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NOTES ON BIMONADS AND HOPF MONADS

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Abstract. For a generalisation of the classical theory of Hopf algebra over fields, A. Bruguières and A. Virelizier study opmonoidal monads on monoidal categories (which they called *bimonads*). In a recent joint paper with S. Lack the same authors define the notion of a *pre-Hopf monad* by requiring only a special form of the fusion operator to be invertible. In previous papers it was observed by the present authors that bimonads yield a special case of an entwining of a pair of functors (on arbitrary categories). The purpose of this note is to show that in this setting the pre-Hopf monads are a special case of Galois entwinings. As a byproduct some new properties are detected which make a (general) bimonad on a Cauchy complete category to a Hopf monad. In the final section applications to cartesian monoidal categories are considered.

1. Introduction

The classical definitions of bialgebras and Hopf algebras over fields (or rings) heavily depend on constructions based on the tensor product. This may have been one of the reasons why first generalisations of this notions were formulated for monoidal categories, or even autonomous monoidal categories when the properties of finite dimensional Hopf algebras were in the focus. This was also the starting point for the definitions of *Hopf monads* by I. Moerdijk in [13]. McCrudden [8] suggested to call these functors *opmonoidal monads* and A. Bruguières and A. Virelizier just called them *bimonads* in [3, Section 2.3].

To be more precise, such a bimonad on a monoidal category $(\mathbb{V}, \otimes, \mathbb{I})$ is a monad $\mathbf{T} = (T, m, e)$ on $\mathbb{V}$ endowed with natural transformations $\chi : T\otimes \to T \otimes T$ and a morphism $\theta : T(\mathbb{I}) \to \mathbb{I}$ subject to certain (compatibility) conditions. These allow to define left and right *fusion operators* by

$$H^l_{V,W} : (T(V) \otimes m_W)\chi_{V,T(W)} : T(V \otimes T(W)) \longrightarrow T(V) \otimes T(W),$$
$$H^r_{V,W} : (m_V \otimes T(W))\chi_{T(V),W} : T(T(V) \otimes W) \longrightarrow T(V) \otimes T(W).$$

As a general form of the *Fundamental Theorem* for Hopf algebras it is described in [2, Theorem 4.6] under which conditions the opmonoidal monads induce an equivalence between the base (autonomous monoidal) category and the category of related bimodules.

It was observed in [11] (see also [1]) that the notions around Hopf algebras can be formulated for any category $\mathbb{A}$ without referring to tensor products. For a *bimonad* on $\mathbb{A}$

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one requires simply a monad and a comonad structure whose compatibility is essentially expressed by *distributive laws* (e.g. [11, Definition 4.1]).

As pointed out in [11, Section 2.2], the opmonoidal monads yield special cases of the entwining of a monad with a comonad on any category: Hereby the monad $\mathbf{T}$ is entwined with the comonad $\mathbf{G}_{T(\mathbb{I})} = - \otimes T(\mathbb{I})$. In [12, Theorem 5.11] the above mentioned [2, Theorem 4.6] is formulated in terms of entwining functors.

In [3] an opmonoidal monad (bimonad) is called a *Hopf monad* provided the left and right fusion operators are isomorphisms and it is called a *left (*resp. *right) pre-Hopf monad* if, for any $V \in \mathbb{V}$, the morphisms $H^l_{\mathbb{I},V}$ (resp. $H^r_{V,\mathbb{I}}$) is invertible.

In this paper we show that the right pre-Hopf monads $\mathbf{T}$ are just those for which the related entwining is $\mathbf{G}_{T(\mathbb{I})}$-Galois in the sense of [11, 3.13]. This leads to an improved version of [3, Theorem 6.11] which describes when a pre-Hopf monad on $\mathbb{V}$ induces an equivalence between $\mathbb{V}$ and the category of left Hopf $\mathbf{T}$-modules (see Theorem 4.7).

In Section 1 we recall some basic notions and can use [3, Lemma 2.19] to improve some of our own results on Galois entwinings (see Theorem 2.12).

This is applied in Section 2 to find new properties of a bimonad in the sense of [11] to make it a Hopf monad, provided the base category is Cauchy complete.

In Section 3 opmonoidal monads $\mathbf{T}$ on $(\mathbb{V}, \otimes, \mathbb{I})$ are investigated. In this case $T(\mathbb{I})$ is a comonoid in $\mathbb{V}$ and we have an entwining between $\mathbf{T}$ and $\mathbf{G}_{T(\mathbb{I})} = - \otimes T(\mathbb{I})$. As mentioned above, the main result in this section is Theorem 2.12 which tells us when pre-Hopf monads induce an equivalence between $\mathbb{V}$ and $\mathbb{V}_T^{G_{T(\mathbb{I})}}$. We also observe (in 4.2) that for any $\mathbb{V}$-comonoid $\mathbf{C} = (C, \delta, \varepsilon)$, $T(C)$ also allows for a $\mathbb{V}$-comonoid structure provided $\mathbf{C}$ allows for a group-like morphism $g : \mathbb{I} \to C$. In this case we get functors from $\mathbb{V}$ to $\mathbb{V}_T^{G_{T(\mathbf{C})}}$ and the question arises under which conditions these induce an equivalence. It is shown in Theorem 4.8 that this is only the case if $g : \mathbb{I} \to C$ is an (comonad) isomorphism.

In the final section we consider applications to cartesian monoidal categories and provide examples of pre-Hopf functors for which the related comparison functor is not an equivalence.

2. Preliminaries

For a monad $\mathbf{T} = (T, m, e)$ on a category $\mathbb{A}$, we write $\mathbb{A}_T$ for the Eilenberg-Moore category of $\mathbf{T}$-modules and write

$$\eta_T, \varepsilon_T : \phi_T \dashv U_T : \mathbb{A}_T \to \mathbb{A}$$

for the corresponding forgetful-free adjunction. Dually, if $\mathbf{G} = (G, \delta, \varepsilon)$ is a comonad on $\mathbb{A}$, we denote by $\mathbb{A}^G$ the Eilenberg-Moore category of $\mathbf{G}$-comodules and by

$$\eta^G, \varepsilon^G : U^G \dashv \phi^G : \mathbb{A} \to \mathbb{A}^G$$

the corresponding forgetful-cofree adjunction.

For convenience we recall some notions and results from [12, Section 3].

2.1. Module functors. For a monad $\mathbf{T} = (T, m, e)$ on $\mathbb{A}$, a (*left*) $\mathbf{T}$-*module* consists of a functor $R : \mathbb{B} \to \mathbb{A}$, equipped with a natural transformation $\alpha : TR \to R$ satisfying $\alpha \cdot eR = 1_R$ and $\alpha \cdot mR = \alpha \cdot T\alpha$.

According to [4, Proposition II.1.1], if (R, α) is a $\mathbf{T}$-module, then the assignment

$$b \longmapsto (R(b), \alpha_b)$$

extends uniquely to a functor $\overline{R} : \mathbb{B} \to \mathbb{A}_T$ with $U_T\overline{R} = R$. This gives a bijection, natural in $\mathbf{T}$, between left $\mathbf{T}$-module structures on $R : \mathbb{B} \to \mathbb{A}$ and functors $\overline{R} : \mathbb{B} \to \mathbb{A}_T$ with $U_T\overline{R} = R$.

It is also shown in [4] that, for any $\mathbf{T}$-module $(R : \mathbb{B} \to \mathbb{A}, \alpha)$ admitting a left adjoint functor $F : \mathbb{A} \to \mathbb{B}$, the composite

$$t_{\overline{R}} : \; T \xrightarrow{T\eta} TRF \xrightarrow{\alpha F} RF \;,$$

where $\eta : 1 \to RF$ is the unit of the adjunction $F \dashv R$, is a monad morphism from $\mathbf{T}$ to the monad on $\mathbb{A}$ generated by the adjunction $F \dashv R$.

2.2. Definition. ([1, 2.19]) *A left* $\mathbf{T}$-*module* $R : \mathbb{B} \to \mathbb{A}$ *with a left adjoint* $F : \mathbb{A} \to \mathbb{B}$ *is said to be* $\mathbf{T}$-Galois *if the corresponding morphism* $t_{\overline{R}} : T \to RF$ *of monads on* $\mathbb{A}$ *is an isomorphism.*

Expressing the dual of [10, Theorem 4.4] in the present situation gives:

2.3. Proposition. *The functor* $\overline{R}$ *is an equivalence of categories if and only if the functor* R *is* $\mathbf{T}$-Galois *and monadic.*

2.4. Comodule functors. Given a comonad $\mathbf{G} = (G, \delta, \varepsilon)$ on $\mathbb{A}$, a *left* $\mathbf{G}$-*comodule* is a functor $F : \mathbb{B} \to \mathbb{A}$ equipped with a natural transformation $\beta : F \to GF$ satisfying $\varepsilon F \cdot \beta = 1_F$ and $\delta F \cdot \beta = G\beta \cdot \beta$.

A left $\mathbf{G}$-comodule structure on $F : \mathbb{B} \to \mathbb{A}$ is equivalent to the existence of a functor $\overline{F} : \mathbb{B} \to \mathbb{A}^G$ (dual to [4, Proposition II.1.1]) with $F = U^G\overline{F}$.

If a $\mathbf{G}$-comodule (F, β) admits a right adjoint $R : \mathbb{A} \to \mathbb{B}$, with counit $\sigma : FR \to 1$, then there is a comonad morphism

$$t_{\overline{F}} : FR \xrightarrow{\beta R} GFR \xrightarrow{G\sigma} G$$

from the comonad generated by the adjunction $F \dashv R$ to the comonad $\mathbf{G}$.

2.5. Definition. ([11, Definition 3.5]) *A left* $\mathbf{G}$-*comodule* $F : \mathbb{B} \to \mathbb{A}$ *with a right adjoint* $R : \mathbb{A} \to \mathbb{B}$ *is said to be* $\mathbf{G}$-Galois *if the corresponding morphism* $t_{\overline{F}} : FR \to G$ *of comonads on* $\mathbb{A}$ *is an isomorphism.*

Now [5, Theorem 2.7] (also [10, Theorem 4.4]) can be rephrased as follows:

2.6. Proposition. *The functor* $\overline{F}$ *is an equivalence of categories if and only if the functor* F *is* $\mathbf{G}$-*Galois and comonadic.*

Recall [12, Definition 1.19]:

2.7. Definition. *Let* $\mathbf{T} = (T, m, e)$ *be a monad and* $\mathbf{G} = (G, \delta, \varepsilon)$ *a comonad on* $\mathbb{A}$. *If* $(G, \alpha : TG \to G)$ *is a left* $\mathbf{T}$*-module, then we say that* (G, α) *is* $\mathbf{T}$-Galois*, if the composite*

$$\gamma^G : TG \xrightarrow{T\delta} TGG \xrightarrow{\alpha G} GG$$

is an isomorphism.

Dually, if $(T, \beta : T \to GT)$ *is a left* $\mathbf{G}$*-comodule, then* (T, β) *is* $\mathbf{G}$-Galois*, if the composite*

$$\gamma_T : TT \xrightarrow{\beta T} GTT \xrightarrow{Gm} GT$$

is an isomorphism

2.8. Entwinings. Recall (for example, from [15]) that an *entwining* or *mixed distributive law* from a monad $\mathbf{T} = (T, m, e)$ to a comonad $\mathbf{G} = (G, \delta, \varepsilon)$ on a category $\mathbb{A}$ is a natural transformation $\lambda : TG \to TG$ with certain commutative diagrams (e.g. [14, 5.3]).

It is well-known (see [15]) that the following structures are in bijective correspondence:

- entwinings $\lambda : TG \to GT$;
- comonads $\widehat{G} = (\widehat{G}, \widehat{\delta}, \widehat{\varepsilon})$ on $\mathbb{A}_T$ that extend $\mathbf{G}$ in the sense that
$$U_T\widehat{G} = GU_T,\ U_T\widehat{\delta} = \delta U_T \text{ and } U_T\widehat{\varepsilon} = \varepsilon U_T;$$
- monads $\widehat{T} = (\widehat{T}, \widehat{m}, \widehat{e})$ on $\mathbb{A}^G$ that extend $\mathbf{T}$ in the sense that
$$U^G\widehat{T} = TU^G,\ U^G\widehat{m} = mU^G \text{ and } U^G\widehat{e} = eU^G.$$

For any entwining $\lambda : TG \to GT$, $(a, h_a) \in \mathbb{A}_T$ and $(a, \theta_a) \in \mathbb{A}^G$ (e.g. [14, Section 5]),

$$\widehat{G}(a, h_a) = (G(a), G(h_a) \cdot \lambda_a), \quad \widehat{\delta}_{(a,h_a)} = \delta_a, \quad \widehat{\varepsilon}_{(a,h_a)} = \varepsilon_a,$$

$$\widehat{T}(a, \theta_a) = (T(a), \lambda_a \cdot T(\theta_a)), \quad \widehat{m}_{(a,\theta_a)} = m_a, \quad \widehat{e}_{(a,\theta_a)} = e_a.$$

We write $\mathbb{A}_T^G(\lambda)$ (or just $\mathbb{A}_T^G$ when λ is understood) for the category whose objects are triples (a, h_a, θ_a), where $(a, h_a) \in \mathbb{A}_T$ and $(a, \theta_a) \in \mathbb{A}^G$, with commuting diagram

$$\begin{tikzcd}
T(a) \arrow[r, "h_a"] \arrow[d, "T(\theta_a)"'] & a \arrow[r, "\theta_a"] & G(a) \\
TG(a) \arrow[rr, "\lambda_a"'] & & GT(a). \arrow[u, "G(h_a)"']
\end{tikzcd}$$

The assignments $(a, h_a, \theta_a) \to ((a, h_a), \theta_a))$ and $((a, h_a), \theta_a)) \to ((a, \theta_a), h_a)$ yield isomorphisms of categories

$$\mathbb{A}_T^G(\lambda) \simeq (\mathbb{A}_T)^{\widehat{G}} \simeq (\mathbb{A}^G)_{\widehat{T}}.$$

We fix now an entwining $\lambda : TG \to GT$ and let $K : \mathbb{A} \to (\mathbb{A}^G)_{\widehat{T}}$ be a functor satisfying $\phi^G = U_{\widehat{T}}K$. Writing $\alpha_K : \widehat{T}\phi^G \to \phi^G$ for the corresponding $\widehat{\mathbf{T}}$-module structure on ϕ^G (see 2.1), the natural transformation

$$U^G(\alpha_K) : TG = TU^G\phi^G = U^G\widehat{T}\phi^G \longrightarrow U^G\phi^G = G$$

provides a left $\mathbf{T}$-module structure on G (see [12, Section 2]).

Similarly, if $K : \mathbb{A} \to (\mathbb{A}_T)^{\widehat{G}}$ is a functor with $\phi_T = U^{\widehat{G}}K$, then the natural transformation

$$U_T(\beta_K) : T = U_T\phi_T \longrightarrow GT = GU_T\phi_T = U_T\widehat{G}\phi_T,$$

where $\beta_K : \phi_T \to \widehat{G}\phi_T$ is the corresponding $\widehat{\mathbf{G}}$-comodule structure on ϕ_T (see 2.4), induces a $\mathbf{G}$-comodule structure on T (see again [12, Section 2]).

The following part of [3, Lemma 2.19] is of use for our investigation.

2.9. Lemma. *Let $\tau : FU_T \to F'U_T$ be a natural transformation, where $F, F' : \mathbb{A} \to \mathbb{B}$ are arbitrary functors. If the natural transformation*

$$\tau\phi_T : FT = FU_T\phi_T \longrightarrow F'U_T\phi_T = F'T$$

is an isomorphism, then so is τ.

2.10. Proposition. *Suppose $K : \mathbb{A} \to (\mathbb{A}_T)^{\widehat{G}}$ to be a functor with $U^{\widehat{G}}K = \phi_T$ and denote by $\beta_K : \phi_T \to \widehat{G}\phi_T$ the corresponding $\widehat{\mathbf{G}}$-comodule structure on ϕ_T. Then (ϕ_T, β_K) is $\widehat{\mathbf{G}}$-Galois if and only if $(T, U_T(\beta_K))$ is $\mathbf{G}$-Galois.*

Proof. By 2.5, (ϕ_T, β_K) is $\widehat{\mathbf{G}}$-Galois if the comonad morphism $t_K : \phi_T U_T \to \widehat{G}$, which is the composite

$$\phi_T U_T \xrightarrow{\beta_K U_T} \widehat{G}\phi_T U_T \xrightarrow{\widehat{G}\varepsilon^T} \widehat{G},$$

is an isomorphism, while, by 2.7, $(T, U_T(\beta_K))$ is $\mathbf{G}$-Galois if the composite

$$\gamma_T : TT \xrightarrow{U_T\beta_K T} GTT \xrightarrow{Gm} GT$$

is an isomorphism. So, we have to show that t_K is an isomorphism if and only if γ_T is so.

Since $U_T\widehat{G} = GU_T$, the natural transformation

$$U_T t_K : U_T\phi_T U_T \xrightarrow{U_T\beta_K U_T} U_T\widehat{G}\phi_T U_T \xrightarrow{U_T\widehat{G}\varepsilon^T} U_T\widehat{G}$$

can be rewritten as

$$TU_T \xrightarrow{U_T\beta_K U_T} GU_T\phi_T U_T \xrightarrow{GU_T\varepsilon^T} GU_T.$$

Then $U_T t_K \phi_T$ is the composite

$$TT = TU_T\phi_T \xrightarrow{U_T\beta_K U_T\phi_T} GU_T\phi_T U_T\phi_T \xrightarrow{GU_T\varepsilon^T\phi_T} GU_T\phi_T,$$

and since $U_T\varepsilon^T\phi_T = m : TT = U_T\phi_T U_T\phi_T \to U_T\phi_T = T$, it follows that $U_T t_K \phi_T$ is just γ_T. Now, if t_K is an isomorphism, it is then clear that $\gamma_T = U_T(t_K)\phi_T$ is also an isomorphism. Conversely, if γ_T is an isomorphism, then by Lemma 2.9, $U_T t_K$ is also an isomorphism. But since U_T is conservative, t_K is an isomorphism too. This completes the proof. ∎

Dually, one has

2.11. Proposition. *Suppose that $K : \mathbb{A} \to (\mathbb{A}^G)_{\widehat{T}}$ is a functor with $U_{\widehat{T}} K = \phi^G$ and let $\alpha_K : \widehat{T}\phi^G \to \phi^G$ be the corresponding $\widehat{\mathbf{T}}$-module structure on ϕ^G. Then (ϕ^G, α_K) is $\widehat{\mathbf{T}}$-Galois if and only if $(G, U^G(\alpha_K))$ is $\mathbf{T}$-Galois.*

In view of Propositions 2.10 and 2.11, we get from Propositions 2.3 and 2.6:

2.12. Theorem. *In the situation of Proposition 2.10, the functor $K : \mathbb{A} \to (\mathbb{A}_T)^{\widehat{G}}$ is an equivalence of categories if and only if $(T, U_T(\beta_K))$ is $\mathbf{G}$-Galois and the monad $\mathbf{T}$ is of effective descent type (i.e. the functor $\phi_T : \mathbb{A} \to \mathbb{A}_T$ is comonadic.)*

Dually, in the situation of Proposition 2.11, the functor $K : \mathbb{A} \to (\mathbb{A}^G)_{\widehat{T}}$ is an equivalence if and only if $(G, U^G(\alpha_K))$ is $\mathbf{T}$-Galois and the comonad $\mathbf{G}$ is of effective codescent type (i.e. the functor $\phi^G : \mathbb{A} \to \mathbb{A}^G$ is monadic).

2.13. Galois entwinings. Let $\mathbf{T} = (T, m, e)$ be a monad and $\mathbf{G} = (G, \delta, \varepsilon)$ a comonad on a category $\mathbb{A}$ with an entwining $\lambda : TG \to GT$. If $\mathbf{G}$ has a group-like morphism $g : 1 \to G$ (in the sense of [12, Definition 3.1]), then T has two left $\mathbf{G}$-comodule structures given by

$$gT : T \to GT \quad \text{and} \qquad \tilde{g} : T \xrightarrow{Tg} TG \xrightarrow{\lambda} GT\ ,$$

and it was shown in [12] that the equaliser (T^g, i) of these natural transformations admits the structure of a monad in such a way that $i : T^g \to T$ becomes a monad morphism. We write $i^* : \mathbb{A}_T \to \mathbb{A}_{T^g}$ for the functor that takes an arbitrary $\mathbf{T}$-algebra $(a, h_a) \in \mathbb{A}_T$ to the $\mathbf{T}^g$-algebra $(a, h_a \cdot i_a) \in \mathbb{A}_{T^g}$. When the category $\mathbb{A}_T$ admits coequalisers of reflexive pairs (which is certainly the case if $\mathbb{A}$ has coequalisers of reflexive pairs and the functor T preserves them), i^* has a left adjoint $i_* : \mathbb{A}_{T^g} \to \mathbb{A}_T$. In this case, according to the results of [12], there is a comparison functor $\bar{i} : \mathbb{A}_{T^g} \to (\mathbb{A}_T)^{\widehat{G}}$ yielding commutativity of the diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "\phi_{T^g}"] \arrow[rr, bend left, "K_{g,\mathbf{G}}"] \arrow[dr, "\phi_T"'] & \mathbb{A}_{T^g} \arrow[r, "\bar{i}"] \arrow[d, "i_*"] & (\mathbb{A}_T)^{\widehat{G}} \arrow[dl, "U^{\widehat{G}}"] \\
 & \mathbb{A}_T &
\end{tikzcd} \tag{1}$$

where $U^{\widehat{G}} : (\mathbb{A}_T)^{\widehat{G}} \to \mathbb{A}_T$ is the evident forgetful functor and $K_{g,\mathbf{G}} : \mathbb{A} \to (\mathbb{A}_T)^{\widehat{G}}$ is the functor that takes $a \in \mathbb{A}$ to $((T(a), m_a), \widetilde{g_a}) \in (\mathbb{A}_T)^{\widehat{G}}$ (see [12, Section 3]).

Let us write $\widetilde{G}$ for the comonad generated by the adjunction $i^* \dashv i_*$ and write

- $S_{K_{g,\mathbf{G}}} : U_T\phi_T \to \widehat{G}$ for the comonad morphism corresponding to the outer diagram in (1),
- $S_{\phi_{T^g}} : U_T\phi_T \to \widetilde{G}$ for the comonad morphism corresponding to the left triangle in (1),

- and $S_{\bar{i}} : \widetilde{G} \to \widehat{G}$ for the comonad morphism corresponding to the right triangle in (1) that exists according to [12, Proposition 1.20].

2.14. Definition. [12] *Under the circumstances above, we call* $(\mathbf{T}, \mathbf{G}, \lambda, g)$ *a* Galois entwining *if the comonad morphism* $S_{\bar{i}} : \widetilde{G} \to \widehat{G}$ *is an isomorphism, or, equivalently, the functor* i_* *is* $\widehat{G}$*-Galois. In this case* $g : 1 \to G$ *is said to be a* Galois group-like morphism.

2.15. Theorem. [12] *Let* $\lambda : TG \to GT$ *be an entwining from a monad* $\mathbf{T}$ *to a comonad* $\mathbf{G}$ *on a category* $\mathbb{A}$*. Suppose that* $g : 1 \to G$ *is a group-like morphism such that the corresponding functor* $i^* : \mathbb{A}_T \to \mathbb{A}_{T^g}$ *admits a left adjoint functor* $i_* : \mathbb{A}_{T^g} \to \mathbb{A}_T$*. Then the comparison functor* $\bar{i} : \mathbb{A}_{T^g} \to (\mathbb{A}_T)^{\widehat{G}}$ *is an equivalence of categories if and only if* $(\mathbf{T}, \mathbf{G}, \lambda, g)$ *is a Galois entwining and the functor* i_* *is comonadic.*

3. Bimonads

The preceding results allow to formulate new conditions which turn bimonads into Hopf monads. Recall from [11, Definition 4.1] that a bimonad $\mathbf{H}$ on any category $\mathbb{A}$ is an endofunctor $H : \mathbb{A} \to \mathbb{A}$ with a monad structure $\underline{H} = (H, m, e)$, a comonad structure $\overline{H} = (H, \delta, \varepsilon)$, and an entwining $\lambda : HH \to HH$ from the monad $\underline{H}$ to the comonad $\overline{H}$ inducing commutativity of the diagrams

$$\begin{tikzcd} HH \arrow[r, shift left, "\varepsilon H"] \arrow[r, shift right, "H\varepsilon"'] \arrow[d, "m"'] & H \arrow[d, "\varepsilon"] \\ H \arrow[r, "\varepsilon"] & 1, \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "e"] \arrow[d, "e"'] & H \arrow[d, "\delta"] \\ H \arrow[r, shift left, "eH"] \arrow[r, shift right, "He"'] & HH, \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "e"] \arrow[dr, "="'] & H \arrow[d, "\varepsilon"] \\ & 1, \end{tikzcd}$$

$$\begin{tikzcd} HH \arrow[r, "m"] \arrow[d, "H\delta"'] & H \arrow[r, "\delta"] & HH \\ HHH \arrow[rr, "\lambda H"'] & & HHH. \arrow[u, "Hm"'] \end{tikzcd}$$

Given a bimonad $\mathbf{H}$, one has the comparison functor

$$K_H : \mathbb{A} \to \mathbb{A}_H^H = \mathbb{A}_{\underline{H}}^{\overline{H}}(\lambda), \quad a \mapsto (H(a), m_a, \delta_a)$$

with commutative diagrams

$$\begin{tikzcd} \mathbb{A} \arrow[r, "K_H"] \arrow[dr, "\phi_{\underline{H}}"'] & \mathbb{A}_H^H \simeq (\mathbb{A}_{\underline{H}})^{\widehat{\overline{H}}} \arrow[d, "U^{\widehat{\overline{H}}}"] \\ & \mathbb{A}_{\underline{H}}, \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} \arrow[r, "K_H"] \arrow[dr, "\phi^{\overline{H}}"'] & \mathbb{A}_H^H \simeq (\mathbb{A}^{\overline{H}})_{\widehat{\underline{H}}} \arrow[d, "U_{\widehat{\underline{H}}}"] \\ & \mathbb{A}^{\overline{H}}. \end{tikzcd}$$

Writing $K_{\underline{H}}$ (resp. $K_{\overline{H}}$) for the composite $\mathbb{A} \xrightarrow{K_H} \mathbb{A}_H^H \simeq (\mathbb{A}_{\underline{H}})^{\widehat{\overline{H}}}$ (resp. $\mathbb{A} \xrightarrow{K_H} \mathbb{A}_H^H \simeq (\mathbb{A}^{\overline{H}})_{\widehat{\underline{H}}}$) and writing $\alpha_{K_{\underline{H}}}$ (resp. $\alpha_{K_{\overline{H}}}$) for the $\widehat{\overline{H}}$-comodule (resp. $\widehat{\underline{H}}$-module) structure on $\phi_{\underline{H}}$ (resp. $\phi^{\overline{H}}$) that exists by 2.4 (resp. 2.1), we know from [11, 4.3] that $U_{\underline{H}}(\alpha_{K_{\underline{H}}}) = \delta : H \to HH$ and that $U^{\overline{H}}(\alpha_{K_{\overline{H}}}) = m : HH \to H$. It then follows from 2.7 that $\gamma_{\underline{H}} : \underline{H}\,\underline{H} \to \overline{H}\underline{H}$ is the composite

$$HH \xrightarrow{\delta H} HHH \xrightarrow{Hm} HH,$$

while $\gamma^{\overline{H}} : \underline{H}\overline{H} \to \overline{H}\,\overline{H}$ is the composite

$$HH \xrightarrow{H\delta} HHH \xrightarrow{mH} HH.$$

Employing the notions considered above we have the following list of

3.1. Characterisations of Hopf monads. *For a bimonad* $\mathbf{H}$ *on a Cauchy complete category* $\mathbb{A}$*, the following are equivalent*:

(a) $(\phi_{\underline{H}}, \alpha_{K_{\underline{H}}})$ *is* $\widehat{\overline{H}}$*-Galois, i.e., the composite* $t_{K_{\underline{H}}} : \phi_{\underline{H}} U_{\underline{H}} \xrightarrow{\alpha_{K_{\underline{H}}} U_{\underline{H}}} \widehat{\overline{H}} \phi_{\underline{H}} U_{\underline{H}} \xrightarrow{\widehat{\overline{H}} \varepsilon_{\underline{H}}} \widehat{\overline{H}}$ *is an isomorphism;*

(b) $(\phi^{\overline{H}}, \alpha_{K_{\overline{H}}})$ *is* $\widehat{\underline{H}}$*-Galois, i.e., the composite* $t_{K_{\overline{H}}} : \widehat{\underline{H}} \xrightarrow{\widehat{\underline{H}} \eta^{\overline{H}}} \widehat{\underline{H}} \phi^{\overline{H}} U^{\overline{H}} \xrightarrow{\alpha_{K_{\overline{H}}} U^{\overline{H}}} \phi^{\overline{H}} U^{\overline{H}}$ *is an isomorphism;*

(c) *the unit* $e : 1 \to H$ *is a Galois group-like morphism;*

(d) *the functor* $K_H : \mathbb{A} \to \mathbb{A}_H^H$ *(hence also* $K_{\underline{H}} : \mathbb{A} \to (\mathbb{A}_{\underline{H}})^{\widehat{\overline{H}}}$ *and* $K_{\overline{H}} : \mathbb{A} \to (\mathbb{A}^{\overline{H}})_{\widehat{\underline{H}}}$*) is an equivalence of categories;*

(e) (H, m) *is* $\overline{H}$*-Galois, i.e.,* $\gamma_{\underline{H}} : HH \xrightarrow{\delta H} HHH \xrightarrow{Hm} HH$ *is an isomorphism;*

(f) (H, δ) *is* $\underline{H}$*-Galois, i.e.,* $\gamma^{\overline{H}} : HH \xrightarrow{H\delta} HHH \xrightarrow{mH} HH$ *is an isomorphism;*

(g) $\mathbf{H}$ *has an antipode, i.e., there exists a natural transformation* $S : H \to H$ *with*

$$m \cdot HS \cdot \delta = e \cdot \varepsilon = m \cdot SH \cdot \delta.$$

Proof. (a), (c) and (d) are equivalent by [12, 4.2], while (e), (f) and (g) are equivalent by [11, 5.5]. Moreover, (a)$\Leftrightarrow$(e) follows by Proposition 2.10 and (b)$\Leftrightarrow$(f) by Proposition 2.11. ∎

3.2. Example. Let $(\mathbb{V}, \tau)$ be a *lax braided* monoidal category (see, for example, [3]) and $\mathbf{A} = (A, m, e, \delta, \varepsilon)$ a bialgebra in $\mathbb{V}$. We write H for the endofunctor $A \otimes - : \mathbb{V} \to \mathbb{V}$. It is easy to verify directly, using the axioms of lax braidings, that the natural transformation $\overline{\tau} = \tau_A \otimes - : HH \to HH$ is a *local prebraiding* (in the sense of [11]) and that

$$(H, \overline{m}, \overline{e}, \overline{\delta}, \overline{\varepsilon}),$$

where $\overline{m} = m \otimes -$, $\overline{e} = e \otimes -$, $\overline{\delta} = \delta \otimes -$ and $\overline{\varepsilon} = \varepsilon \otimes -$, is a τ-bimonad on $\mathbb{V}$. Then, according to [11, Section 6], the composite $\widetilde{\tau} = \overline{m}H \cdot H\overline{\tau} \cdot \overline{\delta}H$ is an entwining from the monad $(H, \overline{m}, \overline{e})$ to the comonad $(H, \overline{\delta}, \overline{\varepsilon})$ that makes $(H, \overline{m}, \overline{e}, \overline{\delta}, \overline{\varepsilon})$ a bimonad on $\mathbb{V}$. Writing $\mathbb{V}^{\mathbf{A}}_{\mathbf{A}}$ for the category $\mathbb{V}^{\overline{H}}_{\underline{H}}(\widetilde{\tau})$, we get from Theorem 3.1 the following generalisation of [11, Theorem 6.12]:

3.3. Proposition. *Let $(\mathbb{V}, \tau)$ be a lax braided category such that $\mathbb{V}$ is Cauchy complete. If $\mathbf{A}$ is a bialgebra in $\mathbb{V}$, then the comparison functor*

$$K : \mathbb{V} \to \mathbb{V}^{\mathbf{A}}_{\mathbf{A}}, \quad V \mapsto (A \otimes V, m \otimes V, \delta \otimes V),$$

is an equivalence of categories if and only if $\mathbf{A}$ is a Hopf algebra, that is, $\mathbf{A}$ has an antipode.

4. Opmonoidal Monads

4.1. Pre-Hopf monads. Recall (for example, from [8]) that an *opmonoidal functor* from a monoidal category $(\mathbb{V}, \otimes, \mathbb{I})$ to a monoidal category $(\mathbb{V}', \otimes', \mathbb{I}')$ is a triple (S, χ, θ), where $S : \mathbb{V} \to \mathbb{V}'$ is a functor, $\chi : S\otimes \to S \otimes' S$ is a natural transformation, and $\theta : S(\mathbb{I}) \to \mathbb{I}'$ is a morphism that are compatible with the tensor structures. Note that opmonoidal functors S take $\mathbb{V}$-comonoids (i.e. comonoids in $\mathbb{V}$) into $\mathbb{V}'$-comonoids in the sense that if $\mathbf{C} = (C, \delta, \varepsilon)$ is a $\mathbb{V}$-comonoid, then the triple $S(\mathbf{C}) = (S(C), \chi_{C,C} \cdot S(\delta), \theta \cdot S(\varepsilon))$ is a $\mathbb{V}'$-comonoid.

Recall also (again from [8]) that an *opmonoidal monad* on a monoidal category $(\mathbb{V}, \otimes, \mathbb{I})$ is a monad $\mathbf{T} = (T, m, e)$ on the category $\mathbb{V}$ whose functor part T is an opmonoidal endofunctor together with natural transformations

$$\chi_{V,W} : T(V \otimes W) \to T(V) \otimes T(W) \text{ for } V, W \in \mathbb{V}$$

and a morphism $\theta : T(\mathbb{I}) \to \mathbb{I}$ that are compatible with the monad structure.

For example, it was pointed out in [2] that any bialgebra $\mathbf{A} = (A, \mu, \eta, \delta, \varepsilon)$ in a braided monoidal category $(\mathbb{V}, \otimes, \mathbb{I})$ with braiding $\tau_{V,W} : V \otimes W \to W \otimes V$ gives rise to an opmonoidal $\mathbb{V}$-monad $A \otimes -$, where the natural transformation $\chi_{V,W} : A \otimes V \otimes W \to A \otimes V \otimes A \otimes W$ is the composite

$$A \otimes V \otimes W \xrightarrow{\delta \otimes V \otimes W} A \otimes A \otimes V \otimes W \xrightarrow{A \otimes \tau_{A,V} \otimes W} A \otimes V \otimes A \otimes W\,,$$

while $\theta : A \to \mathbb{I}$ is just ε.

From now on we shall assume (actually without loss of generality by the coherence theorem in [7]) that all our monoidal categories are strict.

According to [3], an opmonoidal monad $\mathbf{T} = (T, m, e)$ on the monoidal category $(\mathbb{V}, \otimes, \mathbb{I})$ is *left pre-Hopf* if, for any object V of $\mathbb{V}$, the composite

$$H^l_{\mathbb{I},V} : TT(V) = T(\mathbb{I} \otimes T(V)) \xrightarrow{\chi_{\mathbb{I},T(V)}} T(\mathbb{I}) \otimes TT(V) \xrightarrow{T(\mathbb{I})\otimes m_V} T(\mathbb{I}) \otimes T(V)$$

is an isomorphism, and $\mathbf{T}$ is *right pre-Hopf* provided

$$H^r_{V,\mathbb{I}} : TT(V) = T(T(V) \otimes \mathbb{I}) \xrightarrow{\chi_{T(V),\mathbb{I}}} TT(V) \otimes T(\mathbb{I}) \xrightarrow{m_V\otimes T(\mathbb{I})} T(V) \otimes T(\mathbb{I})$$

is an isomorphism. $\mathbf{T}$ is called a *pre-Hopf monad* if it is both left and right pre-Hopf.

For for any $(V, h_V) \in \mathbb{V}_T$ and $W \in \mathbb{V}$, consider the morphisms

$$\mathbb{H}^r_{V,W} : T(V \otimes W) \xrightarrow{\chi_{V,W}} T(V) \otimes T(W) \xrightarrow{h_V\otimes T(W)} V \otimes T(W),$$

and for any $V \in \mathbb{V}$ and $(W, h_W) \in \mathbb{V}_T$, define

$$\mathbb{H}^l_{V,W} : T(V \otimes W) \xrightarrow{\chi_{V,W}} T(V) \otimes T(W) \xrightarrow{T(V)\otimes h_W} T(V) \otimes W.$$

It is shown in [3] that, for any $V \in \mathbb{V}$, $H^r_{-,V}$ (resp. $H^l_{V,-}$) is an isomorphism if and only if $\mathbb{H}^r_{-,V}$ (resp. $\mathbb{H}^l_{V,-}$) is so. In particular, $\mathbf{T}$ is right (resp. left) pre-Hopf monad if and only if for any $(V, h_V) \in \mathbb{V}_T$, the morphism $\mathbb{H}^r_{V,\mathbb{I}}$ (resp. $\mathbb{H}^l_{\mathbb{I},V}$) is an isomorphism.

4.2. Entwined modules. Let $(\mathbb{V}, \otimes, \mathbb{I})$ be a monoidal category and let $\mathbf{T} = (T, m, e)$ be an opmonoidal monad on $\mathbb{V}$. As the functor T is opmonoidal, for any $\mathbb{V}$-comonoid $\mathbf{C} = (C, \delta, \varepsilon)$, the triple $T(\mathbf{C}) = (T(C), \chi_{C,C} \cdot T(\delta), \theta_{\mathbb{I}} \cdot T(\varepsilon))$ is also a $\mathbb{V}$-comonoid. In particular, the triple $T(\mathbb{I}) = (T(\mathbb{I}),\ \chi_{\mathbb{I},\mathbb{I}},\ \theta)$ is a $\mathbb{V}$-comonoid corresponding to the trivial $\mathbb{V}$-comonoid $\mathbf{I} = (\mathbb{I}, 1_{\mathbb{I}}, 1_{\mathbb{I}})$. Given a $\mathbb{V}$-comonoid $\mathbf{C}$, we write $\mathbf{G_C}$ for the comonad on $\mathbb{V}$ whose functor part is $G_{\mathbf{C}} = - \otimes C$.

The compatibility axioms for $\mathbf{T}$ ensure that the natural transformation

$$\lambda^{\mathbf{C}}_- := H^l_{-,C} = (T(-) \otimes m_C) \cdot \chi_{-,T(C)} : T(- \otimes T(C)) \to T(-) \otimes T(C)$$

is a mixed distributive law (entwining) from the monad $\mathbf{T}$ to the comonad $\mathbf{G}_{T(\mathbf{C})}$ and the diagrams in 2.8 come out as

$$\begin{tikzcd} & V\otimes T(C) \arrow[dl, "e_{V\otimes T(C)}"'] \arrow[d, "e_V\otimes T(C)"] \\ T(V\otimes T(C)) \arrow[r, "\lambda^{\mathbf{C}}_{V,T(\mathbf{C})}"'] & T(V)\otimes T(C), \end{tikzcd} \qquad \begin{tikzcd} T(V\otimes T(C)) \arrow[r, "T(V\otimes T(\varepsilon))"] \arrow[d, "\lambda^{\mathbf{C}}_{V,T(\mathbf{C})}"'] & T(V\otimes T(\mathbb{I})) \arrow[dr, "T(V\otimes\theta)"] & \\ T(V)\otimes T(C) \arrow[r, "T(V)\otimes T(\varepsilon)"'] & T(V)\otimes T(\mathbb{I}) \arrow[r, "T(V)\otimes\theta"'] & T(V), \end{tikzcd}$$

$$\begin{tikzcd}
T(V\otimes T(C)) \arrow[r, "T(V\otimes T(\delta))"] \arrow[dd, "\lambda^{\mathbf{C}}_{V,T(\mathbf{C})}"'] & T(V\otimes T(C\otimes C)) \arrow[r, "T(V\otimes \chi_{C,C})"] & T(V\otimes T(C)\otimes T(C)) \arrow[d, "\lambda^{\mathbf{C}}_{V\otimes T(\mathbf{C}),T(C)}"] \\
 & & T(V\otimes T(C))\otimes T(C) \arrow[d, "\lambda^{\mathbf{C}}_{V,T(\mathbf{C})}\otimes T(C)"] \\
T(V)\otimes T(C) \arrow[r, "T(V)\otimes T(\delta)"] & T(V)\otimes T(C\otimes C) \arrow[r, "T(V)\otimes \chi_{C,C}"] & T(V)\otimes T(C)\otimes T(C),
\end{tikzcd}$$

$$\begin{tikzcd}
T(T(V\otimes T(C))) \arrow[r, "T(\lambda^{\mathbf{C}}_{V,T(\mathbf{C})})"] \arrow[d, "m_{V\otimes T(\mathbf{C})}"'] & T(T(V)\otimes T(C)) \arrow[r, "\lambda^{\mathbf{C}}_{T(V),T(\mathbf{C})}"] & TT(V)\otimes T(C) \arrow[d, "m_V\otimes T(C)"] \\
T(V\otimes T(C)) \arrow[rr, "\lambda^{\mathbf{C}}_{V,T(\mathbf{C})}"] & & T(V)\otimes T(C).
\end{tikzcd}$$

The *entwined* $T(\mathbf{C})$*-modules* are objects $V \in \mathbb{V}$ with a T-module structure $h : T(V) \to V$ and a $T(\mathbf{C})$-comodule structure $\rho : V \to V \otimes T(C)$ inducing commutativity of the diagram

$$\begin{tikzcd}
T(V) \arrow[r, "h"] \arrow[d, "T(\rho)"'] & V \arrow[r, "\rho"] & V\otimes T(C) \\
T(V\otimes T(C)) \arrow[r, "\chi_{V,T(C)}"'] & T(V)\otimes TT(C) \arrow[r, "T(V)\otimes m_C"'] & T(V)\otimes T(C). \arrow[u, "h\otimes T(C)"']
\end{tikzcd}$$

They form a category in an obvious way which we denote by $\mathbb{V}_T^{T(\mathbf{C})}$. It is clear that $\mathbb{V}_T^{T(\mathbf{C})}$ is just the category $\mathbb{V}_T^{G_{T(\mathbf{C})}}(\lambda_{\mathbf{C}}) = (\mathbb{V}_T)^{\widehat{G_{T(\mathbf{C})}}}$.

When $\mathbf{C} = \mathbf{I}$ is the trivial $\mathbb{V}$-comonad, the entwined $T(\mathbf{I})$-modules are named *right Hopf T-modules* in [2, Section 4.2] (also [3, 6.5]).

There is another description of the category $\mathbb{V}_T^{T(\mathbf{C})}$. Since $\mathbf{T}$ is opmonoidal, $\mathbb{V}_T$ is a monoidal category, and the functor $\phi_T : \mathbb{V} \to \mathbb{V}_T$ is also opmonoidal. Then, for any $\mathbb{V}$-comonoid $\mathbf{C}$, the triple

$$\phi_T(\mathbf{C}) = ((T(C), m_C),\ \chi_{C,C}\cdot T(\delta),\ \theta_{\mathbb{I}}\cdot T(\varepsilon)))$$

is a $\mathbb{V}_T$-comonoid and it is easy to see that the comonad $\widehat{G_{T(\mathbf{C})}}$ is just the comonad $\mathbf{G}_{\phi_T(\mathbf{C})}$ and that the category $\mathbb{V}_T^{T(\mathbf{C})}$ is just the category $(\mathbb{V}_T)^{\phi_T(\mathbf{C})}$. In particular, if $\phi_T(\mathbf{I}) = ((T(\mathbb{I}), m_{\mathbb{I}})\ \chi_{\mathbb{I},\mathbb{I}},\ \theta)$ is a $\mathbb{V}_T$-comonoid corresponding to the trivial $\mathbb{V}$-comonoid $\mathbf{I} = (\mathbb{I}, 1_{\mathbb{I}}, 1_{\mathbb{I}})$, then $\widehat{G_{T(\mathbf{I})}} = \mathbf{G}_{\phi_T(\mathbf{I})}$ and $\mathbb{V}_T^{T(\mathbf{I})} = (\mathbb{V}_T)^{\phi_T(\mathbf{I})}$.

4.3. Remark. It follows from the results of [12, 5.13] that, for an arbitrary bialgebra $\mathbf{A} = (A, \mu, \eta, \delta, \varepsilon)$ in a braided monoidal category $(\mathbb{V}, \otimes, \mathbb{I})$, the following are equivalent:

(i) *the natural transformation* $\lambda^{\boldsymbol{I}}_{-} = H^{l}_{-,\mathbb{I}} : A \otimes - \otimes A \to A \otimes - \otimes A$, *corresponding to the opmonoidal* $\mathbb{V}$*-monad* $A \otimes -$, *is an isomorphism;*

(ii) *the morphism* $\lambda^{\boldsymbol{I}}_{\mathbb{I}} = H^{l}_{\mathbb{I},\mathbb{I}} : A \otimes A \to A \otimes A$ *is an isomorphism;*

(iii) *the composite* $A \otimes A \xrightarrow{\delta \otimes A} A \otimes A \otimes A \xrightarrow{A \otimes m} A \otimes A$ *is an isomorphism.*

Recall (for example from [10]) that condition (iii) is in turn equivalent to saying that $\mathbf{A}$ has an antipode, i.e. $\mathbf{A}$ is a Hopf algebra. It follows from the equivalence (i)⇔(iii) that, for any $V \in \mathbb{V}$, the natural transformation $H^{l}_{-,V}$, which is easily seen to be just the natural transformation $H^{l}_{-,\mathbb{I}} \otimes V$, is an isomorphism, or equivalently, the monad $A \otimes -$ is left Hopf, if and only if $\mathbf{A}$ is a Hopf algebra. Moreover, if the monad $A \otimes -$ is left pre-Hopf (and hence, in particular, the morphism $H^{l}_{\mathbb{I},\mathbb{I}}$ is an isomorphism), then according to the equivalence (ii)⇔(iii), $\mathbf{A}$ is a Hopf algebra. Putting this information together and using that, quite obviously, any left Hopf monad is left pre-Hopf, we have proved that the following are equivalent:

(i) $\mathbf{A}$ *is a Hopf algebra;*

(ii) *the monad* $A \otimes -$ *is left pre-Hopf;*

(iii) *the monad* $A \otimes -$ *is left Hopf.*

This result may be compared with [3, Proposition 5.4(a)].

4.4. Group-like morphisms. Suppose now that the $\mathbb{V}$-comonoid $\mathbf{C}$ allows for a group-like element $g : \mathbb{I} \to C$ (see [10], [11]). Then direct inspection shows that $\overline{g} : \mathbb{I} \xrightarrow{g} C \xrightarrow{e_C} T(C)$ is a group-like element for the $\mathbb{V}$-comonoid $T(\mathbf{C})$ implying that the natural transformation $- \otimes \overline{g} : 1 \to - \otimes T(C)$ is a group-like morphism. Thus the results of [10] apply. In particular, the composite

$$T(-) \xrightarrow{T(-\otimes \overline{g})} T(- \otimes T(C)) \xrightarrow{\lambda^{\mathbf{C}}_{-}} T(-) \otimes T(C)$$

gives the structure $\vartheta : \phi_T \to \phi_T \widehat{G_{T(\mathbf{C})}}$ of a $\widehat{G_{T(\mathbf{C})}}$-comodule on the functor $\phi_T : \mathbb{V} \to \mathbb{V}_T$. Since in the diagram

$$\begin{tikzcd}
T(-) \arrow[r, "T(-\otimes g)"] \arrow[d, "\chi_{-,\mathbb{I}}"'] & T(- \otimes C) \arrow[r, "T(-\otimes e_C)"] \arrow[d, "\chi_{-,C}"] & T(- \otimes T(C)) \arrow[r, "\chi_{-,T(C)}"] & T(-) \otimes T^2(C) \arrow[d, "T(-)\otimes m_C"] \\
T(-) \otimes T(\mathbb{I}) \arrow[r, "T(-)\otimes T(g)"'] & T(-) \otimes T(C) \arrow[urr, "T(-)\otimes T(e_C)"] \arrow[rr, equal] & & T(-) \otimes T(C)
\end{tikzcd}$$

the rectangle and the top triangle commute by naturality of χ, while the bottom triangle commutes since e is the unit for the multiplication m, it follows that ϑ is just the natural transformation

$$T(-) \xrightarrow{\chi_{-,\mathbb{I}}} T(-) \otimes T(\mathbb{I}) \xrightarrow{T(-)\otimes T(g)} T(-) \otimes T(C).$$

It then follows that the assignment $V \longrightarrow ((T(V), m_V), (T(V) \otimes T(g)) \cdot \chi_{V,\mathbb{I}})$ yields a functor

$$K_{g,\mathbf{C}} := K_{g,G_{T(\mathbf{C})}} : \mathbb{V} \to \mathbb{V}_T^{T(\mathbf{C})} = (\mathbb{V}_T)^{\widehat{G_{T(\mathbf{C})}}}$$

with $\phi_T = U^{\widehat{G_{T(\mathbf{C})}}} K_{g,\mathbf{C}}$.

One calculates that for any $(V, h_V) \in V_T$, the (V, h_V)-component of the induced comonad morphism $S_{K_{g,\mathbf{C}}} : \phi_T U_T \to \widehat{G_{T(\mathbf{C})}}$ is the composite

$$T(V) \xrightarrow{\chi_{V,\mathbb{I}}} T(V) \otimes T(\mathbb{I}) \xrightarrow{T(V)\otimes T(g)} T(V) \otimes T(C) \xrightarrow{h_V \otimes T(C)} V \otimes T(C).$$

In particular, when $\mathbf{C}$ is the trivial $\mathbb{V}$-comonoid $\mathbf{I}$ together with the evident group-like morphism $1_{\mathbb{I}} : \mathbb{I} \to \mathbb{I}$, the morphism $\chi_{-,\mathbb{I}} : T(-) \to T(-) \otimes T(\mathbb{I})$ gives the structure $\vartheta' : \phi_T \to \phi_T \widehat{G_{T(\mathbf{I})}}$ of a $\widehat{G_{T(\mathbf{I})}}$-comodule on the functor $\phi_T : \mathbb{V} \to \mathbb{V}_T$, and then one has $\phi_T = U^{\widehat{G_{T(\mathbf{I})}}} K_{1_{\mathbb{I}},\mathbf{I}}$ with the comparison functor $K_{1_{\mathbb{I}},\mathbf{I}}(V) = ((T(V), m_V), \chi_{V,I})$. Moreover, for any $(V, h_V) \in V_T$, the (V, h_V)-component of the induced comonad morphism $S_{K_{1_{\mathbb{I}},\mathbf{I}}} : \phi_T U_T \to \widehat{G_{T(\mathbf{I})}}$ is the composite

$$T(V) \xrightarrow{\chi_{V,\mathbb{I}}} T(V) \otimes T(\mathbb{I}) \xrightarrow{h_V \otimes T(\mathbb{I})} V \otimes T(\mathbb{I}).$$

Comparing now ϑ and ϑ' gives

$$\vartheta = (T(-) \otimes T(g)) \cdot \vartheta' \tag{2}$$

while comparing $S_{K_{g,C}}$ and $S_{K_{e_{\mathbb{I}},\mathbb{I}}}$ and using that

$$(h_V \otimes T(C)) \cdot (T(V) \otimes T(g)) = (V \otimes T(g)) \cdot (h_V \otimes T(\mathbb{I}))$$

by bifunctoriality of the tensor product, gives

$$S_{K_{g,C}} = (- \otimes T(g)) \cdot S_{K_{e_{\mathbb{I}},\mathbb{I}}}. \tag{3}$$

It is easy to see that $S_{K_{e_{\mathbb{I}},\mathbb{I}}}$ just the composite $\mathbb{H}^r_{V,\mathbb{I}}$. This yields in particular a fact proved in [3, Lemma 6.5]:

4.5. Lemma. $\mathbb{H}^r_{-,\mathbb{I}} : \mathrm{T}(-) \to - \otimes \mathrm{T}(\mathbb{I})$ *is a morphism of comonads* $\phi_{\mathrm{T}} \mathrm{U}_{\mathrm{T}} \to \widehat{\mathrm{G}_{\mathrm{T}(\mathbf{I})}}$.

We already know (see 4.1) that $\mathbf{T}$ is a right pre-Hopf monad iff the natural transformation $\mathbb{H}^r_{-,\mathbb{I}}$ (or, equivalently, the comonad morphism $S_{K_{e_{\mathbb{I}},\mathbb{I}}}$) is an isomorphism. It now follows from Proposition 2.10:

4.6. Proposition. *An opmonoidal monad* $\mathbf{T}$ *on* $\mathbb{V}$ *is a right pre-Hopf monad if and only if* $\mathbf{T}$ *is* $\mathbf{G}_{\mathrm{T}(\mathbf{I})}$*-Galois.*

This allows us to present an improved version of [3, Theorem 6.11].

4.7. THEOREM. *For an opmonoidal monad* $\mathbf{T}$ *on a monoidal category* $(\mathbb{V}, \otimes, \mathbb{I})$, *the following are equivalent:*

(a) the functor $K_{1_{\mathbb{I}},\mathbf{I}} : \mathbb{V} \to \mathbb{V}_{\mathrm{T}}^{\mathrm{T}(\mathbf{I})}$ *is an equivalence of categories;*

(b) (i) $\mathbf{T}$ *is* $G_{\mathrm{T}(\mathbf{I})}$*-Galois,*

(ii) $\mathbf{T}$ *is of effective descent type.*

PROOF. The assertion follows by Proposition 2.6. ∎

4.8. THEOREM. *Let* $\mathbf{T} = (T, m, e)$ *be an opmonoidal monad on a monoidal category* $(\mathbb{V}, \otimes, \mathbb{I})$ *and* $\mathbf{C} = (C, \delta, \varepsilon)$ *a* $\mathbb{V}$*-comonoid with a group-like element* $g : \mathbb{I} \to C$. *The following are equivalent:*

(a) the functor $K_{g,\mathbf{C}} : \mathbb{V} \to (\mathbb{V}_{\mathrm{T}})^{\phi_{\mathrm{T}}(\mathbf{C})}$ *is an equivalence of categories;*

(b) $K_{1_{\mathbb{I}},\mathbf{I}} : \mathbb{V} \to (\mathbb{V}_{\mathrm{T}})^{\phi_{\mathrm{T}}(\mathbf{I})}$ *is an equivalence of categories and* g *is an isomorphism;*

(c) (i) $\mathbf{T}$ *is* $\mathrm{G}_{\mathrm{T}(\mathbf{I})}$*-Galois,*

(ii) $\mathbf{T}$ *is of effective descent type,*

(iii) g *is an isomorphism.*

PROOF. Note first that, being group-like morphisms, g and $T(g)$ are split monomorphisms. Hence the natural transformation $-\otimes T(g) : G_{T(\mathbb{I})} \to G_{T(C)}$ is also a split monomorphism.

Now, if the functor $K_{g,C} : \mathbb{V} \to \mathbb{V}_T^{T(\mathbf{C}} = (\mathbb{V}_T)^{\widehat{G_{T(\mathbf{C})}}}$ is an equivalence of categories, then it follows from Proposition 2.6 that the monad $\mathbf{T}$ is of effective descent type and the comonad morphism $S_{K_{g,\mathbf{C}}} : \phi_T U_T \to \widehat{G_{T(C)}}$ is an isomorphism. Since $S_{K_{g,C}} = (-\otimes T(g)) \cdot S_{K_{e_{\mathbb{I}},\mathbb{I}}}$ by (3) and since the natural transformation $-\otimes T(g) : G_{T(\mathbf{I})} \to G_{T(\mathbf{C})}$ is a split monomorphism, it follows that the natural transformations $-\otimes T(g)$ and $S_{K_{e_{\mathbb{I}},\mathbb{I}}}$ are both isomorphisms. Then, in particular, $T(g)$ is an isomorphism. Since $\mathbf{T}$ is of effective descent type, the functor T is conservative (see, [9, Proposition 3.11]). Thus g is also an isomorphism. Since $\mathbf{T}$ is of effective descent type and since $S_{K_{e_{\mathbb{I}},\mathbb{I}}}$ is an isomorphism, it follows from Proposition 2.6 that the functor $K_{e_{\mathbb{I}},\mathbb{I}} : \mathbb{V} \to \mathbb{V}_T^{T(\boldsymbol{I})}$ is an equivalence of categories. Hence (a) implies (b). Since (b) trivially implies (a), (a) and (b) are equivalent. Finally, (b) and (c) are equivalent by Theorem 4.7. ∎

4.9. GALOIS GROUP-LIKE MORPHISMS. We will assume from now on that our monoidal category $(\mathbb{V}, \otimes, \mathbb{I})$ admits equalisers.

Let $\mathbf{T} = (T, m, e)$ be an opmonoidal monad on $\mathbb{V}$, $\mathbf{C}$ a $\mathbb{V}$-comonoid and $g : \mathbb{I} \to C$ a group-like element. Since $\mathbb{V}$ has equalisers, one can consider the $\mathbb{V}$-monad $\mathbf{T}^{-\otimes\overline{g}}$. We write $\mathbf{T}^g$ for this monad. Let us say that $g : \mathbb{I} \to C$ is a Galois group-like element if the induced group-like morphism $-\otimes \overline{g} : 1 \to G_{T(\mathbf{C})}$ is Galois. In particular, the group-like element $1_{\mathbb{I}} : \mathbb{I} \to \mathbb{I}$ is Galois if the group-like morphism $-\otimes \overline{1_{\mathbb{I}}} = -\otimes e_C : 1 \to G_{T(\mathbf{I})}$ is Galois.

Specialising now Theorem 2.15 to the present situation gives:

4.10. Theorem. *Let* $\mathbf{T}$ *be an opmonoidal monad on a monoidal category* $(\mathbb{V}, \otimes, \mathbb{I})$ *and* $\mathbf{C}$ *a* $\mathbb{V}$*-comonoid. Suppose that* $g : \mathbb{I} \to C$ *is a group-like element such that the corresponding functor* $i^* : \mathbb{V}_T \to \mathbb{V}_{T^g}$ *admits a left adjoint functor* $i_* : \mathbb{V}_{T^g} \to \mathbb{V}_T$*. Then the comparison functor* $\bar{i} : \mathbb{V}_{T^g} \to \mathbb{V}_T^{T(\mathbf{C})} = (\mathbb{V}_T)^{\widehat{G_{T(\mathbf{C})}}}$ *is an equivalence of categories if and only if* g *is a Galois group-like element and the functor* i_* *is comonadic.*

Direct inspection shows (see also [12, Section 5]) that $T^{1_\mathbb{I}}$ is given by the equaliser

$$\begin{tikzcd} T^{1_\mathbb{I}}(-) \arrow[r] & T(-) \arrow[r, shift left, "T(-)\otimes e_\mathbb{I}"] \arrow[r, shift right, "\chi_{-,\mathbb{I}}"'] & T(-) \otimes T(\mathbb{I}), \end{tikzcd}$$

while T^g is given by the equaliser

$$\begin{tikzcd} T^g(-) \arrow[r] & T(-) \arrow[r, shift left, "T(-)\otimes e_\mathbb{I}"] \arrow[r, shift right, "\chi_{-,\mathbb{I}}"'] & T(-) \otimes T(\mathbb{I}) \arrow[r, "T(-)\otimes T(g)"] & T(-) \otimes T(C). \end{tikzcd}$$

Since g, being a group-like morphism, is a split monomorphism, so too is the natural transformation $T(-) \otimes T(g)$. It follows that the monad $\mathbf{T}^{1_\mathbb{I}}$ can be identified with the monad $\mathbf{T}^g$. Since $g : \mathbb{I} \to C$ is nothing but a comonoid morphism from the trivial $\mathbb{V}$-comonoid $\mathbf{I}$ to the $\mathbb{V}$-comonoid $\mathbf{C}$ and since any opmonoidal functor preserves comonoid morphisms, $T(g) : T(\mathbb{I}) \to T(C)$ can be seen as a morphism of $\mathbb{V}$-comonoids $T(\mathbf{I}) \to T(\mathbf{C})$. It is then easy to see that the induced morphism of $\mathbb{V}$-comonads $- \otimes T(g) : G_{T(\mathbb{I})} \to G_{T(C)}$ can be lifted to a morphism $\widehat{- \otimes T(g)} : \widehat{G_{T(\mathbf{I})}} \to \widehat{G_{T(\mathbf{C})}}$ of $\mathbb{V}_T$-comonads. Using that $\vartheta = (T(-) \otimes T(g)) \cdot \vartheta'$ by (2), it follows from [12, Lemma 3.9] that one has the commutative diagram

$$\begin{tikzcd} i_* \arrow[rr, "\alpha"] \arrow[dr, "\alpha'"'] & & \widehat{G_{T(\mathbf{C})}} \cdot i_* \\ & \widehat{G_{T(\mathbf{I})}} \cdot i_* \arrow[ur, "\widehat{-\otimes T(g)} \cdot i_*"'] & \end{tikzcd},$$

where $i : T^g = T^{1_\mathbb{I}} \to T$ is the canonical inclusion, while α (resp. α') is a left $\widehat{G_{T(\mathbf{C})}}$-comodule (resp. $\widehat{G_{T(\mathbf{I})}}$-comodule) structure on i_*. It then follows that one also has commutativity of

$$\begin{tikzcd} \widetilde{G_{T(\mathbf{C})}} \arrow[rr, "S_{\bar{i}}"] \arrow[dr, "S_{\bar{i}}"'] & & \widehat{G_{T(\mathbf{C})}} \\ & \widehat{G_{T(\mathbf{I})}} \arrow[ur, "\widehat{-\otimes T(g)}"'] & \end{tikzcd}, \qquad (4)$$

where $\widetilde{G_{T(\mathbf{C})}}$ denotes the comonad generated by the adjunction $i^* \dashv i_*$ (see 2.13).

4.11. Proposition. *Let* $\mathbf{T}$ *be an opmonoidal monad on a monoidal category* $(\mathbb{V}, \otimes, \mathbb{I})$ *and* $\mathbf{C}$ *a* $\mathbb{V}$*-comonoid.*

(1) *A group-like element $g : \mathbb{I} \to C$ is Galois if and only if the group-like element $1_{\mathbb{I}} : \mathbb{I} \to \mathbb{I}$ is Galois and the morphism $T(g) : T(\mathbb{I}) \to T(C)$ is an isomorphism.*

(2) *If the monad $\mathbf{T}$ is conservative, then any $\mathbb{V}$-comonoid admitting a Galois group-like element is (isomorphic to) the trivial $\mathbb{V}$-comonoid $\mathbf{I}$.*

PROOF. To say that the group-like morphism $g : \mathbb{I} \to C$ (resp. $1_{\mathbb{I}} : \mathbb{I} \to \mathbb{I}$) is Galois is to say that the comonad morphism $S_{\bar{i}} : \widetilde{G_{T(\mathbf{I})}} \to \widehat{G_{T(\mathbf{I})}}$ (resp. $S_{\bar{i}} : \widetilde{G_{T(\mathbf{C})}} \to \widehat{G_{T(\mathbf{C})}}$) is an isomorphism. Now, since $T(g)$ is a split monomorphism, the result follows from the commutativity of diagram (4). This proves (1). Recalling that a monad is called conservative provided that its functor part is conservative, one sees that (2) follows from (1). ∎

Combining Theorem 4.10 and Proposition 4.11 gives:

4.12. THEOREM. *Let $\mathbf{T}$ be an opmonoidal monad on a monoidal category $(\mathbb{V}, \otimes, \mathbb{I})$ such that the functor $i^* : \mathbb{V}_T \to \mathbb{V}_{T^{1_{\mathbb{I}}}}$ admits a left adjoint functor $i_* : \mathbb{V}_{T^{1_{\mathbb{I}}}} \to \mathbb{V}_T$ and $\mathbf{C}$ a $\mathbb{V}$-comonoid. Then, for any group-like element $g : \mathbb{I} \to C$, the following are equivalent:*

(a) *$g : \mathbb{I} \to C$ is a Galois group-like element and the functor i_* is comonadic;*

(b) *the comparison functor $\bar{i} : \mathbb{V}_{T^g} \to (\mathbb{V}_{\mathrm{T}})^{\phi_{\mathrm{T}}(\mathbf{C})}$ is an equivalence of categories;*

(c) *$1_{\mathbb{I}} : \mathbb{I} \to \mathbb{I}$ is a Galois group-like element, the functor i_* is comonadic and the morphism $T(g) : T(\mathbb{I}) \to T(C)$ is an isomorphism;*

(d) *the comparison functor $\bar{\bar{i}} : \mathbb{V}_{T^{1_{\mathbb{I}}}} \to (\mathbb{V}_{\mathrm{T}})^{\phi_{\mathrm{T}}(\mathbf{I})}$ is an equivalence of categories and the morphism $T(g) : T(\mathbb{I}) \to T(C)$ is an isomorphism.*

It is easy to see that, in the case where the monad $\mathbf{T}^{1_{\mathbb{I}}} = \mathbf{T}^g$ is (isomorphic to) the identity monad, the functor $\phi_{T^{1_{\mathbb{I}}}} = \phi_{T^g}$ is (isomorphic to) the identity functor, the functor i_* is (isomorphic to) the functor ϕ_T, while the functor $\bar{i}$ is (isomorphic to) the comparison functor $K_{g,\mathbf{C}}$. Using now that the monad $\mathbf{T}$ is conservative provided that the functor ϕ_T is so, in the light of Proposition 4.11, we get from Theorems 4.8 and 4.12:

4.13. THEOREM. *Let $\mathbf{T}$ be an opmonoidal monad on a monoidal category $(\mathbb{V}, \otimes, \mathbb{I})$ such that the monad $\boldsymbol{T}^{1_{\mathbb{I}}}$ is (isomorphic to) the identity monad and $\mathbf{C}$ a $\mathbb{V}$-comonoid. Then, for any group-like element $g : \mathbb{I} \to C$, the following are equivalent:*

(a) *$g : \mathbb{I} \to C$ is a Galois group-like element and the functor ϕ_T is comonadic;*

(b) *the comparison functor $K_{g,\mathbf{C}} : \mathbb{V} \to (\mathbb{V}_{\mathrm{T}})^{\phi_{\mathrm{T}}(\mathbf{C})}$ is an equivalence of categories;*

(c) *$1_{\mathbb{I}} : \mathbb{I} \to \mathbb{I}$ is a Galois group-like element, the functor ϕ_T is comonadic, and the morphism $g : \mathbb{I} \to C$ is an isomorphism;*

(d) *the comparison functor $K_{g,\boldsymbol{I}} : \mathbb{V} \to (\mathbb{V}_{\mathrm{T}})^{\phi_{\mathrm{T}}(\mathbf{I})}$ is an equivalence of categories and the morphism $g : \mathbb{I} \to C$ is an isomorphism;*

(e) *(i)* $\mathbf{T}$ *is* $\mathrm{G}_{\mathrm{T}(\mathbf{I})}$*-Galois,*

(ii) $\mathbf{T}$ *is of effective descent type,*

(iii) g *is an isomorphism.*

4.14. Definition. *We say that an opmonoidal monad* $\mathbf{T}$ *on a monoidal category* $\mathbb{V}$ *is* augmented *if it is equipped with a monad morphism* $\sigma : T \to 1_{\mathbb{V}}$*. In this case* σ *is said to be an* augmentation.

4.15. Lemma. *Suppose that* $\mathbf{T}$ *is an augmented right-Hopf opmonoidal monad on a monoidal category* $\mathbb{V}$ *with an augmentation* $\sigma : T \to 1_{\mathbb{V}}$*. Then, for any* $V \in \mathbb{V}$*, the composite*

$$\overline{\sigma}_V : T(V) \xrightarrow{\chi_{V,\mathbb{I}}} T(V) \otimes T(\mathbb{I}) \xrightarrow{\sigma_V \otimes T(\mathbb{I})} V \otimes T(\mathbb{I})$$

is an isomorphism.

Proof. Just note that, since $\sigma : T \to 1_{\mathbb{V}}$ is a morphism of monads, for any $V \in \mathbb{V}$, (V, σ_V) is an object of $\mathbb{V}_T$. ∎

4.16. Theorem. *Let* $\mathbf{T} = (T, m, e)$ *be an augmented right-Hopf opmonoidal monad on a Cauchy complete monoidal category* $(\mathbb{V}, \otimes, \mathbb{I})$ *with an augmentation* $\sigma : T \to 1_{\mathbb{V}}$*. Then, for any group-like element* $g : \mathbb{I} \to C$*, the following are equivalent:*

(a) $g : \mathbb{I} \to C$ *is a Galois group-like element;*

(b) the comparison functor $K_{g,\mathbf{C}} : \mathbb{V} \to (\mathbb{V}_{\mathrm{T}})^{\phi_{\mathrm{T}}(\mathbf{C})}$ *is an equivalence of categories;*

(c) the comparison functor $K_{g,\boldsymbol{I}} : \mathbb{V} \to (\mathbb{V}_{\mathrm{T}})^{\phi_{\mathrm{T}}(\mathbf{I})}$ *is an equivalence of categories and the morphism* $g : \mathbb{I} \to C$ *is an isomorphism;*

(d) $1_{\mathbb{I}} : \mathbb{I} \to \mathbb{I}$ *is a Galois group-like element and the morphism* $g : \mathbb{I} \to C$ *is an isomorphism;*

(e) *(i)* $\mathbf{T}$ *is* $\mathrm{G}_{\mathrm{T}(\mathbf{I})}$*-Galois,*

(ii) g *is an isomorphism.*

Proof. Using naturality of χ, it is not hard to check that the diagram

$$\begin{tikzcd}
T(V) \arrow[r, shift left, "T(V)\otimes e_{\mathbb{I}}"] \arrow[r, shift right, "\chi_{V,\mathbb{I}}"'] \arrow[d, "\overline{\sigma}_V"'] & T(V) \otimes T(\mathbb{I}) \arrow[d, "\overline{\sigma}_V \otimes T(\mathbb{I})"] \\
V \otimes T(\mathbb{I}) \arrow[r, shift left, "V\otimes T(\mathbb{I})\otimes e_{\mathbb{I}}"] \arrow[r, shift right, "V\otimes\chi_{\mathbb{I},\mathbb{I}}"'] & V \otimes T(\mathbb{I}) \otimes T(\mathbb{I})
\end{tikzcd}$$

commutes. Since $\overline{\sigma}_V$ is an isomorphism by Lemma 4.15, it follows that $T^{1_{\mathbb{I}}}(V)$ is (isomorphic to) the equaliser of the pair

$$\begin{tikzcd}
V \otimes T(\mathbb{I}) \arrow[r, shift left, "V\otimes T(\mathbb{I})\otimes e_{\mathbb{I}}"] \arrow[r, shift right, "V\otimes\chi_{\mathbb{I},\mathbb{I}}"'] & V \otimes T(\mathbb{I}) \otimes T(\mathbb{I})\,.
\end{tikzcd}$$

Using now that

- $\theta \cdot e_{\mathbb{I}} = 1_{\mathbb{I}}$ and $(\theta \otimes T(\mathbb{I})) \cdot \chi_{\mathbb{I},\mathbb{I}} = 1_{T(\mathbb{I})}$, since the monad $\mathbf{T}$ is opmonoidal, and
- $(\theta \otimes T(\mathbb{I})) \cdot (T(\mathbb{I}) \otimes e_{\mathbb{I}}) = e_{\mathbb{I}} \cdot \theta$ by naturality of composition,

one sees that the diagram

$$\begin{tikzcd} \mathbb{I} \arrow[r, "e_{\mathbb{I}}"] & T(\mathbb{I}) \arrow[l, bend left, "\theta"] \arrow[r, shift left, "T(\mathbb{I})\otimes e_{\mathbb{I}}"] \arrow[r, shift right, "\chi_{\mathbb{I},\mathbb{I}}"'] & T(\mathbb{I}) \otimes T(\mathbb{I}) \arrow[l, bend left, "\theta\otimes T(\mathbb{I})"] \end{tikzcd} .$$

is a split equaliser. Since split equalisers are preserved by any functor the diagram

$$\begin{tikzcd} V \arrow[r, "V\otimes e_{\mathbb{I}}"] & V \otimes T(\mathbb{I}) \arrow[r, shift left, "V\otimes T(\mathbb{I})\otimes e_{\mathbb{I}}"] \arrow[r, shift right, "V\otimes \chi_{\mathbb{I},\mathbb{I}}"'] & V \otimes T(\mathbb{I}) \otimes T(\mathbb{I}) \end{tikzcd}$$

is a (split) equaliser. Thus the monad $\mathbf{T}^{1_{\mathbb{I}}}$ is (isomorphic to) the identity monad.

Next, as $\sigma : T \to 1_{\mathbb{V}}$ is a morphism of monads, $\sigma \cdot e = 1$. Thus the unit of the monad $\mathbf{T}$ is a split monomorphism, and since the category $\mathbb{V}$ is Cauchy complete by hypothesis, it follows from [9, Corollary 3.17] that $\mathbf{T}$ is of effective descent type, i.e. the functor ϕ_T is comonadic. Putting now this information together, the assertions follow by Theorem 4.13. ∎

5. Applications

In this final section we outline some applications of the notions developed.

5.1. Monads on cartesian monoidal categories. Let $\mathbb{A}$ be a category with finite products. Then $\mathbb{A}$ is equipped with the (symmetric) monoidal structure $(\mathbb{A}, \times, \mathtt{1})$ (known as the *cartesian monoidal structure*), where $a \times b$ is some chosen product of a and b, and $\mathtt{1}$ is a chosen terminal object in $\mathbb{A}$. For any object $a \in \mathbb{A}$, we write $!_a$ for the unique morphism $a \to \mathtt{1}$. Given morphisms $f : a \to x$ and $g : a \to y$ in $\mathbb{A}$, we write $< f, g >: a \to x \times y$ for the unique morphism inducing commutativity of the diagram

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[d, "<f,g>"] \arrow[dr, "g"] & \\ x & x \times y \arrow[l, "p_1"] \arrow[r, "p_2"'] & y. \end{tikzcd}$$

Any monad $\mathbf{T}$ on $\mathbb{A}$ has a canonical structure of an opmonoidal monad given by

$$\chi_{a,b} =< T(p_1), T(p_2) >: T(a \times b) \to T(a) \times T(b), \quad \theta =!_{T(\mathtt{1})} : T(\mathtt{1}) \to \mathtt{1}.$$

Thus, for any monad $\mathbf{T}$ on $\mathbb{A}$, the category $\mathbb{A}_T$ is also cartesian.

Since, for any $a \in \mathbb{A}$, the projection $p_1 : a \simeq a \times \mathtt{1} \to a$ is (isomorphic to) the identity morphism $1_a : a \to a$, while the projection $p_2 : a \simeq a \times \mathtt{1} \to \mathtt{1}$ is (isomorphic to) the morphism $!_a : a \to \mathtt{1}$, $\chi_{a,\mathtt{1}} : T(a) \to T(a) \times T(\mathtt{1})$ is just the morphism $< 1_{T(a)}, T(!_a) >$.

An arbitrary object $a \in \mathbb{A}$ has a canonical $\mathbb{A}$-comonoid structure given by the diagonal morphism $\Delta_a =< 1_a, 1_a >: a \to a \times a$. Writing $\mathbf{a}$ for the corresponding $\mathbb{A}$-comonoid, one has that $\mathbb{A}^{\mathbf{a}}$ is (isomorphic to) the comma category $\mathbb{A}{\downarrow}a$ (see, for example, [12]). Modulo this isomorphism, the forgetful functor $U^{\mathbf{a}} : \mathbb{A}^{\mathbf{a}} \to \mathbb{A}$ corresponds to the functor

$$\Sigma_a : \mathbb{A}{\downarrow}a \to \mathbb{A}, \ \ (x \to a) \longrightarrow x,$$

while its right adjoint $\phi_{\mathbf{a}} : \mathbb{A} \to \mathbb{A}^{\mathbf{a}}$ corresponds to the functor

$$a^* : \mathbb{A} \to \mathbb{A}{\downarrow}a, \ \ x \longrightarrow (p_1 : a \times x \to a).$$

Suppose now $\mathbf{T}$ to be a monad on a cartesian category $\mathbb{A}$ such that the category $\mathbb{A}_T$ admits equalisers. Then one can form the monad T^{1_1}. Moreover, modulo the isomorphism of categories $(\mathbb{A}_T)^{\phi_T(\mathtt{1})} \simeq (\mathbb{A}_T \downarrow \phi_T(\mathtt{1}))$, one rewrites Diagram (1) from 2.13 as

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "\phi_{T^{1_1}}"] \arrow[rr, bend left, "K_{1_1,\mathtt{1}}"] \arrow[dr, "\phi_T"'] & \mathbb{A}_{T^{1_1}} \arrow[r, "\bar{i}"] \arrow[d, "i_*"] & (\mathbb{A}_T) \downarrow \phi_T(\mathtt{1}) \arrow[dl, "\Sigma_{\phi_T(\mathtt{1})}"] \\
& \mathbb{A}_T &
\end{tikzcd} \tag{5}$$

Note that, for any $a \in \mathbb{A}$, $K_{1_1,\mathtt{1}}(a) = ((T(a), m_a), T(!_a))$.

5.2. Remark. Obviously, for any $(a, h_a) \in \mathbb{A}_T$, the (a, h_a)-component of the natural transformation $\mathbb{H}^r_{-,\mathtt{1}} : T \to - \times T(\mathtt{1})$ is the composite

$$T(a) \xrightarrow{<1_{T(a)},T(!_a)>} T(a) \times T(\mathtt{1}) \xrightarrow{h_a \times T(\mathtt{1})} a \times T(\mathtt{1}),$$

which is the same as the morphism

$$T(a) \xrightarrow{<h_a,T(!_a)>} a \times T(\mathtt{1}).$$

If $T(\mathtt{1}) \simeq \mathtt{1}$, then $T(!_a) \simeq !_{T(a)}$ and thus $<h_a, T(!_a)>$ can be identified with the morphism $h_a : T(a) \to a$.

Now fix a monad $\mathbf{T} = (T, m, e)$ on a cartesian monoidal category $\mathbb{A}$ with equalisers. Then, for any $a \in \mathbb{A}$, $T^{1_1}(a)$ can be calculated as the equaliser of the diagram

$$T(a) \underset{1_{T(a)} \times e_1}{\overset{<1_{T(a)},T(!_a)>}{\rightrightarrows}} T(a) \times T(\mathtt{1}).$$

But since $1_{T(a)} \times e_1$ can be identified with the morphism $< 1_{T(a)}, e_1 \cdot !_{T(a)} >$, the diagram

$$T^{1_1}(a) \xrightarrow{i_a} T(a) \underset{<1_{T(a)}, e_1 \cdot !_{T(a)}>}{\overset{<1_{T(a)}, T(!_a)>}{\rightrightarrows}} T(a) \times T(1)$$

is an equaliser if and only if so is

$$T^{1_1}(a) \xrightarrow{i_a} T(a) \underset{p_2 \cdot <1_{T(a)}, e_1 \cdot !_{T(a)}>}{\overset{p_2 \cdot <1_{T(a)}, T(!_a)>}{\rightrightarrows}} T(1)\ .$$

As $p_2 \cdot < 1_{T(a)}, T(!_a) >= T(!_a)$ and $p_2 \cdot < 1_{T(a)}, e_1 \cdot !_{T(a)} >= e_1 \cdot !_{T(a)}$, the diagram

$$T^{1_1}(a) \xrightarrow{i_a} T(a) \underset{e_1 \cdot !_{T(a)}}{\overset{T(!_a)}{\rightrightarrows}} T(1)$$

is an equaliser. It follows that if $T(1) \simeq 1$, then i_a is an isomorphism. Conversely, if i_a is an isomorphism, then $T(!_a) = e_1 \cdot !_{T(a)}$. In particular, $T(!_1) = e_1 \cdot !_{T(1)}$. But $T(!_1) = 1_1$, implying that both e_1 and $!_{T(1)}$ are isomorphisms. Thus:

5.3. Lemma. *Let $\mathbf{T}$ be a monad on a cartesian monoidal category $(\mathbb{A}, \times, 1)$. Then the canonical inclusion $i : T^{1_1} \to T$ is an isomorphism if and only if the functor part T preserves the terminal object.*

5.4. Proposition. *Let $(\mathbb{A}, \times, 1)$ be a cartesian monoidal category. For any monad $\mathbf{T}$ on $\mathbb{A}$, whose functor part preserves the terminal object 1, the comparison functor*

$$\bar{i} : \mathbb{A}_{T^{1_1}} \to (\mathbb{A}_T)^{\phi_T(1)}$$

is an equivalence. In particular, $1_1 : 1 \to 1$ is a Galois group-like element (w.r.t. $\mathbf{T}$).

Proof. Since $T(1) \simeq 1$, the monads $\mathbf{T}^{1_1}$ and $\mathbf{T}$ are isomorphic by Lemma 5.3. It then follows from commutativity of diagram (5) that $\bar{i}$ is just the functor $(\phi_T(1))^* : \mathbb{A}_T \to (\mathbb{A}_T) \downarrow \phi_T(1)$. But since $T(1) \simeq 1$, $\phi_T(1)$ is a terminal object in $\mathbb{A}_T$. Thus the functor $(\phi_T(1))^*$ (and hence also $\bar{i}$) is an isomorphism of categories.

Now the last assertion follows from Theorem 4.12. ■

Recall that a monad $\mathbf{T} = (T, m, e)$ on a category $\mathbb{A}$ is said to be *idempotent* if the multiplication $m : TT \to T$ is a natural isomorphism.

5.5. Proposition. *Let $(\mathbb{A}, \times, 1)$ be a cartesian monoidal category. Any idempotent monad on $\mathbb{A}$, whose functor part preserves the terminal object 1, is right pre-Hopf.*

Proof. It is well-known that if $\mathbf{T} = (T, m, e)$ is an idempotent monad on a category $\mathbb{A}$, then for any $(a, h_a) \in \mathbb{A}_T$, the morphism $h_a : T(a) \to a$ is an isomorphism. Thus the result follows from Remark 5.2. ■

5.6. Example. Recall from [7] that a category $\mathbb{A}$ with all finite products is called *cartesian closed* if for each object $a \in \mathbb{A}$, the functor $a \times - : \mathbb{A} \to \mathbb{A}$ has a right adjoint $(-)^a : \mathbb{A} \to \mathbb{A}$.

For any object $a \in \mathbb{A}$, the endofunctor $T_a = (-)^a$ can be made a monad with multiplication and unit

$$m_x = x^{\Delta_a} : T_a T_a(x) = (x^a)^a \simeq x^{a\times a} \to T_a(x) = x^a, \quad e_x = x^{!_a} : x \to x^a = T_a(x).$$

Let $\mathbb{A}$ be a cartesian closed category such that the terminal object $\mathtt{1}$ has a nontrivial proper subobject $u \rightarrowtail \mathtt{1}$ (for example, let $\mathbb{A}$ be the category of set-valued sheaves on a nontrivial topological space). Since $u \times u \simeq u$, the diagonal $\Delta_u : u \to u \times u$ is an isomorphism, whence the monad T_u is idempotent. Since $\mathtt{1}^u = \mathtt{1}$, the functor $(-)^u$ preserves the terminal object and it follows from Proposition 5.5 that the opmonoidal monad T_U is right pre-Hopf.

Note that by Proposition 5.4, the comparison functor $K_{1_1,\mathtt{1}} : \mathbb{V} \to (\mathbb{V}_{T_u})^{\phi_{T_u}(\mathtt{1})}$ is not an equivalence of categories. Thus T_u is an example of an opmonoidal monad which is right pre-Hopf, but the corresponding comparison functor $K_{1_1,\mathtt{1}}$ is not an equivalence of categories.

5.7. Example. Recall that the *covariant power set functor* $\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$ is defined by

$$\mathcal{P}(X) = \mathrm{Sub}(X), \quad \mathcal{P}(f : X \to Y) = \mathcal{P}(X) \xrightarrow{\mathcal{P}(f)} \mathcal{P}(Y),$$

where $\mathrm{Sub}(X)$ is the set of all subsets of X and for each $U \in \mathrm{Sub}(X)$, $\mathcal{P}(f)(U)$ is the image $f(U)$ of U under f. $\mathcal{P}$ is actually the functor part of a monad $(\mathcal{P}, e, m)$ with

$e_X : X \to \mathcal{P}(X)$ the singleton map, $e_X : x \to \{x\}$, and
$m_X : \mathcal{P}\mathcal{P}(X) \to \mathcal{P}(X)$ the union, $m_X(\{X_\alpha\}) = \bigcup_\alpha X_\alpha$.

It is well-known that the Eilenberg-Moore category of $\mathcal{P}$-algebras is isomorphic to the category **CSLat** of *complete (join-)semilattices*. Recall that the category **CSLat** has as its objects partially ordered sets $(X, \leq)$ which admit arbitrary suprema, and as its morphisms $f : X \to Y$ maps which preserve suprema. We write $\mathtt{2}$ for the two-element semilattice $\phi_{\mathcal{P}}(\mathtt{1}) = \{0 \leq 1\}$.

It is not hard to check that $\mathcal{P}^{1_1}$ is just the *proper power set functor* $\mathcal{P}^+$, where $\mathcal{P}^+(X) = \mathcal{P}(X) \setminus \{\varnothing\}$. It is also well-known (see, for example, [6, Problem 1.3.3.]) that the Eilenberg-Moore category of $\mathcal{P}^+$-algebras is isomorphic to the category **ACSLat** of *almost complete (join-)semilattices*, i.e. partially ordered sets $(X, \leq)$ such that the suprema of all non-empty subsets of X exists. Morphisms $f : (X, \leq) \to (Y, \leq)$ of **ACSLat** are non-empty suprema preserving maps.

Writing $i : \mathcal{P}^+ \to \mathcal{P}$ for the canonical inclusion, it is not hard to see that the functor

$$i^* : \mathrm{Set}_{\mathcal{P}} = \mathbf{CSLat} \to \mathrm{Set}_{\mathcal{P}^+} = \mathbf{ACSLat}$$

just forgets about the bottom element, while

$$i_* : \mathbf{ACSLat} \to \mathbf{CSLat}$$

takes an object $X \in \mathbf{ACSLat}$ to the complete semilattice $\overline{X}$ obtained from X by adding a bottom element 0_X. It then follows in particular that the endofunctor $i_* i^* : \mathbf{CSLat} \to \mathbf{CSLat}$ takes a complete semilattice X to the complete semilattice $\overline{X}$ obtained from X by adding a new bottom element $0_{\overline{X}} < 0_X$. Direct inspection shows that, for any $X \in \mathbf{CSLat}$, the X-component of the comonad morphism $S_{\bar{i}} : \mathbf{G}_i \to \mathbf{G}_{\phi_{\mathcal{P}}(\mathtt{1})}$ is the map $\omega : \overline{X} \to X \times \mathtt{2}$ defined by

$$\omega(x) = \begin{cases} (x, 1) & \text{if } x \neq o_{\overline{X}} \\ (0_X, 0) & \text{if } x = o_{\overline{X}}. \end{cases}$$

It is clear that ω is not an isomorphism. Thus $1_{\mathtt{1}} : \mathtt{1} \to \mathtt{1}$ is not a Galois group-like element (w.r.t. the monad $\mathcal{P}$), and hence, by Theorem 4.12, the comparison functor

$$\bar{i} : \mathrm{Set}_{\mathcal{P}^+} = \mathbf{ACSLat} \to (\mathrm{Set}_{\mathcal{P}} \downarrow \phi_{\mathcal{P}}(\mathtt{1})) = (\mathbf{CSLat} \downarrow \mathtt{2}),$$

which sends an object $X \in \mathbf{ACSLat}$ to $(\omega : \overline{X} \to \mathtt{2}) \in (\mathbf{CSLat} \downarrow \mathtt{2})$ with

$$\omega(x) = \begin{cases} 1 & \text{if } x \neq o_{\overline{X}} \\ 0 & \text{if } x = o_{\overline{X}}, \end{cases}$$

is not an equivalence of categories. According to [12, 1.4], $\bar{i}$ admits a right adjoint r: for any $(\omega : \overline{X} \to \mathtt{2}) \in (\mathbf{CSLat} \downarrow \mathtt{2})$, $r(\omega) = (\omega)^{-1}(1)$. It is now easy to see that $r\bar{i} \simeq 1$. Thus $\mathbf{ACSLat}$ is a full coreflective subcategory of $(\mathbf{CSLat} \downarrow \mathtt{2})$.

Note finally that $\mathcal{P}^+(\mathtt{1}) = \mathtt{1}$. Now it follows from Proposition 5.4 that $1_{\mathtt{1}} : \mathtt{1} \to \mathtt{1}$ is a Galois group-like element w.r.t. the monad $\mathcal{P}^+$.

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