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THE POINTED SUBOBJECT FUNCTOR, 3×3 LEMMAS, AND SUBTRACTIVITY OF SPANS

Dedicated to Dominique Bourn on the occasion of his sixtieth birthday

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Abstract. The notion of a *subtractive category* recently introduced by the author, is a pointed categorical counterpart of the notion of a *subtractive variety* of universal algebras in the sense of A. Ursini (recall that a variety is subtractive if its theory contains a constant 0 and a binary term s satisfying $s(x,x) = 0$ and $s(x,0) = x$). Let us call a pointed regular category $\mathbb{C}$ *normal* if every regular epimorphism in $\mathbb{C}$ is a normal epimorphism. It is well known that any *homological category* in the sense of F. Borceux and D. Bourn is both normal and subtractive. We prove that in any subtractive normal category, the upper and lower 3×3 lemmas hold true, which generalizes a similar result for homological categories due to D. Bourn (note that the middle 3×3 lemma holds true if and only if the category is homological). The technique of proof is new: the pointed subobject functor $\mathcal{S} = \mathrm{Sub}(-) : \mathbb{C} \to \mathbf{Set}_*$ turns out to have suitable preservation/reflection properties which allow us to reduce the proofs of these two diagram lemmas to the standard diagram-chasing arguments in $\mathbf{Set}_*$ (alternatively, we could use the more advanced embedding theorem for regular categories due to M. Barr). The key property of $\mathcal{S}$, which allows to obtain these diagram lemmas, is the preservation of *subtractive spans*. Subtractivity of a span provides a weaker version of the *rule of subtraction* — one of the *elementary rules for chasing diagrams* in abelian categories, in the sense of S. Mac Lane. A pointed regular category is subtractive if and only if every span in it is subtractive, and moreover, the functor $\mathcal{S}$ not only preserves but also reflects subtractive spans. Thus, subtractivity seems to be exactly what we need in order to prove the upper/lower 3×3 lemmas in a normal category. Indeed, we show that a normal category is subtractive if and only if these 3×3 lemmas hold true in it. Moreover, we show that for any pointed regular category $\mathbb{C}$ (not necessarily a normal one), we have: $\mathbb{C}$ is subtractive if and only if the lower 3×3 lemma holds true in $\mathbb{C}$.

Introduction

Homological categories in the sense of F. Borceux and D. Bourn [2] provide a convenient non-abelian setting for proving homological lemmas such as the 3×3 lemmas, the short five lemma and the snake lemma (see [4, 2]). The aim of the present paper is to show

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that some of these lemmas can be proved in an even more general setting of *subtractive normal categories* (see below).

The paper is organized as follows: Section 1 recalls the connections between the concepts of homological / subtractive normal categories and some other closely related concepts from categorical and universal algebra. Section 2 introduces notation used throughout the rest of the paper.

In Section 3 we define exact sequences in pointed regular categories and show that for such a category $\mathbb{C}$, the pointed subobject functor $\mathcal{S} = \mathrm{Sub}(-) : \mathbb{C} \to \mathbf{Set}_*$ preserves and reflects exactness. We also establish some other useful properties of the functor $\mathcal{S}$. Note that our definition of an exact sequence is different from the one given in [2]. However, the two notions of exactness coincide for *normal categories*, i.e. pointed regular categories in which every regular epimorphism is normal.

In Section 4 we recall the definition of a subtractive category [17], and obtain one more property of the functor $\mathcal{S}$ (preservation and reflection of "subtractive spans"). In subsequent sections these properties of $\mathcal{S}$ are used for diagram chasing in subtractive regular categories and subtractive normal categories.

In Section 5 we investigate 3×3 lemmas in normal categories. Specifically, we show that the upper/lower 3×3 lemma holds true in a normal category $\mathbb{C}$ if and only if $\mathbb{C}$ is subtractive. We also observe that the middle 3×3 lemma holds true in $\mathbb{C}$ if and only if $\mathbb{C}$ is homological.

In Section 6 we give several characterizations of subtractive regular categories via diagram lemmas. In particular, we show that a pointed regular category is subtractive if and only if the lower 3×3 lemma holds true in it.

In Section 7 we observe that the diagram chasing technique of the present paper is in fact a direct adaptation of the diagram chasing via "members" developed for abelian categories in S. Mac Lane's book [23].

1. Connections between homological / subtractive normal categories and other closely related classes of categories

Recall that a homological category is a pointed regular category [1] which is *protomodular* in the sense of D. Bourn [3]. Any variety of universal algebras is regular, so a variety is homological if and only if it is pointed protomodular, and also, if and only if it is *semi-abelian* in the sense of G. Janelidze, L. Márki and W. Tholen [11] (which is defined as a Barr exact [1] pointed protomodular category having binary sums). As shown in [5], such varieties are also the same as pointed *classically ideal determined varieties* in the sense of A. Ursini [26] (which were introduced as *BIT speciale* varieties in [25]).

We call a category *normal* if it is a pointed regular category [1] and every regular epimorphism in it is a normal epimorphism (which in any finitely complete pointed category is equivalent to every split epimorphism being a normal epimorphism — see [7]). A pointed variety of universal algebras is normal if and only if it is a *variety with ideals* in the sense of K. Fichtner [8], also known in universal algebra as a 0-*regular variety* (see [13]).

A *subtractive normal category* is a normal category which is *subtractive* in the sense of [17] (see Section 4 below). The notion of a subtractive category is a categorical counterpart of the notion of a *subtractive variety* due to A. Ursini [26] (see also [10] and see [17] for further references): a pointed variety is a subtractive variety if and only if it is subtractive as a category. It was shown in [17] that in a subtractive category product projections are normal epimorphisms, however, in general, not every split epimorphism in a subtractive category is normal. As shown in [10], a variety of universal algebras is subtractive and 0-regular, if and only if it is a *BIT* variety in the sense of A. Ursini [24], which in [10] is also called an *ideal-determined variety.* Thus, subtractive normal varieties are exactly the pointed ideal-determined varieties. In [12] a notion of an *ideal-determined category* was introduced, as a categorical counterpart of the notion of an ideal-determined variety (a pointed variety of universal algebras is an ideal-determined variety if and only if it is ideal-determined as a category). Thus for varieties we have "ideal-determined = subtractive + normal", but in general, subtractive normal categories are not the same as ideal-determined categories: as observed in [12], the category of torsion-free abelian groups is a subtractive normal category but not ideal-determined. The precise relationship between these two classes of categories has not been investigated yet — it was proposed in [12] as an open question; more precisely, the following was asked in [12]: is it true that a Barr exact normal category with finite colimits is ideal-determined if and only if it is subtractive? This, however, is beyond the scope of the present paper.

2. Notation

Most of the time we work in a pointed regular category $\mathbb{C}$. By 0 we denote the zero object of $\mathbb{C}$, as well as the zero morphism between any two objects in $\mathbb{C}$. For a morphism $f : X \to Y$ in $\mathbb{C}$, the morphism

$$\mathrm{K}_f \xrightarrow{\mathrm{k}_f} X$$

denotes the kernel of f. We say that f has a *trivial kernel* if k_f is a null morphism. For a morphism $f : X \to Y$ in $\mathbb{C}$, by m_f and e_f we denote the monomorphism and regular epimorphism, respectively, appearing in the image decomposition of f:

$$\begin{tikzcd} & \mathrm{I}_f \arrow[dr, "\mathrm{m}_f"] & \\ X \arrow[ur, "\mathrm{e}_f"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

We write $\mathrm{Im}(f)$ for the subobject of Y represented by m_f. For any two morphisms u and v having the same codomain, we write $u \leqslant v$ if u factors through v, i.e. u and v are part of a commutative diagram

$$\begin{tikzcd} & \bullet \arrow[d, "v"] \\ \bullet \arrow[ur, "w"] \arrow[r, "u"'] & \bullet \end{tikzcd}$$

For any two morphisms $f : C \to X$ and $g : C \to Y$ having the same domain, we write (f, g) for the induced morphism $(f, g) : C \to X \times Y$ to the product $X \times Y$. We write $[s_1, s_2]$ to denote a span

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\ X & & Y \end{tikzcd}$$

Given two spans $[s_1, s_2]$ and $[s'_1, s'_2]$ from X to Y, we write $[s'_1, s'_2] \leqslant [s_1, s_2]$ if these spans are part of a commutative diagram

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\ X & & Y \\ & S' \arrow[ul, "s'_1"] \arrow[ur, "s'_2"'] \arrow[uu] & \end{tikzcd}$$

When the product $X \times Y$ exists, this has the same meaning as $(s'_1, s'_2) \leqslant (s_1, s_2)$.

3. The pointed subobject functor

Let $\mathbb{C}$ be a pointed regular category [1]. We construct a functor

$$\mathcal{S} : \mathbb{C} \to \mathbf{Set}_*$$

which will have a number of useful properties, allowing to transfer certain diagram-chasing arguments from $\mathbf{Set}_*$ to $\mathbb{C}$.

In fact, $\mathcal{S}$ is nothing but the covariant subobject functor (we assume that $\mathbb{C}$ is well-powered)

$$\mathcal{S}(X) = \mathrm{Sub}(X),$$

where the base point in $\mathrm{Sub}(X)$ is the subobject represented by the zero morphism $0 \to X$ (which is clearly a monomorphism). Note that elements of $\mathrm{Sub}(X)$ are not simply monomorphisms with codomain X, but equivalence classes of such monomorphisms, under the following equivalence relation: $w_1 : W_1 \to X$ is equivalent to $w_2 : W_2 \to X$ if w_1 and w_2 factor through each other, i.e. $w_1 \leqslant w_2$ and $w_2 \leqslant w_1$. For any morphism $f : X \to Y$, the map $\mathcal{S}(f) : \mathcal{S}(X) \to \mathcal{S}(Y)$ sends a subobject of X, represented by, say, $w : W \to X$, to the image of the composite $fw : W \to Y$, which is then a subobject of Y. Thus, $\mathcal{S}$ is similar to the *transfer functor* in the sense of M. Grandis [9].

3.1. Remark. If $\mathbb{C}$ is an abelian category, then for any object X in $\mathbb{C}$ the set $\mathcal{S}(X)$ is nothing but the set of equivalence classes of *members* of X in the sense of S. Mac Lane [23] (see Section 4 in Chapter VIII in [23]). As shown in [23], the language of "members" can be used for repeating certain classical diagram-chasing arguments in arbitrary abelian categories (without resorting to more advanced "embedding theorems", see Notes to Chapter VIII in [23]). In the present paper we show that some of these arguments generalize to arbitrary subtractive normal categories (see also Section 7).

We begin studying properties of the functor $\mathcal{S}$ by stating the first few most basic preservation/reflection properties of $\mathcal{S}$. We omit straightforward proofs.

3.2. Proposition. *The functor $\mathcal{S} : \mathbb{C} \to \mathbf{Set}_*$ preserves and reflects the zero objects and the zero morphisms.*

3.3. Proposition. *The functor $\mathcal{S}$ preserves and reflects regular epimorphisms.*

Proof. Regular epimorphisms in $\mathbf{Set}_*$ are surjective morphisms of pointed sets. Using this, it is easy to show that $\mathcal{S}$ preserves and reflects regular epimorphisms. ∎

3.4. Lemma. *The functor $\mathcal{S}$ preserves monomorphisms.*

Proof. The proof is straightforward (recall that in $\mathbf{Set}_*$ monomorphisms are injective morphisms of pointed sets). ∎

The following easy lemma (see Lemma 3.5 below), and the fact that $\mathcal{S}$ preserves regular epimorphisms, allows to extend $\mathcal{S}$ to the categories of relations

$$\overline{\mathcal{S}} : \mathrm{Rel}(\mathbb{C}) \to \mathrm{Rel}(\mathbf{Set}_*).$$

3.5. Lemma. *The functor $\mathcal{S} : \mathbb{C} \to \mathbf{Set}_*$ preserves weak pullbacks.*

Specifically, for an internal relation

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ X & & Y \end{tikzcd}$$

in $\mathbb{C}$, which we write as $[r_1, r_2] : X \rightharpoonup Y$, the relation $\overline{\mathcal{S}}([r_1, r_2]) : \mathcal{S}(X) \rightharpoonup \mathcal{S}(Y)$ is defined as the relation generated by the span

$$\begin{tikzcd} & \mathcal{S}(R) \arrow[dl, "\mathcal{S}(r_1)"'] \arrow[dr, "\mathcal{S}(r_2)"] & \\ \mathcal{S}(X) & & \mathcal{S}(Y) \end{tikzcd}$$

For the sake of formality, we should mention that we identify two relations $[r_1, r_2] : X \rightharpoonup Y$ and $[r_1', r_2'] : X \rightharpoonup Y$ if they define the same subobject of $X \times Y$, i.e. if $\mathrm{Im}(r_1, r_2) =$

$\mathrm{Im}(r_1', r_2')$. This is the case precisely when the two relations are part of a commutative diagram

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ X & A \arrow[u, "e_1"'] \arrow[d, "e_2"'] & Y \\ & R' \arrow[ul, "r_2'"] \arrow[ur, "r_1'"'] & \end{tikzcd}$$

where e_1 and e_2 are regular epimorphisms. Then, since $\mathcal{S}$ preserves regular epimorphisms, $\overline{\mathcal{S}}$ is indeed well defined.

In [2] a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is said to be *exact* at Y, if both the following conditions are satisfied:

(i) m_f is a kernel of e_g,

(ii) and e_g is a cokernel of m_f.

When all regular epimorphisms in $\mathbb{C}$ are normal epimorphisms, this is equivalent to requiring just (i) (since any normal epimorphism is a cokernel of its kernel). In this paper we use (i) as the definition of exactness for general pointed regular categories. Note that (i) is equivalent to the following:

(i′) m_f is a kernel of g.

3.6. Definition. *In a pointed regular category $\mathbb{C}$, a sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of morphisms is said to be exact at B, if m_f is a kernel of g.

In $\mathbf{Set}_*$ exactness at B simply means

$$\forall_{b\in B}[g(b) = 0 \;\Leftrightarrow\; \exists_{a\in A} f(a) = b].$$

3.7. Proposition. *For any pointed regular category $\mathbb{C}$, the functor $\mathcal{S} : \mathbb{C} \to \mathbf{Set}_*$ preserves and reflects exactness.*

3.8. Lemma. *For a morphism $f : X \to Y$ in a pointed regular category the following conditions are equivalent:*

(a) f is a regular epimorphism.

(b) The sequence

$$X \xrightarrow{f} Y \longrightarrow 0$$

is exact at Y.

3.9. PROPOSITION. *For a pointed category $\mathbb{C}$ with pullbacks, the following conditions are equivalent:*

(a) Any morphism having a trivial kernel is a monomorphism.

(b) Any regular epimorphism having a trivial kernel is an isomorphism.

(c) Any split epimorphism having a trivial kernel is an isomorphism.

PROOF. The implications (a)⇒(b)⇒(c) are obvious. To show (c)⇒(a), take any morphism $f : X \to Y$ having a trivial kernel. Then the pullback of f along itself also has a trivial kernel, and being a split epimorphism, by (c) it should be an isomorphism. This implies that f is a monomorphism. ■

3.10. COROLLARY. *For a pointed regular category $\mathbb{C}$ the following conditions are equivalent:*

(a) Any morphism in $\mathbb{C}$ having a trivial kernel is a monomorphism.

(b) A sequence

$$0 \longrightarrow X \xrightarrow{f} Y$$

in $\mathbb{C}$ is exact at X if and only if f is a monomorphism.

(c) A sequence

$$0 \longrightarrow X \xrightarrow{f} Y \longrightarrow 0$$

in $\mathbb{C}$ is exact at both X and Y if and only if f is an isomorphism.

If these conditions are satisfied, then the functor $\mathcal{S} : \mathbb{C} \to \mathbf{Set}_*$ reflects monomorphisms and isomorphisms.

The equivalent conditions of Proposition 3.9 are trivially satisfied when every regular epimorphism is normal. It is natural to call pointed regular categories having this property *normal categories*, since they can be defined in the same style as regular categories, using normal epimorphisms in the place of regular ones:

3.11. DEFINITION. *A finitely complete pointed category $\mathbb{C}$ is said to be normal if every morphism in $\mathbb{C}$ decomposes as a pullback-stable normal epimorphism followed by a monomorphism. In other words, a normal category is a pointed regular category in which every regular epimorphism is normal.*

3.12. PROPOSITION. *For a pointed regular category $\mathbb{C}$ the following conditions are equivalent:*

(a) $\mathbb{C}$ is normal.

(b) $\mathbb{C}$ satisfies equivalent conditions (a,b,c) of Proposition 3.9, and every normal monomorphism in $\mathbb{C}$ has a cokernel.

Proof. (a)⇒(b): Suppose $\mathbb{C}$ is normal. Then 3.9(b) is trivially satisfied. Also, then every normal monomorphism in $\mathbb{C}$ has a cokernel — indeed, for a morphism $f : X \to Y$, the cokernel of the kernel k_f of f is the morphism e_f in the image decomposition of f:

$$\begin{tikzcd} & \mathrm{I}_f \arrow[dr, "\mathrm{m}_f"] & \\ X \arrow[ur, "\mathrm{e}_f"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

(b)⇒(a): Suppose (b) holds true. Let e be a regular epimorphism and let c be the cokernel of its kernel k_e. Then there exists a unique morphism u such that $uc = e$. Since e is a regular epimorphism, so is u. It is easy to show that u has a trivial kernel, which implies that u is an isomorphism. ■

4. Subtractive regular categories

In a pointed category $\mathbb{C}$, call a relation

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ X & & Y \end{tikzcd} \tag{1}$$

subtractive if for any object C in $\mathbb{C}$, and for any two morphisms

$$x : C \to X, \quad y : C \to Y$$

we have

$$([x, y] \leqslant [r_1, r_2] \ \wedge \ [x, 0] \leqslant [r_1, r_2]) \quad \Rightarrow \quad [0, y] \leqslant [r_1, r_2].$$

In other words, $[r_1, r_2]$ is subtractive if it is *(strictly) closed* with respect to the matrix

$$\left(\begin{array}{cc} x & y \\ x & 0 \\ \hline 0 & y \end{array}\right)$$

in the sense of [18]. The use of the term “subtractive” here can be justified by the idea that the pair $[0, y]$ is obtained via the formal subtraction $[0, y] = [x, y] - [x, 0]$ (in fact, this idea can be exploited further — see [6, 7]).

4.1. Theorem. [18] *For a finitely complete pointed category $\mathbb{C}$ the following conditions are equivalent:*

(a) $\mathbb{C}$ is subtractive in the sense of [17], i.e. for any relation (1) we have

$$([1_X, 1_X] \leqslant [r_1, r_2] \ \wedge \ [1_X, 0] \leqslant [r_1, r_2]) \quad \Rightarrow \quad [0, 1_X] \leqslant [r_1, r_2].$$

(b) Every relation in $\mathbb{C}$ is subtractive.

In $\mathbf{Set}_*$, a relation $R \subseteq X \times Y$ is subtractive if and only if for all $x \in X$ and $y \in Y$ we have

$$[(x,y) \in R \ \wedge \ (x,0) \in R] \quad \Rightarrow \quad (0,y) \in R.$$

4.2. Proposition. *For any pointed regular category $\mathbb{C}$, the functor $\overline{\mathcal{S}} : \mathrm{Rel}(\mathbb{C}) \to \mathrm{Rel}(\mathbf{Set}_*)$ preserves and reflects subtractive relations.*

Proof. Subtractivity of an internal relation (1) in any pointed category $\mathbb{C}$ having weak pullbacks is equivalent to having

$$[0, r_2 q_2 s_2] \leqslant [r_1, r_2], \tag{2}$$

where q_2 and s_2 are morphisms from a diagram

$$\begin{tikzcd}
 & & & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & & & \\
 & & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & Q \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & & \\
 & R \arrow[dl, "0"'] \arrow[dr, "0"] & & R \arrow[dl, "r_2"'] \arrow[dr, "r_1"] & & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
X & & Y & & X & & Y
\end{tikzcd}$$

whose all three diamonds are weak pullbacks. The inequality (2) implies

$$[0, \mathcal{S}(r_2)\mathcal{S}(q_2)\mathcal{S}(s_2)] \leqslant [\mathcal{S}(r_1), \mathcal{S}(r_2)].$$

Clearly $[\mathcal{S}(r_1), \mathcal{S}(r_2)] \leqslant \overline{\mathcal{S}}([r_1, r_2])$, and so the above yields

$$[0, \mathcal{S}(r_2)\mathcal{S}(q_2)\mathcal{S}(s_2)] \leqslant \overline{\mathcal{S}}([r_1, r_2]).$$

Using the fact that $\mathcal{S}$ preserves weak pullbacks, it is easy to check that the above inequality is in fact equivalent to subtractivity of $\overline{\mathcal{S}}([r_1, r_2])$. This readily gives that $\overline{\mathcal{S}}$ preserves subtractive relations. To show that it also reflects subtractive relations, it suffices to have the implication

$$[\mathcal{S}(u), \mathcal{S}(v)] \leqslant \overline{\mathcal{S}}([r_1, r_2]) \quad \Rightarrow \quad [u, v] \leqslant [r_1, r_2]$$

for a span $[u, v] : X \rightharpoonup Y$, and specifically, for $u = 0$ and $v = r_2 q_2 s_2$. Although the above implication fails for general u, v, it is indeed satisfied when $u = 0$. To show this, let $w : W \to R$ be a subobject such that $\mathcal{S}(r_1)(w) = \mathcal{S}(u)(1_S)$ and $\mathcal{S}(r_2)(w) = \mathcal{S}(v)(1_S)$, where S is the domain of u and v (such w exists once $[\mathcal{S}(u), \mathcal{S}(v)] \leqslant \overline{\mathcal{S}}([r_1, r_2])$). We can then form a commutative diagram

$$\begin{tikzcd}
 & & T \arrow[dl, "t_1"'] \arrow[dr, "t_2"] & & \\
W \arrow[d, "w"'] & U \arrow[l, "e_1"'] \arrow[dr, "e_2"'] & & V \arrow[r, "e_4"] \arrow[dl, "e_3"] & W \arrow[d, "w"] \\
R \arrow[dr, "r_1"'] & & S \arrow[dl, "u"'] \arrow[dr, "v"] & & R \arrow[dl, "r_2"] \\
 & X & & Y &
\end{tikzcd}$$

where the morphisms e_1, e_2, e_3, e_4 are regular epimorphisms and the diamond is a pullback (hence t_1 and t_2 are also regular epimorphisms). To show $[u, v] \leqslant [r_1, r_2]$, it is sufficient to show $[ue_2t_1, ve_2t_1] = [r_1we_4t_2, r_2we_4t_2]$, which we have as soon as $u = 0$ (since if $u = 0$ then $r_1we_1 = 0$ and hence $r_1w = 0$, because e_1 is an epimorphism). ∎

4.3. Definition. *A span*

$$\begin{tikzcd}
 & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
X & & Y
\end{tikzcd} \tag{3}$$

in a pointed regular category $\mathbb{C}$ is said to be subtractive, if the relation generated by it is subtractive.

From Proposition 4.2 we get:

4.4. Corollary. *For any pointed regular category $\mathbb{C}$, the functor $\mathcal{S} : \mathbb{C} \to \mathbf{Set}_*$ preserves and reflects subtractive spans.*

Notice that a span (3) in $\mathbf{Set}_*$ is subtractive if and only if for any two elements $a, b \in S$ such that $s_1(a) = s_1(b)$ and $s_2(b) = 0$, there exists an element $c \in S$, such that $s_1(c) = 0$ and $s_2(c) = s_2(a)$. If we write c as a formal difference $c = a - b$, then the last two equalities formally follow from the previous ones:

$$s_1(a - b) = s_1(a) - s_1(b) = s_1(a) - s_1(a) = 0,$$

$$s_2(a - b) = s_2(a) - s_2(b) = s_2(a) - 0 = s_2(a).$$

5. 3×3 lemmas and subtractive normal categories

In a pointed regular category, by a 3×3 diagram we mean a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "u_1"] \arrow[d, "f_1"'] & B_1 \arrow[r, "v_1"] \arrow[d, "g_1"'] & C_1 \arrow[r] \arrow[d, "h_1"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "u_2"] \arrow[d, "f_2"'] & B_2 \arrow[r, "v_2"] \arrow[d, "g_2"'] & C_2 \arrow[r] \arrow[d, "h_2"] & 0 \\
0 \arrow[r] & A_3 \arrow[r, "u_3"'] \arrow[d] & B_3 \arrow[r, "v_3"'] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{4}$$

where all columns are exact sequences.

5.1. LEMMA. [The upper 3×3 lemma] *In any subtractive normal category $\mathbb{C}$, for any 3×3 diagram (4), if the second and third rows are exact sequences then so is the first row.*

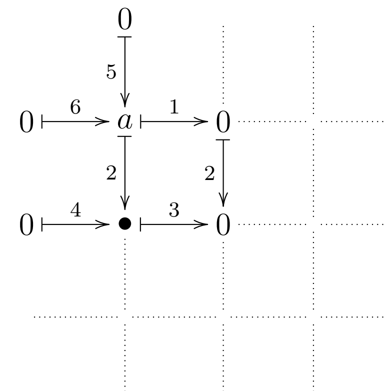
PROOF. Since g_1 and h_1 have trivial kernels, they are monomorphisms by Proposition 3.12. So, by the preservation/reflection properties of $\mathcal{S}$, it suffices to prove that if in $\mathbf{Set}_*$ we have a 3×3 diagram (4) where

(i) g_1 is injective,

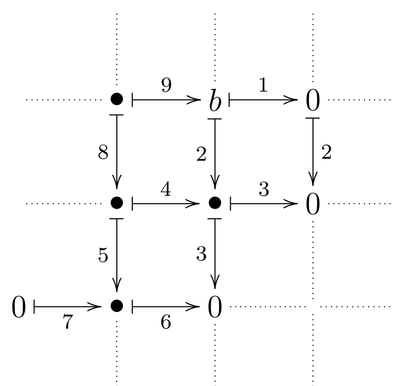
(ii) the span $[g_2, v_2]$ is subtractive,

(iii) and h_1 is injective,

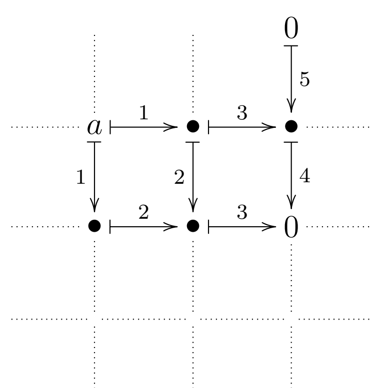
then exactness of the second and the third rows imply exactness of the first row. Now that we are in $\mathbf{Set}_*$ we can carry out the standard diagram-chasing argument. First, let us prove exactness at A_1. That is, we show $\mathrm{k}_{u_1} = 0$. The diagram chasing needed to prove this is summarized by the following display (where numbers indicate the progression of the diagram chase):



Notice that here we did not use any one of the conditions (i), (ii), (iii). Next, we prove exactness at B_1, for which we will use (i). $\mathrm{k}_{v_1} \leqslant \mathrm{m}_{u_1}$ can be proved by the following diagram chase (to get the arrow 9 we use (i)):



$\mathrm{m}_{u_1} \leqslant \mathrm{k}_{v_1}$ (or, equivalently, $v_1 u_1 = 0$) can be proved as follows:



Finally, we prove exactness at C_1 (which amounts to showing that v_1 is surjective). We begin the diagram chase:

$$\begin{tikzcd}
 & & c \arrow[r, mapsto, "1"] \arrow[d, mapsto, "2"] & 0 \\
a \arrow[d, mapsto, "9"'] & b \arrow[r, mapsto, "4"] \arrow[d, mapsto, "5"'] & c' \arrow[r, mapsto, "3"] \arrow[d, mapsto, "3"] & 0 \\
\bullet \arrow[r, mapsto, "7"'] \arrow[d, mapsto, "8"'] & \bullet \arrow[r, mapsto, "6"'] & 0 & \\
0 & & &
\end{tikzcd}$$

To be able to proceed we use (ii). Since $g_2(u_2(a)) = g_2(b)$ and $v_2(u_2(a)) = 0$, we can apply (ii) to get an element $b' = b - u_2(a) \in C_2$, such that $g_2(b') = 0$ and $v_2(b') = v_2(b)$. We then replace b in the above display with b' and continue the diagram chase (we get the last arrow 12 by using (iii)):

$$\begin{tikzcd}
\bullet \arrow[r, mapsto, "12"] \arrow[d, mapsto, "11"'] & c \arrow[d, mapsto, "2"] \\
b - u_2(a) \arrow[r, mapsto, "10"] \arrow[d, mapsto, "10"'] & c' \\
0 &
\end{tikzcd}$$

■

The following 3×3 lemma can be proved in a similar way (see Theorem 6.1):

5.2. LEMMA. [The lower 3×3 lemma] *In any subtractive normal category* $\mathbb{C}$, *for any* 3×3 *diagram (4), if the first and second rows are exact sequences then so is the third row.*

The middle 3×3 lemma fails in a subtractive normal category, unless it is a homological category (the following theorem is essentially well known):

5.3. THEOREM. *For a pointed regular category* $\mathbb{C}$ *the following conditions are equivalent:*

(a) $\mathbb{C}$ *is protomodular.*

(b) *The middle* 3×3 *lemma holds true in* $\mathbb{C}$*, that is, for any* 3×3 *diagram (4), if the first and third rows are exact sequences, then the middle row is exact as soon as we have* $v_2u_2 = 0$.

(c) *The short five lemma holds true in* $\mathbb{C}$*, that is, for any commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d, "u_1"'] & A_2 \arrow[r, "f_2"] \arrow[d, "u_2"'] & A_3 \arrow[r] \arrow[d, "u_3"] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r] & 0
\end{tikzcd}$$

with exact rows, if u_1 *and* u_3 *are isomorphisms, then* u_2 *is also an isomorphism.*

Proof. (a)⇒(b) and (c)⇒(a) are well known (see e.g. [2]). To get (b)⇒(c), first we show that (b) implies the following modified short five lemma: for any commutative diagram as in (c), with exact rows, if u_1 and u_3 are isomorphisms, then u_2 is a regular epimorphism. For any diagram as in (c), construct the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "u_1"] \arrow[d, "f_1"'] & B_1 \arrow[r] \arrow[d, "g_1"'] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "u_2"] \arrow[d, "f_2"'] & B_2 \arrow[r] \arrow[d, "g_2"'] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_3 \arrow[r, "u_3"'] \arrow[d] & B_3 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the assumptions in (c) are satisfied, then this diagram is a 3×3 diagram with top and the bottom rows exact. Applying (b) we get that the middle row is exact. Thus, u_2 is regular epimorphism, proving the modified short five lemma. To get the short five lemma, it suffices to be able to derive from the modified short five lemma the fact that every regular epimorphism having a trivial kernel is an isomorphism. By Proposition 3.9, we have this as soon as every split epimorphism having a trivial kernel is an isomorphism. Let $f : X \to Y$ be a split epimorphism, with a splitting $g : Y \to X$, and having a trivial kernel. Then we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & Y \arrow[r, "1_Y"] \arrow[d, "g"'] & Y \arrow[r] \arrow[d, "1_Y"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & X \arrow[r, "f"'] & Y \arrow[r] & 0
\end{tikzcd}$$

with exact rows. It now follows from the modified short five lemma that f is an isomorphism. ∎

It turns out that for a normal category, the upper and lower 3×3 lemmas are equivalent to subtractivity, like the middle 3×3 lemma is equivalent to protomodularity (see also Theorem 6.1).

5.4. Theorem. *For any normal category $\mathbb{C}$, the following conditions are equivalent:*

(a) $\mathbb{C}$ is subtractive.

(b) The upper 3×3 lemma holds true in $\mathbb{C}$.

(c) The lower 3×3 lemma holds true in $\mathbb{C}$.

Proof. We have (a)⇒(b) and (a)⇒(c) by Lemmas 5.1 and 5.2, respectively.

(b)⇒(a): Consider a relation

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ X & & X \end{tikzcd}$$

in $\mathbb{C}$. Suppose $(1_X, 1_X) \leqslant (r_1, r_2)$ and $(1_X, 0) \leqslant (r_1, r_2)$. The first inequality gives that both r_1 and r_2 are split epimorphisms and hence regular epimorphisms. The second inequality is equivalent to $r_1 \mathrm{k}_{r_2}$ being an isomorphism, and similarly, $r_2 \mathrm{k}_{r_1}$ is an isomorphism if and only if $(0, 1_X) \leqslant (r_1, r_2)$ (note that both $r_1 \mathrm{k}_{r_2}$ and $r_2 \mathrm{k}_{r_1}$ are monomorphisms). So, assuming that $r_1 \mathrm{k}_{r_2}$ is an isomorphism, we want to show that $r_2 \mathrm{k}_{r_1}$ is an isomorphism. If $r_1 \mathrm{k}_{r_2}$ is an isomorphism, then we get a 3×3 diagram where the second and the third rows are exact:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{K}_{r_1} \arrow[r, "r_2 \mathrm{k}_{r_1}"] \arrow[d, "\mathrm{k}_{r_1}"'] & X \arrow[r] \arrow[d, "1_X"] & 0 \\ 0 \arrow[r] & \mathrm{K}_{r_2} \arrow[r, "\mathrm{k}_{r_2}"] \arrow[d, "1_{\mathrm{K}_{r_2}}"'] & R \arrow[r, "r_2"] \arrow[d, "r_1"'] & X \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{K}_{r_2} \arrow[r, "r_1 \mathrm{k}_{r_2}"'] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

By (b), the top row is also exact, which means that $r_2 \mathrm{k}_{r_1}$ is an isomorphism.

The proof of (c)⇒(a) is analogous, and uses the same diagram as above (with the roles of r_1 and r_2 interchanged). ∎

6. Beyond normality

The main goal in this section is to show that in a pointed regular category, subtractivity turns out to be equivalent to certain restricted homological diagram lemmas.

6.1. THEOREM. *For a pointed regular category $\mathbb{C}$, the following conditions are equivalent:*

(a) $\mathbb{C}$ is a subtractive category.

(b) A modified upper 3×3 lemma holds true in $\mathbb{C}$: for any 3×3 diagram (4), where g_1 and h_1 are monomorphisms, if the first and second rows are exact sequences then so is the third row.

(c) The lower 3×3 lemma holds true in $\mathbb{C}$, i.e. for any 3×3 diagram (4), if the first and second rows are exact sequences then so is the third row.

PROOF. The proof of (a)⇒(b) is the same as the proof of the upper 3×3 lemma given in the previous section (see Lemma 5.1). Indeed, the only time we used normality there was to conclude that g_1 and h_1 are monomorphisms, which is now given as an assumption.

The proofs of (b)⇒(a) and (c)⇒(a) are the same as for Theorem 5.4.

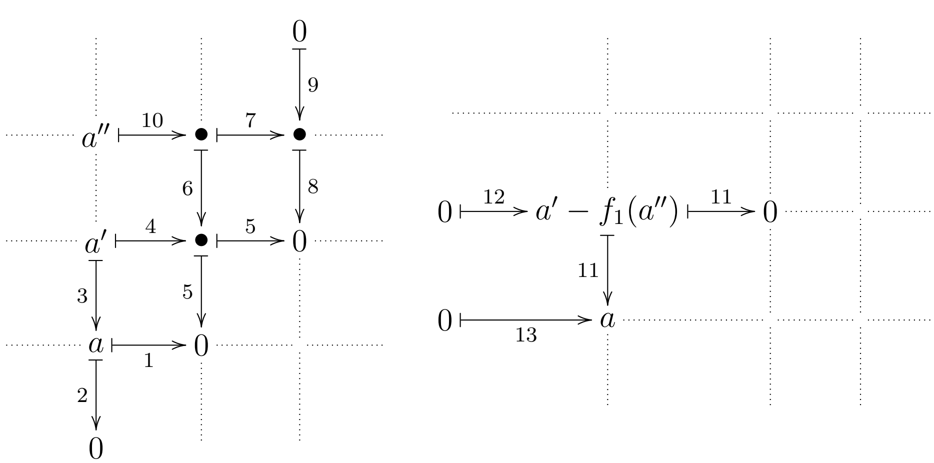
(a)⇒(c): It suffices to show that in $\mathbf{Set}_*$, for any 3×3 diagram (4), if the first and second rows are exact sequences, and

(i) the span $[u_2, f_2]$ is subtractive,

(ii) the span $[g_2, v_2]$ is subtractive,

(iii) and the span $[v_2, g_2]$ is subtractive,

then the third row is exact. Note that since h_2 and v_2 are regular epimorphisms, also v_3 is a regular epimorphism, which implies exactness of the third row at C_3. By the standard diagram chase, exactness at A_3 follows from (i); this diagram chase is summarized in the following displays:



Exactness at B_3 follows from (ii) and (iii). In particular, we use (ii) to show $v_3 u_3 = 0$:

$$
\begin{tikzcd}
a' \arrow[d, mapsto, "3"'] & b \arrow[r, mapsto, "4"] \arrow[d, mapsto, "3"'] & c' \arrow[d, mapsto, "5"] \\
a \arrow[r, mapsto, "1"'] \arrow[d, mapsto, "2"'] & \bullet \arrow[r, mapsto, "2"'] \arrow[d, mapsto, "2"'] & c \\
0 & 0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
b - u_2(a') \arrow[r, mapsto, "6"] \arrow[d, mapsto, "6"'] & c' \arrow[d, mapsto, "5"] \\
0 \arrow[r, mapsto, "7"'] & c
\end{tikzcd}
$$

Finally, we use (iii) to show $\mathrm{k}_{v_3} \leqslant \mathrm{m}_{u_3}$:

$$
\begin{tikzcd}
b'' \arrow[r, mapsto, "8"] & \bullet \arrow[r, mapsto, "7"] \arrow[d, mapsto, "6"] & 0 \\
b' \arrow[r, mapsto, "4"] \arrow[d, mapsto, "3"'] & \bullet \arrow[d, mapsto, "5"] & \\
b \arrow[r, mapsto, "1"'] \arrow[d, mapsto, "2"'] & 0 & \\
0 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, mapsto, "10"] \arrow[d, mapsto, "11"'] & b' - g_1(b'') \arrow[r, mapsto, "9"] \arrow[d, mapsto, "9"'] & 0 \\
\bullet \arrow[r, mapsto, "12"'] & b &
\end{tikzcd}
$$

■

The following interesting question was first answered by A. Ursini for varieties of universal algebras, and the answer was presented in the more general context of regular categories with binary coproducts in [22]: is it possible to obtain subtractivity as a kind of restricted version of the short five lemma? In the theorem below (Theorem 6.2) we refine the answer given in [22].

As defined in [14] (see also [13]), an *ideal* (in a pointed regular category $\mathbb{C}$) is a monomorphism m which is a regular image of a normal monomorphism, i.e. m is part of a commutative diagram

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "m'"'] & \bullet \arrow[d, "m"] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
$$

where horizontal arrows are regular epimorphisms and m' is a normal monomorphism.

6.2. Theorem. *For a pointed regular category $\mathbb{C}$ the following conditions are equivalent:*

(a) $\mathbb{C}$ is subtractive.

(b) Short five lemma for ideals holds true in $\mathbb{C}$: for any commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d, "u_1"'] & A_2 \arrow[r, "f_2"] \arrow[d, "u_2"'] & A_3 \arrow[r] \arrow[d, "u_3"] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r] & 0
\end{tikzcd}$$

with exact rows, if u_1 and u_3 are isomorphisms, then u_2 is an isomorphism provided it is an ideal.

Proof. (a)$\Rightarrow$(b): A monomorphism u_2 is an ideal if and only if it is part of a commutative diagram

$$\begin{tikzcd}
C_1 \arrow[r, "e_1"] \arrow[d, "h_1"'] & A_2 \arrow[r] \arrow[d, "u_2"] & 0 \\
C_2 \arrow[r, "e_2"'] \arrow[d, "h_2"'] & B_2 \arrow[r] & 0 \\
C_3 & &
\end{tikzcd}$$

with exact rows and exact column. We show that u_2 is a regular epimorphism (and hence an isomorphism), by a chase along the diagram

$$\begin{tikzcd}
C_1 \arrow[r, "e_1"] \arrow[d, "h_1"'] & A_2 \arrow[r, "f_2"] \arrow[d, "u_2"'] & A_3 \arrow[d, "u_3"] \\
C_2 \arrow[r, "e_2"'] \arrow[d, "h_2"'] & B_2 \arrow[r, "g_2"'] & B_3 \\
C_3 & &
\end{tikzcd}$$

Below, we use subtractivity of the spans $[g_2e_2, h_2]$, $[e_2, h_2]$ and $[h_2, e_2]$ for the steps 5, 10, and 11, respectively. Apart from this, we only use exactness at C_2, the fact that u_1, u_3, e_1, e_2, f_2 are regular epimorphisms, and the inequality $\mathrm{k}_{g_2} \leqslant \mathrm{m}_{g_1}$.

$$\begin{tikzcd}
a \arrow[r, mapsto, "4"] \arrow[d, dash, dotted] & \bullet \arrow[r, mapsto, "3"] \arrow[d, dash, dotted] & \bullet \arrow[d, mapsto, "2"] \\
c \arrow[r, mapsto, "2"'] \arrow[d, mapsto, "3"'] & b \arrow[r, mapsto, "1"'] & \bullet \\
c' & &
\end{tikzcd}
\qquad
\begin{tikzcd}
a' \arrow[r, mapsto, "9"] \arrow[d, dash, dotted] & \bullet \arrow[r, dash, dotted] \arrow[d, mapsto, "8"] & {} \arrow[d, dash, dotted] \\
c - h_1(a) \arrow[r, mapsto, "6"] \arrow[rr, mapsto, bend right, "5"'] \arrow[d, mapsto, "5"'] & \bullet \arrow[r, mapsto, "7"] & 0 \\
c' & &
\end{tikzcd}$$

$$
\begin{tikzcd}
(c - h_1(a)) - h_1(a') \arrow[r, mapsto, "10"'] \arrow[d, mapsto, "10"'] & 0 \\
c' &
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, mapsto, "13"] \arrow[d, mapsto, "12"'] & \bullet \arrow[d, mapsto, "14"] \\
c - ((c - h_1(a)) - h_1(a')) \arrow[r, mapsto, "11"'] \arrow[d, mapsto, "11"'] & b \\
0 &
\end{tikzcd}
$$

(b)⇒(a) follows directly from Corollary 3.6 in [22]. ■

7. Final remarks

We now explain more elaborately than in Remark 3.1, how the diagram-chasing method for abelian categories presented in Chapter VIII of S. Mac Lane's book [23], can be adapted to subtractive normal categories.

Let $\mathbb{C}$ be a pointed regular category. For each object X in $\mathbb{C}$ define a *member* x of X to be a morphism x with codomain X. When x is a member of X, we write $x \in_m X$, as in [23]. If $x, y \in_m X$, define $x \equiv y$ to mean that there are regular epimorphisms u and v with $xu = yv$. This is the same as to say that $\mathrm{Im}(x) = \mathrm{Im}(y)$. Thus, we can think of elements of $\mathcal{S}(X)$ as equivalence classes of members in X.

Theorem 3 in Chapter VIII of [23] gives "elementary rules for chasing diagrams" in abelian categories. Due to the preservation/reflection properties of $\mathcal{S}$ established earlier, we have the following analogue of this theorem:

7.1. Theorem. [Elementary rules for chasing diagrams in subtractive normal categories] *Let $\mathbb{C}$ be a subtractive normal category. Then members of objects in $\mathbb{C}$ obey the following rules:*

- *(a) $f : X \to Y$ is a monomorphism if and only if, for all $x \in_m X$, $fx \equiv 0$ implies $x \equiv 0$;*
- *(b) $f : X \to Y$ is a monomorphism if and only if, for all $x, x' \in_m X$, $fx \equiv fx'$ implies $x \equiv x'$;*
- *(c) $g : Y \to Z$ is a regular epimorphism if and only if, for each $z \in_m Z$ there exists $y \in_m Y$ such that $gy \equiv z$;*
- *(d) $h : X \to Z$ is a zero morphism if and only if, for all $x \in_m X$, $hx \equiv 0$;*
- *(e) A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact at Y if and only if $gf = 0$ and for each $y \in_m Y$ with $gy \equiv 0$ there exists $x \in_m X$ with $fx \equiv y$;*
- *(f) (Subtraction) Given $g : B \to C$ and $f : B \to D$, for any $x, y \in_m B$ with $gx \equiv gy$ and $fx \equiv 0$, there exists $z \in_m B$ with $gz \equiv 0$ and $fy \equiv fz$.*

As we saw, in some cases we can still perform carry out diagram chasing in subtractive regular categories. But then the first two rules are not available any more:

7.2. Theorem. [Elementary rules for chasing diagrams in subtractive regular categories] *Members of objects in a subtractive regular category* $\mathbb{C}$ *obey the rules 7.1(c-f).*

The author invites the Reader to formulate, as an exercise, elementary rules for chasing diagrams in homological categories (see also [19]).

Notice that the rules 7.1(c,d,e) hold true for members of objects in any pointed regular category (see Section 3). Note also that the condition 7.1(a) is equivalent to the condition 3.10(a), and hence, by Corollary 3.10, it implies 7.1(b). By Corollary 4.4, the condition 7.1(f) is equivalent to the category being subtractive.

It can be easily seen that all results for normal categories that we obtain in Section 5, are also valid for pointed regular categories in which members of objects obey the following rule:

(M) if a morphism f has a trivial kernel then for all $x, x' \in_m X$, $fx \equiv fx'$ implies $x \equiv x'$.

In other words, the above condition states that the pointed subobject functor $\mathcal{S}$ carries morphisms with trivial kernels to monomorphisms in $\mathbf{Set}_*$. In particular, the condition 7.1(a) clearly implies (M). It would be interesting to find examples (if there are any) of pointed regular categories showing that all three conditions — normality, the condition 7.1(a) and the condition (M) — are different from each other.

The true role of normality becomes apparent in the investigation of the snake lemma. In a forthcoming paper it will be shown that for a pointed regular category $\mathbb{C}$ the (suitably formulated) snake lemma is in fact equivalent to $\mathbb{C}$ being a subtractive normal category.

Towards a further generalization: while the use of the pointed subobject functor for diagram chasing is in fact a rather limited technique compared to M. Barr's embedding theorem for regular categories [1], it inspires to lift this theory beyond regular categories where the Grothendieck topology of regular epimorphisms is replaced with a *cover relation* in the sense of [20] (see also [21]); this direction of investigation leads to a unified framework for proving diagram lemmas in homological categories in the sense of F. Borceux and D. Bourn [2], and in homological categories in the sense of M. Grandis [9]. Work in this direction is currently in progress (the proposed general setting will also include that of quasi-pointed categories considered in [4]).

Finally, it would be interesting to investigate "relative" versions of these results in the style of [15] (see also [16]).

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