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HOMOTOPICAL INTERPRETATION OF GLOBULAR COMPLEX BY MULTIPOINTED D-SPACE

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Abstract. Globular complexes were introduced by E. Goubault and the author to model higher dimensional automata. Globular complexes are topological spaces equipped with a globular decomposition which is the directed analogue of the cellular decomposition of a CW-complex. We prove that there exists a combinatorial model category such that the cellular objects are exactly the globular complexes and such that the homotopy category is equivalent to the homotopy category of flows. The underlying category of this model category is a variant of M. Grandis' notion of d-space over a topological space colimit generated by simplices. This result enables us to understand the relationship between the framework of flows and other works in directed algebraic topology using d-spaces. It also enables us to prove that the underlying homotopy type functor of flows can be interpreted up to equivalences of categories as the total left derived functor of a left Quillen adjoint.

1. Introduction

Globular complexes were introduced by E. Goubault and the author to model higher dimensional automata in [GG03], and studied further in [Gau05a]. They are topological spaces modeling a state space equipped with a globular decomposition encoding the temporal ordering which is a directed analogue of the cellular decomposition of a CW-complex.

The fundamental geometric shape of this topological model of concurrency is the topological globe of a space Z, practically of a n-dimensional disk for some $n \geqslant 0$. The underlying state space is the quotient

$$\frac{\{\widehat{0},\widehat{1}\} \sqcup (Z \times [0,1])}{(z,0)=(z',0)=\widehat{0},(z,1)=(z',1)=\widehat{1}}$$

equal to the unreduced suspension of Z if $Z \neq \varnothing$ and equal to the discrete space $\{\widehat{0},\widehat{1}\}$ if the space Z is empty. The segment $[0,1]$ together with the usual total ordering plays in this setting the role of time ordering. The point $\widehat{0}$ is the initial state and the point $\widehat{1}$ is the final state of the globe of Z. The execution paths are the continuous maps $t \mapsto (z,t)$ and all strictly increasing reparametrizations preserving the initial and final states. This

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construction means that, between $\widehat{0}$ and $\widehat{1}$, orthogonally to the time flow, there is, up to strictly increasing reparametrization, a topological space of execution paths Z which represents the geometry of concurrency. By pasting together this kind of geometric shape with Z being a sphere or a disk using attaching maps locally preserving time ordering, it is then possible to construct, up to homotopy, any time flow of any concurrent process. In particular, the time flow of any process algebra can be modeled by a precubical set [Gau08b], and then by a globular complex using a realization functor from precubical sets to globular complexes ([GG03, Proposition 3.9] and [Gau08a, Theorem 5.4.2]). See also in [GG03] several examples of PV diagrams whose first appearance in computer science goes back to [Dij68], and in concurrency theory to [Gun94].

Although the topological model of globular complexes is included in all other topological models [Gou03] introduced for this purpose (local pospace [FGR98], d-space [Gra03], stream [Kri09]), it is therefore expressive enough to contain all known examples coming from concurrency.

However, the category of globular complexes alone does not satisfy any good mathematical property for doing homotopy because it is, in a sense, too small. In particular, it is not complete nor cocomplete. This is one of the reasons for introducing the category of flows in [Gau03] and for constructing a functor associating a globular complex with a flow in [Gau05a] allowing the interpretation of some geometric properties of globular complexes in the model category of flows.

We prove in this work that a variant of M. Grandis' notion of d-space [Gra03] can be used to give a homotopical interpretation of the notion of globular complex. Indeed, using this variant, it is possible to construct a combinatorial model category such that the globular complexes are exactly the cellular objects.

This result must be understood as a directed version of the following fact: The category of cellular spaces, in which the cells are not necessarily attached by following the increasing ordering of dimensions as in CW-complexes, is the category of cellular objects of the usual model category of topological spaces. Moreover, if we choose to work in the category of Δ-generated spaces, i.e. with spaces which are colimits of simplices, then the model category becomes combinatorial.

It turns out that the model category of multipointed d-spaces has a homotopy category which is equivalent to the homotopy category of flows of [Gau03]. So the result of this paper enables us to understand the relationship between the framework of flows and other works in directed algebraic topology using M. Grandis' d-spaces.

As a straightforward application, it is also proved that, up to equivalences of categories, the underlying homotopy type functor of flows introduced in [Gau05a] can be viewed as the total left derived functor of a left Quillen adjoint. This result is interesting since this functor is complicated to use. Indeed, it takes a flow to a homotopy type of topological space. The latter plays the role of the underlying state space which is unique only up to homotopy, not up to homeomorphism, in the framework of flows. This result will simplify future calculations of the underlying homotopy type thanks to the possibility of using homotopy colimit techniques.

Outline of the paper. Section 2 is devoted to a short exposition about topological spaces (k-space, Δ-generated space, compactly generated space). Proposition 2.8 and Proposition 2.9 seem to be new. Section 3 presents the variant of Grandis' notion of d-space which is used in the paper and it is proved that this new category is locally presentable. The new, and short, definition of a globular complex is given in Section 4. It is also proved in this section that the new globular complexes are exactly the ones previously defined in [Gau05a]. Section 5 is a technical section which sketches the theory of inclusions of a strong deformation retract in the category of multipointed d-spaces. The main result, the closure of these maps under pushout, is used in the construction of the model structure. Section 6 constructs the model structure. Section 7 establishes the equivalence between the homotopy category of multipointed d-spaces and the homotopy category of flows of [Gau03]. The same section also explores other connections between multipointed d-spaces and flows. In particular, it is proved that there is a kind of left Quillen equivalence of cofibration categories from multipointed d-spaces to flows. And finally Section 8 is the application interpreting the underlying homotopy type functor of flows as a total left derived functor.

Prerequisites. There are many available references for general topology applied to algebraic topology, e.g., [Mun75] [Hat02]. However, the notion of k-space which is presented is not exactly the good one. In general, the category of k-spaces is unfortunately defined as the coreflective hull of the category of quasi-compact spaces (i.e. spaces satisfying the finite open subcovering property and which are not necessarily Hausdorff) which is not cartesian closed ([Bre71], and [Čin91, Theorem 3.6]). One then obtains a cartesian closed full subcategory by restricting to Hausdorff spaces ([Bor94b, Definition 7.2.5 and Corollary 7.2.6]). However, it is preferable to use the notion of weak Hausdorff space since some natural constructions can lead outside this category. So [May99, Chapter 5] or [FP90] Appendix A must be preferred for a first reading. See also [Bro06] and the appendix of [Lew78]. Section 2 of this paper is an important section collecting the properties of topological spaces used in this work. In particular, the category of k-spaces is defined as the coreflective hull of the full subcategory of compact spaces, i.e. of quasi-compact Hausdorff spaces. The latter category is cartesian closed.

The reading of this work requires some familiarity with model category techniques [Hov99] [Hir03], with category theory [ML98] [Bor94a][GZ67], and especially with locally presentable categories [AR94] and topological categories [AHS06].

Notations. All categories are locally small. The set of morphisms from X to Y in a category $\mathcal{C}$ is denoted by $\mathcal{C}(X,Y)$. The identity of X is denoted by Id_X. Colimits are denoted by $\varinjlim$ and limits by $\varprojlim$. Let $\mathcal{C}$ be a cocomplete category. The class of morphisms of $\mathcal{C}$ that are transfinite compositions of pushouts of elements of a set of morphisms K is denoted by $\mathbf{cell}(K)$. An element of $\mathbf{cell}(K)$ is called a *relative K-cell complex*. The category of sets is denoted by $\mathbf{Set}$. The class of maps satisfying the right lifting property with respect to the maps of K is denoted by $\mathbf{inj}(K)$. The class of maps satisfying the left lifting property with respect to the maps of $\mathbf{inj}(K)$ is denoted by $\mathbf{cof}(K)$. The cofibrant replacement functor of a model category is denoted by $(-)^{cof}$. The notation $\simeq$ means *weak equivalence* or *equivalence of categories*, the notation $\cong$ means *isomorphism*. A combinatorial model category is a cofibrantly generated model category such that the underlying category is locally presentable. The notation Id_A means identity of A. The initial object (resp. final object) of a category is denoted by $\varnothing$ (resp. $\mathbf{1}$). In a cofibrantly generated model category with set of generating cofibrations I, a *cellular object* is an object X such that the map $\varnothing \to X$ belongs to $\mathbf{cell}(I)$. The cofibrant objects are the retracts of the cellular objects in a cofibrantly generated model category.

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2. About topological spaces

We must be very careful in this paper since we are going to work with Δ-generated spaces without any kind of separation condition. However, every compact space is Hausdorff.

Let $\mathcal{T}$ be the category of general topological spaces [Mun75] [Hat02]. This category is complete and cocomplete. Limits are obtained by taking the initial topology, and colimits are obtained by taking the final topology on the underlying (co)limits of sets. This category is the paradigm of topological category because of the existence of the initial and final structures [AHS06].

A one-to-one continuous map $f : X \to Y$ between general topological spaces is a *(resp. closed) inclusion of spaces* if f induces a homeomorphism $X \cong f(X)$ where $f(X)$ is a (resp. closed) subset of Y equipped with the relative topology. If f is a closed inclusion and if moreover for all $y \in Y \backslash f(X)$, $\{y\}$ is closed in Y, then $f : X \to Y$ is called a *closed T_1-inclusion of spaces.*

2.1. Proposition. *(well-known) Let $i : X \to Y$ be a continuous map between general topological spaces. If there exists a retract $r : Y \to X$, i.e. a continuous map r with $r \circ i = \mathrm{Id}_X$, then i is an inclusion of spaces and X is equipped with the final topology with respect to r, i.e. r is a quotient map.*

Proof. The set map i is one-to-one and the set map r is onto. Since i is the equalizer (resp. p is the coequalizer) of the pair of maps $(i \circ r, \mathrm{Id}_Y)$, one has $X \cong \{y \in Y, y = i(r(y))\} \cong i(X)$ and X is a quotient of Y equipped with the final topology. ∎

Let us emphasize two facts related to Proposition 2.1:

1. There does not exist any reason for the map $i : X \to Y$ to be a closed inclusion of spaces without any additional separation condition.

2. If the map $i : X \to Y$ is a closed inclusion of spaces anyway, then there does not exist any reason for the map $i : X \to Y$ to be a closed T_1-inclusion of spaces without any additional separation condition.

Let $\mathcal{B}$ be a full subcategory of the category $\mathcal{T}$ of general topological spaces. A topological space X is *$\mathcal{B}$-generated* if the natural map

$$k_{\mathcal{B}}(X) := \varinjlim_{B\in\mathcal{B}, B\to X} B \longrightarrow X$$

is a homeomorphism (note that the diagram above may be large). The underlying set of the space $k_{\mathcal{B}}(X)$ is equal to the underlying set of the space X. The $\mathcal{B}$-generated spaces assemble to a full coreflective subcategory of $\mathcal{T}$, in fact the coreflective hull of $\mathcal{B}$, denoted by $\mathbf{Top}_{\mathcal{B}}$. The right adjoint to the inclusion functor $\mathbf{Top}_{\mathcal{B}} \subset \mathcal{T}$ is precisely the functor $k_{\mathcal{B}}$, called the *Kelleyfication functor*. The category $\mathbf{Top}_{\mathcal{B}}$ is complete and cocomplete. Colimits in $\mathbf{Top}_{\mathcal{B}}$ and in $\mathcal{T}$ are the same. Limits in $\mathbf{Top}_{\mathcal{B}}$ are obtained by calculating the limits in $\mathcal{T}$ and by applying the Kelleyfication functor $k_{\mathcal{B}}$. See [Vog71] for a proof of all these facts. The category $\mathbf{Top}_{\mathcal{B}}$ is also locally presentable as soon as $\mathcal{B}$ is small by [FR08, Theorem 3.6]. This fact was conjectured by J. H. Smith in unpublished notes.

2.2. Notation. *The binary product in the category $\mathbf{Top}_{\mathcal{B}}$ of $\mathcal{B}$-generated spaces is denoted by $\times_{\mathcal{B}}$, if necessary. The binary product in the category $\mathcal{T}$ is denoted by $\times_{\mathcal{T}}$, if necessary.*

2.3. Proposition. *([Vog71, Proposition 1.5]) Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two full subcategories of the category $\mathcal{T}$ with $\mathcal{B}_1 \subset \mathcal{B}_2$. Then one has $\mathbf{Top}_{\mathcal{B}_1} \subset \mathbf{Top}_{\mathcal{B}_2}$ and the inclusion functor $i_{\mathcal{B}_1}^{\mathcal{B}_2}$ has a right adjoint $k_{\mathcal{B}_1}$.*

2.4. Definition. *(e.g., [Bor94b, Proposition 7.1.5] or [Lew78, p 160]) Let X and Y be two general topological spaces. The space $\mathcal{T}_t(X,Y)$ is the set of continuous maps $\mathcal{T}(X,Y)$ from X to Y equipped with the topology generated by the subbasis $N(h,U)$ where $h : K \to X$ is a continuous map from a compact K to X, where U is an open of Y and where $N(h,U) = \{f : X \to Y, f(h(K)) \subset U\}$. This topology is called the* compact-open topology.

Proposition 2.5 is a slight modification of [Vog71, Section 3], which is stated without proof in [Dug03]. It is important because our theory requires a cartesian closed category of topological spaces.

2.5. Proposition. *(Dugger-Vogt) [Dug03] Let us suppose that every object of $\mathcal{B}$ is compact and that the binary product in $\mathcal{T}$ of two objects of $\mathcal{B}$ is $\mathcal{B}$-generated. Then:*

1. *For any $B \in \mathcal{B}$ and any $X \in \mathbf{Top}_{\mathcal{B}}$, one has $B \times_{\mathcal{T}} X \in \mathbf{Top}_{\mathcal{B}}$.*
2. *The category $\mathbf{Top}_{\mathcal{B}}$ is cartesian closed.*

Sketch of Proof. We follow Vogt's proof. It is well known that for any compact space K, the functor $K \times_{\mathcal{T}} - : \mathcal{T} \to \mathcal{T}$ has the right adjoint $\mathcal{T}_t(K,-) : \mathcal{T} \to \mathcal{T}$. So the partial evaluation set map $\mathcal{T}_t(X,Y) \times_{\mathcal{T}} K \to \mathcal{T}_t(K,Y) \times_{\mathcal{T}} K \to Y$ is continuous for any general topological space X and Y and for any continuous map $K \to X$ with K compact.

Consider the topological space $\mathbf{TOP}_{\mathcal{B}}(X,Y) = k_{\mathcal{B}}(\mathcal{T}_t(X,Y))$, i.e. the set $\mathbf{Top}_{\mathcal{B}}(X,Y)$ equipped with the Kelleyfication of the compact-open topology. Let X and Y be two $\mathcal{B}$-generated topological spaces. For any $B \in \mathcal{B}$, the composite $g : B \to \mathcal{T}_t(X,Y) \times_{\mathcal{T}} X \to Y$ is continuous for any continuous map $B \to \mathcal{T}_t(X,Y) \times_{\mathcal{T}} X$ since g is equal to the composite $B \to \mathcal{T}_t(B,Y) \times_{\mathcal{T}} B \to Y$ where $B \to \mathcal{T}_t(B,Y)$ is the composite $B \to \mathcal{T}_t(X,Y) \times_{\mathcal{T}} X \to \mathcal{T}_t(X,Y) \to \mathcal{T}_t(B,Y)$ and since B is compact. So the set map $\mathbf{TOP}_{\mathcal{B}}(X,Y) \times_{\mathcal{B}} X = k_{\mathcal{B}}(\mathcal{T}_t(X,Y) \times_{\mathcal{T}} X) \to Y$ is continuous.

By hypothesis, if $B_1, B_2 \in \mathcal{B}$, then $B_1 \times_{\mathcal{T}} B_2 \in \mathbf{Top}_{\mathcal{B}}$. Since B_1 is compact, and since colimits in $\mathbf{Top}_{\mathcal{B}}$ and in $\mathcal{T}$ are the same, one deduces that for any $X \in \mathbf{Top}_{\mathcal{B}}$, $B_1 \times_{\mathcal{T}} X \in \mathbf{Top}_{\mathcal{B}}$. So the canonical map $B_1 \times_{\mathcal{B}} X = k_{\mathcal{B}}(B_1 \times_{\mathcal{T}} X) \to B_1 \times_{\mathcal{T}} X$ is a homeomorphism if $B_1 \in \mathcal{B}$ and $X \in \mathbf{Top}_{\mathcal{B}}$. Hence the first assertion.

Let X, Y and Z be three $\mathcal{B}$-generated spaces. Let $f : Y \times_{\mathcal{B}} X \to Z$ be a continuous map. Consider the set map $\widetilde{f} : Y \to \mathbf{TOP}_{\mathcal{B}}(X,Z)$ defined by $\widetilde{f}(y)(x) = f(y,x)$. Let $g : B \to Y$ be a continuous map with $B \in \mathcal{B}$. Then the composite set map $B \to Y \to \mathcal{T}_t(X,Z)$ is continuous since it corresponds by adjunction to the continuous map $B \times_{\mathcal{T}} X \xrightarrow{\cong} B \times_{\mathcal{B}} X \to Y \times_{\mathcal{B}} X \to Z$. Since Y is $\mathcal{B}$-generated, and therefore since $k_{\mathcal{B}}(Y) = Y$, the set map $\widetilde{f} : Y \to \mathbf{TOP}_{\mathcal{B}}(X,Z)$ is therefore continuous. Hence the second assertion. ∎

For the rest of the section, let us suppose that $\mathcal{B}$ satisfies the following properties:

- Every object of $\mathcal{B}$ is compact.
- All simplices $\Delta^n = \{(t_0,\dots,t_n) \in (\mathbb{R}^+)^n, t_0 + \dots + t_n = 1\}$ with $n \geqslant 0$ are objects of $\mathcal{B}$.
- The binary product in $\mathcal{T}$ of two objects of $\mathcal{B}$ is $\mathcal{B}$-generated.

The category $\mathcal{K}$ of all compact spaces satisfies the conditions above. It is the biggest possible choice. An object of $\mathbf{Top}_{\mathcal{K}}$ is called a *k-space* [May99] [FP90] [Lew78].

The full subcategory Δ of $\mathcal{T}$ generated by all topological simplices $\Delta^n = \{(t_0,\dots,t_n) \in (\mathbb{R}^+)^n, t_0+\dots+t_n = 1\}$ with $n \geqslant 0$ is another possible choice. Indeed, one has $\Delta^n \cong |\Delta[n]|$ where $|\Delta[n]|$ is the geometric realization of the n-simplex viewed as a simplicial set. And there is a homeomorphism $\Delta^m \times_{\mathcal{K}} \Delta^n \cong |\Delta[m] \times \Delta[n]|$ by [Hov99, Lemma 3.1.8] or [GZ67, Theorem III.3.1 p49] [1]. At last, since Δ^m is compact, the canonical map $\Delta^m \times_{\mathcal{K}} \Delta^n \to \Delta^m \times_{\mathcal{T}} \Delta^n$ is an isomorphism. This choice is the smallest possible choice. Further details about these topological spaces are available in [Dug03].

As corollary of Proposition 2.5, one has:

2.6. Proposition. *Let X and Y be two $\mathcal{B}$-generated spaces with Y compact. Then the canonical map $X \times_{\mathcal{B}} Y \to X \times_{\mathcal{T}} Y$ is a homeomorphism.*

[1] Note that Gabriel and Zisman's proof is written in the full subcategory of Hausdorff spaces of the coreflective hull of the category of quasi-compact spaces, that is with the wrong notion of k-space.

Proof. Because the colimits in $\mathcal{T}$ and $\mathbf{Top}_{\mathcal{B}}$ are equal, one has:

$$\begin{aligned} X \times_{\mathcal{B}} Y &\cong \varinjlim_{B \to X, B \in \mathcal{B}} (B \times_{\mathcal{B}} Y) && \text{since } - \times_{\mathcal{B}} Y \text{ is a left adjoint by Proposition 2.5 (2)} \\ &\cong \varinjlim_{B \to X, B \in \mathcal{B}} (B \times_{\mathcal{T}} Y) && \text{since } B \times_{\mathcal{T}} Y \in \mathbf{Top}_{\mathcal{B}} \text{ by Proposition 2.5 (1)} \\ &\cong X \times_{\mathcal{T}} Y && \text{since } Y \text{ is compact and since } - \times_{\mathcal{T}} Y \text{ is a left adjoint.} \end{aligned}$$

■

The following two propositions exhibit two striking differences between Δ-generated spaces and k-spaces.

2.7. Proposition. *[Lew78] [Dug03] Let X be a general topological space.*

1. *If X is Δ-generated, then every open subset of X equipped with the relative topology is Δ-generated. There exists a closed subset of the topological 2-simplex Δ^2 which is not Δ-generated when it is equipped with the relative topology.*
2. *If X is a k-space, then every closed subset of X equipped with the relative topology is a k-space. An open subset of a k-space equipped with the relative topology need not be a k-space.*

2.8. Proposition. *Every Δ-generated space is homeomorphic to the disjoint union of its non-empty connected components, which are also its non-empty path-connected components. In particular, a Δ-generated space is connected if and only if it is path-connected.*

Proof. Let X be a Δ-generated space. Let $\widehat{X}$ be the disjoint sum of the connected components (resp. path-connected components) of X which is still a Δ-generated space. There is a canonical continuous map $\widehat{X} \to X$. Let $B \in \Delta$ be a simplex. Then any continuous map $B \to X$ factors uniquely as a composite $B \to \widehat{X} \to X$ since B is connected (resp. path-connected). Thus $\widehat{X} \cong k_\Delta(\widehat{X}) \cong k_\Delta(X) \cong X$. ■

For example, the set of rational numbers $\mathbb{Q}$ equipped with the order topology is a non-discrete totally disconnected space. The latter space is not Δ-generated since $k_\Delta(\mathbb{Q})$ is the set $\mathbb{Q}$ equipped with the discrete topology.

The category of $\mathcal{B}$-generated spaces contains among other things all geometric realizations of simplicial sets, of cubical sets, all simplicial and cubical complexes, the discrete spaces, the n-dimensional sphere $\mathbf{S}^n$ and the n-dimensional disk $\mathbf{D}^n$ for all $n \geqslant 0$, all their colimits in the category of general topological spaces, and so all CW-complexes and all open subspaces of these spaces equipped with the relative topology by Proposition 2.7 (1).

The category of spaces $\mathbf{Top}_{\mathcal{B}}$ has other interesting properties which will be useful for the paper.

2.9. Proposition. *The forgetful functor $\omega_{\mathcal{B}} : \mathbf{Top}_{\mathcal{B}} \to \mathbf{Set}$ is topological and fibre-small in the sense of [AHS06].*

PROOF. The category $\mathbf{Top}_{\mathcal{B}}$ is a concretely coreflective subcategory of $\mathcal{T}$. Therefore it is topological by [AHS06, Theorem 21.33]. ∎

By the topological duality theorem (see [AHS06, Theorem 21.9]), any cocone $(f_i : \omega_{\mathcal{B}}(A_i) \to X)$ of $\mathbf{Top}_{\mathcal{B}}$ has a unique $\omega_{\mathcal{B}}$-final lift $(\overline{f_i} : A_i \to A)$. The space A is the set X equipped with the final topology. In particular, as already mentioned, colimits in $\mathbf{Top}_{\mathcal{B}}$ and in $\mathcal{T}$ are the same. Another consequence is that any quotient of a $\mathcal{B}$-generated space equipped with the final topology is also a $\mathcal{B}$-generated space. In particular, by Proposition 2.1, any retract of a $\mathcal{B}$-generated space is $\mathcal{B}$-generated.

2.10. DEFINITION. *A general topological space X is* weak Hausdorff *if for any continuous map $g : K \to X$ with K compact, $g(K)$ is closed in X. A* compactly generated space *is a weak Hausdorff k-space.*

Note that if X is weak Hausdorff, then any subset of X equipped with the relative topology is weak Hausdorff and that $k_{\mathcal{B}}(X)$ is weak Hausdorff as well since $k_{\mathcal{B}}(X)$ contains more closed subsets than X.

2.11. NOTATION. *The category of compactly generated topological spaces is denoted by* $\mathbf{CGTop}$.

Let us conclude this section by some remarks explaining why the homotopy theory of any of the preceding categories of spaces is the same. By [FR08] [Hov99], the inclusion functor $i_{\Delta}^{\mathcal{K}} : \mathbf{Top}_{\Delta} \to \mathbf{Top}_{\mathcal{K}}$ is a left Quillen equivalence for the Quillen model structure: it is evident that it is a left Quillen adjoint; it is a Quillen equivalence because the natural map $k_{\Delta}(X) \to X$ is a weak homotopy equivalence since for any space $K \in \{\mathbf{S}^n, \mathbf{S}^n \times [0,1], n \geqslant 0\}$, a map $K \to X$ factors uniquely as a composite $K \to k_{\Delta}(X) \to X$. The weak Hausdorffization functor $H : \mathbf{Top}_{\mathcal{K}} \to \mathbf{CGTop}$ left adjoint to the inclusion functor $\mathbf{CGTop} \subset \mathbf{Top}_{\mathcal{K}}$ is also a left Quillen equivalence for the Quillen model structure by [Hov99, Theorem 2.4.23 and Theorem 2.4.25]. All cofibrant spaces of $\mathbf{Top}_{\Delta}$ and $\mathbf{Top}_{\mathcal{K}}$ are weak Hausdorff, and therefore compactly generated because of the inclusion $\mathbf{Top}_{\Delta} \subset \mathbf{Top}_{\mathcal{K}}$.

3. Multipointed d-spaces

3.1. DEFINITION. *A* multipointed space *is a pair $(|X|, X^0)$ where*

- *$|X|$ is a Δ-generated space called the* underlying space *of X.*
- *X^0 is a subset of $|X|$ called the* 0-skeleton *of X.*

A morphism of multipointed spaces $f : X = (|X|, X^0) \to Y = (|Y|, Y^0)$ is a commutative square

$$\begin{tikzcd} X^0 \arrow[r, "f^0"] \arrow[d] & Y^0 \arrow[d] \\ {|X|} \arrow[r, "{|f|}"] & {|Y|}. \end{tikzcd}$$

The corresponding category is denoted by $\mathbf{MTop}_\Delta$.

3.2. NOTATION. *Let* M *be a topological space. Let* ϕ_1 *and* ϕ_2 *be two continuous maps from* $[0,1]$ *to* M *with* $\phi_1(1) = \phi_2(0)$. *Let us denote by* $\phi_1 *_a \phi_2$ *(with* $0 < a < 1$*) the following continuous map: if* $0 \leqslant t \leqslant a$, $(\phi_1 *_a \phi_2)(t) = \phi_1(\frac{t}{a})$ *and if* $a \leqslant t \leqslant 1$, $(\phi_1 *_a \phi_2)(t) = \phi_2(\frac{t-a}{1-a})$.

3.3. DEFINITION. *A* multipointed d-space X *is a triple* $(|X|, X^0, \mathbb{P}^{top}X)$ *where*

- $(|X|, X^0)$ *is a multipointed space.*
- *The set* $\mathbb{P}^{top}X$ *is the disjoint union of the sets* $\mathbb{P}^{top}_{\alpha,\beta}X$ *for* (α,β) *running over* $X^0 \times X^0$ *where* $\mathbb{P}^{top}_{\alpha,\beta}X$ *is a set of continuous paths* ϕ *from* $[0,1]$ *to* $|X|$ *such that* $\phi(0) = \alpha$ *and* $\phi(1) = \beta$ *which is closed under composition* $*_{1/2}$ *and under strictly increasing reparametrization* [2], *that is for every* $\phi \in \mathbb{P}^{top}_{\alpha,\beta}X$ *and for any strictly increasing continuous map* $\psi : [0,1] \to [0,1]$ *with* $\psi(0) = 0$ *and* $\psi(1) = 1$, $\phi \circ \psi \in \mathbb{P}^{top}_{\alpha,\beta}X$. *The element of* $\mathbb{P}^{top}X$ *are called* execution path *or* d-path.

A morphism of multipointed d-spaces $f : X = (|X|, X^0, \mathbb{P}^{top}X) \to Y = (|Y|, Y^0, \mathbb{P}^{top}Y)$ *is a commutative square*

$$\begin{tikzcd} X^0 \arrow[r, "f^0"] \arrow[d] & Y^0 \arrow[d] \\ |X| \arrow[r, "|f|"] & |Y| \end{tikzcd}$$

such that for every $\phi \in \mathbb{P}^{top}X$, *one has* $|f| \circ \phi \in \mathbb{P}^{top}Y$. *Let* $\mathbb{P}^{top}f(\phi) = |f| \circ \phi$. *The corresponding category is denoted by* $\mathbf{MdTop}_\Delta$.

Any set E can be viewed as one of the multipointed d-spaces $(E, E, \varnothing)$, (E, E, E) and $(E, \varnothing, \varnothing)$ with E equipped with the discrete topology.

3.4. CONVENTION. *For the sequel, a set* E *is always associated with the multipointed* d*-space* $(E, E, \varnothing)$.

3.5. THEOREM. *The category* $\mathbf{MdTop}_\Delta$ *is concrete topological and fibre-small over the category of sets, i.e. the functor* $X \mapsto \omega_\Delta(|X|)$ *from* $\mathbf{MdTop}_\Delta$ *to* $\mathbf{Set}$ *is topological and fibre-small. Moreover, the category* $\mathbf{MdTop}_\Delta$ *is locally presentable.*

PROOF. We partially mimic the proof of [FR08, Theorem 4.2] and we use the terminology of [AR94, Chapter 5]. A multipointed d-space X is a concrete structure over the set $\omega_\Delta(|X|)$ which is characterized by a 0-skeleton, a topology and a set of continuous paths satisfying several axioms. The category $\mathbf{Top}_\Delta$ is topological and fibre-small over the category of sets by Proposition 2.9. So by [Ros81, Theorem 5.3], it is isomorphic to the category of models $\mathrm{Mod}(T)$ of a relational universal strict Horn theory T without equality, i.e. all axioms are of the form $(\forall x), \phi(x) \Rightarrow \psi(x)$ where ϕ and ψ are conjunctions of atomic

[2]These two facts imply that the set of paths is closed under $*_t$ for any $t \in]0,1[$.

formulas without equalities. As in the proof of [FR08, Theorem 3.6], let us suppose that T contains all its consequences. It means that a universal strict Horn sentence without equality belongs to T if and only if it holds for all models of T. The theory T may contain a proper class of axioms. Let us construct as in the proof of [FR08, Theorem 3.6] an increasing chain of full coreflective subcategories such that the union is equal to $\mathrm{Mod}(T) = \mathbf{Top}_\Delta$

$$\mathrm{Mod}(T_0) \subset \mathrm{Mod}(T_1) \subset \dots \mathrm{Mod}(T_\alpha) \subset \dots.$$

where each T_α is a subset of axioms of T indexed by an ordinal α. Let α_0 be an ordinal such that the subcategory $\mathrm{Mod}(T_{\alpha_0})$ contains all simplices $\Delta^n = \{(t_0, \dots, t_n) \in (\mathbb{R}^+)^n, t_0 + \dots + t_n = 1\}$ with $n \geqslant 0$. Then one has $\mathrm{Mod}(T) \subset \mathrm{Mod}(T_{\alpha_0})$ since the category $\mathrm{Mod}(T_{\alpha_0})$ is cocomplete. So one obtains the isomorphism of categories $\mathrm{Mod}(T_{\alpha_0}) \cong \mathbf{Top}_\Delta$. The theory T_{α_0} is a universal strict Horn theory without equality containing a set of axioms using $2^{\aleph_0}$-ary relational symbols R_j for $j \in J$ for some set J. See [Man80] for a description of this theory for the category of general topological spaces. One has the isomorphism of categories $\mathbf{MTop}_\Delta \cong \mathrm{Mod}(T_{\alpha_0} \cup \{S\})$ where S is a 1-ary relational symbol whose interpretation is the 0-skeleton. Let us add to the theory $T_{\alpha_0} \cup \{S\}$ a new $2^{\aleph_0}$-ary relational symbol R whose interpretation is the set of execution paths. And let us add the axioms (as in the proof of [FR08, Theorem 4.2]):

1. $(\forall x) R(x) \Rightarrow (S(x_0) \wedge S(x_1))$
2. $(\forall x, y, z) \left(\left(\bigwedge_{0 \leqslant t \leqslant 1/2} x_{2t} = z_t \right) \wedge \left(\bigwedge_{1/2 \leqslant t \leqslant 1} y_{2t-1} = z_{\frac{1}{2}+t} \right) \wedge R(x) \wedge R(y) \right) \Rightarrow R(z)$
3. $(\forall x) R(x) \Rightarrow R(xt)$ where t is a strictly increasing reparametrization
4. $(\forall x) R(x) \Rightarrow R_j(xa)$ where $j \in J$ and T_{α_0} satisfies R_j for a.

The first axiom says that for all execution paths x of a multipointed d-space X, one has $x_0, x_1 \in X^0$. The second axiom says that the set of execution paths is closed under composition $*_{1/2}$, and the third one that the same set is closed under strictly increasing reparametrization. The last axiom says that all execution paths are continuous. Then the new theory T' is a relational universal strict Horn theory without equality containing a set of axioms. So by [AR94, Theorem 5.30], the category $\mathbf{MdTop}_\Delta \cong \mathrm{Mod}(T')$ is locally presentable [3], and by [Ros81, Theorem 5.3], it is fibre-small concrete and topological over the category of sets. ∎

Note that the same proof shows that the category $\mathbf{MTop}_\Delta$ is also concrete topological and locally presentable. It is of course possible to prove directly that $\mathbf{MTop}_\Delta$ and $\mathbf{MdTop}_\Delta$ are concrete fibre-small and topological using Proposition 2.9 and Proposition 3.6 below. But we do not know how to prove that they are locally presentable without using a logical argument.

[3] A relational universal strict Horn theory is a limit theory since the axiom $(\forall x)(A(x) \Rightarrow B(x))$ is equivalent to the axiom $(\forall x)(\exists y)(A(x) \Rightarrow (B(x) \wedge y = x))$: see [Ros81, p 324].

Since all relational symbols appearing in the relational universal strict Horn theory axiomatizing $\mathbf{MdTop}_\Delta$ are of arity at most $2^{\aleph_0}$, the category $\mathbf{MdTop}_\Delta$ is locally λ-presentable, where λ denotes a regular cardinal greater or equal to $2^{\aleph_0}$ ([HJ99, p 160]).

The proof of the following proposition makes explicit, using the $*_t$ composition laws, the construction of colimits in $\mathbf{MdTop}_\Delta$.

3.6. Proposition. *The functor* $U_\Delta : X = (|X|, X^0, \mathbb{P}^{top}X) \mapsto |X|$ *from* $\mathbf{MdTop}_\Delta$ *to* $\mathbf{Top}_\Delta$ *is topological and fibre-small. In particular, it creates limits and colimits.*

Proof. That it is fibre-small is clear. It is very easy to prove that the functor U_Δ is topological by observing that the functor $\omega_\Delta \circ U_\Delta$ is topological by Theorem 3.5 and that the functor ω_Δ is topological by Proposition 2.9. We prefer to give a proof with an explicit construction of the final lift using the $*_t$ composition laws because this proof will be reused several times in the paper. By [AHS06, Theorem 21.9], it then suffices to see that the forgetful functor $U_\Delta : \mathbf{MdTop}_\Delta \to \mathbf{Top}_\Delta$ satisfies: for any cocone $(f_i : U_\Delta(A_i) \to X)$, there is a unique U_Δ-final lift $(\overline{f_i} : A_i \to A)$. Let A^0 be the image of the set map $\varinjlim A_i^0 \to X = |A|$. Let $\mathbb{P}^{top}A$ be the set of all possible finite compositions $*_t$ of $f_i \circ \phi_i$ with $\phi_i \in \mathbb{P}^{top}A_i$ and $t \in]0,1[$. Then one obtains a set of continuous paths which is closed under strictly increasing reparametrization. Indeed, let $\phi_0 *_{t_1} \phi_1 *_{t_2} \cdots *_{t_n} \phi_n$ such a path. Then a strictly increasing reparametrization is of the form $\phi_0' *_{t_1'} \phi_1 *_{t_2'} \cdots *_{t_n'} \phi_n'$ where the ϕ_i' maps are strictly increasing reparametrizations of the ϕ_i maps. Then the cocone $(\overline{f_i} : A_i \to A)$ is the unique U_Δ-final lift. The last sentence is a consequence of [AHS06, Proposition 21.15]. ∎

Let $f : X \to Y$ be a map of multipointed d-spaces. For every $(\alpha,\beta) \in X^0 \times X^0$, the set map $\mathbb{P}^{top}f : \mathbb{P}^{top}_{\alpha,\beta}X \to \mathbb{P}^{top}_{f^0(\alpha),f^0(\beta)}Y$ is continuous if $\mathbb{P}^{top}_{\alpha,\beta}X$ and $\mathbb{P}^{top}_{f^0(\alpha),f^0(\beta)}Y$ are equipped with the Kelleyfication of the relative topology coming from the inclusions $\mathbb{P}^{top}_{\alpha,\beta}X \subset \mathbf{TOP}_\Delta([0,1],|X|)$ and $\mathbb{P}^{top}_{f^0(\alpha),f^0(\beta)}Y \subset \mathbf{TOP}_\Delta([0,1],|Y|)$.

The following proposition will be used in the paper:

3.7. Proposition. *The functor* $\mathbb{P}^{top} : \mathbf{MdTop}_\Delta \to \mathbf{Top}_\Delta$ *is finitely accessible.*

Proof. Since the forgetful functor $\omega_\Delta : \mathbf{Top}_\Delta \to \mathbf{Set}$ is topological by Proposition 2.9, it creates colimits by [AHS06, Proposition 21.15]. So it suffices to prove that the functor $\omega_\Delta \circ \mathbb{P}^{top} : \mathbf{MdTop}_\Delta \to \mathbf{Set}$ is finitely accessible. The latter fact is due to the construction of colimits in $\mathbf{MdTop}_\Delta$ (see Proposition 3.6). ∎

Since the functor $\mathbb{P}^{top} : \mathbf{MdTop}_\Delta \to \mathbf{Top}_\Delta$ preserves limits, it is a right adjoint by [AR94, Theorem 1.66]. The left adjoint $G : \mathbf{Top}_\Delta \to \mathbf{MdTop}_\Delta$ is explicitly constructed in Section 4.

4. Globular complexes

Let Z be a topological space. By definition, the *topological globe of* Z, which is denoted by $\mathrm{Glob}^{top}(Z)$, is the multipointed d-space

$$\left(\frac{\{\widehat{0},\widehat{1}\}\sqcup(Z\times_{\Delta}[0,1])}{(z,0)=(z',0)=\widehat{0},(z,1)=(z',1)=\widehat{1}},\{\widehat{0},\widehat{1}\},\mathbb{P}^{top}\mathrm{Glob}^{top}(Z)\right)$$

where $\mathbb{P}^{top}\mathrm{Glob}^{top}(Z)$ is the closure by strictly increasing reparametrizations of the set of continuous maps $\{t\mapsto(z,t),z\in Z\}$. In particular, $\mathrm{Glob}^{top}(\varnothing)$ is the multipointed d-space $\{\widehat{0},\widehat{1}\}=(\{\widehat{0},\widehat{1}\},\{\widehat{0},\widehat{1}\},\varnothing)$. Let

$$\boxed{I^{gl,top}=\{\mathrm{Glob}^{top}(\mathbf{S}^{n-1})\longrightarrow\mathrm{Glob}^{top}(\mathbf{D}^{n}),n\geqslant 0\}}$$

where the maps are the image by $\mathrm{Glob}(-)$ of the inclusions of spaces $\mathbf{S}^{n-1}\subset\mathbf{D}^n$ with $n\geqslant 0$.

4.1. Definition. *A* globular complex *is a multipointed d-space X such that the map $X^0\to X$ is a relative $I^{gl,top}$-cell complex. The set of cells of $X^0\to X$ is called the* globular decomposition *of the globular complex.*

4.2. Notation. $\overrightarrow{I}^{top}=\mathrm{Glob}^{top}(\mathbf{D}^0)$.

4.3. Proposition. *Let $f:X\to Y$ be an element of* $\mathbf{cell}(I^{gl,top})$. *Then $|f|:|X|\to|Y|$ is a Serre cofibration of spaces. In particular, it is a closed T_1-inclusion of spaces.*

Proof. The continuous map $|\mathrm{Glob}^{top}(\mathbf{S}^{n-1})|\to|\mathrm{Glob}^{top}(\mathbf{D}^{n})|$ is a cofibration by [Gau06, Theorem 8.2] (the fact that [Gau06] is written in the category of compactly generated spaces does not matter here). Since the class of cofibrations of the Quillen model structure of $\mathbf{Top}_{\Delta}$ is closed under pushout and transfinite composition, one obtains the first sentence using Proposition 3.6. The map $|f|:|X|\to|Y|$ is also a cofibration in $\mathcal{T}$ and therefore a closed T_1-inclusion of spaces by [Hov99, Lemma 2.4.5]. ∎

4.4. Proposition. *Let X be a globular complex. Then the topological space $|X|$ is compactly generated.*

Proof. By Proposition 3.6 and Proposition 4.3, the space $|X|$ is cofibrant. Therefore it is weak Hausdorff. ∎

4.5. Proposition. *Let X be a globular complex. Then the space $\mathbb{P}^{top}X$ is compactly generated.*

Proof. The space $\mathcal{T}_t([0,1],|X^{cof}|)$ is weak Hausdorff by [Lew78, Lemma 5.2] since $|X^{cof}|$ is weak Hausdorff by Proposition 4.4. Thus the space $\mathbf{TOP}_{\Delta}([0,1],|X^{cof}|)$ is weak Hausdorff since the Kelleyfication functor adds open and closed subsets. So $\mathbb{P}^{top}X$ which is equipped with the Kelleyfication of the relative topology is weak Hausdorff as well. Thus it is compactly generated by Proposition 2.3. ∎

4.6. Proposition. *([Lew78, Lemma 1.1]) Let X be a weak Hausdorff space. Let $g : K \to X$ be a continuous map from a compact K to X. Then $g(K)$ is compact.*

4.7. Proposition. *Let X be a globular complex of* $\mathbf{MdTop}_\Delta$*. Then X^0 is a discrete subspace of X. Let $\phi \in \mathbb{P}^{top}X$. Then there exist $0 = t_0 < \dots < t_n = 1$ such that*

- *$\phi(t_i) \in X^0$ for all $i \in \{0, 1, \dots, n\}$*
- *$\phi(]t_i, t_{i+1}[) \cap X^0 = \varnothing$ for all $i \in \{0, 1, \dots, n-1\}$*
- *the restriction $\phi\restriction_{]t_i,t_{i+1}[}$ is one-to-one.*

Proof. By definition, the map $X^0 \to X$ is a relative $I^{gl,top}$-cell complex. By Proposition 4.3, the continuous map $X^0 \to |X|$ is a closed T_1-inclusion of spaces. So X^0 is a discrete subspace of X. Since $[0, 1]$ is compact and since $|X|$ is compactly generated by Proposition 4.4, the subset $\phi([0, 1])$ is a closed subset and a compact subspace of $|X|$ by Proposition 4.6. Thus $\phi([0, 1]) \cap X^0$ is finite. The rest of the statement is then clear. ∎

Thus this new definition of globular complex coincides with the (very long !) definition of [Gau05a].

4.8. Notation. *The full subcategory of globular complexes of* $\mathbf{MdTop}_\Delta$ *is denoted by* **glTop**.

As an illustration of the objects of this section, it is now constructed the left adjoint $G : \mathbf{Top}_\Delta \to \mathbf{MdTop}_\Delta$ of the path space functor $\mathbb{P}^{top} : \mathbf{MdTop}_\Delta \to \mathbf{Top}_\Delta$. Let Z be a Δ-generated space. If Z is non-empty connected, one has the natural set bijection

$$\mathbf{MdTop}_\Delta(\mathrm{Glob}^{top}(Z), X) \cong \mathbf{Top}_\Delta(Z, \mathbb{P}^{top}X)$$

because of the cartesian closedness of $\mathbf{Top}_\Delta$. In the general case, the Δ-generated space Z is homeomorphic to the disjoint sum of its non-empty connected components by Proposition 2.8. Denote this situation by $Z \cong \bigsqcup_{Z_i \in \pi_0(Z)} Z_i$. Then the functor G is defined on objects by

$$\boxed{G(Z) = \bigsqcup_{Z_i \in \pi_0(Z)} \mathrm{Glob}^{top}(Z_i)}$$

and in on obvious way on morphisms.

4.9. Proposition. *The functor $G : \mathbf{Top}_\Delta \to \mathbf{MdTop}_\Delta$ is left adjoint to the functor $\mathbb{P}^{top} : \mathbf{MdTop}_\Delta \to \mathbf{Top}_\Delta$.*

Proof. It suffices to use the fact that $\mathbf{Top}_\Delta$ is cartesian closed. ∎

5. S-homotopy and strong deformation retract

The space $\mathbf{MTOP}_\Delta((|X|,X^0),(|Y|,Y^0))$ is defined as the set

$$\mathbf{MTop}_\Delta((|X|,X^0),(|Y|,Y^0))$$

equipped with the Kelleyfication of the relative topology coming from the set inclusion

$$\mathbf{MTop}_\Delta((|X|,X^0),(|Y|,Y^0)) \subset \mathbf{TOP}_\Delta(|X|,|Y|) \times \mathbf{Set}(X^0,Y^0)$$

where $\mathbf{Set}(X^0,Y^0)$ is equipped with the discrete topology. The space

$$\mathbf{MDTOP}_\Delta((|X|,X^0,\mathbb{P}^{top}X),(|Y|,Y^0,\mathbb{P}^{top}Y))$$

is defined as the set $\mathbf{MdTop}_\Delta((|X|,X^0,\mathbb{P}^{top}X),(|Y|,Y^0,\mathbb{P}^{top}Y))$ equipped with the Kelleyfication of the relative topology coming from the set inclusion

$$\mathbf{MdTop}_\Delta((|X|,X^0,\mathbb{P}^{top}X),(|Y|,Y^0,\mathbb{P}^{top}Y)) \subset \mathbf{MTOP}_\Delta((|X|,X^0),(|Y|,Y^0)).$$

5.1. Definition. *(Compare with [Gau03, Definition 7.2 and Proposition 7.5]) Let $f,g : X \rightrightarrows Y$ be two morphisms of multipointed d-spaces. Then f and g are* S-homotopic *if there exists a continuous map $H : [0,1] \to \mathbf{MDTOP}_\Delta(X,Y)$ called a* S-homotopy *such that $H(0) = f$ and $H(1) = g$. This situation is denoted by $\sim_S$. Two multipointed d-spaces X and Y are* S-homotopy equivalent *if and only if there exist two morphisms $f : X \leftrightarrows Y : g$ with $g \circ f \sim_S \mathrm{Id}_X$ and $f \circ g \sim_S \mathrm{Id}_Y$.*

5.2. Definition. *(Compare with [Gau03, Proposition 8.1]) Let X be a multipointed d-space. Let U be a non-empty connected Δ-generated space. Then the multipointed d-space $U \boxtimes X$ is defined as follows:*

- *Let $(U \boxtimes X)^0 = X^0$.*
- *Let $|U \boxtimes X|$ be defined by the pushout diagram of spaces*

$$\begin{tikzcd} U \times_\Delta X^0 \arrow[r] \arrow[d] & U \times_\Delta |X| \arrow[d] \\ X^0 \arrow[r] & {|U \boxtimes X|.} \end{tikzcd}$$

- *The set $\mathbb{P}^{top}_{\alpha,\beta}(U \boxtimes X)$ is the smallest subset of continuous maps from $[0,1]$ to $|U \boxtimes X|$ containing the continuous maps $\phi : [0,1] \to |U \boxtimes X|$ of the form $t \mapsto (u, \phi_2(t))$ where $u \in [0,1]$ and $\phi_2 \in \mathbb{P}^{top}_{\alpha,\beta}X$ and closed under composition $*_{1/2}$ and strictly increasing reparametrization.*

5.3. Definition. *Let X be a multipointed d-space. The multipointed d-space $[0,1] \boxtimes X$ is called the* cylinder *of X.*

5.4. Theorem. *(Compare with [Gau03, Theorem 7.9]) Let U be a non-empty connected space. Let X and Y be two multipointed d-spaces. Then there is a natural bijection of sets*

$$\mathbf{MdTop}_\Delta(U \boxtimes X, Y) \cong \mathbf{Top}_\Delta(U, \mathbf{MDTOP}_\Delta(X,Y)).$$

Proof. Let $f : U \boxtimes X \to Y$ be a map of multipointed d-spaces. Then by definition, one has a set map $f^0 : X^0 \to Y^0$ and a continuous map $|f| : |U \boxtimes X| \to |Y|$. By adjunction, the composite $U \times_\Delta |X| \to |U \boxtimes X| \to |Y|$ gives rise to a continuous map $\widehat{f} : U \to \mathbf{TOP}_\Delta(|X|,|Y|)$. Let $\phi_2 \in \mathbb{P}^{top}_{\alpha,\beta}X$. Then one has the equality

$$(\widehat{f}(u) \circ \phi_2)(t) = f(u, \phi_2(t))$$

for any $(u,t) \in U \times_\Delta [0,1]$. The continuous map $t \mapsto (u, \phi_2(t))$ is an element of $\mathbb{P}^{top}_{\alpha,\beta}(U \boxtimes X)$ by definition of the latter space. So the continuous map $t \mapsto f(u,\phi_2(t))$ is an element of $\mathbb{P}^{top}_{f^0(\alpha),f^0(\beta)}Y$ since $f : U \boxtimes X \to Y$ is a map of multipointed d-spaces. Thus for all $u \in U$, $\widehat{f}(u)$ together with f^0 induce a map of multipointed d-spaces from X to Y. One obtains a set map from U to $\mathbf{MDTOP}_\Delta(X,Y)$. It is continuous if and only if the composite $U \longrightarrow \mathbf{MDTOP}_\Delta(X,Y) \subset \mathbf{TOP}_\Delta(|X|,|Y|) \times_\Delta \mathbf{Set}(X^0,Y^0)$ is continuous by definition of the topology of $\mathbf{MDTOP}_\Delta(X,Y)$. By adjunction, the latter map corresponds to the continuous map $(|f|, f^0) \in \mathbf{TOP}_\Delta(U \times_\Delta |X|, |Y|) \times_\Delta \mathbf{Set}(X^0,Y^0)$. Hence the continuity and the commutative diagram of sets

$$\begin{tikzcd}
\mathbf{MdTop}_\Delta(U \boxtimes X, Y) \arrow[r, "f \mapsto (\widehat{f}, f^0)"] \arrow[d, "\subset"'] & \mathbf{Top}_\Delta(U, \mathbf{MDTOP}_\Delta(X,Y)) \arrow[d, "\subset"'] \\
\mathbf{Top}_\Delta(U \times_\Delta |X|, |Y|) \times_\Delta \mathbf{Set}(X^0,Y^0) \arrow[r, "\cong"] & \mathbf{Top}_\Delta(U, \mathbf{TOP}_\Delta(|X|,|Y|)) \times_\Delta \mathbf{Set}(X^0,Y^0)
\end{tikzcd} \tag{1}$$

Conversely, let $g : U \to \mathbf{MDTOP}_\Delta(X,Y)$ be a continuous map. The inclusion

$$\mathbf{MDTOP}_\Delta(X,Y) \subset \mathbf{TOP}_\Delta(|X|,|Y|) \times_\Delta \mathbf{Set}(X^0,Y^0)$$

gives rise to a set map $\widetilde{g}^0 : X^0 \to Y^0$ since U is connected non-empty and since $\mathbf{Set}(X^0,Y^0)$ is discrete and to a continuous map $|\widetilde{g}| : |U \boxtimes X| \to |Y|$ by adjunction. Let $\phi \in \mathbb{P}^{top}_{\alpha,\beta}(U \boxtimes X)$. By construction, ϕ is a composition of continuous paths of the form $(u, \phi_2(-))$ with $\phi_2 \in \mathbb{P}^{top}_{\alpha,\beta}X$ and $u \in U$. Then $|\widetilde{g}|(u,\phi_2(t)) = g(u) \circ \phi_2(t)$. Since $g(u) \in \mathbf{MdTop}_\Delta(X,Y)$, $|\widetilde{g}| \circ \phi \in \mathbb{P}^{top}_{\widetilde{g}^0(\alpha),\widetilde{g}^0(\beta)}Y$ and therefore $\widetilde{g} \in \mathbf{MdTop}_\Delta(U \boxtimes X, Y)$. Hence the commutative diagram of sets

$$\begin{tikzcd}
\mathbf{Top}_\Delta(U, \mathbf{MDTOP}_\Delta(X,Y)) \arrow[r, "g \mapsto \widetilde{g}"] \arrow[d, "\subset"'] & \mathbf{MdTop}_\Delta(U \boxtimes X, Y) \arrow[d, "\subset"'] \\
\mathbf{Top}_\Delta(U, \mathbf{TOP}_\Delta(|X|,|Y|)) \times_\Delta \mathbf{Set}(X^0,Y^0) \arrow[r, "\cong"] & \mathbf{Top}_\Delta(U \times_\Delta |X|, |Y|) \times_\Delta \mathbf{Set}(X^0,Y^0)
\end{tikzcd} \tag{2}$$

The commutativity of the diagrams (1) and (2) implies that the set maps $f \mapsto (\widehat{f}, f^0)$ and $g \mapsto \widetilde{g}$ are inverse to each other, hence the result. ∎

5.5. Proposition. *Let $f, g : X \rightrightarrows Y$ be two S-homotopic morphisms of multipointed d-spaces. Then $f^0 = g^0$ and the continuous maps $|f|, |g| : |X| \rightrightarrows |Y|$ are homotopic. Moreover, for all $(\alpha, \beta) \in X^0 \times X^0$, the pair of continuous maps $\mathbb{P}^{top} f, \mathbb{P}^{top} g : \mathbb{P}^{top}_{\alpha,\beta} X \rightrightarrows \mathbb{P}^{top}_{f^0(\alpha), f^0(\beta)} Y$ are homotopic.*

Proof. Let $H : [0,1] \to \mathbf{MDTOP}_\Delta(X, Y)$ be a S-homotopy between f and g. Then the composite

$$[0,1] \times_\Delta |X| \longrightarrow |[0,1] \boxtimes X| \xrightarrow{\widetilde{H}} |Y|$$

gives a homotopy between $|f|$ and $|g|$. The mapping $t \mapsto \mathbb{P}^{top}(H(t))$ induces a set map from $[0,1]$ to $\mathbf{TOP}_\Delta(\mathbb{P}^{top}_{\alpha,\beta} X, \mathbb{P}^{top}_{f^0(\alpha), f^0(\beta)} Y)$ for any $(\alpha, \beta) \in X^0 \times X^0$. The latter set map is continuous since it corresponds by adjunction to the continuous mapping $(t, \phi) \mapsto |H(t)| \circ \phi$ from $[0,1] \times_\Delta \mathbb{P}^{top}_{\alpha,\beta} X$ to $\mathbb{P}^{top}_{f^0(\alpha), f^0(\beta)} Y$. So one obtains a homotopy between $\mathbb{P}^{top} f$ and $\mathbb{P}^{top} g$. Finally, the composite

$$[0,1] \xrightarrow{H} \mathbf{MDTOP}_\Delta(X, Y) \subset \mathbf{TOP}_\Delta(|X|, |Y|) \times_\Delta \mathbf{Set}(X^0, Y^0) \longrightarrow \mathbf{Set}(X^0, Y^0)$$

is constant since $[0,1]$ is connected. So $f^0 = g^0$. ∎

5.6. Definition. *A map $i : A \to B$ of multipointed d-spaces is an* inclusion of a strong deformation retract *if there exists a continuous map $H : [0,1] \to \mathbf{MDTOP}_\Delta(B, B)$ called the* deformation retract *such that*

- *$H(0) = \mathrm{Id}_B$, $H(1) = i \circ r$ where $r : B \to A$ is a map of multipointed d-spaces such that $r \circ i = \mathrm{Id}_A$*
- *$|\widetilde{H}|(t, i(a)) = i(a)$ for all $a \in A$ and for all $t \in [0,1]$.*

5.7. Definition. *A continuous map $i : A \to B$ of general topological spaces is an* inclusion of a strong deformation retract of spaces *if there exists a continuous map $H : [0,1] \times B \to B$ called the* deformation retract *such that*

- *$H(0, -) = \mathrm{Id}_B$, $H(1, -) = i \circ r$ where $r : B \to A$ is a map of multipointed d-spaces such that $r \circ i = \mathrm{Id}_A$*
- *$H(t, i(a)) = i(a)$ for all $a \in A$ and for all $t \in [0,1]$.*

5.8. Proposition. *Let $i : A \to B$ be an inclusion of a strong deformation retract of multipointed d-spaces. Then*

- *The set map $i^0 : A^0 \to B^0$ is bijective.*
- *The continuous map $|i| : |A| \to |B|$ is an inclusion of a strong deformation retract of spaces.*
- *The continuous map $\mathbb{P}^{top} i : \mathbb{P}^{top}_{\alpha,\beta} A \to \mathbb{P}^{top}_{f^0(\alpha), f^0(\beta)} B$ is an inclusion of a strong deformation retract of spaces for all $(\alpha, \beta) \in A^0 \times A^0$.*

Proof. Consequence of Proposition 5.5. ∎

5.9. Theorem. *The class of inclusions of a strong deformation retract of multipointed d-spaces is closed under pushout.*

Proof. We mimic the proof given in [Hov99, Proposition 2.4.9]. Consider a pushout diagram of multipointed d-spaces

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "j"] \\ B \arrow[r] & D \end{tikzcd}$$

where i is an inclusion of a strong deformation retract. Let $K : [0,1] \to \mathbf{MDTOP}_{\Delta}(B, B)$ be the corresponding deformation retract and let $r : B \to A$ be the retraction. The commutative diagram of multipointed d-spaces

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] \arrow[drr, equal] & C \arrow[d] \arrow[drr, equal] & & \\ B \arrow[r] \arrow[drr, "r"'] & D \arrow[drr, dashed] & A \arrow[r] \arrow[d, equal] & C \arrow[d] \\ & & A \arrow[r] & C \end{tikzcd}$$

gives the retraction $s : D \to C$. Denote by $\widetilde{\mathrm{Id}_C}$ the map corresponding to the constant map from $[0,1]$ to $\mathbf{MDTOP}_{\Delta}(C, C)$ taking any element of $[0,1]$ to the identity of C. Then consider the commutative diagram of multipointed d-spaces

$$\begin{tikzcd} {[0,1] \boxtimes A} \arrow[r] \arrow[d] & {[0,1] \boxtimes C} \arrow[d] \arrow[ddrr, "\widetilde{\mathrm{Id}_C}"] & & \\ {[0,1] \boxtimes B} \arrow[r] \arrow[ddrr, "\widetilde{K}"'] & {[0,1] \boxtimes D} \arrow[ddrr, dashed] & & \\ & & & C \arrow[d] \\ & & B \arrow[r] & D \end{tikzcd}$$

Since the functor $[0,1] \boxtimes -$ preserves colimits by Theorem 5.4, the universal property of the pushout induces a map of multipointed d-spaces $\widetilde{H} : [0,1] \boxtimes D \to D$. Denote by H the corresponding continuous map from $[0,1]$ to $\mathbf{MDTOP}_{\Delta}(D, D)$. By Proposition 3.6, one obtains the commutative diagram of topological spaces

$$\begin{tikzcd}
{|[0,1] \boxtimes A|} \arrow[r] \arrow[d] & {|[0,1] \boxtimes C|} \arrow[d] \arrow[ddrr, "|\widetilde{\mathrm{Id}_C}|"] & & \\
{|[0,1] \boxtimes B|} \arrow[r] \arrow[ddr, "|\widetilde{K}|"'] & {|[0,1] \boxtimes D|} \arrow[ddrr, dashed] \arrow[ul, phantom, "\ulcorner" very near start] & & \\
 & & & {|C|} \arrow[d] \\
 & & {|B|} \arrow[r] & {|D|.}
\end{tikzcd}$$

By construction, one then has $|\widetilde{H}|(t,c) = j(c)$ for all $c \in C$ and all $t \in [0,1]$ and $|\widetilde{H}|(0,d) = d$ for all $d \in D$. Since $|\widetilde{K}|(1,b) \in |i|(|A|)$ for all $b \in B$, it follows that $|\widetilde{H}|(1,d) \in |j|(|C|)$ for all $d \in D$. Since $|j|$ is an inclusion of spaces by Proposition 5.8, H is a deformation retract as required. ■

6. The combinatorial model structure

6.1. Definition. *(Compare with [Gau03, Definition 11.6]) A morphism $f : X \to Y$ of multipointed d-spaces is a* weak S-homotopy equivalence *if $f^0 : X^0 \to Y^0$ is a bijection of sets and if $\mathbb{P}^{top} f : \mathbb{P}^{top}_{\alpha,\beta} X \to \mathbb{P}^{top}_{f^0(\alpha),f^0(\beta)} Y$ is a weak homotopy equivalence of topological spaces. The class of weak S-homotopy equivalences of multipointed d-spaces is denoted by $\mathcal{W}^{top}$.*

If $C : \varnothing \to \{0\}$ and $R : \{0,1\} \to \{0\}$ are set maps viewed as morphisms of multipointed d-spaces, let $I^{gl,top}_{+} = I^{gl,top} \cup \{C, R\}$. Let

$$\boxed{J^{gl,top} = \{\mathrm{Glob}^{top}(\mathbf{D}^n \times_\Delta \{0\}) \longrightarrow \mathrm{Glob}^{top}(\mathbf{D}^n \times_\Delta [0,1]), n \geqslant 0\}}$$

where the maps are the image by $\mathrm{Glob}(-)$ of the inclusions of spaces $\mathbf{D}^n \times_\Delta \{0\} \subset \mathbf{D}^n \times_\Delta [0,1]$ with $n \geqslant 0$.

6.2. Proposition. *Let $f : X \to Y$ be a morphism of multipointed d-spaces. Then f satisfies the right lifting property with respect to $\{C, R\}$ if and only if $f^0 : X^0 \to Y^0$ is bijective.*

Proof. See [Gau03, Proposition 16.2] or [Gau05b, Lemma 4.4 (5)]. ■

6.3. Proposition. *(Compare with [Gau03, Proposition 13.2]) Let $f : X \to Y$ be a morphism of multipointed d-spaces. Let $g : U \to V$ be a continuous map. Then f satisfies the right lifting property with respect to $\mathrm{Glob}^{top}(g)$ if and only if for all $(\alpha,\beta) \in X^0 \times X^0$, $\mathbb{P}^{top} f : \mathbb{P}^{top}_{\alpha,\beta} X \to \mathbb{P}^{top}_{f^0(\alpha),f^0(\beta)} Y$ satisfies the right lifting property with respect to g.*

PROOF. Consider a commutative diagram of solid arrows of spaces

$$\begin{tikzcd} U \arrow[r] \arrow[d, "g"'] & \mathbb{P}^{top}_{\alpha,\beta}X \arrow[d, "\mathbb{P}^{top}f"] \\ V \arrow[r] \arrow[ur, dashed, "\ell"] & \mathbb{P}^{top}_{f^0(\alpha),f^0(\beta)}Y. \end{tikzcd}$$

Since $\mathbf{Top}_\Delta$ is cartesian closed by Proposition 2.5, the existence of the lift ℓ is equivalent to the existence of the lift ℓ' in the commutative diagram of solid arrows of multipointed spaces

$$\begin{tikzcd} \mathrm{Glob}^{top}(U) \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ \mathrm{Glob}^{top}(V) \arrow[r] \arrow[ur, dashed, "\ell'"] & Y. \end{tikzcd}$$

■

6.4. PROPOSITION. *Let $i : A \to B$ be an inclusion of a strong deformation retract of spaces between Δ-generated spaces. Then $\mathrm{Glob}^{top}(i) : \mathrm{Glob}^{top}(A) \to \mathrm{Glob}^{top}(B)$ is an inclusion of a strong deformation retract of $\mathbf{MdTop}_\Delta$.*

PROOF. Let $r : B \to A$ be the corresponding retraction and let $H : [0,1] \times_\Delta B \to B$ be the corresponding deformation retract. Then $\mathrm{Glob}^{top}(r) : \mathrm{Glob}^{top}(B) \to \mathrm{Glob}^{top}(A)$ is the corresponding retraction of multipointed d-spaces and $\mathrm{Glob}^{top}(H) : \mathrm{Glob}^{top}([0,1]\times_\Delta B) \to \mathrm{Glob}^{top}(B)$ is the corresponding deformation retract of multipointed d-spaces since there is an isomorphism of multipointed d-spaces $\mathrm{Glob}^{top}([0,1] \times_\Delta B) \cong [0,1] \boxtimes \mathrm{Glob}^{top}(B)$. ■

6.5. THEOREM. *There exists a unique cofibrantly generated model structure on the category $\mathbf{MdTop}_\Delta$ such that the weak equivalences are the weak S-homotopy equivalences, such that $I^{gl,top}_+ = I^{gl,top} \cup \{C, R\}$ is the set of generating cofibrations and such that $J^{gl,top}$ is the set of trivial generating cofibrations. Moreover, one has:*

1. *The cellular objects are exactly the globular complexes.*
2. *A map $f : X \to Y$ of multipointed d-spaces is a (resp. trivial) fibration if and only if for all $(\alpha, \beta) \in X^0 \times X^0$, the continuous map $\mathbb{P}^{top}_{\alpha,\beta}X \to \mathbb{P}^{top}_{f^0(\alpha),f^0(\beta)}Y$ is a (resp. trivial) fibration of spaces.*
3. *Every multipointed d-space is fibrant.*
4. *The model structure is right proper and simplicial.*
5. *Two cofibrant multipointed d-spaces are weakly S-homotopy equivalent if and only they are S-homotopy equivalent.*

Proof. We have to prove the following facts [Hov99, Theorem 2.1.19]:

1. The class $\mathcal{W}^{top}$ satisfies the two-out-of-three property and is closed under retracts.
2. The domains of $I^{gl,top}_+$ are small relative to $\mathbf{cell}(I^{gl,top}_+)$.
3. The domains of $J^{gl,top}$ are small relative to $\mathbf{cell}(J^{gl,top})$.
4. $\mathbf{cell}(J^{gl,top}) \subset \mathcal{W}^{top} \cap \mathbf{cof}(I^{gl,top}_+)$.
5. $\mathbf{inj}(I^{gl,top}_+) = \mathcal{W}^{top} \cap \mathbf{inj}(J^{gl,top})$.

That $\mathcal{W}^{top}$ satisfies the two-out-of-three property and is closed under retracts is clear. By [Bek00, Proposition 1.3], every object of $\mathbf{MdTop}_\Delta$ is small relative to the whole class of morphisms since $\mathbf{MdTop}_\Delta$ is locally presentable by Theorem 3.5. Hence Assertions (2) and (3). By Proposition 6.2 and Proposition 6.3, a map of multipointed d-spaces $f : X \to Y$ belongs to $\mathbf{inj}(I^{gl,top}_+)$ if and only if the set map $f^0 : X^0 \to Y^0$ is bijective and for all $(\alpha, \beta) \in X^0 \times X^0$, the map $\mathbb{P}^{top}_{\alpha,\beta}X \to \mathbb{P}^{top}_{f(\alpha),f(\beta)}Y$ is a trivial fibration of spaces. By Proposition 6.3 again, a map of multipointed d-spaces $f : X \to Y$ belongs to $\mathbf{inj}(J^{gl,top})$ if and only if for all $(\alpha, \beta) \in X^0 \times X^0$, the map $\mathbb{P}^{top}_{\alpha,\beta}X \to \mathbb{P}^{top}_{f(\alpha),f(\beta)}Y$ is a fibration of spaces. Hence the fifth assertion.

It remains to prove the fourth assertion which is the most delicate part of the proof. The set inclusion $\mathbf{cell}(J^{gl,top}) \subset \mathbf{cell}(I^{gl,top})$ comes from the obvious set inclusion $J^{gl,top} \subset \mathbf{cof}(I^{gl,top})$. It remains to prove the set inclusion $\mathbf{cell}(J^{gl,top}) \subset \mathcal{W}^{top}$. First of all, consider the case of a map $f : X \to Y$ which is the pushout of one element of $J^{gl,top}$ (for some $n \geqslant 0$):

$$\begin{tikzcd} \mathrm{Glob}^{top}(\mathbf{D}^n \times_\Delta \{0\}) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Glob}^{top}(\mathbf{D}^n \times_\Delta [0,1]) \arrow[r] & Y. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Since each map of $J^{gl,top}$ is an inclusion of a strong deformation retract of multipointed d-spaces by Proposition 6.4, the map $f : X \to Y$ is itself an inclusion of a strong deformation retract by Theorem 5.9. So by Proposition 5.8, the continuous map $\mathbb{P}^{top}f : \mathbb{P}^{top}X \to \mathbb{P}^{top}Y$ is an inclusion of a strong deformation retract of spaces. By Proposition 4.3, the subset $|Y|\backslash|X|$ is open in $|Y|$. Let $\phi \in \mathbb{P}^{top}Y\backslash\mathbb{P}^{top}X$. By Proposition 3.6, one has $\phi = \phi_0 *_{t_1} \phi_1 *_{t_2} \cdots *_{t_n} \phi_n$ with the ϕ_i being execution paths of the images of $\mathrm{Glob}^{top}(\mathbf{D}^n \times_\Delta [0,1])$ or of X in Y. Since $\phi \notin \mathbb{P}^{top}X$, the set $\phi^{-1}(|Y|\backslash|X|)$ is a non-empty open subset of $[0,1]$. Let $[a,b] \subset \phi^{-1}(|Y|\backslash|X|)$ with $0 \leqslant a < b \leqslant 1$. Then ϕ belongs to $N([a,b], |Y|\backslash|X|)$ which is an open of the compact-open topology. Since the Kelleyfication functor adds open subsets, the set $N([a,b], |Y|\backslash|X|)$ is an open neighborhood of ϕ included in $\mathbb{P}^{top}Y\backslash\mathbb{P}^{top}X$. So $\mathbb{P}^{top}Y\backslash\mathbb{P}^{top}X$ is open in $\mathbb{P}^{top}Y$. Therefore, the continuous map $\mathbb{P}^{top}f : \mathbb{P}^{top}X \to \mathbb{P}^{top}Y$ is a closed inclusion of a strong deformation retract of spaces [4]. This class of maps is closed

[4]This step of the proof is necessary. See Proposition 2.1 of this paper.

under transfinite composition in $\mathcal{T}$, and in $\mathbf{Top}_\Delta$, since it is the class of trivial cofibrations of the Strøm model structure on the category of general topological spaces [Str66] [Str68] [Str72]. So for any map $f \in \mathbf{cell}(J^{gl,top})$, the continuous map $\mathbb{P}^{top}f : \mathbb{P}^{top}X \to \mathbb{P}^{top}Y$ is a closed inclusion of a strong deformation retract of spaces since the functor $\mathbb{P}^{top}$ is finitely accessible by Proposition 3.7. Therefore it is a weak homotopy equivalence. Hence the set inclusion $\mathbf{cell}(J^{gl,top}) \subset \mathcal{W}^{top}$.

It remains to prove the last five assertions of the statement of the theorem. The first one is obvious. The second one is explained above. The third one comes from the second one and from the fact that all topological spaces are fibrant (the terminal object of $\mathbf{MdTop}_\Delta$ is $(\{0\},\{0\},\{0\})$). The model structure is right proper since all objects are fibrant. The construction of the simplicial structure of this model category is postponed until Appendix B. The very last assertion is due to the construction of the simplicial structure and to the fact that $X \otimes \Delta[1] \cong [0,1] \boxtimes X$ since $[0,1]$ is non-empty and connected. ∎

Here are some comments about cofibrancy. The set $\{0\} = (\{0\},\{0\},\varnothing)$ is cofibrant. More generally, any set $E = (E, E, \varnothing)$ is cofibrant. The multipointed d-space $(\{0\},\varnothing,\varnothing)$ is not cofibrant. Indeed, its cofibrant replacement is equal to the initial multipointed d-space $\varnothing = (\varnothing,\varnothing,\varnothing)$. Finally, the terminal multipointed d-space $(\{0\},\{0\},\{0\})$ is not cofibrant.

6.6. Conjecture. *The model category* $\mathbf{MdTop}_\Delta$ *is left proper.*

7. Comparing multipointed d-spaces and flows

We describe a functor *cat* from the category of multipointed d-spaces to the category of flows which generalizes the functor constructed in [Gau05a].

7.1. Definition. *[Gau03] A flow X is a small category without identity maps enriched over topological spaces. The composition law of a flow is denoted by $*$. The set of objects is denoted by X^0. The space of morphisms from α to β is denoted by $\mathbb{P}_{\alpha,\beta}X$. Let $\mathbb{P}X$ be the disjoint sum of the spaces $\mathbb{P}_{\alpha,\beta}X$. A morphism of flows $f : X \to Y$ is a set map $f^0 : X^0 \to Y^0$ together with a continuous map $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ preserving the structure. The corresponding category is denoted by* $\mathbf{Flow}(\mathbf{Top}_\Delta)$, $\mathbf{Flow}(\mathbf{Top}_\mathcal{K})$, *etc... depending on the category of topological spaces we are considering.*

Let Z be a topological space. The flow $\mathrm{Glob}(Z)$ is defined by

- $\mathrm{Glob}(Z)^0 = \{\widehat{0},\widehat{1}\}$,
- $\mathbb{P}\mathrm{Glob}(Z) = \mathbb{P}_{\widehat{0},\widehat{1}}\mathrm{Glob}(Z) = Z$,
- $s = \widehat{0}$, $t = \widehat{1}$ and a trivial composition law.

It is called the *globe* of the space Z.

Let X be a multipointed d-space. Consider for every $(\alpha,\beta) \in X^0 \times X^0$ the coequalizer of sets

$$\mathbb{P}_{\alpha,\beta}X = \varinjlim \left(\mathbb{P}^{top}_{\alpha,\beta}X \times_\Delta \mathbb{P}^{top}\overrightarrow{I}^{top} \rightrightarrows \mathbb{P}^{top}_{\alpha,\beta}X\right)$$

where the two maps are $(c,\phi) \mapsto c \circ \mathrm{Id}_{\overrightarrow{I}^{top}} = c$ and $(c,\phi) \mapsto c \circ \phi$. Let $[-]_{\alpha,\beta} : \mathbb{P}^{top}_{\alpha,\beta}X \to \mathbb{P}_{\alpha,\beta}X$ be the canonical set map. The set $\mathbb{P}_{\alpha,\beta}X$ is equipped with the final topology.

7.2. Theorem. *Let X be a multipointed d-space. Then there exists a flow $cat_\Delta(X)$ with $cat_\Delta(X)^0 = X^0$, $\mathbb{P}_{\alpha,\beta}cat_\Delta(X) = \mathbb{P}_{\alpha,\beta}X$ and the composition law $* : \mathbb{P}_{\alpha,\beta}X \times_\Delta \mathbb{P}_{\beta,\gamma}X \to \mathbb{P}_{\alpha,\gamma}X$ is for every triple $(\alpha,\beta,\gamma) \in X^0 \times X^0 \times X^0$ the unique map making the following diagram commutative:*

$$\begin{tikzcd}
\mathbb{P}^{top}_{\alpha,\beta}X \times_\Delta \mathbb{P}^{top}_{\beta,\gamma}X \arrow[r] \arrow[d, "{[-]_{\alpha,\beta} \times_\Delta [-]_{\beta,\gamma}}"'] & \mathbb{P}^{top}_{\alpha,\gamma}X \arrow[d, "{[-]_{\alpha,\gamma}}"'] \\
\mathbb{P}_{\alpha,\beta}X \times_\Delta \mathbb{P}_{\beta,\gamma}X \arrow[r] & \mathbb{P}_{\alpha,\gamma}X
\end{tikzcd}$$

where the map $\mathbb{P}^{top}_{\alpha,\beta}X \times_\Delta \mathbb{P}^{top}_{\beta,\gamma}X \to \mathbb{P}^{top}_{\alpha,\gamma}X$ is the continuous map defined by the concatenation of continuous paths: $(c_1,c_2) \in \mathbb{P}^{top}_{\alpha,\beta}X \times_\Delta \mathbb{P}^{top}_{\beta,\gamma}X$ is sent to $c_1 *_{1/2} c_2$. The mapping $X \mapsto cat_\Delta(X)$ induces a functor from $\mathbf{MdTop}_\Delta$ *to* $\mathbf{Flow}(\mathbf{Top}_\Delta)$.

Proof. The existence and the uniqueness of the continuous map $\mathbb{P}_{\alpha,\beta}X \times_\Delta \mathbb{P}_{\beta,\gamma}X \to \mathbb{P}_{\alpha,\gamma}X$ for any triple $(\alpha,\beta,\gamma) \in X^0 \times X^0 \times X^0$ comes from the fact that the topological space $\mathbb{P}_{\alpha,\beta}X$ is the quotient of the set $\mathbb{P}^{top}_{\alpha,\beta}X$ by the equivalence relation generated by the identifications $c \circ \phi = c$ equipped with the final topology. This defines a strictly associative law because the concatenation of continuous paths is associative up to strictly increasing reparametrization. The functoriality is obvious. ∎

7.3. Proposition. *The functor $cat_\Delta : \mathbf{MdTop}_\Delta \to \mathbf{Flow}(\mathbf{Top}_\Delta)$ does not have any right adjoint.*

Proof. Suppose that the right adjoint exists. Let Z be a Δ-generated space. Then for any multipointed d-space X, one would have the natural isomorphism

$$\mathbf{MdTop}_\Delta(X,Y) \cong \mathbf{Flow}(\mathbf{Top}_\Delta)(cat_\Delta(X), \mathrm{Glob}(Z))$$

for some multipointed d-space Y.

With $X = X^0$, one obtains the equalities

$$\mathbf{MdTop}_\Delta(X,Y) = \mathbf{Set}(X^0, Y^0)$$

and

$$\mathbf{Flow}(\mathbf{Top}_\Delta)(cat_\Delta(X), \mathrm{Glob}(Z)) = \mathbf{Set}(X^0, \mathrm{Glob}(Z)^0).$$

By an application of Yoneda's lemma within the category of sets, one deduces that $Y^0 = \{\widehat{0}, \widehat{1}\}$.

Consider now the case $X = \overrightarrow{I}^{top}$. The set $\mathbb{P}^{top}_{\alpha,\beta}Y$ is non-empty if and only if there exists a map $f : \overrightarrow{I}^{top} \to Y$ with $f^0(\widehat{0}) = \alpha$ and $f^0(\widehat{1}) = \beta$. The map $f : \overrightarrow{I}^{top} \to Y$ corresponds by adjunction to a map $\overrightarrow{I} = \mathrm{Glob}(\{0\}) \to \mathrm{Glob}(Z)$. Thus one obtains $\mathbb{P}^{top}Y = \mathbb{P}^{top}_{\{\widehat{0},\widehat{1}\}}Y$.

Consider now the case $X = \mathrm{Glob}^{top}(T)$ for some Δ-generated space T. Then one has the isomorphisms $\mathbf{MdTop}_\Delta(X,Y) \cong \mathbf{Top}_\Delta(T, \mathbb{P}^{top}_{\{\widehat{0},\widehat{1}\}}Y)$ and

$$\mathbf{Flow}(\mathbf{Top}_\Delta)(cat_\Delta(X), \mathrm{Glob}(Z)) \cong \mathbf{Top}_\Delta(T, Z).$$

So by Yoneda's lemma applied within the category $\mathbf{Top}_\Delta$, one obtains the isomorphism $\mathbb{P}^{top}Y = \mathbb{P}^{top}_{\{\widehat{0},\widehat{1}\}}Y \cong Z$.

Take $Z = \{0\}$. Then Y is a multipointed d-space with unique initial state $\widehat{0}$, with unique final state $\widehat{1}$ and with $\mathbb{P}^{top}Y = \mathbb{P}^{top}_{\{\widehat{0},\widehat{1}\}}Y = \{0\} \neq \varnothing$. Thus there exists a continuous map $\phi : [0,1] \to |Y|$ with $\phi(0) = \widehat{0}$, $\phi(1) = \widehat{1}$ which is an execution path of Y. So any strictly increasing reparametrization of ϕ is an execution path of Y. So $\widehat{0} = \widehat{1}$ in Y^0. Contradiction. ∎

7.4. Corollary. *The functor* $cat_\Delta : \mathbf{MdTop}_\Delta \longrightarrow \mathbf{Flow}_\Delta$ *is not colimit-preserving.*

Proof. The category $\mathbf{MdTop}_\Delta$ is a topological fibre-small category over the category of sets. Therefore, it satisfies the hypothesis of the opposite of the special adjoint functor theorem. Thus, the functor cat_Δ is colimit-preserving if and only if it has a right adjoint. ∎

The colimit-preserving functor $H \circ i^{\mathcal{K}}_\Delta : \mathbf{Top}_\Delta \to \mathbf{CGTop}$ induces a colimit-preserving functor $\mathbf{Flow}(\mathbf{Top}_\Delta) \to \mathbf{Flow}(\mathbf{CGTop})$ from the category of flows enriched over Δ-generated spaces to that of flows enriched over compactly generated topological spaces (the latter category is exactly the category used in [Gau03] and in [Gau05a]). So the composite functor

$$cat : \mathbf{MdTop}_\Delta \xrightarrow{cat_\Delta} \mathbf{Flow}(\mathbf{Top}_\Delta) \longrightarrow \mathbf{Flow}(\mathbf{CGTop})$$

coincides on globular complexes with the functor constructed in [Gau05a] since the globular complexes of [Gau05a] are exactly those defined in Section 4 and since one has the equality $cat(\mathrm{Glob}^{top}(Z)) = \mathrm{Glob}(Z)$ for any compactly generated space Z.

7.5. Theorem. *The composite functor*

$$\mathbf{MdTop}_\Delta \xrightarrow{(-)^{cof}} \mathbf{glTop} \xrightarrow{cat} \mathbf{Ho}(\mathbf{Flow}(\mathbf{CGTop}))$$

induces an equivalence of categories $\mathbf{Ho}(\mathbf{MdTop}_\Delta) \simeq \mathbf{Ho}(\mathbf{Flow}(\mathbf{CGTop}))$ *between the category of multipointed d-spaces up to weak S-homotopy and the category of flows up to weak S-homotopy where a map of flows* $f : X \to Y$ *is a weak S-homotopy if and only if* $f^0 : X^0 \to Y^0$ *is a bijection and* $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ *is a weak homotopy equivalence.*

Proof. By [Gau05a, Theorem V.4.2], the restriction of the functor cat to the category $\mathbf{glTop}$ of globular complexes induces a categorical equivalence $\mathbf{glTop}[(\mathcal{W}^{top})^{-1}] \simeq \mathbf{Ho}(\mathbf{Flow}(\mathbf{CGTop}))$ where $\mathbf{glTop}[(\mathcal{W}^{top})^{-1}]$ is the categorical localization of $\mathbf{glTop}$ by the class of weak S-homotopy equivalences $\mathcal{W}^{top}$. Every multipointed d-space is fibrant. Therefore the localization $\mathbf{glTop}[(\mathcal{W}^{top})^{-1}]$ is isomorphic to the quotient $\mathbf{glTop}/\sim_S$ where $\sim_S$ is the congruence relation on morphisms induced by S-homotopy (see Definition 5.1). The inclusion functor $\mathbf{glTop} \subset (\mathbf{MdTop}_\Delta)_{cof}$ from the category of globular complexes to that of cofibrant multipointed d-spaces induces a full and faithful functor $\mathbf{glTop}/\sim_S \to (\mathbf{MdTop}_\Delta)_{cof}/\sim_S$. The right-hand category is equivalent to $\mathbf{Ho}(\mathbf{MdTop}_\Delta)$ by [Hir03, Section 7.5.6] or [Hov99, Proposition 1.2.3]. ∎

Recall that there exists a unique model structure on $\mathbf{Flow}(\mathbf{CGTop})$ such that [Gau03]

- The set of generating cofibrations is $I^{gl}_+ = I^{gl} \cup \{C, R\}$ with

$$\boxed{I^{gl} = \{\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n), n \geqslant 0\}}$$

  where the maps are the image by $\mathrm{Glob}(-)$ of the inclusions of spaces $\mathbf{S}^{n-1} \subset \mathbf{D}^n$ with $n \geqslant 0$.

- The set of generating trivial cofibration is

$$\boxed{J^{gl} = \{\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n \times [0,1]), n \geqslant 0\}}$$

  where the maps are the image by $\mathrm{Glob}(-)$ of the inclusions of spaces $\mathbf{D}^n \times \{0\} \subset \mathbf{D}^n \times [0,1]$ with $n \geqslant 0$.

- The weak equivalences are the weak S-homotopy equivalences.

7.6. Proposition. *Let X be a cofibrant flow of $\mathbf{Flow}(\mathbf{CGTop})$. Then the path space $\mathbb{P}X$ is cofibrant.*

Sketch of Proof. The cofibrant flow X is a retract of a cellular flow $\overline{X}$ and the space $\mathbb{P}X$ is then a retract of the space $\mathbb{P}\overline{X}$. Thus one can suppose X cellular, i.e. $\varnothing \to X$ belonging to $\mathbf{cell}(I^{gl}_+)$. By [Gau07, Proposition 7.1] (see also Proposition A.1 of this paper and [Gau03, Proposition 15.1]), the continuous map $\varnothing \to \mathbb{P}X$ is a transfinite composition of a λ-sequence $Y : \lambda \to \mathbf{CGTop}$ such that for any $\mu < \lambda$, the map $Y_\mu \to Y_{\mu+1}$ is a pushout of a map of the form $\mathrm{Id}_{X_1} \times \dots \times i_n \times \dots \times \mathrm{Id}_{X_p}$ with $p \geqslant 0$ and with $i_n : \mathbf{S}^{n-1} \subset \mathbf{D}^n$ being the usual inclusion with $n \geqslant 0$. Suppose that the set of ordinals

$$\{\mu \leqslant \lambda, Y_\mu \text{ is not cofibrant}\}$$

is non-empty. Consider the smallest element μ_0. Then the continuous map $\varnothing \to \mathbb{P}Y_{\mu_0}$ is a transfinite composition of pushout of maps of the form $\mathrm{Id}_{X_1} \times \dots \times i_n \times \dots \times \mathrm{Id}_{X_p}$ with all spaces X_i cofibrant since the spaces X_i are built using the path spaces of the flows Y_μ with $\mu < \mu_0$. So the space $\mathbb{P}Y_{\mu_0}$ is cofibrant because the model category $\mathbf{Top}_\Delta$ is monoidal. Contradiction. ∎

Note that the analogous statement for multipointed d-spaces is false. Indeed, the multipointed d-space $\overrightarrow{I}^{top}$ is cofibrant since $\overrightarrow{I}^{top} \cong \mathrm{Glob}^{top}(\mathbf{D}^0)$. And the space $\mathbb{P}^{top}\overrightarrow{I}^{top}$ is the space of strictly increasing continuous maps from $[0,1]$ to itself preserving 0 and 1. The latter space is not cofibrant.

Let us conclude the comparison of multipointed d-spaces and flows by:

7.7. THEOREM. *There exists a unique model structure on* $\mathbf{Flow}(\mathbf{Top}_\Delta)$ *such that the set of generating (resp. trivial) cofibrations is* I^{gl}_+ *(resp.* J^{gl}*) and such that the weak equivalences are the weak S-homotopy equivalences. Moreover:*

1. *The model category* $\mathbf{Flow}(\mathbf{Top}_\Delta)$ *is proper simplicial and combinatorial.*
2. *The categorical adjunction* $\mathbf{Flow}(\mathbf{Top}_\Delta) \leftrightarrows \mathbf{Flow}(\mathbf{CGTop})$ *is a Quillen equivalence.*
3. *The functor* $cat_\Delta : \mathbf{MdTop}_\Delta \to \mathbf{Flow}(\mathbf{Top}_\Delta)$ *preserves cofibrations, trivial cofibrations and weak S-homotopy equivalences between cofibrant objects.*

SKETCH OF PROOF. The construction of the model structure goes as in [Gau03]. It is much easier since the category $\mathbf{Flow}(\mathbf{Top}_\Delta)$ is locally presentable by a proof similar to the one of [Gau05b, Proposition 6.11]. Therefore, we do not have to worry about the problems of smallness by [Bek00, Proposition 1.3]. So a big part of [Gau03] can be removed. It remains to check that [Hov99, Theorem 2.1.19]:

- The class $\mathcal{W}$ of weak S-homotopy equivalences of $\mathbf{Flow}(\mathbf{Top}_\Delta)$ satisfies the two-out-of-three property and is closed under retracts: clear.
- $\mathbf{cell}(J^{gl}) \subset \mathcal{W} \cap \mathbf{cof}(I^{gl}_+)$. The set inclusion $\mathbf{cell}(J^{gl}) \subset \mathbf{cof}(I^{gl}_+)$ comes from the obvious set inclusion $J^{gl} \subset \mathbf{cof}(I^{gl})$. The proof of the set inclusion $\mathbf{cell}(J^{gl}) \subset \mathcal{W}$ is similar to the proof of the same statement in [Gau03]: the crucial facts are that:
  - Proposition A.1 which calculates a pushout by a map $\mathrm{Glob}(U) \to \mathrm{Glob}(V)$.
  - A map of the form $\mathrm{Id}_X \times_\Delta j_n : X \times_\Delta \mathbf{D}^n \times_\Delta \{0\} \to X \times_\Delta \mathbf{D}^n \times_\Delta [0,1]$, where $j_n : \mathbf{D}^n \times_\Delta \{0\} \subset \mathbf{D}^n \times_\Delta [0,1]$ is the usual inclusion of spaces for some $n \geqslant 0$, is a closed inclusion of a strong deformation retract of spaces. Indeed, since $\mathbf{D}^n$ is compact, the map $\mathrm{Id}_A \times_\Delta j_n$ is equal to the map $\mathrm{Id}_A \times_{\mathcal{T}} j_n$ by Proposition 2.6. And we then consider once again the Strøm model structure on the category of general topological spaces $\mathcal{T}$.
  - So for every map $f \in \mathbf{cell}(J^{gl})$, the continuous map $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ is a closed inclusion of a strong deformation retract of spaces since the path space functor $\mathbb{P} : \mathbf{Flow}(\mathbf{Top}_\Delta) \to \mathbf{Top}_\Delta$ is finitely accessible as in Proposition 3.7. Therefore it is a weak homotopy equivalence.
- $\mathbf{inj}(I^{gl}_+) = \mathcal{W} \cap \mathbf{inj}(J^{gl})$. See [Gau03, Proposition 16.2] and [Gau03, Proposition 13.2].

The model category $\mathbf{Flow}(\mathbf{Top}_\Delta)$ is right proper since every object is fibrant. The proof of the left properness of $\mathbf{Flow}(\mathbf{Top}_\Delta)$ is postponed until Appendix A. It is simplicial by the same proof as the one given in [Gau08a]. The second assertion of the statement is then obvious. The functor $cat_\Delta : \mathbf{MdTop}_\Delta \to \mathbf{Flow}(\mathbf{Top}_\Delta)$ is colimit-preserving. Therefore it takes maps of $\mathbf{cell}(I_+^{gl,top})$ (resp. $\mathbf{cell}(J^{gl,top})$) to maps of $\mathbf{cell}(I_+^{gl})$ (resp. $\mathbf{cell}(J^{gl})$). Since every (resp. trivial) cofibration of multipointed d-spaces is a retract of an element of $\mathbf{cell}(I_+^{gl,top})$ (resp. $\mathbf{cell}(J^{gl,top})$), one deduces that the functor $cat_\Delta : \mathbf{MdTop}_\Delta \to \mathbf{Flow}(\mathbf{Top}_\Delta)$ takes (trivial) cofibrations to (trivial) cofibrations. Let $f : X \to Y$ be a weak S-homotopy equivalence between cofibrant multipointed d-spaces. The cofibrant object X is a retract of a globular complex $\overline{X}$. The space $\mathbb{P}^{top}\overline{X}$ is compactly generated by Proposition 4.5. The space $\mathbb{P}\overline{X}$ is the path space of the cofibrant flow $cat(\overline{X})$. Therefore the space $\mathbb{P}\overline{X}$ is cofibrant by Proposition 7.6. Any cofibrant space is weak Hausdorff. So $\mathbb{P}\overline{X}$ is compactly generated and by [Gau05a, Theorem IV.3.10], the map $\mathbb{P}^{top}\overline{X} \to \mathbb{P}\overline{X}$ is a weak homotopy equivalence, and even a trivial Hurewicz fibration. Thus the map $\mathbb{P}^{top}X \to \mathbb{P}X$ is a retract of the weak homotopy equivalence $\mathbb{P}^{top}\overline{X} \to \mathbb{P}\overline{X}$. Hence it is a weak homotopy equivalence as well. The commutative diagram of Δ-generated spaces

$$\begin{tikzcd} \mathbb{P}^{top}X \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathbb{P}^{top}Y \arrow[d, "\simeq"'] \\ \mathbb{P}X \arrow[r] & \mathbb{P}Y \end{tikzcd}$$

and the two-out-of-three property completes the proof. ■

In conclusion, let us say that the functor $cat_\Delta : \mathbf{MdTop}_\Delta \to \mathbf{Flow}(\mathbf{Top}_\Delta)$ is a kind of left Quillen equivalence of cofibration categories.

8. Underlying homotopy type of flows as a total left derived functor

The underlying homotopy type functor of flows is defined in [Gau05a]. Morally speaking, it is the underlying topological space of a flow, but it is unique only up to a weak homotopy equivalence. It is equal, with the notations of this paper, to the composite functor

$$\mathbf{Ho}(\mathbf{Flow}(\mathbf{Top})) \simeq \mathbf{glTop}[(\mathcal{W}^{top})^{-1}] \longrightarrow \mathbf{Ho}(\mathbf{Top}_\Delta) \simeq \mathbf{Ho}(\mathbf{CGTop})$$

where the middle functor is the unique functor making the following diagram commutative:

$$\begin{tikzcd} \mathbf{glTop} \arrow[r, "X\mapsto|X|"] \arrow[d] & \mathbf{Top}_\Delta \arrow[d] \\ \mathbf{glTop}/\sim_S \cong \mathbf{glTop}[(\mathcal{W}^{top})^{-1}] \arrow[r] & \mathbf{Ho}(\mathbf{Top}_\Delta) \end{tikzcd}$$

where both vertical maps are the canonical localization functors.

8.1. Proposition. *The functor* $|-| : X = (|X|, X^0, \mathbb{P}^{top}X) \mapsto |X|$ *from* $\mathbf{MdTop}_\Delta$ *to* $\mathbf{Top}_\Delta$ *is a left Quillen functor.*

PROOF. By Proposition 3.6, this functor is topological. So it has a right adjoint by [AHS06, Proposition 21.12]. In fact, the right adjoint $R : \mathbf{Top}_\Delta \to \mathbf{MdTop}_\Delta$ is defined by:

$$R(Z) = (Z, \omega_\Delta(Z), \mathbf{MTop}_\Delta(([0,1],\{0,1\}),(Z,\omega_\Delta(Z)))).$$

The functor $|-|$ preserves cofibrations and trivial cofibrations by Proposition 4.3. ∎

8.2. COROLLARY. *Up to equivalences of categories, the underlying homotopy type functor of flows is a total left derived functor.*

PROOF. Indeed, the composite functor

$$\mathbf{Ho}(\mathbf{Flow}(\mathbf{Top})) \simeq \mathbf{Ho}(\mathbf{MdTop}_\Delta) \stackrel{X\mapsto X^{cof}}{\longrightarrow} \mathbf{glTop}/{\sim_S} \stackrel{Y\mapsto |Y|}{\longrightarrow} \mathbf{Ho}(\mathbf{Top}_\Delta) \simeq \mathbf{Ho}(\mathbf{CGTop})$$

is equal to the underlying homotopy type functor. ∎

The underlying space functor $X \mapsto |X|$ from $\mathbf{MdTop}_\Delta$ to $\mathbf{Top}_\Delta$ is not invariant with respect to weak S-homotopy equivalences. With the identification $\mathbf{S}^1 = \{z \in \mathbb{C}, |z| = 1\}$, consider the multipointed d-space $X = (\mathbf{S}^1, \{1, \exp(i\pi/2)\}, \mathbb{P}^{top}X)$ where $\mathbb{P}^{top}X$ is the closure by strictly increasing reparametrization of the set of continuous paths $\{t \mapsto \exp(it\pi/2), t \in [0,1]\}$. The multipointed d-space has a unique initial state 1 and a unique final state $\exp(i\pi/2)$. Then consider the map of multipointed d-spaces $f : \overrightarrow{I}^{top} \longrightarrow X$ defined by $|f|(t) = \exp(it\pi/2)$ for $t \in [0,1]$. Then f is a weak S-homotopy equivalence with $|\overrightarrow{I}^{top}|$ contractible whereas $|X|$ is not so.

A. Left properness of $\mathbf{Flow}(\mathbf{Top}_\Delta)$

As the proof of the left properness of $\mathbf{Flow}(\mathbf{CGTop})$ given in [Gau07, Section 7], the proof of the left properness of $\mathbf{Flow}(\mathbf{Top}_\Delta)$ lies in Proposition A.1, Proposition A.2 and Proposition A.6:

A.1. PROPOSITION. *(Compare with [Gau07, Proposition 7.1]) Let $f : U \to V$ be a continuous map between Δ-generated spaces. Let X be a flow enriched over Δ-generated spaces. Consider the pushout diagram of multipointed d-spaces:*

$$\begin{tikzcd} \mathrm{Glob}(U) \arrow[r] \arrow[d, "\mathrm{Glob}(f)"'] & X \arrow[d] \\ \mathrm{Glob}(V) \arrow[r] & Y. \end{tikzcd}$$

Then the continuous map $\mathbb{P}X \to \mathbb{P}Y$ is a transfinite composition of pushouts of maps of the form

$$\mathrm{Id}_{X_1} \times_\Delta \dots \times_\Delta f \times_\Delta \dots \times_\Delta \mathrm{Id}_{X_p}$$

with $p \geqslant 0$.

PROOF. The proof is exactly the same as for $\mathbf{Flow}(\mathbf{CGTop})$. Nothing particular need be assumed on the category of topological spaces we are working with, except that it must be cartesian closed. ∎

A.2. Proposition. *(Compare with [Gau07, Proposition 7.2]) Let $n \geqslant 0$. Let $i_n : \mathbf{S}^{n-1} \subset \mathbf{D}^n$ be the usual inclusion of spaces. Let $X_1, \dots, X_p$ be Δ-generated spaces. Then the pushout of a weak homotopy equivalence along a map of the form a finite product*

$$\mathrm{Id}_{X_1} \times_\Delta \dots \times_\Delta i_n \times_\Delta \dots \times_\Delta \mathrm{Id}_{X_p}$$

with $p \geqslant 0$ is still a weak homotopy equivalence.

Proof. The three ingredients of [Gau07, Proposition 7.2] are

1. The pushout of an inclusion of a strong deformation retract of spaces is an inclusion of a strong deformation retract of spaces. This assertion is true in the category of general topological spaces by [Hov99, Lemma 2.4.5].

2. The Seifert-Van-Kampen theorem for the fundamental groupoid functor which is true in the category of general topological spaces by [Bro67] [Hig05].

3. The Mayer-Vietoris long exact sequence which holds in the category of general topological spaces: it is the point of view of, e.g., [Rot88].

■

The following notion is a weakening of the notion of closed T_1-inclusion, introduced by Dugger and Isaksen.

A.3. Definition. *[DI04, p 686] An inclusion of spaces $f : Y \to Z$ is a* relative T_1-inclusion of spaces *if for any open set U of Y and any point $z \in Z \backslash U$, there is an open set W of Z with $U \subset W$ and $z \notin W$.*

A.4. Proposition. *(Variant of [DI04, Lemma A.2]) Consider a pushout diagram of Δ-generated spaces*

$$\begin{tikzcd} A \times_\Delta \mathbf{S}^{n-1} \arrow[r] \arrow[d] & Y \arrow[d] \\ A \times_\Delta \mathbf{D}^n \arrow[r] & Z. \end{tikzcd}$$

Then the map $Y \to Z$ is a relative T_1-inclusion of spaces.

Proof. By [DI04, Lemma A.2], the cocartesian diagram of $\mathcal{T}$

$$\begin{tikzcd} A \times_\mathcal{T} \mathbf{S}^{n-1} \arrow[r] \arrow[d] & Y \arrow[d] \\ A \times_\mathcal{T} \mathbf{D}^n \arrow[r] & Z'. \end{tikzcd}$$

yields a relative T_1-inclusion of spaces $Y \to Z'$. Since both $\mathbf{S}^{n-1}$ and $\mathbf{D}^n$ are compact, one has $A \times_\mathcal{T} \mathbf{S}^{n-1} \cong A \times_\Delta \mathbf{S}^{n-1}$ and $A \times_\mathcal{T} \mathbf{D}^n \cong A \times_\Delta \mathbf{D}^n$ by Proposition 2.6. Since colimits in $\mathcal{T}$ and in $\mathbf{Top}_\Delta$ are the same, the map $Y \to Z'$ is the map $Y \to Z$. ■

A.5. Proposition. *[DI04, Lemma A.3] Any compact is finite relative to the class of relative T_1-inclusions of spaces.*

A.6. Proposition. *(Compare with [Gau07, Proposition 7.3]) Let λ be an ordinal. Let $M : \lambda \longrightarrow \mathbf{Top}_\Delta$ and $N : \lambda \longrightarrow \mathbf{Top}_\Delta$ be two λ-sequences of topological spaces. Let $s : M \longrightarrow N$ be a morphism of λ-sequences which is also an objectwise weak homotopy equivalence. Finally, let us suppose that for all $\mu < \lambda$, the continuous maps $M_\mu \longrightarrow M_{\mu+1}$ and $N_\mu \longrightarrow N_{\mu+1}$ are pushouts of maps [5] of the form of a finite product*

$$\mathrm{Id}_{X_1} \times_\Delta \ldots \times_\Delta i_n \times_\Delta \ldots \times_\Delta \mathrm{Id}_{X_p}$$

with $p \geqslant 0$, with $i_n : \mathbf{S}^{n-1} \subset \mathbf{D}^n$ being the usual inclusion of spaces for some $n \geqslant 0$. Then the continuous map $\varinjlim s : \varinjlim M \longrightarrow \varinjlim N$ is a weak homotopy equivalence.

Proof. The main ingredient of the proof of [Gau07, Proposition 7.3] is that any map $K \to M_\lambda$ factors as a composite $K \to M_\mu \to M_\lambda$ for some ordinal $\mu < \lambda$ as soon as K is compact if λ is a limit ordinal. More precisely, one needs to apply this fact for K belonging to the set $\{\mathbf{S}^n, \mathbf{S}^n \times_\Delta [0,1], n \geqslant 0\}$. It is not true that the maps $M_\mu \to M_{\mu+1}$ and $N_\mu \to N_{\mu+1}$ are closed T_1-inclusions of spaces since the spaces X_i are not necessarily weak Hausdorff anymore. So we cannot apply [Hov99, Proposition 2.4.2] saying that compact spaces are finite relative to closed T_1-inclusions of spaces, unlike in the proof of [Gau07, Proposition 7.3]. Let $Z = X_1 \times_\Delta \ldots \times_\Delta X_p$. Then each map $M_\mu \longrightarrow M_{\mu+1}$ and $N_\mu \longrightarrow N_{\mu+1}$ is a pushout of a map of the form $Z \times_\Delta \mathbf{S}^{n-1} \to Z \times_\Delta \mathbf{D}^n$. We can then apply Proposition A.4 and Proposition A.5. The proof is therefore complete. ∎

B. The simplicial structure of $\mathbf{MdTop}_\Delta$

The construction is very similar to the one given in [Gau08a] for the category of flows.

B.1. Definition. *Let K be a non-empty connected simplicial set. Let X be an object of $\mathbf{MdTop}_\Delta$. Let $X \otimes K = |K| \boxtimes X$ where $|K|$ means the geometric realization of K [GJ99].*

B.2. Definition. *Let K be a non-empty simplicial set. Let $(K_i)_{i\in I}$ be its set of non-empty connected components. Let $X \otimes K := \bigsqcup_{i\in I} X \otimes K_i$. And let $X \otimes \varnothing = \varnothing$.*

B.3. Proposition. *Let K be a simplicial set. Then the functor $- \otimes K : \mathbf{MdTop}_\Delta \to \mathbf{MdTop}_\Delta$ is a left adjoint.*

Proof. As in [Gau08a], it suffices to prove the existence of the right adjoint

$$(-)^K : \mathbf{MdTop}_\Delta \to \mathbf{MdTop}_\Delta$$

for K non-empty connected and to set:

[5] There is a typo error in the statement of [Gau07, Proposition 7.3]. The expression “pushouts of maps” is missing.

- $X^{\varnothing} = \mathbf{1}$
- for a general simplicial set K with non-empty connected components $(K_i)_{i\in I}$, let $X^K := \prod_{i\in I} X^{K_i}$.

So now suppose that K is non-empty connected. For a given multipointed d-space X, let (compare with [Gau03, Notation 7.6] and [Gau03, Theorem 7.7]):

- $(X^K)^0 = X^0$
- $|X^K| = \mathbf{TOP}_\Delta(|K|, |X|)$
- for $(\alpha,\beta) \in X^0 \times X^0$, $\mathbb{P}^{top}_{\alpha,\beta}(X^K) = \mathbf{TOP}_\Delta(|K|, \mathbb{P}^{top}_{\alpha,\beta}X)$.

We can observe that the functor $(-)^K : \mathbf{MdTop}_\Delta \to \mathbf{MdTop}_\Delta$ commutes with limits and is λ-accessible if $|K|$ is λ-presentable in $\mathbf{Top}_\Delta$ for some regular cardinal λ since the functor $\mathbb{P}^{top} : \mathbf{MdTop}_\Delta \to \mathbf{Set}$ is finitely accessible by Proposition 3.7. So by [AR94, Theorem 1.66], it is a right adjoint. It is easy to check that the left adjoint is precisely $- \otimes K$. ∎

B.4. Proposition. *Let X and Y be two multipointed d-spaces. Let $\Delta[n]$ be the n-simplex. Then there is a natural isomorphism of simplicial sets*

$$\mathbf{MdTop}_\Delta(X \otimes \Delta[*], Y) \cong \operatorname{Sing} \mathbf{MDTOP}_\Delta(X, Y)$$

where Sing *is the singular nerve functor. This simplicial set is denoted by* $\operatorname{Map}(X, Y)$*.*

Proof. Since $\Delta[n]$ is non-empty and connected, one has

$$\operatorname{Sing}(\mathbf{MDTOP}_\Delta(X,Y))_n = \mathbf{Top}_\Delta(|\Delta[n]|, \mathbf{MDTOP}_\Delta(X,Y)) \cong \mathbf{MdTop}_\Delta(X \otimes \Delta[n], Y).$$

∎

B.5. Theorem. *The model category $\mathbf{MdTop}_\Delta$ together with the functors $- \otimes K$, $(-)^K$ and $\operatorname{Map}(-,-)$ assembles to a simplicial model category.*

Proof. Proof analogous to the proof of [Gau08a, Theorem 3.3.15]. ∎

In fact, the category of multipointed d-spaces $\mathbf{MdTop}_\Delta$ is also tensored and cotensored over $\mathbf{Top}_\Delta$ in the sense of [Col06] because of Proposition 2.8 and Theorem 5.4. On the contrary, one has:

B.6. Proposition. *The category $\mathbf{MdTop}_\mathcal{K}$ of multipointed d-spaces over $\mathbf{Top}_\mathcal{K}$ is neither tensored, nor cotensored over $\mathbf{Top}_\mathcal{K}$.*

Proof. Otherwise the functor $\mathbf{MDTOP}_\mathcal{K}(X, -) : \mathbf{MdTop}_\mathcal{K} \to \mathbf{Top}_\mathcal{K}$ would preserve limits. Take $X = \{0\}$. Then for any multipointed d-space Y, $\mathbf{MDTOP}_\mathcal{K}(\{0\}, Y)$ is the discrete space Y^0 by Proposition B.4. But a limit of discrete spaces in $\mathbf{Top}_\mathcal{K}$ is not necessarily discrete (e.g. the p-adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ [Mun75]). Contradiction. ∎

The same phenomenon arises for the category of flows: read the comment [Gau03, p567] after the statement of Theorem 5.10.

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