then $\partial(\iota(A)) \simeq \partial(A)$.*

Proof. By construction. ■

7.10. Proposition. *Let A be an opetope of degree $n \geq 0$. Then $\iota(A)$ is an n-pasting diagram.*

Proof. Straightforward. For Axiom PD8, use Lemma 7.9. ■

7.11. Proposition. *Let A be an opetope of degree $n \geq 0$ and let $B : \mathbf{PD}_n[\partial = \partial(A)]$. Since $\iota(A)_n = \{*_A\}$, we may regard B as a map $(x : \iota(A)_n) \to \mathbf{PD}_n[\partial = \partial(A \downarrow x)]$. Then $\mathbf{Subst}(\iota(A), B) \simeq B$.*

Proof. By construction, $\mathbf{Subst}(\iota(A), B)$ is the pushout of the equivalence $\partial(A) \simeq A_{<n}$ along $\partial(A) = \partial(B) \to B$ and thus equivalent to B. ■

7.12. Proposition. *Let $n \geq 0$ and let A be an n-pasting diagram. Then* $\mathbf{Subst}(A, (x \mapsto \iota(A \downarrow x))) \simeq A$.

Proof. We first note that $\mathbf{Subst}(A, (x \mapsto \iota(A \downarrow x)))$ is well-typed by Lemma 7.9. We have the following commutative square.

$$\begin{array}{ccc} \coprod_{x:A_n} \partial(A \downarrow x) & \longrightarrow & A_{<n} \\ \downarrow & & \downarrow \\ \coprod_{x:A_n} A \downarrow x & \longrightarrow & A \end{array} \tag{4}$$

It suffices to show that Square (4) is a pushout. By Proposition 5.6, the fiber of Square (4) is

$$\begin{array}{ccc} \mathbf{0} & = & \mathbf{0} \\ \downarrow & & \downarrow \\ (x : A_n) \times \{x\} & \xrightarrow[\simeq]{} & A_n, \end{array}$$

which is a pushout. By Proposition 5.5, the restriction of Square (4) to $< n$ is

$$\begin{array}{ccc} \coprod_{x:A_n} (A \downarrow x)_{<n} & \longrightarrow & A_{<n} \\ \| & & \| \\ \coprod_{x:A_n} (A \downarrow x)_{<n} & \longrightarrow & A_{<n}, \end{array}$$

which is a pushout. Therefore, Square (4) is a pushout by Proposition 5.8. ■

We then define grafting.

7.13. Construction. *Let $n \geq 1$ and let A be an n-pasting diagram. We refer to the unique root object in A as $\mathbf{r}_A$ and define an opetope $\mathbf{t}^{\mathrm{PD}}(A)$ to be $A \downarrow \mathbf{r}_A$.*

7.14. Construction. *Let $n \geq 0$, let A be an $(n+1)$-pasting diagram, and let $B : (x : \mathbf{L}(A)) \to \mathbf{PD}_{n+1}[\mathbf{t}^{\mathrm{PD}} = A \downarrow x]$. We define an opetopic set $\mathbf{Graft}(A, B)$ called the* grafting of B onto A *by the following pushout in* $\mathbf{OSet}$*.*

$$\begin{array}{ccc} \coprod_{x:\mathbf{L}(A)} A \downarrow x & \longrightarrow & A \\ \downarrow & \ulcorner & \downarrow \\ \coprod_{x:\mathbf{L}(A)} B(x) & \longrightarrow & \mathbf{Graft}(A, B) \end{array}$$

We extend $\mathbf{Graft}(A, B)$ to an $(n+1)$-prepasting diagram by $\mathbf{L}(\mathbf{Graft}(A, B)) \equiv (x : \mathbf{L}(A)) \times \mathbf{L}(B(x))$ and $\mathbf{R}(\mathbf{Graft}(A, B)) \equiv \mathbf{R}(A)$.

7.15. Lemma. *Let $n \geq 0$, let A be an $(n+1)$-pasting diagram, and let $B : (x : \mathbf{L}(A)) \to \mathbf{PD}_{n+1}[\mathbf{t}^{\mathrm{PD}} = A \downarrow x]$. Then $A_{n+1} + ((x : \mathbf{L}(A)) \times B(x)_{n+1}) \simeq \mathbf{Graft}(A, B)_{n+1}$.*

Proof. By Proposition 5.6. ■

7.16. LEMMA. *Let $n \geq 0$, let A be an $(n+1)$-pasting diagram, and let $B : (x : \mathbf{L}(A)) \to \mathbf{PD}_{n+1}[\mathbf{t}^{\mathrm{PD}} = A \downarrow x]$.*

1. $(A_n \setminus \mathbf{L}(A)) + ((x : \mathbf{L}(A)) \times B(x)_n) \simeq \mathbf{Graft}(A, B)_n$
2. $A_n + ((x : \mathbf{L}(A)) \times (B(x)_n \setminus \mathbf{R}(B(x)))) \simeq \mathbf{Graft}(A, B)_n$

PROOF. By Proposition 5.6, we have the following pushout

$$\begin{array}{ccc} \mathbf{L}(A) & \hookrightarrow & A_n \\ \downarrow & \ulcorner & \downarrow \\ (x : \mathbf{L}(A)) \times B(x)_n & \longrightarrow & \mathbf{Graft}(A, B)_n, \end{array}$$

where the left map sends $x : \mathbf{L}(A)$ to $(x, \mathbf{r}_{B(x)})$. The first claim directly follows from this pushout. Since $\mathbf{L}(A) + ((x : \mathbf{L}(A)) \times (B(x)_n \setminus \{\mathbf{r}_{B(x)}\})) \simeq (x : \mathbf{L}(A)) \times B(x)_n$, the second claim follows. ■

7.17. LEMMA. *Let $n \geq 0$, let A be an $(n+1)$-pasting diagram, and let $B : (x : \mathbf{L}(A)) \to \mathbf{PD}_{n+1}[\mathbf{t}^{\mathrm{PD}} = A \downarrow x]$. Then $\Lambda^{\mathrm{s}}(\mathbf{Graft}(A, B)) = \mathbf{Subst}(\Lambda^{\mathrm{s}}(A), (x \mapsto \Lambda^{\mathrm{s}}(B(x))))$ in the type of n-prepasting diagrams.*

PROOF. We first note that $\Lambda^{\mathrm{s}}(A)_n = \mathbf{L}(A)$ and $\partial(\Lambda^{\mathrm{s}}(B(x))) = \partial(\mathbf{t}^{\mathrm{PD}}(B(x))) = \partial(A \downarrow x) = \partial(\Lambda^{\mathrm{s}}(A) \downarrow x)$ for all $x : \Lambda^{\mathrm{s}}(A)_n$, and thus $\mathbf{Subst}(\Lambda^{\mathrm{s}}(A), (x \mapsto \Lambda^{\mathrm{s}}(B(x))))$ is well-typed. We have the following commutative square in $\mathbf{OSet}$.

$$\begin{array}{ccc} \coprod_{x:\Lambda^{\mathrm{s}}(A)_n} \partial(\Lambda^{\mathrm{s}}(A) \downarrow x) & \longrightarrow & \Lambda^{\mathrm{s}}(A)_{<n} \\ \downarrow & & \downarrow \\ \coprod_{x:\Lambda^{\mathrm{s}}(A)_n} \Lambda^{\mathrm{s}}(B(x)) & \longrightarrow & \Lambda^{\mathrm{s}}(\mathbf{Graft}(A, B)) \end{array} \tag{5}$$

It suffices to show that Square (5) is a pushout. By Proposition 5.6, the fiber of Square (5) over n is

$$\begin{array}{ccc} \mathbf{0} & = & \mathbf{0} \\ \downarrow & & \downarrow \\ (x : \mathbf{L}(A)) \times \mathbf{L}(B(x)) & = & (x : \mathbf{L}(A)) \times \mathbf{L}(B(x)), \end{array}$$

which is a pushout. By Proposition 5.5, the restriction of Square (5) to $< n$ is

$$\begin{array}{ccc} \coprod_{x:\mathbf{L}(A)} (A \downarrow x)_{<n} & \longrightarrow & A_{<n} \\ \downarrow & & \downarrow \\ \coprod_{x:\mathbf{L}(A)} B(x)_{<n} & \longrightarrow & \mathbf{Graft}(A, B)_{<n}, \end{array}$$

which is a pushout by the definition of $\mathbf{Graft}(A, B)$ and Proposition 5.5. Therefore, Square (5) is a pushout by Proposition 5.8. ■

7.18. Proposition. *Let $n \geq 0$, let A be an $(n+1)$-pasting diagram, and let $B : (x : \mathbf{L}(A)) \to \mathbf{PD}_{n+1}[\mathbf{t}^{\mathrm{PD}} = A \downarrow x]$. Then $\mathbf{Graft}(A, B)$ is an $(n+1)$-pasting diagram.*

Proof. Axiom PD1 is by Lemma 7.15. Axioms PD2 and PD3 follow from Lemma 7.16. Axiom PD4 is vacuously true. Axioms PD5 to PD7 follow from Lemma 7.16. Axiom PD8 is by Lemma 7.17. ∎

The grafting operator is associative in the following sense.

7.19. Proposition. *Let $n \geq 0$, let A be an $(n+1)$-pasting diagram, let $B : (x : \mathbf{L}(A)) \to \mathbf{PD}_{n+1}[\mathbf{t}^{\mathrm{PD}} = A \downarrow x]$, and let $C : (x : \mathbf{L}(A)) \to (y : \mathbf{L}(B(x))) \to \mathbf{PD}_{n+1}[\mathbf{t}^{\mathrm{PD}} = B(x) \downarrow y]$. Then*

$$\mathbf{Graft}(A, (x \mapsto \mathbf{Graft}(B(x), C(x)))) \simeq \mathbf{Graft}(\mathbf{Graft}(A, B), ((x, y) \mapsto C(x, y))).$$

Proof. We first note that the both sides of the stated equivalence are well-typed. The left side is well-typed by Lemma 7.17. The right side is well-typed since $B(x) \downarrow y \simeq \mathbf{Graft}(A, B) \downarrow (x, y)$ for all $x : \mathbf{L}(A)$ and $y : \mathbf{L}(B(x))$ by Proposition 3.35. Let X be the following pushout.

$$\begin{array}{ccccc}
 & & \coprod_{x:\mathbf{L}(A)} A \downarrow x & \longrightarrow & A \\
 & & \downarrow & \ulcorner & \downarrow \\
\coprod_{x:\mathbf{L}(A)} \coprod_{y:\mathbf{L}(B(x))} B(x) \downarrow y & \longrightarrow & \coprod_{x:\mathbf{L}(A)} B(x) & \longrightarrow & \mathbf{Graft}(A, B) \\
\downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\
\coprod_{x:\mathbf{L}(A)} \coprod_{y:\mathbf{L}(B(x))} C(x, y) & \to & \coprod_{x:\mathbf{L}(A)} \mathbf{Graft}(B(x), C(x)) & \longrightarrow & X
\end{array}$$

The composite of the upper right and lower right pushouts exhibits X as $\mathbf{Graft}(A, (x \mapsto \mathbf{Graft}(B(x), C(x))))$. The composite of the lower left and lower right pushouts exhibits X as $\mathbf{Graft}(\mathbf{Graft}(A, B), ((x, y) \mapsto C(x, y)))$. ∎

Any opetope of degree n can be turned into an $(n+1)$-pasting diagram, which plays the role of the unit for grafting (Propositions 7.23 and 7.24).

7.20. Construction. *Let A be an opetope of degree $n \geq 0$. We define an $(n+1)$-prepasting diagram $\sigma(A)$ called the* degenerate pasting diagram *on A as follows. The underlying opetopic set of $\sigma(A)$ is A. The terminal object $*_A : A_n$ is both a leaf object and a root object.*

7.21. Lemma. *Let A be an opetope of degree $n \geq 0$. Then $\Lambda^{\mathrm{s}}(\sigma(A)) \simeq A$ and $\mathbf{t}^{\mathrm{PD}}(\sigma(A)) \simeq A$.*

Proof. By construction. ∎

7.22. Proposition. *Let A be an opetope of degree $n \geq 0$. Then $\sigma(A)$ is an $(n+1)$-pasting diagram.*

Proof. Straightforward. For Axiom PD8, use Lemma 7.21. ∎

7.23. Proposition. *Let A be an opetope of degree $n \geq 0$ and let $B : \mathbf{PD}_{n+1}[\mathbf{t}^{\mathrm{PD}} = A]$. Since $\mathbf{L}(\sigma(A)) = \{*_A\}$, we may regard B as a map $(x : \mathbf{L}(\sigma(A))) \to \mathbf{PD}_{n+1}[\mathbf{t}^{\mathrm{PD}} = A \downarrow x]$. Then $\mathbf{Graft}(\sigma(A), B) \simeq B$.*

Proof. By construction, $\mathbf{Graft}(A, B)$ is the pushout of the equivalence $A \downarrow *_A \simeq A$ along $A \downarrow *_A \to B$ and thus equivalent to B. ∎

7.24. Proposition. *Let $n \geq 0$ and let A be an $(n+1)$-pasting diagram. Then $\mathbf{Graft}(A, (x \mapsto \sigma(A \downarrow x))) \simeq A$.*

Proof. We first note that $\mathbf{Graft}(A, (x \mapsto \sigma(A \downarrow x)))$ is well-typed by Lemma 7.21. By construction, $\mathbf{Graft}(A, (x \mapsto \sigma(A \downarrow x)))$ is the pushout of the equivalence $\coprod_{x:\mathbf{L}(A)} A \downarrow x \simeq \coprod_{x:\mathbf{L}(A)} \sigma(A \downarrow x)$ along $\coprod_{x:\mathbf{L}(A)} A \downarrow x \to A$ and thus equivalent to A. ∎

8. Equivalence with existing definitions

We show that our definition of opetopes is equivalent to the polynomial monad definition given by Kock et al. [2010]. We also see that the category of opetopes is presented by the generators and relations described by Ho Thanh [2021].

8.1. Equivalence with the polynomial monad definition. We first review the polynomial monad definition of opetopes given by Kock et al. [2010]. Let I be a set. We define $\mathbb{F}_I$ to be the type of finite sets E equipped with a map $E \to I$. The category $\mathbf{Poly}_I$ of *(finitary) polynomials on I* is defined to be $\mathbf{Set} \downarrow (\mathbb{F}_I \times I)$. Concretely, a polynomial P on I consists of a set $\mathbf{B}(P)$ and two maps

$$\mathbb{F}_I \xleftarrow{\mathbf{E}_P} \mathbf{B}(P) \xrightarrow{\mathbf{t}_P} I.$$

We regard $\mathbf{B}(P)$ as an object in $\mathbf{Set} \downarrow I$ with $\mathbf{t}_P$. We refer to $\coprod_{b:\mathbf{B}(P)} \mathbf{E}_P(b) : \mathbf{Set} \downarrow I$ as $\mathbf{s}_P : \mathbf{E}(P) \to I$. A polynomial on I is thus equivalently presented by three maps

$$I \xleftarrow{\mathbf{s}_P} \mathbf{E}(P) \xrightarrow{\mathbf{p}_P} \mathbf{B}(P) \xrightarrow{\mathbf{t}_P} I,$$

which is a more standard definition of polynomials [Gambino and Kock, 2013]. For a map $f : I \to J$, the postcomposition with f induces a functor $\mathbf{Poly}_I \to \mathbf{Poly}_J$. In this way the map $I \mapsto \mathbf{Poly}_I$ is functorial, and let $\mathbf{I} : \mathbf{Poly} \to \mathbf{Set}$ denote the cocartesian fibration corresponding to it.

For a polynomial P on I, we define a functor $\mathbf{F}_P : \mathbf{Set} \downarrow I \to \mathbf{Set} \downarrow I$ by $\mathbf{F}_P(A)_i \equiv (b : \mathbf{B}(P)_i) \times \mathbf{Arr}_{\mathbf{Set}\downarrow I}(\mathbf{E}_P(b), A)$, which is identical to $(b : \mathbf{B}(P)_i) \times ((e : \mathbf{E}_P(b)) \to A_{\mathbf{s}_P(e)})$. Every morphism $P \to Q$ of polynomials on I induces a natural transformation $\mathbf{F}_P \Rightarrow \mathbf{F}_Q$ which is cartesian in the sense that all the naturality squares are pullbacks. Let $\mathbf{End}^{\mathrm{c}}(I)$ denote the category of endofunctors on $\mathbf{Set} \downarrow I$ and cartesian natural transformations between them. The mapping $P \mapsto \mathbf{F}_P$ defines a fully faithful functor $\mathbf{Poly}_I \to \mathbf{End}^{\mathrm{c}}(\mathbf{Set} \downarrow I)$ [Gambino and Kock, 2013, Lemma 2.15]. The monoidal structure on $\mathbf{End}^{\mathrm{c}}(\mathbf{Set} \downarrow I)$ given by composition of endofunctors restricts to a monoidal structure on $\mathbf{Poly}_I$. The category $\mathbf{PM}_I$ of *polynomial monads on* I is defined to be the category of monoid objects in the monoidal category $\mathbf{Poly}_I$.

Let P be a polynomial monad on a set I. By a *P-polynomial monad* we mean an object in $\mathbf{PM}_I \downarrow P$. By definition, $\mathbf{Poly}_I \downarrow P \simeq \mathbf{Set} \downarrow \mathbf{B}(P)$. We thus regard $\mathbf{PM}_I \downarrow P$ as a category over $\mathbf{Set} \downarrow \mathbf{B}(P)$. The *Baez-Dolan construction* P^+ is the polynomial monad on $\mathbf{B}(P)$ whose algebras are the P-polynomial monads. The set $\mathbb{O}_n^{\mathrm{KJBM}}$ of *Kock-Joyal-Batanin-Mascari (KJBM) opetopes of degree* n and the polynomial monad $\mathbf{Z}_n$ on $\mathbb{O}_n^{\mathrm{KJBM}}$ are inductively defined by $\mathbb{O}_0^{\mathrm{KJBM}} \equiv \mathbf{1}$, $\mathbf{Z}_0 \equiv (\mathbb{F}_{\mathbf{1}} \xleftarrow{\mathbf{1}} \mathbf{1} \to \mathbf{1})$, $\mathbb{O}_{n+1}^{\mathrm{KJBM}} \equiv \mathbf{B}(\mathbf{Z}_n)$, and $\mathbf{Z}_{n+1} \equiv \mathbf{Z}_n^+$.

Let us concretely describe the structure of a polynomial monad P on I. Because the identity polynomial on I is $\mathbb{F}_I \xleftarrow{よ} I \xrightarrow{\mathbf{id}} I$, where よ denotes the Yoneda embedding, the unit of P is a map $\eta_P : (i : I) \to \mathbf{B}(P)_i$ equipped with an equivalence $\bar{\eta}_P :$ よ$(i) \simeq \mathbf{E}_P(\eta_P(i))$. Because the composite P^2 is defined by $\mathbf{B}(P^2) \equiv \mathbf{F}_P(\mathbf{B}(P))$ and $\mathbf{E}_{P^2}(b_1, b_2) \equiv \coprod_{e:\mathbf{E}_P(b_1)} \mathbf{E}_P(b_2(\mathbf{s}_P(e)))$, the multiplication of P is a map $\mu_P : \mathbf{F}_P(\mathbf{B}(P)) \to \mathbf{B}(P)$ over I equipped with an equivalence $\bar{\mu}_P : (\coprod_{e:\mathbf{E}_P(b_1)} \mathbf{E}_P(b_2(\mathbf{s}_P(e)))) \simeq \mathbf{E}_P(\mu_P(b_1, b_2))$. Let $A : \mathbf{Set} \downarrow \mathbf{B}(P)$. The polynomial Q over P corresponding to A is the composite $A \to \mathbf{B}(P) \xrightarrow{(\mathbf{E}_P, \mathbf{t}_P)} \mathbb{F}_I \times I$, and $\mathbf{F}_Q(X)_i = ((b_1, b_2) : \mathbf{F}_P(X)_i) \times A_{b_1}$. A P-polynomial monad structure on A thus consists of a map $\eta_A : (i : I) \to A_{\eta_P(i)}$ and a map $\mu_A : \{(b_1, b_2) : \mathbf{F}_P(\mathbf{B}(P))\} \to A_{b_1} \to ((e : \mathbf{E}_P(b_1)) \to A_{b_2(e)}) \to A_{\mu_P(b_1, b_2)}$ satisfying suitable associativity and unit laws.

We also recall the notion of a P-tree. For a polynomial P, we define a graph $\mathbf{G}^{\mathrm{Poly}}(P)$ as follows. The set of vertices in $\mathbf{G}^{\mathrm{Poly}}(P)$ is $\mathbf{I}(P) + \mathbf{B}(P)$. There is no edge between vertices from $\mathbf{I}(P)$. There is no edge between vertices from $\mathbf{B}(P)$. An edge from $x : \mathbf{I}(P)$ to $y : \mathbf{B}(P)$ is an element $e : \mathbf{E}_P(y)_x$. An edge from $y : \mathbf{B}(P)$ to $x : \mathbf{I}(P)$ is an identification $\mathbf{t}_P(y) = x$. A *polynomial tree* is a polynomial P satisfying the following axioms.

PT1. The sets $\mathbf{I}(P)$ and $\mathbf{B}(P)$ are finite.

PT2. The maps $\mathbf{t}_P$ and $\mathbf{s}_P$ are injective.

PT3. The graph $\mathbf{G}^{\mathrm{Poly}}(P)$ is a tree.

Note that the image of any map between finite sets is decidable. For a polynomial tree P, let $\mathbf{L}^{\mathrm{Tr}}(P)$ denote the complement of the image of $\mathbf{t}_P$ whose elements are called *leaves in* P. The complement of the image of $\mathbf{s}_P$ is the singleton consisting of the root of the tree $\mathbf{G}^{\mathrm{Poly}}(P)$ by Axiom PT3. We refer to the root of $\mathbf{G}^{\mathrm{Poly}}(P)$ as $\mathbf{r}_P$ and called it the *root of* P. For a polynomial P, a *P-tree* is a polynomial tree T equipped with a morphism $\mathbf{d}_T : T \to P$ in $\mathbf{Poly}$. Let $\mathbf{Tr}(P)$ denote the category of P-trees whose morphisms are those morphisms of polynomials over P preserving roots and leaves.

The Baez-Dolan construction P^+ has an explicit construction using P-trees. There is an equivalence $\mathbf{h} : \mathbf{Obj}(\mathbf{Tr}(P)) \simeq \mathbf{B}(P^+)$ characterized as follows. By the definition of P^+, the object $\mathbf{B}(P^+) \simeq \mathbf{F}_{P^+}(\mathbf{1}) : \mathbf{Set} \downarrow \mathbf{B}(P)$ is the free P-polynomial monad over $\mathbf{1}$. For a P-tree T and $i : \mathbf{I}(T)$, we define a polynomial $T \downarrow^* x$ as follows. $\mathbf{I}(T \downarrow^* i)$ is the subset of $\mathbf{I}(T)$ spanned by those i' such that there is a (unique) path in $\mathbf{G}^{\mathrm{Poly}}(T)$ from i' to i. We define $\mathbf{B}(T \downarrow^* i) \equiv \mathbf{B}(T) \times_{\mathbf{I}(T)} \mathbf{I}(T \downarrow^* i)$. The map $\mathbf{s}_T : \mathbf{E}_T(b) \to \mathbf{I}(T)$ factors through $\mathbf{I}(T \downarrow^* i)$ when $b : \mathbf{B}(T \downarrow^* i)$, and thus we can define $\mathbf{E}_{T\downarrow^* i}(b) \equiv \mathbf{E}_T(b)$. One can show that $T \downarrow^* i$ is a polynomial tree. By construction, we have a morphism $T \downarrow^* i \to T \xrightarrow{\mathbf{d}_T} P$ in **Poly** by which we regard $T \downarrow^* i$ as a P-tree. Then $\mathbf{h}(T)$ for a P-tree T is defined by induction on the size of $\mathbf{I}(T)$ as follows.

- If the root $\mathbf{r}_T$ is a leaf, then $\mathbf{h}(T) = \eta_{\mathbf{B}(P^+)}(\mathbf{d}_T(\mathbf{r}_T))$.
- If there is a (unique) $b : \mathbf{B}(T)_{\mathbf{r}_T}$, then $\mathbf{h}(T) = \mu_{\mathbf{B}(P^+)}(\eta_{P^+}(\mathbf{d}_T(b)), (e \mapsto \mathbf{h}(T \downarrow^* \mathbf{s}_T(e))))$.

We now prove the equivalence of opetopes in our sense and KJBM opetopes, that is, $\mathbb{O}_n \simeq \mathbb{O}_n^{\mathrm{KJBM}}$ (Corollary 8.13). We first make $\mathbb{O}_n$'s part of polynomial monads.

8.2. Construction. *Let A be an opetopic set. For $n \geq 0$, we define a polynomial $\mathbf{Y}_n(A)$ to be*

$$\mathbb{F}_{A_n} \xleftarrow{\mathbf{E}_{\mathbf{Y}_n(A)}} A_{n+1} \xrightarrow{\mathbf{t}_A} A_n,$$

where $\mathbf{E}_{\mathbf{Y}_n(A)}(x) \equiv A \downarrow^{\mathrm{s}} x$, which is finite by Axiom O1. A morphism $F : A \to B$ of opetopic sets induces a morphism of polynomials $\mathbf{Y}_n(A) \to \mathbf{Y}_n(B)$ because $A \downarrow^{\mathrm{s}} x \simeq B \downarrow^{\mathrm{s}} F(x)$ by Proposition 3.35. In particular, $\mathbf{Y}_n(A) : \mathbf{Poly} \downarrow \mathbf{Y}_n(\mathbb{O})$ by Proposition 4.7.

8.3. Construction. *Let $n \geq 0$. We define a polynomial $\mathbf{Y}'_n$ to be*

$$\mathbb{F}_{\mathbf{Bd}_n} \xleftarrow{\mathbf{E}_{\mathbf{Y}'_n}} \mathbf{PD}_n \xrightarrow{\partial} \mathbf{Bd}_n,$$

where $\mathbf{E}_{\mathbf{Y}'_n}(A) \equiv A_n$, which is finite by Axiom PD1, with $(x \mapsto \partial(A \downarrow x)) : A_n \to \mathbf{Bd}_n$. We extend $\mathbf{Y}'_n$ to a polynomial monad on $\mathbf{Bd}_n$ as follows. We define $\eta_{\mathbf{Y}'_n}(A) \equiv \iota(\mathbf{Fill}(A))$ and $\mu_{\mathbf{Y}'_n}(A, B) \equiv \mathbf{Subst}(A, B)$. We have $\eta_{\mathbf{Y}'_n}(A)_n \simeq \{*\}$ and $\partial(\eta_{\mathbf{Y}'_n}(A) \downarrow *) \simeq A$ by construction, and thus よ$(A) \simeq \mathbf{E}_{\mathbf{Y}'_n}(\eta_{\mathbf{Y}'_n}(A))$. By Lemma 7.3, $\coprod_{x:\mathbf{E}_{\mathbf{Y}'_n}(A)} \mathbf{E}_{\mathbf{Y}'_n}(B(x)) \simeq \mathbf{E}_{\mathbf{Y}'_n}(\mu_{\mathbf{Y}'_n}(A, B))$. The associativity and unit laws follow from Propositions 7.7, 7.11 and 7.12. By Corollary 6.24 and Proposition 6.20, $\mathbf{Y}'_n \simeq \mathbf{Y}_n(\mathbb{O})$, so the polynomial monad structure on $\mathbf{Y}'_n$ is transported to $\mathbf{Y}_n(\mathbb{O})$. That is, the polynomial monad structure on $\mathbf{Y}_n(\mathbb{O})$ is determined by $\Lambda^{\mathrm{s}}(\eta_{\mathbf{Y}_n(\mathbb{O})}(A)) \simeq \iota(A)$ and $\Lambda^{\mathrm{s}}(\mu_{\mathbf{Y}_n(\mathbb{O})}(A, B)) \simeq \mathbf{Subst}(\Lambda^{\mathrm{s}}(A)), (x \mapsto \Lambda^{\mathrm{s}}(B(x)))$.

It suffices to construct an equivalence $\mathbf{Y}_n(\mathbb{O}) \simeq \mathbf{Z}_n$ (Theorem 8.12). We proceed by induction on $n : \mathbb{N}$. The base case is easy.

8.4. Lemma. $\mathbf{Y}_0(\mathbb{O}) \simeq \mathbf{Z}_0$.

Proof. By Propositions 4.10 and 4.11. ∎

For the successor case, we show that $\mathbf{Y}_{n+1}(\mathbb{O}) \simeq \mathbf{Y}_n(\mathbb{O})^+$ (Lemma 8.11). By the definition of the Baez-Dolan construction, to get a morphism $\mathbf{Y}_n(\mathbb{O})^+ \to \mathbf{Y}_{n+1}(\mathbb{O})$ of polynomial monads on $\mathbb{O}_{n+1}$, it suffices to construct a functor $\mathbf{Alg}(\mathbf{Y}_{n+1}(\mathbb{O})) \to \mathbf{PM}_{\mathbb{O}_n} \downarrow \mathbf{Y}_n(\mathbb{O})$ over $\mathbf{Set} \downarrow \mathbb{O}_{n+1}$, where $\mathbf{Alg}(\mathbf{Y}_{n+1}(\mathbb{O}))$ is the category of $\mathbf{Y}_{n+1}(\mathbb{O})$-algebras.

8.5. Construction. *Let $n \geq 0$ and let A be a $\mathbf{Y}_{n+1}(\mathbb{O})$-algebra. That is, $A : \mathbf{Set} \downarrow \mathbb{O}_{n+1}$ is equipped with an operator $\mathbf{m}_A : (X : \mathbb{O}_{n+2})(a : (x : X \downarrow^{\mathrm{s}} *_X) \to A_{X\downarrow x}) \to A_{\mathbf{t}(X)}$ compatible with the polynomial monad structure on $\mathbf{Y}_{n+1}(\mathbb{O})$. By Corollary 6.24, $\mathbf{m}_A$ is also regarded as an operator $(X : \mathbf{PD}_{n+1})(a : (x : X_{n+1})) \to A_{\mathbf{Fill}(\partial(X))}$. We equip A with a $\mathbf{Y}_n(\mathbb{O})$-polynomial monad structure as follows. For $X : \mathbb{O}_n$, we have $\sigma(X)_{n+1} \simeq \mathbf{0}$ by construction, so let $\eta_A(X) \equiv \mathbf{m}_A(\sigma(X), !)$, where $!$ is the unique map from $\mathbf{0}$. Since $\Lambda^{\mathrm{s}}(\sigma(X)) \simeq \iota(X)$ by construction, we see that $\eta_A(X)$ lies over $\eta_{\mathbf{Y}_n(\mathbb{O})}(X)$. For $X : \mathbb{O}_{n+1}$, $X' : (x : X \downarrow^{\mathrm{s}} *_X) \to \mathbb{O}_{n+1}[\mathbf{t} = X \downarrow x]$, $a : A_X$, and $a' : (x : X \downarrow^{\mathrm{s}} *_X) \to A_{X'(x)}$, we have $(\{*_X\} + (\coprod_{x : X \downarrow^{\mathrm{s}} *_X} \{*_{X'}\})) \simeq \mathbf{Graft}(\iota(X), (x \mapsto \iota(X'(x))))_{n+1}$ by Lemma 7.15, and thus a and a' defines a map $(a, a') : \mathbf{Graft}(\iota(X), (x \mapsto \iota(X'(x))))_{n+1} \to A$. We then define $\mu_A(a, a') \equiv \mathbf{m}_A(\mathbf{Graft}(\iota(X), (x \mapsto \iota(X'(x)))), (a, a'))$. Since $\Lambda^{\mathrm{s}}(\mathbf{Graft}(\iota(X), (x \mapsto \iota(X'(x))))) \simeq \mathbf{Subst}(\Lambda^{\mathrm{s}}(X), (x \mapsto \Lambda^{\mathrm{s}}(X'(x))))$ by Lemma 7.17, we see that $\mu_A(a, a')$ lies over $\mu_{\mathbf{Y}_n(\mathbb{O})}(X, X')$. The associativity and unit laws follow from Propositions 7.19, 7.23 and 7.24. This construction extends to a functor $\mathbf{Alg}(\mathbf{Y}_{n+1}(\mathbb{O})) \to \mathbf{PM}_{\mathbb{O}_n} \downarrow \mathbf{Y}_n(\mathbb{O})$ over $\mathbf{Set} \downarrow \mathbb{O}_{n+1}$, and let $\mathbf{K} : \mathbf{Y}_n(\mathbb{O})^+ \to \mathbf{Y}_{n+1}(\mathbb{O})$ be the corresponding morphism of polynomial monads on $\mathbb{O}_{n+1}$.*

To see that $\mathbf{K}$ is an equivalence, it suffices to show that the composite

$$\mathbf{Obj}(\mathbf{Tr}(\mathbf{Y}_n(\mathbb{O}))) \simeq \mathbf{B}(\mathbf{Y}_n(\mathbb{O})^+) \xrightarrow{\mathbf{K}} \mathbf{B}(\mathbf{Y}_{n+1}(\mathbb{O})) = \mathbb{O}_{n+2} \simeq \mathbf{PD}_{n+1} \tag{6}$$

is an equivalence. We construct an equivalence $\mathbf{Obj}(\mathbf{Tr}(\mathbf{Y}_n(\mathbb{O}))) \simeq \mathbf{PD}_{n+1}$ (Lemma 8.10) and see that it coincides with Eq. (6).

8.6. Lemma. *Let $n \geq 0$ and let A be an $(n+1)$-pasting diagram. Then $\mathbf{Y}_n(A)$ is a $\mathbf{Y}_n(\mathbb{O})$-tree.*

Proof. Axiom PT1 is by Axioms PD1 and O1 and Lemma 6.13. Axiom PT2 follows from Lemmas 6.14 and 6.12. Since $\mathbf{G}^{\mathrm{Poly}}(\mathbf{Y}_n(A)) \simeq \mathbf{G}^{\mathrm{PD}}(A)$ by construction, Axiom PT3 follows from Lemma 6.11. ∎

8.7. Construction. *Let $n \geq 0$ and let T be a $\mathbf{Y}_n(\mathbb{O})$-tree. We construct a graph $\mathbf{G}^{\mathrm{Poly}}(T)'$ from $\mathbf{G}^{\mathrm{Poly}}(T)$ by reversing the directions of the edges $y \to \mathbf{t}_T(y)$ for all $y : \mathbf{B}(T)$. We define a diagram $\mathbf{D}_T : \mathbf{G}^{\mathrm{Poly}}(T)' \to \mathbf{OSet}$ as follows. A vertex x in $\mathbf{G}^{\mathrm{Poly}}(T)'$ is either in $\mathbf{I}(T)$ or $\mathbf{B}(T)$. In both cases, $\mathbf{D}_T(x) \equiv \mathbf{d}_T(x)$ defines an opetope. For an edge of the form $\mathbf{s}(e) \to y$ for $e : \mathbf{E}_T(y)$, let $\mathbf{d}_T$ send it to the source morphism $\mathbf{d}_T(e) : \mathbf{d}_T(\mathbf{s}(e)) \simeq \mathbf{s}(e') \to \mathbf{d}_T(y)$, where e' is the element corresponding to e via the equivalence $\mathbf{E}_{\mathbf{Y}_n(\mathbb{O})}(\mathbf{d}_T(y)) \simeq \mathbf{E}_T(y)$. For an edge of the form $\mathbf{t}_T(y) \to y$ for $y : \mathbf{B}(T)$, let $\mathbf{d}_T$ send it to the target morphism $\mathbf{d}_T(\mathbf{t}_T(y)) \simeq \mathbf{t}_{\mathbf{Y}_n(\mathbb{O})}(\mathbf{d}_T(y)) \to \mathbf{d}_T(y)$. Finally, we define $\mathbf{c}(T)$ to be the colimit of $\mathbf{D}_T$. We further extend $\mathbf{c}(T)$ to an $(n+1)$-prepasting diagram by $\mathbf{L}(\mathbf{c}(T)) \equiv \coprod_{x : \mathbf{L}^{\mathrm{Tr}}(T)} \mathbf{L}(\mathbf{D}_T(x))$ and $\mathbf{R}(\mathbf{c}(T)) \equiv \mathbf{R}(\mathbf{D}_T(\mathbf{r}_T))$.*

8.8. Lemma. *Let $n \geq 0$ and let T be a $\mathbf{Y}_n(\mathbb{O})$-tree. Then $\mathbf{c}(T)$ is an $(n+1)$-pasting diagram. Moreover, the following hold.*

1. *Suppose that the root $\mathbf{r}_T$ is a leaf. Then $\sigma(\mathbf{d}_T(\mathbf{r}_T)) \simeq \mathbf{c}(T)$.*
2. *Suppose that there is a (unique) $b : \mathbf{B}(T)_{\mathbf{r}_T}$. Then $\mathbf{Graft}(\iota(\mathbf{d}_T(b)), (x \mapsto \mathbf{c}(T \downarrow^* \mathbf{s}_T(x)))) \simeq \mathbf{c}(T)$, where we identify $\mathbf{L}(\iota(\mathbf{d}_T(b))) \simeq \mathbf{d}_T(b) \downarrow^{\mathrm{s}} *_{\mathbf{d}_T(b)} \simeq \mathbf{E}_T(b)$.*

Proof. We proceed by induction on the size of $\mathbf{I}(T)$. Suppose that $\mathbf{r}_T$ is a leaf. Then $\mathbf{G}^{\mathrm{Poly}}(T)'$ is the singleton $\{\mathbf{r}_T\}$ with no edge. Then $\sigma(\mathbf{d}_T(\mathbf{r}_T)) \simeq \mathbf{c}(T)$, and thus $\mathbf{c}(T)$ is an $(n+1)$-pasting diagram. Suppose that there is a (unique) $b : \mathbf{B}(T)_{\mathbf{r}_T}$. By induction hypothesis, $\mathbf{c}(T \downarrow^* \mathbf{s}_T(x))$ is an $(n+1)$-pasting diagram for every $x : \mathbf{E}_T(b)$. Observe that $\mathbf{G}^{\mathrm{Poly}}(T)'$ is the following pushout in the category of graphs

$$\begin{array}{ccc} \coprod_{e:\mathbf{E}_T(b)} \{\mathbf{s}_T(e)\} & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ \coprod_{e:\mathbf{E}_T(b)} \mathbf{G}^{\mathrm{Poly}}(T \downarrow^* \mathbf{s}_T(e))' & \longrightarrow & \mathbf{G}^{\mathrm{Poly}}(T)', \end{array}$$

where X is the full subgraph spanned by $\mathbf{r}_T$, b, and $\mathbf{s}_T(e)$ for all $e : \mathbf{E}_T(b)$. Then $\mathbf{c}(T)$ is the following pushout in **OSet**

$$\begin{array}{ccc} \coprod_{x:\mathbf{L}(\iota(\mathbf{d}_T(b)))} \mathbf{d}_T(b) \downarrow x & \longrightarrow & \mathbf{d}_T(b) \\ \downarrow & \ulcorner & \downarrow \\ \coprod_{x:\mathbf{L}(\iota(\mathbf{d}_T(b)))} \mathbf{c}(T \downarrow^* \mathbf{s}_T(x)) & \longrightarrow & \mathbf{c}(T), \end{array}$$

where the colimit of the restriction of $\mathbf{D}_T$ to X is $\mathbf{d}_T(b)$ because b is the terminal object in X. Therefore, $\mathbf{Graft}(\iota(\mathbf{d}_T(b)), (x \mapsto \mathbf{c}(T \downarrow^* \mathbf{s}_T(x)))) \simeq \mathbf{c}(T)$ by the definition of **Graft**, and thus $\mathbf{c}(T)$ is an $(n+1)$-pasting diagram. ∎

8.9. Lemma. *Let P be a polynomial. Then all the morphisms in $\mathbf{Tr}(P)$ are equivalences.*

Proof. Let $h : T \to T'$ be a morphism of P-trees. Since h preserves leaves, we see that $\mathbf{B}(T) \simeq \mathbf{I}(T) \times_{\mathbf{I}(T')} \mathbf{B}(T')$. Thus, it suffices to show that $h_{\mathbf{I}} : \mathbf{I}(T) \to \mathbf{I}(T')$ is an equivalence. We show that the fiber of $h_{\mathbf{I}}$ over $i' : \mathbf{I}(T')$ is contractible by induction on the length of the path in $\mathbf{G}^{\mathrm{Poly}}(T')$ from i' to the root. If i' is the root, then the root of T is the unique element of the fiber of $h_{\mathbf{I}}$ over i'. Suppose that there is a (unique) pair (b', e') of $b' : \mathbf{B}(T')$ and $e' : \mathbf{E}_{T'}(b')_{i'}$. Since h preserves roots, we see that $\mathbf{E}(T) \simeq \mathbf{I}(T) \times_{\mathbf{I}(T')} \mathbf{E}(T')$. Thus, it suffices to show that the fiber of $h_{\mathbf{E}} : \mathbf{E}(T) \to \mathbf{E}(T')$ over (b', e') is contractible. Since $\mathbf{E}(T) \simeq \mathbf{B}(T) \times_{\mathbf{B}(T')} \mathbf{E}(T')$ and $\mathbf{B}(T) \simeq \mathbf{I}(T) \times_{\mathbf{I}(T')} \mathbf{B}(T')$, this follows from the induction hypothesis for $\mathbf{t}_{T'}(b') : \mathbf{I}(T')$. ∎

8.10. Lemma. *Let $n \geq 0$. Then the functors $\mathbf{Y}_n : \mathbf{PD}_{n+1} \to \mathbf{Tr}(\mathbf{Y}_n(\mathbb{O}))$ induced by Lemma 8.6 and $\mathbf{c} : \mathbf{Tr}(\mathbf{Y}_n(\mathbb{O})) \to \mathbf{PD}_{n+1}$ induced by Lemma 8.8 are mutual inverses.*

Proof. For a $\mathbf{Y}_n(\mathbb{O})$-tree T, we have a canonical morphism $T \to \mathbf{Y}_n(\mathbf{c}(T))$ of polynomials over $\mathbf{Y}_n(\mathbb{O})$ by construction. This morphism preserves roots and leaves and thus is an equivalence by Lemma 8.9. For an $(n+1)$-pasting diagram A, we have a canonical morphism $\mathbf{c}(\mathbf{Y}_n(A)) \to A$ of opetopic sets by construction. This morphism preserves root and leaf objects and thus is an equivalence by Corollary 6.25. ∎

8.11. Lemma. *Let $n \geq 0$. Then the morphism $\mathbf{K} : \mathbf{Y}_n(\mathbb{O})^+ \to \mathbf{Y}_{n+1}(\mathbb{O})$ of polynomial monads on $\mathbb{O}_{n+1}$ is an equivalence.*

Proof. By Lemma 8.8 and by the definition of the equivalence $\mathbf{Obj}(\mathbf{Tr}(\mathbf{Y}_n(\mathbb{O}))) \simeq \mathbf{B}(\mathbf{Y}_n(\mathbb{O})^+)$, we see that Eq. (6) is equivalent to $\mathbf{c} : \mathbf{Obj}(\mathbf{Tr}(\mathbf{Y}_n(\mathbb{O}))) \to \mathbf{PD}_{n+1}$, which is an equivalence by Lemma 8.10. ∎

8.12. Theorem. *$\mathbf{Y}_n(\mathbb{O}) \simeq \mathbf{Z}_n$ for all $n : \mathbb{N}$.*

Proof. By Lemmas 8.4 and 8.11. ∎

8.13. Corollary. *$\mathbb{O}_n \simeq \mathbb{O}_n^{\mathrm{KJBM}}$ for all $n : \mathbb{N}$.*

Proof. By Theorem 8.12. ∎

8.14. Equivalence with Ho Thanh's category of opetopes. We compare the canonical presentation of our category $\mathbb{O}$ of opetopes (Construction 3.2 and Proposition 3.15) and the presentation given by Ho Thanh [2021, Definition 3.6]. The set of objects $\mathbf{Obj}(\mathbb{O})$ is $(n : \omega) \times \mathbb{O}_n$, which coincides with [Ho Thanh, 2021, Definition 3.6 (1)] by Corollary 8.13. The generating morphisms for $\mathbb{O}$ are the source and target morphisms, which coincides with [Ho Thanh, 2021, Definition 3.6 (2)]. The relations for $\mathbb{O}$ are all the equations $f_1 \circ g_1 = f_2 \circ g_2$ that hold in $\mathbb{O}$ such that exactly one of the following holds.

1. f_1, f_2, and g_2 are source arrows and g_1 is a target arrow. This corresponds to Eq. (Inner) in [Ho Thanh, 2021, Definition 3.6 (3)].
2. f_1 is a source arrow and g_1, f_2, and g_2 are target arrows. This corresponds to Eq. (Glob1) in [Ho Thanh, 2021, Definition 3.6 (3)].
3. f_1 is a target arrow and g_1, f_2, and g_2 are source arrows. This corresponds to Eq. (Glob2) in [Ho Thanh, 2021, Definition 3.6 (3)].
4. f_1, f_2, and g_2 are target arrows and g_1 is a source arrow. This corresponds to Eq. (Degen) in [Ho Thanh, 2021, Definition 3.6 (3)].

Therefore:

8.15. Theorem. *The category of opetopes $\mathbb{O}$ is equivalent to the one given by Ho Thanh [2021, Definition 3.6].* ∎

References

Benedikt Ahrens, Krzysztof Kapulkin, and Michael Shulman. Univalent categories and the rezk completion. *Mathematical Structures in Computer Science*, 25(5):1010–1039, 2015. URL https://doi.org/10.1017/S0960129514000486.

John C. Baez and James Dolan. Higher-dimensional algebra. III. n-categories and the algebra of opetopes. *Advances in Mathematics*, 135(2):145–206, 1998. URL https://doi.org/10.1006/aima.1997.1695.

Clark Barwick and Christopher Schommer-Pries. On the unicity of the theory of higher categories. *Journal of the American Mathematical Society*, 34(4):1011–1058, 2021. URL https://doi.org/10.1090/jams/972.

Alex Chandler. Thin posets, cw posets, and categorification, 2019. URL https://arxiv.org/abs/1911.05600v2.

Eugenia Cheng. The category of opetopes and the category of opetopic sets. *Theory and Applications of Categories*, 11:No. 16, 353–374, 2003. URL http://www.tac.mta.ca/tac/volumes/11/16/11-16abs.html.

Eugenia Cheng. Weak n-categories: opetopic and multitopic foundations. *Journal of Pure and Applied Algebra*, 186(2):109–137, 2004a. URL https://doi.org/10.1016/S0022-4049(03)00139-7.

Eugenia Cheng. Weak n-categories: comparing opetopic foundations. *Journal of Pure and Applied Algebra*, 186(3):219–231, 2004b. URL https://doi.org/10.1016/S0022-4049(03)00140-3.

Pierre-Louis Curien, Cédric Ho Thanh, and Samuel Mimram. Type theoretical approaches to opetopes. *Higher Structures*, 6(1):80–181, 2022. URL https://doi.org/10.21136/HS.2022.02.

Nicola Gambino and Joachim Kock. Polynomial functors and polynomial monads. *Mathematical Proceedings of the Cambridge Philosophical Society*, 154(1):153–192, 2013. URL https://doi.org/10.1017/S0305004112000394.

Amar Hadzihasanovic. A combinatorial-topological shape category for polygraphs. *Applied Categorical Structures*, 28:419–476, 2020. URL https://doi.org/10.1007/s10485-019-09586-6.

Claudio Hermida, Michael Makkai, and John Power. On weak higher-dimensional categories. I. 3. *Journal of Pure and Applied Algebra*, 166(1-2):83–104, 2002. URL https://doi.org/10.1016/S0022-4049(01)00014-7.

Cédric Ho Thanh. The equivalence between many-to-one polygraphs and opetopic sets, 2021. URL https://arxiv.org/abs/1806.08645v4.

Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. In *Twenty-five years of constructive type theory (Venice, 1995)*, volume 36 of *Oxford Logic Guides*, pages 83–111. Oxford Univ. Press, New York, 1998. URL https://doi.org/10.1093/oso/9780198501275.003.0008.

Joachim Kock, André Joyal, Michael Batanin, and Jean-François Mascari. Polynomial functors and opetopes. *Advances in Mathematics*, 224(6):2690–2737, 2010. URL https://doi.org/10.1016/j.aim.2010.02.012.

Louise Leclerc. A poset-like approach to positive opetopes, 2024a. URL https://arxiv.org/abs/2405.17948v1.

Louise Leclerc. Two equivalent descriptions of opetopes: in terms of zoom complexes and of partial orders, 2024b. URL https://arxiv.org/abs/2409.02744v1.

Tom Leinster. *Higher Operads, Higher Categories*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2004. URL https://doi.org/10.1017/CBO9780511525896.

Peter McMullen and Egon Schulte. *Abstract Regular Polytopes*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2002. URL https://doi.org/10.1017/CBO9780511546686.

Christopher Nguyen. *Parity structure on associahedra and other polytopes*. PhD thesis, Macquarie University, 2018. URL https://doi.org/10.25949/19440344.v1.

Egbert Rijke. Introduction to homotopy type theory, 2022. URL https://arxiv.org/abs/2212.11082v1.

The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study, 2013. URL http://homotopytypetheory.org/book/.

Marek Zawadowski. On positive opetopes, positive opetopic cardinals and positive opetopic set, 2023. URL https://arxiv.org/abs/0708.2658v2.

Graduate School of Informatics, Nagoya University
Furo-cho, Chikusa-ward, Nagoya-City, 464-8601
Email: t.uemura00@gmail.com

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