Theory and Applications of Categories, Vol. 28, No. 21, 2013, pp. 616–695.

ENRICHED INDEXED CATEGORIES

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ABSTRACT. We develop a theory of categories which are simultaneously (1) indexed over a base category $\mathbf{S}$ with finite products, and (2) enriched over an $\mathbf{S}$-indexed monoidal category $\mathscr{V}$. This includes classical enriched categories, indexed and fibered categories, and internal categories as special cases. We then describe the appropriate notion of "limit" for such enriched indexed categories, and show that they admit "free cocompletions" constructed as usual with a Yoneda embedding.

Contents

1. Introduction

It is well-known that ordinary category theory admits several important generalizations, such as the following.

- A category *enriched* in a monoidal category $\mathbf{V}$ has a set of objects, but hom-objects belonging to $\mathbf{V}$.

This material is based upon work supported by the National Science Foundation under a postdoctoral fellowship and agreement No. DMS-1128155. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

Received by the editors 2013-01-16 and, in revised form, 2013-08-02.

Transmitted by Tom Leinster. Published on 2013-08-05, this version 2022-03-28.

2010 Mathematics Subject Classification: 18D20,18D30.

Key words and phrases: monoidal category, enriched category, indexed category, fibered category.

- A category *internal* to a category **S** with pullbacks has both an **S**-object of objects and an **S**-object of morphisms.
- A category *indexed* or *fibered* over a category **S** has sets of objects and morphisms, each of which lives over a specified object or morphism in **S**.

Sometimes, however, we encounter category-like objects which appear simultaneously enriched and internal, or enriched and indexed. Here are a few examples.

(i) *Parametrized homotopy theory* (as developed in [MS06]) studies spaces and spectra parametrized over a given base space. Each of these forms a category indexed over the category of base spaces. In addition, however, parametrized spectra are enriched over parametrized spaces, in a sense which was recognized in [MS06] but not given a general context.

(ii) The free abelian group on a monoid is a ring; applying this functor homwise to an ordinary category, we obtain a category enriched over abelian groups. Similarly, the suspension spectrum of a topological monoid is a ring spectrum, and so the fiberwise suspension spectrum of a topologically-internal category should be a category which is simultaneously internal to spaces and enriched over spectra. Such categories played an important role in [Pon07].

(iii) *Equivariant homotopy theory* studies spaces and spectra with actions of a topological group. As the group in question varies, we find categories simultaneously indexed over groups and enriched over spaces. More recently, *global equivariant homotopy theory* [Boh12] studies equivariant spaces and spectra constructed in a coherent way across all groups of equivariance. Such “global spectra” can be defined just like ordinary diagram spectra, if we work in the context of categories simultaneously indexed over groups and enriched over equivariant spaces.

(iv) In [Bun13], enriched indexed categories (which were discovered independently by Bunge) provide a general context to compare completions such as the Karoubi completion, stack completion, Grothendieck completion, and Cauchy completion.

(v) The category of abelian sheaves is simultaneously indexed over base spaces and enriched over abelian groups. Similarly, chain complexes of sheaves are indexed over spaces and enriched over chain complexes.

(vi) When doing mathematics relative to a base topos, we must replace small categories by internal ones and large categories by indexed ones. Therefore, wherever enriched category theory is used in classical mathematics, in topos-relative mathematics we should expect to combine it with internalization and indexing.

In this paper we show that enriched indexed categories support a category theory as rich and powerful as all three classical cases. To a large extent, this is entirely straightforward. Unsurprisingly, the resulting theory exhibits aspects that are characteristic both of classical enriched category theory and internal and indexed category theory. Notable among the former is the need for a notion of *weighted* limit. Notable among the latter is the nontriviality of the passage from *small* categories (e.g. internal ones) to *large* ones (e.g. indexed ones).

1.1. Some remarks about formal category theory. The theory of enriched indexed categories is clarified by using tools from formal category theory, which are already known to encompass both enriched and internal/indexed categories separately. These tools center around *profunctors* between categories (in the classical case, a profunctor from A to B is a functor $B^{\mathrm{op}} \times A \to \mathrm{Set}$). Every functor gives rise to an adjoint pair of profunctors, and altogether categories, functors, and profunctors of any fixed sort form a *proarrow equipment* [Woo82] (or "framed bicategory" [Shu08]).

The central observations are the following:

(i) In any equipment, there is a formal notion of a *weighted limit* in an object (e.g. a category) weighted by a morphism (e.g. a profunctor); see [SW78, Woo82]. Starting from this we can develop large amounts of category theory purely formally.

(ii) For any well-behaved equipment $\mathbb{W}$, there is an equipment of "$\mathbb{W}$-enriched categories", functors, and profunctors; see e.g. [BCSW83, Str83, GS13]. Moreover, we can remove the qualifier "well-behaved" by generalizing to *virtual equipments* in the sense of [CS10] (in which not all profunctors may be composable). This was originally observed in [Lei07, Lei04, Lei02].

Observation (i) means that in order to automatically obtain a formally well-behaved theory of "enriched indexed categories", essentially all we need is to define suitably related notions of category, functor, and profunctor. And observation (ii) means that for this, we can start with a simpler (virtual) equipment and apply the general enriched-categories construction. Finally, the relevant simpler equipment to begin with was already constructed in [Shu08], starting from an indexed monoidal category $\mathscr{V}$ (the relevant "base for enrichment").

It would seem, then, that there is very little left to do; so why is this paper so long? There are several reasons.

Firstly, for the purposes of exposition, application, and wide accessibility, it seems valuable to have explicit descriptions of what the formal equipment-theoretic notions reduce to in our particular case of interest, not requiring the reader to be familiar with the literature of formal category theory. For this reason, I will minimize references to equipments, generally confining them to remarks and to the proofs of lemmas (all of which could also easily be done "by hand").

Secondly, not all of the formal category theory existing in any equipment has yet been generalized to the virtual case. (The generalization should be entirely straightforward, but for the most part it has not yet been written out.) However, the virtual case is necessary in order to deal with *large* categories, since even when the enriching category is cocomplete, profunctors between large categories may not be composable.[1] A reader who is so inclined can read parts of sections 5, 8, and 10 of this paper as contributions to this theory, since wherever possible, we give proofs that apply in any virtual equipment.

Finally, enriched indexed categories share with classical indexed categories the property of having multiple not-obviously-equivalent definitions. Given an indexed monoidal category $\mathscr{V}$, there is an obvious notion of *small $\mathscr{V}$-category*, which directly generalizes internal categories and small enriched categories. On the other hand, there is also a fairly obvious notion of (large) *indexed $\mathscr{V}$-category*, which generalizes locally small indexed categories (incarnated as "locally internal categories" in the sense of [Pen74]). There are plenty of good examples of both definitions, but it is not entirely trivial how to regard a small $\mathscr{V}$-category as an indexed one!

The equipment-theoretic approach actually yields a *third* notion, which we will call simply a *(large) $\mathscr{V}$-category*. This notion manifestly includes small $\mathscr{V}$-categories as a special case, but its connection to indexed $\mathscr{V}$-categories is not entirely trivial. In the special case of ordinary (unenriched) indexed categories, this relationship was established by [BCSW83, BW87]. Thus, we spend some time producing the analogous correspondence between indexed $\mathscr{V}$-categories and large $\mathscr{V}$-categories. It turns out to behave even better when we work with equipments, rather than merely bicategories as [BCSW83, BW87] did.

The nontriviality of this correspondence also means that it also takes a little work to rephrase equipment-theoretic notions (such as weighted limits) in the language of indexed $\mathscr{V}$-categories. Pleasingly, the results are exactly what one might hope for.

1.2. Historical remarks. Apparently, the idea of enriched indexed categories was first proposed by Lawvere [Law73]. A formal definition was given by Gouzou and Grunig in [GG76], corresponding to what I will call an "indexed $\mathscr{V}$-category". They did not apply general equipment-theoretic methods (which did not exist at that time). Perhaps because of a lack of applications, the definition did not become well-known, and the theory was not extensively developed.

I discovered enriched indexed categories myself around 2007, with examples such as [MS06] and [Pon07] in mind. Michal Przybylek [Prz07] independently invented them at about the same time as well. After a brief discussion on the categories mailing list, Thomas Streicher very kindly sent me a copy of the work of Gouzou and Grunig. Seeing this, and not having any real applications in mind yet, and feeling that it was all a special case of equipment theory, I put the notion aside for a while.

However, recently two new applications have appeared. Firstly, the notion of "global equivariant spectrum" developed in [Boh12] requires categories that are both indexed over groups and enriched in spaces with group actions, and was made possible by an early draft of this paper. Secondly, during Octoberfest 2012 in Montreal, I found that Marta Bunge had also independently arrived at the same notion of "indexed $\mathscr{V}$-category", with the goal of comparing various idempotent completion monads [Bun13].

[1] In some of the literature, such as [SW78], this is avoided by invoking an embedding theorem to make the enriching category into an even larger one. However, an analogous process for enriched indexed categories would be rather more complicated and obscure the important ideas.

This suggests that it is time to publish a careful development of the theory, using modern technology, and with due credit given to everyone who discovered it independently.

1.3. Outline of the paper. We begin in §2 by studying *indexed monoidal categories*, which provide the "base" of enrichment and indexing for our enriched indexed categories. More specifically, there is a category $\mathbf{S}$ which provides the base of the indexing, and an $\mathbf{S}$-indexed category $\mathscr{V}$ with a monoidal structure which provides the enrichment. Much of this theory can be found in [Shu08] and [PS12], but we recall it all here for convenience. We also give a large number of examples.

In §§3–5, we define respectively the three kinds of $\mathscr{V}$-category: small, indexed, and large. In each case we also define the relevant notions of $\mathscr{V}$-functor, $\mathscr{V}$-natural transformation, and $\mathscr{V}$-profunctor, and give several examples. In §3 and §5 we also study profunctors in some more detail, making use of some equipment-theoretic notions.

Then in §6 we compare the notions of $\mathscr{V}$-category. Small $\mathscr{V}$-categories are manifestly a special case of large ones. As for the indexed ones, we identify a particular subclass of large $\mathscr{V}$-categories, called *$\mathscr{V}$-fibrations*, and a subclass of functors between them, called *indexed*, which form a 2-category that is 2-equivalent to the 2-category of indexed $\mathscr{V}$-categories. We regard this correspondence as closely analogous to the classical equivalence between pseudofunctors (indexed categories) and fibrations, although it is not strictly a generalization of it.

Moreover, it turns out that the 2-category of $\mathscr{V}$-fibrations and indexed $\mathscr{V}$-functors is *biequivalent* to the entire 2-category of large $\mathscr{V}$-categories and all $\mathscr{V}$-functors. The equivalence also carries over to profunctors, so we have a complete equivalence of "category theories".

It seems that in applications, the $\mathscr{V}$-categories which act like the "large categories" in classical category theory are always $\mathscr{V}$-fibrations (or, equivalently, indexed $\mathscr{V}$-categories), while those that act like the "small categories" are not always so. (Sometimes they are small in the sense of §3; other times they are small only in a weaker, non-elementary sense.) Thus, it is useful to have the context of large $\mathscr{V}$-categories which includes both.

In §7 we consider "change of enrichment" along a morphism $\mathscr{V} \to \mathscr{V}'$. The definitions are all straightforward and mostly omitted; mainly we give a lot of examples to show the generality of the concept. A particularly important case is that of the "underlying indexed category" of an enriched indexed category, which generalizes the classical "underlying ordinary category" of an enriched category.

In §8–9 we study the very important topic of limits and colimits. In §8 we work purely equipment-theoretically, defining limits in terms of profunctors and proving their basic properties abstractly. Then in §9 we specialize these notions to the case of *indexed $\mathscr{V}$-categories*, where they turn out to reduce exactly to a combination of well-known indexed and enriched notions of limit. In the case of ordinary (unenriched) indexed categories, this perspective on limits was explored in [BW87] and sequels such as [Bet89, Bet00, BW89]. On the other hand, the same combination of indexed and enriched notions of limit was studied in [GG76], but without the equipment-theoretic context for justification and formal properties.

With the basic theory of limits and colimits available, there are of course many different directions in which to develop category theory. We choose only two: presheaf categories in §10, and monoidal structures in §11.

The goal of §10 is to prove that presheaf $\mathscr{V}$-categories are free cocompletions. The arguments are purely formal and equipment-theoretic. Rather than restrict ourselves to presheaves on small categories, we consider more generally *small presheaves* in the sense of [DL07], which form free cocompletions of not-necessarily-small categories.

Finally, in §11 we study monoidal $\mathscr{V}$-categories, using for the first time in an essential way the symmetry of $\mathscr{V}$. We define two tensor products of $\mathscr{V}$-categories, one "indexed" and one not, which extend the biequivalence of §6 to a monoidal biequivalence. Monoidal $\mathscr{V}$-categories are then pseudomonoids with respect to either of these tensor products. We also define *closed* monoidal $\mathscr{V}$-categories by way of profunctors, essentially specializing the general definitions of [DS97, DMS03, Str04] to $\mathscr{V}$-categories. As in the classical case, closed monoidal $\mathscr{V}$-categories correspond closely to monoidal adjunctions involving $\mathscr{V}$. We conclude with the Day convolution monoidal structure for $\mathscr{V}$-presheaf categories, which, combined with the previous theory, yields some of the most important examples, from [MS06] and [Boh12].

1.4. ACKNOWLEDGMENTS. I would like to thank my thesis advisor, Peter May, for many things too numerous to mention. I would also like to thank Anna Marie Bohmann and Marta Bunge, for providing the final impetus for publication. Finally, I would like to thank Hurricane Sandy, for providing a week free from other commitments, in which I was able to polish my notes into a readable paper; and the Institute for Advanced Study, for providing an electricity generator during that week.

2. Indexed monoidal categories

Let $\mathbf{S}$ be a category with finite products. We write $\Delta_X : X \to X \times X$ for the diagonal of $X \in \mathbf{S}$, and for related maps such as $X \times Y \to X \times X \times Y$. Similarly, we write π_X for any projection map in which X is projected away, such as $X \to 1$ or $X \times Y \to Y$. If this would be ambiguous, such as for the product projections $X \times X \to X$, we use numerical subscripts which again denote the copy being projected *away*; thus $\pi_1 : X \times X \to X$ is the projection *onto* the *second* copy.

Our enriched indexed categories will be indexed over $\mathbf{S}$. Their enrichment, on the other hand, will not be over a monoidal category in the classical sense, but over the following type of category.

2.1. DEFINITION. An **$\mathbf{S}$-indexed monoidal category** is a pseudofunctor $\mathscr{V} : \mathbf{S}^{\mathrm{op}} \to$ MonCat, where MonCat is the 2-category of monoidal categories, strong monoidal functors, and monoidal transformations.

As usual, we write the image of $X \in \mathbf{S}$ as $\mathscr{V}^X$, and the image of $f : X \to Y$ as $f^* : \mathscr{V}^Y \to \mathscr{V}^X$. We write the tensor product and unit of $\mathscr{V}^X$ as $\otimes_X$ and $\mathbb{I}_X$. The monoidality of f^* means we have isomorphisms such as

$$f^*A \otimes_X f^*B \;\cong\; f^*(A \otimes_Y B) \qquad \text{and} \qquad \mathbb{I}_X \cong f^*\mathbb{I}_Y.$$

Of course, by applying the "Grothendieck construction" we can equally regard $\mathscr{V}$ as a fibration $\int \mathscr{V} \to \mathbf{S}$. Moreover, the monoidal structures on the fibers $\mathscr{V}^X$ can equivalently be described by giving a monoidal structure on the category $\int \mathscr{V}$ such that

(i) the fibration $\int \mathscr{V} \to \mathbf{S}$ is strict monoidal, and

(ii) the tensor product of $\int \mathscr{V}$ preserves cartesian arrows.

In [Shu08] this is called a **monoidal fibration**; see there for a proof of the equivalence. If we write $\otimes$ and $\mathbb{I}$ for the monoidal structure and unit of $\int \mathscr{V}$, then the relationships between these and the fiberwise monoidal structures are as follows. For $A \in \mathscr{V}^X$ and $B \in \mathscr{V}^Y$, we have

$$A \otimes B = \pi_Y^* A \otimes_{X\times Y} \pi_X^* B \quad \in \mathscr{V}^{X\times Y} \qquad \text{and} \qquad \mathbb{I} = \mathbb{I}_1 \quad \in \mathscr{V}^1,$$

while for $A, B \in \mathscr{V}^X$ we have

$$A \otimes_X B = \Delta_X^*(A \otimes B) \quad \in \mathscr{V}^X \qquad \text{and} \qquad \mathbb{I}_X = \pi_X^*\mathbb{I} \quad \in \mathscr{V}^X.$$

We sometimes call $\otimes$ the **external** product, and $\otimes_X$ the **fiberwise** or **internal** one. Property (ii) gives us isomorphisms such as

$$f^*A \otimes g^*B \;\cong\; (f \times g)^*(A \otimes B).$$

We say that $\mathscr{V}$ is **symmetric** if the pseudofunctor $\mathscr{V} : \mathbf{S}^{\mathrm{op}} \to \mathrm{MonCat}$ lifts to SymMonCat; this is equivalent to asking $\int \mathscr{V}$ and the fibration $\int \mathscr{V} \to \mathbf{S}$ to be symmetric monoidal. We say that $\mathscr{V}$ is **cartesian** if each category $\mathscr{V}^X$ is cartesian monoidal, or equivalently if $\int \mathscr{V}$ is cartesian monoidal.

The two most important examples, corresponding to classical enrichment and classical internalization/indexing, are as follows.

2.2. Example. Let $\mathbf{V}$ be an ordinary monoidal category, let $\mathbf{S} = \mathrm{Set}$, and let $\mathscr{V}^X = \mathbf{V}^X$ be the category of X-indexed families of objects of $\mathbf{V}$ with the pointwise tensor product. We call this the **naive indexing** of $\mathbf{V}$ and write it as $\mathscr{F}am(\mathbf{V})$. Its total category $\int \mathscr{F}am(\mathbf{V})$ is the category $\mathrm{Fam}(\mathbf{V})$ of all set-indexed families of objects of $\mathbf{V}$, where a morphism from $(A_x)_{x\in X}$ to $(B_y)_{y\in Y}$ consists of a function $f : X \to Y$ and a family of morphisms $f_x : A_x \to B_{f(x)}$. The external product is defined by

$$(A \otimes B)_{(x,y)} = A_x \otimes B_y.$$

2.3. Example. Let $\mathbf{S}$ be a category with finite limits, and let $\mathscr{V}^X = \mathbf{S}/X$, with the cartesian product (which is pullback in $\mathbf{S}$). This is a cartesian monoidal fibration called the **self-indexing** of $\mathbf{S}$; we write it as $\mathscr{S}\mathit{elf}(\mathbf{S})$. Its total category $\int \mathscr{S}\mathit{elf}(\mathbf{S})$ is the category $\mathbf{S}^2$ of arrows in $\mathbf{S}$, its external product is just the cartesian product in $\mathbf{S}$.

See [Shu08] and [PS12] for further study of indexed monoidal categories; the latter includes an informal string diagram calculus.

Now, as is the case with classical enriched category theory, we frequently need completeness conditions on $\mathscr{V}$. By a **fiberwise** limit or colimit, we mean a limit or colimit in a fiber category $\mathscr{V}^X$ which is preserved by all functors f^*. If κ is a regular cardinal, we say that $\mathscr{V}$ is **fiberwise κ-complete** if it has all fiberwise limits of cardinality $< \kappa$, and we say $\mathscr{V}$ is **fiberwise complete** if it has all small fiberwise limits. Of course we have similar notions of fiberwise cocompleteness.

The other important sort of (co)limit for indexed categories is the following.

2.4. Definition. $\mathscr{V}$ has **S-indexed coproducts** if

(i) each functor $f^* : \mathscr{V}^Y \to \mathscr{V}^X$ has a left adjoint $f_!$, and

(ii) for any pullback square

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "k"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

in $\mathbf{S}$, the induced Beck-Chevalley transformation $k_! h^* \to g^* f_!$ is an isomorphism.

Dually, $\mathscr{V}$ has **S-indexed products** if each f^* has a right adjoint f_* satisfying an analogous condition.

It is well-known that the adjoints $f_!$ exist if and only if the fibration $\int \mathscr{V} \to \mathbf{S}$ is also an opfibration.

2.5. Example. $\mathscr{F}am(\mathbf{V})$ has any fiberwise limits and colimits that $\mathbf{V}$ has, and has Set-indexed (co)products iff $\mathbf{V}$ has (co)products.

2.6. Example. If $\mathbf{S}$ has finite limits, then $\mathscr{S}\mathit{elf}(\mathbf{S})$ has fiberwise finite limits. It is fiberwise complete if $\mathbf{S}$ is complete, and has any fiberwise colimits that $\mathbf{S}$ has. It always has $\mathbf{S}$-indexed coproducts, and has $\mathbf{S}$-indexed products if and only if $\mathbf{S}$ is locally cartesian closed.

We say that $\mathscr{V}$ is **κ-complete** if it is fiberwise κ-complete and has indexed products, and similarly it is **κ-cocomplete** if it is fiberwise κ-cocomplete and has indexed coproducts.

Now in the case when $\mathscr{V}$ has indexed coproducts, there is a third variant of the monoidal structure. For $A \in \mathscr{V}^{X\times Y}$ and $B \in \mathscr{V}^{Y \times Z}$, we define

$$A \otimes_{[Y]} B = \pi_{Y!} \Delta_Y^* (A \otimes B),$$

which lies in $\mathscr{V}^{X\times Z}$. We can also express this in terms of the fiberwise product as:

$$A \otimes_{[Y]} B \cong \pi_{Y!}\big(\pi_Z^* A \otimes_{X\times Y\times Z} \pi_X^* B\big).$$

We call this the **canceling product** because the object Y no longer appears in the base of the result. We have induced isomorphisms such as

$$(f\times 1)^* A \otimes_{[Y]} (1\times g)^* B \;\cong\; (f\times g)^*(A \otimes_{[Y]} B) \tag{2.7}$$

for $f : X' \to X$ and $g : Z' \to Z$.

2.8. Example. When $\mathscr{V} = \mathscr{F}am(\mathbf{V})$, the canceling product is

$$(A \otimes_{[Y]} B)_{(x,z)} = \coprod_{y\in Y} A_{(x,y)} \otimes B_{(y,z)}.$$

2.9. Example. When $\mathscr{V} = \mathscr{S}elf(\mathbf{S})$, the canceling product is just a pullback in $\mathbf{S}$, but which then forgets the map to the object we pulled back over.

Classically, in a monoidal category one often needs colimits to be preserved by the tensor product in each variable. For fiberwise colimits, we can simply impose this condition fiberwise. For indexed coproducts, the relevant condition is the following.

2.10. Definition. If $\mathscr{V}$ is a monoidal fibration with indexed coproducts, we say that $\otimes$ **preserves indexed coproducts** if for any $f\colon X \to Y$ in $\mathbf{S}$ and any $A \in \mathscr{C}^Y$ and $B \in \mathscr{C}^X$, the canonical map

$$f_!(f^* A \otimes_X B) \to A \otimes_Y f_! B \tag{2.11}$$

is an isomorphism, and symmetrically. (This condition is sometimes called "Frobenius reciprocity", or said to make the adjunction $f_! \dashv f^*$ into a "Hopf adjunction".)

An exercise in pasting mates implies that this condition is equivalent to the external product $\otimes$ preserving opcartesian arrows, yielding isomorphisms such as

$$(f\times g)_!(A\otimes B) \;\cong\; f_! A \otimes g_! B.$$

When this condition holds, the canceling product has its own derived commutativity isomorphisms, namely

$$(1\times f)^* A \otimes_{[X]} B \cong A \otimes_{[Z]} f_! B \tag{2.12}$$
$$(f\times 1)_! A \otimes_{[Y]} B \cong f_!\big(A \otimes_{[Y]} B\big)$$
$$(1\times f)_! A \otimes_{[Z]} B \cong A \otimes_{[X]} f^* B. \tag{2.13}$$

Finally, we consider what it means for a monoidal fibration to be closed. For simplicity, we consider only the symmetric case (in the non-symmetric case, we would have two homs of each type, a "right one" and a "left one").

2.14. Theorem. *Let $\mathscr{V}$ be an $\mathbf{S}$-indexed symmetric monoidal category with indexed products, and indexed coproducts preserved by $\otimes$. Then the following are equivalent.*

(i) Each fiber $\mathscr{V}^X$ is closed symmetric monoidal and each restriction functor f^* is closed symmetric monoidal. This means that for $B, C \in \mathscr{V}^X$ we have $\underline{\mathscr{V}}^X(B,C) \in \mathscr{V}^X$ and isomorphisms

$$\mathscr{V}^X(A \otimes_X B, C) \cong \mathscr{V}^X\left(A, \underline{\mathscr{V}}^X(B,C)\right),$$

natural in A, and moreover the canonical maps

$$f^*\underline{\mathscr{V}}^Y(B,C) \longrightarrow \underline{\mathscr{V}}^X(f^*B, f^*C) \tag{2.15}$$

are isomorphisms.

(ii) For any $B \in \mathscr{V}^Y$ and $C \in \mathscr{V}^{X\times Y}$, we have a $\underline{\mathscr{V}}^{[Y]}(B,C) \in \mathscr{V}^X$ and isomorphisms

$$\mathscr{V}^{X\times Y}(A \otimes B, C) \cong \mathscr{V}^X\left(A, \underline{\mathscr{V}}^{[Y]}(B,C)\right)$$

natural in A, and moreover the resulting canonical maps

$$f^*\underline{\mathscr{V}}^{[Y]}(B,C) \longrightarrow \underline{\mathscr{V}}^{[Y]}(B, (f\times 1)^*C)$$

are isomorphisms.

(iii) For any $B \in \mathscr{V}^Y$ and $C \in \mathscr{V}^X$ we have a $\underline{\mathscr{V}}(B,C) \in \mathscr{V}^{X\times Y}$ and isomorphisms

$$\mathscr{V}^X(A \otimes_{[Y]} B, C) \cong \mathscr{V}^{X\times Y}(A, \underline{\mathscr{V}}(B,C)) \tag{2.16}$$

natural in A, and moreover the resulting canonical maps

$$(f\times g)^*\underline{\mathscr{V}}(B,C) \to \underline{\mathscr{V}}(g^*B, f^*C)$$

are isomorphisms.

When these conditions hold, we say that $\mathscr{V}$ is ***closed****.*

Proof. The relationships between the three kinds of hom-functors are

$$\begin{array}{rclcl}
\underline{\mathscr{V}}^X(B,C) & \cong & \underline{\mathscr{V}}^{[X]}\big(B, \Delta_{X*}C\big) & \cong & \Delta_X^*\underline{\mathscr{V}}(B,C)\\
\underline{\mathscr{V}}(B,C) & \cong & \underline{\mathscr{V}}^{Y\times X}\big(\pi_X^*B, \pi_Y^*C\big) & \cong & \underline{\mathscr{V}}^{[Y]}\big(B, \Delta_{Y*}\pi_Y^*C\big)\\
\underline{\mathscr{V}}^{[Y]}(B,C) & \cong & \pi_{Y*}\underline{\mathscr{V}}^{X\times Y}\big(\pi_X^*B, C\big) & \cong & \pi_{Y*}\Delta_Y^*\underline{\mathscr{V}}(B,C).
\end{array}$$

Checking that the canonical maps coincide is an exercise in diagram chasing. The equivalence of (i) and (ii) can be found in [Shu08]. ∎

When the conditions of Theorem 2.14 hold, we say that $\mathscr{V}$ is **closed**. We call $\underline{\mathscr{V}}^X(-,-)$ the **fiberwise hom**, $\underline{\mathscr{V}}(-,-)$ the **external hom**, and $\underline{\mathscr{V}}^{[X]}(-,-)$ the **canceling hom**.

2.17. Example. If $\mathbf{V}$ is complete and cocomplete closed symmetric monoidal with internal-homs $\underline{\mathbf{V}}(-,-)$, then $\mathscr{F}am(\mathbf{V})$ is closed; we have

$$\begin{aligned}\underline{\mathscr{V}}^X(B,C) &= \Big(\underline{\mathbf{V}}(B_x,C_x)\Big)_{x\in X}\\ \underline{\mathscr{V}}^{[Y]}(B,C) &= \left(\prod_{y\in Y}\underline{\mathbf{V}}(B_y,C_{x,y})\right)_{x\in X}\\ \underline{\mathscr{V}}(B,C) &= \Big(\underline{\mathbf{V}}(B_x,C_y)\Big)_{x\in X,y\in Y}.\end{aligned}$$

2.18. Example. $\mathscr{S}elf(\mathbf{S})$ is closed just when $\mathbf{S}$ is locally cartesian closed.

2.19. Remark. The construction of the canceling hom from the fiberwise or external hom, and vice versa, do require indexed products as assumed. This is natural when looking at Example 2.17, in which the canceling hom involves a product whereas the other two do not.

On the other hand, the *definitions* of the fiberwise and external homs in terms of each other do not require any indexed products or coproducts, although the adjunction isomorphism (2.16) does require indexed coproducts since it involves the canceling tensor product. Thus, in the absence of any completeness or cocompleteness conditions on $\mathscr{V}$, we should define closedness by (i), and we are free to use the external hom defined by $\underline{\mathscr{V}}(B,C) = \underline{\mathscr{V}}^{Y\times X}(\pi_X^*B,\pi_Y^*C)$, although not its universal property (2.16). (In fact, the external hom does have a universal property even in the absence of indexed coproducts, but we defer mention of it until §11, where it will seem more natural.)

Classically, the tensor product in a closed monoidal category preserves colimits in each variable. It is similarly immediate that the tensor product in an indexed closed monoidal category preserves *fiberwise* colimits in each variable, while for indexed colimits we have:

2.20. Lemma. *If $\mathscr{V}$ is closed and has indexed coproducts, then its indexed coproducts are preserved by $\otimes$.*

Proof. The morphism (2.11) is a mate of (2.15), such that each is an isomorphism if and only if the other is. ∎

We can also define combination fiberwise/external/canceling products and homs, which satisfy a more symmetric-looking adjunction. If $A \in \mathscr{V}^{X\times Y\times Z}$, $B \in \mathscr{V}^{Y\times Z\times W}$, and $C \in \mathscr{V}^{X\times Y\times W}$, then we define

$$\begin{aligned}A \otimes_{Y,[Z]} B &= \pi_{Z!}\Delta^*_{Y\times Z}\big(A\otimes B\big) &&\cong \pi_{Z!}(\pi_W^*A \otimes_{X\times Y\times Z\times W} \pi_X^*B)\\ \underline{\mathscr{V}}^{Y,[W]}(B,C) &= \pi_{W*}\Delta^*_{Y\times W}\underline{\mathscr{V}}(B,C) &&\cong \pi_{W*}\underline{\mathscr{V}}^{X\times Y\times Z\times W}(\pi_X^*B,\pi_Z^*C).\end{aligned}$$

We then have

$$\mathscr{V}^{X\times Y\times W}\big(A \otimes_{Y,[Z]} B, C\big) \cong \mathscr{V}^{X\times Y\times Z}\big(A, \underline{\mathscr{V}}^{Y,[W]}(B,C)\big).$$

All the other products, homs, and adjunctions can be seen as special cases of these when X, Y, Z, and/or W are taken to be the terminal object 1, and in such cases we omit them from the notation.

Furthermore, when $\mathscr{V}$ is closed, the base change and tensor-hom adjunctions also automatically become enriched, in a suitable sense.

2.21. Proposition. *For any $f\colon X \to Y$ we have natural isomorphisms*

$$\underline{\mathscr{V}}^Y(f_!B, C) \;\cong\; f_*\underline{\mathscr{V}}^X(B, f^*C) \qquad \textit{and} \tag{2.22}$$

$$\underline{\mathscr{V}}^Y(C, f_*B) \;\cong\; f_*\underline{\mathscr{V}}^X(f^*C, B). \tag{2.23}$$

Proof. The isomorphism (2.23), for any colax/lax adjunction $f^* \dashv f_*$ between closed monoidal categories, is actually equivalent to f^* being strong monoidal (it is a mate of $f^*(A \otimes B) \cong f^*A \otimes f^*B$). The isomorphism (2.22) is perhaps less well-known, but a similar argument shows that given a chain of adjunctions $f_! \dashv f^* \dashv f_*$ between closed monoidal categories with f^* strong, the isomorphism (2.22) is equivalent to f^* being *closed* monoidal. ∎

2.24. Remark. In fact, (2.22) makes sense and is true even if $\mathscr{V}$ lacks indexed products in general, as an assertion that $\underline{\mathscr{V}}^Y(f_!B, C)$ has the universal property that $f_*\underline{\mathscr{V}}^X(B, f^*C)$ would have if it existed. The same is true of the first isomorphism in the following proposition.

2.25. Proposition. *For any $f\colon X \to Y$ we have natural isomorphisms*

$$\begin{aligned}
\underline{\mathscr{V}}(f_!B, C) &\cong (1\times f)_*\underline{\mathscr{V}}(B, C)\\
\underline{\mathscr{V}}(B, f^*C) &\cong (f\times 1)^*\underline{\mathscr{V}}(B, C)\\
\underline{\mathscr{V}}(f^*C, B) &\cong (1\times f)^*\underline{\mathscr{V}}(C, B)\\
\underline{\mathscr{V}}(C, f_*B) &\cong (f\times 1)_*\underline{\mathscr{V}}(C, B).
\end{aligned}$$

Proof. These are mates of the compatibility relations (2.7) and (2.12)–(2.13) for the canceling product. ∎

2.26. Proposition. *For any $f\colon X \to Y$ we have isomorphisms*

$$\begin{aligned}
\underline{\mathscr{V}}^{[Y]}(f_!B, C) &\cong \underline{\mathscr{V}}^{[X]}(B, (1\times f)^*C)\\
\underline{\mathscr{V}}^{[X]}(f^*B, C) &\cong \underline{\mathscr{V}}^{[Y]}(B, (1\times f)_*C)\\
f^*\underline{\mathscr{V}}^{[Z]}(B, C) &\cong \underline{\mathscr{V}}^{[Z]}(B, (f\times 1)^*C)\\
f_*\underline{\mathscr{V}}^{[Z]}(B, C) &\cong \underline{\mathscr{V}}^{[Z]}(B, (f\times 1)_*C)
\end{aligned}$$

Proof. These are the mates under adjunction of the compatibility relations such as $(1\times f)_!(A\otimes B) \cong A \otimes f_!B$. ∎

It is a standard result that in any closed symmetric monoidal category, such as $\mathscr{V}^X$, the ordinary hom-tensor adjunction isomorphism

$$\mathscr{V}^X(A \otimes_X B, C) \cong \mathscr{V}^X\big(A, \underline{\mathscr{V}}^X(B, C)\big)$$

enriches to an isomorphism of internal-hom objects in $\mathscr{V}^X$:

$$\underline{\mathscr{V}}^X(A \otimes_X B, C) \cong \underline{\mathscr{V}}^X\big(A, \underline{\mathscr{V}}^X(B, C)\big).$$

It follows that in a closed monoidal fibration we also have other enriched hom-tensor adjunction isomorphisms, such as the 'purely external' isomorphism

$$\underline{\mathscr{V}}(A \otimes B, C) \cong \underline{\mathscr{V}}(A, \underline{\mathscr{V}}(B, C)).$$

Bénabou has used the word *cosmos* for a complete and cocomplete closed symmetric monoidal category (the ideal situation for classical enriched category theory). By analogy, we define:

2.27. Definition. An **S-indexed cosmos** is an **S**-indexed closed symmetric monoidal category which is ω-complete and ω-cocomplete.

We have chosen ω as the cardinality of completeness and cocompleteness in this definition so as to make the notion an elementary one. That is, at least if we reformulate indexed categories using fibrations, then indexed cosmoi are models of a first-order theory, making them appropriate for foundational contexts such as category theory over a base topos. However, many indexed cosmoi arising in other applications are also fiberwise complete and cocomplete.

Any cosmos **V** in Bénabou's sense gives rise to a Set-indexed cosmos $\mathscr{F}am(\mathbf{V})$, although not every Set-indexed cosmos arises in this way. Moreover, we will see in §5 that any fiberwise cocomplete indexed cosmos gives rise to a cosmos in the sense of [Str81].

We now collect a number of further examples.

2.28. Example. For any **S**, and any ordinary monoidal category **V**, the constant pseudofunctor $X \mapsto \mathbf{V}$ is an **S**-indexed monoidal category with indexed products and coproducts preserved by $\otimes$. Limits and colimits in **V** give fiberwise limits and colimits, and it is closed if **V** is. The fiberwise, external, and canceling products and homs are all identical. We call this a *constant* indexed monoidal category and denote it by $\mathscr{C}onst(\mathbf{S}, \mathbf{V})$. Its total category $\int \mathscr{C}onst(\mathbf{S}, \mathbf{V})$ is $\mathbf{S} \times \mathbf{V}$.

2.29. Example. As a particular case of the previous example, we may take **S** to be the terminal category $\star$, in which case $\int \mathscr{C}onst(\star, \mathbf{V}) \cong \mathbf{V}$.

2.30. Example. Recall that Fam(**S**) denotes the category of all set-indexed families of objects of **S**. For any **S**-indexed category $\mathscr{V}$, there is a Fam(**S**)-indexed category $\mathscr{F}am(\mathscr{V})$, whose total category is $\mathrm{Fam}(\int \mathscr{V})$ and whose fiber over $(X_i)_{i\in I} \in \mathrm{Fam}(\mathbf{S})$ is $\prod_{i\in I} \mathscr{V}^{X_i}$. It inherits a monoidal structure, closedness, and fiberwise limits and colimits from $\mathscr{V}$, while it has indexed (co)products if and only if $\mathscr{V}$ has both indexed (co)products and small fiberwise ones.

Noting that $\mathrm{Fam}(\star) \cong \mathrm{Set}$, we see that by applying this construction to $\mathscr{C}onst(\star, \mathbf{V})$ we reproduce our original example $\mathscr{F}am(\mathbf{V})$ from Example 2.2.

2.31. Example. If $\mathscr{V}$ is an $\mathbf{S}$-indexed monoidal category and $F : \mathbf{S}' \to \mathbf{S}$ is any functor, then there is an $\mathbf{S}'$-indexed monoidal category $F^*\mathscr{V}$ defined by $(F^*\mathscr{V})^X = \mathscr{V}^{F(X)}$: the fiberwise monoidal structure of $\mathscr{V}$ passes immediately to $F^*\mathscr{V}$. (If F does not preserve finite products, then the external product of $F^*\mathscr{V}$ may differ from that of $\mathscr{V}$.)

2.32. Example. The category of topological spaces has pullbacks, but is not locally cartesian closed, so its self-indexing does not have indexed products and is not closed. Its subcategories of "compactly generated spaces" and "k-spaces" are cartesian closed, but still not locally cartesian closed. However, the references given in [MS06, §1.3] show that if we take $\mathbf{S}$ to be the category of *compactly generated* spaces, and for $X \in \mathbf{S}$ we take $\mathscr{K}^X$ to be the category of *k-spaces* over X, then we do obtain an indexed cosmos $\mathscr{K}$. Since not every k-space is compactly generated, this indexed cosmos $\mathscr{K}$ is larger than $\mathscr{S}elf(\mathbf{S})$ (which is not an indexed cosmos).

2.33. Example. If $\mathbf{S}$ is a category with finite limits and finite colimits which are preserved by pullback, then there is an $\mathbf{S}$-indexed monoidal category $\mathscr{S}elf_*(\mathbf{S})$ whose fiber $\mathscr{S}elf_*(\mathbf{S})^X$ is the category of *sectioned* objects over X. For such A and B, the *fiberwise smash product* is the following pushout.

$$\begin{tikzcd}
A \sqcup_X B \arrow[r] \arrow[d] & A \times_X B \arrow[d] \\
X \arrow[r] & A \wedge_X B.
\end{tikzcd}$$

This defines a monoidal structure with the unit object $X \to X \sqcup X \to X$, which has indexed coproducts preserved by $\wedge$. If $\mathbf{S}$ is locally cartesian closed, it is an indexed cosmos.

More generally, a similar construction can be applied to any $\mathscr{V}$ with fiberwise finite limits and colimits, with the fiberwise colimits preserved by $\otimes$. For instance, starting from Example 2.32 we obtain an indexed cosmos $\mathscr{K}_*$ of sectioned topological spaces.

2.34. Example. Let $\mathbf{S}$ be locally cartesian closed with countable colimits, and let $\mathscr{A}b(\mathbf{S})^X$ be the category of abelian group objects in $\mathbf{S}/X$. The countable colimits in $\mathbf{S}$ enable us to define free abelian group objects. Thus by [Joh02, D5.3.2], $\mathscr{A}b(\mathbf{S})$ has indexed products and coproducts and fiberwise finite limits and colimits. The countable colimits also enable us to define a tensor product, making $\mathscr{A}b(\mathbf{S})$ into an indexed cosmos.

More generally, if $\mathscr{V}$ is an $\mathbf{S}$-indexed cartesian cosmos with countable fiberwise colimits, we can define an $\mathbf{S}$-indexed cosmos $\mathscr{A}b(\mathscr{V})$ whose fiber over X is the category of abelian groups in $\mathscr{V}^X$. For example, from Example 2.32 we obtain an indexed cosmos of topological abelian groups.

2.35. Example. Let $\mathscr{V}$ be an $\mathbf{S}$-indexed cosmos and R a commutative monoid object in $\mathscr{V}^1$, so that $\pi_X^* R$ is a commutative monoid in $\mathscr{V}^X$ for any X. Let $({}_R\mathscr{V})^X$ be the category of objects in $\mathscr{V}^X$ with a $\pi_X^* R$-action. This is closed symmetric monoidal with finite limits and colimits; its tensor product is the coequalizer of the two actions

$$A \otimes_X (\pi_X^* R) \otimes_X B \rightrightarrows A \otimes_X B.$$

Since each f^* preserves $\pi^* R$-actions and their limits, tensor product, and hom, ${}_R\mathscr{V}$ is an $\mathbf{S}$-indexed closed symmetric monoidal category with finite fiberwise limits and colimits.

To show that it has indexed products and coproducts, we verify that $f_!$ and f_* preserve $\pi^* R$-actions. Let $f\colon X \to Y$; then if M is a $\pi_X^* R$-object in $\mathscr{V}^X$, we have

$$\begin{aligned} \pi_Y^* R \otimes_Y f_! M &\cong f_!(f^* \pi_Y^* R \otimes_X M) \\ &\cong f_!(\pi_X^* R \otimes_X M) \end{aligned}$$

so we can use the action $\pi_X^* R \otimes_X M \to M$ to induce an action of $\pi_Y^* R$ on $f_! M$. On the other hand, since f_* is lax monoidal (being the right adjoint of the strong monoidal f^*), it preserves monoids and their actions; thus $f_* \pi_X^* R$ is a monoid in $\mathscr{V}^Y$ and $f_* M$ is a $f_* \pi_X^* R$-object. However, $f_* \pi_X^* R = f_* f^* \pi_Y^* R$ and the unit $\pi_Y^* R \to f_* f^* \pi_Y^* R$ is a monoid homomorphism, so this induces a $\pi_Y^* R$-action on M. It is straightforward to check that these functors $f_!$ and f_* are left and right adjoints to f^* on $\pi^* R$-actions. The Beck-Chevalley condition follows from that for $\mathscr{V}$, so ${}_R\mathscr{V}$ is an indexed cosmos.

For example, if $(\mathbf{S}, R)$ is a ringed topos, such as the topos of sheaves on a scheme, then ${}_R\mathscr{A}b(\mathbf{S})$ is an indexed cosmos of sheaves of modules over the structure sheaf.

2.36. Example. Example 2.34 also works with the theory of abelian groups replaced by any other *finitary commutative theory* (see [Bor94a, 3.10]), such as the theory of R-modules for a fixed commutative ring R (in sets), or the theory of pointed sets. Applying the latter case to $\mathscr{S}elf(\mathbf{S})$, we recover Example 2.33.

In fact, we can also consider *internal* finitary commutative theories in $\mathbf{S}$, such as modules for an internal commutative ring object in $\mathbf{S}$. This gives another way to approach Example 2.35.

2.37. Example. On the other hand, if $\mathscr{V}$ is an $\mathbf{S}$-indexed *cartesian* indexed cosmos, and $G \in \mathscr{V}^1$ is any monoid (not necessarily commutative), then each category of $(\pi_X^* G)$-modules in $\mathscr{V}^X$ is also cartesian monoidal, with $(\pi_X^* G)$-action induced on the products by the diagonal of G. We denote this indexed monoidal category by $G\mathscr{V}$; note that when G is commutative, the underlying indexed category of $G\mathscr{V}$ is the same as that of ${}_G\mathscr{V}$. For instance, G could be a topological group in $\mathscr{K}$, yielding the indexed monoidal category of equivariant unsectioned spaces from [MS06]. Applying Example 2.33 we obtain the sectioned versions.

2.38. Example. For $\mathbf{S}$ with finite products, let $\mathrm{Grp}(\mathbf{S})$ denote the category of group objects in $\mathbf{S}$. For any such group object G, let $\mathscr{A}ct(\mathbf{S})^G = G\mathbf{S}$ denote the category of objects

$$\begin{tikzcd} G \times K \arrow[r, "f\times 1_K"] \arrow[d, "1_G\times g"'] & H \times K \arrow[d, "1_H\times g"] \\ G \times L \arrow[r, "f\times 1_L"'] & H \times L \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & G \times G \arrow[d, "1_G\times\Delta"] \\ G \times G \arrow[r, "\Delta\times 1_G"'] & G \times G \times G \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "(1_G{,}f)"] \arrow[d, "f"'] & G \times H \arrow[d, "f\times 1_H"] \\ H \arrow[r, "\Delta_H"'] & H \times H \end{tikzcd}$$

Figure 1: Homotopy pullback squares

with a G-action. With fiberwise cartesian monoidal structures (and G-actions induced by the diagonal), this yields a $\mathrm{Grp}(\mathbf{S})$-indexed cartesian monoidal category $\mathscr{A}ct(\mathbf{S})$.

Since $G\mathbf{S}$ is monadic over $\mathbf{S}$ with monad $(G \times -)$, $\mathscr{A}ct(\mathbf{S})$ has any fiberwise limits that $\mathbf{S}$ has, and any fiberwise colimits that $\mathbf{S}$ has and that are preserved by $\times$ in each variable. And if $\mathbf{S}$ is cartesian closed, then $\mathscr{A}ct(\mathbf{S})$ is closed, with the exponentials in $G\mathbf{S}$ being those of $\mathbf{S}$ with a conjugation action.

Now if $\mathbf{S}$ has coequalizers preserved by $\times$ in each variable, then the restriction functors $f^* : G \to H$ have left adjoints. Namely, for a group homomorphism $f : G \to H$ and $X \in G\mathbf{S}$, we define $f_!X$ to be the coequalizer of the two maps

$$H \times G \times X \rightrightarrows H \times X$$

induced by the action of G on X, and by f followed by the multiplication of H. Similarly, if $\mathbf{S}$ is cartesian closed and has equalizers, then f^* has a right adjoint, with f_*X defined to be the equalizer of the analogous pair of maps

$$X^H \rightrightarrows X^{G\times H}.$$

Unfortunately, these adjoints do *not* satisfy the Beck-Chevalley condition for all pullback squares in $\mathbf{S}$. However, we do have the Beck-Chevalley condition for three important classes of pullback squares, shown in Figure 1. (For example, this follows from [Joh02, B2.5.11].) Note that these squares are all pullbacks in any category with finite products, whether or not it has all pullbacks.

Only once or twice in this paper will we use the Beck-Chevalley condition for fully general pullback squares; in most cases we will only need it for these particular ones, along with their transposes, and their cartesian products with fixed objects. In [PS12], we said that an indexed category had **indexed homotopy coproducts** if its restriction functors f^* have left adjoints satisfying the Beck-Chevalley condition for these pullback squares. But since $\mathscr{A}ct(\mathbf{S})$ is the only example we will discuss in this paper[2] which fails

[2]There are other examples discussed in [PS12], in which the fiber categories $\mathscr{V}^X$ are "homotopy categories". However, these examples are usually not very interesting to enrich in, both because they tend to lack fiberwise limits and colimits, and because the resulting enriched categories would be only "up to unspecified homotopy", not up to *coherent* homotopy.

to have true indexed (co)products, we will not bother to add the adjective "homotopy" everywhere in this paper. Instead we will merely remark on the one or two places where the fully general Beck-Chevalley condition is used.

2.39. Example. If $\mathscr{V}$ is an $\mathbf{S}$-indexed monoidal category and $\mathbf{D}$ a small category, define $(\mathscr{V}^{\mathbf{D}})^X = (\mathscr{V}^X)^{\mathbf{D}}$, with the pointwise monoidal structure. Then $\mathscr{V}^{\mathbf{D}}$ is an $\mathbf{S}$-indexed monoidal category, which inherits closedness, fiberwise limits and colimits, and indexed products and coproducts from $\mathscr{V}$.

For example, if G is an ordinary monoid, we can consider it as a one-object category and obtain an indexed cosmos $\mathscr{V}^G$ of objects in $\mathscr{V}$ with a G-action. On the other hand, taking $\mathbf{D} = \mathbb{\Delta}^{\mathrm{op}}$ to be the simplex category and $\mathbf{S}$ a topos of sheaves, we obtain an indexed cosmos $\mathscr{S}\!\mathit{elf}(\mathbf{S})^{\mathbb{\Delta}^{\mathrm{op}}}$ of simplicial sheaves.

2.40. Example. Now let $\mathbf{D}$ be a small *monoidal* category and $\mathscr{V}$ a fiberwise complete and cocomplete $\mathbf{S}$-indexed cosmos. Then the category $(\mathscr{V}^X)^{\mathbf{D}}$ is also closed monoidal under the *Day convolution product*; see [Day70]. This gives a different monoidal structure to the indexed category $\mathscr{V}^{\mathbf{D}}$. Since the Day monoidal structure and internal-homs are constructed out of limits and colimits from those in $\mathscr{V}^X$, and f^* preserves all of these, the transition functors f^* are also strong and closed with respect to the convolution product. Thus we have a second indexed cosmos structure on $\mathscr{V}^{\mathbf{D}}$ in this case.

For example, if we take Σ to be the category of finite sets and permutations and $\mathscr{V}$ to be the indexed cosmos $\mathscr{K}_*$ of sectioned spaces, then $\mathscr{K}_*^{\Sigma}$ is the indexed cosmos of symmetric sequences of sectioned spaces. The spheres give a canonical commutative monoid S in $(\mathscr{K}_*^{\Sigma})^1$ with respect to the convolution product, so by Example 2.35 we have an indexed cosmos ${}_S(\mathscr{K}_*^{\Sigma})$ of *parametrized (topological) symmetric spectra*. We will consider parametrized orthogonal spectra in §11.

2.41. Example. Suppose $\mathbf{S}$ is equipped with an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ which is *stable* in that $\mathcal{E}$ is preserved by pullback. Then defining $\mathcal{M}(\mathbf{S})^X = \mathcal{M}/X$ gives an $\mathbf{S}$-indexed monoidal category. For instance, if $\mathbf{S}$ is a regular category, then $(\mathcal{E}, \mathcal{M})$ could be (regular epi, mono), in which case $\mathcal{M}(\mathbf{S})^X$ is the poset of subobjects of X.

2.42. Example. Let $\mathbf{S}$ have pullbacks and $\mathbf{V}$ be an ordinary monoidal category (such as Set), and define $\mathscr{P}\!\mathit{sh}(\mathbf{S}, \mathbf{V})^X = \mathbf{V}^{(\mathbf{S}/X)^{\mathrm{op}}}$, with the pointwise monoidal structure. The functor f^* is defined by precomposition with $f : \mathbf{S}/X \to \mathbf{S}/Y$. Since $\mathbf{S}$ has pullbacks, $f : \mathbf{S}/X \to \mathbf{S}/Y$ has a right adjoint, hence so does f^*. The Beck-Chevalley condition follows from that for pullbacks in $\mathbf{S}$, so the resulting $\mathbf{S}$-indexed monoidal category $\mathscr{P}\!\mathit{sh}(\mathbf{S}, \mathbf{V})$ has indexed products. It also inherits fiberwise limits and colimits from $\mathbf{V}$.

If $\mathbf{S}$ is small and $\mathbf{V}$ is complete and cocomplete, then $\mathscr{P}\!\mathit{sh}(\mathbf{S}, \mathbf{V})$ is closed and has indexed coproducts, hence is an $\mathbf{S}$-indexed cosmos. If $\mathbf{S}$ is not small, then $\mathscr{P}\!\mathit{sh}(\mathbf{S}, \mathbf{V})$ may not have these properties, but if $\mathbf{S}$ is *locally* small and $\mathbf{V}$ is complete, cocomplete, and closed (i.e. a classical Bénabou cosmos), then it has an important class of them.

Namely, for any $Z \xrightarrow{g} X$, consider the *representable* presheaf $F_g \in \mathbf{V}^{(\mathbf{S}/X)^{\mathrm{op}}}$, defined by $F_g(W \xrightarrow{h} X) = (\mathbf{S}/X)(h, g) \cdot \mathbb{I}$, a copower of copies of the unit object $\mathbb{I}$ of $\mathbf{V}$. Now as in [DL07], we define an object of $\mathbf{V}^{(\mathbf{S}/X)^{\mathrm{op}}}$ to be **small** if it is a small ($\mathbf{V}$-weighted) colimit of such representables. Representable objects are closed under restriction, since $f^*(F_g) \cong F_{f^*g}$; hence so are small objects. Similarly, representable objects are closed under tensor products, since $F_g \otimes_X F_h \cong F_{g\times_X h}$; hence so are small objects.

Now $f^* : \mathbf{V}^{(\mathbf{S}/Y)^{\mathrm{op}}} \to \mathbf{V}^{(\mathbf{S}/X)^{\mathrm{op}}}$ has a partial left adjoint $f_!$ defined on small objects, which takes them to small objects: we define $f_!(F_g) = F_{fg}$ and extend cocontinuously. Similarly, all homs $\underline{\mathscr{V}}^?(A, B)$ exist when A is small: we define $\underline{\mathscr{V}}^X(F_g, B)(W \xrightarrow{h} X) = B(g \times_X h)$ and extend cocontinuously, and construct the other homs from this as usual.

3. Small $\mathscr{V}$-categories

Let $\mathbf{S}$ be a category with finite products and $\mathscr{V}$ an $\mathbf{S}$-indexed monoidal category, with corresponding fibration $\int\mathscr{V} \to \mathbf{S}$. In this section we describe a notion of "small $\mathscr{V}$-category" which directly generalizes internal categories and small enriched categories. (In §5 we will see that there is also another, less elementary, notion of "smallness" for $\mathscr{V}$-categories.)

We will use the following notation:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, mapsto] & B \arrow[d, mapsto] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

to indicate that $\phi : A \to B$ is a morphism in $\int\mathscr{V}$ lying over $f : X \to Y$ in $\mathbf{S}$. Of course, to give such a ϕ is equivalent to giving a morphism $A \to f^*B$ in $\mathscr{V}^X$, but using morphisms in $\int\mathscr{V}$ often makes commutative diagrams less busy (since there are fewer f^*'s to notate).

3.1. Definition. A **small $\mathscr{V}$-category** A consists of:

(i) An object $\epsilon A \in \mathbf{S}$.

(ii) An object $\underline{A} \in \mathscr{V}^{\epsilon A\times\epsilon A}$.

(iii) A morphism in $\int\mathscr{V}$:

$$\begin{tikzcd} \mathbb{I}_{\epsilon A} \arrow[r, "\mathrm{ids}"] \arrow[d, mapsto] & \underline{A} \arrow[d, mapsto] \\ \epsilon A \arrow[r, "\Delta"'] & \epsilon A \times \epsilon A \end{tikzcd}$$

(iv) A morphism in $\int\mathscr{V}$:

$$\begin{tikzcd} \underline{A} \otimes_{\epsilon A} \underline{A} \arrow[r, "\mathrm{comp}"] \arrow[d, mapsto] & \underline{A} \arrow[d, mapsto] \\ \epsilon A \times \epsilon A \times \epsilon A \arrow[r, "\pi_2"'] & \epsilon A \times \epsilon A \end{tikzcd}$$

in which the fiberwise tensor product is over the middle copy of ϵA.

(v) The following diagrams commute:

$$\begin{tikzcd}
\underline{A} \otimes_{\epsilon A} (\underline{A} \otimes_{\epsilon A} \underline{A}) \arrow[r, "\cong"] \arrow[d, "\mathrm{comp}"'] & (\underline{A} \otimes_{\epsilon A} \underline{A}) \otimes_{\epsilon A} \underline{A} \arrow[r, "\mathrm{comp}"] & \underline{A} \otimes_{\epsilon A} \underline{A} \arrow[d, "\mathrm{comp}"] \\
\underline{A} \otimes_{\epsilon A} \underline{A} \arrow[rr, "\mathrm{comp}"'] & & \underline{A}
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{A} \arrow[r, "\cong"] \arrow[drr, bend right, "="'] & \mathbb{I}_{\epsilon A} \otimes_{\epsilon A} \underline{A} \arrow[r, "\mathrm{ids}"] & \underline{A} \otimes_{\epsilon A} \underline{A} \arrow[d, "\mathrm{comp}"] & \underline{A} \otimes_{\epsilon A} \mathbb{I}_{\epsilon A} \arrow[l, "\mathrm{ids}"'] & \underline{A} \arrow[l, "\cong"'] \arrow[dll, bend left, "="] \\
& & \underline{A} & &
\end{tikzcd}$$

3.2. Example. If $\mathbf{V}$ is an ordinary monoidal category, then a small $\mathscr{F}am(\mathbf{V})$-category is precisely a small $\mathbf{V}$-enriched category. It has a set ϵA, an $(\epsilon A \times \epsilon A)$-indexed family

$$(\underline{A}(a,b))_{(b,a)\in\epsilon A\times\epsilon A}$$

of objects of $\mathbf{V}$, an identities map with components $\mathbb{I} \to \underline{A}(a,a)$, and a composition map with components $\underline{A}(b,c) \otimes \underline{A}(a,b) \longrightarrow \underline{A}(a,c)$.

3.3. Remark. In $\epsilon A \times \epsilon A$, we interpret the *first* copy of ϵA as the *codomain* and the *second* copy as the *domain*; hence the reversal of order in the subscript in (3.2). This is so that we end up composing morphisms in the usual order.

3.4. Examples. If $\mathbf{S}$ has finite limits, then a small $\mathscr{S}elf(\mathbf{S})$-category is precisely a category internal to $\mathbf{S}$. Similarly, if $\mathcal{M}$ is the class of monomorphisms in $\mathbf{S}$ as in Example 2.41, then a small $\mathscr{M}(\mathbf{S})$-category is precisely an internal poset in $\mathbf{S}$.

When $\mathbf{S}$ satisfies the hypotheses of Example 2.33, then a small $\mathscr{S}elf_*(\mathbf{S})$-category may be called a *pointed* $\mathbf{S}$*-internal category*. It consists of an $\mathbf{S}$-internal category $A_1 \rightrightarrows A_0$ together with a morphism $A_0 \times A_0 \to A_1$ assigning to every pair of objects a "zero morphism" between them, which is preserved by composition on each side.

3.5. Example. For $\mathbf{S}$ and $\mathbf{V}$ as in Example 2.28, a small $\mathscr{C}onst(\mathbf{S},\mathbf{V})$-category consists of an object of $\mathbf{S}$ together with a monoid in $\mathbf{V}$.

3.6. Example. A small $\mathscr{A}b(\mathbf{S})$-category consists of an internal category $A_1 \rightrightarrows A_0$ in $\mathbf{S}$, together with the structure of an abelian group on the object A_1 of $\mathbf{S}/(A_0 \times A_0)$ which is preserved by composition in each variable.

3.7. Examples. A small ${}_R\mathscr{V}$-category is a small $\mathscr{V}$-category with an action of $(\pi_{(\epsilon A\times\epsilon A)})^*R$ on $\underline{A} \in \mathscr{V}^{\epsilon A\times\epsilon A}$, which is suitably preserved by the composition in each variable. By contrast, a small $G\mathscr{V}$-category is a small $\mathscr{V}$-category with an action of $(\pi_{(\epsilon A\times\epsilon A)})^*G$ on $\underline{A}$, which is preserved by the composition in both variables together.

3.8. Definition. If $\mathscr{V}$ has indexed coproducts preserved by $\otimes$, then for any object $X \in \mathbf{S}$, there is a small $\mathscr{V}$-category δX with $\epsilon(\delta X) = X$ and $\underline{\delta X} = (\Delta_X)_! \mathbb{I}_X$. We call it the **discrete $\mathscr{V}$-category on X**.

3.9. Definition. Let A and B be small $\mathscr{V}$-categories. A **$\mathscr{V}$-functor** $f\colon A \to B$ consists of:

(i) A morphism $\epsilon f\colon \epsilon A \to \epsilon B$ in $\mathbf{S}$.

(ii) A morphism in $\int\mathscr{V}$:

$$\begin{tikzcd}
\underline{A} \arrow[r, "f"] \arrow[d, mapsto] & \underline{B} \arrow[d, mapsto] \\
\epsilon A \times \epsilon A \arrow[r, "\epsilon f \times \epsilon f"'] & \epsilon B \times \epsilon B.
\end{tikzcd}$$

(iii) The following diagrams commute:

$$\begin{tikzcd}
\mathbb{I}_{\epsilon A} \arrow[r, "\mathrm{ids}"] \arrow[d] & \underline{A} \arrow[d, "f"] \\
\mathbb{I}_{\epsilon B} \arrow[r, "\mathrm{ids}"'] & \underline{B}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\underline{A} \otimes_{\epsilon A} \underline{A} \arrow[r, "\mathrm{comp}"] \arrow[d, "f \otimes f"'] & \underline{A} \arrow[d, "f"] \\
\underline{B} \otimes_{\epsilon B} \underline{B} \arrow[r, "\mathrm{comp}"'] & \underline{B}.
\end{tikzcd}$$

3.10. Examples. Evidently a $\mathscr{F}am(\mathbf{V})$-functor is a $\mathbf{V}$-enriched functor, and a $\mathscr{S}elf(\mathbf{S})$-functor is an $\mathbf{S}$-internal functor. The other specific examples are similar.

3.11. Remark. If δX is the discrete $\mathscr{V}$-category on $X \in \mathbf{S}$ as in Definition 3.8, then a $\mathscr{V}$-functor $f : \delta X \to A$ is uniquely determined by its underlying morphism $\epsilon f : X \to \epsilon A$ in $\mathbf{S}$ (the unit axiom forces its action on homs to be induced by $\mathrm{ids} : \mathbb{I}_{\epsilon A} \to \underline{A}$).

If $\mathscr{V}$ lacks indexed coproducts, in which case δX may not exist as a $\mathscr{V}$-category, it is nevertheless often convenient to abuse language and allow the phrase "$\mathscr{V}$-functor $\delta X \to A$" to refer simply a morphism $X \to \epsilon A$ in $\mathbf{S}$.

3.12. Definition. Let $f, g\colon A \to B$ be $\mathscr{V}$-functors. A **$\mathscr{V}$-natural transformation** $\alpha\colon f \to g$ consists of:

(i) A morphism

$$\begin{tikzcd}
\mathbb{I}_{\epsilon A} \arrow[r, "\alpha"] \arrow[d, mapsto] & \underline{B} \arrow[d, mapsto] \\
\epsilon A \arrow[r, "(\epsilon g, \epsilon f)"'] & \epsilon B \times \epsilon B.
\end{tikzcd} \tag{3.13}$$

(ii) The following diagram commutes.

$$\begin{tikzcd}
\underline{A} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \underline{A} \otimes_{\epsilon A} \mathbb{I}_{\epsilon A} \arrow[r, "f \otimes \alpha"] & \underline{B} \otimes_{\epsilon B} \underline{B} \arrow[d, "\mathrm{comp}"] \\
\mathbb{I}_{\epsilon A} \otimes_{\epsilon A} \underline{A} \arrow[r, "\alpha \otimes g"'] & \underline{B} \otimes_{\epsilon B} \underline{B} \arrow[r, "\mathrm{comp}"'] & \underline{B}.
\end{tikzcd} \tag{3.14}$$

3.15. Examples. A $\mathscr{S}elf(\mathbf{S})$-natural transformation is an $\mathbf{S}$-internal natural transformation, and a $\mathscr{F}am(\mathbf{V})$-natural transformation is a $\mathbf{V}$-enriched one. The other specific examples are similar.

3.16. Remark. If δX is a discrete $\mathscr{V}$-category as in Definition 3.8 and $f, g : \delta X \to A$ are determined by morphisms $\epsilon f, \epsilon g : X \to \epsilon A$ as in Remark 3.11, then a $\mathscr{V}$-natural transformation $\alpha : f \to g$ consists solely of a morphism (3.13) (the axiom (3.14) is automatic). Thus, as in Remark 3.11, we may abuse language by referring to a morphism (3.13) as a "$\mathscr{V}$-natural transformation" between "$\mathscr{V}$-functors $\delta X \to A$" even when $\mathscr{V}$ lacks indexed coproducts.

3.17. Theorem. *Small $\mathscr{V}$-categories, $\mathscr{V}$-functors, and $\mathscr{V}$-natural transformations form a 2-category $\mathscr{V}$-$\mathcal{C}at$.*

Proof. The composition of $\mathscr{V}$-functors is obvious. The composition of $\mathscr{V}$-natural transformations $\alpha : f \to g$ and $\beta : g \to h$ has components

$$\mathbb{I}_{\epsilon A} \cong \mathbb{I}_{\epsilon A} \otimes_{\epsilon A} \mathbb{I}_{\epsilon A} \xrightarrow{\alpha \otimes \beta} \underline{B} \otimes_{\epsilon B} \underline{B} \xrightarrow{\text{comp}} \underline{B}.$$

The "whiskering" of a natural transformation on either side by a functor is likewise obvious; we leave the verification of the axioms to the reader. ∎

3.18. Example. When $\mathscr{V}$ has indexed coproducts preserved by $\otimes$, the "discrete $\mathscr{V}$-category" operation defines a 2-functor $\delta : \mathbf{S} \to \mathscr{V}\text{-}\mathcal{C}at$. Thus, any small $\mathscr{V}$-category A induces an indexed category

$$\mathscr{V}\text{-}\mathcal{C}at(\delta -, A) : \mathbf{S}^{\mathrm{op}} \to \mathrm{Cat}.$$

The same construction works formally even if $\mathscr{V}$ lacks indexed coproducts, using the conventions of Remarks 3.11 and 3.16. In §7 we will identify this "underlying indexed category" with a special case of a general "change of cosmos" construction.

3.19. Definition. Let A and B be small $\mathscr{V}$-categories. A **$\mathscr{V}$-profunctor** $H : A \nrightarrow B$ consists of:

(i) An object $\underline{H} \in \mathscr{V}^{\epsilon A \times \epsilon B}$.

(ii) Morphisms in $\int \mathscr{V}$:

$$\begin{tikzcd} \underline{A} \otimes_{\epsilon A} \underline{H} \arrow[r, "\text{act}"] \arrow[d, mapsto] & \underline{H} \arrow[d, mapsto] \\ \epsilon A \times \epsilon A \times \epsilon B \arrow[r, "1 \times \pi \times 1"'] & \epsilon A \times \epsilon B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \underline{H} \otimes_{\epsilon B} \underline{B} \arrow[r, "\text{act}"] \arrow[d, mapsto] & \underline{H} \arrow[d, mapsto] \\ \epsilon A \times \epsilon B \times \epsilon B \arrow[r, "1 \times \pi \times 1"'] & \epsilon A \times \epsilon B \end{tikzcd}$$

(iii) The following diagrams commute.

$$\begin{tikzcd}
\underline{A} \otimes_{\epsilon A} \underline{A} \otimes_{\epsilon A} \underline{H} \arrow[r, "1\otimes \mathrm{act}"] \arrow[d, "\mathrm{comp}\otimes 1"'] & \underline{A} \otimes_{\epsilon A} \underline{H} \arrow[d, "\mathrm{act}"] \\
\underline{A} \otimes_{\epsilon A} \underline{H} \arrow[r, "\mathrm{act}"'] & \underline{H}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{H} \otimes_{\epsilon B} \underline{B} \otimes_{\epsilon B} \underline{B} \arrow[r, "\mathrm{act}\otimes 1"] \arrow[d, "1\otimes \mathrm{comp}"'] & \underline{H} \otimes_{\epsilon B} \underline{B} \arrow[d, "\mathrm{act}"] \\
\underline{H} \otimes_{\epsilon B} \underline{B} \arrow[r, "\mathrm{act}"'] & \underline{H}
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{H} \arrow[r, "\mathrm{ids}"] \arrow[dr, "="'] & \underline{A} \otimes_{\epsilon A} \underline{H} \arrow[d, "\mathrm{act}"] \\
& \underline{H}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{H} \arrow[r, "\mathrm{ids}"] \arrow[dr, "="'] & \underline{H} \otimes_{\epsilon B} \underline{B} \arrow[d, "\mathrm{act}"] \\
& \underline{H}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{A} \otimes_{\epsilon A} \underline{H} \otimes_{\epsilon B} \underline{B} \arrow[r, "\mathrm{act}\otimes 1"] \arrow[d, "1\otimes \mathrm{act}"'] & \underline{H} \otimes_{\epsilon B} \underline{B} \arrow[d, "\mathrm{act}"] \\
\underline{A} \otimes_{\epsilon A} \underline{H} \arrow[r, "\mathrm{act}"'] & \underline{H}.
\end{tikzcd}$$

We have an obvious notion of **morphism of profunctors**, yielding a category which we denote $\mathscr{V}\text{-}\mathcal{P}rof(A, B)$.

3.20. Examples. A $\mathscr{F}am(\mathbf{V})$-profunctor is equivalent to a $\mathbf{V}$-functor $B^{\mathrm{op}} \otimes A \to \mathbf{V}$, which is the classical notion of enriched profunctor; see for example [Bor94b]. Similarly, $\mathscr{S}elf(\mathbf{S})$-profunctors give the usual notion of internal profunctor; see e.g. [Joh02, §B2.7].

3.21. Example. For any small $\mathscr{V}$-category A, there is a **unit** $\mathscr{V}$-profunctor $A \colon A \nrightarrow A$ defined by the hom-object $\underline{A}$. The action maps are simply composition in $\underline{A}$.

3.22. Remark. If A or B is a discrete $\mathscr{V}$-category as in Definition 3.8, then the corresponding action map is necessarily the unitality isomorphism. In particular, a profunctor $\delta X \nrightarrow \delta Y$ is simply an object of $\mathscr{V}^{X\times Y}$, while a profunctor $\delta X \nrightarrow B$ or $A \nrightarrow \delta Y$ is simply an object of $\mathscr{V}^{X\times \epsilon B}$ or $\mathscr{V}^{\epsilon A\times Y}$ with a one-sided action of B or A, as appropriate. Thus, as in Remarks 3.11 and 3.16, we allow ourselves to abuse language by referring to such data as "$\mathscr{V}$-profunctors" even if $\mathscr{V}$ lacks indexed coproducts.

3.23. Example. If $H : A \nrightarrow B$ is a $\mathscr{V}$-profunctor and $f : A' \to A$ and $g : B' \to B$ are $\mathscr{V}$-functors, then there is a $\mathscr{V}$-profunctor $H(g, f) : A' \nrightarrow B'$ defined by

$$\underline{H(g, f)} = (\epsilon g \times \epsilon f)^* \underline{H}.$$

The action maps are defined by composing the action maps of $\underline{H}$ with f.

In particular, from any $\mathscr{V}$-functor $f : A \to B$ and the unit profunctor $B : B \nrightarrow B$, we obtain **representable profunctors** $B(1, f) : A \nrightarrow B$ and $B(f, 1) : B \nrightarrow A$.

Classically, profunctors can be composed with a 'tensor product of functors'. In the enriched case, the composite of $H \colon B^{\mathrm{op}} \otimes A \to \mathbf{V}$ and $K \colon C^{\mathrm{op}} \otimes B \to \mathbf{V}$ is the coend

$$\begin{aligned}
(H \odot K)(c, a) &= \int^B H(b, a) \otimes K(c, b) \\
&= \mathrm{coeq}\left(\coprod_{b_1, b_2 \in B} H(b_2, a) \otimes B(b_1, b_2) \otimes K(c, b_1) \rightrightarrows \coprod_{b\in B} H(b, a) \otimes K(c, b) \right).
\end{aligned}$$

3.24. Remark. We write the composite of profunctors in "diagrammatic" order, so that $H : A \nrightarrow B$ and $K : B \nrightarrow C$ yield $H \odot K : A \nrightarrow C$.

By making these colimits indexed or fiberwise, as appropriate, we obtain:

3.25. Lemma. *If $\mathscr{V}$ has indexed coproducts preserved by $\otimes$ and fiberwise coequalizers then any two $\mathscr{V}$-profunctors $H : A \nrightarrow B$ and $K : B \nrightarrow C$ have a* ***composite*** *defined by*

$$\underline{H \odot K} = \operatorname{coeq}\Big(\underline{H} \otimes_{[\epsilon B]} \underline{B} \otimes_{[\epsilon B]} \underline{K} \rightrightarrows \underline{H} \otimes_{[\epsilon B]} \underline{K}\Big).$$

Composition of profunctors is associative up to coherent isomorphism, with units as in Example 3.21, yielding a bicategory $\mathscr{V}$-$\mathcal{P}rof$.

Proof. Straightforward. The compatibility conditions for indexed coproducts (including the Beck-Chevalley condition), and the preservation of fiberwise coequalizers by restriction, are necessary to give $\underline{H \odot K}$ actions by A and C, and to show associativity. ∎

In fact, this is a formal consequence of known results. It is shown in [Shu08] that if $\mathscr{V}$ has indexed coproducts preserved by $\otimes$, then it gives rise to a "framed bicategory", or equivalently a "proarrow equipment" in the sense of [Woo82, Woo85], which we may denote $\mathbb{M}\mathsf{at}(\mathscr{V})$. If $\mathscr{V}$ furthermore has fiberwise coequalizers, then we can form the further equipment $\mathbb{M}\mathsf{od}(\mathbb{M}\mathsf{at}(\mathscr{V}))$ as defined in [Shu08] (which according to [GS13] is the free cocompletion of $\mathbb{M}\mathsf{at}(\mathscr{V})$ under tight Kleisli objects). The equipment $\mathbb{M}\mathsf{od}(\mathbb{M}\mathsf{at}(\mathscr{V}))$ consists of small $\mathscr{V}$-categories, $\mathscr{V}$-functors, and $\mathscr{V}$-profunctors, and so we might denote it $\mathscr{V}$-$\mathbb{P}\mathsf{rof}$. Its bicategory of proarrows is precisely $\mathscr{V}$-$\mathcal{P}rof$.

If $\mathscr{V}$ lacks indexed coproducts, then a construction analogous to that in [Shu08] produces instead a *virtual equipment* in the sense of [CS10], and thereby another virtual equipment $\mathscr{V}$-$\mathbb{P}\mathsf{rof}$. We leave the details to the reader, but we note the following corollaries. We refer the proofs to the cited references, but no knowledge of equipments or framed bicategories will be necessary for the rest of this paper, so the reader is free to take these results on faith or to re-prove them by hand (which is not difficult).

3.26. Lemma. *For $\mathscr{V}$-functors $f, g : A \to B$, there are natural bijections*

$$\begin{aligned}\mathscr{V}\text{-}\mathcal{C}at(A,B)(f,g) &\cong \mathscr{V}\text{-}\mathcal{P}rof(A,B)(B(1,f),B(1,g))\\ &\cong \mathscr{V}\text{-}\mathcal{P}rof(B,A)(B(g,1),B(f,1)).\end{aligned}$$

When $\mathscr{V}$ has indexed coproducts preserved by $\otimes$ and fiberwise coequalizers, the first of these is the action on 2-cells of a locally fully faithful pseudofunctor $\mathscr{V}$-$\mathcal{C}at \to \mathscr{V}$-$\mathcal{P}rof$, which is the identity on objects and sends $f : A \to B$ to $B(1,f) : A \nrightarrow B$. Furthermore, we have an adjunction $B(1,f) \dashv B(f,1)$ in $\mathscr{V}$-$\mathcal{P}rof$.

Proof. The first statement is [CS10, Cor. 7.22] applied in the virtual equipment $\mathscr{V}$-$\mathcal{P}rof$, together with [CS10, Prop. 6.2] to identify the left-hand side with $\mathscr{V}$-natural transformations as we have defined them. The rest is [Shu08, Props. 4.5 and 5.3]. ∎

3.27. Lemma. *If $\mathscr{V}$ is an indexed cosmos, then $\mathscr{V}$-$\mathcal{P}rof$ is* closed *in that we have natural isomorphisms*

$$\mathscr{V}\text{-}\mathcal{P}rof(A,C)(H \odot K, L) \cong \mathscr{V}\text{-}\mathcal{P}rof(A,B)(H, K \rhd L) \cong \mathscr{V}\text{-}\mathcal{P}rof(B,C)(K, L \lhd H).$$

Proof. By [Shu08, Theorems 14.2 and 11.5], or as a direct construction using fiberwise equalizers:

$$\underline{K \rhd L} = \mathrm{eq}\Big(\mathscr{V}^{[\epsilon C]}(\underline{K},\underline{L}) \rightrightarrows \mathscr{V}^{[\epsilon C]}(\underline{K}, \mathscr{V}^{[\epsilon C]}(\underline{C},\underline{L}))\Big)$$
$$\underline{L \lhd H} = \mathrm{eq}\Big(\mathscr{V}^{[\epsilon A]}(\underline{H},\underline{L}) \rightrightarrows \mathscr{V}^{[\epsilon A]}(\underline{H}, \mathscr{V}^{[\epsilon A]}(\underline{A},\underline{L}))\Big).$$ ■

We also have a couple versions of the Yoneda lemma.

3.28. Lemma. *For $f : A \to B$, $H : C \nrightarrow B$, and $K : B \nrightarrow C$, there are natural bijections*

$$\mathscr{V}\text{-}\mathcal{P}rof(B(1,f), H) \cong \mathscr{V}\text{-}\mathcal{P}rof(A, H(f,1)) \qquad \textit{and}$$
$$\mathscr{V}\text{-}\mathcal{P}rof(B(f,1), K) \cong \mathscr{V}\text{-}\mathcal{P}rof(A, K(1,f)).$$

Proof. By [CS10, Theorems 7.16 and 7.20]. ■

3.29. Lemma. *If $\mathscr{V}$ is an indexed cosmos, $f\colon A \to B$, $H\colon B \nrightarrow C$, and $K : C \nrightarrow B$ then we have canonical isomorphisms*

$$\begin{array}{ccccccc} H(1,f) & \cong & H \lhd B(f,1) & \cong & B(1,f) \odot H & \textit{and} \\ K(f,1) & \cong & B(1,f) \rhd K & \cong & K \odot B(f,1). \end{array}$$

Proof. By [Shu08, Prop. 5.11]. ■

In particular, the Yoneda lemma implies the usual sort of hom-object characterization of adjunctions.

3.30. Proposition. *For $\mathscr{V}$-functors $f\colon A \to B$ and $g\colon B \to A$, there is a bijection between*

(i) Adjunctions $f \dashv g$ in $\mathscr{V}$-$\mathcal{C}at$, and

(ii) Isomorphisms $B(f,1) \cong A(1,g)$ of $\mathscr{V}$-profunctors $B \nrightarrow A$.

Proof. Since $B(f,1)$ is always right adjoint to $B(1,f)$, to give $B(f,1) \cong A(1,g)$ is equivalent to giving data exhibiting $A(1,g)$ as right adjoint to $B(1,f)$. By Lemma 3.26, this is equivalent to data exhibiting g as right adjoint to f. (As stated, this argument requires the bicategory $\mathscr{V}$-$\mathcal{P}rof$ to exist, but for general $\mathscr{V}$ we can translate it into the language of the virtual equipment $\mathscr{V}$-$\mathbb{P}\mathsf{rof}$.) ■

We postpone further development of the theory of $\mathscr{V}$-categories until we have a good notion of *large* $\mathscr{V}$-category, to minimize repetition.

4. Indexed $\mathscr{V}$-categories

As in §3, let $\mathbf{S}$ be a category with finite products and $\mathscr{V}$ an $\mathbf{S}$-indexed monoidal category, with corresponding fibration $\int\mathscr{V} \to \mathbf{S}$. In this section we describe a notion of "indexed $\mathscr{V}$-category" which directly generalizes classical indexed categories.

Recall that if $F\colon \mathbf{V} \to \mathbf{W}$ is a lax monoidal functor and A is a $\mathbf{V}$-category, there is an induced $\mathbf{W}$-category $F_\bullet A$ with the same objects as A and hom-objects defined by $\underline{F_\bullet A}(x,y) = F(\underline{A}(x,y))$. Note moreover that if $F\colon \mathbf{V} \to \mathbf{W}$ is a *closed* monoidal functor, then it can be regarded as a fully faithful $\mathbf{W}$-functor $F_\bullet : F_\bullet\mathbf{V} \to \mathbf{W}$.

4.1. Definition. An **indexed $\mathscr{V}$-category** $\mathscr{A}$ consists of:

(i) For each $X \in \mathbf{S}$, a $\mathscr{V}^X$-enriched category $\mathscr{A}^X$.

(ii) For each $f\colon X \to Y$ in $\mathbf{S}$, a fully faithful $\mathscr{V}^X$-enriched functor $f^*\colon (f^*)_\bullet\mathscr{A}^Y \to \mathscr{A}^X$.

(iii) For each $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{S}$, a $\mathscr{V}^X$-natural isomorphism
$$(gf)^* \xrightarrow{\sim} f^* \circ (f^*)_\bullet(g^*)$$
(where we implicitly identify $(f^*)_\bullet(g^*)_\bullet\mathscr{A}^Z$ with $((gf)^*)_\bullet\mathscr{A}^Z$ in the domains of these functors).

(iv) For each $X \in \mathbf{S}$, a $\mathscr{V}^X$-natural isomorphism $(1_X)^* \cong 1_{\mathscr{A}^X}$.

(v) The following diagrams of isomorphisms commute:
$$\begin{tikzcd}
(hgf)^* \arrow[rr] \arrow[d] & & f^* \circ (f^*)_\bullet((hg)^*) \arrow[d] \\
(gf)^* \circ ((gf)^*)_\bullet(h^*) \arrow[d] & & f^* \circ (f^*)_\bullet(g^* \circ (g^*)_\bullet(h^*)) \arrow[d] \\
f^* \circ (f^*)_\bullet(g^*) \circ ((gf)^*)_\bullet(h^*) \arrow[rr] & & f^* \circ (f^*)_\bullet(g^*) \circ (f^*)_\bullet(g^*)_\bullet(h^*).
\end{tikzcd}$$
$$\begin{tikzcd}
(f1_X)^* \arrow[r] \arrow[dr, equal] & (1_X)^* \circ ((1_X)^*)_\bullet(f^*) \arrow[d] & f^* \circ (f^*)_\bullet((1_Y)^*) \arrow[d] & (1_Y f)^* \arrow[l] \arrow[dl, equal] \\
& f^* & f^* &
\end{tikzcd}$$
(These are the same coherence conditions as for an ordinary indexed category or pseudofunctor, with some $(f^*)_\bullet$'s added to make things make sense.)

An **indexed $\mathscr{V}$-functor** $F\colon \mathscr{A} \to \mathscr{B}$ consists of $\mathscr{V}^X$-enriched functors $F^X\colon \mathscr{A}^X \to \mathscr{B}^X$ together with isomorphisms

$$F^X \circ f^* \cong f^* \circ (f^*)_\bullet(F^Y)$$

such that the following diagrams of isomorphisms commute:

$$\begin{tikzcd}
F^X \circ (gf)^* \arrow[r] \arrow[d] & F^X \circ f^* \circ (f^*)_\bullet(g^*) \arrow[r] & f^* \circ (f^*)_\bullet(F^Y) \circ (f^*)_\bullet(g^*) \arrow[d] \\
(gf)^* \circ ((gf)^*)_\bullet(F^Z) \arrow[d] & & f^* \circ (f^*)_\bullet(F^Y \circ g^*) \arrow[d] \\
f^* \circ (f^*)_\bullet(g^*) \circ (f^*)_\bullet(g^*)_\bullet(F^Z) \arrow[rr] & & f^* \circ (f^*)_\bullet(g^* \circ (g^*)_\bullet(F^Z))
\end{tikzcd}$$

$$\begin{tikzcd}
F^X \circ (1_X)^* \arrow[r] \arrow[dr] & (1_X)^* \circ ((1_X)^*)_\bullet(F^X) \arrow[d] \\
& F^X.
\end{tikzcd}$$

Finally, an **indexed $\mathscr{V}$-natural transformation** consists of $\mathscr{V}^X$-natural transformations $\alpha^X : F^X \to G^X$ such that the following diagram of isomorphisms commutes:

$$\begin{tikzcd}
F^X \circ f^* \arrow[r] \arrow[d] & f^* \circ (f^*)_\bullet(F^Y) \arrow[d] \\
G^X \circ f^* \arrow[r] & f^* \circ (f^*)_\bullet(G^Y).
\end{tikzcd}$$

We denote the resulting 2-category by $\mathscr{V}$-$\mathcal{C}$AT.

The required fully-faithfulness of $f^*\colon (f^*)_\bullet\mathscr{A}^Y \to \mathscr{A}^X$ may seem odd. The following example should help clarify the intent.

4.2. EXAMPLE. Let $\mathbf{V}$ be an ordinary monoidal category and $\mathbf{A}$ a (large) $\mathbf{V}$-enriched category. We define an indexed $\mathscr{F}am(\mathbf{V})$-category $\mathscr{F}am(\mathbf{A})$ where, for a set X, $\mathscr{F}am(\mathbf{A})^X$ is the $\mathbf{V}^X$-enriched category of X-indexed families of objects of $\mathbf{A}$. That is, for $(A_x)_{x\in X}$ and $(B_x)_{x\in X}$ with each $A_x, B_x \in \mathbf{A}$, the hom-object in $\mathbf{V}^X$ is defined by

$$\underline{\mathscr{F}am(\mathbf{A})}^X(A,B)_x = \underline{\mathbf{A}}(A_x, B_x).$$

For a function $f : Y \to X$, the $\mathbf{V}^Y$-enriched category $(f^*)_\bullet\mathscr{A}$ has hom-objects in $\mathbf{V}^Y$:

$$\underline{\mathscr{F}am(\mathbf{A})}^X(A,B)_y = \underline{\mathbf{A}}(A_{f(y)}, B_{f(y)}).$$

Finally, the functor $f^*\colon (f^*)_\bullet\mathscr{A}^X \to \mathscr{A}^Y$ sends an X-indexed family $(A_x)_{x\in X}$ to the Y-indexed family defined by $(f^*A)_y = A_{f(y)}$. Thus we have

$$\underline{\mathscr{F}am(\mathbf{A})}^Y(f^*A, f^*B)_y = \underline{\mathbf{A}}((f^*A)_y, (f^*B)_y) = \underline{\mathbf{A}}(A_{f(y)}, B_{f(y)}) = \underline{\mathscr{F}am(\mathbf{A})}^X(A,B)_{f(y)}$$

and hence $\underline{\mathscr{F}am(\mathbf{A})}^Y(f^*A, f^*B) = f^*(\underline{\mathscr{F}am(\mathbf{A})}^X(A,B))$, so that $f^*\colon (f^*)_\bullet \mathscr{A}^X \to \mathscr{A}^Y$ is fully faithful. This construction defines a 2-functor

$$\mathscr{F}am : \mathbf{V}\text{-}\mathcal{C}AT \to \mathscr{F}am(\mathbf{V})\text{-}\mathcal{C}\mathrm{AT}.$$

It is not essentially surjective, but it induces an equivalence on hom-categories (i.e. it is bicategorically fully-faithful). We could characterize its essential image by imposing "stack" conditions.

Here are a few other important examples.

4.3. Example. If $\mathscr{V}$ is symmetric and closed as in Theorem 2.14(i), then we can regard it as an indexed $\mathscr{V}$-category, by regarding each closed symmetric monoidal category $\mathscr{V}^X$ as enriched over itself, and each closed monoidal functor $f^* : \mathscr{V}^Y \to \mathscr{V}^X$ as a fully faithful $\mathscr{V}^X$-enriched functor $(f^*)_\bullet \mathscr{V}^Y \to \mathscr{V}^X$.

4.4. Example. If $\mathbf{S}$ has finite limits, then an indexed $\mathscr{S}elf(\mathbf{S})$-category is precisely a *locally internal category* over $\mathbf{S}$, as defined in [Pen74] or [Joh02, §B2.2], and similarly for functors and transformations.

4.5. Example. Indexed $\mathscr{K}$-categories and $\mathscr{K}_*$-categories, where $\mathscr{K}$ and $\mathscr{K}_*$ are the indexed cosmoi from Examples 2.32 and 2.33, are ubiquitous throughout [MS06].

4.6. Example. Let $\mathscr{A}$ be an indexed $\mathscr{V}$-category, and T a monad on $\mathscr{A}$ in the 2-category $\mathscr{V}$-$\mathcal{C}$at. This is easily seen to consist of

(i) A $\mathscr{V}^X$-enriched monad T^X on $\mathscr{C}^X$, for every X, and

(ii) Isomorphisms $f^* \circ (f^*)_\bullet(T^Y) \cong T^X \circ f^*$ which simultaneously make T into an indexed $\mathscr{V}$-functor and f^* into a morphism of $\mathscr{V}^X$-enriched monads from $(f^*)_\bullet(T^Y)$ to T^X.

We can then form the Eilenberg-Moore object $\mathscr{A}lg(T)$ in $\mathscr{V}$-$\mathcal{C}$at. Explicitly, $\mathscr{A}lg(T)^X = \mathscr{A}lg(T^X)$, with transition functors induced by the above morphisms of monads.

For instance, if $\mathscr{V}$ is an indexed cosmos and R is a monoid in $\mathscr{V}^1$, then there is a $\mathscr{V}$-monad on $\mathscr{V}$ defined by $R^X A = \pi_X^* R \otimes_X A$, whose algebras in $\mathscr{V}^X$ are $\pi_X^* R$-modules.

As another example, if $\mathscr{V}$ is an indexed cartesian cosmos with fiberwise countable colimits, then there is a $\mathscr{V}$-monad T on $\mathscr{V}$ for which $\mathscr{A}lg(T)^X$ is the $\mathscr{V}^X$-enriched category of monoids in $\mathscr{V}^X$. More generally, we can consider algebras for any finite-product theory.

Finally, if $\mathscr{V}$ is a cartesian cosmos with countable coproducts and C is an *operad* in $\mathscr{V}^1$ as in [May72, Kel05], then $\pi_X^* C$ is an operad in $\mathscr{V}^X$, for any X. The induced monad $\widehat{\pi_X^* C}$ on $\mathscr{V}^X$ is $\mathscr{V}^X$-enriched, because $\mathscr{V}$ is cartesian, and as X varies these fit together into a $\mathscr{V}$-monad $\widehat{C}$ on $\mathscr{V}$. We thus obtain a $\mathscr{V}$-category of C-algebras. Taking $\mathscr{V}$ to be $\mathscr{K}$ as in Example 2.32, we obtain $\mathscr{V}$-categories of parametrized A_∞- and E_∞-spaces.

4.7. Example. If $\mathbf{S}$ has finite products and $\mathbf{V}$ is an ordinary monoidal category as in Example 2.42, then an indexed $\mathscr{P}sh(\mathbf{S},\mathbf{V})$-category $\mathscr{A}$ consists of, in particular, for each $X \in \mathbf{S}$, a $\mathbf{V}^{(\mathbf{S}/X)^{\mathrm{op}}}$-enriched category $\mathscr{A}^X$. However, a $\mathbf{V}^{(\mathbf{S}/X)^{\mathrm{op}}}$-enriched category is equivalently a functor $\mathscr{A}^X : (\mathbf{S}/X)^{\mathrm{op}} \to \mathbf{V}\text{-}\mathcal{C}AT$ whose image consists of functors that are the identity on objects.

Moreover, by definition of the functor $f^* : \mathbf{V}^{(\mathbf{S}/X)^{\mathrm{op}}} \to \mathbf{V}^{(\mathbf{S}/Y)^{\mathrm{op}}}$ and by full-faithfulness of $f^* : (f^*)_\bullet \mathscr{A}^Y \to \mathscr{A}^X$, for any objects a, b of $\mathscr{A}^Y$ the hom-object $\underline{\mathscr{A}}^Y(f)(a,b) \in \mathbf{V}$ must be isomorphic to $\underline{\mathscr{A}}^X(1_X)(f^*a, f^*b)$, and similarly for all the category structure. Thus, the $\mathbf{V}$-category $\mathscr{A}^Y(f)$ is completely determined by the $\mathbf{V}$-category $\mathscr{A}^X(1_X)$ and the function $f^* : \mathrm{ob}(\mathscr{A}^Y(1_Y)) \to \mathrm{ob}(\mathscr{A}^X(1_X))$.

Furthermore, the action on hom-objects of the functors in the image of the functor $\mathscr{A}^X : (\mathbf{S}/X)^{\mathrm{op}} \to \mathbf{V}\text{-}\mathcal{C}AT$ assemble exactly into an extension of this function on objects to a $\mathbf{V}$-functor $\mathscr{A}^Y(1_Y) \to \mathscr{A}^X(1_X)$. Adding in the pseudofunctoriality constraints, we see that an indexed $\mathscr{P}sh(\mathbf{S},\mathbf{V})$-category is equivalently an ordinary pseudofunctor $\mathbf{S}^{\mathrm{op}} \to \mathbf{V}\text{-}\mathcal{C}AT$. In fact, the 2-category $\mathscr{P}sh(\mathbf{S},\mathbf{V})\text{-}\mathcal{C}\mathrm{AT}$ is 2-equivalent to the 2-category $[\mathbf{S}^{\mathrm{op}}, \mathbf{V}\text{-}\mathcal{C}AT]$ of pseudofunctors, pseudonatural transformations, and modifications.

In particular, an indexed $\mathscr{P}sh(\mathbf{S},\mathrm{Set})$-category is merely an ordinary $\mathbf{S}$-indexed category (with locally small fibers). It is known (e.g. [Joh02, B2.2.2]) that locally internal categories can be identified with indexed categories that are "locally small" in an indexed sense. This corresponds to identifying indexed $\mathscr{S}elf(\mathbf{S})$-categories with indexed $\mathscr{P}sh(\mathbf{S},\mathrm{Set})$-categories whose hom-presheaves $\underline{\mathscr{A}}^X(a,b) \in \mathrm{Set}^{(\mathbf{S}/X)^{\mathrm{op}}}$ are all representable. (The connection with the usual definition of "locally small indexed category" will be more evident in §5.)

4.8. Remark. In [GG76] indexed $\mathscr{V}$-categories are called *$\mathscr{V}$-enriched fibrations*. We have chosen a different terminology because indexed $\mathscr{V}$-categories are more analogous to pseudofunctors than to fibrations. In §6 we will see a more 'fibrational' approach.

It is straightforward to define indexed $\mathscr{V}$-profunctors as well.

4.9. Definition. For indexed $\mathscr{V}$-categories $\mathscr{A}$ and $\mathscr{B}$, an **indexed $\mathscr{V}$-profunctor** consists of a $\mathscr{V}^X$-enriched profunctor $H^X : \mathscr{A}^X \nrightarrow \mathscr{B}^X$ for each X, together with isomorphisms $(f^*)_\bullet H^Y \cong H^X$ satisfying evident coherence axioms. We obtain a category $\mathscr{V}\text{-}\mathcal{P}\mathrm{ROF}(\mathscr{A},\mathscr{B})$ of indexed $\mathscr{V}$-profunctors.

However, rather than develop the theory of indexed $\mathscr{V}$-categories any further here, we will instead move on to a more general notion which includes both small $\mathscr{V}$-categories and indexed $\mathscr{V}$-categories as special cases.

5. Large $\mathscr{V}$-categories

Continuing with our minimal assumptions that $\mathbf{S}$ has finite products and $\mathscr{V}$ is an $\mathbf{S}$-indexed monoidal category, we will now define a different sort of "large $\mathscr{V}$-category" which more obviously includes the small ones from §3. The relationship of these large

$\mathscr{V}$-categories with indexed $\mathscr{V}$-categories is less immediately clear, but will turn out to be similar to the relationship between fibrations and classical indexed categories.

5.1. DEFINITION. A **large $\mathscr{V}$-category** $\mathscr{A}$ consists of

(i) A collection of objects $x, y, z, \ldots$.

(ii) For each object x, an object $\epsilon x \in \mathbf{S}$, called its *extent*.

(iii) For each x, y, an object $\underline{\mathscr{A}}(x, y)$ of $\mathscr{V}^{\epsilon y \times \epsilon x}$.

(iv) For each x, a morphism in $\int \mathscr{V}$:

$$\begin{tikzcd} \mathbb{I}_{\epsilon x} \arrow[r, "\mathrm{ids}"] \arrow[d, mapsto] & \underline{\mathscr{A}}(x,x) \arrow[d, mapsto] \\ \epsilon x \arrow[r, "\Delta"'] & \epsilon x \times \epsilon x \end{tikzcd}$$

(v) For each x, y, z, a morphism in $\int \mathscr{V}$:

$$\begin{tikzcd} (\underline{\mathscr{A}}(y,z) \otimes_{\epsilon y} \underline{\mathscr{A}}(x,y)) \arrow[r, "\mathrm{comp}"] \arrow[d, mapsto] & \underline{\mathscr{A}}(x,z) \arrow[d, mapsto] \\ \epsilon z \times \epsilon y \times \epsilon x \arrow[r, "\pi_{\epsilon y}"'] & \epsilon z \times \epsilon x \end{tikzcd}$$

(vi) Composition is associative and unital, just as in Definition 3.1.

If $\mathscr{A}$ and $\mathscr{B}$ are large $\mathscr{V}$-categories, a **$\mathscr{V}$-functor** $f \colon \mathscr{A} \to \mathscr{B}$ consists of:

(i) For each object x of $\mathscr{A}$, an object fx of $\mathscr{B}$ and a morphism $\epsilon f_x \colon \epsilon x \to \epsilon(fx)$ in $\mathbf{S}$.

(ii) For each pair x, y in $\mathscr{A}$, a morphism in $\int \mathscr{V}$:

$$\begin{tikzcd} \underline{\mathscr{A}}(x,y) \arrow[r, "f_{xy}"] \arrow[d, mapsto] & \underline{\mathscr{B}}(fx,fy) \arrow[d, mapsto] \\ \epsilon y \times \epsilon x \arrow[r, "\epsilon f_y \times \epsilon f_x"'] & \epsilon(fy) \times \epsilon(fx). \end{tikzcd}$$

(iii) Composition and identities are preserved, as in Definition 3.9.

If $f, g \colon \mathscr{A} \to \mathscr{B}$ are $\mathscr{V}$-functors between large $\mathscr{V}$-categories, a **$\mathscr{V}$-natural transformation** $\alpha \colon f \to g$ consists of, for each object x, a morphism

$$\begin{tikzcd} \mathbb{I}_{\epsilon x} \arrow[r, "\alpha_x"] \arrow[d, mapsto] & \underline{\mathscr{B}}(fx,gx) \arrow[d, mapsto] \\ \epsilon x \arrow[r, "(\epsilon g_x, \epsilon f_x)"'] & \epsilon(gx) \times \epsilon(fx) \end{tikzcd}$$

such that for all x, y a diagram analogous to that in Definition 3.12 commutes.

5.2. Lemma. *Large $\mathscr{V}$-categories form a 2-category $\mathscr{V}$-$\mathcal{CAT}$, which contains the 2-category $\mathscr{V}$-$\mathcal{C}at$ of small $\mathscr{V}$-categories as the full sub-2-category of large $\mathscr{V}$-categories with exactly one object.*

Proof. Just like Theorem 3.17. ∎

The most obvious non-small example comes from $\mathscr{V}$ itself.

5.3. Example. If $\mathscr{V}$ is closed as in Theorem 2.14(iii), then we can define a large $\mathscr{V}$-category whose objects are the objects of $\int \mathscr{V}$ (that is, the disjoint union of the objects of the categories $\mathscr{V}^X$) and whose hom-objects are the external ones $\underline{\mathscr{V}}(x, y)$. (Recall from Remark 2.19 that the external-homs can be defined in terms of the fiberwise ones for arbitrary $\mathscr{V}$. This explicit definition suffices to make them into a large $\mathscr{V}$-category.)

5.4. Remark. When $\mathscr{V}$ has indexed coproducts preserved by $\otimes$, we have the "discrete $\mathscr{V}$-category" 2-functor

$$\delta : \mathbf{S} \to \mathscr{V}\text{-}\mathcal{C}at \hookrightarrow \mathscr{V}\text{-}\mathcal{CAT}.$$

As in Remarks 3.11 and 3.16, we can make sense of "$\mathscr{V}$-functors $f : \delta X \to \mathscr{A}$" and "$\mathscr{V}$-natural transformations $\alpha : f \to g$" between them, even when $\mathscr{V}$ lacks indexed coproducts. Namely, the former is simply a choice of an object $a \in \mathscr{A}$ and a morphism $\epsilon f_a : X \to \epsilon a$ in $\mathbf{S}$, while the latter is simply a morphism $\mathbb{I}_X \to \underline{\mathscr{A}}(a, b)$ in $\int \mathscr{V}$ lying over $(\epsilon g_b, \epsilon f_a)$. In particular, any large $\mathscr{V}$-category $\mathscr{A}$ induces an $\mathbf{S}$-indexed category

$$\mathscr{V}\text{-}\mathcal{CAT}(\delta -, \mathscr{A}) : \mathbf{S}^{\mathrm{op}} \to \mathrm{Cat}.$$

Again, in §7 we will identify this with a special case of "change of cosmos".

5.5. Definition. If κ is a regular cardinal, then we say a large $\mathscr{V}$-category $\mathscr{A}$ is **κ-small** if its collection of objects is a set of cardinality $< \kappa$. We say $\mathscr{A}$ is **∞-small** or **set-small** if its collection of objects is a small set (of any cardinality).

Note that if $\mathbf{S}$ is a small ordinary category, then a $\mathscr{V}$-category $\mathscr{A}$ is set-small if and only if for each $X \in \mathbf{S}$, there is a small set of objects of $\mathscr{A}$ having extent X.

5.6. Remark. If we allow κ to be an "arity class" in the sense of [Shu12], then according to this definition the "small $\mathscr{V}$-categories" of §3 may be called "$\{1\}$-small".

5.7. Remark. Recall from Example 2.30 that an $\mathbf{S}$-indexed monoidal category $\mathscr{V}$ gives rise to a Fam($\mathbf{S}$)-indexed monoidal category $\mathscr{F}am(\mathscr{V})$. It is straightforward to identify set-small $\mathscr{V}$-categories, as defined above, with small $\mathscr{F}am(\mathscr{V})$-categories, as defined in §3. (Of course, by allowing "large families" we could include all large $\mathscr{V}$-categories.)

In particular, for a classical monoidal category $\mathbf{V}$, we can identify set-small $\mathscr{C}onst(\star, \mathbf{V})$-categories with small $\mathbf{V}$-enriched categories in the classical sense—while we have already observed in Example 2.30 that $\mathscr{F}am(\mathbf{V}) = \mathscr{F}am(\mathscr{C}onst(\star, \mathbf{V}))$, and in Example 3.2 that small $\mathscr{F}am(\mathbf{V})$-categories (in the sense of §3) can also be identified with small $\mathbf{V}$-enriched categories.

Thus, although on the one hand small $\mathscr{V}$-categories are evidently a special case of large ones, we could equivalently regard large $\mathscr{V}$-categories as a special case of small ones, by the expedient of changing $\mathscr{V}$.

The theory of profunctors from §3 also generalizes to large $\mathscr{V}$-categories, as long as we pay appropriate attention to size questions.

5.8. Definition. For large $\mathscr{V}$-categories $\mathscr{A}$ and $\mathscr{B}$, a **$\mathscr{V}$-profunctor** $H : \mathscr{A} \rightarrow\!\!\!\!\!\!\mapsto \mathscr{B}$ consists of:

(i) For each pair of objects a of $\mathscr{A}$ and b of $\mathscr{B}$, an object $\underline{H}(b,a) \in \mathscr{V}^{\epsilon a \times \epsilon b}$.

(ii) Morphisms in $\int \mathscr{V}$:

$$\begin{tikzcd} \underline{A}(a,a') \otimes_{\epsilon a} \underline{H}(b,a) \arrow[r, "\mathrm{act}"] \arrow[d, mapsto] & \underline{H}(b,a') \arrow[d, mapsto] \\ \epsilon a' \times \epsilon a \times \epsilon b \arrow[r, "\pi_{\epsilon a}"'] & \epsilon a' \times \epsilon b \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} \underline{H}(b,a) \otimes_{\epsilon b} \underline{B}(b',b) \arrow[r, "\mathrm{act}"] \arrow[d, mapsto] & \underline{H}(b',a) \arrow[d, mapsto] \\ \epsilon a \times \epsilon b \times \epsilon b' \arrow[r, "\pi_{\epsilon b}"'] & \epsilon a \times \epsilon b \end{tikzcd}$$

(iii) The following diagrams commute.

$$\begin{tikzcd} \underline{A}(a',a'') \otimes_{\epsilon a'} \underline{A}(a,a') \otimes_{\epsilon a} \underline{H}(b,a) \arrow[r, "1\otimes\mathrm{act}"] \arrow[d, "\mathrm{comp}\otimes 1"'] & \underline{A}(a',a'') \otimes_{\epsilon a'} \underline{H}(b,a') \arrow[d, "\mathrm{act}"] \\ \underline{A}(a,a'') \otimes_{\epsilon a} \underline{H}(b,a) \arrow[r, "\mathrm{act}"'] & \underline{H}(b,a'') \end{tikzcd}$$

$$\begin{tikzcd} \underline{H}(b,a) \otimes_{\epsilon b} \underline{B}(b',b) \otimes_{\epsilon b'} \underline{B}(b'',b') \arrow[r, "\mathrm{act}\otimes 1"] \arrow[d, "1\otimes\mathrm{comp}"'] & \underline{H}(b',a) \otimes_{\epsilon b'} \underline{B}(b'',b') \arrow[d, "\mathrm{act}"] \\ \underline{H}(b,a) \otimes_{\epsilon b} \underline{B}(b'',b) \arrow[r, "\mathrm{act}"'] & \underline{H}(b'',a) \end{tikzcd}$$

$$\begin{tikzcd} \underline{H}(b,a) \arrow[r, "\mathrm{ids}"] \arrow[dr, "="'] & \underline{A}(a,a) \otimes_{\epsilon a} \underline{H}(b,a) \arrow[d, "\mathrm{act}"] \\ & \underline{H}(b,a) \end{tikzcd} \qquad \begin{tikzcd} \underline{H}(b,a) \arrow[r, "\mathrm{ids}"] \arrow[dr, "="'] & \underline{H}(b,a) \otimes_{\epsilon b} \underline{B}(b,b) \arrow[d, "\mathrm{act}"] \\ & \underline{H}(b,a) \end{tikzcd}$$

$$\begin{tikzcd} \underline{A}(a,a') \otimes_{\epsilon a} \underline{H}(b,a) \otimes_{\epsilon b} \underline{B}(b',b) \arrow[r, "\mathrm{act}\otimes 1"] \arrow[d, "1\otimes\mathrm{act}"'] & \underline{H}(b,a') \otimes_{\epsilon b} \underline{B}(b',b) \arrow[d, "\mathrm{act}"] \\ \underline{A}(a,a') \otimes_{\epsilon a} \underline{H}(b',a) \arrow[r, "\mathrm{act}"'] & \underline{H}(b',a). \end{tikzcd}$$

We have an obvious notion of **morphism of profunctors**, yielding a category which we denote $\mathscr{V}\text{-}\mathcal{P}ROF(A,B)$.

Of course, when $\mathscr{A}$ and $\mathscr{B}$ are small, this reduces exactly to Definition 3.19.

5.9. Example. Any large $\mathscr{V}$-category $\mathscr{A}$ has a *unit profunctor* $\mathscr{A} : \mathscr{A} \nrightarrow \mathscr{A}$ which is made up of its hom-objects.

5.10. Example. If $H : \mathscr{A} \nrightarrow \mathscr{B}$ is a $\mathscr{V}$-profunctor and $f : \mathscr{A}' \to \mathscr{A}$ and $g : \mathscr{B}' \to \mathscr{B}$ are $\mathscr{V}$-functors, then there is a $\mathscr{V}$-profunctor $H(g, f) : \mathscr{A}' \nrightarrow \mathscr{B}'$ defined by

$$\underline{H(g, f)}(b, a) = (\epsilon f \times \epsilon g)^*\big(\underline{H}(gb, fa)\big)$$

In particular, for a $\mathscr{V}$-functor $f : \mathscr{A} \to \mathscr{B}$, we have the *representable $\mathscr{V}$-profunctors* $\mathscr{B}(1, f) : \mathscr{A} \nrightarrow \mathscr{B}$ and $\mathscr{B}(f, 1) : \mathscr{B} \nrightarrow \mathscr{A}$.

To define the composite of $\mathscr{V}$-profunctors $H : \mathscr{A} \nrightarrow \mathscr{B}$ and $K : \mathscr{B} \nrightarrow \mathscr{C}$ in general, we need $\mathscr{V}$ to have fiberwise colimits of the size of the collection of objects of $\mathscr{B}$. Therefore, if this collection is itself large (in the sense of the ambient set theory), we cannot expect such composites to exist. For this reason, it is useful to introduce the following notion.

5.11. Definition. For $\mathscr{V}$-profunctors $H : \mathscr{A} \nrightarrow \mathscr{B}$, $K : \mathscr{B} \nrightarrow \mathscr{C}$, and $L : \mathscr{A} \nrightarrow \mathscr{C}$, a **bimorphism** $\phi : H, K \to L$ consists of

(i) For each a, b, c, a morphism in $\int \mathscr{V}$:

$$\begin{tikzcd}
\underline{H}(b, a) \otimes_{\epsilon b} \underline{K}(c, b) \arrow[r, "\phi_{abc}"] \arrow[d, mapsto] & \underline{L}(c, a) \arrow[d, mapsto] \\
\epsilon a \times \epsilon b \times \epsilon c \arrow[r] & \epsilon a \times \epsilon c
\end{tikzcd}$$

(ii) The following diagrams commute:

$$\begin{tikzcd}
\underline{H}(b, a) \otimes_{\epsilon b} \underline{K}(c, b) \otimes_{\epsilon c} \underline{\mathscr{C}}(c', c) \arrow[r, "1\otimes\mathrm{act}"] \arrow[d, "\phi\otimes 1"'] & \underline{H}(b, a) \otimes_{\epsilon b} \underline{K}(c', b) \arrow[d, "\phi"] \\
\underline{L}(c, a) \otimes_{\epsilon c} \underline{\mathscr{C}}(c', c) \arrow[r, "\mathrm{act}"'] & \underline{L}(c', a)
\end{tikzcd} \tag{5.12}$$

$$\begin{tikzcd}
\underline{\mathscr{A}}(a, a') \otimes_{\epsilon a} \underline{H}(b, a) \otimes_{\epsilon b} \underline{K}(c, b) \arrow[r, "\mathrm{act}\otimes 1"] \arrow[d, "1\otimes\phi"'] & \underline{H}(b, a') \otimes_{\epsilon b} \underline{K}(c, b) \arrow[d, "\phi"] \\
\underline{\mathscr{A}}(a, a') \otimes_{\epsilon a} \underline{L}(c, a) \arrow[r, "\mathrm{act}"'] & \underline{L}(c, a')
\end{tikzcd} \tag{5.13}$$

$$\begin{tikzcd}
\underline{H}(b, a) \otimes_{\epsilon b} \underline{\mathscr{B}}(b', b) \otimes_{\epsilon b'} \underline{K}(c, b') \arrow[r, "1\otimes\mathrm{act}"] \arrow[d, "\mathrm{act}\otimes 1"'] & \underline{H}(b, a) \otimes_{\epsilon b} \underline{K}(c, b) \arrow[d, "\phi"] \\
\underline{H}(b', a) \otimes_{\epsilon b'} \underline{K}(c, b') \arrow[r, "\phi"'] & \underline{L}(c, a)
\end{tikzcd} \tag{5.14}$$

We write $\mathscr{V}\text{-}\mathcal{B}imor(H, K; L)$ for the set of bimorphisms $H, K \to L$. It is evident that such bimorphisms can be composed with ordinary morphisms of profunctors $L \to L'$, $H' \to H$, and $K' \to K$, yielding a functor

$$\mathscr{V}\text{-}\mathcal{B}imor(-,-;-) : \mathscr{V}\text{-}\mathcal{P}ROF(\mathscr{A},\mathscr{B})^{\mathrm{op}} \times \mathscr{V}\text{-}\mathcal{P}ROF(\mathscr{B},\mathscr{C})^{\mathrm{op}} \times \mathscr{V}\text{-}\mathcal{P}ROF(\mathscr{A},\mathscr{C}) \longrightarrow \mathrm{Set}$$

We can now characterize some conditions under which composites exist.

5.15. Lemma. *If $\mathscr{V}$ has indexed coproducts, fiberwise coequalizers, and fiberwise κ-small coproducts, all preserved by $\otimes$, and $\mathscr{B}$ is κ-small, then for any $H : \mathscr{A} \nrightarrow \mathscr{B}$ and $K : \mathscr{B} \nrightarrow \mathscr{C}$, the functor $\mathscr{V}\text{-}\mathcal{B}imor(H, K; -)$ is representable by some $H \odot K : \mathscr{A} \nrightarrow \mathscr{C}$.*

Proof. As in Lemma 3.25, define $\underline{H \odot K}(c, a)$ to be the coequalizer of the parallel pair

$$\coprod_{b,b' \in \mathrm{ob}\mathscr{B}} \big(\underline{H}(b, a) \otimes_{[\epsilon b]} \underline{\mathscr{B}}(b', b) \otimes_{[\epsilon b']} \underline{K}(c, b')\big) \rightrightarrows \coprod_{b \in \mathrm{ob}\mathscr{B}} \big(\underline{H}(b, a) \otimes_{[\epsilon b]} \underline{K}(c, b)\big). \qquad \blacksquare$$

We also have dual constructions of right and left homs, under analogous hypotheses.

5.16. Lemma. *Suppose $\mathscr{V}$ is symmetric and closed and has indexed products, fiberwise equalizers, and fiberwise κ-small products, and we have $\mathscr{V}$-profunctors $H : \mathscr{A} \nrightarrow \mathscr{B}$, $K : \mathscr{B} \nrightarrow \mathscr{C}$, and $L : \mathscr{A} \nrightarrow \mathscr{C}$.*

(i) If $\mathscr{C}$ is κ-small, then $\mathscr{V}\text{-}\mathcal{B}imor(-, K; L)$ is representable by some $K \rhd L : \mathscr{A} \nrightarrow \mathscr{B}$.

(ii) If $\mathscr{A}$ is κ-small, then $\mathscr{V}\text{-}\mathcal{B}imor(H, -; L)$ is representable by some $L \lhd H : \mathscr{B} \nrightarrow \mathscr{C}$.

Proof. We define $\underline{K \rhd L}(b, a)$ to be the fiberwise equalizer of the following parallel pair of maps between fiberwise products:

$$\prod_{c \in \mathrm{ob}\mathscr{C}} \mathscr{V}^{[\epsilon c]}\big(\underline{K}(c, b), \underline{L}(c, a)\big) \ \rightrightarrows \ \prod_{c,c' \in \mathrm{ob}\mathscr{C}} \mathscr{V}^{[\epsilon c]}\Big(\underline{K}(c, b), \mathscr{V}^{[\epsilon c']}\big(\underline{\mathscr{C}}(c', c), \underline{L}(c', a)\big)\Big) \tag{5.17}$$

and similarly for $\underline{L \lhd H}(c, b)$ we use:

$$\prod_{a \in \mathrm{ob}\mathscr{A}} \mathscr{V}^{[\epsilon a]}\big(\underline{H}(b, a), \underline{L}(c, a)\big) \ \rightrightarrows \ \prod_{a,a' \in \mathrm{ob}\mathscr{A}} \mathscr{V}^{[\epsilon a]}\Big(\underline{H}(b, a), \mathscr{V}^{[\epsilon a']}\big(\underline{\mathscr{A}}(a, a'), \underline{L}(c, a')\big)\Big). \qquad \blacksquare$$

5.18. Remark. If $\mathscr{C}$ is a discrete $\mathscr{V}$-category δZ, then its actions on K and L are trivial, and so the two morphisms in (5.17) are in fact equal. Thus, in this case we have $\underline{K \rhd L}(b, a) = \mathscr{V}^{[Z]}(\underline{K}(\star, b), \underline{L}(\star, a))$. Of course, similar observations hold when $\mathscr{A}$ is discrete, or in Lemma 5.15 when $\mathscr{B}$ is discrete.

5.19. Example. Recall from Example 2.42 that when $\mathscr{V} = \mathscr{P}sh(\mathbf{S}, \mathbf{V})$, with $\mathbf{S}$ a locally small category with pullbacks and $\mathbf{V}$ a classical cosmos, we do not have all indexed coproducts and homs, but only those satisfying some smallness conditions. Let us say that a $\mathscr{P}sh(\mathbf{S}, \mathbf{V})$-category $\mathscr{A}$ is **locally small** if each hom-object $\underline{\mathscr{A}}(a, a')$ is small in the sense of Example 2.42, and likewise that a $\mathscr{P}sh(\mathbf{S}, \mathbf{V})$-profunctor $H : \mathscr{A} \nrightarrow \mathscr{B}$ is **locally small** if each $\underline{H}(b, a)$ is small. Then the proof of Lemma 5.15 goes through as long as we assume additionally that $\mathscr{B}$, H, and K are locally small. Likewise, Lemma 5.16(i) holds when $\mathscr{C}$ and K are locally small, and (ii) holds when $\mathscr{A}$ and H are locally small.

The $\mathscr{P}sh(\mathbf{S}, \mathbf{V})$-categories that arise "in nature" are generally not locally small. However, we will see in §9 that locally small $\mathscr{P}sh(\mathbf{S}, \mathbf{V})$-categories and profunctors are useful for describing weighted limits and colimits.

These representing objects actually have a stronger universal property, which is necessary (for instance) to show that they are associative. To express this property, we first observe that for $\mathscr{V}$-profunctors $H_i : \mathscr{A}_i \nrightarrow \mathscr{A}_{i+1}$ and $K : \mathscr{A}_0 \nrightarrow \mathscr{A}_n$, there is a more general notion of a **multimorphism** $\phi : H_1, \ldots, H_n \to K$. This has components

$$\begin{tikzcd}
\underline{H}_1(a_1, a_0) \otimes_{\epsilon a_1} \cdots \otimes_{\epsilon a_n} \underline{H}_n(a_n, a_{n-1}) \arrow[r, "\phi_{a_0,\ldots,a_n}"] \arrow[d, mapsto] & \underline{K}(a_n, a_0) \arrow[d, mapsto] \\
\epsilon a_0 \times \cdots \times \epsilon a_n \arrow[r] & \epsilon a_0 \times \epsilon a_n
\end{tikzcd}$$

satisfying axioms similar to (5.12) and (5.13) for the actions of $\mathscr{A}_0$ and $\mathscr{A}_n$, and axioms similar to (5.14) for the actions of $\mathscr{A}_1$ through $\mathscr{A}_{n-1}$. In the case $n = 0$, the components of a multimorphism $\phi : () \to K$ are

$$\begin{tikzcd}
\mathbb{I}_{\epsilon a} \arrow[r, "\phi_a"] \arrow[d, mapsto] & \underline{K}(a, a) \arrow[d, mapsto] \\
\epsilon a \arrow[r, "\Delta"'] & \epsilon a \times \epsilon a
\end{tikzcd}$$

and its only axioms are of the form (5.12) and (5.13). Write $\mathscr{V}\text{-}\mathcal{M}ultimor(H_1, \ldots, H_n; K)$ for the set of multimorphisms $H_1, \ldots, H_n \to K$. Multimorphisms can obviously also be composed with ordinary morphisms, and also with each other in a multicategory-like way, e.g. given $\phi : H_1, H_2 \to K_1$ and $\psi : K_1, K_2 \to L$ we have $\psi(\phi, 1) : H_1, H_2, K_2 \to L$.

Formally, multimorphisms are the 2-cells of another virtual equipment, which we denote $\mathscr{V}\text{-}\mathbb{P}\mathsf{ROF}$. The following is [CS10, Def. 5.1].

5.20. Definition. A **composite** of $\mathscr{V}$-profunctors $H : \mathscr{A} \nrightarrow \mathscr{B}$ and $K : \mathscr{B} \nrightarrow \mathscr{C}$ is a $\mathscr{V}$-profunctor $H \odot K : \mathscr{A} \nrightarrow \mathscr{C}$ and a bimorphism $\phi : H, K \to H \odot K$ such that composing with ϕ induces bijections

$$\mathscr{V}\text{-}\mathcal{M}ultimor(L_1, \ldots, L_n, H \odot K, M_1, \ldots, M_m) \xrightarrow{\sim} \mathscr{V}\text{-}\mathcal{M}ultimor(L_1, \ldots, L_n, H, K, M_1, \ldots, M_m)$$

for all well-typed L_i, M_j.

When $n = m = 0$, this just says that $H \odot K$ represents the functor $\mathscr{V}\text{-}\mathcal{B}imor(H, K; -)$, so being a composite is a strengthening of that universal property. It is easy to verify that the proof of Lemma 5.15 actually constructs composites in the sense of Definition 5.20.

5.21. Definition. A **left hom** of $\mathscr{V}$-profunctors $H : \mathscr{A} \rightarrow\!\!\!\!\!\!+\; \mathscr{B}$ and $K : \mathscr{A} \rightarrow\!\!\!\!\!\!+\; \mathscr{C}$ is a $\mathscr{V}$-profunctor $K \lhd H : \mathscr{B} \rightarrow\!\!\!\!\!\!+\; \mathscr{C}$ and a bimorphism $\phi : H, (K \lhd H) \to K$ such that composing with ϕ induces bijections

$$\mathscr{V}\text{-}\mathcal{M}ultimor(L_1, \dots, L_n; K \lhd H) \xrightarrow{\sim} \mathscr{V}\text{-}\mathcal{M}ultimor(H, L_1, \dots, L_n; K)$$

for all well-typed L_i. Similarly, a **right hom** of $H : \mathscr{B} \rightarrow\!\!\!\!\!\!+\; \mathscr{C}$ and $K : \mathscr{A} \rightarrow\!\!\!\!\!\!+\; \mathscr{C}$ is a $\mathscr{V}$-profunctor $H \rhd K : \mathscr{A} \rightarrow\!\!\!\!\!\!+\; \mathscr{B}$ and a bimorphism $\phi : (H \rhd K), H \to K$ which induces bijections

$$\mathscr{V}\text{-}\mathcal{M}ultimor(L_1, \dots, L_n; H \rhd K) \xrightarrow{\sim} \mathscr{V}\text{-}\mathcal{M}ultimor(L_1, \dots, L_n, H; K).$$

Again, when $n = 1$ these definitions reproduce the universal property stated in Lemma 5.16, whereas the proof of that lemma produces objects with this stronger universal property.

5.22. Remark. The omitted verification in Lemmas 5.15 and 5.16 that the given objects do, in fact, form a profunctor with the desired property, applies verbatim to show that if $H : \mathscr{A} \rightarrow\!\!\!\!\!\!+\; \mathscr{B}$ and $K : \mathscr{B} \rightarrow\!\!\!\!\!\!+\; \mathscr{C}$ are $\mathscr{V}$-profunctors such that the composite $H(1, a) \odot K(c, 1)$ exists for all $a \in \mathscr{A}$ and $c \in \mathscr{C}$, then the composite $H \odot K$ also exists. There is a similar result for homs.

We also note that the unit profunctors $\mathscr{A} : \mathscr{A} \rightarrow\!\!\!\!\!\!+\; \mathscr{A}$ from Example 5.9 have an analogous universal property.

5.23. Lemma. *For any large $\mathscr{V}$-category $\mathscr{A}$, there is a multimorphism $\phi : () \to \mathscr{A}$ such that composing with ϕ induces bijections*

$$\mathscr{V}\text{-}\mathcal{M}ultimor(\vec{L}, \mathscr{A}, \vec{M}; N) \xrightarrow{\sim} \mathscr{V}\text{-}\mathcal{M}ultimor(\vec{L}, \vec{M}; N)$$

for all well-typed $\vec{L} = L_1, \dots, L_n$ and $\vec{M} = M_1, \dots, M_m$ and N.

Proof. By [CS10, Prop. 5.5]. ∎

One value of these stronger universal properties is that they automatically imply associativity and unitality of composites and homs.

5.24. Lemma. *If all necessary composites and homs exist, then we have the following isomorphisms:*

$$\begin{aligned}
(H \odot K) \odot L &\cong H \odot (K \odot L) \qquad (5.25)\\
(H \odot K) \rhd L &\cong H \rhd (K \rhd L)\\
(H \lhd K) \lhd L &\cong H \lhd (K \odot L)\\
(H \rhd K) \lhd L &\cong H \rhd (K \lhd L).
\end{aligned}$$

PROOF. We prove only (5.25); the others are similar. For any appropriately typed M, we have

$$\begin{aligned}\mathscr{V}\text{-}\mathcal{PROF}((H \odot K) \odot L, M) &\cong \mathscr{V}\text{-}\mathcal{B}imor((H \odot K), L; M)\\ &\cong \mathscr{V}\text{-}\mathcal{M}ultimor(H, K, L; M)\\ &\cong \mathscr{V}\text{-}\mathcal{B}imor(H, K \odot L; M)\\ &\cong \mathscr{V}\text{-}\mathcal{PROF}(H \odot (K \odot L), M).\end{aligned}$$

Thus, the Yoneda lemma gives (5.25). ∎

5.26. LEMMA. *For $H : \mathscr{A} \nrightarrow \mathscr{B}$, the composites $H \odot \mathscr{B}$ and $\mathscr{A} \odot H$, and the homs $\mathscr{B} \rhd H$ and $H \lhd \mathscr{A}$, all exist and are canonically isomorphic to H.*

PROOF. Just like the previous lemma. ∎

5.27. REMARK. We will henceforth abuse language by writing "$H \odot K \cong L$" to mean that the composite $H \odot K$ exists and is canonically isomorphic to L, and similarly for left and right homs.

5.28. LEMMA. *For a $\mathscr{V}$-profunctor $H : \mathscr{A} \nrightarrow \mathscr{B}$ and a $\mathscr{V}$-functor $f : \mathscr{A}' \to \mathscr{A}$, we have $\mathscr{A}(1, f) \odot H \cong H(1, f)$. Similarly, for $g : \mathscr{B}' \to \mathscr{B}$, we have $H \odot \mathscr{B}(g, 1) \cong H(g, 1)$.*

PROOF. By [CS10, Theorem 7.16]. ∎

In particular, for $\mathscr{A} \xrightarrow{f} \mathscr{B} \xrightarrow{g} \mathscr{C}$, we have $\mathscr{B}(1, f) \odot \mathscr{C}(1, g) \cong \mathscr{C}(1, gf)$, so that representable profunctors are "pseudofunctorial", even though in general, $\mathscr{V}$-profunctors between large $\mathscr{V}$-categories do not form a bicategory. The same proof as in Lemma 3.26 shows that this functor is fully faithful, i.e. we have natural bijections

$$\begin{aligned}\mathscr{V}\text{-}\mathcal{CAT}(\mathscr{A}, \mathscr{B})(f, g) &\cong \mathscr{V}\text{-}\mathcal{PROF}(\mathscr{A}, \mathscr{B})(\mathscr{B}(1, f), \mathscr{B}(1, g))\\ &\cong \mathscr{V}\text{-}\mathcal{PROF}(\mathscr{B}, \mathscr{A})(\mathscr{B}(g, 1), \mathscr{B}(f, 1)).\end{aligned}$$

The dual statement for homs is the second Yoneda Lemma, as in 3.29.

5.29. LEMMA. *For a $\mathscr{V}$-profunctor $H : \mathscr{A} \nrightarrow \mathscr{B}$ and a $\mathscr{V}$-functor $f : \mathscr{A}' \to \mathscr{A}$, we have $H \lhd \mathscr{A}(f, 1) \cong H(1, f)$. Similarly, for $g : \mathscr{B}' \to \mathscr{B}$, we have $\mathscr{B}(1, g) \rhd H \cong H(g, 1)$.*

PROOF. This follows immediately from [CS10, Theorem 7.20]. ∎

In particular, although in the statement of Lemma 3.29 we assumed $\mathscr{V}$ to be an indexed cosmos so that the homs $\lhd$ and $\rhd$ would exist *a priori*, this version of it shows that that assumption is unnecessary; the particular homs in question *automatically* exist.

The first Yoneda Lemma 3.28 follows immediately.

5.30. LEMMA. *For any $\mathscr{V}$-functor $f : \mathscr{A} \to \mathscr{B}$ and $\mathscr{V}$-profunctors $H : \mathscr{A} \nrightarrow \mathscr{B}$ and $K : \mathscr{B} \nrightarrow \mathscr{A}$, there are natural bijections*

$$\begin{aligned}&\mathscr{V}\text{-}\mathcal{PROF}(\mathscr{A}, \mathscr{B})(\mathscr{B}(1, f), H) \cong \mathscr{V}\text{-}\mathcal{PROF}(\mathscr{A}, \mathscr{A})(\mathscr{A}, H(f, 1)) \qquad \textit{and}\\ &\mathscr{V}\text{-}\mathcal{PROF}(\mathscr{B}, \mathscr{A})(\mathscr{B}(f, 1), K) \cong \mathscr{V}\text{-}\mathcal{PROF}(\mathscr{A}, \mathscr{A})(\mathscr{A}, K(1, f)).\end{aligned}$$

PROOF. For the first, we have

$$\begin{aligned}\mathscr{V}\text{-}\mathcal{PROF}(\mathscr{A},\mathscr{B})(\mathscr{B}(1,f),H) &\cong \mathscr{V}\text{-}\mathcal{B}imor(\mathscr{A},\mathscr{B}(1,f);H)\\ &\cong \mathscr{V}\text{-}\mathcal{PROF}(\mathscr{A},\mathscr{A})(\mathscr{A},H(f,1))\end{aligned}$$

by Lemma 5.26 and Lemma 5.29. The second is analogous. ■

Finally, to generalize the remaining lemmas about profunctors from §3 we require the following weakened notion of adjunction, which does not require all composites to exist.

5.31. DEFINITION. An **adjunction** $H \dashv K$ between $\mathscr{V}$-profunctors $H : \mathscr{A} \nrightarrow \mathscr{B}$ and $K : \mathscr{B} \nrightarrow \mathscr{A}$ consists of a composite $H \odot K$ together with multimorphisms $\eta : () \to H \odot K$ and $\varepsilon : K, H \to \mathscr{B}$ such that the composites

$$\begin{aligned}&H \xrightarrow{\eta,1} H \odot K, H \xrightarrow{\hat{\varepsilon}} H \qquad \text{and}\\ &K \xrightarrow{1,\eta} K, H \odot K \xrightarrow{\check{\varepsilon}} K\end{aligned}$$

are identities. Here $\hat{\varepsilon}$ denotes the unique bimorphism such that

$$H, K, H \to H \odot K, H \xrightarrow{\hat{\varepsilon}} H \qquad \text{is equal to} \qquad H, K, H \xrightarrow{1,\varepsilon} H, \mathscr{B} \to H,$$

and similarly for $\check{\varepsilon}$.

It is easy to show that such adjunctions have all the same formal properties as ordinary adjunctions in a bicategory. We can now duplicate essentially the same proofs from §3 of the following.

5.32. LEMMA. *For any $\mathscr{V}$-functor $f : \mathscr{A} \to \mathscr{B}$ there is an adjunction $\mathscr{A}(1, f) \dashv \mathscr{A}(f, 1)$.*

5.33. LEMMA. *For $\mathscr{V}$-functors $f : \mathscr{A} \to \mathscr{B}$ and $g : \mathscr{B} \to \mathscr{A}$, there is a bijection between adjunctions $f \dashv g$ in $\mathscr{V}$-$\mathcal{CAT}$ and isomorphisms $\mathscr{B}(f, 1) \cong \mathscr{A}(1, g)$ of $\mathscr{V}$-profunctors.*

We also note, for future reference:

5.34. LEMMA. *Suppose $H \dashv K$ and $L \dashv M$ are adjunctions as in Definition 5.31, and that the composites $H \odot L$ and $M \odot K$ and $(H \odot L) \odot (M \odot K)$ exist. Then there is an adjunction $H \odot L \dashv M \odot K$.*

6. $\mathscr{V}$-fibrations

At first glance, indexed $\mathscr{V}$-categories and large $\mathscr{V}$-categories appear very different, but it turns out that both are 'loose enough' notions that they are essentially equivalent. A starting point for this equivalence is to recall from Theorem 2.14 that the fiberwise-homs, which make $\mathscr{V}$ into an indexed $\mathscr{V}$-category, and the external-homs, which make it into a large $\mathscr{V}$-category, are related as follows:

$$\begin{aligned}\underline{\mathscr{V}}(x, y) &\cong \underline{\mathscr{V}}^{Y\times X}\big(\pi_X^* x, \pi_Y^* y\big)\\ \underline{\mathscr{V}}^X(x, y) &\cong \Delta_X^* \underline{\mathscr{V}}(x, y)\end{aligned}$$

Thus, it is natural to try to extend these operations to compare large and indexed $\mathscr{V}$-categories. In one direction this is straightforward: given an indexed $\mathscr{V}$-category $\mathscr{A}$, we define a large $\mathscr{V}$-category $\Theta\mathscr{A}$ whose objects are the objects of the fiber categories $\mathscr{A}^X$, and whose hom-objects are

$$\underline{\Theta\mathscr{A}}(x,y) = \underline{\mathscr{A}}^{Y\times X}\left(\pi_X^* x, \pi_Y^* y\right)$$

(where $x \in \mathscr{A}^X$, $y \in \mathscr{A}^Y$). It is easy to verify that this does, in fact, give a large $\mathscr{V}$-category. Similarly, if $F\colon \mathscr{A} \to \mathscr{B}$ is an indexed $\mathscr{V}$-functor, we define ΘF to take each object $x \in \mathscr{A}^X$ to $Fx \in \mathscr{B}^X$, with $\epsilon(\Theta F)_x = 1_X$ and the obvious action on hom-objects. We leave to the reader the definition of Θ on natural transformations and the verification that it defines a 2-functor

$$\Theta\colon \mathscr{V}\text{-}\mathcal{C}\mathrm{AT} \to \mathscr{V}\text{-}\mathcal{C}AT.$$

from indexed $\mathscr{V}$-categories to large $\mathscr{V}$-categories.

Now Θ is nothing like a 2-equivalence of 2-categories. Any large $\mathscr{V}$-category of the form $\Theta\mathscr{A}$ has lots of objects, and the number of objects is clearly preserved by isomorphisms in $\mathscr{V}\text{-}\mathcal{C}AT$. In particular, no *small* $\mathscr{V}$-category can be isomorphic to anything in the image of Θ. Similarly, the induced functors

$$\Theta\colon \mathscr{V}\text{-}\mathcal{C}\mathrm{AT}(\mathscr{A},\mathscr{B}) \to \mathscr{V}\text{-}\mathcal{C}AT(\Theta\mathscr{A},\Theta\mathscr{B})$$

are clearly not isomorphisms, since every functor of the form ΘF preserves extents strictly (that is, the maps $\epsilon(\Theta F)_x$ are identities).

However, as we will prove shortly, Θ is nevertheless a *biequivalence*. We will approach this by trying to construct an inverse to Θ. One obvious place to start, given a large $\mathscr{V}$-category $\mathscr{B}$, is to try to define an indexed $\mathscr{V}$-category $\Lambda\mathscr{B}$ as follows. We take the objects of $\Lambda\mathscr{B}^X$ to be the objects of $\mathscr{B}$ of extent X, and set

$$\underline{\Lambda\mathscr{B}}^X(x,y) = \Delta_X^* \underline{\mathscr{B}}(x,y).$$

It is easy to check that this defines a $\mathscr{V}^X$-enriched category $\Lambda\mathscr{B}^X$, but when we come to try to define the reindexing functors f^* we are stuck. Just because x is an object of $\mathscr{B}$ with extent Y and we have an arrow $f\colon X \to Y$ in $\mathbf{S}$, there need not be any object at all with extent X; this is glaringly obvious when $\mathscr{B}$ is a small $\mathscr{V}$-category.

This should be regarded as similar to the problem we might encounter when trying to define a inverse to the classical "Grothendieck construction" which makes a pseudofunctor $\mathscr{A} : \mathbf{S}^{\mathrm{op}} \to \mathrm{Cat}$ into a functor $\int \mathscr{A} \to \mathbf{S}$. In that case, the answer is that we need the input functor $\mathbf{A} \to \mathbf{S}$ to be a *fibration*; thus it makes sense to look for "fibrational" conditions on large $\mathscr{V}$-categories.

If x is an object of a large $\mathscr{V}$-category $\mathscr{A}$, we will write x also for the $\mathscr{V}$-functor $\delta(\epsilon x) \to \mathscr{A}$ induced by x and $1_{\epsilon x} : \epsilon x \to \epsilon x$. (Recall from Remark 5.4 that we can make sense of this even if $\mathscr{V}$ lacks indexed coproducts.)

6.1. Definition. Let $\mathscr{A}$ be a large $\mathscr{V}$-category, x an object of $\mathscr{A}$, and $f\colon Y \to \epsilon x$ a morphism in $\mathbf{S}$. A **restriction** of x along f is an object f^*x of $\mathscr{A}$ such that $\epsilon(f^*x) = Y$, together with an isomorphism between the $\mathscr{V}$-functors

$$\delta Y \xrightarrow{f^*x} \mathscr{A} \qquad \text{and} \qquad \delta Y \xrightarrow{\delta f} \delta X \xrightarrow{x} \mathscr{A}.$$

Of course, by the Yoneda lemma, this can equivalently be expressed by isomorphisms of profunctors

$$\begin{aligned} \mathscr{A}(1, f^*x) &\cong \mathscr{A}(1, \delta f \circ x) \qquad \text{or} \\ \mathscr{A}(f^*x, 1) &\cong \mathscr{A}(\delta f \circ x, 1) \end{aligned}$$

If we note that $\mathscr{A}(1, \delta f \circ x) \cong \mathscr{A}(1, x)(1, \delta f)$ and similarly, and evaluate these profunctors at some $y \in \mathscr{A}$, we obtain isomorphisms

$$\begin{aligned} \underline{\mathscr{A}}(y, f^*x) &\cong (f \times 1)^* \underline{\mathscr{A}}(y, x) \qquad \text{and} \\ \underline{\mathscr{A}}(f^*x, y) &\cong (1 \times f)^* \underline{\mathscr{A}}(x, y). \end{aligned}$$

In the case $\mathscr{V} = \mathscr{S}\!elf(\mathbf{S})$, this idea is due to [BW87], where restrictions are called *substitutions*.

We can now characterize the large $\mathscr{V}$-categories in the image of Θ.

6.2. Definition. A **$\mathscr{V}$-fibration** is a large $\mathscr{V}$-category such that for each object x and each $f\colon Y \to \epsilon x$, there exists a restriction f^*x.

6.3. Remark. The phrase "$\mathscr{V}$-fibration" is, of course, motivated by the remarks above comparing Θ to the Grothendieck construction. Moreover, just as ordinary fibrations replace the "algebraic" reindexing functors of an ordinary indexed category by cartesian arrows with a universal property, $\mathscr{V}$-fibrations replace the reindexing functors of an indexed $\mathscr{V}$-category by "restrictions" as defined above, which are objects with a sort of universal property. In particular, it is no longer necessary to specify the coherence isomorphisms in Definition 4.1; they follow automatically from the universal property.

However, the analogy is just an analogy: there is no $\mathbf{S}$-indexed monoidal category $\mathscr{V}$ such that large $\mathscr{V}$-categories can be identified with arbitrary functors into $\mathbf{S}$. (Example 4.7 might suggest that $\mathscr{P}sh(\mathbf{S}, \mathrm{Set})$ should have this property, but it does not). Large $\mathscr{V}$-categories contain more data than an arbitrary functor into $\mathbf{S}$, which as we will see is in fact sufficient to characterize a corresponding $\mathscr{V}$-fibration.

6.4. Remark. On the other hand, $\mathscr{V}$-fibrations share the virtue of ordinary fibrations that for fixed $\mathscr{V}$, they are an elementary (first-order) notion, as contrasted with indexed $\mathscr{V}$-categories and classical indexed categories which are not.

6.5. Proposition. *A large $\mathscr{V}$-category is isomorphic to one of the form $\Theta\mathscr{A}$ if and only if it is a $\mathscr{V}$-fibration.*

Proof. If $\mathscr{A}$ is an indexed $\mathscr{V}$-category with transition functors $f^* : \mathscr{A}^X \to \mathscr{A}^Y$, then it is easy to check that for any $x \in \mathscr{A}^X$, the object $f^*x \in \mathscr{A}^Y$ is a restriction of x along f in $\Theta\mathscr{A}$. Thus $\Theta\mathscr{A}$, and anything isomorphic to it, is a $\mathscr{V}$-fibration.

Conversely, given a $\mathscr{V}$-fibration $\mathscr{B}$, we complete the above construction of an indexed $\mathscr{V}$-category $\Lambda\mathscr{B}$ as follows. We choose, for every x and f, a restriction f^*x, and define the functor $f^*\colon (f^*)_\bullet\Lambda\mathscr{B}^Y \to \Lambda\mathscr{B}^X$ to take x to f^*x. The definition of restriction ensures that this can be extended to a fully faithful $\mathscr{V}^Y$-functor, and the essential uniqueness of restrictions ensures that they are coherent. Finally, it is straightforward to check that $\Theta\Lambda\mathscr{B} \cong \mathscr{B}$ in $\mathscr{V}\text{-}\mathcal{CAT}$. ∎

As remarked above, in order to fully characterize the image of Θ, we will also need to limit the functors we consider.

6.6. Definition. A $\mathscr{V}$-functor $f : \mathscr{A} \to \mathscr{B}$ between large $\mathscr{V}$-categories is called **indexed** if ϵf_x is an identity for all x.

If we now let $\mathscr{V}\text{-}\mathcal{FIB}$ denote the sub-2-category of $\mathscr{V}\text{-}\mathcal{CAT}$ consisting of the $\mathscr{V}$-fibrations, the indexed $\mathscr{V}$-functors between them, and all the $\mathscr{V}$-natural transformations between those, then it is easy to extend Λ to a 2-functor $\mathscr{V}\text{-}\mathcal{FIB} \to \mathscr{V}\text{-}\mathcal{C}\textsc{at}$. Our double use of the word 'indexed' is unproblematic because of the following result.

6.7. Theorem. *The 2-functors Θ and Λ are inverse 2-equivalences between $\mathscr{V}\text{-}\mathcal{C}\textsc{at}$ and $\mathscr{V}\text{-}\mathcal{FIB}$.*

Proof. Left to the reader. ∎

Since 2-equivalences preserve all 2-categorical structure, we can use indexed $\mathscr{V}$-categories and $\mathscr{V}$-fibrations interchangeably, just as we do for ordinary fibrations and pseudofunctors, and we will rarely distinguish notationally between them.

6.8. Remark. This 2-equivalence also extends to profunctors. We could define a virtual equipment of indexed $\mathscr{V}$-profunctors and show it is equivalent to the restriction of $\mathscr{V}\text{-}\mathbb{P}\mathsf{ROF}$ to the $\mathscr{V}$-fibrations and indexed functors. However, for our purposes it will suffice to note that for $\mathscr{V}$-fibrations $\mathscr{A}$ and $\mathscr{B}$, we have an equivalence of categories

$$\mathscr{V}\text{-}\mathcal{P}\textsc{rof}(\mathscr{A},\mathscr{B}) \simeq \mathscr{V}\text{-}\mathcal{PROF}(\mathscr{A},\mathscr{B})$$

connecting indexed $\mathscr{V}$-profunctors, as in Definition 4.9, to $\mathscr{V}$-profunctors as considered in §5. This equivalence is constructed just as for the hom-objects of categories, by restricting along diagonals and projections.

In contrast to the classical case, however, it turns out that by including the *non-indexed* $\mathscr{V}$-functors, we can put back in the large $\mathscr{V}$-categories that *aren't* $\mathscr{V}$-fibrations and still maintain a biequivalence. We first observe the following.

6.9. Proposition. *If $\mathscr{B}$ is a $\mathscr{V}$-fibration, then any $\mathscr{V}$-functor $F\colon \mathscr{A} \to \mathscr{B}$ is naturally isomorphic to an indexed one.*

PROOF. Given F, with components $\epsilon F_x\colon \epsilon x \to \epsilon(Fx)$, we define $F'\colon \mathscr{A} \to \mathscr{B}$ by choosing $F'x$ to be a restriction $(\epsilon F_x)^*(Fx)$ of Fx along ϵF_x. It is easy to check that F' is an indexed $\mathscr{V}$-functor, and that the isomorphisms $(\epsilon F_x)^*(Fx) \cong (Fx) \circ \delta(\epsilon F_x)$ from Definition 6.1 assemble into a natural isomorphism $F \cong F'$. ∎

It follows that $\mathscr{V}$-$\mathcal{FIB}$, while not a full sub-2-category of $\mathscr{V}$-$\mathcal{CAT}$, is a 'full sub-bicategory' in the sense that the inclusions

$$\mathscr{V}\text{-}\mathcal{FIB}(\mathscr{A},\mathscr{B}) \hookrightarrow \mathscr{V}\text{-}\mathcal{CAT}(\mathscr{A},\mathscr{B})$$

are equivalences of categories. Thus, to prove that $\mathscr{V}$-$\mathcal{FIB} \hookrightarrow \mathscr{V}$-$\mathcal{CAT}$ is a biequivalence, it suffices to check that every large $\mathscr{V}$-category is equivalent, in $\mathscr{V}$-$\mathcal{CAT}$, to a $\mathscr{V}$-fibration. This is included in the following theorem.

6.10. THEOREM. *The (non-full) inclusion $\mathscr{V}$-$\mathcal{FIB} \hookrightarrow \mathscr{V}$-$\mathcal{CAT}$ has a right 2-adjoint Γ; this means that for a $\mathscr{V}$-fibration $\mathscr{A}$ and a large $\mathscr{V}$-category $\mathscr{B}$, we have natural isomorphisms of hom-categories*

$$\mathscr{V}\text{-}\mathcal{FIB}(\mathscr{A},\Gamma\mathscr{B}) \cong \mathscr{V}\text{-}\mathcal{CAT}(\mathscr{A},\mathscr{B}).$$

Moreover, the unit and counit $\mathscr{A} \to \Gamma\mathscr{A}$ and $\Gamma\mathscr{B} \to \mathscr{B}$ are internal equivalences, so this 2-adjunction is actually a biequivalence.

PROOF. We define the objects of $\Gamma\mathscr{B}$ to be 'formal restrictions' f^*x, where x is an object of $\mathscr{B}$ with extent X and $f\colon Y \to X$ is a map in $\mathbf{S}$. Of course, we set $e(f^*x) = Y$, and we define $\underline{\Gamma\mathscr{B}}(f^*x, g^*y)$ to be $(g \times f)^*\underline{\mathscr{B}}(x,y)$. We leave it to the reader to define the rest of the structure and check that $\Gamma\mathscr{B}$ is a $\mathscr{V}$-fibration.

Now, an indexed $\mathscr{V}$-functor $\mathscr{A} \to \Gamma\mathscr{B}$ sends each object a of $\mathscr{A}$ to a formal restriction f^*x in $\Gamma\mathscr{B}$, where $f\colon \epsilon a \to \epsilon x$ is a map in $\mathbf{S}$. On the other hand, a non-indexed $\mathscr{V}$-functor $\mathscr{A} \to \mathscr{B}$ sends each object a to an object x and chooses a map $f\colon \epsilon a \to \epsilon x$, so at this level the bijection is obvious. It is easy to check that it carries over to the action on hom-objects and to natural transformations, so that Γ defines a right 2-adjoint to the inclusion.

Now since the inclusion is *bicategorically* fully faithful, it follows automatically that the unit $\mathscr{A} \to \Gamma\mathscr{A}$ is an *equivalence* (though not an isomorphism). Thus, it remains to check that the counit $\varepsilon\colon \Gamma\mathscr{B} \to \mathscr{B}$ is an equivalence for any large $\mathscr{V}$-category $\mathscr{B}$.

Of course, the counit $\varepsilon\colon \Gamma\mathscr{B} \to \mathscr{B}$ sends f^*x to $\varepsilon(f^*x) = x$ with $\epsilon(\varepsilon_{f^*x}) = f$. We define a $\mathscr{V}$-functor $\xi\colon \mathscr{B} \to \Gamma\mathscr{B}$ by sending each object x of $\mathscr{B}$ to its formal restriction $1_{\epsilon x}^*x$. Clearly $\varepsilon\xi$ is the identity on $\mathscr{B}$. The composition $\xi\varepsilon$ sends the formal restriction f^*x to $1_{\epsilon x}^*x$, with $\epsilon(\xi\varepsilon)_{f^*x} = f$. It suffices to show that $\xi\varepsilon \cong \mathrm{Id}_{\Gamma\mathscr{B}}$ in $\mathscr{V}$-$\mathcal{CAT}$, which we can do by assembling the isomorphisms from Definition 6.1, as we did in Proposition 6.9. ∎

6.11. EXAMPLE. If $\mathbf{V}$ is a classical monoidal category and $\mathbf{C}$ is a small $\mathbf{V}$-enriched category, regarded as a small $\mathscr{F}am(\mathbf{V})$-category as in Example 3.2, then $\Gamma\mathbf{C}$ is the $\mathscr{F}am(\mathbf{V})$-category $\mathscr{F}am(\mathbf{C})$ constructed in Example 4.2.

6.12. Example. If A is an $\mathbf{S}$-internal category regarded as a small $\mathscr{S}\!\mathit{elf}(\mathbf{S})$-category, then ΓA is the locally internal category classically associated to A.

6.13. Remark. On the other hand, if we write $\mathscr{V}\text{-}\mathcal{CAT}_{\mathrm{ind}}$ for the sub-2-category of $\mathscr{V}\text{-}\mathcal{CAT}$ containing all the $\mathscr{V}$-categories but only the indexed $\mathscr{V}$-functors, then the non-full inclusion $\mathscr{V}\text{-}\mathcal{CAT}_{\mathrm{ind}} \hookrightarrow \mathscr{V}\text{-}\mathcal{CAT}$ is *not* a biequivalence. This is relevant because if we were to restrict ourselves to data contained in the *bicategory* constructed from $\mathscr{V}$ (rather than the whole equipment), then the indexed $\mathscr{V}$-functors would be the only type of morphism available. For this reason, the authors of [BCSW83, BW87, Bet89, Bet00] had to impose extra conditions at least as strong as being a $\mathscr{V}$-fibration in order to obtain an equivalence with indexed $\mathscr{V}$-categories (in the case $\mathscr{V} = \mathscr{S}\!\mathit{elf}(\mathbf{S})$, which is the only one they considered).

Most $\mathscr{V}$-categories which arise "in nature" are either small or are $\mathscr{V}$-fibrations. We can regard the other large $\mathscr{V}$-categories as a technical tool which makes it easier to relate these two most important types. (As we will see in §8, set-sized $\mathscr{V}$-categories are also convenient to use as diagram shapes.)

6.14. Remark. We can also define an anologue of the general hom-functors from the end of §2 for any $\mathscr{V}$-fibration $\mathscr{A}$; we set

$$\begin{aligned}\underline{\mathscr{A}}^{Y,[W]}(B,C) &= \pi_{W*}\Delta^*_{Y\times W}\underline{\mathscr{A}}(B,C)\\ &\cong \pi_{W*}\underline{\mathscr{A}}^{X\times Y\times Z\times W}(\pi_X^* B, \pi_Z^* C).\end{aligned}$$

When we consider tensors, cotensors, and monoidal structures for $\mathscr{V}$-categories, we will also find analogues for $\mathscr{V}$-fibrations of the various types of monoidal structure on $\mathscr{V}$.

7. Change of cosmos and underlying indexed categories

If $\mathscr{V}$ is an $\mathbf{S}$-indexed monoidal category and $\mathscr{W}$ is a $\mathbf{T}$-indexed one, then by a **lax monoidal morphism** $\Phi : \mathscr{V} \to \mathscr{W}$ we mean a commutative square

$$\begin{array}{ccc} \int\mathscr{V} & \xrightarrow{\Phi} & \int\mathscr{W} \\ \downarrow & & \downarrow \\ \mathbf{S} & \xrightarrow[\Phi]{} & \mathbf{T} \end{array}$$

such that $\Phi : \mathbf{S} \to \mathbf{T}$ preserves finite products (hence is strong cartesian monoidal), $\Phi : \int\mathscr{V} \to \int\mathscr{W}$ is lax monoidal and preserves cartesian arrows, and the square commutes in the 2-category of lax monoidal functors. If $\Phi : \mathbf{S} \to \mathbf{T}$ is an identity, as is often the case, we say that Φ is a morphism **over S.**

In this situation, we have induced operations $\Phi_\bullet$ from small, large, and indexed $\mathscr{V}$-categories to the corresponding sort of $\mathscr{W}$-categories, and similarly for functors, transformations, profunctors, multimorphisms, and so on, which we call **change of cosmos**.

Formally, $\Phi_\bullet$ is a (normal, lax) equipment functor $\mathscr{V}\text{-}\mathbb{P}\mathsf{ROF} \to \mathscr{W}\text{-}\mathbb{P}\mathsf{ROF}$, which in particular induces 2-functors $\mathscr{V}\text{-}\mathcal{C}at \to \mathscr{W}\text{-}\mathcal{C}at$, $\mathscr{V}\text{-}\mathcal{CAT} \to \mathscr{W}\text{-}\mathcal{CAT}$, and so on.

If Φ is strong monoidal and preserves indexed coproducts, fiberwise coequalizers, and fiberwise coproducts of the appropriate cardinalities, then $\Phi_\bullet$ preserves composition of profunctors. Similarly, if Φ is closed monoidal and preserves indexed products, fiberwise equalizers, and fiberwise products of the appropriate cardinalities, then $\Phi_\bullet$ preserves right and left homs of profunctors.

We can furthermore assemble the operations $(-)\text{-}\mathbb{P}\mathsf{ROF}$ and $(-)_\bullet$ into a 2-functor from a 2-category of indexed monoidal categories as in [Shu08] into the 2-category $v\mathcal{E}quip$ of [CS10, 7.6]. This can be decomposed into the 2-functor $\mathbb{F}\mathrm{r}$ of [Shu08, 14.9] (suitably generalized to the virtual case) followed by a many-object version of the 2-functor $\mathbb{M}\mathsf{od}$ of [CS10, 3.9]. In particular, any monoidal adjunction between indexed monoidal categories induces an adjunction between 2-categories (or equipments) of enriched indexed categories.

We omit the details of all of this since we will not need them here; instead we merely mention some important special cases.

7.1. Example. Any lax monoidal functor $\mathbf{V} \to \mathbf{W}$ gives rise to a lax monoidal morphism $\mathscr{F}am(\mathbf{V}) \to \mathscr{F}am(\mathbf{W})$. When we identify $\mathbf{V}$-enriched categories with certain indexed $\mathscr{F}am(\mathbf{V})$-categories as in Example 4.2, the induced operations from $\mathscr{F}am(\mathbf{V})$-categories to $\mathscr{F}am(\mathbf{W})$-categories agree with the classical change-of-enrichment operations.

7.2. Example. Any pullback-preserving functor $F\colon \mathbf{S} \to \mathbf{T}$ gives rise to a strong monoidal morphism $\mathscr{S}elf(\mathbf{S}) \to \mathscr{S}elf(\mathbf{T})$. The induced operations on internal categories and locally internal categories agree with the obvious ones.

7.3. Example. If $\mathscr{V}$ has indexed coproducts preserved by $\otimes$, then there is a strong monoidal morphism $\Sigma : \mathscr{S}elf(\mathbf{S}) \longrightarrow \mathscr{V}$, which takes an object $A \xrightarrow{a} X$ of $\mathscr{S}elf(\mathbf{S})^X = \mathbf{S}/X$ to the object $a_!\mathbb{I}_A$ of $\mathscr{V}^X$. The pseudonaturality isomorphism $\Sigma(f^*A) \cong f^*(\Sigma A)$ is just the Beck-Chevalley condition for indexed coproducts in $\mathscr{V}$, together with the fact that $f^*\mathbb{I}_Y \cong \mathbb{I}_X$. And for a pullback square

$$\begin{tikzcd} C \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, "c"] & B \arrow[d, "b"] \\ A \arrow[r, "a"'] & X \end{tikzcd}$$

in $\mathbf{S}$, regarded as the fiberwise product $C = A \times_X B$ in $\mathscr{S}elf(\mathbf{S})^X$, the comparison isomorphism $\Sigma(A) \otimes_X \Sigma(B) \xrightarrow{\sim} \Sigma(C)$ is the composite

$$\begin{aligned} a_!\mathbb{I}_A \otimes_X b_!\mathbb{I}_B &\cong a_!(\mathbb{I}_A \otimes_A a^*b_!\mathbb{I}_B) && \text{(since $\otimes$ preserves indexed coproducts)} \\ &\cong a_!(\mathbb{I}_A \otimes_A p_!q^*\mathbb{I}_B) && \text{(by the Beck-Chevalley condition)} \\ &\cong a_!p_!(p^*\mathbb{I}_A \otimes_C q^*\mathbb{I}_B) && \text{(since $\otimes$ preserves indexed coproducts)} \\ &\cong c_!(\mathbb{I}_C \otimes_C \mathbb{I}_C) && \text{(since p^* and q^* are strong monoidal)} \\ &\cong c_!(\mathbb{I}_C). \end{aligned}$$

It is evident that Σ also preserves indexed coproducts.

We say that the induced functor $\mathscr{S}elf(\mathbf{S})$-$\mathcal{CAT} \to \mathscr{V}$-$\mathcal{CAT}$ builds the **free $\mathscr{V}$-category** on a $\mathscr{S}elf(\mathbf{S})$-category, and write it as $\mathscr{V}[-]$. As a special case, the discrete $\mathscr{V}$-category δX is the free $\mathscr{V}$-category on the discrete $\mathbf{S}$-internal category on X.

(This is the primary place in this paper where we use the fully general Beck-Chevalley condition for $\mathscr{V}$, as opposed to the limited version described in Example 2.38. Thus we cannot expect to build free $\mathscr{A}ct(\mathbf{S})$-categories.)

7.4. Example. Assuming $\mathbf{S}$ is locally small, the fiberwise Yoneda embedding gives a strong monoidal morphism $\mathscr{S}elf(\mathbf{S}) \to \mathscr{P}sh(\mathbf{S}, \mathrm{Set})$. Since this morphism is fiberwise fully faithful, the induced functor from $\mathscr{S}elf(\mathbf{S})$-categories to $\mathscr{P}sh(\mathbf{S}, \mathrm{Set})$-categories is 2-fully-faithful. Hence, we can regard $\mathscr{S}elf(\mathbf{S})$-categories as $\mathscr{P}sh(\mathbf{S}, \mathrm{Set})$-categories—that is, classical $\mathbf{S}$-indexed categories with locally small fibers—with the property that all their hom-presheaves are representable.

When expressed in terms of the fiberwise-homs $\underline{\mathscr{A}}^X(x, y)$ of an indexed $\mathscr{P}sh(\mathbf{S}, \mathrm{Set})$-category, this requires that for any $x, y \in \mathscr{A}^X$ the functor

$$\begin{array}{rcl} \mathbf{S}/X & \longrightarrow & \mathrm{Set} \\ (Z \xrightarrow{f} X) & \mapsto & \mathscr{A}^Z(f^*x, f^*y) \end{array}$$

is representable. And when expressed equivalently in terms of the external-homs $\underline{\mathscr{A}}(x, y)$ of a large $\mathscr{P}sh(\mathbf{S}, \mathrm{Set})$-category, it requires that for any $x \in \mathscr{A}^X$ and $y \in \mathscr{A}^Y$, the functor

$$\begin{array}{rcl} \mathbf{S}/(X \times Y) & \longrightarrow & \mathrm{Set} \\ (Z \xrightarrow{(f,g)} X \times Y) & \mapsto & \mathscr{A}^Z(f^*x, g^*y) \end{array}$$

is representable. But this latter condition is exactly the usual notion of when an indexed category is "locally small". Thus, we recover the theorem identifying locally small indexed categories with "locally internal categories" (which, recall, are the same as indexed $\mathscr{S}elf(\mathbf{S})$-categories).

7.5. Example. In fact, for *any* $\mathscr{V}$ with locally small fibers, there is a lax monoidal morphism $\mathscr{V} \to \mathscr{P}sh(\mathbf{S}, \mathrm{Set})$, which takes $A \in \mathscr{V}^X$ to the functor

$$\begin{array}{rcl} (\mathbf{S}/X)^{\mathrm{op}} & \longrightarrow & \mathrm{Set} \\ (Y \xrightarrow{f} X) & \mapsto & \mathscr{V}^X(\mathbb{I}_Y, f^*A). \end{array} \tag{7.6}$$

In the case $\mathscr{V} = \mathscr{S}elf(\mathbf{S})$ this reduces to the fiberwise Yoneda embedding. In general, it implies that any indexed $\mathscr{V}$-category $\mathscr{A}$ has an underlying ordinary $\mathbf{S}$-indexed category, which we denote $\mathscr{A}_{\mathrm{o}}$. We call $\mathscr{A}_{\mathrm{o}}$ the **underlying indexed category** of $\mathscr{A}$; it should be regarded as an indexed version of the classical "underlying ordinary category of an enriched category". In particular, if $\mathscr{V}$ is closed, then by applying this construction to the $\mathscr{V}$-category $\mathscr{V}$ we recover the original $\mathbf{S}$-indexed category $\mathscr{V}$.

Tracing through the identification of indexed $\mathscr{P}sh(\mathbf{S}, \mathrm{Set})$-categories with classical indexed categories, we see that the fiber category $(\mathscr{A}_{\mathrm{o}})^X$ over X is the underlying ordinary category of the classical $\mathscr{V}^X$-enriched category $\mathscr{A}^X$, generally denoted $(\mathscr{A}^X)_\mathrm{o}$. Thus, there is no ambiguity in writing $\mathscr{A}_\mathrm{o}^X$ for this fiber.

If $\mathscr{A}$ is a (large or small) $\mathscr{V}$-category that is not a $\mathscr{V}$-fibration, then it still has an underlying $\mathbf{S}$-indexed category, namely $(\Gamma\mathscr{A})_\mathrm{o}$ where Γ is the functor from Theorem 6.10. It is easy to see that this is precisely the underlying indexed category constructed in Example 3.18 and Remark 5.4. However, if $\mathscr{A}$ *is* a $\mathscr{V}$-fibration, then $(\Gamma\mathscr{A})_\mathrm{o}$ is rather larger than $\mathscr{A}_\mathrm{o}$ (though still equivalent to it).

7.7. Example. If $\mathscr{V}$ has the property that each functor (7.6) is representable, then the morphism $\mathscr{V} \to \mathscr{P}sh(\mathbf{S}, \mathrm{Set})$ factors through $\mathscr{S}elf(\mathbf{S})$, and so $\mathscr{A}_\mathrm{o}$ is a $\mathscr{S}elf(\mathbf{S})$-category for any $\mathscr{V}$-category $\mathscr{A}$. When $\mathscr{V}$ is cartesian, this property of $\mathscr{V}$ is precisely the **comprehension schema** of [Law70], so we will henceforth extend that terminology to the non-cartesian case.

In particular, if $\mathscr{V}$ is closed, then the $\mathscr{V}$-category $\mathscr{V}$ itself has an underlying $\mathscr{S}elf(\mathbf{S})$-category. It is easy to see that this means the $\mathbf{S}$-indexed category $\mathscr{V}$ is locally small. Indeed, for a closed $\mathscr{V}$, local smallness is equivalent to the comprehension schema.

If $\mathscr{V}$ satisfies the comprehension schema and also has indexed coproducts preserved by $\otimes$, then the "underlying" morphism $\mathscr{V} \to \mathscr{S}elf(\mathbf{S})$ is right adjoint to the "free" morphism $\mathscr{S}elf(\mathbf{S}) \to \mathscr{V}$ from Example 7.3. Thus, the induced functors on enriched indexed categories are likewise adjoint.

For instance, when $\mathscr{V} = \mathscr{S}elf_*(\mathbf{S})$, we see that every pointed internal category (as in Examples 3.4) has an underlying ordinary internal category, and that this operation has a left adjoint which "adjoins a disjoint section" to the hom-objects. Similarly, if $\mathbf{S}$ satisfies the hypotheses of Example 2.34, then any $\mathscr{A}b(\mathbf{S})$-category has an underlying $\mathscr{S}elf(\mathbf{S})$-category, and this operation has a left adjoint which builds free abelian group objects on the hom-objects.

7.8. Example. Here is an example in which the base category changes. Let $\mathbf{S}$ have finite limits; then there is a lax monoidal morphism $\mathscr{S}elf(\mathbf{S}) \to \mathscr{F}am(\mathbf{S})$, which takes $X \in \mathbf{S}$ to the set $\mathbf{S}(1, X)$ and an object $A \in \mathbf{S}/X$ to its family of fibers. Thus, any $\mathbf{S}$-internal category gives rise to a small $\mathbf{S}$-enriched category. For example, any internal topological category gives rise to a topologically enriched category by forgetting the topology on the set of objects.

If $\mathbf{S}$ has small coproducts preserved by pullback, then this morphism has a strong monoidal left adjoint, which sends a set X to $\coprod_{x\in X} 1$ and an X-indexed family $\{A_x\}$ of objects of $\mathbf{S}$ to $\coprod_{x\in X} A_x$. The corresponding operation on small $\mathbf{S}$-enriched categories regards them as $\mathbf{S}$-internal categories whose object-of-objects is "discrete" (i.e. a coproduct of copies of the terminal object).

8. Limits and colimits

We now begin studying limits and colimits for $\mathscr{V}$-categories. Here is where the equipment-theoretic machinery of profunctors is most helpful, because it automatically gives us a general definition of weighted limit with many good formal properties. In this section we recall this definition and some of these good properties; in the next section we translate some examples into more concrete terms for indexed $\mathscr{V}$-categories.

8.1. DEFINITION. Let $J\colon K \nrightarrow A$ be a $\mathscr{V}$-profunctor and $f\colon A \to \mathscr{C}$ a $\mathscr{V}$-functor. A **J-weighted colimit of f** consists of a $\mathscr{V}$-functor $\ell\colon K \to \mathscr{C}$ together with an isomorphism

$$\mathscr{C}(\ell, 1) \cong J \rhd \mathscr{C}(f, 1) \tag{8.2}$$

of profunctors $\mathscr{C} \nrightarrow K$. (Recall Remark 5.27.)

If instead $J\colon A \nrightarrow K$, then a **J-weighted limit of f** consists of a $\mathscr{V}$-functor $\ell\colon K \to X$ together with an isomorphism

$$\mathscr{C}(1, \ell) \cong \mathscr{C}(1, f) \lhd J.$$

of profunctors $K \nrightarrow \mathscr{C}$.

In general, K, A, and $\mathscr{C}$ could be any large $\mathscr{V}$-categories. However, in our examples, most often K and A will be small or set-small, while $\mathscr{C}$ will be a $\mathscr{V}$-fibration (hence our choice of typefaces).

We will consider many examples in §9, but we should at least mention the following one here, to clarify why this is a reasonable definition of "weighted limit".

8.3. EXAMPLE. If $\mathscr{V} = \mathscr{F}am(\mathbf{V})$ and we take K to be the unit $\mathscr{V}$-category $\delta 1$ and A a small $\mathscr{V}$-category, then J is simply a diagram on A. If $\mathscr{C}$ is the indexed $\mathscr{V}$-category constructed from a (possibly large) $\mathbf{V}$-enriched category $\mathbf{C}$ as in Example 4.2, then the above definitions reduce to the usual notion of weighted limit and colimit. Specifically, if we assume ℓ to be an indexed $\mathscr{V}$-functor, then it is just an object of $\mathbf{C}$, and (in the colimit case) the isomorphism (8.2) means that

$$\mathbf{C}(\ell, x) \cong [A^{\mathrm{op}}, \mathbf{C}](J, \mathbf{C}(f-, x))$$

for all $x \in \mathbf{C}$, which is the usual definition of ℓ being a J-weighted colimit of f.

By contrast with the above classical situation, in general it turns out to be very useful to allow K to be an arbitrary $\mathscr{V}$-category. Here is one example of what such more general "limits" include.

8.4. EXAMPLE. Let $j\colon K \to A$ and take $J = A(1, j)$. Then for any $f\colon A \to \mathscr{C}$, we have

$$A(1, j) \rhd \mathscr{C}(f, 1) \cong \mathscr{C}(fj, 1)$$

by Lemma 5.29. Hence fj is always a $A(1, j)$-weighted colimit of f. Dually, fj is also always an $A(j, 1)$-weighted limit of f.

In particular, if j is the identity functor of A, then $A(1, 1) = A$ is the identity profunctor of A, and f is its own A-weighted (co)limit.

One real advantage of allowing arbitrary profunctors as weights is that a given profunctor can be used both as a weight for limits and a weight for colimits. This symmetry is necessary in order to even *state* the following fact, which will be very useful.

8.5. Proposition. *For a $\mathscr{V}$-profunctor $J\colon A \nrightarrow B$ and $\mathscr{V}$-functors $f\colon B \to \mathscr{C}$ and $g\colon A \to \mathscr{C}$ which have respectively a J-weighted colimit $A \xrightarrow{\operatorname{colim}^J f} \mathscr{C}$ and a J-weighted limit $B \xrightarrow{\lim^J g} \mathscr{C}$, we have a natural isomorphism*

$$\mathscr{V}\text{-}\mathcal{CAT}(\operatorname{colim}^J f, g) \cong \mathscr{V}\text{-}\mathcal{CAT}(f, \lim^J g).$$

Proof. We calculate

$$\begin{aligned}
\mathscr{V}\text{-}\mathcal{CAT}(\operatorname{colim}^J f, g) &\cong \mathscr{V}\text{-}\mathcal{PROF}(\mathscr{C}(g,1), \mathscr{C}(\operatorname{colim}^J f, 1)) \\
&\cong \mathscr{V}\text{-}\mathcal{B}imor(\mathscr{C}(g,1), J; \mathscr{C}(f,1)) \\
&\cong \mathscr{V}\text{-}\mathcal{B}imor(J, \mathscr{C}(1,f); \mathscr{C}(1,g)) \\
&\cong \mathscr{V}\text{-}\mathcal{PROF}(\mathscr{C}(1,f), \mathscr{C}(1, \lim^J g)) \\
&\cong \mathscr{V}\text{-}\mathcal{CAT}(f, \lim^J g).
\end{aligned}$$
∎

We immediately obtain a description of Kan extensions as particular weighted limits and colimits.

8.6. Corollary. *Let $j\colon A \to K$ and $f\colon A \to X$. Then any $K(j,1)$-weighted colimit of f is an internal left extension of f in $\mathscr{V}$-$\mathcal{CAT}$, and any $K(1,j)$-weighted limit of f is an internal right extension of f in $\mathscr{V}$-$\mathcal{CAT}$.*

Proof. Internal left extension is defined to be a (partial) left adjoint to composition, and we observed in Example 8.4 that $\underline{K}(j,1)$-weighted limits are composites with j. ∎

In general, not all 2-categorical left extensions have the stronger universal property of a $K(1,j)$-weighted colimit. Sometimes extensions with this additional property are called *pointwise* (see [ML98, §X.5]), but we follow [Kel82, Ch. 4] in reserving the simple term *Kan extension* for the pointwise ones.

8.7. Definition. A **left Kan extension** of $f\colon A \to \mathscr{C}$ along $j\colon A \to K$ is a $K(j,1)$-weighted colimit of f. Dually, a **right Kan extension** of f along j is a $K(1,j)$-weighted limit of f.

The following theorem, which will also be very useful, also requires the use of arbitrary profunctors as weights.

8.8. Theorem. *Let $J_2\colon A \nrightarrow B$ and $J_1\colon B \nrightarrow C$ be weights and let $f\colon C \to \mathscr{C}$ be a $\mathscr{V}$-functor. Suppose that ℓ_1 is a J_1-weighted colimit of f and ℓ_2 is a J_2-weighted colimit of ℓ_1, and that the composite $J_2 \odot J_1$ exists. Then ℓ_2 is a $(J_2 \odot J_1)$-weighted colimit of f.*

Proof. For any $\vec{H} = H_1, \ldots, H_n$, we have

$$\begin{aligned}
\mathscr{V}\text{-}\mathcal{M}ultimor(\vec{H}, J_2 \odot J_1; \mathscr{C}(f,1)) &\cong \mathscr{V}\text{-}\mathcal{M}ultimor(\vec{H}, J_2, J_1; \mathscr{C}(f,1)) \\
&\cong \mathscr{V}\text{-}\mathcal{M}ultimor(\vec{H}, J_2; \mathscr{C}(\ell_1,1)) \\
&\cong \mathscr{V}\text{-}\mathcal{M}ultimor(\vec{H}; \mathscr{C}(\ell_2,1)).
\end{aligned}$$
∎

Here is one example of the usefulness of Theorem 8.8. We say that a $\mathscr{V}$-functor $f : \mathscr{A} \to \mathscr{B}$ is **fully faithful** if each morphism $\mathscr{A}(x, y) \to \mathscr{B}(f(x), f(y))$ is cartesian over $\epsilon f_x \times \epsilon f_y$. It is easy to see that this is equivalent to the induced morphism $\mathscr{A} \to \mathscr{B}(f, f)$ of profunctors $\mathscr{A} \nrightarrow \mathscr{A}$ being an isomorphism, or equivalently that $\mathscr{B}(1, f) \odot \mathscr{B}(f, 1) \cong \mathscr{A}$.

8.9. COROLLARY. *Left and right Kan extensions along fully faithful $\mathscr{V}$-functors are honest extensions. In other words, if $j\colon A \to K$ is fully faithful and $\ell\colon K \to \mathscr{C}$ is a left Kan extension of $f\colon A \to \mathscr{C}$ along j, then $\ell j \cong f$.*

PROOF. Left Kan extensions are $K(j, 1)$-weighted colimits, while precomposition with j is a $K(1, j)$-weighted colimit. Thus, by Theorem 8.8, ℓj is a $(K(1, j) \odot K(j, 1))$-weighted colimit. However, since j is fully faithful, $(K(1, j) \odot K(j, 1))$ is isomorphic to the identity profunctor, for which a weighted colimit of f is just f itself. The case of right Kan extensions is dual. ∎

We now consider what it means for a $\mathscr{V}$-functor to preserve or reflect limits. Let $J\colon K \nrightarrow A$ be a weight and $d\colon A \to \mathscr{C}$ a $\mathscr{V}$-functor, and suppose given a bimorphism

$$\mathscr{C}(\ell, 1), J \xrightarrow{\psi} \mathscr{C}(d, 1) \tag{8.10}$$

Let $f\colon \mathscr{C} \to \mathscr{D}$ be a $\mathscr{V}$-functor, and consider the unique morphism

$$\mathscr{C}(f\ell, 1), J \longrightarrow \mathscr{C}(fd, 1) \tag{8.11}$$

whose composite with the universal bimorphism $\mathscr{C}(f, 1), \mathscr{C}(\ell, 1) \xrightarrow{\phi} \mathscr{C}(f\ell, 1)$ is

$$\mathscr{C}(f, 1), \mathscr{C}(\ell, 1), J \xrightarrow{1,\psi} \mathscr{C}(f, 1), \mathscr{C}(d, 1) \xrightarrow{\phi} \mathscr{C}(fd, 1).$$

8.12. DEFINITION. In the above situation, if (8.10) exhibits ℓ as a J-weighted colimit of d, we say that f **preserves** this colimit if (8.11) exhibits $f\ell$ as a J-weighted colimit of fd. Similarly, if (8.11) exhibits $f\ell$ as a J-weighted colimit of fd, we say that f **reflects** this colimit if (8.10) exhibits ℓ as a J-weighted colimit of d.

Dually, we define what it means for a $\mathscr{V}$-functor to preserve and reflect a weighted limit. The following observations are expected.

8.13. PROPOSITION. *If $f\colon \mathscr{C} \to \mathscr{D}$ is a left adjoint, then f preserves any colimits which exist in $\mathscr{C}$. Dually, right adjoints preserve all limits.*

PROOF. Recall that an adjunction $f \dashv g$ implies an isomorphism $\mathscr{D}(f, 1) \cong \mathscr{C}(1, g)$. Therefore, if $\ell\colon K \to \mathscr{C}$ is a colimit of $d\colon A \to \mathscr{C}$ weighted by $J\colon K \nrightarrow A$, then for any well-typed $\vec{H} = H_1, \ldots, H_n$, we have

$$\begin{aligned}
\mathscr{V}\text{-}\mathcal{M}ultimor(\vec{H}, J; \mathscr{D}(fd, 1)) &\cong \mathscr{V}\text{-}\mathcal{M}ultimor(\vec{H}, J, \mathscr{D}(1, d); \mathscr{D}(f, 1)) \\
&\cong \mathscr{V}\text{-}\mathcal{M}ultimor(\vec{H}, J, \mathscr{D}(1, d); \mathscr{C}(1, g)) \\
&\cong \mathscr{V}\text{-}\mathcal{M}ultimor(\mathscr{C}(g, 1), \vec{H}, J; \mathscr{D}(d, 1)) \\
&\cong \mathscr{V}\text{-}\mathcal{M}ultimor(\mathscr{C}(g, 1), \vec{H}; \mathscr{D}(\ell, 1)) \\
&\cong \mathscr{V}\text{-}\mathcal{M}ultimor(\vec{H}, \mathscr{D}(1, \ell); \mathscr{C}(1, g)) \\
&\cong \mathscr{V}\text{-}\mathcal{M}ultimor(\vec{H}, \mathscr{D}(1, \ell); \mathscr{D}(f, 1)) \\
&\cong \mathscr{V}\text{-}\mathcal{P}ROF(M, \mathscr{D}(f\ell, 1)).
\end{aligned}$$

The case of right adjoints is dual. ∎

8.14. Proposition. *A fully faithful $\mathscr{V}$-functor reflects all limits and colimits.*

Proof. Let $f\colon \mathscr{C} \to \mathscr{D}$ be fully faithful and let $f\ell$ be a J-weighted colimit of fd. Suppose we have a multimorphism $\vec{H}, J \to \mathscr{C}(d, 1)$; we want to show that it factors uniquely through a multimorphism $\vec{H} \to \mathscr{C}(\ell, 1)$. We can compose on the left with $\mathscr{D}(f, 1)$ to obtain a multimorphism

$$\mathscr{D}(f, 1), \vec{H}, J \longrightarrow \mathscr{D}(f, 1) \odot \mathscr{C}(d, 1) \cong \mathscr{D}(fd, 1),$$

and since $\mathscr{D}(f\ell, 1) \cong J \rhd \mathscr{D}(fd, 1)$, this factors uniquely through $\mathscr{D}(f\ell, 1)$ via a multimorphism $\mathscr{D}(f, 1), \vec{H} \to \mathscr{D}(f\ell, 1)$. Now composing on the left with $\mathscr{D}(1, f)$ gives a multimorphism

$$\mathscr{D}(1, f) \odot \mathscr{D}(f, 1), \vec{H} \longrightarrow \mathscr{D}(1, f) \odot \mathscr{D}(f, 1) \odot \mathscr{C}(\ell, 1).$$

But since f is fully faithful, we have $\mathscr{D}(1, f) \odot \mathscr{D}(f, 1) \cong \mathscr{C}$, so this is equivalent to a multimorphism $\vec{H} \to \mathscr{C}(\ell, 1)$. We leave it to the reader to verify that this is the desired factorization, and that it is unique. The case of limits is analogous. ∎

If $i\colon \mathscr{D} \to \mathscr{C}$ has a left adjoint $r\colon \mathscr{C} \to \mathscr{D}$ whose counit $\varepsilon\colon ri \to 1$ is an isomorphism, we say that i exhibits $\mathscr{D}$ as a **reflective sub-$\mathscr{V}$-category** of $\mathscr{C}$.

8.15. Proposition. *If $\mathscr{D}$ is a reflective sub-$\mathscr{V}$-category of $\mathscr{C}$, then $\mathscr{D}$ admits all limits and colimits which $\mathscr{C}$ does.*

Proof. For the case of colimits, if $J\colon K \nrightarrow A$ is a $\mathscr{V}$-profunctor and $f\colon A \to \mathscr{D}$ a $\mathscr{V}$-functor, we can consider the composite if. If $\mathscr{C}$ admits the J-weighted colimit of if, then since r is a left adjoint, it preserves this colimit; thus $r(\operatorname{colim}^J if)$ is a J-weighted colimit of $rif \cong f$.

For the case of limits, let $J\colon K \nrightarrow A$ again be a weight and $g\colon K \to \mathscr{D}$ a $\mathscr{V}$-functor, and suppose that $\ell\colon A \to \mathscr{C}$ is a J-weighted limit of ig; thus we have

$$\begin{aligned}
\mathscr{C}(1,\ell) &\cong \mathscr{C}(1,ig) \lhd J\\
&\cong (\mathscr{D}(1,g) \odot \mathscr{C}(1,i)) \lhd J\\
&\cong (\mathscr{D}(1,g) \odot \mathscr{D}(r,1)) \lhd J\\
&\cong (\mathscr{D}(1,r) \rhd \mathscr{D}(1,g)) \lhd J\\
&\cong \mathscr{D}(1,r) \rhd (\mathscr{D}(1,g) \lhd J)\\
&\cong (\mathscr{D}(1,g) \lhd J) \odot \mathscr{D}(r,1).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathscr{D}(1,r\ell) &\cong \mathscr{C}(1,\ell) \odot \mathscr{D}(1,r)\\
&\cong (\mathscr{D}(1,g) \lhd J) \odot \mathscr{D}(r,1) \odot \mathscr{D}(1,r)\\
&\cong (\mathscr{D}(1,g) \lhd J) \odot \mathscr{C}(1,i) \odot \mathscr{D}(1,r)\\
&\cong \mathscr{D}(1,g) \lhd J
\end{aligned}$$

since $ri \cong 1$. Thus $r\ell$ is a J-weighted limit of g. Note that since i is a right adjoint, it preserves all limits, so we can say more strongly that $\mathscr{D}$ is 'closed under limits' in $\mathscr{C}$. ∎

9. Limits in indexed $\mathscr{V}$-categories

All the definitions and theorems in §8 make sense in any (virtual) equipment. Now, however, we truly specialize to the case of enriched indexed categories, considering several special types of limits and colimits and their relationship to more familiar ones. We will see that many such limits and colimits can be described as 'fiberwise' limits together with conditions ensuring the limits are (1) compatible with the enrichment and (2) preserved by restriction. For classical indexed categories, the conditions (1) tend to be automatic and the conditions (2) are significant, while for classical enriched categories, the situation is reversed. For general $\mathscr{V}$, both conditions will be nontrivial.

As mentioned in the introduction, in the case $\mathscr{V} = \mathscr{S}\!elf(\mathbf{S})$ this reduction of the abstract limit-notions to familiar indexed ones is due to [BW87]; while the combination of indexed and enriched universal properties for a general $\mathscr{V}$ was first considered (in an *ad hoc* manner) in [GG76].

For simplicity, in this section we generally assume $\mathscr{V}$ to be an $\mathbf{S}$-indexed cosmos. Most of the results could be rephrased with some care under weaker assumptions on $\mathscr{V}$ (in particular, symmetry is never really necessary), but we mostly leave this to the interested reader.

We will also assume that the $\mathscr{V}$-category $\mathscr{C}$ in which we consider limits is a $\mathscr{V}$-fibration. In particular, by Proposition 6.9 this implies that up to isomorphism, we may as well assume that any $\mathscr{V}$-functor with codomain $\mathscr{C}$ is indexed, and we will generally do so.

First, suppose that $K = \delta X$ and $A = \delta Y$ are small discrete $\mathscr{V}$-categories. In this case, a weight $J\colon \delta X \nrightarrow \delta Y$ is simply an object of $\mathscr{V}^{X\times Y}$. Since (indexed) $\mathscr{V}$-functors $\delta X \to \mathscr{C}$ are equivalent to objects of the fiber $\mathscr{C}^X$, J-weighted colimits take the fiber $\mathscr{C}^X$ to the fiber $\mathscr{C}^Y$, and J-weighted limits take $\mathscr{C}^Y$ to $\mathscr{C}^X$.

For such a $J \in \mathscr{V}^{X\times Y}$, we call a J-weighted colimit of $x \in \mathscr{C}^X$ a **global $\mathscr{V}$-tensor** of x with J, and write it as $J \circledast_{[X]} x$. (We use the word 'global' to distinguish these tensors from the 'fiberwise' ones that we will consider later.) Invoking the definition of colimits and Remark 5.18, we find that $J \circledast_{[X]} x$ is an object of $\mathscr{C}^Y$ characterized by an isomorphism of profunctors $\mathscr{C} \nrightarrow \delta Y$:

$$\underline{\mathscr{C}}\big(J \circledast_{[X]} x, 1\big) \cong \underline{\mathscr{V}}^{[X]}\big(J, \underline{\mathscr{C}}(x,1)\big).$$

Dually, we call a J-weighted limit of $y \in \mathscr{C}^Y$ a **global $\mathscr{V}$-cotensor** with J and write it as $\{J, y\}^{[Y]}$. By definition and Remark 5.18, it is an object of $\mathscr{C}^X$ characterized by an isomorphism of profunctors $\delta X \nrightarrow \mathscr{C}$:

$$\underline{\mathscr{C}}\big(1, \{J,y\}^{[Y]}\big) \cong \underline{\mathscr{V}}^{[Y]}\big(J, \underline{\mathscr{C}}(1,y)\big).$$

9.1. Example. Suppose $\mathscr{V} = \mathscr{F}am(\mathbf{V})$ and $\mathscr{C} = \mathscr{F}am(\mathbf{C})$ as in Example 4.2. Then when $X = Y = 1$, global $\mathscr{V}$-tensors are the same as classical tensors in enriched category theory.

For general X and Y, global tensors combine classical tensors with coproducts. Namely, if $J = (J_{y,x})_{(y,x)\in Y\times X}$ is an object of $\mathbf{C}^{Y\times X}$ and $M = (M_x)_{x\in X}$ is an object of $\mathscr{D}^X$ for a $\mathbf{C}$-enriched category $\mathscr{D}$, then we have

$$(J \circledast_{[X]} M)_y \cong \coprod_{x\in X} J_{y,x} \circledast M_x.$$

In particular, if $Y = X$ and $J_{y,x} = \emptyset$ is an initial object for $y \neq x$ (which is to say that $J = \Delta_! J'$ for some $J' \in \mathscr{D}^X$), then $(J \circledast_{[X]} M)_x = J_{x,x} \circledast M_x$ involves no coproducts. This is also the tensor of M with J' in the $\mathbf{V}^X$-enriched category $\mathbf{C}^X$.

On the other hand, if we have a function $g\colon X \to Y$ and we define

$$J_{y,x} = \begin{cases} \mathbb{I} & \text{if } f(x) = y \\ \emptyset & \text{otherwise} \end{cases},$$

then $(J \circledast_{[X]} M)_y = \coprod_{f(x)=y} M_x$ involves *only* coproducts.

This example suggests that for general $\mathscr{V}$ we may also profitably split up the study of global $\mathscr{V}$-tensors into those of the form $\Delta_! J'$ and those induced by morphisms in $\mathbf{S}$. We start by considering the latter.

Suppose that $f\colon X \to Y$ is a morphism in $\mathbf{S}$. Then it gives rise to profunctors $Y(1,f) : \delta X \nrightarrow \delta Y$ and $Y(f,1) : \delta Y \nrightarrow \delta X$. Explicitly, we have

$$\underline{Y(1,f)} = (f\times 1)^*(\Delta_Y)_! \mathbb{I}_Y \qquad \text{and}$$
$$\underline{Y(f,1)} = (1\times f)^*(\Delta_Y)_! \mathbb{I}_Y.$$

By Lemma 5.28 and Lemma 5.29, for $M \in \mathscr{V}\text{-}\mathcal{P}rof(\delta Y, \delta Z) = \mathscr{V}^{Y\times Z}$, we have natural isomorphisms

$$Y(1,f) \odot M \;\cong\; (f\times 1)^* M \;\cong\; M \lhd Y(f,1).$$

Similarly, for $N \in \mathscr{V}\text{-}\mathcal{P}rof(\delta X, \delta Z) = \mathscr{V}^{X\times Z}$ we have

$$\begin{aligned} Y(f,1) \odot N &\cong (f\times 1)_! N \\ N \lhd Y(1,f) &\cong (f\times 1)_* N. \end{aligned} \tag{9.2}$$

and symmetrically in all cases.

Now by Example 8.4, for any $y \in \mathscr{C}^Y$, the $Y(1,f)$-weighted colimit and the $Y(f,1)$-weighted limit of the corresponding indexed $\mathscr{V}$-functor $y : \delta Y \to \mathscr{C}$ are just the composite $fy : \delta X \to \mathscr{C}$. (If we demand the (co)limit to be an *indexed* $\mathscr{V}$-functor, then it must be precisely a *restriction* of y along f as in Definition 6.1.)

9.3. Theorem. *If $\mathscr{C}$ is an indexed $\mathscr{V}$-category admitting $Y(f,1)$-weighted colimits for all $f : X \to Y$ in $\mathbf{S}$, then the $\mathbf{S}$-indexed category $\mathscr{C}_o$ has $\mathbf{S}$-indexed coproducts. Dually, if $\mathscr{C}$ admits $Y(1,f)$-weighted limits for all $f : X \to Y$, then $\mathscr{C}_o$ has $\mathbf{S}$-indexed products.*

Proof. By Proposition 8.5 and the above observation, $Y(f,1)$-weighted colimits define a left adjoint $f_! : \mathscr{C}_o^X \to \mathscr{C}_o^Y$ to the restriction functor f^*, and dually. So it remains only to check the Beck-Chevalley condition. Thus, suppose that

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "k"] \\ Y \arrow[r, "h"'] & Z \end{tikzcd}$$

is a pullback square in $\mathbf{S}$; the question is whether the canonical transformation

$$\operatorname{colim}^{Y(g,1)} \operatorname{colim}^{X(1,f)} \longrightarrow \operatorname{colim}^{Z(1,h)} \operatorname{colim}^{Z(k,1)}$$

is an isomorphism. By Theorem 8.8, this can be reduced to the question of whether the canonical transformation

$$Y(g,1) \odot X(1,f) \longrightarrow Z(1,h) \odot Z(k,1) \tag{9.4}$$

is an isomorphism. However, by the above remarks, the functors

$$\begin{aligned} &Y(g,1) \odot X(1,f) \odot - \qquad \text{and} \\ &Z(1,h) \odot Z(k,1) \odot - \end{aligned}$$

are naturally isomorphic to $(g\times 1)_!(f\times 1)^*$ and $(h\times 1)^*(k\times 1)_!$, respectively, and under these isomorphisms (9.4) is identified with the Beck-Chevalley morphism in $\mathscr{V}$. Since $\mathscr{V}$ has indexed coproducts, this transformation is an isomorphism; hence by the bicategorical Yoneda lemma for $\mathscr{V}\text{-}\mathcal{P}rof$, so is (9.4). The case of indexed products is dual. ∎

Of course, if $\mathscr{V}$ only satisfies the Beck-Chevalley condition for some pullback squares in $\mathbf{S}$, as in Example 2.38, then $\mathscr{C}_o$ only inherits the Beck-Chevalley condition for those same pullback squares.

9.5. Definition. We say that $\mathscr{C}$ has **S-indexed $\mathscr{V}$-coproducts** if it admits $Y(f,1)$-weighted colimits for all $f\colon X \to Y$ in $\mathbf{S}$. Similarly, we say it has **S-indexed $\mathscr{V}$-products** if it admits all $Y(1,f)$-weighted limits.

Just as a limit in the underlying category of a classical enriched category need not be an enriched limit, it is not necessarily true that indexed (co)products in $\mathscr{C}_o$ imply indexed $\mathscr{V}$-(co)products in $\mathscr{C}$. We need to also require that the adjunction $f_! \dashv f^*$ or $f^* \dashv f_*$ is "enriched" in a suitable sense. To explain this condition, suppose we have an adjunction $f_! \dashv f^*$ relating $\mathscr{C}_o^X$ and $\mathscr{C}_o^Y$; then for $x \in \mathscr{C}^X$ and $y \in \mathscr{C}^Y$ we have a transformation

$$f^*\left(\underline{\mathscr{C}}^Y(f_!x,y)\right) \xrightarrow{\sim} \underline{\mathscr{C}}^X(f^*f_!x,f^*y) \to \underline{\mathscr{C}}^X(x,f^*y) \tag{9.6}$$

in which the second map is precomposition with the unit $x \to f^*f_!x$ of the adjunction $f_! \dashv f^*$. The mate of (9.6) under the adjunction $f^* \dashv f_*$ in $\mathscr{V}$ is a transformation

$$\underline{\mathscr{C}}^Y(f_!x,y) \longrightarrow f_*\left(\underline{\mathscr{C}}^X(x,f^*y)\right). \tag{9.7}$$

9.8. Theorem. *A $\mathscr{V}$-fibration $\mathscr{C}$ has indexed $\mathscr{V}$-coproducts if and only if $\mathscr{C}_o$ has indexed coproducts and every map (9.7) is an isomorphism. In this case, $f_!$ extends to a $\mathscr{V}^Y$-enriched functor $(f_*)_\bullet\mathscr{C}^X \to \mathscr{C}^Y$.*

Dually, $\mathscr{C}$ has indexed $\mathscr{V}$-products if and only if $\mathscr{C}_o$ has indexed products and every canonical map

$$\underline{\mathscr{C}}^Y(y,f_*x) \longrightarrow f_*\left(\underline{\mathscr{C}}^X(f^*y,x)\right)$$

is an isomorphism, in which case f_* extends to a $\mathscr{V}^Y$-enriched functor $(f_*)_\bullet\mathscr{C}^X \to \mathscr{C}^Y$.

Proof. First, assume that $\mathscr{C}$ has indexed $\mathscr{V}$-coproducts. We have already shown that then $\mathscr{C}_o$ has indexed coproducts. And for $x \in \mathscr{C}^X$ and $y \in \mathscr{C}^Y$ we have

$$\begin{aligned}
\underline{\mathscr{C}}^Y(f_!x,y) &\cong \Delta_Y^*\underline{\mathscr{C}}(f_!x,y) \\
&\cong \Delta_Y^*\big(Y(f,1) \rhd \underline{\mathscr{C}}(x,y)\big) && \text{(by definition of indexed } \mathscr{V}\text{-coproducts)} \\
&\cong \Delta_Y^*(1\times f)_*\underline{\mathscr{C}}(x,y) && \text{(by the dual of (9.2))} \\
&\cong f_*\Delta_X^*(f\times 1)^*\underline{\mathscr{C}}(x,y) && \text{(by the Beck-Chevalley condition)} \\
&\cong f_*\Delta_X^*\underline{\mathscr{C}}(x,f^*y) && \text{(since } f^*y \text{ is a restriction of } y) \\
&\cong f_*\underline{\mathscr{C}}^X(x,f^*y).
\end{aligned}$$

We leave it to the reader to check that this isomorphism is in fact the canonical map (9.7). Now using this isomorphism, we can define a morphism

$$f_*\underline{\mathscr{C}}^X(x,w) \longrightarrow f_*\underline{\mathscr{C}}^X(x,f^*f_!w) \cong \underline{\mathscr{C}}^Y(f_!x,f_!w)$$

and check that it makes $f_!$ into a $\mathscr{C}^Y$-enriched functor $(f_*)_\bullet\mathscr{C}^X \to \mathscr{C}^Y$, and the isomorphism (9.7) into an enriched adjunction.

For the other direction, suppose that $\mathscr{C}_o$ has indexed coproducts and that (9.7) is always an isomorphism. Then for any $z \in \mathscr{C}$, we have

$$\begin{aligned}
\underline{\mathscr{C}}(f_!x, z) &\cong \underline{\mathscr{C}}^{Z\times Y}(\pi_Z^* f_! x, \pi_Y^* z) \\
&\cong \underline{\mathscr{C}}^{Z\times Y}\big((1\times f)_!\pi_Z^* x, \pi_Y^* z\big) && \text{(by the Beck-Chevalley condition)}\\
&\cong (1\times f)_*\underline{\mathscr{C}}^{Z\times X}\big(\pi_Z^* x, (1\times f)^*\pi_Y^* z\big) && \text{(by (9.7) for } 1\times f)\\
&\cong (1\times f)_*\underline{\mathscr{C}}^{Z\times X}\big(\pi_Z^* x, \pi_X^* z\big)\\
&\cong (1\times f)_*\underline{\mathscr{C}}(x, z)\\
&\cong Y(f,1)\rhd \underline{\mathscr{C}}(x,z) && \text{(by the dual of (9.2)).}
\end{aligned}$$

Since this isomorphism is suitably natural in z, by definition $f_!x$ is an indexed $\mathscr{V}$-coproduct. The case of f_* is similar. ∎

9.9. Example. If $\mathscr{V} = \mathscr{F}am(\mathbf{V})$ and $\mathscr{C} = \mathscr{F}am(\mathbf{C})$ as in Example 4.2, then $\mathscr{C}$ has indexed $\mathscr{F}am(\mathbf{V})$-coproducts just when $\mathbf{C}$ has small $\mathbf{V}$-coproducts. Here the Beck-Chevalley condition is automatic, but (9.7) gives the coproducts their $\mathbf{V}$-enriched universal property, beyond merely being coproducts in $\mathbf{C}_o$.

9.10. Example. On the other hand, for an indexed $\mathscr{S}elf(\mathbf{S})$-category, one can show that condition (9.7) is automatic; here all the content is in the Beck-Chevalley condition. This is also true for indexed $\mathscr{P}sh(\mathbf{S}, \mathrm{Set})$-categories (that is, ordinary indexed categories). More generally, for indexed $\mathscr{P}sh(\mathbf{S}, \mathbf{V})$-categories, we need the adjunctions $f_! \dashv f^*$ to be $\mathbf{V}$-enriched, in addition to the Beck-Chevalley condition.

We now consider the other principal type of global tensors. If $J \in \mathscr{V}^X$ and $x \in \mathscr{C}^X$, we write $J \circledast_X x$ for the $\Delta_! J$-weighted colimit of x. By definition, it is characterized by an isomorphism

$$\begin{aligned}
\underline{\mathscr{C}}(J\circledast_X x, z) &\cong \underline{\mathscr{V}}^{[X]}(\Delta_{X!}J, \underline{\mathscr{C}}(x,z))\\
&\cong \pi_{X*}\Delta_X^*\big(\underline{\mathscr{V}}(\Delta_{X!}J, \underline{\mathscr{C}}(x,z))\big)\\
&\cong \pi_{X*}\Delta_X^*(1\times\Delta_X)_*\big(\underline{\mathscr{V}}(J, \underline{\mathscr{C}}(x,z))\big) && \text{(by Proposition 2.25)}\\
&\cong \pi_{X*}\Delta_{X*}(\Delta_X\times 1)^*\big(\underline{\mathscr{V}}(J, \underline{\mathscr{C}}(x,z))\big) && \text{(by Beck-Chevalley)}\\
&\cong (\Delta_X\times 1)^*\big(\underline{\mathscr{V}}(J, \underline{\mathscr{C}}(x,z))\big)\\
&\cong \underline{\mathscr{V}}^X(J, \underline{\mathscr{C}}(x,z)).
\end{aligned}\tag{9.11}$$

If we choose z such that $z \in \mathscr{C}^X$, and apply Δ_X^* to the above isomorphism, we obtain as a special case

$$\begin{aligned}
\underline{\mathscr{C}}^X(J \circledast_X x, z) &= \Delta_X^*\big(\underline{\mathscr{C}}(J \circledast_X x, z)\big) \\
&\cong \Delta_X^*(\Delta_X \times 1)^*\big(\underline{\mathscr{V}}(J, \underline{\mathscr{C}}(x,z))\big) \\
&\cong \Delta_X^*(1 \times \Delta_X)^*\big(\underline{\mathscr{V}}(J, \underline{\mathscr{C}}(x,z))\big) \\
&\cong \Delta_X^*\big(\underline{\mathscr{V}}(J, \Delta_X^*\underline{\mathscr{C}}(x,z))\big) \\
&\cong \underline{\mathscr{V}}^X\left(J, \underline{\mathscr{C}}^X(x,z)\right). \qquad (9.12)
\end{aligned}$$

By definition, this isomorphism (natural in z) says that $J \circledast_X x$ is a $\mathscr{V}^X$-enriched tensor of x by J in the fiber category $\mathscr{C}^X$. (This is what we expected, from Example 9.1.)

9.13. DEFINITION. We say $\mathscr{C}$ has **fiberwise $\mathscr{V}$-tensors** if it admits all colimits with weights of the form $\Delta_! J$. Dually, we say $\mathscr{C}$ has **fiberwise $\mathscr{V}$-cotensors** if it has global cotensors with all weights of the form $\Delta_! J$.

Fiberwise cotensors are characterized by an isomorphism

$$\underline{\mathscr{C}}(z, \{J, x\}^X) \cong \underline{\mathscr{V}}^X(J, \underline{\mathscr{C}}(z,x)).$$

and are, in particular, cotensors in the fibers.

Of course, if the fiber categories $\mathscr{C}^X$ have $\mathscr{V}^X$-enriched tensors, it doesn't necessarily follow that $\mathscr{C}$ has fiberwise $\mathscr{V}$-tensors. Here what is missing is not the enrichment (which is already there in the definition of tensors) but stability under restriction (which corresponds to the Beck-Chevalley condition for indexed coproducts).

9.14. THEOREM. *$\mathscr{C}$ has fiberwise $\mathscr{V}$-tensors if and only if each fiber $\mathscr{C}^X$ has $\mathscr{V}^X$-enriched tensors, and for any $f\colon X \to Y$, $J \in \mathscr{V}^Y$, and $y \in \mathscr{C}^Y$ the canonical map*

$$f^*J \circledast_X f^*y \longrightarrow f^*(J \circledast_Y y)$$

is an isomorphism. Dually, $\mathscr{C}$ has fiberwise $\mathscr{V}$-cotensors if and only if each fiber $\mathscr{C}^X$ has $\mathscr{V}^X$-enriched cotensors preserved by restriction.

PROOF. We have already shown that if $\mathscr{C}$ has fiberwise $\mathscr{V}$-tensors, then the fibers have tensors, so for the 'only if' direction it suffices to show that they are preserved by restriction. Like Theorem 9.3, this follows from Theorem 8.8, the fact that f^* is given by a $Y(1,f)$-weighted colimit, and the composite isomorphism

$$Y(1,f) \odot \Delta_{Y!}J \cong (f \times 1)^*\Delta_{Y!}J \cong (1 \times f)_!\Delta_{X!}f^*J \cong \Delta_{X!}f^*J \odot Y(1,f).$$

Conversely, if we suppose that the fibers have tensors preserved by restriction, then we can calculate

$$\begin{aligned}\underline{\mathscr{C}}(J \circledast_X x, z) &\cong \underline{\mathscr{C}}^{Z\times X}(\pi_Z^*(J \circledast_X x), \pi_X^* z)\\ &\cong \underline{\mathscr{C}}^{Z\times X}(\pi_Z^* J \circledast_{X\times Z} \pi_Z^* x, \pi_X^* z)\\ &\cong \underline{\mathscr{V}}^{X\times Z}\Big(\pi_Z^* J, \underline{\mathscr{C}}^{Z\times X}(\pi_Z^* x, \pi_X^* z)\Big)\\ &\cong \underline{\mathscr{V}}^{X\times Z}\big(\pi_Z^* J, \underline{\mathscr{C}}(x,z)\big)\\ &\cong \underline{\mathscr{V}}^{X}(J, \underline{\mathscr{C}}(x,z))\end{aligned}$$

which is (9.12). The case of cotensors is dual. ∎

9.15. Example. The $\mathscr{F}am(\mathbf{V})$-category $\mathscr{F}am(\mathbf{C})$ has fiberwise $\mathscr{F}am(\mathbf{V})$-tensors exactly when $\mathbf{C}$ has $\mathbf{V}$-enriched tensors in the usual sense.

9.16. Example. On the other hand, if a $\mathscr{S}elf(\mathbf{S})$-category $\mathscr{C}$ has indexed coproducts (hence indexed $\mathscr{S}elf(\mathbf{S})$-coproducts, by Example 9.10), then it automatically has fiberwise $\mathscr{S}elf(\mathbf{S})$-tensors. Namely, if $x \in \mathscr{C}^X$ and $p\colon J \to X$ is an object of $\mathscr{S}elf(\mathbf{S})^X = \mathbf{S}/X$, then $p_! p^* x$ is a fiberwise tensor of x by J. For if $z \in \mathscr{C}^Z$, we have

$$\begin{aligned}\underline{\mathscr{S}elf(\mathbf{S})}^X(J, \underline{\mathscr{C}}(x,z)) &\cong (1\times p)_*(1\times p)^*\underline{\mathscr{C}}(x,z)\\ &\cong (1\times p)_*\underline{\mathscr{C}}(p^*x,z)\\ &\cong X(p,1) \rhd \underline{\mathscr{C}}(p^*x,z)\\ &\cong \underline{\mathscr{C}}(p_!p^*x,z).\end{aligned}$$

Dually, if $\mathscr{C}$ has indexed products, it has fiberwise $\mathscr{S}elf(\mathbf{S})$-cotensors.

9.17. Example. For size reasons, it is unreasonable to expect an indexed $\mathscr{P}sh(\mathbf{S},\mathbf{V})$-category to have all fiberwise tensors or cotensors. However, we can ask for fiberwise tensors by *small* objects in the sense of Example 2.42. As in Example 9.16, a fiberwise tensor of $x \in \mathscr{C}^X$ by the representable object F_g is given by $g_! g^* x$, and fiberwise tensors by small objects are $\mathbf{V}$-weighted colimits of these preserved by restriction. In particular, fiberwise tensors by $V \otimes F_{1_X}$, for $V \in \mathbf{V}$, are tensors by V in the $\mathbf{V}$-enriched category $\mathscr{C}^X$ which are preserved by restriction.

9.18. Remark. Of course, our use of $\circledast_{[X]}$ for global tensors and $\circledast_X$ for fiberwise tensors is not an accident. The canceling product is, in fact, a global tensor in the $\mathscr{V}$-category $\mathscr{V}$, while the fiberwise product is a fiberwise tensor. Dually, the canceling hom is a global cotensor and the fiberwise hom is a fiberwise cotensor. (We could also, if we wished, define 'external' tensors and cotensors in arbitrary $\mathscr{V}$-categories.)

For our next example, suppose that $\mathbf{A}$ is a $\mathscr{V}^X$-enriched category, in the classical sense. Then we can construct a large $\mathscr{V}$-category $X[\mathbf{A}]$ whose objects are those of $\mathbf{A}$, all with extent X, and with hom-objects

$$\underline{X[\mathbf{A}]}(a,b) = \Delta_{X!}\big(\underline{\mathbf{A}}(a,b)\big).$$

Then for any $\mathscr{V}$-fibration $\mathscr{C}$, indexed $\mathscr{V}$-functors $X[\mathbf{A}] \to \mathscr{C}$ are equivalent to $\mathscr{V}^X$-enriched functors $\mathbf{A} \to \mathscr{C}^X$.

Now suppose additionally that $J\colon \mathbf{A}^{\mathrm{op}} \to \mathscr{V}^X$ is a $\mathscr{V}^X$-enriched functor. Then we can talk about J-weighted colimits in any $\mathscr{V}^X$-enriched category, and in particular in $\mathscr{C}^X$. On the other hand, we can build a $\mathscr{V}$-profunctor $X[J]\colon \delta X \nrightarrow X[\mathbf{A}]$ by setting $X[J](\star, a) = \Delta_{X!}(Ja)$, and ask about $X[J]$-weighted colimits. It should no longer be surprising that $X[J]$-weighted colimits will turn out to be J-weighted colimits which are preserved by restriction.

To make the latter precise in this case, let $f : Y \to X$ be a morphism in $\mathbf{S}$, and observe that any $\mathscr{V}^X$-enriched functor $d : \mathbf{A} \to \mathscr{C}^X$ gives rise to a $\mathscr{V}^Y$-enriched functor

$$(f^*)_\bullet \mathbf{A} \xrightarrow{(f^*)_\bullet d} (f^*)_\bullet \mathscr{C}^X \xrightarrow{f^*} \mathscr{C}^Y$$

which we denote $\widehat{d}$. Similarly, $J\colon \mathbf{A}^{\mathrm{op}} \to \mathscr{V}^X$ gives rise to a $\mathscr{V}^Y$-enriched functor

$$(f^*)_\bullet \mathbf{A}^{\mathrm{op}} \xrightarrow{(f^*)_\bullet J} (f^*)_\bullet \mathscr{V}^X \xrightarrow{f^*} \mathscr{V}^Y$$

which we denote $\widehat{J}$, and any J-weighted cocone under d in $\mathscr{C}^X$ induces a $\widehat{J}$-weighted cocone under $\widehat{d}$ in $\mathscr{C}^Y$.

9.19. Theorem. *In the above situation, a $\mathscr{V}$-fibration $\mathscr{C}$ admits $X[J]$-weighted colimits if and only if the fiber $\mathscr{C}^X$ admits J-weighted colimits and moreover for any $f : Y \to X$, the functor $f^* : (f^*)_\bullet \mathscr{C}^X \to \mathscr{C}^Y$ takes J-colimiting cocones to $\widehat{J}$-colimiting ones.*

Proof. For simplicity we assume that $\mathbf{A}$ is κ-small and that $\mathscr{V}$ has fiberwise κ-small products, so that homs over $X[\mathbf{A}]$ can be constructed as in Lemma 5.16. Then by definition, an $X[J]$-weighted colimit ℓ of $d : X[\mathbf{A}] \to \mathscr{C}$ is characterized by an equalizer

$$\underline{\mathscr{C}}(\ell, z) \longrightarrow \prod_{a\in\mathbf{A}} \underline{\mathscr{V}}^{[X]}(\Delta_{X!}Ja, \underline{\mathscr{C}}(da, z)) \rightrightarrows \prod_{a,a'\in\mathbf{A}} \underline{\mathscr{V}}^{[X]}(\Delta_{X!}Ja \otimes_{[X]} \Delta_{X!}\mathbf{A}(a, a'), \underline{\mathscr{C}}(da', z)).$$

As in (9.11), this is equivalent to an equalizer

$$\underline{\mathscr{C}}(\ell, z) \longrightarrow \prod_{a\in\mathbf{A}} \underline{\mathscr{V}}^{X}(Ja, \underline{\mathscr{C}}(da, z)) \rightrightarrows \prod_{a,a'\in\mathbf{A}} \underline{\mathscr{V}}^{X}(Ja \otimes_X \mathbf{A}(a, a'), \underline{\mathscr{C}}(da', z)).$$

Again, choosing $z \in \mathscr{C}^X$ and applying Δ_X^* yields an equalizer

$$\underline{\mathscr{C}}^X(\ell, z) \longrightarrow \prod_{a\in\mathbf{A}} \underline{\mathscr{V}}^{X}(Ja, \underline{\mathscr{C}}^X(da, z)) \rightrightarrows \prod_{a,a'\in\mathbf{A}} \underline{\mathscr{V}}^{X}(Ja \otimes_X \mathbf{A}(a, a'), \underline{\mathscr{C}}^X(da', z))$$

whence ℓ is the J-weighted colimit of $d : \mathbf{A} \to \mathscr{C}^X$. Preservation by restriction follows exactly as in the proof of Theorem 9.14, as does the converse. ∎

In particular, if $\mathbf{A}$ is an unenriched category and $J = \Delta 1$ the standard conical weight $\mathbf{A}^{\mathrm{op}} \to \mathrm{Set}$, then for any $X \in \mathbf{S}$ we can first take the free $\mathscr{V}^X$-enriched category $\mathscr{V}^X[\mathbf{A}]$ and weight $\mathscr{V}^X[J]$ and then apply this construction. We refer to the resulting $X[\mathscr{V}^X[J]]$-weighted limits and colimits as **fiberwise $\mathscr{V}$-limits and colimits**. Thus we have notions of *fiberwise $\mathscr{V}$-equalizers*, *fiberwise $\mathscr{V}$-products*, and so on. These can all be expressed more explicitly; for example, the fiberwise $\mathscr{V}$-product of two objects $x, y \in \mathscr{C}^X$ is an object $z \in \mathscr{C}^X$ together with an isomorphism

$$\mathscr{C}(1, z) \cong \mathscr{C}(1, x) \times \mathscr{C}(1, y)$$

of profunctors $\delta X \nrightarrow \mathscr{C}$. Of course, $\mathscr{C}(1, x) \times \mathscr{C}(1, y)$ denotes the profunctor whose value at z is $\mathscr{C}(z, x) \times \mathscr{C}(z, y)$, the product taking place in $\mathscr{V}^{\epsilon z \times X}$.

Since $f^* : \mathscr{V}^X \to \mathscr{V}^Y$ is a strong monoidal left adjoint, it commutes with the free enriched category construction up to isomorphism: $(f^*)_\bullet \mathscr{V}^X[\mathbf{A}] \cong \mathscr{V}^Y[\mathbf{A}]$. Thus, to give any sort of fiberwise conical limit is equivalently to give a $\mathscr{V}^X$-enriched conical limit in $\mathscr{C}^X$ which is preserved by restriction, in the appropriate sense.

9.20. Example. Clearly, the indexed $\mathscr{F}am(\mathbf{V})$-category $\mathscr{F}am(\mathbf{C})$ has fiberwise $\mathscr{F}am(\mathbf{V})$-limits and colimits just when $\mathbf{C}$ has the relevant $\mathbf{V}$-enriched ones. Here again the preservation by restriction is automatic, but the fact that the limits are enriched in each fiber is crucial.

9.21. Example. As usual, by contrast, in an indexed $\mathscr{S}elf(\mathbf{S})$-category or $\mathscr{P}sh(\mathbf{S}, \mathrm{Set})$-category, it is the preservation by restriction which contains the content. Once we know that limits exist in fibers and are preserved by restriction, the fact that they are enriched in each fiber follows automatically. Similarly, fiberwise $\mathscr{P}sh(\mathbf{S}, \mathbf{V})$-limits are $\mathbf{V}$-enriched limits in fibers preserved by restriction.

Finally, it is well-known in classical enriched category theory that if cotensors exist, then the distinction between enriched and unenriched ordinary limits disappears (see [Kel82, §3.8]). The analogue of this for $\mathscr{V}$-categories is the following.

9.22. Theorem. *Let $\mathscr{C}$ be a $\mathscr{V}$-fibration with fiberwise $\mathscr{V}$-cotensors.*

(i) $\mathscr{C}$ has fiberwise $\mathscr{V}$-colimits of a given (conical) type if and only if $\mathscr{C}_o$ has fiberwise colimits of that type.

(ii) $\mathscr{C}$ has indexed $\mathscr{V}$-coproducts if and only if $\mathscr{C}_o$ has indexed coproducts.

Of course, there is a dual result for tensors and limits.

Proof. Both 'only if' statements have already been proven. By Theorem 9.19, for (i) it suffices to show that fiberwise colimits in $\mathscr{C}_o^Y$ are actually $\mathscr{V}^Y$-enriched for any Y. But this follows from the classical version of this theorem, since $\mathscr{C}^X$ is $\mathscr{V}^X$-enriched and cotensored.

For (ii), by Theorem 9.8 it suffices to show that (9.7) is an isomorphism. For $f\colon X \to Y$, $x \in \mathscr{C}^X$, $y \in \mathscr{C}^Y$, and any $J \in \mathscr{V}^X$, we compute

$$\begin{aligned}
\mathscr{V}^Y\left(J, \underline{\mathscr{C}}^Y(f_! x, y)\right) &\cong \mathscr{C}^Y\left(f_! x, \{J, y\}^Y\right)\\
&\cong \mathscr{C}^X\left(x, f^*\{J, y\}^Y\right)\\
&\cong \mathscr{C}^X\left(x, \{f^* J, f^* y\}^X\right)\\
&\cong \mathscr{V}^X\left(f^* J, \underline{\mathscr{C}}^X(x, f^* y)\right)\\
&\cong \mathscr{V}^Y\left(J, f_* \underline{\mathscr{C}}^X(x, f^* y)\right)
\end{aligned}$$

so the desired isomorphism (9.7) follows by the Yoneda lemma in $\mathscr{V}^Y$. ∎

We now turn to the question of constructing general $\mathscr{V}$-limits and colimits out of basic ones such as those we have just studied. Our first observation is that the generality in allowing K to be a non-discrete or large $\mathscr{V}$-category comes for free.

9.23. Theorem. *Let $J\colon \mathscr{K} \to\!\!\!\!\!\!+\; A$ be a $\mathscr{V}$-profunctor, let $f\colon A \to \mathscr{C}$ be a $\mathscr{V}$-functor and suppose that the $J(1,k)$-weighted colimit of f exists for all objects k of $\mathscr{K}$. Then the J-weighted colimit of f also exists, and agrees with* $\operatorname{colim}^{J(1,k)} f$ *upon restriction to $\delta(\epsilon k)$ for each k.*

Proof. We must define a $\mathscr{V}$-functor $\operatorname{colim}^J f : \mathscr{K} \to \mathscr{C}$. Its action on objects is fixed: we send k to $\operatorname{colim}^{J(1,k)} f$. What remains of the data is morphisms

$$\underline{\mathscr{K}}(k, k') \to \underline{\mathscr{C}}(\operatorname{colim}^{J(1,k)} f, \operatorname{colim}^{J(1,k')} f).$$

But if we consider this to be a morphism of profunctors $\delta(\epsilon k') \to\!\!\!\!\!\!+\; \delta(\epsilon k)$, then we can obtain it by passing across the isomorphism

$$\begin{aligned}
&\mathscr{V}\text{-}\mathcal{P}rof(\underline{\mathscr{K}}(k, k'), \mathscr{C}(\operatorname{colim}^{J(1,k)} f, \operatorname{colim}^{J(1,k')} f))\\
&\qquad\cong \mathscr{V}\text{-}\mathcal{M}ultimor(\mathscr{C}(\operatorname{colim}^{J(1,k')} f, 1), \underline{\mathscr{K}}(k, k'); \mathscr{C}(\operatorname{colim}^{J(1,k)} f, 1))\\
&\qquad\cong \mathscr{V}\text{-}\mathcal{M}ultimor(\mathscr{C}(\operatorname{colim}^{J(1,k')} f, 1), \underline{\mathscr{K}}(k, k'), J(1,k); \mathscr{C}(f, 1))
\end{aligned}$$

from the composite multimorphism

$$\mathscr{C}(\operatorname{colim}^{J(1,k')} f, 1), \underline{\mathscr{K}}(k, k'), J(1,k) \longrightarrow \mathscr{C}(\operatorname{colim}^{J(1,k')} f, 1), J(1,k') \longrightarrow \mathscr{C}(f, 1)$$

built out of the action of $\mathscr{K}$ on J and the universal bimorphism of $\operatorname{colim}^{J(1,k')} f$. It is straightforward to show that this defines a $\mathscr{V}$-functor, and since isomorphisms of profunctors $\mathscr{C} \to\!\!\!\!\!\!+\; \mathscr{K}$ are detected at each object of $\mathscr{K}$, this functor must be the J-weighted colimit of f. ∎

9.24. Corollary. *For a fixed $\mathscr{V}$-category A, if $\mathscr{C}$ admits all colimits with weights $J\colon \delta X \to\!\!\!\!\!\!+\; A$, then it admits all colimits with weights $J\colon \mathscr{K} \to\!\!\!\!\!\!+\; A$.* ∎

Next, we observe that the two special cases of tensors considered above actually suffice to reconstruct all global $\mathscr{V}$-tensors.

9.25. THEOREM. *If $\mathscr{C}$ is a $\mathscr{V}$-category with indexed $\mathscr{V}$-coproducts and fiberwise $\mathscr{V}$-tensors, then $\mathscr{C}$ admits all global tensors. Dually, if $\mathscr{C}$ has indexed $\mathscr{V}$-products and fiberwise $\mathscr{V}$-cotensors, then it admits all global cotensors.*

PROOF. Let $J \in \mathscr{V}^{Y\times X}$ be a weight for a global tensor. Then we can define

$$J' = \Delta_{(Y\times X)!}J \in \mathscr{V}^{Y\times X\times Y\times X},$$

which we can regard as a profunctor $J' \colon \delta(Y\times X) \nrightarrow \delta(Y\times X)$. Now since $(\pi_X\times\pi_Y)\circ\Delta_{Y\times X}$ is the identity, we have

$$\begin{aligned} J &\cong (\pi_X \times \pi_Y)_!(J') \\ &\cong Y(\pi_X, 1) \odot J' \odot X(1, \pi_Y). \end{aligned}$$

Thus, by Theorem 8.8, J-weighted colimits can be built from $X(1,\pi_Y)$-weighted colimits, J'-weighted colimits, and $Y(\pi_X, 1)$-weighted colimits. However, $X(1,\pi_Y)$-weighted colimits are restrictions, J'-weighted colimits are fiberwise $\mathscr{V}$-tensors, and $Y(\pi_X, 1)$-weighted colimits are indexed $\mathscr{V}$-coproducts. ∎

Finally, we have an enriched indexed version of the classical construction of colimits out of tensors, coproducts, and coequalizers.

9.26. THEOREM. *If $\mathscr{C}$ admits global $\mathscr{V}$-tensors, fiberwise $\mathscr{V}$-coequalizers, and fiberwise $\mathscr{V}$-coproducts of the size of the set of objects of A, then it admits all colimits with weights $J \colon K \nrightarrow A$.*

PROOF. By Theorem 9.23 it suffices to assume that K is a discrete small $\mathscr{V}$-category δY. We then have to show that for any $f \colon A \to \mathscr{C}$ there is an object $\operatorname{colim}^J f \in \mathscr{C}^X$ with an isomorphism

$$\underline{\mathscr{C}}(\operatorname{colim}^J f, 1) \cong J \rhd \underline{\mathscr{C}}(f, 1)$$

of profunctors $\mathscr{C} \nrightarrow \delta Y$. Now, by the construction of $\rhd$ in Lemma 5.16, we have

$$J \rhd \underline{\mathscr{C}}(f, -) \cong \operatorname{eq}\left(\prod_{a\in A} \underline{J}(\star, a) \rhd \underline{\mathscr{C}}(fa, -) \rightrightarrows \prod_{a,b\in A} (\underline{J}(\star, b) \odot \underline{A}(a, b)) \rhd \underline{\mathscr{C}}(fa, -)\right).$$

Since $\mathscr{C}$ admits global $\mathscr{V}$-tensors, for each $a \in A$ we have an object $x_a \in \mathscr{C}^X$ such that

$$\underline{\mathscr{C}}(x_a, -) \cong \underline{J}(\star, a) \rhd \underline{\mathscr{C}}(fa, -).$$

Similarly, for each pair $a, b \in A$ we have an object $y_{a,b} \in \mathscr{C}^X$ such that

$$\underline{\mathscr{C}}(y_{a,b}, -) \cong (\underline{J}(\star, b) \odot \underline{A}(a, b)) \rhd \underline{\mathscr{C}}(fa, -).$$

Therefore, we have

$$J \rhd \underline{\mathscr{C}}(f,-) \cong \mathrm{eq}\left(\prod_{a\in A} \underline{\mathscr{C}}(x_a,-) \rightrightarrows \prod_{a,b\in A} \underline{\mathscr{C}}(y_{a,b},-)\right).$$

where the two maps are induced by canonical morphisms $y_{a,b} \to x_a$ and $y_{a,b} \to x_b$ in $\mathscr{C}^X$. Similarly, since $\mathscr{C}$ has κ-sized fiberwise $\mathscr{V}$-coproducts, there is an object z such that

$$\underline{\mathscr{C}}(z,-) \cong \prod_{a\in A} \underline{\mathscr{C}}(x_a,-),$$

and an object w such that

$$\underline{\mathscr{C}}(w,-) \cong \prod_{a,b\in A} \underline{\mathscr{C}}(y_{a,b},-).$$

Thus we have

$$J \rhd \underline{\mathscr{C}}(f,-) \cong \mathrm{eq}\Big(\underline{\mathscr{C}}(z,-) \rightrightarrows \underline{\mathscr{C}}(w,-)\Big).$$

Finally, because $\mathscr{C}$ has fiberwise $\mathscr{V}$-coequalizers, there is an object $\mathrm{colim}^J f$ such that

$$\underline{\mathscr{C}}(\mathrm{colim}^J f,-) \cong \mathrm{eq}\Big(\underline{\mathscr{C}}(z,-) \rightrightarrows \underline{\mathscr{C}}(w,-)\Big),$$

which completes the proof. ∎

9.27. Corollary. *If $\mathscr{C}$ admits indexed $\mathscr{V}$-coproducts, fiberwise $\mathscr{V}$-tensors, and fiberwise $\mathscr{V}$-coequalizers, then it admits all $\mathscr{V}$-colimits with weights $J\colon K \nrightarrow A$ where A is small in the sense of §3.* ∎

In [GG76], an indexed $\mathscr{V}$-category satisfying the hypotheses of Corollary 9.27 was called *cocomplete*. Unfortunately, however, the converse of Corollary 9.27 fails, since fiberwise $\mathscr{V}$-coequalizers are not a small $\mathscr{V}$-colimit: the relevant $\mathscr{V}$-category $X[\mathscr{V}^X[\mathbf{A}]]$ has two objects. But if we add at least finite fiberwise coproducts, we do get an equivalence.

9.28. Corollary. *Let κ be an infinite regular cardinal. The following are equivalent for a $\mathscr{V}$-category $\mathscr{C}$.*

(i) $\mathscr{C}$ admits indexed $\mathscr{V}$-coproducts, fiberwise $\mathscr{V}$-tensors, and fiberwise $\mathscr{V}$-colimits of cardinality $< \kappa$.

(ii) $\mathscr{C}$ admits all colimits with weights $J\colon K \nrightarrow A$ where A is κ-small. ∎

In the situation of Corollary 9.28, we will say that $\mathscr{C}$ is **κ-cocomplete**. The two most important cases are when $\kappa = \omega$, since ω-cocompleteness is an elementary condition, and when $\kappa = \infty$ is the size of the universe, in which case we simply say $\mathscr{C}$ is **cocomplete**.

Finally, combining Corollary 9.28 with Theorem 9.22, we see that when $\mathscr{C}$ is tensored and cotensored, it suffices to construct colimits in the underlying fibration.

9.29. COROLLARY. *If $\mathscr{C}$ has fiberwise $\mathscr{V}$-tensors and $\mathscr{V}$-cotensors, and $\mathscr{C}_o$ has indexed coproducts and fiberwise κ-small colimits, then $\mathscr{C}$ is κ-cocomplete.*

Of course, everything we have proven applies dually to limits as well, and we have a notion of **κ-completeness**.

9.30. EXAMPLE. The same proofs show that for a locally small $\mathbf{S}$ with pullbacks and a classical cosmos $\mathbf{V}$, an indexed $\mathscr{P}sh(\mathbf{S}, \mathbf{V})$-category admits colimits with all weights $J : K \nrightarrow A$, where A is set-small and locally small (in the sense of Example 5.19) and J is locally small (in the same sense), if and only if it has indexed coproducts as in Example 9.10, fiberwise tensors by small objects as in Example 9.17, and small fiberwise colimits as in Example 9.21. This is equivalent to asking for $\mathbf{V}$-enriched adjunctions $f_! \dashv f^*$ satisfying the Beck-Chevalley condition, plus that each $\mathbf{V}$-enriched category $\mathscr{C}^X$ is $\mathbf{V}$-cocomplete, with colimits preserved by the restriction functors f^*.

10. Presheaf $\mathscr{V}$-categories

Our goal is now to define presheaf $\mathscr{V}$-categories. Here again it is convenient to use the machinery of profunctors. In particular, for bicategory-enrichment the construction was already done by [Str83]. We will refine it slightly, so as to simultaneously give a notion of "small-presheaf category" as in [DL07] and a version suitable for an elementary context.

Let $\mathbf{S}$ have finite products, and let $\mathscr{V}$ be an $\mathbf{S}$-indexed cosmos which is κ-complete and κ-cocomplete for some chosen regular cardinal κ. We also include the case "$\kappa = \infty$" for which every (small) set is κ-small. (As in §9, with appropriate care we could weaken these assumptions. In particular, symmetry is not really needed.) The cases of most interest are $\kappa = \omega$ (for when we care about being first-order) and $\kappa = \infty$.

10.1. DEFINITION. A $\mathscr{V}$-profunctor $H : \mathscr{B} \nrightarrow \mathscr{A}$ is **κ-small** if for every $b \in \mathscr{B}$, there exist a κ-small $\mathscr{V}$-category A', a $\mathscr{V}$-functor $i : A' \to \mathscr{A}$, and a $\mathscr{V}$-profunctor $H' : \delta(\epsilon b) \nrightarrow A'$ such that $H(1, b) \cong H' \odot \mathscr{A}(1, i)$.

Note that since $\mathscr{V}$ is κ-cocomplete and A' is κ-small, the composite $H' \odot \mathscr{A}(1, i)$ automatically exists.

10.2. EXAMPLE. The unit profunctor $\mathscr{A} : \mathscr{A} \nrightarrow \mathscr{A}$ is always κ-small; for each $a \in \mathscr{A}$ we may take $A' = \delta(\epsilon a)$, H' the unit profunctor, and $i : \delta(\epsilon a) \to \mathscr{A}$ the inclusion.

10.3. EXAMPLE. If $\mathscr{A}$ is itself κ-small, then every profunctor $\mathscr{B} \nrightarrow \mathscr{A}$ is κ-small, as for any b we may take $A' = \mathscr{A}$ and $i = 1_{\mathscr{A}}$.

10.4. Example. If $H : \mathscr{B} \to\!\!\!\!\!\!\mapstochar\;\; \mathscr{A}$ is κ-small, then so is $H(1,f)$ for any $f : \mathscr{C} \to \mathscr{B}$, since $H(1,f)(1,c) = H(1,f(c))$.

10.5. Example. Finally, the connection with the most classical case is a little bit surprising. Suppose $\mathscr{V} = \mathscr{F}am(\mathbf{V})$ and $\mathscr{A} = \mathscr{F}am(\mathbf{A})$ as in Example 4.2, and that H is likewise induced from a $\mathbf{V}$-enriched profunctor $\mathbf{B} \to\!\!\!\!\!\!\mapstochar\;\; \mathbf{A}$. Note that a κ-small $\mathscr{F}am(\mathbf{V})$-category is equivalently a small $\mathbf{V}$-enriched category with a partition of its objects into a family of sets with a κ-small indexing set (but no cardinality restrictions on the individual sets in the partition). Profunctors between such categories are, up to equivalence, just $\mathbf{V}$-enriched profunctors, and functors are those that respect the partitions. It follows that H is κ-small in the sense of Definition 10.1 if and only if it is small in the sense of [DL07]—in particular, the cardinal κ is completely irrelevant!

This makes more sense if we realize that κ does not exactly measure the "size" of a $\mathscr{V}$-category or profunctor, but rather its "departure from elementarity", the case $\kappa = \omega$ being the purely elementary one. The point is that for Set-indexed categories of families, sets of arbitrary cardinality are already built into the indexing and have "become elementary".

10.6. Remark. In Definition 10.1 we are free to assume that i is fully faithful and indexed. For if not, define a new κ-small $\mathscr{V}$-category A'' with one object $\widehat{x}$ for every object $x \in A'$, and with $\epsilon\widehat{x} = \epsilon(ix)$ and $\underline{A''}(\widehat{x},\widehat{y}) = \underline{\mathscr{A}}(ix,iy)$. Then i factors as $A' \xrightarrow{k} A'' \xrightarrow{j} \mathscr{A}$, where j is fully faithful and indexed, and we have

$$H' \odot \mathscr{A}(1,i) \cong (H' \odot A''(1,k)) \odot \mathscr{A}(1,j),$$

the composite $H' \odot A''(1,k)$ existing since A' is κ-small.

We also observe that when the domain of a small profunctor is a small category, then the decompositions of Definition 10.1 can be assembled into a single one.

10.7. Lemma. *If $H : B \to\!\!\!\!\!\!\mapstochar\;\; \mathscr{A}$ is a κ-small profunctor, where B is a κ-small $\mathscr{V}$-category, then there exists a κ-small $\mathscr{V}$-category A', a $\mathscr{V}$-profunctor $H' : B \to\!\!\!\!\!\!\mapstochar\;\; A'$, and a $\mathscr{V}$-functor $i : A' \to \mathscr{A}$ such that $H \cong H' \odot \mathscr{A}(1,i)$.*

Proof. By assumption, for each $b \in B$ we have a κ-small $\mathscr{V}$-category A_b, a profunctor $H_b : \delta(\epsilon b) \to\!\!\!\!\!\!\mapstochar\;\; A_b$, and a functor $i_b : A_b \to A$ with $H(1,b) \cong H_b \odot \mathscr{A}(1,i_b)$. By Remark 10.6 we may assume each i_b to be fully faithful and indexed.

Define A' to have as objects the disjoint union of the objects of the A_b, for all b. This is a κ-small set since each A_b is κ-small and so is B. We let these objects inherit their extents from A_b (and hence from $\mathscr{A}$), and take their hom-objects to be

$$\underline{A'}(a,a') = \begin{cases} \underline{A_b}(a,a') & \text{if } a,a' \in A_b \\ \emptyset & \text{if } a \in A_b \text{ and } a' \in A_{b'} \text{ with } b \neq b' \end{cases}$$

where $\emptyset$ denotes a fiberwise initial object. It is easy to check that this defines a $\mathscr{V}$-category and that we have functors $i : A' \to \mathscr{A}$ and $j_b : A_b \to A'$ with $ij_b = i_b$.

Similarly, we define $H' : B \nrightarrow A'$ by

$$\underline{H'}(a,b) = \begin{cases} \underline{H_b}(a,\star) & \text{if } a \in A_b \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $H'(1,b) = H_b \odot A'(1,j_b)$. Thus, the isomorphisms $H_b \odot \mathscr{A}(1,i_b) \xrightarrow{\sim} H(1,b)$ assemble into a morphism $H' \odot \mathscr{A}(1,i) \to H$, which restricts to an isomorphism at each b and hence is itself an isomorphism. ■

The functor $i : A' \to \mathscr{A}$ constructed in the proof of Lemma 10.7 is not fully faithful, but we may apply the argument of Remark 10.6 to make it so.

Note also that the converse of Lemma 10.7 is universally valid: if A' is κ-small, then for any $i : A' \to \mathscr{A}$ and $H : \mathscr{B} \nrightarrow A'$, the composite $H \odot \mathscr{A}(1,i)$ is κ-small, since $(H \odot \mathscr{A}(1,i))(1,b) \cong H(1,b) \odot \mathscr{A}(1,i)$.

10.8. Lemma. *If $K : B \nrightarrow C$ is a κ-small $\mathscr{V}$-profunctor, B is a κ-small $\mathscr{V}$-category, and $H : A \nrightarrow B$ is any $\mathscr{V}$-profunctor, then the composite $H \odot K$ is κ-small.*

Proof. Write $K = K' \odot C(1,i)$ as in Lemma 10.7. Then $H \odot K \cong (H \odot K') \odot C(1,i)$. ■

10.9. Lemma. *If a $\mathscr{V}$-category $\mathscr{C}$ is κ-cocomplete (i.e. admits all colimits with weights $J : K \nrightarrow A$ where A is κ-small) then it admits all colimits with κ-small weights. Similarly, any $\mathscr{V}$-functor that preserves κ-small colimits preserves all colimits with κ-small weights.*

Proof. By Theorem 9.23, it suffices to show that $\mathscr{C}$ admits J-weighted colimits for any κ-small profunctor $J : \delta X \nrightarrow A$. But then we have $J \cong J' \odot A(1,i)$ for some $J' : \delta X \nrightarrow A'$ and $i : A' \to A$ with A' being κ-small, and thus for any $f : A \to \mathscr{C}$,

$$\operatorname{colim}^J f \cong \operatorname{colim}^{J' \odot A(1,i)} f \cong \operatorname{colim}^{J'} \operatorname{colim}^{A(1,i)} f \cong \operatorname{colim}^{J'} fi$$

(using Theorem 8.8), which exists because A' is κ-small. The second statement follows immediately. ■

10.10. Lemma. *For any κ-small profunctor $H : \mathscr{B} \nrightarrow \mathscr{A}$ and any profunctor $K : \mathscr{C} \nrightarrow \mathscr{A}$, the hom $H \rhd K$ exists.*

Proof. By Remark 5.22, it suffices to show that $H(1,b) \rhd K(1,c)$ exists for all $b \in \mathscr{B}$ and $c \in \mathscr{C}$. Fixing such, let $H(1,b) \cong H' \odot \mathscr{A}(1,i)$, for $i : A' \to \mathscr{A}$ and $j : A'' \to \mathscr{A}$ with A' κ-small. Then for any well-typed $\vec{L} = L_1, \ldots, L_n$, we have

$$\begin{aligned} \mathscr{V}\text{-}\mathcal{B}imor(\vec{L}, H(1,b); K(1,c)) &\cong \mathscr{V}\text{-}\mathcal{M}ultimor(\vec{L}, H', \mathscr{A}(1,i); K(1,c)) \\ &\cong \mathscr{V}\text{-}\mathcal{M}ultimor(\vec{L}, H'; K(1,c) \odot \mathscr{A}(i,1)) \\ &\cong \mathscr{V}\text{-}\mathcal{M}ultimor(\vec{L}; H' \rhd (K(1,c) \odot \mathscr{A}(i,1))). \end{aligned}$$

The composite $K(1,c) \odot \mathscr{A}(i,1)$ exists by Lemma 5.28, and the hom $H' \rhd (K(1,c) \odot \mathscr{A}(i,1))$ exists because A' is κ-small. ■

Finally, we are ready to define presheaf $\mathscr{V}$-categories.

10.11. DEFINITION. Let A be any $\mathscr{V}$-category. Then there is a $\mathscr{V}$-fibration $\mathcal{P}_\kappa A$ defined as follows.

(i) Its objects are κ-small $\mathscr{V}$-profunctors $H : \delta X \nrightarrow A$, for some $X \in \mathbf{S}$.

(ii) The extent of $H : \delta X \nrightarrow A$ is X.

(iii) The hom-object $\underline{\mathcal{P}_\kappa A}(H, K) \in \mathscr{V}^{X\times Y}$ is $H \rhd K$ (which exists by Lemma 10.10).

(iv) The unit morphism $\mathbb{I}_X \to H \rhd H$ is adjunct to the identity $H \to H$.

(v) The composition morphism $(K \rhd L) \odot (H \rhd K) \to (H \rhd L)$ is adjunct to

$$(K \rhd L) \odot (H \rhd K) \odot H \to (K \rhd L) \odot K \to L.$$

(vi) The restriction of $H : \delta X \nrightarrow A$ along $f : Y \to X$ is $H(1, f) : \delta Y \nrightarrow A$.

10.12. REMARK. Recall that if A is κ-small, then so is every profunctor into it. In particular, if $\kappa' \geq \kappa$ then $\mathcal{P}_{\kappa'} A = \mathcal{P}_\kappa A$. Again we see that κ measures not the “size” of the cocompletion *per se*, but its non-elementariness, and once it is above the level of A no further change takes place. When κ has “stabilized” in this sense, we may write merely $\mathcal{P}A$.

10.13. EXAMPLE. If $\mathscr{V} = \mathscr{F}am(\mathbf{V})$ and $\mathscr{A} = \mathscr{F}am(\mathbf{A})$, then we have remarked that a $\mathscr{V}$-profunctor into $\mathscr{A}$ is small just when it is induced by a small $\mathbf{V}$-enriched profunctor into $\mathbf{A}$. This makes it easy to identify $\mathcal{P}_\kappa\mathscr{A}$ with $\mathscr{F}am(\mathcal{P}\mathbf{A})$, where $\mathcal{P}\mathbf{A}$ is the category of small $\mathbf{V}$-enriched presheaves from [DL07].

10.14. EXAMPLE. If $\mathscr{V} = \mathscr{S}elf(\mathbf{S})$ and A is an $\mathbf{S}$-internal category, regarded as a small $\mathscr{V}$-category, then every profunctor into A is κ-small, and $\mathcal{P}_\kappa A$ is the usual locally internal category of internal presheaves on A.

10.15. EXAMPLE. If $\mathscr{V} = \mathscr{S}elf(\mathbf{S})$ and $\mathbf{A}$ is a κ-small category *enriched* over $\mathbf{S}$ with its cartesian monoidal structure, then we can regard $\mathbf{A}$ as a κ-small $\mathscr{S}elf(\mathbf{S})$ category all of whose objects have extent 1. In this case, $\mathcal{P}_\kappa\mathbf{A}$ is a locally internal category of $\mathbf{S}$-enriched presheaves on $\mathbf{A}$, with $(\mathcal{P}_\kappa\mathbf{A})^X = (\mathbf{S}/X)^{\mathbf{A}^{\mathrm{op}}}$.

More generally, for any $\mathscr{V}$ we may perform the same construction with $\mathbf{A}$ being a $\mathscr{V}^1$-enriched category, obtaining a $\mathscr{V}$-fibration $\mathcal{P}_\kappa\mathbf{A}$ with $(\mathcal{P}_\kappa\mathbf{A})^X = (\mathscr{V}^X)^{\mathbf{A}^{\mathrm{op}}}$. In particular, $\mathbf{A}$ might be freely generated by an $\mathbf{S}$-internal category as in Example 7.3.

These presheaf categories have a universal property relating to the following universal profunctors.

10.16. DEFINITION. For any A, there is a $\mathscr{V}$-profunctor $Y_A : \mathcal{P}_\kappa A \nrightarrow A$ defined for $a \in A$ and $H : \delta X \nrightarrow A$ by

$$\underline{Y}(a, H) = \underline{H}(a, 1) \in \mathscr{V}^{X \times \epsilon a}.$$

Its action by A is determined by the action of A on the H's, while its action by $\mathcal{P}_\kappa A$ is determined by adjunction from the universal property of the homs $H \rhd K$. Since each $H \in \mathcal{P}_\kappa A$ is κ-small, so is Y_A.

Thus, any $\mathscr{V}$-functor $f : \mathscr{B} \to \mathcal{P}_\kappa A$ induces a κ-small profunctor $Y_A(1, f) : \mathscr{B} \nrightarrow A$. Since $Y_A(1, f) \cong \mathcal{P}_\kappa A(1, f) \odot Y_A$, we have a canonical multimorphism

$$\mathcal{P}_\kappa A(1, g), \mathcal{P}_\kappa A(f, 1), Y_A(1, f) \longrightarrow \mathcal{P}_\kappa A(1, f), Y_A \longrightarrow Y_A(1, f).$$

And since $\mathcal{P}_\kappa A(f, g) \cong \mathcal{P}_\kappa A(1, g) \odot \mathcal{P}_\kappa A(f, 1)$, this multimorphism factors uniquely through a bimorphism

$$\mathcal{P}_\kappa A(f, g), Y_A(1, f) \longrightarrow Y_A(1, g). \tag{10.17}$$

10.18. PROPOSITION. *For any $\mathscr{V}$-functors $f : \mathscr{B} \to \mathcal{P}_\kappa A$ and $g : \mathscr{C} \to \mathcal{P}_\kappa A$, the bimorphism (10.17) exhibits an isomorphism*

$$\mathcal{P}_\kappa A(f, g) \cong Y_A(1, f) \rhd Y_A(1, g).$$

PROOF. Follows directly from the definition of hom-objects in $\mathcal{P}_\kappa A$ and Remark 5.22. ∎

10.19. PROPOSITION. *For any $\mathscr{V}$-category $\mathscr{B}$, the functor*

$$\begin{array}{ccc} \mathscr{V}\text{-}\mathcal{CAT}(\mathscr{B}, \mathcal{P}_\kappa A) & \longrightarrow & \mathscr{V}\text{-}\mathcal{PROF}(\mathscr{B}, A) \\ \big[f : \mathscr{B} \to \mathcal{P}_\kappa A\big] & \mapsto & \big[Y_A(1, f) \colon \mathscr{B} \nrightarrow A\big] \end{array} \tag{10.20}$$

is fully faithful, and its image consists of the κ-small profunctors.

PROOF. Invoking Proposition 10.18, for $f, g : \mathscr{B} \to \mathcal{P}_\kappa A$ we have

$$\begin{aligned} \mathscr{V}\text{-}\mathcal{CAT}(\mathscr{B}, \mathcal{P}_\kappa A)(f, g) &\cong \mathscr{V}\text{-}\mathcal{PROF}(\mathscr{B}, \mathcal{P}_\kappa A(f, g)) \\ &\cong \mathscr{V}\text{-}\mathcal{B}imor(\mathscr{B}, Y_A(1, f); Y_A(1, g)) \\ &\cong \mathscr{V}\text{-}\mathcal{PROF}(Y_A(1, f); Y_A(1, g)). \end{aligned}$$

It is straightforward to verify that this isomorphism is the action of (10.20) on homs; thus (10.20) is fully faithful.

Now, this functor certainly takes values in κ-small profunctors. Conversely, suppose $H : \mathscr{B} \nrightarrow A$ is κ-small. Of course, H consists of objects $\underline{H}(a, b) \in \mathscr{V}^{\epsilon a \times \epsilon b}$, for each pair of objects $a \in A$ and $b \in \mathscr{B}$, together with action maps

$$\underline{H}(a, b) \odot \underline{A}(a', a) \to \underline{H}(a', b) \quad \text{and} \tag{10.21}$$

$$\underline{\mathscr{B}}(b, b') \odot \underline{H}(a, b) \to \underline{H}(a, b'). \tag{10.22}$$

The maps (10.21) make each $\underline{H}(-, b)$ into a profunctor $\delta(\epsilon b) \nrightarrow A$, which is κ-small since H is; thus it is an object of $\mathcal{P}_\kappa A$ with extent ϵb. And the maps (10.22) have adjuncts

$$\underline{\mathscr{B}}(b, b') \longrightarrow \underline{H}(a, b) \rhd \underline{H}(a, b') = \underline{\mathcal{P}_\kappa A}(\underline{H}(-, b), \underline{H}(-, b')),$$

so we can define an indexed $\mathscr{V}$-functor $\mathscr{B} \to \mathcal{P}_\kappa A$ sending each b to $\underline{H}(-, b)$. We leave it to the reader to check that this works. ∎

Since the unit $\mathscr{V}$-profunctor $A\colon A \nrightarrow A$ is κ-small, it has a classifying $\mathscr{V}$-functor $y_A\colon A \to \mathcal{P}_\kappa A$, which we call the **Yoneda embedding**. Thus, by definition, we have $Y_A(1, y_A) \cong A$. On the other hand, we can recover Y_A from y_A, since by Proposition 10.18

$$\mathcal{P}_\kappa(y_A, 1) \;\cong\; Y_A(1, y_A) \rhd Y_A \;\cong\; A \rhd Y_A \;\cong\; Y_A. \tag{10.23}$$

10.24. Lemma. *The Yoneda embedding is fully faithful.*

Proof. Taking $f = g = y_A$ in Proposition 10.18 yields

$$\mathcal{P}_\kappa A(y_A, y_A) \;\cong\; Y_A(1, y_A) \rhd Y_A(1, y_A) \;\cong\; A \rhd A \;\cong\; A. \qquad \blacksquare$$

We now move on to familiar completeness properties of presheaf categories.

10.25. Theorem. *For any B, the $\mathscr{V}$-category $\mathcal{P}_\kappa B$ is κ-cocomplete.*

Proof. Suppose A is κ-small, and let $J\colon K \nrightarrow A$ be a weight and $f\colon A \to \mathcal{P}_\kappa B$ a functor. Then f classifies a κ-small profunctor $Y_B(1, f)\colon A \nrightarrow B$, and we define $\ell\colon K \to \mathcal{P}_\kappa B$ to be the classifying map of the composite $J \odot Y_B(1, f)$. This composite exists because A is κ-small, and the composite is itself κ-small by Lemma 10.8. We then have

$$\begin{aligned}
\mathcal{P}_\kappa B(\ell, 1) &\cong Y_B(1, \ell) \rhd Y_B && \text{(by (10.18))}\\
&\cong \big(J \odot Y_B(1, f)\big) \rhd Y_B\\
&\cong J \rhd \big(Y_B(1, f) \rhd Y_B\big)\\
&\cong J \rhd \mathcal{P}_\kappa B(f, 1) && \text{(by (10.18) again),}
\end{aligned}$$

so ℓ is a J-weighted colimit of f. $\blacksquare$

10.26. Corollary. *For any $f : B \to \mathcal{P}_\kappa A$ we have*

$$f \cong \operatorname{colim}^{Y_A(1,f)} y_A,$$

In other words, every presheaf is a colimit of representables.

Proof. By the construction of colimits in Theorem 10.25, we have

$$Y_A\Big(1, \operatorname{colim}^{Y_A(1,f)} y_A\Big) \;\cong\; Y_A(1, f) \odot Y_A(1, y_A) \;\cong\; Y_A(1, f) \odot A \;\cong\; Y_A(1, f).$$

Thus, by the essential uniqueness of classifying arrows, $f \cong \operatorname{colim}^{Y_A(1,f)} y_A$. $\blacksquare$

10.27. Corollary. *The Yoneda embedding is dense, i.e. $1_{\mathcal{P}_\kappa A}$ is the left Kan extension of y_A along itself.*

Proof. Take $f = 1_{\mathcal{P}_\kappa A}$ in Corollary 10.26 and use (10.23). $\blacksquare$

10.28. THEOREM. *The Yoneda embedding $y\colon B \to \mathcal{P}_\kappa B$ preserves all limits.*

PROOF. Let $J\colon A \nrightarrow K$ be any weight, $g\colon A \to B$ a functor, and $\ell\colon K \to B$ a J-weighted limit of g; thus $B(1,\ell) \cong B(1,g) \lhd J$. We then have

$$\begin{aligned}\mathcal{P}_\kappa B(1,y\ell) \;\cong\; Y_B \rhd Y_B(1,y\ell) \;\cong\; Y_B \rhd Y_B(1,y)(1,\ell)\qquad&\\ \cong\; Y_B \rhd B(1,\ell) \;\cong\; Y_B \rhd \Big(B(1,g) \lhd J\Big) \;\cong\; \Big(Y_B \rhd B(1,g)\Big) \lhd J&\\ \cong\; \Big(Y_B \rhd Y_B(1,yg)\Big) \lhd J \;\cong\; \mathcal{P}_\kappa B(1,yg) \lhd J,&\end{aligned}$$

so $y\ell$ is a J-weighted limit of yg. ∎

The following is a generalization of [DL07, Prop. 3.2].

10.29. THEOREM. *For κ-small A and any $J : A \nrightarrow K$, a functor $f : A \to \mathcal{P}_\kappa B$ has a J-weighted limit if and only if the profunctor $Y_B(1,f) \lhd J : A \nrightarrow B$ is κ-small.*

PROOF. Note that $Y_B(1,f) \lhd J$ exists since A is κ-small. Consider a functor $\ell\colon K \to \mathcal{P}_\kappa B$; we will show that ℓ is a J-weighted limit of f if and only if it is a classifying map for $Y_B(1,f) \lhd J$. On the one hand, we have

$$\mathcal{P}_\kappa B(1,\ell) \cong Y_B \rhd Y_B(1,\ell) \tag{10.30}$$

while on the other we have

$$\begin{aligned}\mathcal{P}_\kappa B(1,f) \lhd J &\cong \Big(Y_B \rhd Y_B(1,f)\Big) \lhd J\\ &\cong Y_B \rhd \Big(Y_B(1,f) \lhd J\Big).\end{aligned} \tag{10.31}$$

Now if ℓ classifies $Y_B(1,f) \lhd J$, then by definition $Y_B(1,\ell) \cong Y_B(1,f) \lhd J$, and thus (10.30) and (10.31) are isomorphic; hence ℓ is a J-weighted limit of f. Conversely, if (10.30) and (10.31) are isomorphic, we have

$$\begin{aligned}Y_B(1,\ell) \cong B \rhd Y_B(1,\ell) \cong Y_B(1,y) \rhd Y_B(1,\ell)\qquad&\\ \cong \big(Y_B \rhd Y_B(1,\ell)\big)(y,1) \cong \Big(Y_B \rhd \big(Y_B(1,f) \lhd J\big)\Big)(y,1)&\\ \cong \big(\mathcal{P}_\kappa B(1,f) \lhd J\big)(y,1) \cong \mathcal{P}_\kappa B(y,f) \lhd J \cong Y_B(1,f) \lhd J&\end{aligned}$$

so that ℓ classifies $Y_B(1,f) \lhd J$. ∎

10.32. COROLLARY. *If B is κ-small, then $\mathcal{P}_\kappa B$ is κ-cocomplete.* ∎

Finally, we prove the familiar theorem that presheaf objects are free cocompletions. For $\mathscr{V}$-categories B and C, we write $\mathscr{V}\text{-}\mathcal{CAT}_{\kappa\text{-colim}}(B,C)$ for the full subcategory of $\mathscr{V}\text{-}\mathcal{CAT}(B,C)$ determined by the $\mathscr{V}$-functors which preserve κ-small colimits.

10.33. THEOREM. *If A is any $\mathscr{V}$-category and $\mathscr{B}$ is a κ-cocomplete $\mathscr{V}$-category, then composition with $y_A\colon A \to \mathcal{P}_\kappa A$ defines an equivalence of categories*

$$\mathscr{V}\text{-}\mathcal{CAT}_{\kappa\text{-colim}}(\mathcal{P}_\kappa A, \mathscr{B}) \longrightarrow \mathscr{V}\text{-}\mathcal{CAT}(A, \mathscr{B}).$$

In other words, $\mathcal{P}_\kappa A$ is the free κ-small cocompletion of A.

PROOF. Since $\mathscr{B}$ is κ-cocomplete, by Lemma 10.9 any functor $f : A \to \mathscr{B}$ admits a Y_A-weighted colimit, which is to say a left Kan extension along $y_A : A \to \mathcal{P}_\kappa A$. Thus we have a functor

$$\mathrm{Lan}_y\colon \mathscr{V}\text{-}\mathcal{CAT}(A, \mathscr{B}) \to \mathscr{V}\text{-}\mathcal{CAT}(\mathcal{P}_\kappa A, \mathscr{B}).$$

We claim that for any $f\colon A \to \mathscr{B}$, the functor $\mathrm{Lan}_{y_A} f : \mathcal{P}_\kappa A \to \mathscr{B}$ preserves κ-small colimits. Suppose that $J\colon K \nrightarrow C$ is a weight, where C is κ-small, and $d\colon C \to \mathcal{P}_\kappa A$ is a functor. Let $\ell = \mathrm{colim}^J d\colon K \to \mathcal{P}_\kappa A$. Using again the construction of colimits in $\mathcal{P}_\kappa A$ in Theorem 10.25, we compute

$$\begin{aligned}
\mathscr{B}\big((\mathrm{Lan}_{y_A} f)\ell, 1\big) &\cong \mathscr{B}(\mathrm{Lan}_{y_A} f, 1) \odot \mathcal{P}_\kappa A(\ell, 1)\\
&\cong \mathcal{P}_\kappa A(1, \ell) \rhd \mathscr{B}(\mathrm{Lan}_{y_A} f, 1)\\
&\cong \Big(J \odot \mathcal{P}_\kappa A(1, d)\Big) \rhd \mathscr{B}(\mathrm{Lan}_{y_A} f, 1)\\
&\cong J \rhd \Big(\mathcal{P}_\kappa A(1, d) \rhd \mathscr{B}(\mathrm{Lan}_{y_A} f, 1)\Big)\\
&\cong J \rhd \mathscr{B}((\mathrm{Lan}_{y_A} f)d, 1)
\end{aligned}$$

as desired. Therefore, we have an induced functor

$$\mathrm{Lan}_{y_A}\colon \mathscr{V}\text{-}\mathcal{CAT}(A, \mathscr{B}) \to \mathscr{V}\text{-}\mathcal{CAT}_{\kappa\text{-colim}}(\mathcal{P}_\kappa A, \mathscr{B}),$$

which we claim is an inverse equivalence to (10.33).

On the one hand, since y_A is fully faithful by Lemma 10.24, by Corollary 8.9 we have $(\mathrm{Lan}_{y_A} f)y_A \cong f$. On the other hand, by Corollary 10.27, $1_{\mathcal{P}_\kappa A}$ is the left Kan extension of y along itself. Therefore, if $g\colon \mathcal{P}_\kappa A \to \mathscr{B}$ preserves κ-small colimits, hence (by Lemma 10.9) also colimits with κ-small weights, it must preserve left Kan extensions along y_A. Thus we must have $g \cong \mathrm{Lan}_{y_A}(gy_A)$. ∎

10.34. REMARK. In the case $\mathscr{V} = \mathscr{P}sh(\mathbf{S}, \mathbf{V})$, we can repeat all the above arguments but adding "local smallness" conditions (in the sense of Example 5.19) to all κ-small categories and profunctors. This yields a free cocompletion of any pseudofunctor $\mathbf{S}^{\mathrm{op}} \to \mathbf{V}\text{-}\mathcal{CAT}$ under fiberwise $\mathbf{V}$-enriched colimits and $\mathbf{S}$-indexed $\mathbf{V}$-coproducts.

As usual, we can also find more general free cocompletions inside $\mathcal{P}_\kappa A$ by closing up the image of y_A under various types of colimits. This is the subject of [Bun13].

11. Monoidal $\mathscr{V}$-categories and iterated enrichment

In previous sections we have assumed for simplicity that $\mathscr{V}$ was symmetric, but so far everything could also be done in the non-symmetric case, simply by keeping more careful track of right versus left homs. Now, however, we consider monoidal structures on $\mathscr{V}$-categories, for which we do need a symmetry on $\mathscr{V}$ (or at least a braiding, although we will not consider that case).

Thus, let $\mathbf{S}$ have finite products and $\mathscr{V}$ be, for now, an $\mathbf{S}$-indexed symmetric monoidal category. This enables us to define opposites and tensor products of $\mathscr{V}$-categories.

11.1. Definition. For a $\mathscr{V}$-category $\mathscr{A}$, its **opposite** $\mathscr{A}^{\mathrm{op}}$ has the same objects and extents as $\mathscr{A}$, with $\underline{\mathscr{A}^{\mathrm{op}}}(x,y) = \mathfrak{s}^*\underline{\mathscr{A}}(y,x)$, where $\mathfrak{s} : \epsilon x \times \epsilon y \xrightarrow{\sim} \epsilon y \times \epsilon x$ is the twist isomorphism. Its identities are obvious, and its composition morphism is

$$\mathfrak{s}^*\underline{\mathscr{A}}(z,y) \otimes_{\epsilon y} \mathfrak{s}^*\underline{\mathscr{A}}(y,x) \xrightarrow{\sim} \mathfrak{s}^*\big(\underline{\mathscr{A}}(y,x) \otimes_{\epsilon y} \underline{\mathscr{A}}(z,y)\big) \xrightarrow{\mathrm{comp}} \mathfrak{s}^*\underline{\mathscr{A}}(z,x)$$

using the symmetry of $\mathscr{V}$ and the composition morphism of $\mathscr{A}$.

If $\mathscr{A}$ is a $\mathscr{V}$-fibration, then so is $\mathscr{A}^{\mathrm{op}}$, and we have $(\mathscr{A}^{\mathrm{op}})^X = (\mathscr{A}^X)^{\mathrm{op}}$ (the latter opposite being as a $\mathscr{V}^X$-enriched category). On the other hand, for tensor products this may not be the case. In fact, we have two seemingly different tensor products for $\mathscr{V}$-categories.

11.2. Definition. For $\mathscr{V}$-categories $\mathscr{A}$ and $\mathscr{B}$, their **tensor product** $\mathscr{A} \otimes \mathscr{B}$ has as objects pairs (a,b) where a is an object of $\mathscr{A}$ and b is an object of $\mathscr{B}$, with $\epsilon(a,b) = \epsilon a \times \epsilon b$, and hom-objects

$$\underline{\mathscr{A} \otimes \mathscr{B}}\big((a,b),(a',b')\big) = \mathfrak{s}^*\big(\underline{\mathscr{A}}(a,a') \otimes \underline{\mathscr{B}}(b,b')\big)$$

where $\mathfrak{s}$ is the isomorphism

$$\epsilon a' \times \epsilon b' \times \epsilon a \times \epsilon b \xrightarrow{\sim} \epsilon a' \times \epsilon a \times \epsilon b' \times \epsilon b.$$

Its identities are induced by those of $\mathscr{A}$ and $\mathscr{B}$ in an obvious way, while its composition morphism is

$$\begin{aligned}\mathfrak{s}^*\big(\underline{\mathscr{A}}(a',a'') \otimes \underline{\mathscr{B}}(b',b'')\big) &\otimes_{\epsilon a' \times \epsilon b'} \mathfrak{s}^*\big(\underline{\mathscr{A}}(a,a') \otimes \underline{\mathscr{B}}(b,b')\big)\\ &\xrightarrow{\sim} \mathfrak{s}^*\Big(\big(\underline{\mathscr{A}}(a',a'') \otimes_{\epsilon a'} \underline{\mathscr{A}}(a,a')\big) \otimes \big(\underline{\mathscr{B}}(b',b'') \otimes_{\epsilon b'} \underline{\mathscr{B}}(b,b')\big)\Big)\\ &\xrightarrow{\mathrm{comp}\otimes\mathrm{comp}} \mathfrak{s}^*\big(\underline{\mathscr{A}}(a,a'') \otimes \underline{\mathscr{B}}(b,b'')\big).\end{aligned}$$

11.3. Definition. For indexed $\mathscr{V}$-categories $\mathscr{A}$ and $\mathscr{B}$, their **indexed tensor product** $\mathscr{A} \otimes_{\mathbf{S}} \mathscr{B}$ is defined by

$$(\mathscr{A} \otimes_{\mathbf{S}} \mathscr{B})^X = \mathscr{A}^X \otimes_X \mathscr{B}^X,$$

the right-hand side being the tensor product of $\mathscr{V}^X$-enriched categories. Thus, the objects of $(\mathscr{A} \otimes_{\mathbf{S}} \mathscr{B})^X$ are pairs (a,b) with $a \in \mathscr{A}^X$ and $b \in \mathscr{B}^X$.

The tensor product $\otimes$ makes $\mathscr{V}$-$\mathcal{CAT}$ into a symmetric monoidal 2-category, with unit object $\delta 1$, while $\otimes_{\mathbf{S}}$ makes $\mathscr{V}$-$\mathcal{C}\mathrm{AT}$ into a symmetric monoidal 2-category, with a unit object $\mathcal{I}$ having $\mathcal{I}^X$ the unit $\mathscr{V}^X$-enriched category for all X. The two tensor products are different, but as in §6, both are "loose enough" that they agree up to equivalence.

11.4. THEOREM. *The biequivalence* $\Theta : \mathscr{V}\text{-}\mathcal{C}\mathrm{AT} \simeq \mathscr{V}\text{-}\mathcal{CAT} : \Gamma$ *is a symmetric monoidal biequivalence.*

PROOF. First of all, evidently $\mathcal{I} \cong \Gamma(\delta 1)$, hence $\delta 1 \simeq \Theta\mathcal{I}$. Now let $\mathscr{A}$ and $\mathscr{B}$ be indexed $\mathscr{V}$-categories; we define an equivalence $\Theta\mathscr{A} \otimes \Theta\mathscr{B} \simeq \Theta(\mathscr{A} \otimes_{\mathbf{S}} \mathscr{B})$. In one direction we have an indexed functor:

$$F : \Theta\mathscr{A} \otimes \Theta\mathscr{B} \longrightarrow \Theta(\mathscr{A} \otimes_{\mathbf{S}} \mathscr{B})$$

defined on objects by $F(a,b) = (\pi^*_{\epsilon b} a, \pi^*_{\epsilon a} b)$. In the other direction we have a non-indexed functor

$$G : \Theta(\mathscr{A} \otimes_{\mathbf{S}} \mathscr{B}) \longrightarrow \Theta\mathscr{A} \otimes \Theta\mathscr{B}$$

defined at an object $(a,b) \in (\mathscr{A} \otimes_{\mathbf{S}} \mathscr{B})^X$ by $G(a,b) = (a,b)$ with $G_{(a,b)} = \Delta_X$. We leave it to the reader to verify that these are inverse equivalences and support the additional coherent structure of a symmetric monoidal biequivalence. ∎

Now recall that a *monoidal object*, or *pseudomonoid*, in a monoidal 2-category consists of an object W with a multiplication $m\colon W \otimes W \to W$ and a unit $e\colon I \to W$ together with the usual coherent associativity and unit isomorphisms. We thus obtain notions of **monoidal $\mathscr{V}$-category** and **indexed monoidal $\mathscr{V}$-category** by using the tensor products $\otimes$ and $\otimes_{\mathbf{S}}$ respectively.

Theorem 11.4 tells us that if $\mathscr{W}$ is a $\mathscr{V}$-fibration, then up to equivalence there is no difference between these two notions. However, on the surface they look quite different.

On the one hand, a monoidal $\mathscr{V}$-category has an *external product* $\boxtimes : \mathscr{W} \otimes \mathscr{W} \to \mathscr{W}$. If $\mathscr{W}$ is a $\mathscr{V}$-fibration, then we may assume $\boxtimes$ to be indexed, so that $\epsilon(a \boxtimes b) = \epsilon a \times \epsilon b$, just like for the external product of $\mathscr{V}$ itself.

On the other hand, an indexed monoidal $\mathscr{V}$-category is equipped with a *fiberwise product* $\mathscr{W} \otimes_{\mathbf{S}} \mathscr{W} \to \mathscr{W}$, which takes two objects $a, b \in \mathscr{W}^X$ to $a \boxtimes_X b \in \mathscr{W}^X$, just as for the fiberwise product of $\mathscr{V}$. Indeed, an indexed monoidal $\mathscr{V}$-category is easily seen to be just an indexed $\mathscr{V}$-category for which each fiber $\mathscr{W}^X$ is a monoidal $\mathscr{V}^X$-enriched category and the transition functors f^* and their coherence isomorphisms are strong monoidal.

It should not now be surprising that the equivalence between these two types of monoidal structure on a given $\mathscr{V}$-fibration $\mathscr{W}$ exactly parallels the equivalence between external and fiberwise products for $\mathscr{V}$ described in §2. This is easy to see concretely by tracing through the equivalence constructed in Theorem 11.4. (If $\mathscr{W}$ moreover admits indexed $\mathscr{V}$-coproducts as in §9, then we can define a canceling product for it as well.)

11.5. EXAMPLE. In fact, if we take $\mathscr{V} = \mathscr{P}sh(\mathbf{S}, \mathrm{Set})$ so that indexed $\mathscr{V}$-categories are precisely $\mathbf{S}$-indexed categories, then this equivalence between the two types of monoidal structure on a $\mathscr{V}$-fibration reduces more or less exactly to our development in §2.

Of course, it is easy to define *symmetric* monoidal $\mathscr{V}$-categories, but *closed* ones require the machinery of profunctors yet again, as pioneered by [DS97, DMS03, Str04]. First we note the following.

11.6. Lemma. *For $\mathscr{V}$-categories A, B, C, there is an equivalence of categories*

$$\mathfrak{Z} : \mathscr{V}\text{-}\mathcal{P}ROF(A, B \otimes C) \xrightarrow{\sim} \mathscr{V}\text{-}\mathcal{P}ROF(B^{\mathrm{op}} \otimes A, C) \qquad \blacksquare$$

We will also need to know that $\mathfrak{Z}$ respects composites and homs.

11.7. Lemma. *For $\mathscr{V}$-profunctors $H : A \nrightarrow B \otimes C$, $K : A' \nrightarrow A$, $L : B \nrightarrow B'$, and $M : C \nrightarrow C'$ we have*

$$\mathfrak{Z}\big(K \odot H \odot (L \otimes M)\big) \cong (L^{\mathrm{op}} \otimes K) \odot \mathfrak{Z}H \odot M$$

where L^{op} is the evident induced profunctor $(B')^{\mathrm{op}} \nrightarrow B^{\mathrm{op}}$. $\blacksquare$

11.8. Lemma. *For $\mathscr{V}$-profunctors $H : A \nrightarrow B \otimes C$, $K : A \nrightarrow A'$, $L : B' \nrightarrow B$, and $M : C' \nrightarrow C$ we have*

$$\mathfrak{Z}\big((L \otimes M) \rhd H \lhd K\big) \cong M \rhd \mathfrak{Z}H \lhd (L^{\mathrm{op}} \otimes K). \qquad \blacksquare$$

Now suppose $\mathscr{W}$ is a symmetric monoidal $\mathscr{V}$-category (as usual, symmetry of $\mathscr{W}$ is not required, but keeping track of right versus left homs is tedious). If $m : \mathscr{W} \otimes \mathscr{W} \to \mathscr{W}$ is its tensor product $\mathscr{V}$-functor, we have an induced profunctor $\mathscr{W}(m, 1) : \mathscr{W} \nrightarrow \mathscr{W} \otimes \mathscr{W}$, and hence a profunctor $\mathfrak{Z}(\mathscr{W}(m, 1)) : \mathscr{W}^{\mathrm{op}} \otimes \mathscr{W} \nrightarrow \mathscr{W}$.

11.9. Definition. A symmetric monoidal $\mathscr{V}$-category $\mathscr{W}$ is **closed** if there is a $\mathscr{V}$-functor $h : \mathscr{W}^{\mathrm{op}} \otimes \mathscr{W} \to \mathscr{W}$ and an isomorphism

$$\mathfrak{Z}(\mathscr{W}(m, 1)) \cong \mathscr{W}(1, h). \tag{11.10}$$

If $\mathscr{W}$ is a $\mathscr{V}$-fibration, we may take h to be an *indexed* $\mathscr{V}$-functor $\mathscr{W}^{\mathrm{op}} \otimes_{\mathbf{S}} \mathscr{W} \to \mathscr{W}$, and by Remark 6.8 we may consider (11.10) an isomorphism of *indexed* $\mathscr{V}$-profunctors as in Definition 4.9. If we unravel this explicitly, what it says is merely that the symmetric monoidal $\mathscr{V}^X$-enriched category $\mathscr{W}^X$ is closed, for each X, and that the transition functors $(f^*)_\bullet \mathscr{W}^X \to \mathscr{W}^Y$ are closed monoidal (this is encoded in the $\mathscr{V}$-functoriality of h). We denote these *fiberwise homs* in $\mathscr{W}^X$ by $\underline{\underline{\mathscr{W}}}^X(-,-)$ (by contrast with the $\mathscr{V}$-valued fiberwise hom $\underline{\mathscr{W}}^X(x, y) \in \mathscr{V}^X$).

11.11. Example. In particular, for $\mathscr{V} = \mathscr{P}sh(\mathbf{S}, \mathrm{Set})$ as in Example 11.5, then this notion of closedness for indexed monoidal $\mathscr{V}$-categories reduces essentially to Theorem 2.14(i).

On the other hand, we might remain in the world of non-indexed $\mathscr{V}$-profunctors, but still assume that $\mathscr{W}$ is a $\mathscr{V}$-fibration, so that h might as well be indexed. In this case, h has the right type to be an external-hom such as in Theorem 2.14(iii). Namely, for objects x and y with extents ϵx and ϵy, $h(x, y)$ is an object of $\mathscr{W}^{\epsilon x \times \epsilon y}$, which we denote

$\underline{\underline{\mathscr{W}}}(x,y)$ (by contrast with the $\mathscr{V}$-valued external hom $\underline{\mathscr{W}}(x,y) \in \mathscr{V}^{\epsilon x \times \epsilon y}$). The universal property of these homs is expressed by an isomorphism

$$\underline{\mathscr{W}}(x \boxtimes y, z) \cong \underline{\mathscr{W}}\big(x, \underline{\underline{\mathscr{W}}}(y,z)\big) \tag{11.12}$$

natural in x, y, z. And as usual, given objects $\underline{\underline{\mathscr{W}}}(y,z)$ with isomorphisms (11.12) that are natural in x, then we can construct a unique (indexed) $\mathscr{V}$-functor $\mathscr{W}^{\mathrm{op}} \otimes \mathscr{W} \to \mathscr{W}$ making (11.12) natural in y and z as well. In particular, since this functor preserves restrictions like any indexed $\mathscr{V}$-functor, we have

$$(g \times f)^* \underline{\underline{\mathscr{W}}}(y,z) \cong \underline{\underline{\mathscr{W}}}(f^* y, g^* z). \tag{11.13}$$

which is a version of the compatibility condition from Theorem 2.14(iii).

However, the universal property (11.12) itself looks surprisingly different from that in Theorem 2.14(iii). But if $\mathscr{W}$ has indexed $\mathscr{V}$-coproducts preserved by $\boxtimes$, then (11.12) is equivalent to a universal property looking more like 2.14(iii). Namely, given (11.12), for $x \in \mathscr{W}^{X \times Y}$, $y \in \mathscr{W}^Y$, and $z \in \mathscr{W}^X$, we have

$$\begin{aligned}
\underline{\mathscr{W}}^X(x \boxtimes_{[Y]} y, z) &\cong \Delta_X^* \underline{\mathscr{W}}(\pi_{Y!} \Delta_Y^*(x \boxtimes y), z) \\
&\cong \Delta_X^* \pi_{Y*} \Delta_Y^* \underline{\mathscr{W}}(x \boxtimes y, z) \\
&\cong \Delta_X^* \pi_{Y*} \Delta_Y^* \underline{\mathscr{W}}(x, \underline{\underline{\mathscr{W}}}(y,z)) \\
&\cong \pi_{Y*} \Delta_{X \times Y}^* \underline{\mathscr{W}}(x, \underline{\underline{\mathscr{W}}}(y,z)) \\
&\cong \pi_{Y*} \underline{\mathscr{W}}^{X \times Y}(x, \underline{\underline{\mathscr{W}}}(y,z))
\end{aligned} \tag{11.14}$$

which is a version of (2.16), enhanced for compatibility with $\mathscr{V}$, analogously to (9.7). Conversely, if we assume (11.14) and also (11.13), then for $x \in \mathscr{W}^X$, $y \in \mathscr{W}^Y$, and $z \in \mathscr{W}^Z$ we have (omitting the symbol $\times$ in most places, for conciseness):

$$\begin{aligned}
\underline{\mathscr{W}}(x \boxtimes y, z) &\cong \underline{\mathscr{W}}^{XYZ}\big(\pi_Z^*(x \boxtimes y), \pi_{XY}^* z\big) \\
&\cong \underline{\mathscr{W}}^{XYZ}\big(\pi_{XY!} \Delta_{XY!} \Delta_{XY}^*(\pi_{YZ}^* x \boxtimes \pi_X^* y), \pi_{XY}^* z\big) \\
&\cong \underline{\mathscr{W}}^{XYZ}\big(\pi_{XY!}(1 \times \Delta_{XY})^* (\Delta_{XY} \times 1)_! (\pi_{YZ}^* x \boxtimes \pi_X^* y), \pi_{XY}^* z\big) \\
&\cong \underline{\mathscr{W}}^{XYZ}\big(\Delta_{XY!} \pi_{YZ}^* x \boxtimes_{[XY]} \pi_X^* y, \pi_{XY}^* z\big) \\
&\cong \pi_{XY*} \underline{\mathscr{W}}^{XYXYZ}\big(\Delta_{XY!} \pi_{YZ}^* x, \underline{\underline{\mathscr{W}}}(\pi_X^* y, \pi_{XY}^* z)\big) \\
&\cong \pi_{XY*} \Delta_{XY*} \underline{\mathscr{W}}^{XYZ}\big(\pi_{YZ}^* x, \Delta_{XY}^* (\pi_X \pi_{XY})^* \underline{\underline{\mathscr{W}}}(y,z)\big) \\
&\cong \underline{\mathscr{W}}^{XYZ}\big(\pi_{YZ}^* x, \pi_X^* \underline{\underline{\mathscr{W}}}(y,z)\big) \\
&\cong \underline{\mathscr{W}}(x, \underline{\underline{\mathscr{W}}}(y,z)).
\end{aligned}$$

Of course, the equivalence between the two kinds of $\mathscr{V}$-profunctors implies that the fiberwise homs $\underline{\underline{\mathscr{W}}}^X(-,-)$ and external homs $\underline{\underline{\mathscr{W}}}(-,-)$ for a symmetric monoidal $\mathscr{V}$-category are interderivable, with formulas just like those in Theorem 2.14. And if $\mathscr{W}$ has indexed $\mathscr{V}$-products, we can also define a *canceling hom* $\underline{\underline{\mathscr{W}}}^{[X]}(-,-)$ just as in Theorem 2.14.

11.15. Example. As promised in §2, interpreting (11.12) for $\mathscr{P}sh(\mathbf{S}, \mathrm{Set})$-categories yields a characterization of external-homs for ordinary indexed monoidal categories that doesn't require indexed coproducts. Recall that the $\mathscr{P}sh(\mathbf{S}, \mathrm{Set})$-valued external-hom of an $\mathbf{S}$-indexed category $\mathscr{V}$ is given at $x \in \mathscr{V}^X$ and $y \in \mathscr{V}^Y$ by

$$\begin{array}{rcl} (\mathbf{S}/(X \times Y))^{\mathrm{op}} & \longrightarrow & \mathrm{Set} \\ (Z \xrightarrow{(f,g)} X \times Y) & \mapsto & \mathscr{V}^Z(f^*x, g^*y). \end{array}$$

Thus, (11.12) consists of isomorphisms

$$\mathscr{V}^U\big((f,g)^*(x \boxtimes y), h^*z\big) \cong \mathscr{V}^U\big(f^*x, (g,h)^*\underline{\mathscr{V}}(y,z)\big)$$

for $x \in \mathscr{V}^X$, $y \in \mathscr{V}^Y$, $z \in \mathscr{V}^Z$, $f : U \to X$, $g : U \to Y$, and $h : U \to Z$, satisfying appropriate sorts of naturality. In particular, with this characterization the compatibility condition $(g \times f)^*\underline{\mathscr{V}}(x,y) \cong \underline{\mathscr{V}}(f^*x, g^*y)$ is automatic.

Finally, we have all the ingredients of the following.

11.16. Definition. If $\mathscr{V}$ is an $\mathbf{S}$-indexed cosmos, then a **$\mathscr{V}$-cosmos** is a closed symmetric monoidal indexed $\mathscr{V}$-category which is ω-complete and ω-cocomplete (in the sense of Corollary 9.28).

11.17. Example. Since indexed $\mathscr{S}elf(\mathbf{S})$-categories are just ordinary indexed categories with the property of being "locally small", a $\mathscr{S}elf(\mathbf{S})$-cosmos is just an ordinary $\mathbf{S}$-indexed cosmos with this property.

11.18. Example. An indexed $\mathscr{F}am(\mathbf{V})$-category of the form $\mathscr{F}am(\mathbf{C})$ is a $\mathscr{F}am(\mathbf{V})$-cosmos just when $\mathbf{C}$ is a $\mathbf{V}$-cosmos in a classical sense, namely a complete and cocomplete closed symmetric monoidal $\mathbf{V}$-enriched category.

11.19. Example. Since $\mathscr{P}sh(\mathbf{S}, \mathbf{V})$ is not in general a cosmos, Definition 11.16 is not quite right for it. Instead, we may reasonably define an **S-indexed V-enriched cosmos** to be a closed symmetric monoidal indexed $\mathscr{P}sh(\mathbf{S}, \mathbf{V})$-category with "locally small" limits and colimits in the sense of Example 9.30.

For instance, if $\mathbf{S}$ is locally cartesian closed, complete, and cocomplete, then the cosmos $\mathscr{A}b(\mathbf{S})$ is enriched over abelian groups in this sense.

11.20. Example. If $\mathbf{S}$ is locally cartesian closed, complete and cocomplete, then $\mathscr{S}elf(\mathbf{S})$ is an $\mathbf{S}$-indexed $\mathbf{S}$-enriched cosmos, where in addition to being the base of the indexing, we regard $\mathbf{S}$ as a classical cartesian monoidal category. Similarly, the cosmoi $\mathscr{K}$ and $\mathscr{K}_*$ from Examples 2.32 and 2.33 are indexed enriched cosmoi over a good category of topological spaces. The interaction of these iterated enrichments on $\mathscr{K}$-categories and $\mathscr{K}_*$-categories, along with variations with an action by a fixed topological group (as in Example 2.37) is discussed in detail in [MS06, Ch. 10].

Now, if $\mathscr{W}$ is a symmetric monoidal indexed $\mathscr{V}$-category, then there is a lax symmetric monoidal morphism $\mathscr{W}_{\mathrm{o}} \to \mathscr{V}$ over $\mathbf{S}$, defined on the fiber over X by $W \mapsto \underline{\mathscr{W}}^X(\mathbb{I}_X, W)$. Moreover, the following triangle commutes up to isomorphism:

$$\begin{tikzcd} \mathscr{V} \arrow[dr] & & \mathscr{W}_{\mathrm{o}} \arrow[ll] \arrow[dl] \\ & \mathscr{P}sh(\mathbf{S}, \mathrm{Set}). & \end{tikzcd} \tag{11.21}$$

Conversely, if $\mathscr{W}_{\mathrm{o}}$ is a symmetric monoidal $\mathbf{S}$-indexed category equipped with a lax morphism satisfying (11.21), and moreover $\mathscr{W}_{\mathrm{o}}$ is *closed*, then by applying the induced change-of-cosmos functor to the $\mathscr{W}$-category $\mathscr{W}$, we obtain a closed symmetric monoidal indexed $\mathscr{V}$-category structure on $\mathscr{W}$. These two constructions are inverses, so just as for classical monoidal categories, we have an equivalence between

(i) lax symmetric monoidal morphisms of fibrations $\mathscr{W} \to \mathscr{V}$ over $\mathbf{S}$, where $\mathscr{W}$ is closed symmetric monoidal, satisfying (11.21), and

(ii) closed symmetric monoidal $\mathscr{V}$-fibrations.

Similarly, we can show that the lax morphism $\mathscr{W}_{\mathrm{o}} \to \mathscr{V}$ has a strong monoidal left adjoint if and only if the $\mathscr{V}$-category $\mathscr{W}$ has fiberwise tensors, and that in this case $\mathscr{W}_{\mathrm{o}}$ is a cosmos if and only if $\mathscr{W}$ is a $\mathscr{V}$-cosmos. This gives an alternative approach to many of our examples from §2, but we will not revisit them all here.

We end with a version of the Day convolution monoidal structure [Day70] for $\mathscr{V}$-categories. Note that the monoidal $\mathscr{V}$-category A appearing below is κ-small, hence probably not a $\mathscr{V}$-fibration; thus its tensor product and unit morphisms may not be indexed $\mathscr{V}$-functors.

11.22. THEOREM. *Let $\mathscr{V}$ be an indexed cosmos which is κ-complete and κ-cocomplete, and let A be a κ-small symmetric monoidal $\mathscr{V}$-category. Then $\mathcal{P}_\kappa A$ is a closed symmetric monoidal $\mathscr{V}$-category.*

PROOF. We define the product $\widehat{m} : \mathcal{P}_\kappa A \otimes \mathcal{P}_\kappa A \to \mathcal{P}_\kappa A$ to be the classifying map of the profunctor

$$\mathcal{P}_\kappa A \otimes \mathcal{P}_\kappa A \xrightarrow{(Y_A \otimes Y_A) \odot A(1,m)} A,$$

where $m : A \otimes A \to A$ is the tensor product of A. The displayed composite exists since $A \otimes A$ is κ-small. The unit $\delta 1 \to \mathcal{P}_\kappa A$ is the classifying map of $A(1, i)$, where $i : \delta 1 \to A$ is the unit of A. By full-faithfulness and pseudofunctoriality of representable profunctors, the coherence data for A lift automatically to the corresponding profunctors, and thence to their classifying functors; thus $\mathcal{P}_\kappa A$ is symmetric monoidal. For closedness, we use Proposition 10.18 to compute

$$\begin{aligned} \mathcal{P}_\kappa A(\widehat{m}, 1) &\cong \big((Y_A \otimes Y_A) \odot A(1, m)\big) \rhd Y_A \\ &\cong (Y_A \otimes Y_A) \rhd \big(A(1, m) \rhd Y_A\big) \\ &\cong (Y_A \otimes Y_A) \rhd Y_A(m, 1). \end{aligned}$$

Thus, by Lemma 11.8 and Proposition 10.18 again, we have

$$\begin{aligned}\mathfrak{Z}(\mathcal{P}_\kappa A(\widehat{m},1)) &\cong Y_A \rhd \big(\mathfrak{Z}(Y_A(m,1)) \lhd (\mathcal{P}_\kappa A \otimes Y_A^{\mathrm{op}})\big)\\ &\cong \mathcal{P}_\kappa(1,\widehat{h})\end{aligned}$$

where $\widehat{h} : (\mathcal{P}_\kappa A)^{\mathrm{op}} \times \mathcal{P}_\kappa A \to \mathcal{P}_\kappa A$ is the classifying functor of $\mathfrak{Z}(Y_A(m,1)) \lhd (\mathcal{P}_\kappa A \otimes Y_A^{\mathrm{op}})$. Thus, $\mathcal{P}_\kappa A$ is closed. ■

11.23. Remark. In fact, for Theorem 11.22 it suffices for A to be a symmetric monoidal object in $\mathscr{V}$-$\mathcal{PROF}$, rather than $\mathscr{V}$-$\mathcal{CAT}$ — that is, a *promonoidal* $\mathscr{V}$-category. As in the classical case [Day70, Day74], promonidal structures on A are actually equivalent to closed symmetric monoidal structures on $\mathcal{P}_\kappa A$.

This allows us to produce easily two of the most important topological examples.

11.24. Example. The cosmos $\mathscr{K}_*$ of sectioned topological spaces from Example 2.33 is enriched and fiberwise-tensored over the cosmos $\mathscr{K}$ of topological spaces from Example 2.32, so we have a monoidal adjunction $\mathscr{K} \rightleftarrows \mathscr{K}_*$. Let $\mathscr{I}$ be the topologically enriched category of finite-dimensional inner product spaces and linear isometric isomorphisms, and regard it as a set-small $\mathscr{K}$-category with all objects having extent 1, as in Example 10.15.

Now change cosmos along the left adjoint $\mathscr{K} \to \mathscr{K}_*$ (which adds disjoint sections) to obtain a set-small $\mathscr{K}_*$-category $\mathscr{I}_+$. Then $\mathcal{P}(\mathscr{I}_+^{\mathrm{op}})$, with its Day convolution monoidal structure from Theorem 11.22, is the $\mathscr{K}_*$-cosmos of *parametrized $\mathscr{I}$-spaces* described in [MS06, §11.1].

Finally, one-point compactifications yield a sphere object $S \in \mathcal{P}(\mathscr{I}_+^{\mathrm{op}})^1$ which is a commutative monoid. Thus, regarding $\mathcal{P}(\mathscr{I}_+^{\mathrm{op}})$ as an ordinary cosmos as above, and applying Example 2.35, we obtain the $\mathscr{K}_*$-cosmos of parametrized orthogonal spectra from [MS06]. (In [MS06], everything has an additional action by a fixed topological group; this can easily be added using Example 2.37.)

11.25. Example. Recall from Example 2.38 that we have a Grp(Top)-indexed monoidal category $\mathscr{A}ct(\mathrm{Top})$, where Grp(Top) is the category of topological groups and $\mathscr{A}ct(\mathrm{Top})^G$ is the cartesian monoidal category of G-spaces. From Example 2.33 we obtain a Grp(Top)-indexed monoidal category $\mathscr{A}ct(\mathrm{Top})_*$ of based spaces with group actions.

Let $\mathcal{G} \subseteq \mathrm{Grp(Top)}$ be the full subcategory of finite groups, and $\mathscr{T}op_{\mathcal{G}}$ the restriction of $\mathscr{A}ct(\mathrm{Top})_*$ to $\mathcal{G}$. This is a fiberwise complete and cocomplete $\mathcal{G}$-indexed cosmos.

Let $\mathcal{I}_{\mathcal{G}}$ be the $\mathscr{T}op_{\mathcal{G}}$-category with objects $(G \in \mathcal{G}, n \in \mathbb{N}, \rho : G \to O(n))$; that is, finite-dimensional representations of finite groups. The extent of such a representation is of course G, while $\underline{\mathcal{I}_{\mathcal{G}}}(\rho,\rho') \in (\mathscr{T}op_{\mathcal{G}})^{G\times G'}$ is the space of linear isometric isomorphisms $\mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^{n'}$ (which is of course empty unless $n = n'$) with a disjoint basepoint added, with $(G \times G')$-action by conjugation. Of course, this may be obtained by change of cosmos from an unbased version.

Now $\mathcal{I}_{\mathcal{G}}$ is a set-small $\mathscr{T}op_{\mathcal{G}}$-fibration, and it is moreover symmetric monoidal under the direct sum of representations. Therefore, by Theorem 11.22, we have a $\mathscr{T}op_{\mathcal{G}}$-cosmos

$\mathcal{P}(\mathcal{I}_{\mathcal{G}}^{\mathrm{op}})$ of "$\mathcal{I}_{\mathcal{G}}$-spaces". (In [Boh12], only the objects of $\mathcal{P}(\mathcal{I}_{\mathcal{G}}^{\mathrm{op}})$ of extent 1 are called $\mathcal{I}_{\mathcal{G}}$-spaces; those of extent $G \in \mathcal{G}$ have an additional G-action on each of their spaces.)

Finally, one-point compactifications yield a canonical sphere object $S \in \mathcal{P}(\mathcal{I}_{\mathcal{G}}^{\mathrm{op}})^1$ which is a commutative monoid, and so again from Example 2.35 we obtain a $\mathscr{T}op_{\mathcal{G}}$-cosmos whose objects of extent 1 are the *orthogonal $\mathcal{G}$-spectra* of [Boh12].

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