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ON PURE MONOMORPHISMS AND PURE EPIMORPHISMS IN ACCESSIBLE CATEGORIES

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ABSTRACT. In all κ-accessible additive categories, κ-pure monomorphisms and κ-pure epimorphisms are well-behaved, as shown in our previous paper [L. Positselski, "Locally coherent exact categories", Appl. Categorical Struct. 32, 2024]. This is known to be not always true in κ-accessible nonadditive categories. Nevertheless, mild assumptions on a κ-accessible category are sufficient to prove good properties of κ-pure monomorphisms and κ-pure epimorphisms. In particular, in a κ-accessible category with finite products, all κ-pure monomorphisms are κ-directed colimits of split monomorphisms, while in a κ-accessible category with finite coproducts, all κ-pure epimorphisms are κ-directed colimits of split epimorphisms. We also discuss what we call Quillen exact classes of monomorphisms and epimorphisms, generalizing the additive concept of one-sided exact category.

Contents

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Introduction

The concept of κ-purity is an important technical tool in locally presentable and accessible category theory [2, Section 2.D]. The properties of κ-pure monomorphisms and κ-pure epimorphisms have been studied by Adámek, Rosický, and collaborators in the papers [1, 3, 4], which offer an assortment of theorems and counterexamples (see also the paper by Hu and Pelletier [18] and the recent prepint of Kanalas [19]). However, the basic results about κ-purity in κ-accessible categories seem to only have been proved under assumptions that are somewhat restrictive. In this paper, we explain how to relax some assumptions.

Let us list what we consider the basic results. In any κ-accessible category with pushouts, all κ-pure monomorphisms are κ-directed colimits of split monomorphisms [2, Corollary and Remark 2.30]. In any κ-accessible category with pushouts, all κ-pure monomorphisms are regular monomorphisms [1, Corollary 1]. In any κ-accessible category with pushouts, the class of κ-pure monomorphisms is stable under pushouts [1, Corollary 2], [3, Proposition 15(i)].

In any κ-accessible category with pullbacks, all κ-pure epimorphisms are κ-directed colimits of split epimorphisms [3, Proposition 3]. In any κ-accessible category with pullbacks, all κ-pure epimorphisms are regular epimorphisms [3, Proposition 4(b)]. In any locally κ-presentable category, the class of κ-pure epimorphisms is stable under pullbacks [3, Proposition 15(ii)].

In a slightly different setting of the paper [4], some results similar to the above ones are stated under milder assumptions. In particular, according to [4, Lemma 2.2], existence of weak pushouts in the full subcategory of finitely presentable objects is sufficient for the pure monomorphisms in a finitely accessible category to be directed colimits of split monomorphisms. By [4, Lemma 3.1], existence of weak pullbacks in the full subcategory of finitely presentable objects is sufficient for the pure epimorphisms in a finitely accessible category to be directed colimits of split epimorphisms.

For comparison, in any κ-accessible additive category A, all κ-pure monomorphisms are κ-directed colimits of split monomorphisms, and all κ-pure epimorphisms are κ-directed colimits of split epimorphisms. All κ-pure monomorphisms in A are regular monomorphisms, and all κ-pure epimorphisms are regular epimorphisms. All pushouts of κ-pure monomorphisms always exist in A, and the class of κ-pure monomorphisms is stable under pushouts. All pullbacks of κ-pure epimorphisms exist in A, and the class of κ-pure epimorphisms is stable under pullbacks.

Moreover, in any κ-accessible additive category, all κ-pure epimorphisms have kernels, and all κ-pure monomorphisms have cokernels. The κ-pure monomorphisms are precisely the kernels of the κ-pure epimorphisms, and the κ-pure epimorphisms are precisely the cokernels of the κ-pure monomorphisms (this is a generalization of [3, Proposition 5]). All results mentioned in this and the previous paragraph follow from the exposition in [25, Section 4], particularly from the existence of the κ-pure exact structure (in the sense of Quillen) together with [25, Propositions 4.2 and 4.4].

From our perspective, even such assumptions as existence of weak pushouts and weak pullbacks are too restrictive, and unnecessarily so, for the purity theory. In particular, an additive category A *need not* have weak pushouts or weak pullbacks. For example, the existence of weak pullbacks in the category $\mathsf{A} = R\text{–}\mathsf{Mod}_{\mathsf{inj}}$ of injective left modules over a ring R is equivalent to the existence of injective precovers of all left R-modules (in the sense of the paper [14]). By [14, Propositions 2.1 and 2.2], injective precovers exist in $R\text{–}\mathsf{Mod}$ if and only if R is left Noetherian. Dually, the existence of weak pushouts in the category $\mathsf{A} = R\text{–}\mathsf{Mod}_{\mathsf{proj}}$ of projective left R-modules is equivalent to the existence of projective preenvelopes of all left R-modules. By the argument of [14, proofs of Propositions 2.1 and 5.1], the latter condition implies that the infinite direct products of projective left R-modules are projective, which does not hold for most rings R (cf. [8, Theorem P], [11, Theorem 3.3]).

The aim of this paper is to spell out reasonable conditions on a κ-accessible category A that (1) hold for all κ-accessible additive categories, and (2) imply good properties of κ-pure monomorphisms and κ-pure epimorphisms. The reader will see that the resulting conditions are indeed quite mild.

Let us emphasize that *some* assumptions are certainly necessary for the purity theory in nonadditive categories. In particular, [4, Example 2.5] provides an example of a finitely accessible category with a pure monomorphism that is *not* a directed colimit of split monomorphisms and *not* a regular monomorphism. In Examples 13.4 and 15.4, we present an essentially trivial example of an accessible preadditive (but not additive!) category in which all monomorphisms and epimorphisms are split, but some pushouts of monomorphisms and some pullbacks of epimorphisms do not exist.

In the context of a κ-accessible category A, we use the terminology *strongly κ-pure monomorphisms* for the morphisms in A that can be obtained as κ-directed colimits of split monomorphisms of κ-presentable objects in A. Similarly, the *strongly κ-pure epimorphisms* are the κ-directed colimits of split epimorphisms between κ-presentable objects. We start with establishing very mild sufficient conditions for all κ-pure monomorphisms and κ-pure epimorphisms to be strongly κ-pure. Then we proceed to provide further, also mild sufficient conditions for strongly κ-pure mono/epimorphisms to be regular and preserved by pushouts/pullbacks.

As a generalization of κ-pure monomorphisms and κ-pure epimorphisms, we discuss what we call *QE-mono* and *QE-epi* classes of morphisms (where QE means "Quillen exact"). These are nonadditive generalizations of *right exact* and *left exact* categories inroduced by Rump [29, Definition 4 in Section 5] and studied by Bazzoni and Crivei [9]. In the terminology of Henrard and van Roosmalen [16], the latter (additive categories with additional structure) are called *inflation-exact* and *deflation-exact* categories. See Rosenberg's preprints [27, Section 1.1], [28, Chapter I] for prior art in the context of nonadditive categories.

Given a κ-accessible category A with the full subcategory of κ-presentable objects $\mathsf{A}_{<\kappa} \subset \mathsf{A}$, and given a QE-mono class $\mathcal{M}$ or QE-epi class $\mathcal{P}$ in $\mathsf{A}_{<\kappa}$, we prove that the class of all κ-directed colimits of morphisms from $\mathcal{M}$ (respectively, from $\mathcal{P}$) is a QE-mono (resp., QE-epi) class of morphisms in A. This provides a nonadditive generalization of the results of [25, Sections 1–2], and simultaneously their extension from the setting of exact categories (in the sense of Quillen [10]) to that of right exact and left exact (additive) categories.

Notice that the classical notion of an exact category in the sense of Quillen does not seem to make much sense in the nonadditive setting. The point is that exact categories are additive categories with a class of *admissible short exact sequences* $0 \longrightarrow K \xrightarrow{f} L \xrightarrow{g} M \longrightarrow 0$, which, first of all, have to be *kernel-cokernel pairs*: $f = \ker g$ and $g = \operatorname{coker} f$. In the context of nonadditive categories, one does not usually consider kernel-cokernel pairs of morphisms.

In the general (nonadditive) category theory, there is a natural construction of the *(co)equalizer* of a parallel pair of morphisms $\bullet \rightrightarrows \bullet$, which is a single morphism $\bullet \longrightarrow \bullet$. Conversely, to a single morphism $\bullet \longrightarrow \bullet$, one assigns it *(co)kernel pair*, which is a parallel pair of morphisms $\bullet \rightrightarrows \bullet$. So, instead of a single self-dual concept of a kernel-cokernel pair, in the nonadditive realm there are two concepts, dual to each other, represented by diagrams of the shape

$$\bullet \longrightarrow \bullet \rightrightarrows \bullet \quad \text{or} \quad \bullet \rightrightarrows \bullet \longrightarrow \bullet.$$

Accordingly, it seems to be natural to split the single concept of an exact category in the sense of Quillen into two halves (the forementioned right exact and left exact categories) before extending in to the nonadditive world. This is the approach that we follow in the present paper.

Let us mention that, in spite of our discussion above, a self-dual nonadditive version of Quillen exact categories exists in the literature, introduced by Dyckerhoff and Kapranov under the name of *proto-exact categories* [13, Section 2.4]. We do not consider this concept in the present paper.

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1. Preliminaries on Accessible Categories

We use the book [2] as the background reference source on accessible categories. In particular, we refer to [2, Definition 1.4, Theorem and Corollary 1.5, Definition 1.13(1), and Remark 1.21] for a discussion of λ-*directed* vs. λ-*filtered* colimits. For an earlier exposition avoiding a small mistake in [2, proof of Theorem 1.5], see [5].

Let κ be a regular cardinal and A be a category with κ-directed (equivalently, κ-filtered) colimits. An object $S \in \mathsf{A}$ is called κ-*presentable* [2, Definition 1.13(2)] if the covariant functor $\mathrm{Hom}_{\mathsf{A}}(S,-)\colon \mathsf{A}\longrightarrow \mathsf{Sets}$ from A to the category of sets Sets preserves κ-directed colimits. We denote the full subcategory of κ-presentable objects by $\mathsf{A}_{<\kappa}\subset\mathsf{A}$.

The category A is called *κ-accessible* [2, Definition 2.1] if there is a *set* of κ-presentable objects $\mathsf{S}\subset\mathsf{A}$ such that all the objects of A are κ-directed colimits of objects from S. If this is the case, then the κ-presentable objects of A are precisely all the retracts of the objects from S.

A category is called *accessible* if it is κ-accessible for some regular cardinal κ. In the case of the countable cardinal $\kappa=\aleph_0$, one speaks of *finitely accessible categories* [2, Remark 2.2(1)].

Given a class of objects $\mathsf{T}\subset\mathsf{A}$, we denote by $\varinjlim_{(\kappa)}\mathsf{T}\subset\mathsf{A}$ the class (or the full subcategory) of all objects of A that can be obtained as κ-directed colimits of objects from T. The following proposition is well-known.

1.1. Proposition. *Let A be a κ-accessible category and $\mathsf{T}\subset\mathsf{A}$ be a set of (some) κ-presentable objects. Then the full subcategory $\mathsf{B}=\varinjlim_{(\kappa)}\mathsf{T}\subset\mathsf{A}$ is closed under κ-directed colimits in A. The category B is κ-accessible, and the κ-presentable objects of B are precisely all the retracts of the objects from T. Equivalently, the κ-presentable objects of B are precisely all the objects of B that are κ-presentable in A. An object $A\in\mathsf{A}$ belongs to B if and only if, for every object $S\in\mathsf{A}_{<\kappa}$, every morphism $S\longrightarrow A$ in A factorizes through an object from T.*

Proof. In the context of finitely accessible additive categories, this result goes back to [21, Proposition 2.1], [12, Section 4.1], and [20, Proposition 5.11]. For the full generality, see, e. g., [24, Proposition 1.2]. ∎

Let A, B, and C be three categories, and let $F\colon\mathsf{A}\longrightarrow\mathsf{C}$ and $G\colon\mathsf{B}\longrightarrow\mathsf{C}$ be two functors. Following [2, Notation 2.42], we denote by $F\downarrow G$ the category of all triples (A,B,h), where $A\in\mathsf{A}$ and $B\in\mathsf{B}$ are two objects and $h\colon F(A)\longrightarrow G(B)$ is a morphism in C. Morphisms in the category $F\downarrow G$ are defined in the obvious way.

1.2. Proposition. *Let A, B, and C be κ-accessible categories, and let $F\colon\mathsf{A}\longrightarrow\mathsf{C}$ and $G\colon\mathsf{B}\longrightarrow\mathsf{C}$ be functors preserving κ-directed colimits and taking κ-presentable objects to κ-presentable objects. Then the category $F\downarrow G$ is κ-accessible. An object $(S,T,u)\in F\downarrow G$ is κ-presentable if and only if the object S is κ-presentable in A and the object T is κ-presentable in B.*

Proof. This is [2, proof of Theorem 2.43]; see also [25, Proposition A.3]. ∎

The following proposition is a slightly stronger version of Proposition 1.2.

1.3. Proposition. *Let A, B, and C be κ-accessible categories, and let $F\colon\mathsf{A}\longrightarrow\mathsf{C}$ and $G\colon\mathsf{B}\longrightarrow\mathsf{C}$ be functors preserving κ-directed colimits and taking κ-presentable objects to κ-presentable objects. Let $\mathsf{S}\subset\mathsf{A}_{<\kappa}$ and $\mathsf{T}\subset\mathsf{B}_{<\kappa}$ be some chosen subsets of κ-presentable objects in A and B such that all objects of A are κ-directed colimits of objects from S and all objects of B are κ-directed colimits of objects from T. Then all objects of $F\downarrow G$ are κ-directed colimits of objects $(S,T,u)\in F\downarrow G$ with $S\in\mathsf{S}$ and $T\in\mathsf{T}$.*

Proof. This is what is actually proved in [2, proof of Theorem 2.43]. ■

A category D is said to be *finite* if the set of all morphisms in D is finite. More generally, a category D is said to be κ-*small* if the cardinality of the set of all morphisms in D is smaller than κ.

1.4. Proposition. *Let A be a κ-accessible category, and let D be a finite category in which all endomorphisms of objects are identity morphisms. Then the category A^D of all (covariant) functors $D \longrightarrow \mathsf{A}$ is κ-accessible. A functor $F\colon D \longrightarrow \mathsf{A}$ is κ-presentable as an object of A^D if and only if, for every object $d \in D$, the object $F(d)$ is κ-presentable in A.*

Proof. In the case of finitely accessible categories A, this result goes back to [6, Exposé I, Proposition 8.8.5] and [23, page 55]. For an arbitrary regular cardinal κ, the desired assertion is a particular case of [17, Theorem 1.3]. See also [25, Proposition A.5]. ■

We use the notation $\varprojlim$ and $\varinjlim$ for limits and colimits in categories. The upper index, such as in $\varprojlim^{\mathsf{A}}$ and $\varinjlim^{\mathsf{A}}$, is used to indicate that the (co)limit is taken in the category A. By κ-*small (co)limits* one means (co)limits of diagrams indexed by κ-small indexing categories D.

1.5. Lemma. *Let A be a κ-accessible category. Then the full subcategory $\mathsf{A}_{<\kappa} \subset \mathsf{A}$ of all κ-presentable objects in A is closed under all κ-small colimits that exist in A. Furthermore, the fully faithful inclusion functor $\mathsf{A}_{<\kappa} \longrightarrow \mathsf{A}$ preserves all κ-small colimits that exist in $\mathsf{A}_{<\kappa}$.*

Proof. This follows from the fact that κ-directed colimits commute with κ-small limits in the category of sets. For the first assertion, see [2, Proposition 1.16]. To prove the second claim, let D be a κ-small category and let $F\colon D \longrightarrow \mathsf{A}_{<\kappa}$ be a D-indexed diagram in $\mathsf{A}_{<\kappa}$ with the colimit $A = \varinjlim_{d\in D}^{\mathsf{A}_{<\kappa}} F(d) \in \mathsf{A}_{<\kappa}$ computed in the category $\mathsf{A}_{<\kappa}$. Let $B \in \mathsf{A}$ be an arbitrary object, and let $B = \varinjlim_{\xi\in\Xi}^{\mathsf{A}} S_\xi$ be a representation of B as the colimit of a diagram of objects $S_\xi \in \mathsf{A}_{<\kappa}$, indexed by a κ-directed poset Ξ, the colimit being computed in the category A. Then in the category of sets we have

$$\begin{aligned}\varprojlim\nolimits_{d\in D}^{\mathsf{Sets}} \operatorname{Hom}_{\mathsf{A}}(F(d), B) &= \varprojlim\nolimits_{d\in D}^{\mathsf{Sets}} \varinjlim\nolimits_{\xi\in\Xi}^{\mathsf{Sets}} \operatorname{Hom}_{\mathsf{A}_{<\kappa}}(F(d), S_\xi) \\ &= \varinjlim\nolimits_{\xi\in\Xi}^{\mathsf{Sets}} \varprojlim\nolimits_{d\in D}^{\mathsf{Sets}} \operatorname{Hom}_{\mathsf{A}_{<\kappa}}(F(d), S_\xi) = \varinjlim\nolimits_{\xi\in\Xi}^{\mathsf{Sets}} \operatorname{Hom}_{\mathsf{A}_{<\kappa}}(A, S_\xi) = \operatorname{Hom}_{\mathsf{A}}(A, B),\end{aligned}$$

as desired. ■

In the terminology of [2, Example 6.38], full subcategories $\mathsf{S} \subset \mathsf{A}$ satisfying the assumptions of the next lemma are called *weakly colimit-dense*.

1.6. Lemma. *Let A be a category and $\mathsf{S} \subset \mathsf{A}$ be a full subcategory such that the minimal full subcategory of A containing S and closed under those colimits that exist in A, coincides with A. Then the fully faithful inclusion functor $\mathsf{S} \longrightarrow \mathsf{A}$ preserves all those limits that exist in S.*

Proof. This follows from the fact that limits commute with limits (in any category, and in particular) in the category of sets. Let D be a small category, let $G\colon D \longrightarrow \mathsf{S}$ be a diagram indexed by D, and let $B = \varprojlim^{\mathsf{S}}_{d\in D} G(d) \in \mathsf{S}$ be the limit of the diagram G computed in the category S. We have to prove that the natural map $\mathrm{Hom}_{\mathsf{A}}(A,B) \longrightarrow \varprojlim^{\mathsf{Sets}}_{d\in D} \mathrm{Hom}_{\mathsf{A}}(A,G(d))$ is a bijection of sets for all objects $A\in\mathsf{A}$.

Denote by E the full subcategory of A consisting of all objects E for which the map of sets $\mathrm{Hom}_{\mathsf{A}}(E,B) \longrightarrow \varprojlim^{\mathsf{Sets}}_{d\in D} \mathrm{Hom}_{\mathsf{A}}(E,G(d))$ is bijective. By assumption, we know that $\mathsf{S}\subset\mathsf{E}$, and it remains to check that the full subcategory $\mathsf{E}\subset\mathsf{A}$ is closed under those colimits that exist in A.

Let C be a small category, let $F\colon C \longrightarrow \mathsf{E}$ be a diagram indexed by C, and let $A = \varinjlim^{\mathsf{A}}_{c\in C} F(c) \in \mathsf{A}$ be the colimit of the diagram F computed in the category A. So the map $\mathrm{Hom}_{\mathsf{A}}(F(c),B) \longrightarrow \varprojlim^{\mathsf{Sets}}_{d\in D} \mathrm{Hom}_{\mathsf{A}}(F(c),G(d))$ is a bijection of sets for all $c\in C$. Then it follows that the map $\mathrm{Hom}_{\mathsf{A}}(A,B) \longrightarrow \varprojlim^{\mathsf{Sets}}_{d\in D} \mathrm{Hom}_{\mathsf{A}}(A,G(d))$ is a bijection as well. Indeed, we have

$$\varprojlim\nolimits^{\mathsf{Sets}}_{d\in D} \mathrm{Hom}_{\mathsf{A}}(A,G(d)) = \varprojlim\nolimits^{\mathsf{Sets}}_{d\in D} \varprojlim\nolimits^{\mathsf{Sets}}_{c\in C} \mathrm{Hom}_{\mathsf{A}}(F(c),G(d)) \\ = \varprojlim\nolimits^{\mathsf{Sets}}_{c\in C} \varprojlim\nolimits^{\mathsf{Sets}}_{d\in D} \mathrm{Hom}_{\mathsf{A}}(F(c),G(d)) = \varprojlim\nolimits^{\mathsf{Sets}}_{c\in C} \mathrm{Hom}_{\mathsf{A}}(F(c),B) = \mathrm{Hom}_{\mathsf{A}}(A,B).$$

∎

1.7. Lemma. *Let A be a κ-accessible category and $A \longrightarrow B$ be a morphism in A such that the induced map of sets $\mathrm{Hom}_{\mathsf{A}}(S,A) \longrightarrow \mathrm{Hom}_{\mathsf{A}}(S,B)$ is bijective for all κ-presentable objects $S\in\mathsf{A}$. Then the morphism $A\longrightarrow B$ is an isomorphism in A.*

Proof. This follows from the fact that every object $C\in\mathsf{A}$ is the colimit of the canonical diagram of morphisms into C from κ-presentable objects of A [2, Section 0.6, Definition and Remark 1.23, Remark 2.2(4), and Proposition 2.8(i)]. See also [24, Lemma 1.1]. ∎

1.8. Lemma. *In any κ-accessible category A, κ-directed colimits commute with those κ-small limits that exist in A. Specifically, if Ξ is κ-directed poset, D is a κ-small category, and $F\colon \Xi\times D \longrightarrow \mathsf{A}$ is a functor such that the limit $\varprojlim^{\mathsf{A}}_{d\in D} F(\xi,d)$ exists in A for all $\xi\in\Xi$, then*

$$\varprojlim\nolimits^{\mathsf{A}}_{d\in D} \varinjlim\nolimits^{\mathsf{A}}_{\xi\in\Xi} F(\xi,d) = \varinjlim\nolimits^{\mathsf{A}}_{\xi\in\Xi} \varprojlim\nolimits^{\mathsf{A}}_{d\in D} F(\xi,d). \tag{1}$$

Proof. This is the generalization of [2, Proposition 1.59] from locally presentable to accessible categories. As in Lemma 1.5, the basic explanation for why this assertion holds is because κ-directed colimits commute with κ-small limits in the category of sets. If the limit in the left-hand side of (1) exists, then the assertion that the natural morphism from the right-hand side to the left-hand side is an isomorphism follows easily by applying Lemma 1.7. Notice that the limits in the right-hand side of (1) exist by the assumptions of the present lemma.

When one wants to prove the existence of the limit in the left-hand side of (1) rather than assume it, the following argument works. For every object $A\in\mathsf{A}$, we need to show that the natural map of sets

$$f_A\colon \operatorname{Hom}_{\mathsf{A}}(A, \varinjlim\nolimits^{\mathsf{A}}_{\xi\in\Xi} \varprojlim\nolimits^{\mathsf{A}}_{d\in D} F(\xi,d)) \longrightarrow \varprojlim\nolimits^{\mathsf{Sets}}_{d\in D} \operatorname{Hom}_{\mathsf{A}}(A, \varinjlim\nolimits^{\mathsf{A}}_{\xi\in\Xi} F(\xi,d))$$

is a bijection. When the object $A \in \mathsf{A}$ is κ-presentable, we use the facts that the covariant functor $\operatorname{Hom}_{\mathsf{A}}(A,-)$ takes both limits and κ-directed colimits in A to the respective (co)limits in Sets in order to reduce the question to the previously mentioned assertion that κ-directed colimits commute with κ-small limits in the category of sets. In the general case, the object A is a (κ-directed) colimit of κ-presentable objects, and it remains to point out that both the domain and the codomain of the map f_A, viewed as contravariant functors $\mathsf{A}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$ of the varying object $A \in \mathsf{A}$, take colimits in A to limits in Sets. ■

2. Very Weak Cokernel Pairs

Let C be a category. Given a pair of morphisms $i\colon A \longrightarrow B$ and $p\colon B \longrightarrow A$ in C such that the composition $p \circ i = \mathrm{id}_A$ is the identity morphism, one says that i is a *split monomorphism* and p is a *split epimorphism* in C.

By a *pushout* in C one means the colimit of a diagram of the shape

$$\begin{array}{ccc} B & & \\ \uparrow{\scriptstyle f} & & \\ A & \xrightarrow{\;g\;} & C \end{array} \qquad (2)$$

A *cokernel pair* is a pushout of the diagram as above with $B = C$ and $f = g$. So the cokernel pair of a morphism $f\colon A \longrightarrow B$ in C is a parallel pair of morphisms k_1, $k_2\colon B \rightrightarrows K$ such that $k_1 \circ f = k_2 \circ f$ and the triple (K, k_1, k_2) is universal with this property in the category C.

The definition of a *weak colimit* is obtained from the usual definition of a colimit by dropping the condition of uniqueness of the required morphism and keeping only the existence. Specifically, let D be a small category and $F\colon D \longrightarrow \mathsf{C}$ be a D-indexed diagram in C. Let $A \in \mathsf{C}$ be an object and $F \longrightarrow A$ be a compatible cocone (i. e., in other words, a morphism from F to the constant D-indexed diagram in C corresponding to the object A). Then one says that A is a weak colimit of F if, for every object $B \in \mathsf{C}$ and any compatible cocone $F \longrightarrow B$, there exists a (not necessarily unique) morphism $A \longrightarrow B$ in C making the triangular diagram $F(d) \longrightarrow A \longrightarrow B$ commutative in C for all $d \in D$.

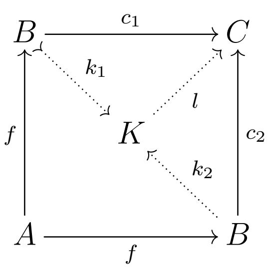
As particular cases of the general definition of a weak colimit, one can speak about weak pushouts, weak cokernel pairs, etc.

Let $f\colon A \longrightarrow B$ be a morphism and k_1, $k_2\colon B \rightrightarrows K$ be a weak cokernel pair of f. The parallel pair of identity morphisms id_B, $\mathrm{id}_B\colon B \rightrightarrows B$ obviously has the property that the two morphisms have equal compositions with the morphism f. Consequently, there exists a morphism $s\colon K \longrightarrow B$ such that $s \circ k_1 = \mathrm{id}_B = s \circ k_2$. Thus both the morphisms k_1 and $k_2\colon B \longrightarrow K$ are split monomorphisms.

2.1. Definition. Let $f\colon A \longrightarrow B$ be a morphism in C and c_1, $c_2\colon B \rightrightarrows C$ be a parallel pair of morphisms such that $c_1 \circ f = c_2 \circ f$. We will say that a parallel pair of morphisms k_1, $k_2\colon B \rightrightarrows K$ in C is a *very weak cokernel pair* of f *with respect to* (c_1, c_2) if the following three conditions hold:

- one has $k_1 \circ f = k_2 \circ f$;
- there exists a morphism $l\colon K \longrightarrow C$ such that $c_1 = l \circ k_1$ and $c_2 = l \circ k_2$;
- the morphism $k_1\colon B \longrightarrow K$ is a split monomorphism (i. e., there exists a morphism $s\colon K \longrightarrow B$ such that $s \circ k_1 = \mathrm{id}_B$).

The commutative diagram described in Definition 2.1 can be drawn as



(3)

Here the splitting s of the split monomorphism k_1 is not depicted on the diagram (3); instead, the condition that k_1 is a split monomorphism is expressed by the tail at the beginning of the dotted arrow showing k_1.

2.2. Examples. (1) If the morphism $f\colon A \longrightarrow B$ has a weak cokernel pair k_1, $k_2\colon B \rightrightarrows K$, then (k_1, k_2) is a very weak cokernel pair of f with respect to every parallel pair of morphisms (c_1, c_2) such that $c_1 \circ f = c_2 \circ f$. This is clear from the discussion above. In this sense, our terminology is consistent.

(2) Let $f\colon A \longrightarrow B$ and c_1, $c_2\colon B \rightrightarrows C$ be three morphisms such that $c_1 \circ f = c_2 \circ f$. Assume that the product $K = B \times C$ exists in C, and denote by $p_B\colon K \longrightarrow B$ and $p_C\colon K \longrightarrow C$ the product projections. Let $k_i\colon B \longrightarrow K$, $i = 1$, 2, be the morphisms for which $p_B \circ k_i = \mathrm{id}_B$ and $p_C \circ k_i = c_i$. Then (k_1, k_2) is a very weak cokernel pair of f with respect to (c_1, c_2). Indeed, the equation $c_1 \circ f = c_2 \circ f$ implies $k_1 \circ f = k_2 \circ f$ by the uniqueness condition in the universal property of the product. In the notation of Definition 2.1, it remains to put $l = p_C$ and $s = p_B$.

We will say that a category C *has very weak cokernel pairs* if for any three morphisms $f\colon A \longrightarrow B$ and c_1, $c_2\colon B \rightrightarrows C$ such that $c_1 \circ f = c_2 \circ f$ in C there exists a very weak cokernel pair of f with respect to (c_1, c_2) in C.

2.3. Remark. By Example 2.2(2), any category with finite products has very weak cokernel pairs. In particular, any additive category has very weak cokernel pairs.

Notice, however, that an accessible additive category *need not* have weak cokernel pairs in general. For example, let R be an associative ring, and consider the additive category of flat left R-modules $\mathsf{A} = R\text{–}\mathsf{Mod}_{\mathsf{flat}}$. It is well known that the category A is finitely accessible.

Given an arbitrary left R-module M, pick a morphism of flat left R-modules $f\colon A \longrightarrow B$ such that M is the cokernel of f in the abelian category $R\text{–}\mathsf{Mod}$. Denote by $m\colon B \longrightarrow M$ the natural epimorphism in $R\text{–}\mathsf{Mod}$. Let k_1, $k_2\colon B \rightrightarrows K$ be a weak cokernel pair of f in A. Then we have $(k_2 - k_1) \circ f = 0$, hence there exists a morphism $e\colon M \longrightarrow K$ in $R\text{–}\mathsf{Mod}$ such that $k_2 - k_1 = e \circ m$. We claim that the morphism e is a flat preenvelope of M, in the sense of [14].

Indeed, let $g\colon M \longrightarrow L$ be a morphism from M to a flat left R-module L. Consider the pair of morphisms $l_1 = 0\colon B \longrightarrow L$ and $l_2 = g \circ m\colon B \longrightarrow L$. Then we have $l_1 \circ f = 0 = g \circ m \circ f = l_2 \circ f$. By assumption, there exists a morphism $h\colon K \longrightarrow L$ such that $l_1 = h \circ k_1$ and $l_2 = h \circ k_2$. Hence $g \circ m = l_2 - l_1 = h \circ (k_2 - k_1) = h \circ e \circ m$. As the morphism m is an epimorphism in $R\text{–}\mathsf{Mod}$, it follows that $g = h \circ e$. Thus the morphism g factorizes through e, as desired.

Conversely, if flat preenvelopes exist in $R\text{–}\mathsf{Mod}$, then all weak colimits exist in $\mathsf{A} = R\text{–}\mathsf{Mod}_{\mathsf{flat}}$. Indeed, given a diagram $F\colon D \longrightarrow \mathsf{A}$, denote by M the colimit of F is $R\text{–}\mathsf{Mod}$. Then any flat preenvelope of M is a weak colimit of F in A.

We have shown that weak cokernel pairs exist in A if and only if flat preenvelopes exist in $R\text{–}\mathsf{Mod}$. The latter property holds if and only if the ring R is right coherent [14, Proposition 5.1]. Taking a ring R that is *not* right coherent, we obtain an example of a finitely accessible additive category A without weak cokernel pairs.

2.4. Example. Here is an example of a preadditive but not additive category (i. e., a category enriched in abelian groups but not having finite products or finite coproducts) which does *not* even have very weak cokernel pairs. Let $\mathbb{k}$ be a field, $n \geq 1$ be an integer, $\mathbb{k}\text{–}\mathsf{Vect}$ be the category of $\mathbb{k}$-vector spaces, and $\mathsf{A} \subset \mathbb{k}\text{–}\mathsf{Vect}$ be the full subcategory of $\mathbb{k}$-vector spaces of finite dimension not exceeding n. For any nonnegative integer i, let $\mathbb{k}^i$ denote the $\mathbb{k}$-vector space of dimension i. Let $A = 0$, $B = \mathbb{k}^n$, and $C = \mathbb{k}^i \in \mathsf{A}$, where $0 < i \leq n$. Let $f\colon A \longrightarrow B$ be the zero morphism and c_1, $c_2\colon B \rightrightarrows C$ be a parallel pair of morphisms such that $c_1 = 0$ and $c_2 \neq 0$. Then, of course, $c_1 \circ f = c_2 \circ f$. However, the morphism f does *not* have a very weak cokernel pair with respect to (c_1, c_2). Indeed, assume for the sake of contradiction that k_1, $k_2\colon B \rightrightarrows K$ is such a very weak cokernel pair. Let $l\colon K \longrightarrow C$ and $s\colon K \longrightarrow B$ be the related morphisms. So k_1 is a split monomorphism, $s \circ k_1 = \mathrm{id}_B$. Since $B = \mathbb{k}^n$ and the category A contains no vector spaces of dimension greater than n, the morphism k_1 has to be an isomorphism. Then the equation $0 = c_1 = l \circ k_1$ implies $l = 0$, which makes the equation $0 \neq c_2 = l \circ k_2$ impossible to satisfy.

2.5. Lemma. *Let κ be a regular cardinal and A be a κ-accessible category with very weak cokernel pairs. Then the full subcategory $\mathsf{A}_{<\kappa} \subset \mathsf{A}$ of κ-presentable objects in A also has very weak cokernel pairs.*

Proof. Let $f\colon A \longrightarrow B$ and c_1, $c_2\colon B \rightrightarrows C$ be three morphisms in $\mathsf{A}_{<\kappa}$ such that $c_1 \circ f = c_2 \circ f$. Let x_1, $x_2\colon B \rightrightarrows X$ be a very weak cokernel pair of f with respect to (c_1, c_2)

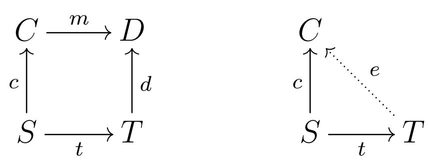
in the category A, and let $y\colon X \longrightarrow C$ and $t\colon X \longrightarrow B$ be the related morphisms. Let $X = \varinjlim_{\xi\in\Xi} K_\xi$ be a representation of X as a κ-directed colimit of κ-presentable objects K_ξ in A, indexed by a κ-directed poset Ξ. Denote by $w_{\eta\xi}\colon K_\xi \longrightarrow K_\eta$ (where $\xi \le \eta$ in Ξ) the transition morphisms and by $z_\xi\colon K_\xi \longrightarrow X$ the canonical morphisms into the colimit.

Since Ξ is κ-directed and B is κ-presentable, there exists an index $\xi \in \Xi$ such that both the morphisms x_1 and $x_2\colon B \rightrightarrows X$ factorize through the morphism z_ξ. So we have a parallel pair of morphisms k_1', $k_2'\colon B \rightrightarrows K_\xi$ such that $x_i = z_\xi \circ k_i'$ for $i = 1, 2$. By assumption, we have $x_1 \circ f = x_2 \circ f$, so $z_\xi \circ k_1' \circ f = z_\xi \circ k_2' \circ f$. Since Ξ is κ-directed and A is κ-presentable, there exists an index $\eta \in \Xi$, $\xi \le \eta$, such that $w_{\eta\xi} \circ k_1' \circ f = w_{\eta\xi} \circ k_2' \circ f$. Put $k_i = w_{\eta\xi} \circ k_i'$ for $i = 1, 2$, and $K = K_\eta$.

Then k_1, $k_2\colon B \rightrightarrows K$ is a very weak cokernel pair of f with respect to (c_1, c_2) in the category $\mathsf{A}_{<\kappa}$. Indeed, we have already seen that $k_1 \circ f = k_2 \circ f$. In the notation of Definition 2.1, it remains to put $l = y \circ z_\eta$ and $s = t \circ z_\eta$. ∎

3. Strongly Pure Monomorphisms

Let κ be a regular cardinal and A be a κ-accessible category. A morphism $m\colon C \longrightarrow D$ is said to be a *κ-pure monomorphism* [2, Definition 2.27] in A if, for every morphism $S \longrightarrow T$ in $\mathsf{A}_{<\kappa}$ and any commutative square diagram



(4)

in A, there exists a morphism $e\colon T \longrightarrow C$ making the lower triangle commutative.

Let $I = (\bullet \to \bullet)$ be the category with two objects and one nonidentity morphism (acting from one object of I to the other one). Given a category C, we denote by $\mathsf{C}^{\to} = \mathsf{C}^I$ the category of functors $I \longrightarrow \mathsf{C}$, i. e., the category of morphisms in C. So the objects of $\mathsf{C}^{\to}$ are all the morphisms in C, and the morphisms in $\mathsf{C}^{\to}$ are all the commutative squares in C.

3.1. Lemma. (a) *All split monomorphisms in A are κ-pure monomorphisms.*

(b) *All κ-pure monomorphisms are monomorphisms in A.*

(c) *The class of κ-pure monomorphisms is closed under compositions of morphisms in A.*

(d) *If i, j is a composable pair of morphisms in A and $i \circ j$ is a κ-pure monomorphism, then j is a κ-pure monomorphism.*

(e) *The class of κ-pure monomorphisms in A is closed under κ-directed colimits in $\mathsf{A}^{\to}$.*

Proof. All the assertions are well-known. Parts (a) and (c–d), mentioned in [2, Example 2.28(1) and Remarks 2.28(1–2)], are elementary. Part (b) is [2, Proposition 2.29]. In part (e), which is [2, Proposition 2.30(i)], one needs to use the fact that all the objects of $(\mathsf{A}_{<\kappa})^{\to}$ are κ-presentable in $\mathsf{A}^{\to}$. ∎

It follows from Lemma 3.1(a,e) that κ-directed colimits of split monomorphisms are κ-pure monomorphisms in A. The aim of this section is to provide a mild sufficient condition for the inverse implication. In fact, we will prove a little bit more.

Let us say that a morphism $m\colon C \longrightarrow D$ in A is a *strongly κ-pure monomorphism* if m is a κ-directed colimit in $\mathsf{A}^{\to}$ of split monomorphisms between κ-presentable objects in A.

3.2. Proposition. *A morphism m in A is a strongly κ-pure monomorphism if and only if any morphism into m from a morphism in $\mathsf{A}_{<\kappa}$ factorizes through a split monomorphism in $\mathsf{A}_{<\kappa}$ in the category $\mathsf{A}^{\to}$. In other words, m is a strongly κ-pure monomorphism if and only if, for any morphism t in $\mathsf{A}_{<\kappa}$ and any morphism $t \longrightarrow m$ in $\mathsf{A}^{\to}$ there exists a split monomorphism s in $\mathsf{A}_{<\kappa}$ such that the morphism $t \longrightarrow m$ factorizes as $t \longrightarrow s \longrightarrow m$ for some morphisms $t \longrightarrow s$ and $s \longrightarrow m$ in $\mathsf{A}^{\to}$.*

Proof. According to Proposition 1.4 for $D = I$ (or to Proposition 1.2 for $\mathsf{A} = \mathsf{B} = \mathsf{C}$), the category $\mathsf{A}^{\to}$ is κ-accessible, and the κ-presentable objects of $\mathsf{A}^{\to}$ are precisely all the morphisms between κ-presentable objects in A, that is $(\mathsf{A}^{\to})_{<\kappa} = (\mathsf{A}_{<\kappa})^{\to}$. (This observation can be found in [2, Exercise 2.c].) The desired assertion is now provided by Proposition 1.1 applied to the κ-accessible category $\mathsf{A}^{\to}$ and the set of κ-presentable objects T consisting of all the (representatives of isomorphism classes) of split monomorphisms in $\mathsf{A}_{<\kappa}$. ∎

3.3. Lemma. *The class of strongly κ-pure monomorphisms in A is closed under κ-directed colimits in $\mathsf{A}^{\to}$.*

Proof. This is another assertion from Proposition 1.1, applicable to the situation at hand as explained in the proof of Proposition 3.2. ∎

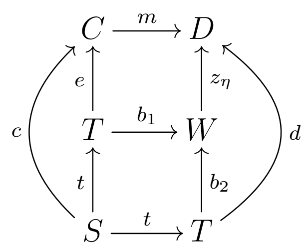
3.4. Theorem. *In any κ-accessible category A with very weak cokernel pairs, the classes of κ-pure monomorphisms and strongly κ-pure monomorphisms coincide. In other words, all κ-pure monomorphisms in A are κ-directed colimits of split monomorphisms between κ-presentable objects of A.*

Proof. Let $m\colon C \longrightarrow D$ be a κ-pure monomorphism in A, and let $t\colon S \longrightarrow T$ be a morphism in $\mathsf{A}_{<\kappa}$. Suppose we are given a morphism $t \longrightarrow m$ in $\mathsf{A}^{\to}$; this means a commutative square diagram as in (4). In view of Proposition 3.2, we need to prove that the morphism $(c,d)\colon t \longrightarrow m$ factorizes through some split monomorphism $u\colon U \longrightarrow V$ in $\mathsf{A}_{<\kappa}$, viewed as an object of $\mathsf{A}^{\to}$.

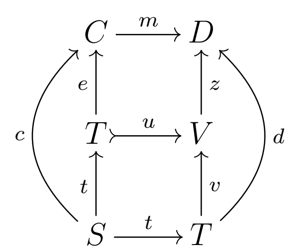
The following argument is a nonadditive version of [25, proof of Lemma 4.3]. By assumption, there exists a lifting $e\colon T \longrightarrow C$ such that $c = e \circ t$, as on the triangular diagram in (4). Consider the parallel pair of morphisms $m \circ e$, $d\colon T \rightrightarrows D$. We have $m \circ e \circ t = m \circ c = d \circ t$.

Let $D = \varinjlim_{\xi\in\Xi} W_\xi$ be a representation of D as a κ-directed colimit of κ-presentable objects W_ξ, indexed by a κ-directed poset Ξ. Denote by $w_{\eta\xi}\colon W_\xi \longrightarrow W_\eta$ the transition morphisms (for ξ, $\eta \in \Xi$, $\xi \le \eta$) and by $z_\xi\colon W_\xi \longrightarrow D$ the canonical morphisms to the colimit. Since $T \in \mathsf{A}_{<\kappa}$, there exists an index $\xi \in \Xi$ such that both the morphisms $m \circ e$

and $d\colon T \rightrightarrows D$ factorize through the morphism $z_\xi\colon W_\xi \longrightarrow D$. So we have a parallel pair of morphisms b'_1, $b'_2\colon T \rightrightarrows W_\xi$ such that $m \circ e = z_\xi \circ b'_1$ and $d = z_\xi \circ b'_2$. Now $z_\xi \circ b'_1 \circ t = m \circ e \circ t = d \circ t = z_\xi \circ b'_2 \circ t$. Since $S \in \mathsf{A}_{<\kappa}$, there exists an index $\eta \in \Xi$, $\xi \le \eta$, such that $w_{\eta\xi} \circ b'_1 \circ t = w_{\eta\xi} \circ b'_2 \circ t$. Put $b_i = w_{\eta\xi} \circ b'_i$, $i = 1$, 2, and $W = W_\eta$. We have constructed a parallel pair of morphisms b_1, $b_2\colon T \rightrightarrows W$ such that $m \circ e = z_\eta \circ b_1$ and $d = z_\eta \circ b_2$, where $W \in \mathsf{A}_{<\kappa}$ and $z_\eta\colon W \longrightarrow D$. Furthermore, we have $b_1 \circ t = b_2 \circ t$. So we obtain a commutative diagram



Finally, by assumption, very weak cokernel pairs exist in A, and by Lemma 2.5 it follows that very weak cokernel pairs exist in $\mathsf{A}_{<\kappa}$ as well. Let k_1, $k_2\colon T \rightrightarrows V$ be a very weak cokernel pair of the morphism $t\colon S \longrightarrow T$ with respect to the parallel pair of morphisms b_1, $b_2\colon T \rightrightarrows W$ in the category $\mathsf{A}_{<\kappa}$. Let $l\colon V \longrightarrow W$ and $s\colon V \longrightarrow T$ be the related morphisms, as per Definition 2.1. Put $u = k_1$ and $v = k_2$, and also $z = z_\eta l$ and $U = T$. So, in particular, u is a split monomorphism, $s \circ u = \mathrm{id}_T$. We have arrived to a commutative diagram



providing the desired factorization $t \longrightarrow u \longrightarrow m$, in the category $\mathsf{A}^{\to}$, of the morphism $(c, d)\colon t \longrightarrow m$ through a split monomorphism of κ-presentable objects $u\colon U = T \longrightarrow V$ in the category A. ∎

3.5. Corollary. *In any κ-accessible category A with finite products, the classes of κ-pure monomorphisms and strongly κ-pure monomorphisms coincide. In other words, all κ-pure monomorphisms in A are κ-directed colimits of split monomorphisms between κ-presentable objects of A.*

Proof. This is a particular case of Theorem 3.4, in view of Example 2.2(2). ∎

4. Very Weak Split Pullbacks

Let C be a category. Dually to the discussion of weak colimits in Section 2, one defines the notion of a *weak limit* of a diagram in C.

A diagram of the shape

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ & & \uparrow{\scriptstyle g} \\ & & C \end{array} \tag{5}$$

is called a *cospan*. The limits of cospans are called the *pullbacks*, and accordingly weak limits of cospans are called weak pullbacks.

We will say that a cospan (5) is *split* if a morphism $h\colon C \longrightarrow B$ exists making the triangular diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ & \nwarrow{\scriptstyle h} & \uparrow{\scriptstyle g} \\ & & C \end{array} \tag{6}$$

commutative. We will say that *weak split pullbacks exist* in a category C if all split cospans have weak pullbacks.

Let P be a weak pullback of a split cospan (6), and let $p_B\colon P \longrightarrow B$ and $p_C\colon P \longrightarrow C$ be the canonical morphisms from the weak limit. Then it is clear from the definitions that there exists a morphism $s\colon C \longrightarrow P$ such that $p_B \circ s = h$ and $p_C \circ s = \mathrm{id}_C$. Therefore, the morphism p_C is a split epimorphism.

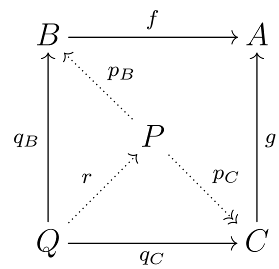
4.1. Definition. Suppose we are given a split cospan diagram (6) in C together with a commutative square as on the diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ & \nwarrow{\scriptstyle h} & \uparrow{\scriptstyle g} \\ & & C \end{array} \qquad\qquad \begin{array}{ccc} B & \xrightarrow{f} & A \\ {\scriptstyle q_B}\uparrow & & \uparrow{\scriptstyle g} \\ Q & \xrightarrow[q_C]{} & C \end{array} \tag{7}$$

We will say that an object $P \in \mathsf{C}$ together with a pair of morphisms $p_B\colon P \longrightarrow B$ and $p_C\colon P \longrightarrow C$ is a *very weak split pullback* of (f, g) *with respect to* (q_B, q_C) if the following three conditions hold:

- one has $f \circ p_B = g \circ p_C$;
- there exists a morphism $r\colon Q \longrightarrow P$ such that $q_B = p_B \circ r$ and $q_C = p_C \circ r$;
- the morphism $p_C\colon P \longrightarrow C$ is a split epimorphism (i. e., there exists a morphism $s\colon C \longrightarrow P$ such that $p_C \circ s = \mathrm{id}_C$).

The commutative diagram described in Definition 4.1 can be drawn as



(8)

Here the splitting s of the split epimorphism p_C is not depicted on the diagram (8); instead, the condition that p_C is a split epimorphism is expressed by the double head at the end of the dotted arrow showing p_C.

4.2. Remark. Dually to cospans, a diagram of the shape (2) is called a *span*. Notice that any span consisting of two equal morphisms, $B = C$ and $f = g$, is a split span (in the sense dual to our definition of a split cospan in this section). For this reason, using the terminology *very weak split pushouts* for the notion dual to very weak split pullbacks, one can observe that the very weak cokernel pairs from Definition 2.1 are a special case of very weak split pushouts.

4.3. Examples. (1) If a split cospan $f\colon B \longrightarrow A$, $g\colon C \longrightarrow A$ has a weak pullback $p_B\colon P \longrightarrow B$, $p_C\colon P \longrightarrow C$, then (p_B, p_C) is a very weak split pullback of (f, g) with respect to any pair of morphisms (q_B, q_C) such that $f \circ q_B = g \circ q_C$. This is clear from the discussion above. In this sense, our terminology is consistent.

(2) Suppose we are given a commutative triangle and a commutative square as in (7). Assume that the coproduct $P = Q \sqcup C$ exists in C, and denote by $i_Q\colon Q \longrightarrow P$ and $i_C\colon C \longrightarrow P$ the coproduct injections. Let $p_B\colon P \longrightarrow B$ be the morphism for which $p_B \circ i_Q = q_B$ and $p_B \circ i_C = h$, and let $p_C\colon P \longrightarrow C$ be the morphism for which $p_C \circ i_Q = q_C$ and $p_C \circ i_C = \mathrm{id}_C$. Then (p_B, p_C) is a very weak split pullback of (f, g) with respect to (q_B, q_C). Indeed, the equations $f \circ q_B = g \circ q_C$ and $f \circ h = g$ imply $f \circ p_B = g \circ p_C$ by the uniqueness condition in the universality property of the coproduct. In the notation of Definition 4.1, it remains to put $r = i_Q$ and $s = i_C$.

We will say that a category C *has very weak split pullbacks* if for every commutative triangle and commutative square as in (7) there exists a very weak split pullback of (f, g) with respect to (q_B, q_C) in C.

4.4. Remark. By Example 4.3(2), any category with finite coproducts has very weak split pullbacks. In particular, any additive category has very weak split pullbacks.

Notice, however, that an accessible additive category *need not* have weak split pullbacks in general. For example, dually to Remark 2.3, let R be an associative ring, and consider the additive category of injective left R-modules $\mathsf{A} = R\text{–}\mathsf{Mod}_{\mathsf{inj}}$. It is well known that the category A is accessible. In fact, A is κ-accessible whenever λ is a regular cardinal such every left ideal in R has less than λ generators, ν is any infinite cardinal greater than or equal to the cardinality of R such that the set $\nu^{<\lambda}$ of all subsets of ν of the cardinality smaller than λ has cardinality equal to ν, that is $\nu^{<\lambda} = \nu$, and $\kappa = \nu^+$ is the successor cardinal of ν (see, e. g., [26, Corollary 3.7]).

Given an arbitrary left R-module M, pick a morphism of injective left R-modules $f\colon B \longrightarrow A$ such that M is the kernel of f in the abelian category R–Mod. Denote by $m\colon M \longrightarrow B$ the natural monomorphism in R–Mod. Put $C = 0$, and let $g\colon C \longrightarrow A$ be the zero morphism. Then (f, g) is obviously a split cospan in A. Let $p_B\colon P \longrightarrow B$ and $p_C\colon P \longrightarrow C$ be a weak pullback of (f, g) in A. Then we have $f \circ p_B = 0$, hence there exists a morphism $c\colon P \longrightarrow M$ in R–Mod such that $p_B = m \circ c$. We claim that the morphism c is an injective precover of M, in the sense of [14].

Indeed, let $k\colon Q \longrightarrow M$ be a morphism into M from an injective left R-module Q. Consider the pair of morphisms $q_B = m \circ k\colon Q \longrightarrow B$ and $q_C = 0\colon Q \longrightarrow C$. Then we have $f \circ q_B = f \circ m \circ k = 0 = g \circ q_C$. By assumption, there exists a morphism $r\colon Q \longrightarrow P$ such that $q_B = p_B \circ r$. Hence $m \circ k = q_B = p_B \circ r = m \circ c \circ r$. As the morphism m is a monomorphism in R–Mod, it follows that $k = c \circ r$. Thus the morphism k factorizes through c, as desired.

Conversely, if injective precovers exist in R–Mod, then all weak limits exist in $\mathsf{A} = R$–$\mathsf{Mod}_{\mathsf{inj}}$. Indeed, given a diagram $F\colon D \longrightarrow \mathsf{A}$, denote by M the limit of F in R–Mod. Then any injective precover of M is a weak limit of F in A.

We have shown that weak split pullbacks exist in A if and only if injective precovers exist in R–Mod. The latter property holds if and only if the ring R is left Noetherian [14, Propositions 2.1 and 2.2]. Taking a ring R that is *not* left Noetherian, we obtain an example of an accessible additive category A without weak split pullbacks.

4.5. Example. Dually to Example 2.4, the preadditive category A of $\Bbbk$-vector spaces of dimension not exceeding n (where $n \ge 1$ is a fixed integer) does *not* have very weak split pullbacks. Specifically, put $A = 0$, $B = \Bbbk^j$ for some $0 < j \le n$, $C = \Bbbk^n$, and $Q = \Bbbk^i$ for some $0 < i \le n$. Let $f\colon B \longrightarrow A$ and $g\colon C \longrightarrow A$ be the zero morphisms, and let $q_B\colon Q \longrightarrow B$ and $q_C\colon Q \longrightarrow C$ be any morphisms such that $q_C = 0$ and $q_B \ne 0$. Then $f \circ q_B = g \circ q_C$, but the split cospan (f, g) has no very weak split pullback with respect to (q_B, q_C) in A. Indeed, if $p_C\colon P \longrightarrow C$ is a split epimorphism in A, then p_C is an isomorphism, so the equation $0 = q_C = p_C \circ r$ implies $r = 0$, which is incompatible with $0 \ne q_B = p_B \circ r$.

4.6. Lemma. *Let κ be a regular cardinal and A be a κ-accessible category with very weak split pullbacks. Then the full subcategory $\mathsf{A}_{<\kappa} \subset \mathsf{A}$ of κ-presentable objects in A also has very weak split pullbacks.*

Proof. Suppose we are given a commutative triangle and a commutative square (7) in the category $\mathsf{A}_{<\kappa}$. Let $x_B\colon X \longrightarrow B$ and $x_C\colon X \longrightarrow C$ be a very weak split pullback of (f, g) with respect to (q_B, q_C) in the category A, and let $y\colon Q \longrightarrow X$ and $t\colon C \longrightarrow X$ be the related morphisms. Let $X = \varinjlim_{\xi\in\Xi} P_\xi$ be a representation of X as a κ-directed colimit of κ-presentable objects P_ξ in A, indexed by a κ-directed poset Ξ. Denote by $z_\xi\colon P_\xi \longrightarrow X$ the canonical morphisms into the colimit.

Since Ξ is κ-directed and Q and C are κ-presentable, there exists an index $\xi \in \Xi$ such that both the morphisms y and t factorize through the morphism z_ξ. So we have morphisms $r\colon Q \longrightarrow P_\xi$ and $s\colon C \longrightarrow P_\xi$ such that $y = z_\xi \circ r$ and $t = z_\xi \circ s$. Put $p_B = x_B \circ z_\xi$ and $p_C = x_C \circ z_\xi$, and $P = P_\xi$.

Then $p_B\colon P \longrightarrow B$ and $p_C\colon P \longrightarrow C$ is a very weak split pullback of (f, g) with respect to (q_B, q_C) in the category $\mathsf{A}_{<\kappa}$. Indeed, we have $f \circ p_B = f \circ x_B \circ z_\xi = g \circ x_C \circ z_\xi = g \circ p_C$. Furthermore, $p_B \circ r = x_B \circ z_\xi \circ r = x_B \circ y = q_B$ and $p_C \circ r = x_C \circ z_\xi \circ r = x_C \circ y = q_C$. Finally, $p_C \circ s = x_C \circ z_\xi \circ s = x_C \circ t = \mathrm{id}_C$. ■

5. Strongly Pure Epimorphisms

Let κ be a regular cardinal and A be a κ-accessible category. A morphism $p\colon D \longrightarrow E$ is said to be a *κ-pure epimorphism* [3, Definition 1] in A if for every κ-presentable object S and any morphism $e\colon S \longrightarrow E$ in A there exists a morphism $l\colon S \longrightarrow D$ making the triangular diagram

$$\begin{array}{ccc} D & \xrightarrow{\ p\ } & E \\ & \nwarrow{\scriptstyle l} & \uparrow{\scriptstyle e} \\ & & S \end{array} \qquad (9)$$

commutative.

We refer to Section 3 for the notation $\mathsf{C}^{\to} = \mathsf{C}^{I}$ for a category C.

5.1. Lemma. (a) *All split epimorphisms in A are κ-pure epimorphisms.*

(b) *All κ-pure epimorphisms are epimorphisms in A.*

(c) *The class of κ-pure epimorphisms is closed under compositions of morphisms in A.*

(d) *If p, q is a composable pair of morphisms in A and $p \circ q$ is a κ-pure epimorphism, then p is a κ-pure epimorphism.*

(e) *The class of κ-pure epimorphisms in A is closed under κ-directed colimits in $\mathsf{A}^{\to}$.*

Proof. All the assertions are well-known and easy to prove. Parts (a) and (e) are [3, Example 2(a–b)]. Part (b) is [3, Proposition 4(a)]. ■

It follows from Lemma 5.1(a,e) that κ-directed colimits of split epimorphisms are κ-pure epimorphisms in A. The aim of this section is to provide a mild sufficient condition for the inverse implication. In fact, we will prove a little bit more.

Let us say that a morphism $p\colon D \longrightarrow E$ in A is a *strongly κ-pure epimorphism* if p is a κ-directed colimit in $\mathsf{A}^{\to}$ of split epimorphisms between κ-presentable objects in A.

5.2. Proposition. *A morphism p in A is a strongly κ-pure epimorphism if and only if any morphism into p from a morphism in $\mathsf{A}_{<\kappa}$ factorizes through a split epimorphism in $\mathsf{A}_{<\kappa}$ in the category $\mathsf{A}^{\to}$. In other words, p is a strongly κ-pure epimorphism if and only if, for any morphism t in $\mathsf{A}_{<\kappa}$ and any morphism $t \longrightarrow p$ in $\mathsf{A}^{\to}$ there exists a split epimorphism s in $\mathsf{A}_{<\kappa}$ such that the morphism $t \longrightarrow m$ factorizes as $t \longrightarrow s \longrightarrow p$ for some morphisms $t \longrightarrow s$ and $s \longrightarrow p$ in $\mathsf{A}^{\to}$.*

PROOF. Similar to the proof of Proposition 3.2. ∎

5.3. LEMMA. *The class of strongly κ-pure epimorphisms in* A *is closed under κ-directed colimits in* $\mathsf{A}^{\to}$.

PROOF. Similar to the proof of Lemma 3.3. ∎

5.4. THEOREM. *In any κ-accessible category* A *with very weak split pullbacks, the classes of κ-pure epimorphisms and strongly κ-pure epimorphisms coincide. In other words, all κ-pure epimorphisms in* A *are κ-directed colimits of split epimorphisms between κ-presentable objects of* A.

PROOF. Let $p\colon D \longrightarrow E$ be a κ-pure epimorphism in A, and let $t\colon T \longrightarrow S$ be a morphism in $\mathsf{A}_{<\kappa}$. Suppose we are given a morphism $t \longrightarrow p$ in $\mathsf{A}^{\to}$; this means a commutative square diagram

$$\begin{array}{ccc} D & \xrightarrow{p} & E \\ {\scriptstyle d}\uparrow & & \uparrow{\scriptstyle e} \\ T & \xrightarrow[t]{} & S \end{array} \tag{10}$$

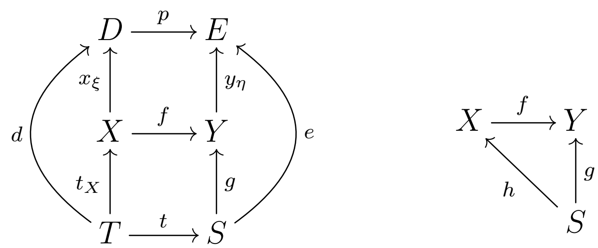
In view of Proposition 5.2, we need to prove that the morphism $(d, e)\colon t \longrightarrow p$ factorizes through some split epimorphism $u\colon U \longrightarrow V$ in $\mathsf{A}_{<\kappa}$, viewed as an object of $\mathsf{A}^{\to}$.

The following argument is a nonadditive version of [25, proofs of Lemmas 1.5, 2.3, and 4.1, and Proposition 4.2]. By assumption, there exists a lifting $l\colon S \longrightarrow D$ such that $e = p \circ l$, as on diagram (9). In other words, this means that the pair of morphisms (p, e) is a split cospan in A.

Following the proof of Proposition 3.2 or [2, Exercise 2.c], any morphism in A, viewed as an object of $\mathsf{A}^{\to}$, is a κ-directed colimit of morphisms between κ-presentable objects. Let $p = \varinjlim_{\xi\in\Xi} w_\xi$ be a representation of the morphism $p\colon D \longrightarrow E$ as a κ-directed colimit of morphisms $w_\xi\colon X_\xi \longrightarrow Y_\xi$, with κ-presentable objects X_ξ and Y_ξ, indexed by a κ-directed poset Ξ. Denote by $x'_{\eta\xi}\colon X_\xi \longrightarrow X_\eta$ and $y'_{\eta\xi}\colon Y_\xi \longrightarrow Y_\eta$ the components of the transition morphisms $(x'_{\eta\xi}, y'_{\eta\xi})$, for all $\xi, \eta \in \Xi$, $\xi \le \eta$. Denote also by $x_\xi\colon X_\xi \longrightarrow D$ and $y_\xi\colon Y_\xi \longrightarrow E$ the components of the canonical morphisms to the colimit $(x_\xi, y_\xi)\colon w_\xi \longrightarrow p$.

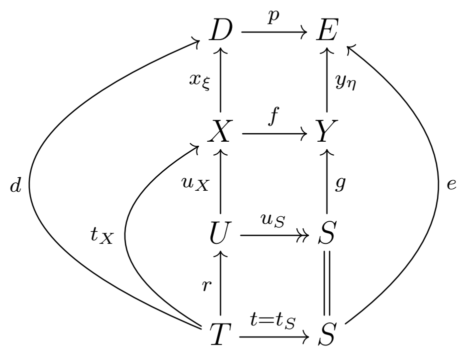
Since Ξ is κ-directed and T and S are κ-presentable, there exists an index $\xi \in \Xi$ such that both the morphisms d and l factorize through x_ξ, while the morphism e factorizes through y_ξ. So we have morphisms $t_X\colon T \longrightarrow X_\xi$, $h\colon S \longrightarrow X_\xi$, and $g'\colon S \longrightarrow Y_\xi$ such that $d = x_\xi \circ t_X$, $l = x_\xi \circ h$, and $e = y_\xi \circ g'$. Hence $y_\xi \circ w_\xi \circ t_X = p \circ x_\xi \circ t_X = p \circ d = e \circ t = y_\xi \circ g' \circ t$ and $y_\xi \circ w_\xi \circ h = p \circ x_\xi \circ h = p \circ l = e = y_\xi \circ g'$. Since Ξ is κ-directed and T and S are κ-presentable, there exists an index $\eta \in \Xi$, $\xi \le \eta$, such that $y'_{\eta\xi} \circ w_\xi \circ t_X = y'_{\eta\xi} \circ g' \circ t$ and $y'_{\eta\xi} \circ w_\xi \circ h = y'_{\eta\xi} \circ g'$.

Put $X = X_\xi$, $Y = Y_\eta$, $f = y'_{\eta\xi} \circ w_\xi\colon X \longrightarrow Y$, and $g = y'_{\eta\xi} \circ g'\colon S \longrightarrow Y$. Then we have commutative diagrams in A



with objects S, T, X, $Y \in \mathsf{A}_{<\kappa}$. Now the pair of morphisms (f, g) is a split cospan in $\mathsf{A}_{<\kappa}$. For convenience, put $t_S = t$.

Finally, by assumption, very weak split pullbacks exist in A, and by Lemma 4.6 it follows that very weak split pullbacks exist in $\mathsf{A}_{<\kappa}$ as well. Let $u_X : U \longrightarrow X$ and $u_S : U \longrightarrow S$ be a very weak split pullback of (f, g) with respect to (t_X, t_S) in the category $\mathsf{A}_{<\kappa}$. Let $r : T \longrightarrow U$ and $s : S \longrightarrow U$ be the related morphisms, as per Definition 4.1. So, in particular, u_S is a split epimorphism, $u_S \circ s = \mathrm{id}_S$. We have arrived to a commutative diagram



providing the desired factorization $t \longrightarrow u \longrightarrow p$, in the category $\mathsf{A}^{\to}$, of the morphism $(d, e) : t \longrightarrow p$ through a split epimorphism of κ-presentable objects $u = u_S : U \longrightarrow V = S$ in the category A. ∎

5.5. Corollary. *In any κ-accessible category A with finite coproducts, the classes of κ-pure epimorphisms and strongly κ-pure epimorphisms coincide. In other words, all κ-pure epimorphisms in A are κ-directed colimits of split epimorphisms between κ-presentable objects of A.*

Proof. This is a particular case of Theorem 5.4, in view of Example 4.3(2). ∎

5.6. Remark. The additional assumptions in Theorems 3.4 and 5.4 (on top of the assumption that A is a κ-accessible category) may be mild, but they *cannot* be completely dropped. Indeed, the counterexample in [4, Example 2.5] shows that a pure (i. e., $\aleph_0$-pure) monomorphism in a finitely accessible (i. e., $\aleph_0$-accessible) category A need not be a colimit of split monomorphisms in general.

A similar construction provides an example showing that a pure epimorphism in a finitely accessible category A need not be a directed colimit of split epimorphisms, generally speaking. In order to obtain the desired counterexample, it suffices to modify [4, Example 2.5] as follows. In the notation of [2, Example 2.5], when freely adding morphisms $e_i : B_i \longrightarrow A_{i+1}$, instead of imposing the relations $e_i m_i = a_{i,i+1}$, impose the relations $m_{i+1} e_i = b_{i,i+1}$ for all $i = 0, 1, 2, \ldots$ Then the morphism $m = \varinjlim_i m_i : \varinjlim_i A_i \longrightarrow \varinjlim_i B_i$ becomes a pure epimorphism that is not a colimit of split epimorphisms (in fact, all split epimorphisms are isomorphisms in the resulting finitely accessible category A).

6. QE-Mono Classes

Let C be a category. A morphism $m : C \longrightarrow D$ in C is called a *regular monomorphism* if m is the equalizer of a parallel pair of morphisms $e_1, e_2 : D \rightrightarrows E$.

Clearly, every regular monomorphism is a monomorphism. Every split monomorphism is regular: if $s : D \longrightarrow C$ is a morphism such that $s \circ m = \mathrm{id}_C$, then m is the equalizer of the pair of morphisms $m \circ s$ and $\mathrm{id}_D : D \rightrightarrows D$.

A morphism $m : C \longrightarrow D$ is said to be an *effective monomorphism* if m has a cokernel pair $k_1, k_2 : D \rightrightarrows K$ in C and m is the equalizer of (k_1, k_2). One can easily see that if a regular monomorphism m has a cokernel pair (k_1, k_2), then m is the equalizer of (k_1, k_2). So a monomorphism is effective if and only if it is regular *and* has a cokernel pair.

6.1. Lemma. *For any κ-accessible category A, the class of effective monomorphisms is closed under κ-directed colimits in $\mathsf{A}^{\to}$.*

Proof. In any category, colimits commute with colimits; in particular, the κ-directed colimits preserve cokernel pairs. So, if a morphism m is a κ-directed colimit of effective monomorphisms m_ξ, then the colimit of the cokernel pairs of m_ξ is the cokernel pair of m. Here we are assuming that the index ξ ranges over a κ-directed poset Ξ. By Lemma 1.8, in a κ-accessible category, the κ-directed colimits commute with κ-small limits; in particular, the κ-directed colimits preserve equalizers. Thus m is the equalizer of its cokernel pair. ∎

Let $\mathcal{M}$ be a class of morphisms in a category C. We will say that $\mathcal{M}$ is a *QE-mono class* in C if the following conditions are satisfied:

i. All pushouts of all morphisms from $\mathcal{M}$ exists in C, and the class $\mathcal{M}$ is stable under pushouts. In other words, for any span (i. e., a pair of morphisms with common domain) $m : C \longrightarrow D$, $f : C \longrightarrow C'$ such that $m \in \mathcal{M}$, the pushout D' exists, and the morphism $m' : C' \longrightarrow D'$ belongs to $\mathcal{M}$,

$$\begin{array}{ccc} C' & \xrightarrow{m'} & D' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ C & \xrightarrow[m]{} & D \end{array} \qquad (11)$$

ii. In particular, condition (i) implies that all morphisms from $\mathcal{M}$ have cokernel pairs in C. It is further required that every morphism from $\mathcal{M}$ is the equalizer of its cokernel pair. In other words, all the morphisms from $\mathcal{M}$ must be effective monomorphisms.

iii. All the identity morphisms in C belong to $\mathcal{M}$, and the class of morphisms $\mathcal{M}$ is closed under compositions.

It is clear from the preceding discussion that, assuming condition (i), condition (ii) is equivalent to the condition that all morphisms from $\mathcal{M}$ are regular monomorphisms in C.

6.2. Examples. (1) If C is an additive category, then a QE-mono class of morphisms in C is the same thing as a structure of *right exact category* on C in the sense of [9, Definition 3.1]. In the terminology of [16, Definition 2.2], such additive categories with an additional structure are called *inflation-exact categories*.

(2) If the category C^{op} opposite to C is regular (in the sense of [7, 15]), then the class of all regular monomorphisms in C is a QE-mono class. See Example 8.2(2) below.

(3) The compositions of split monomorphisms are always split monomorphisms, and all split monomorphisms are regular (as explained above). Furthermore, if the morphism m on the pushout diagram (11) is a split monomorphism with a splitting $s\colon D \longrightarrow C$, $s \circ m = \mathrm{id}_C$, then, by the definition of a pushout, there exists a unique morphism $s'\colon D' \longrightarrow C'$ such that $s' \circ m' = \mathrm{id}_{C'}$ and $s' \circ f' = f \circ s$ (because $\mathrm{id}_{C'} \circ f = f = f \circ s \circ m$; cf. the discussion of split pullbacks in Section 4). So all pushouts of split monomorphisms are split monomorphisms.

Therefore, the class $\mathcal{M}$ of all split monomorphisms in a category C is a QE-mono class if and only all pushouts of split monomorphisms exist in C.

6.3. Remark. One can easily see that all pushouts of split monomorphisms exist in an additive category if and only if the category is *weakly idempotent-complete* (in the sense of [10, Section 7]). However, the following simple example shows that a *preadditive* category that is idempotent-complete (in the sense of [10, Section 6]; or which is the same, has split idempotents in the sense of [2, Observation 2.4]) still need not have pushouts of split monomorphisms.

Let $\Bbbk$ be a field, $n \geq 1$ be an integer, $\Bbbk\text{–}\mathsf{Vect}$ be the category of $\Bbbk$-vector spaces, and $\mathsf{A} \subset \Bbbk\text{–}\mathsf{Vect}$ be the full subcategory of $\Bbbk$-vector spaces of finite dimension not exceeding n. For any nonnegative integer i, let $\Bbbk^i$ denote the $\Bbbk$-vector space of dimension i. Then the pair of split monomorphisms (actually, direct summand injections) $b\colon A = \Bbbk^{n-1} \longrightarrow \Bbbk^n = B$ and $c\colon A = \Bbbk^{n-1} \longrightarrow \Bbbk^n = C$ does *not* have a pushout in A. The pushout of b and c in $\Bbbk\text{–}\mathsf{Vect}$ is isomorphic to $\Bbbk^{n+1}$, which does not belong to A. Morever, the pair of morphisms b and c does not have a weak pushout in A; so weak cokernel pairs do not exist in A. In fact, the morphism $b = c$ does not even have a very weak cokernel pair in A; cf. Example 2.4.

Let $\mathcal{M}$ be a QE-mono class in a category C. For every morphism $m\colon C \longrightarrow D$ belonging to $\mathcal{M}$, consider its cokernel pair k_1, $k_2\colon D \rightrightarrows K$. By an $\mathcal{M}$-*sequence* we mean a diagram

$$C \xrightarrow{m} D \underset{k_2}{\overset{k_1}{\rightrightarrows}} K$$

arising from some morphism $m \in \mathcal{M}$ in this way.

Denote by J the category with three objects 1, 2, and 3, and four nonidentity morphisms $1 \longrightarrow 2$, $2 \rightrightarrows 3$, and $1 \longrightarrow 3$. So both the compositions $1 \longrightarrow 2 \rightrightarrows 3$ are equal to one and the same morphism $1 \longrightarrow 3$. Given a category C, we are interested in the category of diagrams C^J. In particular, for any QE-mono class $\mathcal{M}$ in C, the $\mathcal{M}$-sequences form a subclass of objects of C^J.

Let A be a κ-accessible category. We will say that a QE-mono class $\mathcal{M}$ in A is *locally κ-coherent* if the $\mathcal{M}$-sequences in A are precisely all the κ-directed colimits of $\mathcal{M}$-sequences in A with all the three terms C, D, K belonging to $\mathsf{A}_{<\kappa}$. Here the κ-directed colimits are taken in the diagram category A^J. The terminology "locally κ-coherent" comes from the paper [25, Section 1].

6.4. Lemma. *For any locally κ-coherent QE-mono class $\mathcal{M}$ in a κ-accessible category A, the class of all $\mathcal{M}$-sequences is closed under κ-directed colimits in A^J.*

Proof. By Proposition 1.4, the category A^J is κ-accessible, and its κ-presentable objects are precisely all the J-shaped diagrams in $\mathsf{A}_{<\kappa}$. Now the desired assertion follows from Proposition 1.1 applied to the κ-accessible category A^J. ∎

6.5. Lemma. *A QE-mono class $\mathcal{M}$ in a κ-accessible category A is locally κ-coherent if and only if $\mathcal{M}$ is precisely the class of all κ-directed colimits of the morphisms from $\mathcal{M}$ whose domains and codomains are κ-presentable. Here the κ-directed colimits are taken in the category of morphisms $\mathsf{A}^{\to}$.*

Proof. The point is that the full subcategory of κ-presentable objects $\mathsf{A}_{<\kappa}$ is closed under finite colimits in A (meaning those finite colimits that exist in A); see Lemma 1.5. In particularly, $\mathsf{A}_{<\kappa}$ is closed in A under the cokernel pairs of those morphisms that have cokernel pairs in A. So, in the notation above, if C, $D \in \mathsf{A}_{<\kappa}$, then $K \in \mathsf{A}_{<\kappa}$. Furthermore, all existing colimits commute with all existing colimits in any category; in particular, κ-directed colimits preserve cokernel pairs in A. ∎

6.6. Proposition. *Let $\mathcal{M}$ be a locally κ-coherent QE-mono class in a κ-accessible category A. Then the class $\mathcal{M} \cap \mathsf{A}_{<\kappa}^{\to}$ of all morphisms from $\mathcal{M}$ with κ-presentable domains and codomains is a QE-mono class in the category $\mathsf{A}_{<\kappa}$. Furthermore, the class $\mathcal{M} \cap \mathsf{A}_{<\kappa}^{\to}$ is closed under retracts in the category $\mathsf{A}_{<\kappa}^{\to} = (\mathsf{A}^{\to})_{<\kappa} = (\mathsf{A}_{<\kappa})^{\to}$.*

Proof. Condition (iii) obviously holds for $\mathcal{M} \cap \mathsf{A}_{<\kappa}^{\to}$ whenever it holds for $\mathcal{M}$. Condition (i) holds for $\mathcal{M} \cap \mathsf{A}_{<\kappa}^{\to}$ whenever it holds for $\mathcal{M}$, because the full subcategory $\mathsf{A}_{<\kappa}$ is closed in A under all the pushouts that exist in A (by Lemma 1.5). Finally, condition (ii) holds for $\mathcal{M} \cap \mathsf{A}_{<\kappa}^{\to}$ whenever conditions (i) and (ii) hold for $\mathcal{M}$, because any diagram in $\mathsf{A}_{<\kappa}$ that is an equalizer diagram in A is also an equalizer diagram in $\mathsf{A}_{<\kappa}$. These arguments do not even use the assumption of local κ-coherence of the QE-mono class $\mathcal{M}$

in A. The class $\mathcal{M}\cap\mathsf{A}^{\to}_{<\kappa}$ is closed under retracts in $\mathsf{A}^{\to}_{<\kappa}$ because retracts are special cases of κ-directed colimits in $\mathsf{A}^{\to}$, see [2, Observation 2.4]. ■

7. Construction of Locally Coherent QE-Mono Classes

Let D be the finite category

$$\begin{array}{ccc} 3 & & \\ \uparrow & & \\ 1 & \longrightarrow & 2 \end{array}$$

i. e., the category with three objects 1, 2, 3 and two nonidentity morphisms $1\longrightarrow 2$ and $1\longrightarrow 3$. By the *category of spans* in a category C we mean the category of D-shaped diagrams in C, that is, the category of functors C^D.

7.1. Theorem. *Let A be a κ-accessible category and $\mathcal{N}$ be a QE-mono class in the category $\mathsf{A}_{<\kappa}$. Then the class $\varinjlim_{(\kappa)}\mathcal{N}\subset\mathsf{A}^{\to}$ of all κ-directed colimits of morphisms from $\mathcal{N}$ (the colimits being taken in $\mathsf{A}^{\to}$) is a locally κ-coherent QE-mono class in the category A.*

Proof. Put $\mathcal{M}=\varinjlim_{(\kappa)}\mathcal{N}$. In order to check that condition (i) for the class $\mathcal{N}$ in $\mathsf{A}_{<\kappa}$ implies condition (i) for the class $\mathcal{M}$ in A, let us show that all spans (m,f) in A (where $m\colon C\longrightarrow D$ and $f\colon C\longrightarrow C'$) such that $m\in\mathcal{M}$, are κ-directed colimits, in the category of spans in A, of spans (n,g) in $\mathsf{A}_{<\kappa}$ such that $n\in\mathcal{N}$.

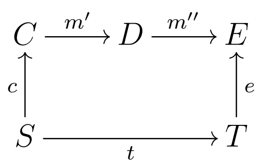
Once again, we use the fact that the category of morphisms $\mathsf{A}^{\to}$ is κ-accessible, and its κ-presentable objects are precisely all the morphisms with κ-presentable domains and codomains. Let M be the full subcategory in $\mathsf{A}^{\to}$ whose objects are all the morphisms belonging to $\mathcal{M}$. By Proposition 1.1 applied to the category $\mathsf{A}^{\to}$, the category M is κ-accessible, and its κ-presentable objects are precisely all the retracts of the morphisms belonging to $\mathcal{N}$. Now let $F\colon\mathsf{M}\longrightarrow\mathsf{A}$ be the functor taking every morphism $m\colon C\longrightarrow D$ to its domain C, and let $G\colon\mathsf{A}\longrightarrow\mathsf{A}$ be the identity functor. Then the category $F\downarrow G$ defined in Section 1 is precisely the category of all spans (m,f) in A with $m\in\mathcal{M}$. By Proposition 1.2, it follows that the category $F\downarrow G$ is κ-accessible, and its κ-presentable objects are precisely all the spans (n',g') in $\mathsf{A}_{<\kappa}$ such that n' is a retract of a morphism belonging to $\mathcal{N}$.

Moreover, let $\mathsf{S}\subset\mathsf{M}_{<\kappa}$ be a set of representatives of the isomorphism classes of morphisms belonging to $\mathcal{N}$, and let $\mathsf{T}\subset\mathsf{A}_{<\kappa}$ be a set of representatives of the isomorphism classes of κ-presentable objects of A. Then Proposition 1.3 applied to the sets of κ-presentable objects S and T tells us that all spans $(m,f)\in\mathsf{A}^D$ with $m\in\mathcal{M}$ are κ-directed colimits, in the category of spans A^D, of spans $(n,g)\in(\mathsf{A}_{<\kappa})^D$ with $n\in\mathcal{N}$ (as desired).

By the second assertion of Lemma 1.5, any finite colimit in $\mathsf{A}_{<\kappa}$ is also a colimit in A. In particular, this applies to pushouts. So pushouts of the spans (n,g) in $\mathsf{A}_{<\kappa}$ remain pushouts in A. Since κ-directed colimits always preserve pushouts, we have shown that condition (i) for $\mathcal{N}$ in $\mathsf{A}_{<\kappa}$ implies condition (i) for $\mathcal{M}$ in A.

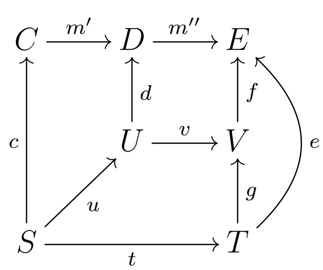
The proof of the assertion that conditions (i) and (ii) for $\mathcal{N}$ in $\mathsf{A}_{<\kappa}$ imply condition (ii) for $\mathcal{M}$ in A is somewhat similar, and based on the arguments above together with Lemmas 1.6 and 6.1. By assumption, every morphism $n \in \mathcal{N}$ is the equalizer in $\mathsf{A}_{<\kappa}$ of its cokernel pair in $\mathsf{A}_{<\kappa}$. We have already seen that the cokernel pair of n in $\mathsf{A}_{<\kappa}$ is also the cokernel pair of n in A. By Lemma 1.6, any limit that exists in $\mathsf{A}_{<\kappa}$ is also a limit in A. Therefore, n is the equalizer in A of its cokernel pair in A. So n is an effective monomorphism in A. Now any morphism $m \in \mathcal{M}$ is a κ-directed colimit in $\mathsf{A}^{\to}$ of morphisms $n \in \mathcal{N}$, and κ-directed colimits of effective monomorphisms in A are effective monomorphisms in A by Lemma 6.1.

Let us prove that conditions (i) and (iii) for $\mathcal{N}$ in $\mathsf{A}_{<\kappa}$ imply condition (iii) for $\mathcal{M}$ in A. Obviously, every identity morphism in A is a κ-directed colimit of identity morphisms in $\mathsf{A}_{<\kappa}$. It remains to show that the class $\mathcal{M}$ is closed under compositions. Let $m'\colon C \longrightarrow D$ and $m''\colon D \longrightarrow E$ be two morphisms belonging to $\mathcal{M}$. By Proposition 1.1, in order to prove that the composition $m = m'' \circ m'$ belongs to $\mathcal{M}$, we need to check that every morphism $t \longrightarrow m$ in $\mathsf{A}^{\to}$ from a morphism t with κ-presentable domain and codomain, $t \in (\mathsf{A}^{\to})_{<\kappa} = (\mathsf{A}_{<\kappa})^{\to}$, into the morphism m factorizes as $t \longrightarrow n \longrightarrow m$, where $n \in \mathcal{N}$. So we have a commutative diagram



in A with κ-presentable objects S and T.

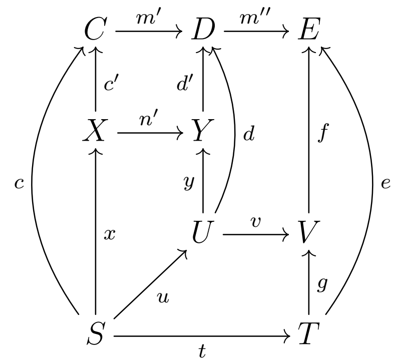
Considering the composition $m' \circ c\colon S \longrightarrow D$ of two morphisms $c\colon S \longrightarrow C$ and $m'\colon C \longrightarrow D$, have a morphism $(m' \circ c,\ e)\colon t \longrightarrow m''$ in the category $\mathsf{A}^{\to}$. Since the morphism $m''\colon D \longrightarrow E$ belongs to $\mathcal{M}$, by Proposition 1.1 the morphism $(m' \circ c,\ e)$ factorizes as $t \longrightarrow v \longrightarrow m''$, where $v\colon U \longrightarrow V$ is some morphism belonging to $\mathcal{N}$. So we have a commutative diagram



in A with κ-presentable objects S, T, U, and V.

Now we have a morphism $(c, d)\colon u \longrightarrow m'$ in the category $\mathsf{A}^{\to}$, where u is a morphism with κ-presentable domain and codomain. Since the morphism $m'\colon C \longrightarrow D$ belongs to $\mathcal{M}$, by Proposition 1.1 the morphism (c, d) factorizes as $u \longrightarrow n' \longrightarrow m'$, where

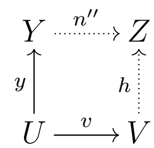
$n'\colon X \longrightarrow Y$ is some morphism belonging to $\mathcal{N}$. Hence we have a commutative diagram



(12)

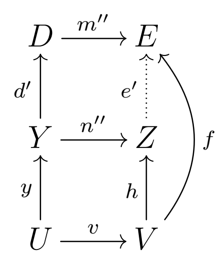
in A with κ-presentable objects S, T, U, V, X, and Y.

Finally, by condition (i) for the class $\mathcal{N}$ in $\mathsf{A}_{<\kappa}$, the span $v\colon U \longrightarrow V$, $y\colon U \longrightarrow Y$ has a pushout in $\mathsf{A}_{<\kappa}$, which by Lemma 1.5 is also a pushout in A. Denote the resulting pushout square by

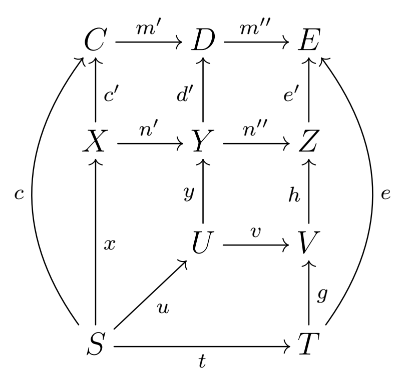


Condition (i) for the class $\mathcal{N}$ in $\mathsf{A}_{<\kappa}$ also tells us that $n'' \in \mathcal{N}$ (since $v \in \mathcal{N}$).

We have a pair of morphisms $m'' \circ d'\colon Y \longrightarrow E$ and $f\colon V \longrightarrow E$ such that $m'' \circ d' \circ y = f \circ v$. Hence there exists a unique morphism $e'\colon Z \longrightarrow E$ making the diagram



commutative. We have arrived to the commutative diagram



proving that the morphism $t \longrightarrow m = m'' \circ m'$ in $\mathsf{A}^{\to}$ factorizes as $t \longrightarrow n \longrightarrow m$, where $n = n'' \circ n'$.

As both the morphisms n' and n'' belong to $\mathcal{N}$ by construction, so does their composition $n'' \circ n'$, by condition (iii) for the class $\mathcal{N}$ in $\mathsf{A}_{<\kappa}$. This finishes the proof of condition (iii) for the class $\mathcal{M}$ in A.

It remains to say that one obviously has $\mathcal{N} \subset \mathcal{M} \cap \mathsf{A}_{<\kappa}^{\to}$. In fact, by Proposition 1.1, $\mathcal{M} \cap \mathsf{A}_{<\kappa}^{\to}$ is precisely the class of all retracts of the morphisms from $\mathcal{N}$ (the retracts being taken in the category $\mathsf{A}^{\to}$ or $\mathsf{A}_{<\kappa}^{\to}$). So the QE-mono class $\mathcal{M}$ in A is locally κ-coherent by Lemmas 6.4 and 6.5. ∎

7.2. Corollary. *For any κ-accessible category A, there is a bijective correspondence between locally κ-coherent QE-mono classes in A and QE-mono classes in the category $\mathsf{A}_{<\kappa}$ closed under retracts in $\mathsf{A}_{<\kappa}^{\to}$. The bijection assigns to every locally κ-coherent QE-mono class $\mathcal{M}$ in A the retraction-closed QE-mono class $\mathcal{N} = \mathcal{M} \cap \mathsf{A}_{<\kappa}^{\to}$ in $\mathsf{A}_{<\kappa}$. Conversely, to every retraction-closed QE-mono class $\mathcal{N}$ in $\mathsf{A}_{<\kappa}$, the locally κ-coherent QE-mono class $\mathcal{M} = \varinjlim_{(\kappa)} \mathcal{N}$ in A is assigned.*

Proof. For every locally κ-coherent QE-mono class $\mathcal{M}$ in A, the class $\mathcal{N} = \mathcal{M} \cap \mathsf{A}_{<\kappa}^{\to}$ is a retraction-closed QE-mono class in $\mathsf{A}_{<\kappa}$ by Proposition 6.6. For every QE-mono class $\mathcal{N}$ in $\mathsf{A}_{<\kappa}$, the class $\mathcal{M} = \varinjlim_{(\kappa)} \mathcal{N}$ is a locally κ-coherent QE-mono class in A by Theorem 7.1. For any locally κ-coherent QE-mono class $\mathcal{M}$ in A, one has $\mathcal{M} = \varinjlim_{(\kappa)}(\mathcal{M} \cap \mathsf{A}_{<\kappa}^{\to})$ by Lemma 6.5. For any retraction-closed QE-mono class $\mathcal{N}$ in $\mathsf{A}_{<\kappa}$, one has $\mathcal{N} = (\varinjlim_{(\kappa)} \mathcal{N}) \cap \mathsf{A}_{<\kappa}^{\to}$ by Proposition 1.1, as it was already mentioned in the last paragraph of the proof of Theorem 7.1. ∎

7.3. Remark. For any κ-accessible category A, the full subcategory $\mathsf{A}_{<\kappa} \subset \mathsf{A}$ has split idempotents, because the category A has split idempotents by [2, Observation 2.4] and $\mathsf{A}_{<\kappa}$ is closed under retracts in A. Conversely, for any small category S with split idempotents and any regular cardinal κ, there exists a unique (up to natural equivalence) κ-accessible category A such that the category $\mathsf{A}_{<\kappa}$ is equivalent to S [2, Theorem 2.26 and Remark 2.26(1)].

A discussion of the retraction-closedness property of QE-mono or QE-epi classes in additive categories, including in particular idempotent-complete and weakly idempotent-complete additive categories, can be found in [16, Theorems 1.1 and 1.2]. In the nonadditive context, we will continue this discussion below in Section 10. At the moment, we restrict ourselves to the following simple counterexample.

7.4. Example. A QE-mono class in an additive category S with split idempotents *need not* be closed under retracts in general. Indeed, let $\mathsf{S} = \Bbbk\text{–}\mathsf{vect}$ be the category of finite-dimensional vector spaces over a field $\Bbbk$, and let $\mathcal{N}$ be the class of all monomorphisms n in S with the dimension of the cokernel $\dim_{\Bbbk}(\operatorname{coker} n)$ divisible by a fixed integer $q \geq 2$. One can easily check that conditions (i–iii) are satisfied for the class $\mathcal{N}$ in the category S (in particular, because the cokernels are not changed by pushouts), but $\mathcal{N}$ is *not* closed under retracts in the category $\mathsf{S}^{\to}$. Obviously, one has $\mathsf{S} = \mathsf{A}_{<\aleph_0}$, where $\mathsf{A} = \Bbbk\text{–}\mathsf{Vect}$ is the finitely accessible category of $\Bbbk$-vector spaces. The class $\mathcal{M} = \varinjlim_{(\kappa)} \mathcal{N} \subset \mathsf{A}^{\to}$ consists of all monomorphisms in A.

8. QE-Epi Classes

Let C be a category. Dually to the discussion in Section 6, a morphism $p\colon D \longrightarrow E$ in C is called a *regular epimorphism* if p is the coequalizer of a parallel pair of morphisms d_1, $d_2\colon C \rightrightarrows D$.

Clearly, every regular epimorphism is an epimorphism. Every split epimorphism is regular: if $s\colon E \longrightarrow D$ is a morphism such that $p \circ s = \mathrm{id}_E$, then p is the coequalizer of the pair of morphisms $s \circ p$ and $\mathrm{id}_D\colon D \rightrightarrows D$.

Dually to the definition in Section 2, by the *kernel pair* of a morphism p one means the pullback of the cospan (p, p), cf. diagram (5). A morphism $p\colon D \longrightarrow E$ in C is said to be an *effective epimorphism* if p has a kernel pair k_1, $k_2\colon K \rightrightarrows D$ and p is the coequalizer of (k_1, k_2). One can easily see that if a regular epimorphism p has a kernel pair (k_1, k_2), then p is the coequalizer of (k_1, k_2). So an epimorphism is effective if and only if it is regular *and* has a kernel pair.

8.1. Lemma. (a) *Let C be a small category and A be a category such that the colimits of all diagrams indexed by C exist in A. Let $P\colon C \longrightarrow \mathsf{A}^{\to}$ be a diagram such that $P(c)$ is an effective epimorphism in A for all objects $c \in C$. Then the colimit of P, computed in $\mathsf{A}^{\to}$, is a regular epimorphism in A.*

(b) *For any κ-accessible category A, the class of effective epimorphisms is closed under κ-directed colimits in $\mathsf{A}^{\to}$.*

Proof. Part (a): the kernel pairs of the morphisms $P(c)$, $c \in C$, form a diagram $K\colon C \longrightarrow \mathsf{A}^{\rightrightarrows}$ in the category $\mathsf{A}^{\rightrightarrows}$ of parallel pairs of morphisms in A. By assumption, the morphism $P(c)$ is the coequalizer of the parallel pair of morphisms $K(c)$ in A for every object $c \in C$. The colimit of K computed in $\mathsf{A}^{\rightrightarrows}$, which exists by assumption, is a parallel pair of morphisms in A whose coequalizer is the colimit of P computed in $\mathsf{A}^{\to}$. Indeed, colimits commute with colimits in any category; so, in particular, colimits indexed by C preserve coequalizers in A.

Part (b): by Lemma 1.8, in a κ-accessible category, the κ-directed colimits commute with κ-small limits; in particular, the κ-directed colimits preserve kernel pairs. So, if a morphism p is a κ-directed colimit of effective epimorphisms p_ξ, then the colimit of the kernel pairs of p_ξ is the kernel pair of p. Here we are assuming that the index ξ ranges over a κ-directed poset Ξ. In any category, colimits commute with colimits; in particular, the κ-directed colimits preserve coequalizers. Thus p is the coequalizer of its kernel pair. ∎

Let $\mathcal{P}$ be a class of morphism in a category C. We will say that $\mathcal{P}$ is a *QE-epi class* in C if the following conditions are satisfied:

- i*. All pullbacks of all morphisms from $\mathcal{P}$ exists in C, and the class $\mathcal{P}$ is stable under pulbacks. In other words, for any cospan $p\colon D \longrightarrow E$, $f\colon E' \longrightarrow E$ such that

$p \in \mathcal{P}$, the pullback D' exists, and the morphism $p' \colon D' \longrightarrow E'$ belongs to $\mathcal{P}$,

$$\begin{array}{ccc} D & \xrightarrow{p} & E \\ {\scriptstyle f'}\uparrow & & \uparrow{\scriptstyle f} \\ D' & \xrightarrow[p']{} & E' \end{array} \tag{13}$$

ii*. In particular, condition (i*) implies that all morphisms from $\mathcal{P}$ have kernel pairs in C. It is further required that every morphism from $\mathcal{P}$ is the coequalizer of its kernel pair. In other words, all the morphisms from $\mathcal{P}$ must be effective epimorphisms.

iii*. All the identity morphisms in C belong to $\mathcal{P}$, and the class of morphisms $\mathcal{P}$ is closed under compositions.

It is clear from the preceding discussion that, assuming condition (i*), condition (ii*) is equivalent to the condition that all morphisms from $\mathcal{P}$ are regular epimorphisms in C.

8.2. Examples. (1) If C is an additive category, then a QE-epi class of morphisms in C is the same thing as a structure of *left exact category* on C in the sense of [9, Definition 3.1]. In the terminology of [16, Definition 2.2], such additive categories with an additional structure are called *deflation-exact categories*.

(2) In any regular category C (in the sense of [7, 15]), the class of all regular epimorphisms is a QE-epi class [15, Definition 1.10 and Proposition 1.13(3)].

(3) The compositions of split epimorphisms are always split epimorphisms, and all split epimorphisms are regular (as explained above). Furthermore, all pushouts of split epimorphisms are split epimorphisms by the argument dual to the one in Example 6.2(3). Therefore, the class $\mathcal{P}$ of all split epimorphisms in a category C is a QE-epi class if and only all pullbacks of split epimorphisms exist in C.

8.3. Remark. Dually to Remark 6.3, all pullbacks of split epimorphisms exist in an additive category if and only if the category is weakly idempotent-complete. However, the idempotent-complete preadditive category A of $\Bbbk$-vector spaces of finite dimension $\le n$ (where $n \ge 1$ is a fixed integer) does not have pullbacks of split epimorphisms. In fact, the category A does not even have very weak split pullbacks of split epimorphisms; see Example 4.5. It is also clear from Example 4.5 (take $j = n$) that the category A does not have (even weak) kernel pairs of split epimorphisms.

Let $\mathcal{P}$ be a QE-epi class in a category C. For every morphism $p \colon D \longrightarrow E$ belonging to $\mathcal{P}$, consider its kernel pair $k_1, k_2 \colon K \rightrightarrows D$. By a *$\mathcal{P}$-sequence* we mean a diagram

$$K \underset{k_2}{\overset{k_1}{\rightrightarrows}} D \xrightarrow{p} E$$

arising from some morphism $p \in \mathcal{P}$ in this way.

Let A be a κ-accessible category. We will say that a QE-epi class $\mathcal{P}$ in A is *locally κ-coherent* if the $\mathcal{P}$-sequences in A are precisely all the κ-directed colimits of $\mathcal{P}$-sequences in A with all the three terms K, D, E belonging to $\mathsf{A}_{<\kappa}$. Here the κ-directed colimits are taken in the diagram category $\mathsf{A}^{J^{\mathsf{op}}}$, where J is the finite category defined in Section 6. The terminology "locally κ-coherent" comes from the paper [25, Section 1].

8.4. Lemma. *For any locally κ-coherent QE-epi class $\mathcal{P}$ in a κ-accessible category A, the class of all $\mathcal{P}$-sequences is closed under κ-directed colimits in $\mathsf{A}^{J^{\mathsf{op}}}$.*

Proof. This is completely similar to Lemma 6.4. ■

Notice the difference between the formulations of the following lemma and its version for QE-mono classes (Lemma 6.5 above).

8.5. Lemma. *A QE-epi class $\mathcal{P}$ in a κ-accessible category A is locally κ-coherent if and only if the following two conditions hold:*

1. *$\mathcal{P}$ is precisely the class of all κ-directed colimits of the morphisms from $\mathcal{P}$ whose domains and codomains are κ-presentable. Here the κ-directed colimits are taken in the category of morphisms $\mathsf{A}^{\to}$.*
2. *For any morphism $p \in \mathcal{P}$ whose domain and codomain are κ-presentable, the domain of the kernel pair of p is κ-presentable as well.*

Proof. The proof of this lemma only uses condition (i*) from the definition of a QE-epi class; conditions (ii*) and (iii*) are not used. Let $\mathcal{P}$ be a locally κ-coherent QE-epi class in A. To check condition (2), assume that a morphism p in A is a κ-directed colimit of some morphisms p_ξ, the colimit being taken in the category $\mathsf{A}^{\to}$, that is $p = \varinjlim_{\xi\in\Xi}^{\mathsf{A}^{\to}} p_\xi$, where Ξ is a κ-directed poset. If the domain and codomain of p are κ-presentable, then p is a κ-presentable object of $\mathsf{A}^{\to}$, and it follows that p is a retract of one of the morphisms p_ξ. Now if p has a kernel pair (k_1, k_2) in A and p_ξ has a kernel pair $(k_{\xi,1}, k_{\xi,2})$ in A, then (k_1, k_2) is a retract of $(k_{\xi,1}, k_{\xi,2})$. In particular, the domain K of (k_1, k_2) is a retract of the domain K_ξ of $(k_{\xi,1}, k_{\xi,2})$. Hence if K_ξ is κ-presentable, then K is κ-presentable, too. After condition (2) is proved, condition (1) becomes obvious. Conversely, if conditions (1) and (2) hold then, in order to check that $\mathcal{P}$ is locally κ-coherent, one needs to use the fact that κ-directed colimits preserve finite limits (in particular, kernel pairs) in A. This is Lemma 1.8. ■

Let A be a κ-accessible category. We will say that a locally κ-coherent QE-epi class $\mathcal{P}$ in A is *strongly locally κ-coherent* if it satisfies the following stronger version of condition (2) from Lemma 8.5:

2′. for any pullback diagram (13) in A with κ-presentable objects D, E, E' and a morphism $p \in \mathcal{P}$, the object D' is also κ-presentable.

8.6. Remark. Under a natural additional assumption, any locally κ-coherent QE-epi class in a κ-accessible *additive* category A is strongly locally κ-coherent. Indeed, given a morphism $p\colon D \longrightarrow E$ with a kernel pair $k_1, k_2\colon K \rightrightarrows D$ in an idempotent-complete additive category A, the kernel $k'\colon K' \longrightarrow D$ of the morphism p can be constructed as the image of a suitable idempotent endomorphism $K \longrightarrow K$. In fact, one has $K \simeq D \oplus K'$. Given a morphism $f\colon E' \longrightarrow E$, the pullback D' of the pair of morphisms $p\colon D \longrightarrow E$ and $f\colon E' \longrightarrow E$ can be constructed as the kernel of the induced morphism $(p, f)\colon D \oplus E' \longrightarrow E$. Hence, in the situation at hand, condition (2) implies $D' \in \mathsf{A}_{<\kappa}$ provided that D, E, $E' \in \mathsf{A}_{<\kappa}$ and $(p, f) \in \mathcal{P}$.

It remains to make sure that $(p, f) \in \mathcal{P}$ whenever $p \in \mathcal{P}$. Denoting by $i_D\colon D \longrightarrow D \oplus E'$ the direct summand injection, we have $(p, f) \circ i_D = p$. By the pullback axiom (i*) above, the pullback D' of the pair of morphisms p and f exists in A; so the morphism (p, f) has a kernel in A. Assuming the axiom dual to [9, axiom [R3] from Definition 3.2], or which is the same, [16, axiom R3 from Definition 2.3], it follows that $(p, f) \in \mathcal{P}$ whenever $p \in \mathcal{P}$. See Section 10 below for a further discussion.

We are not aware of any example of a locally κ-coherent QE-epi class (in any κ-accessible category) that is not strongly locally κ-coherent.

8.7. Proposition. *Let $\mathcal{P}$ be a locally κ-coherent QE-epi class in a κ-accessible category A. Then the class $\mathcal{P} \cap \mathsf{A}^{\to}_{<\kappa}$ of all morphisms from $\mathcal{P}$ with κ-presentable domains and codomains is closed under retracts in the category $\mathsf{A}^{\to}_{<\kappa}$. The locally κ-coherent QE-epi class $\mathcal{P}$ is strongly locally κ-coherent if and only if the class $\mathcal{P} \cap \mathsf{A}^{\to}_{<\kappa}$ is a QE-epi class in the category $\mathsf{A}_{<\kappa}$.*

Proof. The first assertion is provable similarly to the proof of the second assertion of Proposition 6.6. Let us prove the second assertion. "If": assuming that the class $\mathcal{P} \cap \mathsf{A}^{\to}_{<\kappa}$ satisfies condition (i*) in the category $\mathsf{A}_{<\kappa}$, condition (2$'$) for the class $\mathcal{P}$ in A follows, because all limits (in particular, pullbacks) that exist in $\mathsf{A}_{<\kappa}$ are also limits in A by Lemma 1.6. "Only if": condition (iii*) obviously holds for $\mathcal{P} \cap \mathsf{A}^{\to}_{<\kappa}$ whenever it holds for $\mathcal{P}$. Condition (i*) holds for $\mathcal{P} \cap \mathsf{A}^{\to}_{<\kappa}$ whenever it holds for $\mathcal{P}$ *and* $\mathcal{P}$ is strongly locally κ-coherent, because any square diagram in $\mathsf{A}_{<\kappa}$ that is a pullback diagram in A is also a pullback diagram in $\mathsf{A}_{<\kappa}$. Finally, condition (ii*) holds for $\mathcal{P} \cap \mathsf{A}^{\to}_{<\kappa}$ whenever conditions (i*) and (ii*) hold for $\mathcal{P}$ and $\mathcal{P}$ is strongly locally κ-coherent, because any diagram in $\mathsf{A}_{<\kappa}$ that is a coequalizer diagram in A is also a coequalizer diagram in $\mathsf{A}_{<\kappa}$. ∎

9. Construction of Strongly Locally Coherent QE-Epi Classes

Let D be the finite category defined in Section 7. By *the category of cospans* in a category C we mean the category of D^{op}-shaped diagrams in C, i. e., the category of functors $\mathsf{C}^{D^{\mathsf{op}}}$.

9.1. Theorem. *Let A be a κ-accessible category and $\mathcal{Q}$ be a QE-epi class in the category $\mathsf{A}_{<\kappa}$. Then the class $\varinjlim_{(\kappa)} \mathcal{Q} \subset \mathsf{A}^{\to}$ of all κ-directed colimits of morphisms from $\mathcal{Q}$ (the colimits being taken in $\mathsf{A}^{\to}$) is a strongly locally κ-coherent QE-epi class in the category A.*

Proof. The argument is largely similar to the proof of Theorem 7.1, but there are some differences. Put $\mathcal{P} = \varinjlim_{(\kappa)} \mathcal{Q}$. In order to check that condition (i*) for the class $\mathcal{Q}$ in $\mathsf{A}_{<\kappa}$ implies condition (i*) for the class $\mathcal{P}$ in A, let us show that all cospans (p, f) in A (where $p\colon D \longrightarrow E$ and $f\colon E' \longrightarrow E$) such that $p \in \mathcal{P}$, are κ-directed colimits, in the category of cospans in A, of cospans (q, g) in $\mathsf{A}_{<\kappa}$ such that $q \in \mathcal{Q}$.

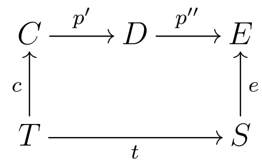
As usual, we keep in mind the fact that the category of morphisms $\mathsf{A}^{\to}$ is κ-accessible, and its κ-presentable objects are precisely all the morphisms with κ-presentable domains and codomains. Let P be the full subcategory in $\mathsf{A}^{\to}$ whose objects are all the morphisms belonging to $\mathcal{P}$. By Proposition 1.1 applied to the category $\mathsf{A}^{\to}$, the category P is κ-accessible, and its κ-presentable objects are precisely all the retracts of the morphisms belonging to $\mathcal{Q}$. Now let $F\colon \mathsf{A} \longrightarrow \mathsf{A}$ be the identity functor, and let $G\colon \mathsf{P} \longrightarrow \mathsf{A}$ be the functor taking every morphism $p\colon D \longrightarrow E$ to its codomain E. Then the category $F \downarrow G$ defined in Section 1 is precisely the category of all cospans (p, f) in A with $p \in \mathcal{P}$. By Proposition 1.2, it follows that the category $F \downarrow G$ is κ-accessible, and its κ-presentable objects are precisely all the cospans (q', g') in $\mathsf{A}_{<\kappa}$ such that q' is a retract of a morphism belonging to $\mathcal{Q}$.

Moreover, let $\mathsf{S} \subset \mathsf{A}_{<\kappa}$ be a set of representatives of the isomorphism classes of κ-presentable objects of A, and let $\mathsf{T} \subset \mathsf{P}_{<\kappa}$ be a set of representatives of the isomorphism classes of morphisms belonging to $\mathcal{Q}$. Then Proposition 1.3 applied to the sets of κ-presentable objects S and T tells us that all cospans $(p, f) \in \mathsf{A}^{D^{\mathsf{op}}}$ with $p \in \mathcal{P}$ are κ-directed colimits, in the category of cospans $\mathsf{A}^{D^{\mathsf{op}}}$, of cospans $(q, g) \in (\mathsf{A}_{<\kappa})^{D^{\mathsf{op}}}$ with $q \in \mathcal{Q}$ (as desired).

By Lemma 1.6, any limit in $\mathsf{A}_{<\kappa}$ is also a limit in A. In particular, this applies to pullbacks. So pullbacks of the cospans (q, g) in $\mathsf{A}_{<\kappa}$ remain pullbacks in A. Since κ-directed colimits in A preserve pullbacks by Lemma 1.8, we have shown that condition (i*) for $\mathcal{Q}$ in $\mathsf{A}_{<\kappa}$ implies condition (i*) for $\mathcal{P}$ in A.

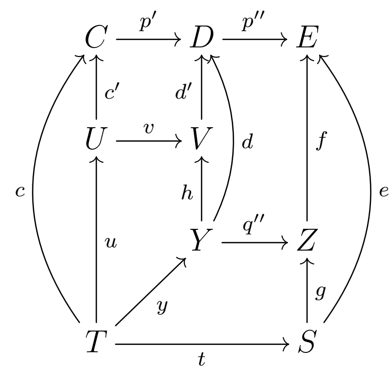
The proof of the assertion that conditions (i*) and (ii*) for $\mathcal{Q}$ in $\mathsf{A}_{<\kappa}$ imply condition (ii*) for $\mathcal{P}$ in A is somewhat similar, and based on the arguments above together with Lemmas 1.5 and 8.1. By assumption, every morphism $q \in \mathcal{Q}$ is the coequalizer in $\mathsf{A}_{<\kappa}$ of its kernel pair in $\mathsf{A}_{<\kappa}$. We have already seen that the kernel pair of q in $\mathsf{A}_{<\kappa}$ is also the kernel pair of q in A. By the second assertion of Lemma 1.5, any finite colimit that exists in $\mathsf{A}_{<\kappa}$ is also a colimit in A. Therefore, q is the coequalizer in A of its kernel pair in A. So q is an effective epimorphism in A. Now any morphism $p \in \mathcal{P}$ is a κ-directed colimit in $\mathsf{A}^{\to}$ of morphisms $q \in \mathcal{Q}$, and κ-directed colimits of effective epimorphisms in A are effective epimorphisms in A by Lemma 8.1(b).

Let us prove that conditions (i*) and (iii*) for $\mathcal{Q}$ in $\mathsf{A}_{<\kappa}$ imply condition (iii*) for $\mathcal{P}$ in A. The assertion concerning the identity morphisms is obvious. We need to show that the class $\mathcal{P}$ is closed under compositions. Let $p'\colon C \longrightarrow D$ and $p''\colon D \longrightarrow E$ be two morphisms belonging to $\mathcal{P}$. By Proposition 1.1, in order to prove that the composition $p = p'' \circ p'$ belongs to $\mathcal{P}$, we need to check that every morphism $t \longrightarrow p$ in $\mathsf{A}^{\to}$ from a morphism t with κ-presentable domain and codomain, $t \in (\mathsf{A}^{\to})_{<\kappa} = (\mathsf{A}_{<\kappa})^{\to}$, into the morphism p factorizes as $t \longrightarrow q \longrightarrow p$, where $q \in \mathcal{Q}$. So we have a commutative diagram



in A with κ-presentable objects S and T.

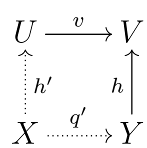
Arguing exactly as in the proof of Theorem 7.1, we construct a commutative diagram similar to (12). Let us redraw it here in our current notation:



(14)

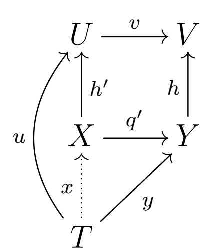
So (14) is a commutative diagram in A with κ-presentable objects S, T, U, V, X, and Y, and morphisms v, $q'' \in \mathcal{Q}$.

By condition (i*) for the class $\mathcal{Q}$ in $\mathsf{A}_{<\kappa}$, the cospan $v\colon U \longrightarrow V$, $h\colon Y \longrightarrow V$ has a pullback in $\mathsf{A}_{<\kappa}$, which by Lemma 1.6 is also a pullback in A. Denote the resulting pullback square by

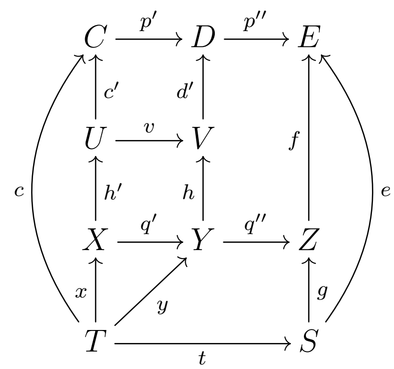


Condition (i*) for the class $\mathcal{Q}$ in $\mathsf{A}_{<\kappa}$ also tells us that $q' \in \mathcal{Q}$ (since $v \in \mathcal{Q}$).

We have a pair of morphisms $u\colon T \longrightarrow U$ and $y\colon T \longrightarrow Y$ such that $v \circ u = h \circ y$. Hence there exists a unique morphism $x\colon T \longrightarrow X$ making the diagram



commutative. We have arrived to the commutative diagram



proving that the morphism $t \longrightarrow p = p'' \circ p'$ in $\mathsf{A}^{\to}$ factorizes as $t \longrightarrow q \longrightarrow p$, where $q = q'' \circ q'$.

As both the morphisms q' and q'' belong to $\mathcal{Q}$ by construction, so does their composition $q'' \circ q'$, by condition (iii*) for the class $\mathcal{Q}$ in $\mathsf{A}_{<\kappa}$. This finishes the proof of condition (iii*) for the class $\mathcal{P}$ in A.

It remains to prove that the QE-epi class $\mathcal{P}$ is strongly locally κ-coherent. In fact, by Proposition 1.1, $\mathcal{P} \cap \mathsf{A}_{<\kappa}^{\to}$ is precisely the class of all retracts of the morphisms from $\mathcal{Q}$ (the retracts being taken in the category $\mathsf{A}^{\to}$ or $\mathsf{A}_{<\kappa}^{\to}$). So the QE-epi class $\mathcal{P}$ in A satisfies condition (1) of Lemma 8.5 in view of Lemma 8.4. To check condition (2′) from Section 8 for the class $\mathcal{P}$, notice that, in view of the proof of condition (i*) for the class $\mathcal{P}$ above, every cospan $(p, f) \in (\mathsf{A}_{<\kappa})^{D^{\mathsf{op}}}$ in $\mathsf{A}_{<\kappa}$ with a morphism $p \in \mathcal{P}$ is a retract of a cospan $(q, g) \in (\mathsf{A}_{<\kappa})^{D^{\mathsf{op}}}$ in $\mathsf{A}_{<\kappa}$ with a morphism $q \in \mathcal{Q}$. Furthermore, the pullback of (q, g) in $\mathsf{A}_{<\kappa}$ is also the pullback of (q, g) in A. It follows that the pullback of (p, f) in A is a retract of the pullback of (q, g), and it remains to point out that the full subcategory $\mathsf{A}_{<\kappa} \subset \mathsf{A}$ is closed under retracts in A. ∎

9.2. Corollary. *For any κ-accessible category A, there is a bijective correspondence between strongly locally κ-coherent QE-epi classes in A and QE-epi classes in the category $\mathsf{A}_{<\kappa}$ closed under retracts in $\mathsf{A}_{<\kappa}^{\to}$. The bijection assigns to every strongly locally κ-coherent QE-epi class $\mathcal{P}$ in A the retraction-closed QE-epi class $\mathcal{Q} = \mathcal{P} \cap \mathsf{A}_{<\kappa}^{\to}$ in $\mathsf{A}_{<\kappa}$. Conversely, to every retraction-closed QE-epi class $\mathcal{Q}$ in $\mathsf{A}_{<\kappa}$, the strongly locally κ-coherent QE-epi class $\mathcal{P} = \varinjlim_{(\kappa)} \mathcal{Q}$ in A is assigned.*

Proof. For every strongly locally κ-coherent QE-epi class $\mathcal{P}$ in A, the class $\mathcal{Q} = \mathcal{P} \cap \mathsf{A}_{<\kappa}^{\to}$ is a retraction-closed QE-epi class in $\mathsf{A}_{<\kappa}$ by Proposition 8.7. For every QE-epi class $\mathcal{Q}$ in $\mathsf{A}_{<\kappa}$, the class $\mathcal{P} = \varinjlim_{(\kappa)} \mathcal{Q}$ is a strongly locally κ-coherent QE-epi class in A by Theorem 9.1. For any strongly locally κ-coherent QE-epi class $\mathcal{P}$ in A, one has $\mathcal{P} = \varinjlim_{(\kappa)}(\mathcal{P} \cap \mathsf{A}_{<\kappa}^{\to})$ by Lemma 8.5(1). For any retraction-closed QE-epi class $\mathcal{Q}$ in $\mathsf{A}_{<\kappa}$, one has $\mathcal{Q} = (\varinjlim_{(\kappa)} \mathcal{Q}) \cap \mathsf{A}_{<\kappa}^{\to}$ by Proposition 1.1, as it was already mentioned in the last paragraph of the proof of Theorem 9.1. ∎

9.3. Example. Dually to Example 7.4, a QE-epi class in an additive category S with split idempotents *need not* be closed under retracts in general. Indeed, let $\mathsf{S} = \Bbbk\text{–}\mathsf{vect}$ be the category of finite-dimensional vector spaces over a field $\Bbbk$, and let $\mathcal{Q}$ be the class of all epimorphisms q in S with the dimension of the kernel $\dim_{\Bbbk}(\ker q)$ divisible by a fixed integer $n \geq 2$. Then conditions (i*–iii*) are satisfied for the class $\mathcal{Q}$ in the category S, but $\mathcal{Q}$ is *not* closed under retracts in the category $\mathsf{S}^{\to}$. Furthermore, one has $\mathsf{S} = \mathsf{A}_{<\aleph_0}$, where $\mathsf{A} = \Bbbk\text{–}\mathsf{Vect}$ is the finitely accessible category of $\Bbbk$-vector spaces. The class $\mathcal{P} = \varinjlim_{(\kappa)} \mathcal{Q} \subset \mathsf{A}^{\to}$ consists of all epimorphisms in A.

10. Strong QE-Epi Classes

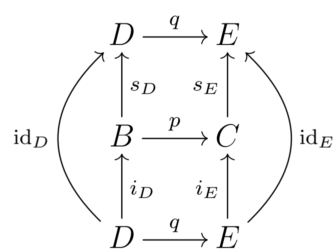
Let C be a category. We will say that a QE-epi class $\mathcal{P}$ in C (as defined in Section 8) is a *strong QE-epi class* if it satisfies the following additional condition:

iv*. If p, q is a composable pair of morphisms in C and $p \circ q \in \mathcal{P}$, then $p \in \mathcal{P}$.

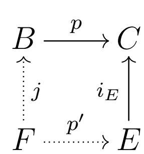
In the context of additive categories, our axiom (iv*) coincides with [16, axiom R3$^+$ from Definition 3.1]. A similar but slightly more general definition of a *strongly left exact* additive category can be found in [9, Definition 3.2].

10.1. Lemma. *Any strong QE-epi class in a category* C *is closed under retracts in the category* $\mathsf{C}^{\to}$.

Proof. Let $p\colon B \longrightarrow C$ be a morphism belonging to $\mathcal{P}$, and let $q\colon D \longrightarrow E$ be a retract of the morphism p. So we have a commutative diagram



By condition (i*), there exists a pullback diagram



in the category C, and the morphism $p' \colon F \longrightarrow E$ belongs to $\mathcal{P}$. Now we have commutative diagrams

$$
\begin{array}{ccc}
D & \xrightarrow{\ q\ } & E \\
\uparrow{\scriptstyle s_D} & & {\scriptstyle s_E}\uparrow \\
B & \xrightarrow{\ p\ } & C \\
\uparrow{\scriptstyle j} & & {\scriptstyle i_E}\uparrow \\
F & \xrightarrow{\ p'\ } & E
\end{array}
\quad (\mathrm{id}_E\colon E \longrightarrow E)
\qquad\qquad
\begin{array}{ccc}
D & \xrightarrow{\ q\ } & E \\
{\scriptstyle s_D}\uparrow & \nearrow{\scriptstyle p'} & \\
B & & \\
{\scriptstyle j}\uparrow & & \\
F & &
\end{array}
$$

with $p' \in \mathcal{P}$. By condition (iv*), it follows that $q \in \mathcal{P}$. ∎

The following proposition is a weak nonadditive version of [16, Theorem 1.2].

10.2. Proposition. *Let C be a category with finite coproducts and $\mathcal{P}$ be a QE-epi class in C. Then $\mathcal{P}$ is a strong QE-epi class in C if and only if the following two conditions hold:*

1. *the class $\mathcal{P}$ is closed under retracts in $\mathsf{C}^{\to}$;*
2. *for any three objects $A, B, C \in \mathsf{C}$ and any two morphisms $p \colon A \longrightarrow C$ and $f \colon B \longrightarrow C$ such that $p \in \mathcal{P}$, the induced morphism from the coproduct $(p, f) \colon A \sqcup B \longrightarrow C$ also belongs to $\mathcal{P}$.*

Proof. "Only if": condition (1) holds by Lemma 10.1 (this implication does not depend on the assumption of existence of finite coproducts in C). Condition (2) follows from (iv*), since the composition $A \longrightarrow A \sqcup B \longrightarrow C$ is equal to $p \in \mathcal{P}$.

"If": Let $q \colon C \longrightarrow D$ and $p \colon D \longrightarrow E$ be a pair of morphisms in C such that the composition $r = p \circ q$ belongs to $\mathcal{P}$. Then, by (2), the morphism $(r, p) \colon C \sqcup D \longrightarrow E$ belongs to $\mathcal{P}$. It remains to point out that the morphism p is a retract of the morphism (r, p), in view of commutativity of the diagram

$$
\begin{array}{ccc}
D & \xrightarrow{\ p\ } & E \\
{\scriptstyle (q,\mathrm{id}_D)}\uparrow & & \| \\
C \sqcup D & \xrightarrow{\ (r,p)\ } & E \\
{\scriptstyle i_D}\uparrow & & \| \\
D & \xrightarrow{\ p\ } & E
\end{array}
$$

where $i_D \colon D \longrightarrow C \sqcup D$ is the coproduct injection. ∎

11. Characterization of Strongly Locally Coherent Strong QE-Epi Class

Let A be a κ-accessible category. By a strongly locally κ-coherent strong QE-epi class in A we mean a strong QE-epi class $\mathcal{P}$ in A (in the sense of Section 10) that is strongly locally κ-coherent as a QE-epi class in A (in the sense of Section 8).

11.1. PROPOSITION. *Let* A *be a* κ*-accessible category with finite coproducts and* $\mathcal{Q}$ *be a strong QE-epi class in the category* $\mathsf{A}_{<\kappa}$*. Let* $\mathcal{P} = \varinjlim_{(\kappa)} \mathcal{Q} \subset \mathsf{A}^{\to}$ *be the related (strongly locally* κ*-coherent) QE-epi class in the category* A*, as per Theorem 9.1. Then a morphism* $p\colon D \longrightarrow E$ *in* A *belongs to* $\mathcal{P}$ *if and only if, for every object* $S \in \mathsf{A}_{<\kappa}$ *and every morphism* $e\colon S \longrightarrow E$ *there exists a commutative square diagram*

$$\begin{array}{ccc} D & \xrightarrow{\;p\;} & E \\ {\scriptstyle d}\uparrow & & \uparrow{\scriptstyle e} \\ T & \xrightarrow[\;q\;]{} & S \end{array} \tag{15}$$

with an object $T \in \mathsf{A}_{<\kappa}$ *and a morphism* $q \in \mathcal{Q}$*.*

PROOF. This is a nonadditive version of [25, Lemmas 1.5 and 2.3]. "Only if": this implication does not depend on the assumption of existence of finite coproducts in A. The assumption that $\mathcal{Q}$ is a strong QE-epi class is not needed for this implication, either; it is only important that the class $\mathcal{Q}$ satisfies condition (i*) in the category $\mathsf{A}_{<\kappa}$.

Suppose that $p = \varinjlim^{\mathsf{A}^{\to}}_{\xi\in\Xi} u_\xi$, where $u_\xi\colon X_\xi \longrightarrow Y_\xi$ are some morphisms belonging to $\mathcal{Q}$ and Ξ is a κ-directed poset. Let $x_\xi\colon X_\xi \longrightarrow D$ and $y_\xi\colon Y_\xi \longrightarrow E$ be the natural morphisms to the colimit. Then, since the object S is κ-accessible, the morphism $e\colon S \longrightarrow E$ factorizes through the morphism y_ξ for some index $\xi \in \Xi$. So we have a morphism $v\colon S \longrightarrow Y_\xi$ such that $e = y_\xi \circ v$. Put $X = X_\xi$, $Y = Y_\xi$, $x = x_\xi$, $y = y_\xi$, and $u = u_\xi$.

Applying condition (i*) to the pair of morphisms $u\colon X \longrightarrow Y$ and $v\colon S \longrightarrow Y$ in the category $\mathsf{A}_{<\kappa}$, with the morphism u belonging to $\mathcal{Q}$, we obtain a pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{\;u\;} & Y \\ {\scriptstyle v'}\uparrow & & \uparrow{\scriptstyle v} \\ T & \xrightarrow[\;q\;]{} & S \end{array}$$

in the category $\mathsf{A}_{<\kappa}$ with the morphism q belonging to $\mathcal{Q}$. Now we have the commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\;p\;} & E \\ {\scriptstyle x}\uparrow & & \uparrow{\scriptstyle y} \\ X & \xrightarrow{\;u\;} & Y \\ {\scriptstyle v'}\uparrow & & \uparrow{\scriptstyle v} \\ T & \xrightarrow[\;q\;]{} & S \end{array} \qquad e\colon S \longrightarrow E$$

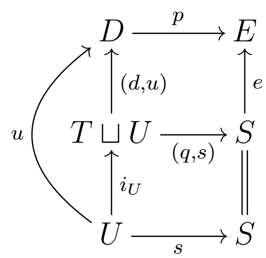
leading to the desired commutative square diagram (15).

"If": given a morphism $s\colon U \longrightarrow S$ in $\mathsf{A}_{<\kappa}$ and a morphism $s \longrightarrow p$ in $\mathsf{A}^{\to}$, we need to find a morphism $r \in \mathcal{Q}$ such that the morphism $s \longrightarrow p$ factorizes as $s \longrightarrow r \longrightarrow p$

in $\mathsf{A}^{\to}$. So suppose we are given a commutative square diagram

$$\begin{array}{ccc} D & \xrightarrow{\ p\ } & E \\ \uparrow{\scriptstyle u} & & \uparrow{\scriptstyle e} \\ U & \xrightarrow[\ s\ ]{} & S \end{array}$$

in A with κ-presentable objects U and S. By assumption, we can extend the morphisms p and e to a commutative square diagram (15) with a morphism $q \in \mathcal{Q}$. Now we have a commutative diagram



where $i_U : U \longrightarrow T \sqcup U$ is the coproduct injection. Here the coproduct $T \sqcup U$ computed in A belongs to $\mathsf{A}_{<\kappa}$ by Lemma 1.5; so it is also the coproduct in $\mathsf{A}_{<\kappa}$. By Proposition 10.2(2), the morphism $r = (q, s) : T \sqcup U \longrightarrow S$ belongs to $\mathcal{Q}$, providing the desired factorization $s \longrightarrow r \longrightarrow p$. ∎

11.2. Theorem. *Let A be a κ-accessible category with finite coproducts. Then, for any strong QE-epi class $\mathcal{Q}$ in the category $\mathsf{A}_{<\kappa}$, the (strongly locally κ-coherent) QE-epi class $\varinjlim_{(\kappa)} \mathcal{Q}$ is a strong QE-epi class in the category A. Conversely, for any strongly locally κ-coherent strong QE-epi class $\mathcal{P}$ in A, the QE-epi class $\mathcal{P} \cap \mathsf{A}_{<\kappa}^{\to}$ in the category $\mathsf{A}_{<\kappa}$ is a strong QE-epi class.*

Proof. To prove the first assertion, we use the characterization of the class $\mathcal{P} = \varinjlim_{(\kappa)} \mathcal{Q}$ provided by Proposition 11.1. Let $p' : C \longrightarrow D$ and $p'' : D \longrightarrow E$ be two morphisms in A such that the composition $p'' \circ p'$ belongs to $\mathcal{P}$. Suppose we are given an object $S \in \mathsf{A}_{<\kappa}$ and a morphism $e : S \longrightarrow E$. By Proposition 11.1, there is a commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{\ p'\ } & D & \xrightarrow{\ p''\ } & E \\ {\scriptstyle c}\uparrow & & & & \uparrow{\scriptstyle e} \\ T & & \xrightarrow[\ q\ ]{} & & S \end{array}$$

with a morphism $q \in \mathcal{Q}$. Setting $d = p' \circ c : T \longrightarrow D$, we obtain the desired commutative diagram (15) for the morphisms p'' and e. Applying Proposition 11.1 again, we can conclude that $p'' \in \mathcal{P}$.

The second assertion of the theorem is obvious and does not depend on the assumption that A has finite coproducts. ∎

11.3. Remark. One can say that a QE-mono class $\mathcal{M}$ in a category C is a *strong QE-mono class* if it satisfies condition (iv) dual to condition (iv*) from Section 10. It would be interesting to know whether a version of Theorem 11.2 holds for strong QE-mono classes, or what assumptions on a κ-accessible category A are needed for it to hold. Let $\mathcal{N}$ be a strong QE-mono class in $\mathsf{A}_{<\kappa}$; does it follow that $\mathcal{M} = \varinjlim_{(\kappa)} \mathcal{N}$ is a strong QE-mono class in A?

12. Regularity of Strongly Pure Monomorphisms

Let κ be a regular cardinal and A be a κ-accessible category. The definition of a strongly κ-pure monomorphism in A was given in Section 3. For the definition of a regular monomorphism, see Section 6.

12.1. Proposition. *Let A be a κ-accessible category such that every split monomorphism in A has a cokernel pair. Then all strongly κ-pure monomorphisms in A are regular monomorphisms.*

Proof. The assumption of the proposition means that all split monomorphisms in A are effective (since all split monomorphisms are always regular; see Section 6). By Lemma 6.1, it follows that all the κ-directed colimits of split monomorphisms in A are effective monomorphisms, too. For an alternative argument applicable under slightly more restrictive assumptions, see Proposition 13.1 below. ∎

The definition of a very weak cokernel pair was given in Section 2.

12.2. Corollary. *Let A be a κ-accessible category with very weak cokernel pairs (e. g., this holds if A has finite products). Assume further that every split monomorphism in $\mathsf{A}_{<\kappa}$ has a cokernel pair (in $\mathsf{A}_{<\kappa}$, or equivalently, in A). Then all κ-pure monomorphisms in A are regular monomorphisms.*

Proof. In any κ-accessible category with very weak cokernel pairs, the classes of κ-pure and strongly κ-pure monomorphisms coincide by Theorem 3.4. Any category with finite products has very weak cokernel pairs by Example 2.2(2). For a morphism i in $\mathsf{A}_{<\kappa}$, the cokernel pair of i in $\mathsf{A}_{<\kappa}$ is the same thing as the cokernel pair of i in A by Lemma 1.5. The rest is clear from the proof of Proposition 12.1. ∎

12.3. Example. Any accessible category has split idempotents [2, Observation 2.4]. In particular, any accessible *additive* category is idempotent-complete. By Remark 6.3, it follows that all pushouts of split monomorphisms exist in any accessible additive category A. In particular, all split monomorphisms have cokernel pairs in A. Thus it follows from Corollary 12.2 that all κ-pure monomorphisms are regular in any κ-accessible additive category A. This result also follows from the discussion of the *κ-pure exact structure* on a κ-accessible additive category in [25, Section 4]. The specific references are [25, Proposition 2.5] (for the existence of the κ-pure exact structure) and [25, Proposition 4.4] (for the description of the admissible monomorphisms in the κ-pure exact structure).

13. Pushouts of Strongly Pure Monomorphisms

The discussion in this section is just a special case of Sections 6–7.

13.1. Proposition. *Let A be a κ-accessible category such that pushouts of split monomorphisms exist in A. Then all pushouts of strongly κ-pure monomorphisms exist in A, and all such pushouts are strongly κ-pure monomorphisms themselves.*

Proof. Let $\mathcal{N}$ be the class of all split monomorphisms in $\mathsf{A}_{<\kappa}$. For any span in $\mathsf{A}_{<\kappa}$, if the pushout exists in A, then it belongs to $\mathsf{A}_{<\kappa}$ by Lemma 1.5. So all pushouts of split monomorphisms exist in $\mathsf{A}_{<\kappa}$ under out assumptions. By Example 6.2(3), it follows that $\mathcal{N}$ is a QE-mono class in $\mathsf{A}_{<\kappa}$. Applying Theorem 7.1, we conclude that the class of strongly κ-pure monomorphisms $\mathcal{M} = \varinjlim_{(\kappa)} \mathcal{N}$ is a QE-mono class in A. In particular, condition (i) from Section 6 is satisfied for $\mathcal{M}$, as desired. ∎

13.2. Corollary. *Let A be a κ-accessible category with very weak cokernel pairs (e. g., this holds if A has finite products). Assume further that pushouts of split monomorphisms exist in $\mathsf{A}_{<\kappa}$. Then all pushouts of κ-pure monomorphisms exist in A, and all such pushouts are κ-pure monomorphisms themselves.*

Proof. Follows from Theorem 3.4, Example 2.2(2), and the proof of Proposition 13.1 (cf. the proof of Corollary 12.2). ∎

13.3. Example. According to Example 12.3, all pushouts of split monomorphisms exist in any accessible additive category A. So it follows from Corollary 13.2 that κ-pure monomorphisms are stable under pushouts in any κ-accessible additive category A. This result also follows from the discussion of the κ-pure exact structure on A in [25, Section 4]; see Example 12.3 for specific references.

13.4. Example. Any small category with split idempotents is accessible [22, Theorem 2.2.2]. Therefore, Remark 6.3 provides an example of an accessible preadditive category that does not have pushouts (or even cokernel pairs, or even very weak cokernel pairs) of split monomorphisms. So pushouts of pure monomorphisms need not exist in an accessible preadditive category, generally speaking.

14. Regularity of Strongly Pure Epimorphisms

Let κ be a regular cardinal and A be a κ-accessible category. The definition of a strongly κ-pure epimorphism in A was given in Section 5. For the definition of a regular epimorphism, see Section 8.

14.1. Proposition. *Let A be a κ-accessible category such that every split epimorphism in A has a kernel pair. Then all strongly κ-pure epimorphisms in A are regular epimorphisms.*

Proof. The assumption of the proposition means that all split epimorphisms in A are effective (since all split epimorphisms are always regular; see Section 8). By Lemma 8.1(b), it follows that all the κ-directed colimits of split epimorphisms in A are effective epimorphisms, too. For an alternative argument applicable under slightly more restrictive (or slightly different) assumptions, see Proposition 15.1 below. ∎

The definition of a very weak split pullback was given in Section 4.

14.2. Corollary. *Let A be a κ-accessible category with very weak split pullbacks (e. g., this holds if A has finite coproducts). Assume further that every split epimorphism in $\mathsf{A}_{<\kappa}$ has a kernel pair in A (in particular, this holds if every split epimorphism has a kernel pair in $\mathsf{A}_{<\kappa}$). Then all κ-pure epimorphisms in A are regular epimorphisms.*

Proof. In any κ-accessible category with very weak split pullbacks, the classes of κ-pure and strongly κ-pure epimorphisms coincide by Theorem 5.4. Any category with finite coproducts has very weak split pullbacks by Example 4.3(2). For a morphism p in $\mathsf{A}_{<\kappa}$, if the kernel pair of p exists in $\mathsf{A}_{<\kappa}$, then it is also the kernel pair of p in A by Lemma 1.6. The rest is clear from the proof of Proposition 14.1. ∎

14.3. Example. Similarly to Example 12.3, all pullbacks of split epimorphisms exist in any accessible *additive* category A by Remark 8.3. In particular, all split epimorphisms have kernel pairs in A. Similarly, all split epimorphisms have kernel pairs in $\mathsf{A}_{<\kappa}$, since $\mathsf{A}_{<\kappa}$ is idempotent-complete by Remark 7.3. Thus it follows from Corollary 14.2 that all κ-pure epimorphisms are regular in any κ-accessible additive category A. This result also follows from the discussion of the *κ-pure exact structure* on a κ-accessible additive category in [25, Section 4]. The specific references are [25, Proposition 2.5] (for the existence of the κ-pure exact structure) and [25, Proposition 4.2] (for the description of the admissible epimorphisms in the κ-pure exact structure).

15. Pullbacks of Strongly Pure Epimorphisms

15.1. Proposition. *Let A be a κ-accessible category such that pullbacks of split epimorphisms exist in the category $\mathsf{A}_{<\kappa}$. Then all pullbacks of strongly κ-pure epimorphisms exist in A, and all such pullbacks are strongly κ-pure epimorphisms themselves.*

Proof. Let $\mathcal{Q}$ be the class of all split epimorphisms in $\mathsf{A}_{<\kappa}$. By Example 8.2(3), $\mathcal{Q}$ is a QE-epi class in $\mathsf{A}_{<\kappa}$. Applying Theorem 9.1, we conclude that the class of strongly κ-pure epimorphisms $\mathcal{P} = \varinjlim_{(\kappa)} \mathcal{Q}$ is a QE-epi class in A. In particular, condition (i*) from Section 8 is satisfied for $\mathcal{P}$, as desired. ∎

15.2. Corollary. *Let A be a κ-accessible category with very weak split pullbacks (e. g., this holds if A has finite coproducts). Assume further that pullbacks of split epimorphisms exist in $\mathsf{A}_{<\kappa}$. (More generally, it suffices to assume that every split epimorphism in $\mathsf{A}_{<\kappa}$ has a pullback in A along every morphism in $\mathsf{A}_{<\kappa}$.) Then all pullbacks of κ-pure epimorphisms exist in A, and all such pullbacks are κ-pure epimorphisms themselves.*

Proof. The main assertion is obtained by combining Theorem 5.4, Example 4.3(2), and Proposition 15.1 (cf. the proof of Corollary 14.2). For the more general assertion under the assumption in parentheses, one needs to follow the proof of condition (i*) for the class $\mathcal{P}$ in Theorem 9.1. ∎

15.3. Example. According to Example 14.3, all pullbacks of split epimorphisms exist in any κ-accessible additive category A, as well as in its full subcategory $\mathsf{A}_{<\kappa}$. So it follows from Corollary 15.2 that κ-pure epimorphisms are stable under pullbacks in any κ-accessible additive category A. This result also follows from the discussion of the κ-pure exact structure on A in [25, Section 4]; see Example 14.3 for specific references.

15.4. Example. Any small category with split idempotents is accessible [22, Theorem 2.2.2]. Therefore, Remark 8.3 provides an example of an accessible preadditive category that does not have pullbacks (or even kernel pairs) of split epimorphisms. So pullbacks of pure epimorphisms need not exist in an accessible preadditive category in general.

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