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COEXPONENTIABILITY AND PROJECTIVITY: RIGS, RINGS, AND QUANTALES

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Abstract. We show that a commutative monoid A is coexponentiable in $\mathrm{CMon}(\mathcal{V})$ if and only if $-\otimes A\colon \mathcal{V} \longrightarrow \mathcal{V}$ has a left adjoint, when $\mathcal{V}$ is a cocomplete symmetric monoidal closed category with finite biproducts and in which every object is a quotient of a free. Using a general characterization of the latter, we show that an algebra over a rig or ring R is coexponentiable if and only if it is finitely generated and projective as an R-module. Omitting the finiteness condition, the same result (and proof) is obtained for algebras over a quantale.

1. Introduction

Recall that an object A of a category $\mathcal{A}$ with finite products is *exponentiable* if and only if $- \times A\colon \mathcal{A} \longrightarrow \mathcal{A}$ has a right adjoint. In [Niefield, 1982], the first author showed that for an algebra A over a commutative ring R, the spectrum $\mathrm{Spec}(A)$ is exponentiable in the category of affine schemes over $\mathrm{Spec}(R)$ if and only if A is finitely generated and projective as an R-module, and later showed in [Niefield, 2016] that essentially the same proof gave a characterization of coexponentiable morphisms of quantales (with the finiteness condition omitted). After a presentation of the latter, Lawvere and Menni asked if this characterization also generalized to rigs and, in particular, idempotent rigs.

A rig (or "ring without negatives") is another name for a commutative semiring, and an idempotent rig is one in which $1{+}1 = 1$. See, for example [Schanuel, 1991], where the Burnside rig of a distributive category is introduced, [Lawvere/Schanuel] for the study of rigs in "Objective Number Theory," or more recently [Castiglioni et al]. Note that idempotent rigs are $\mathbb{2}$-rigs (i.e., rigs under $\mathbb{2}$), or alternately, commutative monoids in the category of join-semilattices with 0.

Since rigs, rings, and quantales are commutative monoids in an appropriate monoidal category, such a generalization seemed reasonable. However, the proof for quantales (respectively, rings) used a property of modules that does not appear to hold for modules over a rig, namely, every flat (respectively, finitely presented flat) module is projective (respectively, finitely generated projective). After consulting the vast semiring literature, the ring/quantale approach did not seem feasible for rigs. Then, a 1981 letter from the second author surfaced including an alternate proof that for a module M over a ring R,

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the endofunctor $-\otimes_R M$ has a left adjoint if and only if the canonical morphism

$$M \otimes_R \mathrm{Hom}_R(M,R) \longrightarrow \mathrm{Hom}_R(M,M)$$

is an isomorphism, and that this is equivalent to M being finitely generated and projective.

In this paper, we show that the above property (and the construction of coexponentials of rings) generalizes to any cocomplete symmetric monoidal closed category with finite biproducts and in which every object is a quotient of a free, and hence gives a characterization of coexponentiable rigs.

2. Left Adjoints to Tensor

Throughout this section, we assume $(\mathcal{V}, I, \otimes, [\cdot,\cdot], \ldots)$ is a symmetric monoidal closed category in which every object is a quotient of a free, i.e., for every V there is a regular epimorphism of the form $\oplus_\alpha I \longrightarrow V$, where the domain is a coproduct of copies of the unit I indexed by α.

2.1. Proposition. *The following are equivalent for an object V of $\mathcal{V}$.*

(a) The functor $-\otimes V\colon \mathcal{V} \longrightarrow \mathcal{V}$ has a left adjoint.

(b) The functor $-\otimes V\colon \mathcal{V} \longrightarrow \mathcal{V}$ preserves limits.

(c) The canonical morphism $\theta_V\colon V\otimes[V,I] \longrightarrow [V,V]$ is an isomorphism.

(d) The functor $-\otimes V$ is left and right adjoint to $-\otimes[V,I]$.

Proof. Clearly, $(d) \Rightarrow (a) \Rightarrow (b)$. To show that $(b) \Rightarrow (c)$, suppose $\oplus_\beta I \rightrightarrows \oplus_\alpha I \longrightarrow V$ is a coequalizer, and consider the commutative diagram

$$\begin{array}{ccccc}
V\otimes[V,I] & \longrightarrow & V\otimes[\oplus_\alpha I, I] & \rightrightarrows & V\otimes[\oplus_\beta I, I] \\
{\scriptstyle\theta_V}\downarrow & & \downarrow{\scriptstyle\theta_\alpha} & & \downarrow{\scriptstyle\theta_\beta} \\
[V,V] & \longrightarrow & [\oplus_\alpha I, V] & \rightrightarrows & [\oplus_\beta I, V]
\end{array}$$

where the rows are equalizers, since $[-,W]$ takes coequalizers to equalizers, for all W, and $V\otimes -$ preserves equalizers by assumption (b). Since $[-,V]$ takes coproducts to products, the canonical morphism $[\oplus_\alpha I, V] \longrightarrow \prod_\alpha [I,V] \cong \prod_\alpha V$ is an isomorphism. Likewise, for $[\oplus_\alpha I, I] \longrightarrow \prod_\alpha I$, and so θ_α is an isomorphism, since $-\otimes V$ preserves products by assumption. Similarly, θ_β is an isomorphism, and it follows that θ_V is, as well.

For $(c) \Rightarrow (d)$, we will show that $-\otimes V$ is left adjoint to $-\otimes[V,I]$. The other adjunction follows by symmetry of $\otimes$. Consider $\varepsilon_I\colon [V,I]\otimes V \xrightarrow{\varepsilon_{V,I}} I$, the counit of $-\otimes V \dashv [V,-]$, and $\eta_I\colon I \xrightarrow{\iota} [V,V] \xrightarrow{\theta_V^{-1}} V\otimes[V,I]$, where ι is the transpose of the identity map on V. Tensoring on the left with W, we get

$$\varepsilon_W\colon W\otimes[V,I]\otimes V \longrightarrow W \quad \text{and} \quad \eta_W\colon W \longrightarrow W\otimes V\otimes[V,I]$$

To see that the adjunction identities hold, it suffices to show they do when $W = I$. Since

$$\begin{tikzcd}
V \otimes [V,I] \otimes V \arrow[r, "\theta_V \otimes V"] \arrow[d, "V \otimes \varepsilon_{V,I}"'] & {[V,V] \otimes V} \arrow[dl, "\varepsilon_{V,V}"] \\
V &
\end{tikzcd}$$

commutes, by definition of θ_V, it follows that

$$\begin{tikzcd}
V \arrow[r, "\iota \otimes V"] \arrow[drr, "id_V"'] & {[V,V] \otimes V} \arrow[r, "\theta_V^{-1} \otimes V"] \arrow[dr, "\varepsilon_{V,V}"] & V \otimes [V,I] \otimes V \arrow[d, "V \otimes \varepsilon_{V,I}"] \\
& & V
\end{tikzcd}$$

commutes, and so $(V \otimes \varepsilon_I)(\eta_I \otimes V) = id_V$. Also,

$$\begin{tikzcd}
{[V,I] \otimes V \otimes [V,I]} \arrow[r, "{[V,I] \otimes \theta_V}"] \arrow[d, "{\varepsilon_{V,I} \otimes [V,I]}"'] & {[V,I] \otimes [V,V]} \arrow[dl, "\circ"] \\
{[V,I]} &
\end{tikzcd}$$

commutes, by definition of $\circ$ and θ_V. Thus, we get a commutative diagram

$$\begin{tikzcd}
{[V,I]} \arrow[r, "{[V,I] \otimes \iota}"] \arrow[drr, "{id_{[V,I]}}"'] & {[V,I] \otimes [V,V]} \arrow[r, "{[V,I] \otimes \theta_V^{-1}}"] \arrow[dr, "\circ"] & {[V,I] \otimes V \otimes [V,I]} \arrow[d, "{\varepsilon_{V,I} \otimes [V,I]}"] \\
& & {[V,I]}
\end{tikzcd}$$

and it follows that $(\varepsilon_I \otimes [V,I])([V,I] \otimes \eta_I) = id_{[V,I]}$, as desired. ∎

Proposition 2.1 applies when $\mathcal{V}$ is the category **Ab** of abelian groups, the category **Sup** of sup lattices, the category **CMon** of commutative monoids, and more generally in each case, the category $R\mathbf{Mod}(\mathcal{V})$ of modules over a commutative monoid R in $\mathcal{V}$ (i.e., a ring, quantale, or rig). Taking $R = I$, we get $I\mathbf{Mod}(\mathcal{V}) = \mathcal{V}$, and so the latter case includes each of the first three. Note that the tensor product $\otimes$ in **CMon** is similar to that of **Ab** and **Sup**.

Suppose $\theta_V : V \otimes [V,I] \longrightarrow [V,V]$ is an isomorphism, and write

$$\theta_V^{-1}(id_V) = \sum_{\alpha \in S} v_\alpha \otimes \varphi_\alpha$$

where S is finite in all cases but $\mathcal{V} = \mathbf{Sup}$. Applying θ_V, we see that $\sum_\alpha \phi_\alpha(v) v_\alpha = v$, for all $v \in V$. Thus, the v_α generate V and we get a morphism $r : \oplus_\alpha I \longrightarrow V$. Since $\oplus_\alpha I \cong \prod_\alpha I$, in each case, the ϕ_α induce a morphism $i\colon V \longrightarrow \oplus_\alpha I$ such that $ri = id_V$, and it follows that V is projective.

Conversely, suppose $F \underset{r}{\overset{i}{\leftrightarrows}} V$ is a retraction of F onto V, where $F = \oplus_\alpha I \cong \prod_\alpha I$, indexed by a finite set A in all cases but $\mathcal{V} = \mathbf{Sup}$, and consider the diagram

$$\begin{tikzcd}
F \otimes [F, I] \arrow[r, shift right] \arrow[d, "\theta_F"'] & V \otimes [V, I] \arrow[l, shift right] \arrow[d, "\theta_V"] \\
{[F, F]} \arrow[r, shift left] & {[V, V]} \arrow[l, shift left]
\end{tikzcd}$$

where the horizontal morphisms are the retractions induced by i and r, and the squares commutes. Since $F = \oplus_\alpha I \cong \prod_\alpha I$, it follows that θ_F is an isomorphism, and so θ_V is as well.

Thus, we get the following corollaries which are well known when $\mathcal{V}$ is **Ab** and **Sup** (see [Niefield, 1982], [Joyal/Tierney, 1984]).

2.2. COROLLARY. *Suppose $\mathcal{V}$ is* **CMon** *or* **Ab**, *and R is a commutative monoid in $\mathcal{V}$, i.e., a rig or ring. Then the following are equivalent for an R-module M in $\mathcal{V}$.*

(a) The functor $- \otimes_R M\colon R\mathbf{Mod} \longrightarrow R\mathbf{Mod}$ has a left adjoint.

(b) M is finitely generated and projective.

(c) The functor $- \otimes_R M$ is left and right adjoint to $- \otimes_R \mathrm{Hom}_R(M, R)$.

Since the category of $\mathbb{2}$-modules in **CMon** is isomorphic to the category **Semi** of join-semilattices (with 0), Corollary 2.2 becomes:

2.3. COROLLARY. *The following are equivalent for X in* **Semi**.

(a) The functor $- \otimes X\colon \mathbf{Semi} \longrightarrow \mathbf{Semi}$ has a left adjoint.

(b) X is finitely generated and projective.

(c) The functor $- \otimes X$ is left and right adjoint to $- \otimes \mathbf{Semi}(X, \mathbb{2})$.

2.4. COROLLARY. *Suppose Q is a commutative quantale. Then the following are equivalent for a Q-module M.*

(a) The functor $- \otimes_Q M\colon Q\mathbf{Mod} \longrightarrow Q\mathbf{Mod}$ has a left adjoint.

(b) M is projective

(c) The functor $- \otimes_Q M$ is left and right adjoint to $- \otimes_Q \mathrm{Hom}_Q(M, Q)$.

3. Coexponentiable Commutative Monoids

Throughout this section, we assume $\mathcal{V}$ is a cocomplete symmetric monoidal closed category with finite biproducts (in the sense of [Mac Lane, 1971]) and in which every object is a quotient of a free. To simplify notation, let $\mathcal{C}$ denote the category of commutative monoids in $\mathcal{V}$. Then $\mathcal{C}$ has finite coproducts given by the tensor $\otimes$ of $\mathcal{V}$, and so one can consider coexponentiable commutative monoids $\mathcal{V}$.

Every object V of $\mathcal{V}$ gives rise to two objects of $\mathcal{C}$. In addition to the free commutative monoid SV (see below), there is a monoid structure on $I \times V$ defined as follows. Take $\eta: I \longrightarrow I \times V$ to be the morphism whose first projection is the identity on I and second projection is the composite $I \longrightarrow 0 \longrightarrow V$, where 0 is the initial and terminal object of $\mathcal{V}$, and let $\mu: (I \times V) \otimes (I \times V) \longrightarrow I \times V$ denote the morphism with first projection given by

$$(I \times V) \otimes (I \times V) \xrightarrow{\pi_1 \otimes \pi_1} I \otimes I \cong I$$

and second projection by

$$(I \times V) \otimes (I \times V) \xrightarrow{(\pi_1 \otimes \pi_2, \pi_2 \otimes \pi_1)} (I \otimes V) \times (V \otimes I) \cong (I \otimes V) \oplus (V \otimes I) \xrightarrow{\binom{\lambda}{\rho}} V$$

where λ and ρ are the structure isomorphism in $\mathcal{V}$. Then, one can show that $I \times V$ is a commutative monoid in $\mathcal{V}$, and every morphism $f: V \longrightarrow W$ of $\mathcal{V}$ induces a homomorphism $I \times f: I \times V \longrightarrow I \times W$. The operation μ can be thought of as a "generalized derivation" since for modules over a ring or rig R, it is defined by $(r, v)(s, w) = (rs, rw + sv)$.

Recall that the free commutative monoid SV (with unit $i: V \longrightarrow SV$) is defined via the coequalizer

$$TV \otimes TV \rightrightarrows TV \longrightarrow SV$$

imposing commutativity on the free monoid $TV = \oplus_{n \geq 0} V^{\otimes n}$. One can show that the counit $\varepsilon: SC \longrightarrow C$ is a regular epimorphism in $\mathcal{C}$, since it is a retraction in $\mathcal{V}$ which induces a coequalizer

$$SC \underset{id_{SC}}{\overset{i\varepsilon}{\rightrightarrows}} SC \xrightarrow{\varepsilon} C$$

in $\mathcal{V}$ and, since the tensor product of coequalizers of this form is a coequalizer in $\mathcal{V}$, the corresponding diagram

$$S(SC) \rightrightarrows SC \xrightarrow{\varepsilon} C$$

is a coequalizer in $\mathcal{C}$.

3.1. THEOREM. *Let C be a commutative monoid in $\mathcal{V}$. Then C is coexponentiable in $\mathcal{C}$ if and only if $- \otimes C: \mathcal{V} \longrightarrow \mathcal{V}$ has a left adjoint.*

PROOF. By Proposition 2.1 it suffices to show that C is coexponentiable in $\mathcal{C}$ if and only if $-\otimes C$ preserves limits in $\mathcal{V}$.

Suppose C is coexponentiable in $\mathcal{C}$. Given a diagram $\{V_\alpha\}$ in $\mathcal{V}$, there is a retraction $I \times V_\alpha \underset{\pi_2}{\overset{i_2}{\rightleftarrows}} V_\alpha$ in $\mathcal{V}$, for each α, and thus we get a commutative diagram

$$\begin{tikzcd}
(\lim_\alpha V_\alpha) \otimes C \arrow[r, shift left, leftarrow] \arrow[r, shift right] \arrow[d] & (\lim_\alpha (I \times V_\alpha)) \otimes C \arrow[d, "\theta"] \\
\lim_\alpha (V_\alpha \otimes C) \arrow[r, shift left] \arrow[r, shift right, leftarrow] & \lim_\alpha ((I \times V_\alpha) \otimes C)
\end{tikzcd}$$

where the horizontal morphisms are retractions in $\mathcal{V}$. Since C is coexponentiable and θ is a homomorphism, we know that θ is an isomorphism in $\mathcal{C}$, and it follows that $-\otimes C$ preserves limits in $\mathcal{V}$.

Conversely, suppose $-\otimes C$ preserves limits in $\mathcal{V}$. Then $-\otimes [C, I] \dashv -\otimes C$ in $\mathcal{V}$, by Proposition 2.1. Then we have the following bijections, natural in V and B,

$$\mathcal{C}(SV, B \otimes C) \cong \mathcal{V}(V, B \otimes C) \cong \mathcal{V}(V \otimes [C, I], B) \cong \mathcal{C}(S(V \otimes [C, I]), B)$$

Given A in $\mathcal{C}$, there is a coequalizer $S(SA) \underset{g}{\overset{f}{\rightrightarrows}} SA \longrightarrow A$ in $\mathcal{C}$, and so defining $L_C A$ to be the coequalizer in $\mathcal{C}$

$$S(SA \otimes [C, I]) \rightrightarrows S(A \times [C, I]) \longrightarrow L_C A$$

of the morphisms induced by f and g, yields the desired natural bijection

$$\mathcal{C}(A, B \otimes C) \cong \mathcal{C}(L_C A, B)$$

■

Theorem 3.1 applies when $\mathcal{V} = R\mathbf{Mod}$, where R is a commutative ring, rig, or quantale. In each case, $\mathcal{C}$ is the category $R\mathbf{Alg}$ of (commutative) R-algebras. Thus, by the corollaries to Proposition 2.1 we get:

3.2. COROLLARY. *Suppose R is a commutative ring or rig. Then A is coexponentiable in $R\mathbf{Alg}$ if and only if A is finitely generated and projective as an R-module.*

3.3. COROLLARY. *Suppose Q is a commutative quantale. Then A is coexponentiable in $Q\mathbf{Alg}$ if and only if A is projective as a Q-module.*

Recall that C is a monoid in $\mathcal{V}$, then $C\mathbf{Alg}$ is isomorphic to the category of commutative monoids under C in $\mathcal{V}$, and so these corollaries become the following characterizations of coexponentiable morphisms in $\mathcal{C}$. Note that, since every rig under an idempotent rig R is again idempotent, the category of rigs under such an R is isomorphic to the full subcategory of idempotent ones.

3.4. Corollary. *A morphism $R \longrightarrow A$ of commutative rings, rigs, or idempotent rigs is coexponentiable if and only if A is finitely generated and projective as an R-module.*

3.5. Corollary. *A morphism $Q \longrightarrow A$ of commutative quantales is coexponentiable if and only if A is projective as a Q-module.*

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