Theory and Applications of Categories, Vol. 33, No. 20, 2018, pp. 537–582.

HIGHER ISOTROPY

J. FUNK, P. HOFSTRA, AND S. KHAN

Abstract. This paper is about an invariant of small categories called *isotropy*. Every small category $\mathbb{C}$ has associated with it a presheaf of groups on $\mathbb{C}$, called its *isotropy group*, which in a sense solves the problem of making the assignment $C \mapsto \mathsf{Aut}(C)$ functorial. Consequently, every category has a canonical congruence that annihilates the isotropy; however, it turns out that the resulting quotient may itself have non-trivial isotropy. This phenomenon, which we term *higher order isotropy*, is the subject of our investigation. We show that with each category $\mathbb{C}$ we may associate a sequence of groups called its higher isotropy groups, and that these give rise to a sequence of quotients of $\mathbb{C}$. This sequence leads us to an ordinal invariant for small categories, which we call *isotropy rank*: the isotropy rank of a small category is the ordinal at which the sequence of quotients stabilizes. Our main results state that each small category has a well-defined isotropy rank, and moreover, that for each small ordinal one may construct a small category with precisely that rank. It happens that isotropy rank of a small category is an instance of the same concept for Grothendieck toposes, for which corresponding statements hold. Most of the technical work in the paper is concerned with the development of tools that allow us to compute (higher) isotropy groups of categories in terms of those of certain suitable subcategories.

1. Introduction

If a small category $\mathbb{C}$ is a groupoid, then the assignment

$$\mathsf{Aut}(-) : \mathbb{C}^{\mathrm{op}} \to \mathsf{Grp}\ ; \qquad C \mapsto \mathsf{Aut}(C) \tag{1}$$

is functorial in a unique way. In fact, transition between isotropy groups

$$f^* : \mathsf{Aut}(C) \to \mathsf{Aut}(D)\ ; \qquad f^*(\alpha) = f^{-1}\alpha f$$

P. Hofstra is partially supported by an NSERC Discovery Grant. Part of the research for this paper and of the writing of the manuscript took place during his sabbatical leave. He is grateful to Steve Awodey for inviting him to CMU, and to him and the research group for providing a stimulating work environment. Support was provided by the Air Force Office of Scientific Research through MURI grant FA9550-15-1-0053. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the AFOSR.

Received by the editors 2017-04-10 and, in final form, 2018-05-26.

Transmitted by Richard Blute. Published on 2018-06-03.

2010 Mathematics Subject Classification: 08A35, 18A23, 18A32, 18B25, 18D35.

Key words and phrases: Automorphism groups, algebraic invariants of categories, toposes.

along a morphism $f : D \to C$ of $\mathbb{C}$ must be given by conjugation with f, so that each diagram

$$\begin{tikzcd} D \arrow[r, "f^*(\alpha)"] \arrow[d, "f"'] & D \arrow[d, "f"] \\ C \arrow[r, "\alpha"'] & C \end{tikzcd} \tag{2}$$

commutes. Without the assumption that $\mathbb{C}$ is a groupoid (1) fails to be a functor. The basic idea of isotropy of small categories is to remedy this failure. The solution coincides with (1) in the case of a groupoid; however, in general also it must account for the fact that for some $f : D \to C$ and $\alpha \in \mathsf{Aut}(C)$ there may not exist $\beta \in \mathsf{Aut}(D)$ for which $f\beta = \alpha f$, that α may lift to various different β along the same f, and that even though α may lift along each possible $D \to C$ it may not be possible to do this coherently.

Once we associate with a category $\mathbb{C}$ its isotropy group

$$\mathsf{Z} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Grp}$$

we may show that for each object C of $\mathbb{C}$ there is a canonical group homomorphism $\mathsf{Z}(C) \to \mathsf{Aut}(C)$, which is in general neither injective nor surjective. We then consider the collection of those automorphisms $\alpha \in \mathsf{Aut}(C)$ that are part of the isotropy group in the sense that they are in the image of $\mathsf{Z}(C) \to \mathsf{Aut}(C)$. The automorphisms in the image generate a congruence I on $\mathbb{C}$ with associated quotient $\psi : \mathbb{C} \to \mathbb{C}/\mathsf{I}$, which we term the *isotropy quotient* of $\mathbb{C}$. Perhaps it is surprising that the quotient category may itself have a non-trivial isotropy group: apparently the quotient has annihilated the obstructions to some non-trivial automorphisms of $\mathbb{C}$ being part of isotropy, a phenomenon we call *higher isotropy*. Therefore, it makes sense to continue the process of forming isotropy quotients transfinitely many times if necessary. Our goal is to study this sequence of iterated isotropy quotients of a category.

1.1. Objectives and main results. The sequence of iterated isotropy quotients of a category raises several questions. Does the sequence stabilize, and if so, then how can we tell when? Is the process of forming isotropy quotients functorial? Is the length of the sequence an invariant of categories? Can we obtain categories for which the associated sequence has an arbitrarily large prescribed length? We shall answer these questions and more.

1. We develop the elementary theory of isotropy groups and isotropy quotients of small categories, and discuss aspects of functoriality. We explain how the isotropy groups of two categories $\mathbb{C}$ and $\mathbb{D}$ are related when the categories are related by a functor $\mathbb{D} \to \mathbb{C}$. We also present sufficient conditions on functors to behave well with respect to (higher) isotropy.

2. We present a sequence of categories $\mathbb{X}[\kappa]$ for each small ordinal κ, with the property that the isotropy quotient sequence of $\mathbb{X}[\kappa]$ stabilizes after exactly κ steps. This shows that isotropy of arbitrarily high ordinal order is possible.

3. Because the examples and proofs that exhibit the desired behaviour would be too involved to construct and calculate by hand, we seek to develop methods for the analysis of isotropy of a category. These are somewhat analogous to ideas from algebraic topology where one seeks to describe invariants of a space in terms of suitable decompositions of that space. In our setting we obtain results about isotropy of categories arising as sequential colimits of categories and as collages of certain profunctors. (The collage of a profunctor $\mathcal{P} : \mathbb{C} \nrightarrow \mathbb{D}$ is the canonical cospan of functors $\mathbb{C} \xleftarrow[G]{} \mathbb{P} \xrightarrow[F]{} \mathbb{D}$ for which $\mathcal{P} \cong G^* \circ F_*$ [Bénabou, 2000].)

4. Isotropy theory for small categories is an instance of the same concept for toposes [Funk et al., 2012]. Propositions 9.2 establishes the fact that the isotropy group of a presheaf topos $\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ agrees with that of the category $\mathbb{C}$. The phenomenon of higher isotropy also occurs for Grothendieck toposes, so that each Grothendieck topos has a well-defined isotropy rank: this is a new invariant of toposes. Moreover, for each ordinal we may find a Grothendieck topos with exactly that isotropy rank.

1.2. Organization. In § 2 we present the basic definitions involving isotropy groups and isotropy quotients. We also present some elementary results concerning the isotropy congruence and isotropy quotients. § 3 turns to aspects of functoriality; it introduces various notions of preservation of isotropy, and establishes some results concerning a class of functors here called sieve inclusions. These play a central role in the construction of our main examples. In § 4 we present a small category with second order isotropy. This key ingredient, while relatively simple, is a building block for the more complicated examples later on. Higher order isotropy is introduced and studied in § 5. Here we prove that every small category has a well-defined isotropy rank, introduce higher isotropy groups, and also develop some tools that help determine lower bounds for isotropy ranks. The family of examples of categories $\mathbb{X}[\kappa]$ is constructed in § 6. After explaining the idea behind the construction informally we turn to profunctors and their collages in order to make things precise. The detailed analysis of the behaviour of isotropy under the formation of collages in § 7 is the main technical work of the paper. § 8 analyzes isotropy of colimits and establishes the central results of this paper. In § 9 we apply some of the main results to the theory of Grothendieck toposes. Finally, we briefly discuss related work and further directions for investigation in § 10. With the exception of § 9 the only background knowledge required for this paper is basic category theory.

2. Basic isotropy

We present the basic theory of (first order) isotropy for small categories.

2.1. Definition of the isotropy group. Fix a small category $\mathbb{C}$. For any object C of $\mathbb{C}$, we may consider the slice category $\mathbb{C}/C$ together with its projection functor $\pi_C : \mathbb{C}/C \to \mathbb{C}$, which sends an object $X \to C$ to its domain X. We define

$$\mathsf{Z}(C) =_{\mathrm{def}} \mathsf{Aut}(\pi_C : \mathbb{C}/C \to \mathbb{C}) \ .$$

Thus $\mathsf{Z}(C)$ is the group of natural automorphisms of the projection functor. An element τ of $\mathsf{Z}(C)$ is a natural transformation from π_C to π_C all of whose components are isomorphisms. This means that for any object $f : X \to C$ of the slice $\mathbb{C}/C$ the component of τ at f, written τ_f, is an automorphism of X. Naturality of τ amounts to the requirement that for any commutative triangle

$$\begin{tikzcd} Y \arrow[rr, "k"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & C & \end{tikzcd}$$

the square

$$\begin{tikzcd} Y \arrow[r, "\tau_g"] \arrow[d, "k"'] & Y \arrow[d, "k"] \\ X \arrow[r, "\tau_f"'] & X \end{tikzcd} \tag{3}$$

commutes. We write τ_C for τ_{1_C}. In particular, the diagram

$$\begin{tikzcd} X \arrow[r, "\tau_f"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ C \arrow[r, "\tau_C"'] & C \end{tikzcd} \tag{4}$$

commutes. Thus, we may regard the component τ_f as a lift of the automorphism τ_C along f. Diagram (3) expresses that these lifts are coherent. The group $\mathsf{Z}(C)$ is called the *isotropy group* of $\mathbb{C}$ at C. We refer to an element $\tau \in \mathsf{Z}(C)$ as *an element of isotropy at* C.

Whiskering makes Z into a contravariant functor on $\mathbb{C}$. Indeed, consider the functor

$$\begin{tikzcd} \mathbb{C}/D \arrow[rr, "\mathbb{C}/p"] \arrow[dr, "\pi_D"'] & & \mathbb{C}/C \arrow[dl, "\pi_C"] \\ & \mathbb{C} & \end{tikzcd} \qquad (\mathbb{C}/p)\left(X \xrightarrow{f} D \right) = \left(X \xrightarrow{f} D \xrightarrow{p} C \right)$$

associated with a morphism $p : D \to C$ of $\mathbb{C}$. We may whisker an element $\tau \in \mathsf{Z}(C)$ with $\mathbb{C}/p$ providing a natural automorphism $p^*(\tau)$ of π_D, also sometimes written $\mathsf{Z}(p)(\tau)$. The component of $p^*(\tau)$ at an object $f : X \to D$ is $\tau_{pf} : X \to X$.

2.2. Definition. [Isotropy group of a category] *The* isotropy group *of a small category* $\mathbb{C}$ *is the presheaf of groups*

$$\mathsf{Z}_{\mathbb{C}} = \mathsf{Z} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Grp}\ ; \qquad \mathsf{Z}(C) = \mathsf{Aut}\,(\pi_C : \mathbb{C}/C \to \mathbb{C})\,.$$

2.3. Elementary theory and examples. We have seen that an element $\tau \in \mathsf{Z}(C)$ of isotropy specifies a coherent system of lifts of the automorphism $\tau_C : C \to C$ along morphisms $f : X \to C$ (4). Forgetting all these lifts and remembering only the automorphism τ_C gives a group homomorphism

$$\nu_C : \mathsf{Z}(C) \to \mathsf{Aut}(C) \ ; \ \ \nu_C(\tau) = \tau_C \ . \tag{5}$$

This comparison homomorphism partly makes precise the idea conveyed in the introduction that the isotropy group remedies the failure of $\mathsf{Aut}(-)$ to be functorial: we have replaced the automorphism group with the functorial Z.

Generally, ν_C is neither injective nor surjective. Three simple examples help illustrate this.

1. Let $\mathbb{C}$ denote the category with two objects C, D, with $\mathbb{C}(C,C) = \{1, \alpha\}$, so that α is an automorphism of order 2, $\mathbb{C}(D,D) = 1$ and $\mathbb{C}(D,C) = \{f, \alpha f\}$. This means that there is no lift
$$\begin{tikzcd} D \arrow[r, dashed, "?"] \arrow[d, "f"'] & D \arrow[d, "f"] \\ C \arrow[r, "\alpha"'] & C \end{tikzcd}$$
of α along f. Hence, α is not in the image of ν_C.

2. Let $\mathbb{C}$ denote the category with again two objects C, D, but this time $\mathbb{C}(C,C) = 1$, $\mathbb{C}(D,D) = \{1, \beta\}$ and $\mathbb{C}(D,C) = \{f\}$. This means that the following diagram commutes:
$$\begin{tikzcd} D \arrow[r, "\beta"] \arrow[d, "f"'] & D \arrow[d, "f"] \\ C \arrow[r, "1_C"'] & C \end{tikzcd}$$
Hence, there is a non-trivial lift of the identity along f, so that $\nu_C : \mathsf{Z}(C) \to \mathsf{Aut}(C) = 1$ is not injective: the group $\mathsf{Z}(C)$ has two elements, namely the trivial one (in which the identity lifts to the identity) and a non-trivial one, where the identity lifts to f. We thus see that elements in the kernel of ν_C give rise to "deck transformations" of $D \to C$.

3. It is possible for ν_C to be neither injective nor surjective: take once more $\mathbb{C}$ with objects C, D, let $\mathbb{C}(C,C) = \{1, \alpha\}$, $\mathbb{C}(D,D) = \{1, \beta\}$ and $\mathbb{C}(D,C) = \{f, \alpha f\}$. This combines the previous two examples in that α does not lift along f, while the identity lifts to β.

2.4. Definition. *Let $\mathbb{C}$ be a category with isotropy group Z. We shall say that the isotropy of $\mathbb{C}$ is:*

(i) free *if the maps ν_C are injective;*

(ii) effective *if the maps* ν_C *are surjective;*

(iii) trivial *if* $\mathsf{Z} = 1$ *, in which case we also say that* $\mathbb{C}$ *is anisotropic.*

The typical example of a category with free and effective isotropy is a groupoid: in that case, each possible lifting problem has a unique solution, presented by conjugation. A rigid category, meaning that it has no non-trivial automorphisms, is another example. Rigid categories are precisely those which have effective and trivial isotropy.

2.5. Isotropy quotient. We wish to form a quotient of a category $\mathbb{C}$ in which none of the isotropy of $\mathbb{C}$ survives. We start by factoring the group homomorphisms $\mathsf{Z}(C) \to \mathsf{Aut}(C)$:

$$\begin{tikzcd} \mathsf{Z}(C) \arrow[r, two heads, "q_C"] \arrow[rr, bend right, "\nu_C"'] & \mathsf{I}(C) \arrow[r, "\subseteq"] & \mathsf{Aut}(C) \end{tikzcd} .$$

Thus the group $\mathsf{I}(C)$ is the image of ν_C in $\mathsf{Aut}(C)$.

2.6. Definition. [Isotropy Maps, Isotropy Congruence] *An automorphism of* C *is said to be* isotropy *if it is in the image of* ν_C*. The* isotropy congruence *on* $\mathbb{C}$ *is the smallest congruence* $\sim$ *for which* $\alpha \sim 1_C$ *for each isotropy automorphism* $\alpha \in \mathsf{Aut}(C)$ *.*

Thus, the isotropy congruence is the smallest equivalence relation on the hom-sets of $\mathbb{C}$ containing $\alpha \sim 1$, which is closed under pre- and postcomposition. It has the following simple description.

2.7. Lemma. *For any two maps* $f, g : C \to D$ *, we have*

$$f \sim g \iff \exists \alpha \in \mathsf{I}(C)\ f\alpha = g\ .$$

Proof. If f and g satisfy the statement (right hand side), then of course $f \sim g$. For the other implication, observe first that the statement holds for the generating cases $\alpha \sim 1$, and that the statement is preserved by postcomposition. It thus remains to show that it is preserved by precomposition. For that it suffices to show that the statement holds for $\beta f \sim f$, for β isotropy. However, since $\beta \in Im(\nu_D)$ we know that there is an element of isotropy $\tau \in \mathsf{Z}(D)$ for which $\tau_D = \beta$. Thus, τ_f is an isotropy automorphism of C with $\beta f = f\tau_f$. ∎

We shall write

$$\psi : \mathbb{C} \to \mathbb{C}/\mathsf{I}$$

for the quotient map. We call it the *isotropy quotient* of $\mathbb{C}$. It is the identity on objects, and it sends an arrow to its $\sim$ equivalence class.

2.8. Lemma. *The quotient map* $\psi : \mathbb{C} \to \mathbb{C}/\mathsf{I}$ *is conservative* (reflects isomorphisms).

Proof. Suppose $\psi(f)$ is an isomorphism with inverse $\psi(g)$. By Lemma 2.7 we have isotropy isomorphisms α, β such that $gf = \alpha$ and $fg = \beta$. It follows that $\alpha^{-1}g$ is a two-sided inverse for f because $gf = \alpha$ gives $\alpha^{-1}gf = \alpha^{-1}\alpha = 1$, and so on. ∎

As a consequence we find that for each object C of $\mathbb{C}$, the following square is a pullback.

$$\begin{tikzcd} \mathsf{Aut}_{\mathbb{C}}(C) \arrow[r, two heads, "\overline{\psi}"] \arrow[d] & \mathsf{Aut}_{\mathbb{C}/\mathsf{I}}(C) \arrow[d] \\ \mathbb{C}(C,C) \arrow[r, two heads, "\psi"] & \mathbb{C}/\mathsf{I}(C,C) \end{tikzcd}$$

2.9. Lemma. *We have a short exact sequence*

$$1 \longrightarrow \mathsf{I}(C) \longrightarrow \mathsf{Aut}_{\mathbb{C}}(C) \overset{\overline{\psi}}{\twoheadrightarrow} \mathsf{Aut}_{\mathbb{C}/\mathsf{I}}(C) \longrightarrow 1 \;\; .$$

Therefore, the isotropy maps are the only automorphisms of $\mathbb{C}$ *that get annihilated by the quotient map.*

Proof. $\mathsf{I}(C)$ is a normal subgroup of $\mathsf{Aut}(C)$: suppose that $\alpha = \tau_C$ for $\tau \in \mathsf{Z}(C)$, and let $\beta \in \mathsf{Aut}(C)$. We deduce that $\beta^{-1}\alpha\beta \in \mathsf{I}(C)$ because $\beta^{-1}\alpha\beta = q_C(\beta^*\tau)$, which is so because reindexing an element of isotropy along β is conjugation. Thus, the map $\overline{\psi}$ is in fact the cokernel of ν_C. ∎

The following facts, which we state for future reference, follow directly from the definitions.

2.10. Proposition. *Let* $\mathbb{C}$ *be a category with isotropy quotient* $\psi : \mathbb{C} \to \mathbb{C}/\mathsf{I}$.

(i) *The isotropy of* $\mathbb{C}$ *is effective if and only if* $\mathbb{C}/\mathsf{I}$ *is rigid;*

(ii) $\mathbb{C}$ *is anisotropic if and only if* ψ *is an equivalence* (equivalently, an isomorphism).

2.11. Universal actions. We conclude this section on basic isotropy of small categories by considering the universal action on presheaves by the isotropy group. Let $\mathbb{C}$ be a small category, and Z its isotropy group. For objects C, D of $\mathbb{C}$, the group homomorphism $\nu_C : \mathsf{Z}(C) \to \mathsf{Aut}(C)$ induces an action of $\mathsf{Z}(C)$ on $\mathbb{C}(C,D)$:

$$\mathbb{C}(C,D) \times \mathsf{Z}(C) \to \mathbb{C}(C,D) \; ; \qquad (f,\tau) \mapsto f\nu_C(\tau) = f\tau_C \; .$$

Varying the object C, we find that Z acts on the representable presheaf $\mathbb{C}(-,D)$:

$$\mathbb{C}(-,D) \times \mathsf{Z} \to \mathbb{C}(-,D) \; .$$

When we also vary D we find that for a morphism $f : D \to D'$ we obtain a commutative square

$$\begin{tikzcd} \mathbb{C}(-,D) \times \mathsf{Z} \arrow[r] \arrow[d, "f \times \mathsf{Z}"'] & \mathbb{C}(-,D) \arrow[d, "f"] \\ \mathbb{C}(-,D') \times \mathsf{Z} \arrow[r] & \mathbb{C}(-,D'). \end{tikzcd}$$

This canonical action of Z on representable presheaves is an instance of an action

$$\theta_X : X \times \mathsf{Z} \to X$$

on arbitrary ones X via the formula

$$X(C) \times \mathsf{Z}(C) \to X(C)\,; \qquad (x, \tau) \mapsto X(\tau_C)(x)\,. \tag{6}$$

Every morphism $f : X \to Y$ of presheaves is equivariant with respect to these canonical actions in the sense that we have a commutative square

$$\begin{tikzcd} X \times \mathsf{Z} \arrow[r, "\theta_X"] \arrow[d, "f \times \mathsf{Z}"'] & X \arrow[d, "f"] \\ Y \times \mathsf{Z} \arrow[r, "\theta_Y"'] & Y\,. \end{tikzcd}$$

Also, θ_{Z}, the canonical action of the isotropy group on itself, is the conjugation action. Moreover, there is a precise sense in which the isotropy group is universal with these properties. We refer the reader to [Funk et al., 2012] for more details.

3. Functoriality

So far we have considered the isotropy group and the isotropy quotient of a fixed small category $\mathbb{C}$. We now investigate what happens when we vary $\mathbb{C}$.

3.1. Comparison spans. Consider a functor $F : \mathbb{D} \to \mathbb{C}$. We may pull back the isotropy group $\mathsf{Z}_{\mathbb{C}}$ of $\mathbb{C}$ along F to $\mathbb{D}$:

$$F^*\mathsf{Z}_{\mathbb{C}} : \mathbb{D}^{\mathrm{op}} \to \mathsf{Grp}\,; \qquad F^*\mathsf{Z}_{\mathbb{C}}(D) = \mathsf{Z}_{\mathbb{C}}(FD)\,.$$

The following diagram shows how to compare this group with $\mathsf{Z}_{\mathbb{D}}$.

$$\begin{tikzcd} \mathbb{D}/D \arrow[r, "F/D"] \arrow[d, "\pi_D"'] & \mathbb{C}/FD \arrow[d, "\pi_{FD}"] \\ \mathbb{D} \arrow[r, "F"'] & \mathbb{C} \end{tikzcd} \tag{7}$$

Generally, there is no reason why an automorphism of π_D would factor through one of π_{FD} or vice versa. However, nothing stops us from considering compatible pairs

$$(\sigma, \tau) \in \mathsf{Z}_{\mathbb{D}}(D) \times \mathsf{Z}_{\mathbb{C}}(FD)$$

in the sense that $F\sigma = \tau(F/D)$. This amounts to

$$\begin{tikzcd} \mathbb{D}/D \arrow[r, "F/D"] \arrow[d, bend right, "\pi_D"'] \arrow[d, bend left, "\pi_D"] \arrow[d, phantom, "\overset{\sigma}{\Rightarrow}"] & \mathbb{C}/FD \arrow[d, bend right, "\pi_{FD}"'] \arrow[d, bend left, "\pi_{FD}"] \arrow[d, phantom, "\overset{\tau}{\Rightarrow}"] \\ \mathbb{D} \arrow[r, "F"'] & \mathbb{C} \end{tikzcd} \tag{8}$$

being commutative. Let us write $\mathsf{Z}_F(D)$ for the group of compatible pairs (σ, τ) as above. We have a pullback diagram

$$\begin{tikzcd}
\mathsf{Z}_F(D) \arrow[r, "m_D"] \arrow[d, "\rho_D"'] & \mathsf{Z}_{\mathbb{C}}(FD) \arrow[d] \\
\mathsf{Z}_{\mathbb{D}}(D) \arrow[r] & \mathsf{Aut}(\mathbb{D}/D \to \mathbb{D} \to \mathbb{C})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Z}_F \arrow[r, "m"] \arrow[d, "\rho"'] & F^*\mathsf{Z}_{\mathbb{C}} \arrow[d] \\
\mathsf{Z}_{\mathbb{D}} \arrow[r] & \mathsf{Z}^F
\end{tikzcd} \tag{9}$$

where the bottom and right-hand maps are given by whiskering with F and with F/D, respectively. Everything is natural in D, so that we obtain a pullback diagram of presheaves of groups on $\mathbb{D}$ above (right). We refer to the (jointly monic) span of groups (m, ρ) in (9) as the *isotropy span of* F.

We sometimes need the following diagram associated with the isotropy span of F and each object D of $\mathbb{D}$.

$$\begin{tikzcd}
& \mathsf{Z}_F(D) \arrow[dl, "\rho_D"'] \arrow[dr, "m_D"] & \\
\mathsf{Z}_{\mathbb{D}}(D) \arrow[d, "\nu_D"'] & & \mathsf{Z}_{\mathbb{C}}(FD) \arrow[d, "\nu_{FD}"] \\
\mathsf{Aut}(D) \arrow[rr, "F"'] & & \mathsf{Aut}(FD)\,.
\end{tikzcd} \tag{10}$$

The following proposition is not needed for the rest of the developments in this paper, but we include it as it further elucidates the nature of the isotropy spans. Its statement refers to the so-called *Peiffer Identity* which is part of the definition of a crossed module: a group homomorphism $d : M \to G$ from a G-module M to G is said to satisfy the Peiffer identity when $h \cdot d(k) = k^{-1}hk$.

3.2. Proposition. *The isotropy span of a functor $F : \mathbb{D} \to \mathbb{C}$ respects universal actions* (6) *in the sense that for every presheaf X on $\mathbb{C}$ the following square* (left) *commutes.*

$$\begin{tikzcd}
F^*X \times \mathsf{Z}_F \arrow[r, "F^*X\times m"] \arrow[d, "F^*X\times\rho"'] & F^*(X \times \mathsf{Z}_{\mathbb{C}}) \arrow[d, "F^*\theta_X"] \\
F^*X \times \mathsf{Z}_{\mathbb{D}} \arrow[r, "\theta_{F^*X}"] & F^*X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Z}_F \times \mathsf{Z}_F \arrow[d, "\mathsf{Z}_F\times\rho"'] \arrow[dr, "\mathrm{conj}"] & \\
\mathsf{Z}_F \times \mathsf{Z}_{\mathbb{D}} \arrow[r, "\theta_{\mathsf{Z}_F}"'] & \mathsf{Z}_F
\end{tikzcd}$$

Moreover, ρ satisfies the Peiffer identity in the sense that the diagram above (right) *commutes. Hence, ρ is a crossed $\mathbb{C}$-module, and (F, m) is a morphism of crossed modules* ([Funk et al., 2012]).

Proof. The instance $X = \mathsf{Z}_{\mathbb{C}}$ of $F \cdot \rho = \theta \cdot m$

$$\begin{tikzcd}
F^*\mathsf{Z}_{\mathbb{C}} \times \mathsf{Z}_F \arrow[r, "F^*\mathsf{Z}_{\mathbb{C}}\times m"] \arrow[d, "F^*\mathsf{Z}_{\mathbb{C}}\times\rho"'] & F^*(\mathsf{Z}_{\mathbb{C}} \times \mathsf{Z}_{\mathbb{C}}) \arrow[d, "\mathrm{conj}"] \\
F^*\mathsf{Z}_{\mathbb{C}} \times \mathsf{Z}_{\mathbb{D}} \arrow[r, "\theta_{F^*\mathsf{Z}_{\mathbb{C}}}"] & F^*\mathsf{Z}_{\mathbb{C}}
\end{tikzcd}$$

shows that for any $(\sigma,\tau) \in \mathsf{Z}_F$ and $\gamma \in F^*\mathsf{Z}_{\mathbb{C}}$ we have

$$\gamma \cdot \sigma = \theta(\gamma, \rho(\sigma,\tau)) = \tau^{-1}\gamma\tau \ .$$

Then for any $(\alpha,\beta) \in \mathsf{Z}_F$ we have

$$(\alpha,\gamma)\cdot\rho(\sigma,\tau) = (\alpha,\gamma)\cdot\sigma = (\alpha\cdot\sigma,\gamma\cdot\sigma) = (\sigma^{-1}\alpha\sigma,\tau^{-1}\gamma\tau) = (\sigma,\tau)^{-1}(\alpha,\gamma)(\sigma,\tau) \ .$$

This establishes the Peiffer identity. ∎

It is interesting to analyze the behavior of isotropy spans with respect to composition of functors, but we omit this aspect.

3.3. Preservation. Generally, the isotropy span associated with a functor is the best we can do in terms of comparing the two isotropy groups. However, there are important special cases where one, or both, of the legs of the span is a surjection, or even an isomorphism. Since the isotropy span of a functor can be thought of as a relation between the isotropy groups, we adopt some terminology from the calculus of relations.

3.4. Definition. *Consider a functor $F : \mathbb{D} \to \mathbb{C}$ with associated isotropy span* (9). *Then the associated span is called*

(i) total *when ρ is an epimorphism;*

(ii) functional *when ρ is an isomorphism;*

(iii) surjective *when m is an epimorphism;*

(iv) co-functional *when m is an isomorphism;*

When both legs of the span are isomorphisms we say that F is stable*.*

We are mostly interested in situations where F induces a well-behaved functor between the quotient categories. Since the isotropy quotient of a category is defined purely in terms of the image $\mathsf{I}(C)$ of ν_C (5) it makes sense to consider the following properties of functors.

3.5. Definition. *A functor $F : \mathbb{D} \to \mathbb{C}$ is said to:*

(i) preserve isotropy *if for any $\alpha \in \mathsf{Aut}(D)$, if α is isotropy then $F(\alpha)$ is isotropy;*

(ii) reflect isotropy *if for any $\beta \in \mathsf{Aut}(FD)$, if $\beta \in \mathsf{I}(FD)$ then there exists an $\alpha \in \mathsf{I}(D)$ with $F(\alpha) = \beta$.*

We may characterize the first condition as follows:

3.6. PROPOSITION. *For $F : \mathbb{D} \to \mathbb{C}$, the following are equivalent:*

- *F preserves isotropy;*
- *F descends to the isotropy quotient in the sense that there is a (necessarily unique) functor $F/\mathsf{I} : \mathbb{D}/\mathsf{I} \to \mathbb{C}/\mathsf{I}$ making*

$$\begin{tikzcd} \mathbb{D} \arrow[r, "F"] \arrow[d, two heads, "\psi_{\mathbb{D}}"'] & \mathbb{C} \arrow[d, two heads, "\psi_{\mathbb{C}}"] \\ \mathbb{D}/\mathsf{I} \arrow[r, dashed, "F/\mathsf{I}"'] & \mathbb{C}/\mathsf{I} \end{tikzcd}$$

  commute.
- *For each D , the map $F : \mathsf{Aut}(D) \to \mathsf{Aut}(FD)$ restricts to a map $\mathsf{I}(D) \to \mathsf{I}(FD)$.*

We now wish to relate properties of the isotropy span of F to its preservation properties. Our first observation is the following.

3.7. LEMMA. *Let $F : \mathbb{D} \to \mathbb{C}$ be a functor with isotropy span (9). Then:*

(i) *F preserves isotropy if the span of F is total;*

(ii) *F reflects isotropy if the span of F is surjective.*

PROOF. (i) Consider

$$\begin{tikzcd} \mathsf{Z}_{\mathbb{D}}(D) \arrow[d, two heads, "q_D"'] & \mathsf{Z}_F(D) \arrow[l, two heads, "\rho_D"'] \arrow[r, "m_D"] & \mathsf{Z}_{\mathbb{C}}(FD) \arrow[d, two heads, "q_{FD}"] \\ \mathsf{I}(D) \arrow[rr, dashed, "?"] \arrow[d, tail, "i_D"'] & & \mathsf{I}(FD) \arrow[d, tail, "i_{FD}"] \\ \mathsf{Aut}(D) \arrow[rr, "F"'] & & \mathsf{Aut}(FD) \end{tikzcd} \tag{11}$$

Commutativity of the outer diagram entails that $i_{FD}\, q_{FD}\, m_D$ coequalizes the kernel pair of $q_D\, \rho_D$, and since i_{FD} is injective, so does $q_{FD}\, m_D$. Hence, it factors through the quotient map $q_D\, \rho_D$.

(ii) If m_D is surjective, then any element of $\mathsf{I}(FD)$ lifts to $\mathsf{Z}_F(D)$, and then projects to $\mathsf{I}(D)$. ∎

3.8. Example. This example illustrates that a functor F may preserve isotropy but the span of F may not be total. Let $F : \mathbb{D} \to \mathbb{C}$ be the non-full inclusion

$$
\begin{tikzcd} Y \arrow[loop left, "\alpha"] \arrow[r, "f"] & X \arrow[r, "h"] & D \end{tikzcd}
\qquad\qquad
\begin{tikzcd} Y \arrow[loop left, "\alpha"] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & X \arrow[r, "h"] & D \end{tikzcd}
$$

where we have $f\alpha = f$, so that there is non-trivial isotropy at X and at D in $\mathbb{D}$. In $\mathbb{C}$ we impose $hf = hg$, but stipulate that $g\alpha \neq g$. Thus, at D the identity can only lift along $hf = hg$ to the identity, and not to α . This makes the isotropy group of $\mathbb{C}$ at D trivial. Thus, F preserves isotropy but not every element of isotropy of $\mathbb{D}$ can be extended to one in $\mathbb{C}$.

On the other hand, a functor may be stable (induce an isomorphism of isotropy groups), but not give an isomorphism between automorphism groups.

3.9. Example. Let $\mathbb{D}$ be the category with two objects X, D , and with hom-sets

$$\mathbb{D}(D, D) = \{1, \alpha\}\ ;\ \ \mathbb{D}(X, X) = \{1, \beta\}\ ,$$

so that both automorphism groups are isomorphic to $\mathbb{Z}_2$, and $\mathbb{D}(X, D) = \{f, \alpha f\}$, where $\alpha f = f\beta$. Let $F : \mathbb{D} \to \mathbb{C}$ be the quotient by the least congruence identifying α with 1 .

$$
\begin{tikzcd}
X \arrow[loop above, "\beta"] \arrow[d, "f"'] & FX \arrow[loop above, "F(\beta)"] \arrow[d, "Ff"] \\
D \arrow[loop below, "\alpha"] & FD \arrow[loop below, "F\alpha=1"] \\
\mathbb{D} \arrow[r, "F"'] & \mathbb{C}
\end{tikzcd}
$$

Then F is stable, but F identifies automorphisms of D .

Since it is important to understand under what conditions properties of F are inherited by its quotient, we record the following result.

3.10. Lemma. *Let $F : \mathbb{D} \to \mathbb{C}$ preserve isotropy. Then if F is full, so is F/I . Moreover, if F is faithful and its span is surjective, then F/I is faithful.*

Proof. The first part is immediate. For the second part, suppose $F/\mathsf{I}[f] = F/\mathsf{I}[g]$, i.e., $Ff \sim Fg$. Then we have $Ff\,\beta = Fg$ for some $\beta \in \mathsf{I}(FD)$. By surjectivity there is $\alpha \in \mathsf{I}(D)$ such that $F\alpha = \beta$. This gives $F(f\alpha) = Fg$. Since F is faithful it follows that $f\alpha = g$, whence $[f] = [g]$. ∎

3.11. Sieves. For our purposes, the case of a full inclusion is particularly important. In this section we collect a few elementary but useful results concerning the isotropy behavior of such functors.

3.12. Proposition. *Let $F : \mathbb{D} \to \mathbb{C}$ be a full and faithful functor. Then the span of F is co-functional.*

Proof. We may define an inverse to m as follows: an automorphism τ of π_{FD} induces by whiskering with F/D an automorphism of the composite

$$\mathbb{D}/D \xrightarrow{F/D} \mathbb{C}/FD \xrightarrow{\pi_{FD}} \mathbb{C} \ .$$

Because F is full and faithful this automorphism factors uniquely as

$$\begin{tikzcd} \mathbb{D}/D \arrow[r, bend left, "\pi_D"] \arrow[r, bend right, "\pi_D"'] \arrow[r, phantom, "\Downarrow F^*\tau"] & \mathbb{D} \arrow[r, "F"] & \mathbb{C} \end{tikzcd} \ .$$

Thus, an element $\tau \in \mathsf{Z}_{\mathbb{C}}(FD)$ induces a compatible element $F^*\tau \in \mathsf{Z}_D(D)$. It is easily verified that this operation is inverse to m_D. ∎

For full and faithful F we identify Z_F and $F^*\mathsf{Z}_C$, and refer to $\rho : F^*\mathsf{Z}_C \to \mathsf{Z}_{\mathbb{D}}$ as the *canonical comparison map*. Generally, ρ is neither injective nor surjective.

In general, we cannot expect a full inclusion to be stable. One situation where we indeed have stability is when $F : \mathbb{D} \to \mathbb{C}$ is a *sieve*, by which we mean F is a full inclusion with the property that $\mathbb{C}(C, FD) = \emptyset$ whenever C is not in the image of F.

3.13. Proposition. *A sieve inclusion is stable.*

Proof. Let $F : \mathbb{C} \to \mathbb{D}$ be a sieve inclusion. Then for any object D of $\mathbb{D}$, the functor F/D is an isomorphism. Therefore, whiskering with it is a bijective operation. ∎

3.14. Lemma. *If $F : \mathbb{C} \to \mathbb{D}$ is a sieve inclusion, then F/I exists and is also a sieve inclusion.*

Proof. It is immediate from Lemma 3.10 that F/I is full and faithful. The sieve property is not changed by factoring out congruences. ∎

In preparation for the technical work in § 8, we briefly consider transporting isotropy along right adjoints. This has significant practical value for the computation of isotropy groups. The pullback functor F^* associated with a functor $F : \mathbb{D} \to \mathbb{C}$ of small categories has a right adjoint

$$\begin{tikzcd} \mathsf{Set}^{\mathbb{D}^{\mathrm{op}}} \arrow[r, bend right, "F_*"'] \arrow[r, phantom, "\perp"] & \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}} \arrow[l, bend right, "F^*"'] \end{tikzcd}$$

that we denote F_* (right Kan extension). Explicitly, given a presheaf B on $\mathbb{D}$ we have

$$F_*B\,(C) = \mathsf{Ran}_F B\,(C) = \lim_{f:FD \to C} B(D) \ ,$$

where the limit is taken over the comma category $F \downarrow C$. This limit coincides with the set of natural transformations

$$\mathbb{C}(F(\_), C) \to B .$$

The unit η of the adjunction $F^* \dashv F_*$ at a presheaf A on $\mathbb{C}$ has the form

$$\eta_A : A \to F_*F^*(A) ; \qquad \eta_{A,C} : A(C) \to F_*F^*(A)(C) ; \qquad y \mapsto \langle f^*(x) \rangle_{f:FD \to C} .$$

Applying this to the isotropy group $\mathsf{Z}_{\mathbb{C}}$, and because right adjoints preserve group objects, we obtain a group $F_*F^*\mathsf{Z}_{\mathbb{C}}$ and a homomorphism

$$\eta : \mathsf{Z}_{\mathbb{C}} \to F_*F^*(\mathsf{Z}_{\mathbb{C}}) ; \qquad \eta_C(\tau) = \langle f^*\tau \rangle_{f:FD \to C} .$$

3.15. Lemma. *For a sieve inclusion $F : \mathbb{D} \to \mathbb{C}$, the unit of $F^* \dashv F_*$ at $\mathsf{Z}_{\mathbb{C}}$ may be identified with the group homomorphism whose component at an object C of $\mathbb{C}$ is*

$$\eta_C : \mathsf{Z}_{\mathbb{C}}(C) \to \lim_{f:FD \to C} \mathsf{Z}_{\mathbb{D}}(D) .$$

Proof. If F is a sieve inclusion, then $F^*\mathsf{Z}_{\mathbb{C}} \cong \mathsf{Z}_{\mathbb{D}}$. ∎

We think of the unit η_C as restricting an element of isotropy at C to a compatible family of elements of isotropy at the objects of $\mathbb{D}$. It is natural to wonder from what additional data one may recover $\mathsf{Z}_{\mathbb{C}}$ from $F_*\mathsf{Z}_{\mathbb{D}}$. This question is answered in § 7.1.

4. Key example

We present a key example of a category with higher order isotropy. We do this by hand partly because we believe this is insightful, and partly to convince the reader that once we seek to develop more complex examples, the combinatorics become so elaborate that a more conceptual approach is justified.

4.1. The category $\mathbb{X}$. We construct a category $\mathbb{X}$ with the following properties:

- $\mathbb{X}$ has non-trivial isotropy;
- $\mathbb{X}/\mathsf{I}$ has non-trivial isotropy (*second order isotropy*);
- $(\mathbb{X}/\mathsf{I})/\mathsf{I}$ is anisotropic (has trivial isotropy).

The category $\mathbb{X}$ is constructed as a non-full subcategory of Set on the objects

$$A = \{0, 1\}, \quad B = \{a, b, c, d\}, \quad C = \{p, q, r, s\} .$$

Let $\mathbb{X}(A, A) = \{1, \alpha\}$ where α is the twist map. We set $\mathbb{X}(B, B) = \{1, \beta\}$ where β is the permutation $(ac)(bd)$. Finally, $\mathbb{X}(C, C) = \{1, \gamma, \delta, \gamma\delta\}$ where

$$\gamma = (pr)(qs), \qquad \delta = (ps)(qr) .$$

It follows that $\gamma\delta = \delta\delta = (pq)(rs)$, and that $\mathbb{X}(C,C) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

These are all the isomorphisms in $\mathbb{X}$. We also add non-isomorphisms

$$\begin{array}{ll} f : B \to A & f(a) = f(b) = 0, f(c) = f(d) = 1 \\ g : C \to B & g(p) = a, g(q) = b, g(r) = c, g(s) = d \\ h : C \to B & h(p) = b, h(q) = a, h(r) = c, h(s) = d \end{array}$$

This gives the following picture:

$$\begin{tikzcd} C \arrow[r, shift left, bend left, "g"] \arrow[r, shift right, bend right, "h"'] \arrow[d, shift right, bend right, "\gamma"'] \arrow[d, shift left, bend left, "\delta"] & B \arrow[r, "f"] \arrow[d, "\beta"] & A \arrow[d, "\alpha"] \\ C \arrow[r, shift left, bend left, "g"] \arrow[r, shift right, bend right, "h"'] & B \arrow[r, "f"'] & A \end{tikzcd}$$

where the vertical maps are the (generating) automorphisms, while the horizontal maps are the (generating) non-isomorphisms. Of course, adding these maps results in new composites, which we exhaustively describe below. Observe that:

- $fg = fh$
- $\alpha f = f\beta$, so that β is a lift of α along f (and is unique with that property)
- $\beta g = g\gamma$, so that γ is a lift of β along g (and is unique with that property)
- $\beta h = h\delta$, so that δ is a lift of β along h (and is unique with that property)
- $\beta g \neq g\delta$ and $\beta h \neq h\gamma$.

The last item is key as it implies that α cannot be isotropy: no possible choice of lift of α along $fg = fh$ is coherent with the (forced) lifts of β. Note that $\mathbb{X}(B,A) = \{f, \alpha f\}$. Indeed, the maps f and αf are clearly different, and there are no other maps $B \to A$ because every such map is a composite of a map in $\mathbb{X}(B,B)$ with f and a map in $\mathbb{X}(A,A)$, and the only other non-trivial possibility is $f\beta = \alpha f$. We also verify by hand that

$$\mathbb{X}(C,B) = \{g, h, g\gamma, g\delta, h\gamma, h\delta, g\gamma\delta, h\gamma\delta\}$$

and that $\mathbb{X}(C,A) = \{fg, \alpha fg\}$. It follows that these sets are closed under pre- and postcomposition by automorphisms. This completes the description of $\mathbb{X}$.

The maps g and h have the unique lifting property, but the composite $fg = fh$ does not because α lifts to both γ and δ; equivalently, the identity on A lifts to $\gamma\delta$.

4.2. Properties of $\mathbb{X}$. We claim that $\mathbb{X}$ has the desired properties. We begin by computing the isotropy of $\mathbb{X}$.

4.3. Lemma. *For the category* $\mathbb{X}$ *we have*

- $\mathsf{I}(C) = \{1, \gamma, \delta, \gamma\delta\} = \mathbb{X}(C, C)$
- $\mathsf{I}(B) = \{1, \beta\} = \mathbb{X}(B, B)$
- $\mathsf{I}(A) = 1$.

Proof. All automorphisms of C are isotropy because $\mathbb{X}(C, C)$ is a group and because there are no other maps with codomain C . As for β we have already indicated that it lifts uniquely along g and h to give γ and δ, respectively. But then β lifts uniquely along any map $C \to B$ because we can first lift along g (or h depending on the map), and then conjugate with the remaining isomorphism.

Finally, we must show that α is not isotropy. But that is so because the only way to lift α along f is to β , and β has two distinct lifts along g and h . ∎

We may now describe the quotient category $\mathbb{X}/\mathsf{I}$. Its objects are of course the same as those of $\mathbb{X}$. We get $\mathbb{X}/\mathsf{I}(C, C) = 1$, and $\mathbb{X}/\mathsf{I}(B, B) = 1$ because all endomorphisms of B and of C are isotropy. Consequently, we have $\mathbb{X}/\mathsf{I}(C, B) = \{[g], [h]\}$. These two maps are not identified because we don't have $g = h\zeta$ for any ζ . And we have $\mathbb{X}/\mathsf{I}(A, A) = \{1, [\alpha]\}$ because α is not isotropy. We also have $\mathbb{X}/\mathsf{I}(B, A) = \{[f]\}$ because $[\alpha f] = [f\beta] = [f][\beta] = [f]$. Similarly, we have $\mathbb{X}/\mathsf{I}(C, A) = \{[fg]\}$.

4.4. Proposition. *The map* $[\alpha]$ *is a non-trivial isotropy map in* $\mathbb{X}/\mathsf{I}$.

Proof. The map $[\alpha]$ is not the identity because α is not isotropy in $\mathbb{X}$. Along $[f]$ it lifts to the identity, as it does along $[fg]$. ∎

Note how the obstruction to α being isotropy in $\mathbb{X}$, which is the fact that γ and δ are different possible lifts along fg , is removed once we pass to the first isotropy quotient: both γ and δ are identified with the identity. We may think of α being "isotropy up to isotropy." The picture for the isotropy quotient is therefore as follows.

$$\begin{tikzcd}
C \arrow[r, shift left, "{[g]}"] \arrow[r, shift right, "{[h]}"'] \arrow[d, "{[\gamma]=[\delta]=1}"'] & B \arrow[r, "{[f]}"] \arrow[d, "{[\beta]=1}"] & A \arrow[d, "{[\alpha]}"] \\
C \arrow[r, shift left, "{[g]}"] \arrow[r, shift right, "{[h]}"'] & B \arrow[r, "{[f]}"'] & A
\end{tikzcd}$$

The quotient $\mathbb{X}/\mathsf{I}$ has full isotropy as the only isotropy automorphism is $[\alpha]$. Thus, the second quotient $(\mathbb{X}/\mathsf{I})/\mathsf{I}$ is rigid.

4.5. Remark. The category $\mathbb{X}$ is in a sense the smallest possible because its automorphism groups are based on $\mathbb{Z}_2$. We may use any group G as a parameter in the construction; then we get $\mathbb{X}(A, A) \cong \mathbb{X}(B, B) \cong G$ and $\mathbb{X}(C, C) \cong G \times G$. The isotropy behavior is then exactly the same.

5. Isotropy rank

Having seen that it is possible for a category to have non-trivial higher order isotropy we wish to study this phenomenon more systematically. Our main tool is what we call the isotropy rank of a category.

5.1. Isotropy rank of a small category. Consider a small category $\mathbb{C}$ with its isotropy quotient $\psi : \mathbb{C} \to \mathbb{C}/\mathsf{I}$. The quotient $\mathbb{C}/\mathsf{I}$ may itself have non-trivial isotropy.

$$\mathbb{C} \to \mathbb{C}/\mathsf{I} \to (\mathbb{C}/\mathsf{I})/\mathsf{I} \to \cdots$$

To study this process we first define the following sequence of categories and congruences:

- $\mathbb{C}^{\{0\}} = \mathbb{C}$, and $\mathsf{I}^{\{0\}} = \mathsf{I}$ (the isotropy congruence on $\mathbb{C}$)
- $\mathbb{C}^{\{\gamma+1\}} = \mathbb{C}^{\{\gamma\}}/\mathsf{I}^{\{\gamma\}}$, and $\mathsf{I}^{\{\gamma+1\}}$ is the isotropy congruence on $\mathbb{C}^{\{\gamma+1\}}$
- $\mathbb{C}^{\{\kappa\}} = \mathrm{Colim}_{\lambda<\kappa}\mathbb{C}^{\{\lambda\}}$, where this colimit is taken in $\mathfrak{Cat}$. Then $\mathsf{I}^{\{\kappa\}}$ is the isotropy congruence on the colimit, or equivalently it is the union of the congruences $\mathsf{I}^{\{\lambda\}}$ for $\lambda < \kappa$.

We thus obtain a diagram, indexed by the ordinals, of which the first bit looks like

$$\mathbb{C} = \mathbb{C}^{\{0\}} \to \mathbb{C}^{\{1\}} \to \mathbb{C}^{\{2\}} \to \cdots \to \mathbb{C}^{\{\omega\}} \to \mathbb{C}^{\{\omega+1\}} \to \cdots$$

Our notation for the quotient functor is $\psi_{\lambda<\kappa} : \mathbb{C}^{\{\lambda\}} \to \mathbb{C}^{\{\kappa\}}$, but when confusion is unlikely to arise we simply write ψ, or even leave these quotient maps unnamed.

5.2. Notation. We denote the image of a morphism $f : C \to D$ of $\mathbb{C}$ under the quotient $\psi_{0<\kappa} : \mathbb{C} \to \mathbb{C}^{\{\kappa\}}$ by

$$\psi_{0<\kappa}(f) = [f]_\kappa .$$

For an ordinal κ, we may describe $\mathbb{C}^{\{\kappa\}}$ directly as the quotient of $\mathbb{C}$ under the congruence $\sim_\kappa$. This congruence is given by

- $f \sim_0 g \Leftrightarrow f = g$
- $f \sim_{\lambda+1} g \Leftrightarrow [f]_\lambda \sim [g]_\lambda$, where $\sim$ is the isotropy congruence on $\mathbb{C}^{\{\lambda\}}$
- $f \sim_\kappa g \Leftrightarrow f \sim_\lambda g$ for some $\lambda < \kappa$, where κ is a limit ordinal.

We turn to the question of stabilization of this chain.

5.3. Definition. *The* isotropy rank *of a small category $\mathbb{C}$ is the least ordinal κ for which the quotient map $\mathbb{C}^{\{\kappa\}} \to \mathbb{C}^{\{\kappa+1\}}$ is an equivalence. In this case, we write $\mathsf{IR}(\mathbb{C}) = \kappa$.*

Any small category has an isotropy rank. Indeed, this follows from a simple cardinality argument as each quotient of a small category $\mathbb{C}$ is induced by a congruence on $\mathbb{C}$, and there is only a small set of such congruences. However, we argue Cor. 5.6 in terms of the following refinement.

5.4. Definition. *Let $\alpha : C \to C$ be an automorphism of a small category $\mathbb{C}$. The* isotropy rank *of α is defined by:*

$$\mathsf{IR}(\alpha) = \begin{cases} 0 & \textit{if } \alpha = 1 \\ \textit{the least } \kappa \textit{ s.t. } [\alpha]_\kappa = 1 & \textit{if such } \kappa \textit{ exists} \\ -\infty & \textit{if no such } \kappa \textit{ exists.} \end{cases}$$

Def. 5.4 formalizes the idea that every automorphism either becomes isotropy at some (necessarily unique) stage, or never becomes isotropy. The isotropy rank of α is never a limit ordinal: if α becomes the identity under $\mathbb{C} \to \mathbb{C}^{\{\kappa\}}$ where κ is a limit ordinal, then $\alpha \sim_\kappa 1$, whence $\alpha \sim_\lambda 1$ for some $\lambda < \kappa$. Thus α already becomes the identity under $\mathbb{C} \to \mathbb{C}^{\{\lambda\}}$ and hence must have isotropy rank λ or less.

5.5. Proposition. *We have* $\mathsf{IR}(\mathbb{C}) = \bigvee \{\mathsf{IR}(\alpha) \mid \alpha \in \mathsf{Aut}(C)\,, C \in \mathrm{Ob}(\mathbb{C})\}$.

Proof. Denoting the supremum by κ we observe that $\mathbb{C}^{\{\kappa\}} = \mathbb{C}^{\{\kappa+1\}}$. Indeed, suppose $[\alpha]_\kappa$ is isotropy. Then because $0 \leq \mathsf{IR}(\alpha) \leq \kappa$ we have $[\alpha]_\kappa = 1$, showing that $\mathbb{C}^{\{\kappa\}}$ has no isotropy. The ordinal κ is the least one for which $\mathbb{C}^{\{\kappa\}} = \mathbb{C}^{\{\kappa+1\}}$. In fact, if there were $\lambda < \kappa$ with this property, then for all automorphisms α we would have $\mathsf{IR}(\alpha) \leq \lambda$. But then κ would be below λ as well, which is a contradiction. ∎

Of course, a small category has only a set of automorphisms. Therefore, we have the following.

5.6. Corollary. *Every small category has an isotropy rank.*

5.7. Higher isotropy groups. The chain of quotients of $\mathbb{C}$ provides a sequence of groups Z_κ, called the *higher order isotropy groups* of $\mathbb{C}$, defined by

$$\mathbb{C}^{\mathrm{op}} \twoheadrightarrow \mathbb{C}^{\{\kappa\}\mathrm{op}} \xrightarrow{\mathsf{Z}^{\{\kappa\}}} \mathsf{Grp} \;.$$

In other words, Z_κ is the presheaf of groups on $\mathbb{C}$ obtained by pulling back the isotropy group $\mathsf{Z}^{\{\kappa\}}$ of $\mathbb{C}^{\{\kappa\}}$ to $\mathbb{C}$. Here and in what follows if N denotes a concept relating to a category $\mathbb{C}$, then $N^{\{\kappa\}}$ denotes the corresponding concept for $\mathbb{C}^{\{\kappa\}}$.

At an object C we have the canonical maps

$$\nu_C^{\{\kappa\}} : \mathsf{Z}^{\{\kappa\}}(C) \to \mathsf{Aut}^{\{\kappa\}}(C)$$

with image $\mathsf{I}^{\{\kappa\}}(C)$. We may pull these groups back along the quotient $\mathsf{Aut}(C) \to \mathsf{Aut}^{\{\kappa\}}(C)$ as in

$$\begin{tikzcd}
 & \mathsf{Z}^{\{\kappa\}}(C) \arrow[d] \arrow[dd, bend left=60, "\nu_C^{\{\kappa\}}"] \\
\mathsf{I}_\kappa(C) \arrow[r, two heads] \arrow[d, tail] & \mathsf{I}^{\{\kappa\}}(C) \arrow[d, tail] \\
\mathsf{Aut}(C) \arrow[r, two heads] & \mathsf{Aut}^{\{\kappa\}}(C)
\end{tikzcd}$$

to get an increasing chain of subgroups of $\mathsf{Aut}(C)$:

$$\mathsf{I}(C) = \mathsf{I}_0(C) \subseteq \mathsf{I}_1(C) \subseteq \cdots \subseteq \mathsf{I}_\lambda(C) \subseteq \mathsf{I}_{\lambda+1}(C) \subseteq \cdots . \tag{12}$$

Moreover, for each λ we have a short exact sequence

$$1 \longrightarrow \mathsf{I}_\lambda(C) \rightarrowtail \mathsf{Aut}(C) \twoheadrightarrow \mathsf{Aut}^{\{\lambda+1\}}(C) \longrightarrow 1 .$$

After we have developed some more tools in § 8 we will be in a position to give for each ordinal κ an example of a small category with an object C such that each inclusion $\mathsf{I}_\lambda(C) \subseteq \mathsf{I}_{\lambda+1}(C)$ in the chain of subgroups (12) is strict, for $\lambda < \kappa$.

5.8. Definition. *The* preservation rank *of a functor $F : \mathbb{D} \to \mathbb{C}$ is the largest ordinal $\kappa \leq max\{\mathsf{IR}(\mathbb{C}), \mathsf{IR}(\mathbb{D})\}$ such that for all $\lambda < \kappa$, the induced $F^{\{\lambda\}} : \mathbb{C}^{\{\lambda\}} \to \mathbb{D}^{\{\lambda\}}$ preserves isotropy. Moreover, this rank is said to be* maximal *when $\kappa = max\{\mathsf{IR}(\mathbb{C}), \mathsf{IR}(\mathbb{D})\}$.*

This definition is sensible because $F^{\{\lambda\}}$ preserving isotropy is equivalent to $F^{\{\lambda+1\}}$ being well-defined. The statement that F has preservation rank κ means that for each $\lambda < \kappa$ and each object D of $\mathbb{D}$ we have a restriction of F as in

$$\begin{tikzcd}
\mathsf{I}_\lambda(D) \arrow[r, dashed, "\overline{F}"] \arrow[d, tail] & \mathsf{I}_\lambda(FD) \arrow[d, tail] \\
\mathsf{Aut}(D) \arrow[r, "F"'] & \mathsf{Aut}(FD) .
\end{tikzcd}$$

We may also consider stability properties of functors, the most important one being the following.

5.9. Definition. *The* stability rank *of a functor $F : \mathbb{D} \to \mathbb{C}$ is the largest ordinal $\kappa \leq max\{\mathsf{IR}(\mathbb{C}), \mathsf{IR}(\mathbb{D})\}$ such that for all $\lambda < \kappa$, the induced $F^{\{\lambda\}} : \mathbb{C}^{\{\lambda\}} \to \mathbb{D}^{\{\lambda\}}$ is stable. Moreover, this rank is said to be* maximal *when $\kappa = max\{\mathsf{IR}(\mathbb{C}), \mathsf{IR}(\mathbb{D})\}$.*

Our work in § 8 depends on the following result.

5.10. Theorem. *A sieve inclusion has maximal stability rank.*

Proof. Transfinite induction is valid because sieve inclusions are stable (Prop. 3.13), and the quotient of a sieve inclusion is again a sieve inclusion (Lemma 3.14). Moreover, it is easily seen that if every $F^{\{\lambda\}}$ is a sieve inclusion for $\lambda < \kappa$, then so is $F^{\{\kappa\}}$. ∎

5.11. Dense families. We are able to further reason about isotropy rank with a relatively simple concept that helps in the proofs of the main results concerning isotropy groups of collages of profunctors and of colimits of categories (7).

5.12. Definition. *Let* $A = \{\alpha_i \in \mathsf{Aut}(X_i) \mid i \in I\}$ *be a set of automorphisms of a category* $\mathbb{C}$. *Then* A *is said to be* dense (for $\mathbb{C}$) *when the equation*

$$\mathsf{IR}(\mathbb{C}) = \bigvee_{i \in I} \mathsf{IR}(\alpha_i)$$

holds.

By Prop. 5.5, the family of all automorphisms of $\mathbb{C}$ is dense. However, we are mostly interested in finding more economical dense families. For example, if a category $\mathbb{C}$ has only finitely many automorphisms, then one may simply choose an α with $\mathsf{IR}(\mathbb{C}) = \mathsf{IR}(\alpha)$, so that $A = \{\alpha\}$ is dense. For example, in the case of the category $\mathbb{X}$ (§ 4.1) the twist automorphism α has rank 2, which is maximal. Therefore $\{\alpha\}$ is a dense family in $\mathbb{X}$.

5.13. Lemma. *Let* $\mathbb{C}$ *be a category with isotropy rank* κ*, and suppose* $A = \{\alpha_i \mid i \in I\}$ *is dense.*

(i) *When* κ *is a successor ordinal there exists an automorphism* $i \in I$ *such that* $\{\alpha_i\}$ *is dense.*

(ii) *When* κ *is a limit ordinal there exists a subset* $J \subseteq I$ *with the property that* $A' = \{\alpha_i \mid i \in J\}$ *is also dense, and* $\mathsf{IR}(\alpha_i) = \mathsf{IR}(\alpha_j)$ *implies* $i = j$.

Proof. (i) If $\mathbb{C}^{\{\kappa\}}$ has non-trivial isotropy, then we may pick an automorphism α such that $\alpha \in \mathsf{I}_\kappa$, i.e., $\mathsf{IR}(\alpha) = \kappa + 1$.

(ii) We know that for each $\lambda < \kappa$ there must exist α with $\lambda \leq \mathsf{IR}(\alpha) < \kappa$. ∎

The main technical point about density that we wish to exploit later is the fact that the image of a dense family under the iterated quotient maps is again dense.

5.14. Theorem. *Let* $A = \{\alpha_i \mid i \in I\}$ *be a dense family in* $\mathbb{C}$. *Then for each* λ*, the image of* A *under* $\mathbb{C} \to \mathbb{C}^{\{\lambda\}}$ *is dense in* $\mathbb{C}^{\{\lambda\}}$.

Proof. We argue by transfinite induction on $\mathsf{IR}(\mathbb{C})$. When $\mathbb{C}$ is anisotropic the result is trivial (as when $\lambda > \kappa$). If $\mathsf{IR}(\mathbb{C})$ is a successor ordinal, then using Lemma 5.13 we may pick $i \in I$ for which $\mathsf{IR}(\alpha_i) = \kappa + 1$. Then for each $\lambda \leq \kappa$, α_i is non-trivial and dense.

When $\mathsf{IR}(\mathbb{C}) = \kappa$ is a limit ordinal, then we may assume by the induction hypothesis that the result holds for all $\kappa' < \kappa$. This means in particular that for all $\lambda < \kappa$, the image of A under $\mathbb{C} \to \mathbb{C}^{\{\lambda\}}$ is dense. It therefore remains to show this for $\lambda = \kappa$, but this is trivial since $\mathbb{C}^{\{\kappa\}}$ is anisotropic. ∎

6. Construction

Thus far we have seen an example of a category with rank 2, but it remains to be shown that higher ranks, and even infinite rank, are indeed manifested. We shall construct for every ordinal κ a category $\mathbb{X}[\kappa]$ with isotropy rank κ. It may help to first provide an intuitive idea behind the construction of $\mathbb{X}[3]$ without proofs. In order to make this construction precise we may realize $\mathbb{X}[3]$ as the collage of a suitable profunctor. This allows us to precisely define the other categories $\mathbb{X}[\kappa]$ as well. Finally, we isolate the kind of profunctors being used here, and turn to the general question of how to compute isotropy of categories arising from such profunctors. This analysis yields the desired results about the categories $\mathbb{X}[\kappa]$.

6.1. Informal sketch. An informal explanation of how to construct a category $\mathbb{X}[3]$ with isotropy rank 3 follows. The category $\mathbb{X}$ (4.1) looks roughly as follows:

$$\begin{tikzcd}
C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[d, shift right, "\gamma"'] \arrow[d, shift left, "\delta"] & B \arrow[r, "f"] \arrow[d, "\beta"] & A \arrow[d, "\alpha"] \\
C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & B \arrow[r, "f"'] & A
\end{tikzcd}$$

where the horizontal morphisms are non-isomorphisms satisfying $fg = fh$, the vertical maps are automorphisms, and α has isotropy rank 2. The idea is to attach a copy of $\mathbb{X}$ to itself, creating an obstruction for the isotropy maps in the second copy. We achieve this by formally adding two new morphisms v, w from the object A in the first copy of $\mathbb{X}$ to the object C in the second. The category $\mathbb{X}[3]$ looks something like this, where we rename the objects and arrow in the second copy of $\mathbb{X}$.

$$\begin{tikzcd}
C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[d, shift right, "\gamma"'] \arrow[d, shift left, "\delta"] & B \arrow[r, "f"] \arrow[d, "\beta"] & A \arrow[r, bend left, "v"] \arrow[r, bend right, "w"'] \arrow[d, "\alpha"] & C' \arrow[r, shift left, "g'"] \arrow[r, shift right, "h'"'] \arrow[d, shift right, "\gamma'"'] \arrow[d, shift left, "\delta'"] & B \arrow[r, "f'"] \arrow[d, "\beta'"] & A' \arrow[d, "\alpha'"] \\
C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & B \arrow[r, "f"'] & A \arrow[r, bend left, "v"] \arrow[r, bend right, "w"'] & C' \arrow[r, shift left, "g'"] \arrow[r, shift right, "h'"'] & B \arrow[r, "f'"'] & A'
\end{tikzcd}$$

We define $\mathbb{X}[3]$ in such a way that the two morphisms v, w create an obstruction for the automorphisms of C' to be isotropy. This we do by imposing the relations

$$v\gamma' = \alpha v,\ w\gamma' = w,\ w\delta' = \alpha w,\ v\delta' = v$$

so that γ' and δ' must lift to a non-isotropy map and hence cannot be of rank lower than the rank of α. This forces the following facts about the isotropy ranks of the automorphisms, where we list only the generating maps and omit their composites.

- Rank 1: γ, δ (at the object C), and β (at the object B).
- Rank 2: α (at A), γ', δ' (at C'), and β' (at B').
- Rank 3: α' (at A').

In the first isotropy quotient, $[\alpha]_1$ is isotropy, and consequently so are $[\gamma']_1$, $[\delta']_1$ and $[\beta']_1$. However, $[\alpha']_1$ has rank 2.

Passing to the next quotient we find that all automorphisms except for $[\alpha']_2$ become identities, while $[\alpha']_2$ itself is isotropy. This implies that $\mathbb{X}[3]$ has isotropy rank 3. It is necessary to adjoin two morphisms, and not just a single morphism, because it is impossible for a single morphism to satisfy lifting conditions which not only make γ', δ' lift non-trivially, but also make their difference $\gamma'\delta'$ lift non-trivially. The latter is necessary because it is this difference which is the obstruction to α' being isotropy, and hence we must make sure that this difference does not vanish after one quotient.

Stacking copies of $\mathbb{X}$ onto itself can be repeated, producing categories $\mathbb{X}[n]$ with isotropy rank n for any finite n. The colimit of the $\mathbb{X}[n]$ produces a category with isotropy rank ω. We continue into the transfinite by stacking copies of $\mathbb{X}$ onto $\mathbb{X}[\omega]$.

6.2. Collages. We wish to make the above construction more formal and more general. A profunctor $\mathcal{P} : \mathbb{C} \nrightarrow \mathbb{D}$ is a functor

$$\mathcal{P} : \mathbb{C}^{op} \times \mathbb{D} \to \mathsf{Set} .$$

It has a corresponding two-sided discrete fibration $\int \mathcal{P} \to \mathbb{C} \times \mathbb{D}$, where $\int \mathcal{P}$ denotes the category of elements of the profunctor. By definition, *the collage of* $\mathcal{P}$ is the following cocomma category.

$$\begin{tikzcd}
 & \int \mathcal{P} \arrow[dl, "\rho"'] \arrow[dr, "\pi"] & \\
\mathbb{C} \arrow[dr, "\iota"'] \arrow[rr, phantom, "\Rightarrow"] & & \mathbb{D} \arrow[dl, "\upsilon"] \\
 & \mathbb{C} *_{\mathcal{P}} \mathbb{D} &
\end{tikzcd}$$

When $\mathcal{P}$ is understood we sometimes simply write $\mathbb{C} * \mathbb{D}$ for the collage. Being a cocomma category, the set of objects of the collage is the disjoint union of those of $\mathbb{C}$ and $\mathbb{D}$. For morphisms we have:

$$\begin{array}{lcll}
\mathbb{C} * \mathbb{D}(C, C') & = & \mathbb{C}(C, C') & \text{for } C, C' \text{ in } \mathbb{C} \\
\mathbb{C} * \mathbb{D}(D, D') & = & \mathbb{D}(D, D') & \text{for } D, D' \text{ in } \mathbb{D} \\
\mathbb{C} * \mathbb{D}(C, D) & = & P(C, D) & \text{for } C \text{ in } \mathbb{C}, D \text{ in } \mathbb{D}.
\end{array}$$

An element of $\mathcal{P}(C, D)$ is realized as a morphism of $\mathbb{C} * \mathbb{D}$. We refer to such a morphism as a *heteromorphism* if we wish to distinguish it from one coming from $\mathbb{C}$ or $\mathbb{D}$. The inclusion ι is a sieve, and υ is a cosieve inclusion.

We seek to identify classes of profunctors for which computing the isotropy group of the collage is manageable. The free profunctor $\mathcal{F}(P)$ on a family of sets $P(X, Y)$ indexed by the objects of $\mathbb{C} \times \mathbb{D}$ is explicitly described as

$$\mathcal{F}(P)(C, D) = \coprod_{X,Y} \mathbb{C}(C, X) \times P(X, Y) \times \mathbb{D}(Y, D) .$$

Consequently, the collage of $\mathcal{F}(P)$ has the property that for any two heteromorphisms (f,m,g) and (f',m',g'), we have

$$(f,m,g) = (f',m',g') \iff f = f'\,, m = m'\,, g = g'\ .$$

It is easily seen that in the collage of such a profunctor, non-trivial automorphisms never lift along heteromorphisms, making the isotropy of such a collage uninteresting. Consequently, we wish to impose relations specifying lifts along heteromorphisms. The data necessary to do so is formalized in the following definition.

6.3. Definition. [$\mathcal{A}$-system] *Let $\mathbb{C},\mathbb{D}$ be small categories. An* $\mathcal{A}$-system $P = (P,p,\Phi)$ *from $\mathbb{C}$ to $\mathbb{D}$ consists of:*

- *for each pair of objects (X,Y) of $\mathbb{C}\times\mathbb{D}$, a set $P(X,Y)$ (whose elements we think of and refer to as* generating heteromorphisms*); we may equivalently regard this data as a map $p : P \to Ob(\mathbb{C}) \times Ob(\mathbb{D})$ such that $P(X,Y) = p^{-1}(X,Y)$.*
- *for each generating heteromorphism $m \in P(X,Y)$, a group homomorphism $\phi_m : \mathsf{Aut}(Y) \to \mathsf{Aut}(X)$; this may be regarded as a functor Φ making the following commute:*

$$\begin{tikzcd}
\coprod_{X,Y} P(X,Y) = P \arrow[r, "\Phi"] \arrow[d, "p"'] & \mathsf{Grp}^{\to} \arrow[d] \\
Ob(\mathbb{C}) \times Ob(\mathbb{D}) \arrow[r, "\mathsf{Aut}(-)\times\mathsf{Aut}(-)"'] & \mathsf{Grp}\times\mathsf{Grp}
\end{tikzcd}$$

where the right vertical morphism forgets the homomorphism.

A morphism *of $\mathcal{A}$-systems from (P,p,Φ) to (Q,q,Ψ) is a function $t : P \to Q$ making the two diagrams below commute:*

$$\begin{tikzcd}
P \arrow[rr, "t"] \arrow[dr, "p"'] & & Q \arrow[dl, "q"] \\
& Ob(\mathbb{C}) \times Ob(\mathbb{D}) &
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[rr, "t"] \arrow[dr, "\Phi"'] & & Q \arrow[dl, "\Psi"] \\
& \mathsf{Grp}^{\to} &
\end{tikzcd}$$

$\mathcal{A}$-systems and their morphisms form a category $\mathcal{A}\text{–}\mathsf{Sys}(\mathbb{C},\mathbb{D})$.

The categories $\mathcal{A}\text{–}\mathsf{Sys}(\mathbb{C},\mathbb{D})$ are easily seen to be complete and cocomplete: limits and colimits are simply computed as in the slice category $\mathsf{Set}/(Ob(\mathbb{C}) \times Ob(\mathbb{D}) \times Ob(\mathsf{Grp}^{\to}))$.

6.4. Construction. *From an $\mathcal{A}$-system $P = (P,p,\Phi)$ we construct a profunctor $\mathcal{P} = \mathcal{R}(P,p,\phi) : \mathbb{C} \nrightarrow \mathbb{D}$ by first forming the free profunctor on the indexed set $\{P(X,Y) \mid X \in Ob(\mathbb{C}), Y \in Ob(\mathbb{D})\}$, and then quotienting by the relations $m\,\phi_m(\beta) = \beta\, m$ for all $m \in P(X,Y), \beta \in \mathsf{Aut}(Y)$. Precisely:*

$$\mathcal{P}(C,D) = \coprod_{X,Y} [\mathbb{C}(C,X) \times P(X,Y) \times \mathbb{D}(Y,D)]/\sim$$

such that

$$(f,m,g) \sim (f',m',g') \iff m = m' \text{ and } \exists \beta \in \mathsf{Aut}(Y)\,.\; f\,\phi_m(\beta) = f'\,,\; \beta\, g' = g\,. \tag{13}$$

The assignment $(P,p,\Phi) \mapsto \mathcal{R}(P,p,\Phi)$ is a functor

$$\mathcal{R} : \mathcal{A}{-}\mathsf{Sys}(\mathbb{C},\mathbb{D}) \to \mathfrak{Prof}(\mathbb{C},\mathbb{D})\,.$$

There are two key points about the general construction that are important for analyzing the isotropy of the collage in § 7. The first is the behaviour of generating heteromorphisms regarding isotropy: these maps have the *unique lifting property* with respect to automorphisms in the sense that for every $m \in P(X,Y)$ and every automorphism $\beta \in \mathsf{Aut}(Y)$, there is exactly one way $\phi_m(\beta)$ to lift β along m. The second point is that when m, m' are two generating heteromorphisms with $m \neq m'$, there is never a commutative diagram containing both m and m'.

We shall need one further ingredient. Fixing the category $\mathbb{D}$ but considering a functor $F : \mathbb{C} \to \mathbb{C}'$, we obtain an induced functor

$$F_* : \mathcal{A}{-}\mathsf{Sys}(\mathbb{C},\mathbb{D}) \to \mathcal{A}{-}\mathsf{Sys}(\mathbb{C}',\mathbb{D})$$

given by postcomposition: for any object $P = (P,p,\Phi)$ of $\mathcal{A}{-}\mathsf{Sys}(\mathbb{C},\mathbb{D})$ we consider

$$F_*P = (P, (|F| \times 1)p, F\Phi)\,,$$

where $F\Phi$ takes $m \in P(X,Y)$ to the composite

$$\mathsf{Aut}(Y) \xrightarrow{\phi_m} \mathsf{Aut}(X) \xrightarrow{F} \mathsf{Aut}(FX)\,.$$

The induced profunctor $\mathcal{R}(F_*P) : \mathbb{C}' \nrightarrow \mathbb{D}$ satisfies

$$\mathcal{R}(F_*P)(C',D) = \coprod_{X,Y} [\mathbb{C}'(C',FX) \times P(X,Y) \times \mathbb{D}(Y,D)]/\sim$$

with

$$(f,m,g) \sim (f',m',g') \iff m = m' \text{ and } \exists \beta \in \mathsf{Aut}(Y).\; f\,F(\phi_m(\beta)) = f'\,,\; \beta\, g' = g\,. \tag{14}$$

6.5. Lemma. *Let $F_\bullet : \mathbb{C}' \nrightarrow \mathbb{C}$ denote the induced corepresentable profunctor associated with a functor $F : \mathbb{C} \to \mathbb{C}'$. Then the square*

$$\begin{tikzcd} \mathcal{A}{-}\mathsf{Sys}(\mathbb{C},\mathbb{D}) \arrow[r, "\mathcal{R}"] \arrow[d, "F_*"'] & \mathfrak{Prof}(\mathbb{C},\mathbb{D}) \arrow[d, "-\circ F_\bullet"] \\ \mathcal{A}{-}\mathsf{Sys}(\mathbb{C}',\mathbb{D}) \arrow[r, "\mathcal{R}"'] & \mathfrak{Prof}(\mathbb{C}',\mathbb{D}) \end{tikzcd}$$

commutes. Moreover, the collage of $\mathcal{R}(F_*P)$ is the pushout

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\iota"] \arrow[d, "F"'] & \mathbb{C} *_P \mathbb{D} \arrow[d] \\ \mathbb{C}' \arrow[r, "\iota'"'] & \mathbb{C}' *_{F_*P} \mathbb{D}\,. \end{tikzcd}$$

Proof. In fact, the two statements are equivalent because precomposing a profunctor with $F_\bullet$ corresponds on the level of collages to forming the pushout. We therefore only prove the second statement. Suppose we are given functors $M : \mathbb{C}' \to \mathbb{B}$ and $N : \mathbb{C}*\mathbb{D} \to \mathbb{B}$ for which $MF = N\iota$. We define the extension $R : \mathbb{C}' * \mathbb{D} \to \mathbb{B}$ as follows. R sends an object X in the image of ι' to MX and an object Y in the image of $\upsilon' : \mathbb{D} \to \mathbb{C}' * \mathbb{D}$ to NY. For any morphism $[f, m, g] : C' \to D$ in $\mathbb{C}' * \mathbb{D}$, i.e., $f : C' \to FX$, $m \in P(X, Y)$, and $g : Y \to D$, define $R[f, m, g] = N[1, m, g]\, Mf$. This evidently makes the relevant triangles commute and is unique with that property. ∎

6.6. Definition of the categories $\mathbb{X}[\kappa]$. By transfinite induction we shall define for each ordinal κ a small category $\mathbb{X}[\kappa]$ as well as full sieve inclusion functors

$$\iota = \iota_{\lambda<\kappa} : \mathbb{X}[\lambda] \to \mathbb{X}[\kappa]$$

for each $\lambda < \kappa$; together these form a functor $\mathbb{X}[-] : \mathfrak{Ord} \to \mathfrak{Cat}$. In addition, we define for each $\kappa > 1$ (except for $\kappa = \lambda+1$ where λ is a limit ordinal) profunctors $\mathcal{P}[\kappa] : \mathbb{X}[\kappa] \looparrowright \mathbb{X}[2]$ and $\mathcal{Q}[\kappa] : \mathbb{X}[\kappa] \looparrowright \mathbb{X}[1]$, arising from $\mathcal{A}$-systems. We define:

- $\mathbb{X}[0] = \emptyset$, the empty category.
- $\mathbb{X}[1]$ is the category with a single object C, and $\mathbb{X}[1](C, C) = \{1, \lambda_1, \lambda_2, \lambda_1\lambda_2\}$ where each of these maps squares to the identity; hence $\mathbb{X}[1](C, C) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
- $\mathbb{X}[2] = \mathbb{X}$ is the category described in § 4. The functor $\iota : \mathbb{X}[1] \to \mathbb{X}[2]$ is the evident full inclusion.

Consider the profunctor $\mathcal{P}[2] = \mathcal{S} : \mathbb{X} \looparrowright \mathbb{X}$ given by the $\mathcal{A}$-system $S = (S, s, \sigma)$ such that

- $s^{-1}(A, C) = \{v, w\}$ (and all other fibers are empty);
- $\sigma_v, \sigma_w : \mathsf{Aut}(C) \to \mathsf{Aut}(A)$ are given on generators by

$$\sigma_v(\gamma) = \alpha, \sigma_v(\delta) = 1\, ; \qquad \sigma_w(\gamma) = 1, \sigma_w(\delta) = \alpha\, .$$

The profunctor $\mathcal{T} = \mathcal{Q}[2] : \mathbb{X} \looparrowright \mathbb{X}[1]$ is given by the $\mathcal{A}$-system from $\mathbb{X}$ to $\mathbb{X}[1]$ given by simply restricting S to this smaller category.

Successor case, standard: Suppose that $\mathbb{X}[\kappa]$ is defined, where κ is not a limit ordinal or a successor of a limit ordinal. By the induction hypothesis we have a profunctor $\mathcal{P}[\kappa] : \mathbb{X}[\kappa] \looparrowright \mathbb{X}$. We define $\mathbb{X}[\kappa+1] = \mathbb{X}[\kappa] *_{\mathcal{P}[\kappa]} \mathbb{X}$:

$$\mathbb{X}[\kappa] \xrightarrow{\iota_\kappa^{\kappa+1}} \mathbb{X}[\kappa+1] = \mathbb{X}[\kappa] * \mathbb{X} \xleftarrow{\upsilon_\kappa^{\kappa+1}} \mathbb{X}$$

To define the profunctors $\mathcal{P}[\kappa+1] : \mathbb{X}[\kappa+1] \looparrowright \mathbb{X}$ and $\mathcal{Q}[\kappa+1] : \mathbb{X}[\kappa+1] \looparrowright \mathbb{X}[1]$ we use the inclusion $\upsilon_\kappa^{\kappa+1} : \mathbb{X} \to \mathbb{X}[\kappa+1]$, allowing us to set $\mathcal{P}[\kappa+1] = \mathcal{S} \circ (\upsilon_\kappa^{\kappa+1})_\bullet$ (Lemma 6.5). Similarly we let $\mathcal{Q}[\kappa+1] = \mathcal{T} \circ (\upsilon_\kappa^{\kappa+1})_\bullet$.

Successor case, exceptions: If κ is a limit ordinal, then we shall define $\mathbb{X}[\kappa+1]$ and $\mathbb{X}[\kappa+2]$ as the collages

- $\mathbb{X}[\kappa+1] = \mathbb{X}[\kappa] *_{\mathcal{Q}[\kappa]} \mathbb{X}[1]$ and
- $\mathbb{X}[\kappa+2] = \mathbb{X}[\kappa] *_{\mathcal{P}[\kappa]} \mathbb{X}$.

We also specify the profunctors $\mathcal{P}[\kappa+2] : \mathbb{X}[\kappa+2] \nrightarrow \mathbb{X}$ and $\mathcal{Q}[\kappa+2] : \mathbb{X}[\kappa+2] \nrightarrow \mathbb{X}[1]$ using the cosieve inclusion $\upsilon_\kappa^{\kappa+2} : \mathbb{X} \to \mathbb{X}[\kappa+2]$ and setting $\mathcal{P}[\kappa+2] = \mathcal{S} \circ (\upsilon_\kappa^{\kappa+2})_\bullet$ (Lemma 6.5). Similarly we let $\mathcal{Q}[\kappa+2] = \mathcal{T} \circ (\upsilon_\kappa^{\kappa+2})_\bullet$.
Limit case: Suppose that κ is a limit ordinal. By induction, we have constructed a chain

$$\mathbb{X}[0] \to \mathbb{X}[1] \to \cdots \to \mathbb{X}[\lambda] \to \mathbb{X}[\lambda+1] \to \cdots$$

Then define

- $\mathbb{X}[\kappa] = \mathrm{Colim}_{\lambda<\kappa}\mathbb{X}[\lambda]$.

It remains to define the profunctors $\mathcal{P}[\kappa] : \mathbb{X}[\kappa] \nrightarrow \mathbb{X}$ and $\mathcal{Q}[\kappa] : \mathbb{X}[\kappa] \nrightarrow \mathbb{X}[1]$. We remark first that we also have $\mathbb{X}[\kappa] = \mathrm{Colim}_{\lambda=\lambda'+2<\kappa}\mathbb{X}[\lambda]$. For each $\lambda = \lambda' + 2 < \kappa$, by the induction hypothesis we are given a profunctor $\mathcal{P}[\lambda] : \mathbb{X}[\lambda] \nrightarrow \mathbb{X}$, arising from an $\mathcal{A}$-system say as $\mathcal{P}[\lambda] = \mathcal{R}(P[\lambda])$. Using the cocone inclusion $\iota_\lambda^\kappa : \mathbb{X}[\lambda] \to \mathbb{X}[\kappa]$ we may consider the induced $\mathcal{A}$-system $(\iota_\lambda^\kappa)_*(P[\lambda])$. Doing this for each λ thus gives a family of $\mathcal{A}$-systems from $\mathbb{X}[\kappa]$ to $\mathbb{X}$. The profunctor corresponding to the coproduct in $\mathcal{A}-\mathsf{Sys}(\mathbb{X}[\kappa], \mathbb{X})$ of this family is the desired $\mathcal{P}[\kappa]$:

$$\mathcal{P}[\kappa] = \mathcal{R}(\coprod_{\lambda=\lambda'+2<\kappa} P[\lambda]) \ .$$

Informally, this system contains two heteromorphisms to connect the new copy of $\mathbb{X}$ with each of the copies of $\mathbb{X}$ occurring in $\mathbb{X}[\kappa]$. We define $\mathcal{Q}[\kappa]$ in exactly the same way, replacing the system $P[\lambda]$ by $Q[\lambda]$. This concludes the definition of the sequence of categories $\mathbb{X}[\kappa]$.

Ultimately in Prop. 8.8 we establish that for any ordinal κ , the isotropy rank of $\mathbb{X}[\kappa]$ is κ . The proof is by transfinite induction; the base cases $\kappa \leq 2$ are taken care of in § 4. The inductive steps involve an analysis of the behaviour of isotropy with respect to taking sequential colimits of categories and of collages, carried out in § 7.

7. Isotropy and profunctors

Having constructed the categories $\mathbb{X}[\kappa]$ using collages of profunctors we are led to investigate the isotropy of collages. We focus on profunctors arising from $\mathcal{A}$-systems as these are the ones that give us some control over the isotropy. We begin by describing the isotropy group of the collage as a certain pullback. We investigate the isotropy quotient, and then look at the sequence of higher quotients obtaining sufficient conditions on the profunctor for finding the isotropy rank of a collage.

7.1. The isotropy group of a collage. We are able to develop a manageable description of the isotropy of the collage when the profunctor arises from an $\mathcal{A}$-system (§ 6.2). Throughout, we consider such a fixed profunctor $\mathcal{P} = \mathcal{R}(P, p, \Phi)$ whose collage we denote

$$\mathbb{C} \xrightarrow{\iota} \mathbb{C} * \mathbb{D} \xleftarrow{\upsilon} \mathbb{D} .$$

Recall from Section 3.11 that a functor is a *sieve inclusion* if it is full and faithful and if there are no morphisms from objects outside its image to objects within its image. Therefore the inclusion functor ι is a sieve inclusion, and by Proposition 3.13 every object C of $\mathbb{C}$ we have

$$\mathsf{Z}_{\mathbb{C}*\mathbb{D}}(C) \cong \mathsf{Z}_{\mathbb{C}}(C) ,$$

where we identify C with $\iota(C)$. It thus remains to describe $\mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D)$ for objects D of $\mathbb{D}$.

For any object D of $\mathbb{D}$ we have a group homomorphism

$$\rho_D : \mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D) \to \mathsf{Z}_{\mathbb{D}}(D) \ \text{ (Prop. 3.12) } .$$

We also have the unit of $\iota^* \dashv \iota_*$ at $\mathsf{Z}_{\mathbb{C}*\mathbb{D}}$, which according to Lemma 3.15 may be identified with

$$\eta_D : \mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D) \to \lim_{v:C\to D} \mathsf{Z}_{\mathbb{C}}(C) ,$$

where the limit is taken over the comma category $\iota{\downarrow}D$ of heteromorphisms $v : C \to D$. The following analysis of $\iota \downarrow D$ ultimately takes us to Theorem 7.2, stating that the isotropy group $\mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D)$ is recovered as the pullback (15).

A heteromorphism $v : C \to D$ has the form $v = [f, m, g]$ with $f : C \to X$, $m \in P(X, Y)$ and $g : Y \to D$. The equivalence relation on such triples is $(\phi_m(\beta)f, m, g) \sim (f, m, g\beta)$ where $\beta \in \mathsf{Aut}(Y)$. We observe first that the objects of the form $[1, m, g] : C \to D$ form a final subcategory of $\iota \downarrow D$. Moreover, this subcategory is a groupoid: if

$$t : [1, m, g] \to [1, m', g']$$

is a morphism of $\iota \downarrow D$, i.e., $[1, m, g] = [1, m', g']\,t$, then $m = m'$ and we obtain an automorphism β with $g'\beta = g$ and $t = \phi_m(\beta)$. Thus, t is an isomorphism.

Putting things together we find that the limit defining $\iota_*\iota^*\mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D)$ may be identified with the subgroup

$$H(D) \subseteq \prod_{\substack{g:Y\to D\\ m\in P(X,Y)}} Z_{\mathbb{C}}(X) \ ; \ \ \langle\theta_{m,g}\rangle \in H(D) \Leftrightarrow \forall\beta \in \mathsf{Aut}(Y)\, \theta_{m,g\beta} = \phi_m(\beta)^*(\theta_{m,g})$$

and the unit η_D with the map $\mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D) \to H(D)$ that sends an element τ to the tuple

$$\eta_D(\tau) = \langle [1, m, g]^*(\tau) \rangle_{m,g} \in H(D) .$$

Now consider the following diagram in which the left hand vertical map is the unit η_D just discussed, and where the top horizontal map is the comparison map ρ_D .

$$
\begin{tikzcd}
\mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D) \arrow[rr, "\rho_D"] \arrow[ddd, "\eta_D"'] & & \mathsf{Z}_{\mathbb{D}}(D) \arrow[d, "j"] \\
& & \prod_{g:Y\to D} \mathsf{Z}_{\mathbb{D}}(Y) \arrow[d, "\prod_{g:Y\to D} \nu_Y"] \\
& & \prod_{g:Y\to D} \mathsf{Aut}(Y) \arrow[d, "\Phi"] \\
H(D) \arrow[r, "\subseteq"] & \prod_{\substack{g:Y\to D \\ m\in P(X,Y)}} \mathsf{Z}_{\mathbb{C}}(X) \arrow[r, "\prod_{g,m} \nu_X"] & \prod_{\substack{g:Y\to D \\ m\in P(X,Y)}} \mathsf{Aut}(X)
\end{tikzcd}
\tag{15}
$$

The map j sends $\tau \in \mathsf{Z}_{\mathbb{D}}(D)$ to $\langle g^*\tau \rangle_{g:Y\to D}$ and the map Φ is defined through its components

$$\Phi_{m,g} : \prod_{g:Y\to D} \mathsf{Aut}(Y) \to \mathsf{Aut}(X); \qquad \Phi_{m,g}(\langle \beta_k \rangle_{k:Y\to D}) = \phi_m(\beta_g) \in \mathsf{Aut}(X) \;.$$

We offer the following intuitive picture and explanation of the commutativity of diagram (15).

$$
\begin{tikzcd}
X \arrow[r, "m"] \arrow[d, "\phi_m(\tau_g)"'] & Y \arrow[r, "g"] \arrow[d, "\tau_g"'] & D \arrow[d, "\tau_D"] \\
X \arrow[r, "m"'] & Y \arrow[r, "g"'] & D
\end{tikzcd}
$$

Suppose we have an element $\tau \in \mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D)$. For any $g : Y \to D$, we get a component $\tau_g \in \mathsf{Aut}(Y)$. By definition, τ_g is part of isotropy, and hence for a given heteromorphism $m : X \to Y$, τ_g must lift against m . But by definition of the profunctor giving rise to the collage there is exactly one such lift, namely $\phi_m(\tau_g)$: this lift does not depend on any information about τ other than the component τ_g . Of course, the automorphism $\tau_{[1,m,g]} = \phi_m(\tau_g)$ must itself be isotropy, but since X is an object of $\mathbb{C}$ and $\mathbb{C}$ is a sieve, this simply means that it is in the image of $\mathsf{Z}_{\mathbb{C}}(X) \to \mathsf{Aut}(X)$.

7.2. Theorem. *The square* (15) *is a pullback.*

Proof. We have already explained why the diagram commutes. Let us therefore consider an element $\tau \in \mathsf{Z}_{\mathbb{D}}(D)$ and an element $\theta = \langle \theta_{m,g} \rangle \in H(D)$ which get mapped to the same element of $\prod_{m,g} \mathsf{Aut}(X)$. This agreement means that for each $g : Y \to D$ and each $m \in P(X,Y)$, the underlying component $(\theta_{m,g})_X \in \mathsf{Aut}(X)$ is equal to $\phi_m(\tau_g)$. This data may be used to construct an element ζ of $\mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D)$: we define ζ through its components ζ_v , where $v : U \to D$ is a morphism in $\mathbb{C} * \mathbb{D}$.

$$\zeta_v = \begin{cases} \tau_g & \text{if } v \text{ is in } \mathbb{D} \\ (\theta_{m,g})_f & \text{if } v = [f,m,g] \end{cases}$$

We must show that this is well-defined: if $[f, m, g\beta] = [f\phi_m(\beta), m, g]$, then we wish to obtain $(\theta_{m,g\beta})_f = (\theta_{m,g})_{f\phi_m(\beta)}$. Since $\theta_{m,g}$ is an element of isotropy and hence natural, we have $(\theta_{m,g})_{f\phi_m(\beta)} = (\phi_m(\beta)^*(\theta_{m,g}))_f$, which equals $(\theta_{m,g\beta}))_f$ because $\theta \in H(D)$.

We wish to show that ζ is natural, i.e., that given $w : V \to U$ and $v : U \to D$ we have $\zeta_v w = w\zeta_{vw}$. This is immediate from the naturality of τ when both v, w are in $\mathbb{D}$. There are two remaining cases: where v is in $\mathbb{D}$ and w is a heteromorphism, and where v is a heteromorphism and w is in $\mathbb{C}$. Suppose first that $v = [f, m, g]$, and $w : C' \to C$ is a morphism of $\mathbb{C}$. Then $\zeta_{vw} = (\theta_{m,g})_{fw}$, and naturality of θ gives

$$w\zeta_{vw} = w(\theta_{m,g})_{fw} = (\theta_{m,g})_f w = \zeta_v w$$

as required. Now suppose $v : D \to D'$ is in $\mathbb{D}$ and $w = [f, m, g]$ is a heteromorphism. Then $\zeta_{vw} = (\theta_{m,vg})_f$. Consider the diagram

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\zeta_{vw}=(\theta_{m,vg})_f"'] & X \arrow[r, "m"] \arrow[d, "\phi_m(\tau_{vg})"] & Y \arrow[r, "g"] \arrow[d, "\tau_{vg}"] & D \arrow[r, "v"] \arrow[d, "\tau_v"] & D' \arrow[d, "\tau_{D'}"] \\
C \arrow[r, "f"'] & X \arrow[r, "m"'] & Y \arrow[r, "g"'] & D \arrow[r, "v"'] & D'
\end{tikzcd}$$

in which the two squares on the right hand side commute by naturality of τ, the second square commutes by definition of the equivalence relation on maps in $\mathbb{C} * \mathbb{D}$, and the first square commutes because θ is assumed to agree with τ. We thus get

$$w\zeta_{vw} = [f, m, g]\zeta_{vw} = \tau_v[f, m, g] = \zeta_v w$$

as needed. ∎

7.3. The isotropy quotient of a collage. The restriction homomorphisms

$$\rho_D : \mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D) \to \mathsf{Z}_{\mathbb{D}}(D)$$

of a general profunctor $\mathcal{P} : \mathbb{C} \nrightarrow \mathbb{D}$ induces a congruence $\sim_{\mathcal{P}}$ on $\mathbb{D}$ via the composite

$$\mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D) \xrightarrow{\rho_D} \mathsf{Z}_{\mathbb{D}}(D) \xrightarrow{\nu_D} \mathsf{Aut}_{\mathbb{D}}(D)\,. \tag{16}$$

Thus we only quotient out those isotropy maps in $\mathbb{D}$ that are also isotropy when considered in $\mathbb{C} * \mathbb{D}$. Explicitly, for $g, h : D \to E$ in $\mathbb{D}$,

$$g \sim_P h \;\Leftrightarrow\; \exists \tau \in \mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D).\, g \circ \tau_D = h\,.$$

Consequently, we have the following commutative diagram.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\iota"] \arrow[d] & \mathbb{C} *_{\mathcal{P}} \mathbb{D} \arrow[d] & \mathbb{D} \arrow[l, "\upsilon"'] \arrow[d] \\
\mathbb{C}^{\{1\}} \arrow[r, "\iota^{\{1\}}"'] & (\mathbb{C} *_{\mathcal{P}} \mathbb{D})^{\{1\}} & \mathbb{D}/{\sim_{\mathcal{P}}} \arrow[l, "\upsilon'"']
\end{tikzcd} \tag{17}$$

Alternatively, $\mathbb{D}/{\sim_{\mathcal{P}}}$ is the bijective-on-objects, fully-faithful factorization of the composite

$$\mathbb{D} \to \mathbb{C} * \mathbb{D} \to (\mathbb{C} * \mathbb{D})^{\{1\}} \,.$$

In general, $\mathbb{D}/{\sim_{\mathcal{P}}}$ is not the isotropy quotient of $\mathbb{D}$ because the restriction maps ρ_D need not be surjective. The right hand square in diagram (17) is in fact a pullback. However, the bottom row of the diagram is itself a collage of an induced profunctor

$$\mathcal{P}^{\{1\}} : \mathbb{C}^{\{1\}} \nrightarrow \mathbb{D}/{\sim_{\mathcal{P}}} \,,$$

to which we refer as the *quotient* of $\mathcal{P}$. Explicitly, we may consider the left Kan extension of $\mathcal{P}$ along the quotient map depicted vertically in the following diagram.

$$\begin{tikzcd}
\mathbb{C}^{\mathrm{op}} \times \mathbb{D} \arrow[r, "\mathcal{P}"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathsf{Set} \\
\mathbb{C}^{\{1\}^{\mathrm{op}}} \times \mathbb{D}/{\sim_{\mathcal{P}}} \arrow[ur, bend right, "\mathcal{P}^{\{1\}} = \mathrm{Lan}(\mathcal{P})"'] &
\end{tikzcd}$$

A straightforward calculation reveals that $\mathcal{P}^{\{1\}}$ is given as $\mathcal{P}^{\{1\}}(C, D) = \mathcal{P}(C, D)/\sim$, where $\sim$ is the isotropy congruence on $\mathbb{C} * \mathbb{D}$. We easily obtain:

7.4. Lemma. *Let $\mathcal{P} : \mathbb{C} \nrightarrow \mathbb{D}$ be a profunctor. Then the bottom row of* (17) *is isomorphic to the collage of the induced profunctor $\mathcal{P}^{\{1\}}$.*

We return to the case where a profunctor $\mathcal{P}$ is constructed from an $\mathcal{A}$-system $P = (P, p, \Phi)$ as in 6.4. The quotient of $\mathcal{P}$ comes from the $\mathcal{A}$-system $P^{\{1\}}$, which has exactly the same generating heteromorphisms as P, and with induced $\phi_m^{\{1\}}$ depicted below.

$$\begin{tikzcd}
\mathsf{Aut}_{\mathbb{D}}(Y) \arrow[r, "\phi_m"] \arrow[d] & \mathsf{Aut}_{\mathbb{C}}(X) \arrow[d] \\
\mathsf{Aut}_{\mathbb{D}^{\{1\}}}(Y) \arrow[r, "\phi_m^{\{1\}}"'] & \mathsf{Aut}_{\mathbb{C}^{\{1\}}}(Y)
\end{tikzcd}$$

The map $\phi_m^{\{1\}}$ is well-defined because if $\beta \in \mathsf{Aut}_{\mathbb{D}}(Y)$ is isotropy, then so is $\phi_m(\beta)$. We conclude the following.

7.5. Lemma. *Let $\mathcal{P} = \mathcal{R}(P) : \mathbb{C} \nrightarrow \mathbb{D}$ be a profunctor arising from $P = (P, p, \Phi)$. Then the quotient profunctor satisfies $\mathcal{P}^{\{1\}} = \mathcal{R}(P^{\{1\}})$.*

We refer to the $\mathcal{A}$-system $P^{\{1\}}$ obtained in this manner as the *quotient* of P.

We aim to identify specific cases where the congruence $\sim_{\mathcal{P}}$ on $\mathbb{D}$ induced by $\mathcal{P}$ is either trivial or the entire isotropy congruence. It is convenient to rewrite the pullback diagram (15) for $\mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D)$. Observe that the bottom composite of (15) factors through the subgroup

$$K(D) = \{\langle \sigma_{m,g} \rangle_{m,g} \mid \forall \beta \in \mathsf{Aut}(Y).\, \phi_m(\beta)^{-1} \sigma_{m,g} \phi_m(\beta) = \sigma_{m,g}\}$$

of $\prod_{m,g} \mathsf{Aut}(X)$. In fact, it is easily seen that this makes the square

$$\begin{tikzcd} H(D) \arrow[r] \arrow[d, tail] & K(D) \arrow[d, tail] \\ \prod_{m,g} \mathsf{Z}_{\mathbb{C}}(X) \arrow[r] & \prod_{m,g} \mathsf{Aut}(X) \end{tikzcd} \tag{18}$$

a pullback. The vertical composite of (15) clearly factors through $K(D)$, so that the following diagram is a pullback as well.

$$\begin{tikzcd} \mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D) \arrow[r] \arrow[d] & \mathsf{Z}_{\mathbb{D}}(D) \arrow[d] \\ H(D) \arrow[r] & K(D) \end{tikzcd} \tag{19}$$

We exploit this in the following lemma. A category is called *rigid* when it has no non-trivial automorphisms. In particular, such a category is anisotropic.

7.6. Lemma. [Rigidity Lemma] *Let $\mathcal{P} = \mathcal{R}(P, p, \Phi) : \mathbb{C} \looparrowright \mathbb{D}$.*

(i) *Suppose that $\mathbb{C}$ is rigid. Then the canonical map $\upsilon^*\mathsf{Z}_{\mathbb{C}*\mathbb{D}} \to \mathsf{Z}_{\mathbb{D}}$ is an isomorphism, and the quotient collage is*

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d, equal] & \mathbb{C} * \mathbb{D} \arrow[d] & \mathbb{D} \arrow[l, "\upsilon"'] \arrow[d] \\ \mathbb{C} \arrow[r] & \mathbb{C} * \mathbb{D}^{\{1\}} & \mathbb{D}^{\{1\}} \arrow[l] \end{tikzcd} \tag{20}$$

(ii) *Let $\mathbb{C}$ be such that $\mathbb{C}^{\{1\}}$ is rigid. Then the canonical map $\upsilon^* Z_{\mathbb{C}*\mathbb{D}} \to Z_{\mathbb{D}}$ is a surjection, and the quotient collage of $\mathcal{P}$ is*

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d] & \mathbb{C} * \mathbb{D} \arrow[d] & \mathbb{D} \arrow[l, "\upsilon"'] \arrow[d] \\ \mathbb{C}^{\{1\}} \arrow[r] & \mathbb{C}^{\{1\}} * \mathbb{D}^{\{1\}} & \mathbb{D}^{\{1\}} \arrow[l, "\upsilon^{\{1\}}"'] \end{tikzcd} . \tag{21}$$

Proof. (i) is an immediate consequence of (ii). To see why (ii) is true, note that it suffices to show that the map $H(D) \to K(D)$ is surjective because diagram (19) is a pullback. In turn, for this it suffices that $\prod_{m,g} \mathsf{Z}_{\mathbb{C}}(X) \to \prod_{m,g} \mathsf{Aut}(X)$ is surjective because diagram (18) is a pullback. However, each of the components $\nu_X : Z_{\mathbb{C}}(X) \to \mathsf{Aut}(X)$ is trivially surjective because the cokernel $\mathsf{Aut}(X)/\mathsf{I}$ must be trivial since the quotient of $\mathbb{C}$ is rigid. ■

While the rigidity lemma gives sufficient conditions for the induced congruence on $\mathbb{D}$ to be the full isotropy congruence, the following lemma gives a condition for it to be trivial.

7.7. LEMMA. *Let $\mathcal{P} = \mathcal{R}(P, p, \Phi) : \mathbb{C} \nrightarrow \mathbb{D}$, and suppose that for each $\tau \in \mathsf{Z}_{\mathbb{D}}(D)$ with $\tau_D \neq 1$ there are $g : Y \to D$ and $m \in P(X, Y)$ with $\mathsf{IR}(\phi_m(\tau_g)) > 1$. Then $\mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D) = 1$. Consequently, the quotient collage is as follows.*

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d, two heads] & \mathbb{C} * \mathbb{D} \arrow[d, two heads] & \mathbb{D} \arrow[l, "\upsilon"'] \arrow[d, equal] \\
\mathbb{C}^{\{1\}} \arrow[r] & \mathbb{C}^{\{1\}} * \mathbb{D} & \mathbb{D} \arrow[l]
\end{tikzcd} \tag{22}$$

PROOF. This is a consequence of Theorem 7.2: an element τ of $\mathsf{Z}_{\mathbb{D}}(D)$ with the stated property induces an element of $\prod_{m,g} \mathsf{Aut}(X)$ that cannot be in the image of $\prod_{m,g} \mathsf{Z}_{\mathbb{C}}(X)$. Thus, the pullback (15) is trivial. ∎

For future reference we shall call a profunctor $\mathcal{P}$ *obstructed* if $\sim_{\mathcal{P}}$ is trivial, and *unobstructed* if $\sim_{\mathcal{P}}$ is the full isotropy congruence. Of course, these are two extremes as in general the congruence $\sim_{\mathcal{P}}$ is somewhere between the two.

7.8. HIGHER QUOTIENTS OF A COLLAGE. We consider the iterated isotropy quotients of $\mathbb{C} *_{\mathcal{P}} \mathbb{D}$ where $\mathcal{P} = \mathcal{R}(P, p, \Phi)$ is a profunctor coming from an $\mathcal{A}$-system. The pattern we have discovered, namely that the quotient of the collage of $\mathcal{P}$ is the collage of the induced $\mathcal{P}^{\{1\}}$, persists into the transfinite.

Consider the case of a limit ordinal κ. On the one hand, by definition the κ-th isotropy quotient of $\mathbb{C}*\mathbb{D}$ is $\mathrm{Colim}_{\lambda<\kappa}(\mathbb{C}*\mathbb{D})^{\{\lambda\}}$. On the other hand, we have a congruence $\sim_{\mathcal{P},\kappa}$ on $\mathbb{D}$ induced by the composite $\mathbb{D} \to \mathbb{C}*\mathbb{D} \to (\mathbb{C}*\mathbb{D})^{\{\kappa\}}$, making the following commutative diagram.

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d, two heads] & \mathbb{C} * \mathbb{D} \arrow[d, two heads] & \mathbb{D} \arrow[l] \arrow[d, two heads] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\mathbb{C}^{\{\lambda\}} \arrow[r] \arrow[d, two heads] & (\mathbb{C} * \mathbb{D})^{\{\lambda\}} \arrow[d, two heads] & \mathbb{D}/\sim_{\mathcal{P},\lambda} \arrow[l] \arrow[d, two heads] \\
\mathbb{C}^{\{\lambda+1\}} \arrow[r] \arrow[d, two heads] & (\mathbb{C} * \mathbb{D})^{\{\lambda+1\}} \arrow[d, two heads] & \mathbb{D}/\sim_{\mathcal{P},\lambda+1} \arrow[l] \arrow[d, two heads] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\mathbb{C}^{\{\kappa\}} \arrow[r] & (\mathbb{C} * \mathbb{D})^{\{\kappa\}} & \mathbb{D}/\sim_{\mathcal{P},\kappa} \arrow[l]
\end{tikzcd}$$

It is easily seen that because each of the squares on the right side of the sequence is a pullback, the sequence of quotients of $\mathbb{D}$ is a colimit diagram, so that $\mathbb{D}/\sim_{\mathcal{P},\kappa}$ is the least congruence on $\mathbb{D}$ containing all the approximating $\sim_{\mathcal{P},\lambda}$. The bottom row is again a collage; the corresponding profunctor $\mathcal{P}^{\{\kappa\}}$ may be given by

$$\mathcal{P}^{\{\kappa\}}(C,D) = (\mathbb{C} * \mathbb{D})^{\{\kappa\}}(C,D) .$$

Moreover, using the same construction as in Lemma 7.5 there is an $\mathcal{A}$-system $P^{\{\kappa\}}$ from $\mathbb{C}[\kappa]$ to $\mathbb{D}$ for which $\mathcal{P}[\kappa] = \mathcal{R}(P^{\{\kappa\}})$: the set of generating heteromorphisms from X to Y is $P(X,Y)$, while to $m \in P(X,Y)$ we associate the unique group homomorphism $\phi_m^{\{\kappa\}}$ making the following diagram commute:

$$\begin{tikzcd}
\mathsf{Aut}_{\mathbb{D}}(Y) \arrow[r, "\phi_m"] \arrow[d] & \mathsf{Aut}_{\mathbb{C}}(X) \arrow[d] \\
\mathsf{Aut}_{\mathbb{D}/\sim_{\mathcal{P},\kappa}}(Y) \arrow[r, "\phi_m^{\{\kappa\}}"'] & \mathsf{Aut}_{\mathbb{C}^{\{\kappa\}}}(Y)
\end{tikzcd}$$

This is well-defined by virtue of the definition of $\sim_{\mathcal{P},\kappa}$.

7.9. Lemma. *For each ordinal λ, the λ-th row of the quotient sequence of $\mathcal{P} = \mathcal{R}(P)$ is isomorphic to the collage of $\mathcal{P}^{\{\lambda\}} : \mathbb{C}^{\{\lambda\}} \looparrowright \mathbb{D}/\sim_{\mathcal{P},\lambda}$. Moreover, $\mathcal{P}[\kappa]$ arises from the induced $\mathcal{A}$-system $P^{\{\kappa\}}$.*

From this picture we can derive an upper bound for the isotropy rank of a collage: we always have

$$\mathsf{IR}(\mathbb{C} * \mathbb{D}) \leq \mathsf{IR}(\mathbb{C}) + \mathsf{IR}(\mathbb{D}) \qquad \text{(ordinal sum)} .$$

Under additional assumptions on P we can be more specific. For example, consider the following condition, which should be compared with that of Lemma 7.7.

7.10. Definition. *An $\mathcal{A}$-system $P = (P, p, \Phi)$ from $\mathbb{C}$ to $\mathbb{D}$ (and its associated profunctor) is called* maximally obstructive *if for each D in $\mathbb{D}$ and each $\tau \in \mathsf{Z}_{\mathbb{D}}(D)$ with $\tau_D \neq 1$ we have*

$$\bigvee_{\substack{g:Y\to D\\ m\in P(X,Y)}} \mathsf{IR}(\phi_m(\tau_g)) = \mathsf{IR}(\mathbb{C}) .$$

It immediately follows from Def. 7.10 that if P is maximally obstructive and $\mathsf{IR}(\mathbb{C}) > 1$, then $\mathcal{P} = \mathcal{R}(P)$ is obstructed. Therefore, by Lemma 7.7 it induces the trivial congruence on $\mathbb{D}$.

7.11. Proposition. *Let P be maximally obstructive, and let $\mathcal{P} = \mathcal{R}(P)$. Then:*

(i) *all $P^{\{\kappa\}}$ are maximally obstructive;*

(ii) *for all $\lambda < \mathsf{IR}(\mathbb{C})$, the induced $\mathcal{P}^{\{\lambda\}}$ are obstructed;*

(iii) *when $0 \leq \mathsf{IR}(\alpha) \leq 1$ for each automorphism of $\mathbb{C}$, the induced isotropy congruence on $\mathbb{D}$ is the full isotropy congruence;*

PROOF. P is maximally obstructive if and only if for each $\tau \in \mathsf{Z}_{\mathbb{D}}(D)$, the family

$$\{\phi_m(\tau_g) \mid g : Y \to D, m \in P(X,Y)\}$$

is dense in $\mathbb{C}$. Being a dense family descends to all higher quotients (Theorem 5.14). This implies immediately that each $P^{\{\kappa\}}$ is maximally obstructive as well.

If also $\mathsf{IR}(\mathbb{C}) > 1$, then Theorem 7.2 implies that no element $\tau \in \mathsf{Z}_{\mathbb{D}}$ can be isotropy in $\mathbb{C} * \mathbb{D}$ because a necessary condition is that each $\phi_m(\tau_g)$ is isotropy in $\mathbb{C}$.

Finally, if all automorphisms of $\mathbb{C}$ are isotropy, then again by Theorem 7.2 we find that all elements $\tau \in \mathsf{Z}_{\mathbb{D}}$ are in the image of $\mathsf{Z}_{\mathbb{C}*\mathbb{D}}(D) \to \mathsf{Z}_{\mathbb{D}}(D)$. ∎

We stress that in (iii) we cannot weaken the hypothesis to $0 \le \mathsf{IR}(\mathbb{C}) \le 1$ for it could happen that there are automorphisms α with $\mathsf{IR}(\alpha) = -\infty$. If we had $\phi_m(\tau) = \alpha$ for such α, then $\mathsf{IR}(\tau) = -\infty$ as well. The following definition captures the condition we need.

7.12. DEFINITION. *An $\mathcal{A}$-system P from $\mathbb{C}$ to $\mathbb{D}$ is* suitable *if it is maximally obstructive and if for all $m \in P(X,Y)$ we have $\mathsf{IR}(\phi_m(\beta)) \ge 0$ for all $\beta \in \mathsf{Aut}_{\mathbb{D}}(Y)$.*

All the categories that interest us in this paper are *eventually rigid* in the sense that all automorphisms have non-negative isotropy rank. If $\mathbb{C}$ is an eventually rigid category, any maximally obstructive profunctor with domain $\mathbb{C}$ is automatically suitable.

Let $\mathcal{P}$ be suitable, and suppose $\kappa = \mathsf{IR}(\mathbb{C}), \kappa' = \mathsf{IR}(\mathbb{D})$. The isotropy sequence of $\mathcal{P}$ can take one of two forms depending on whether κ is a limit ordinal or a successor. When κ is a limit ordinal we have

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d, two heads] & \mathbb{C} * \mathbb{D} \arrow[d, two heads] & \mathbb{D} \arrow[l] \arrow[d, equal] \\
\mathbb{C}^{\{\kappa\}} \arrow[r] \arrow[d, equal] & (\mathbb{C} * \mathbb{D})^{\{\kappa\}} \arrow[d, two heads] & \mathbb{D} \arrow[l] \arrow[d, two heads] \\
\mathbb{C}^{\{\kappa\}} \arrow[r] & (\mathbb{C} * \mathbb{D})^{\{\kappa+\kappa'\}} & \mathbb{D}^{\{\kappa'\}} \arrow[l]
\end{tikzcd}$$

so that $\mathsf{IR}(\mathbb{C} * \mathbb{D}) = \kappa + \kappa'$, with end result $(\mathbb{C} * \mathbb{D})^{\{\kappa+\kappa'\}} \cong \mathbb{C}^{\{\kappa\}} * \mathbb{D}^{\{\kappa'\}}$. On the other hand, when $\mathsf{IR}(\mathbb{C}) = \kappa = \lambda + 1$, the sequence is as follows.

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d, two heads] & \mathbb{C} * \mathbb{D} \arrow[d, two heads] & \mathbb{D} \arrow[l] \arrow[d, equal] \\
\mathbb{C}^{\{\lambda\}} \arrow[r] \arrow[d, two heads] & (\mathbb{C} * \mathbb{D})^{\{\lambda\}} \arrow[d, two heads] & \mathbb{D} \arrow[l] \arrow[d, two heads] \\
\mathbb{C}^{\{\lambda+1\}} \arrow[r] \arrow[d, equal] & (\mathbb{C} * \mathbb{D})^{\{\lambda+1\}} \arrow[d, two heads] & \mathbb{D}^{\{1\}} \arrow[l] \arrow[d, two heads] \\
\mathbb{C}^{\{\lambda+1\}} \arrow[r] & (\mathbb{C} * \mathbb{D})^{\{\lambda+\kappa'\}} & \mathbb{D}^{\{\kappa'\}} \arrow[l]
\end{tikzcd}$$

The only aspect that needs commenting is the fact that at stage λ the induced congruence on $\mathbb{D}$ is the full isotropy congruence. However, this follows from Proposition 7.11. We have therefore established the following.

7.13. Theorem. *Let $\mathcal{P} = \mathcal{R}(P) : \mathbb{C} \nrightarrow \mathbb{D}$ be suitable (Def. 7.12). Then*

$$\mathsf{IR}(\mathbb{C} * \mathbb{D}) = \begin{cases} \kappa + \mathsf{IR}(\mathbb{D}) & \text{if } \kappa = \mathsf{IR}(\mathbb{C}) \text{ is a limit} \\ \lambda + \mathsf{IR}(\mathbb{D}) & \text{if } \lambda + 1 = \mathsf{IR}(\mathbb{C}) \text{ is a successor}. \end{cases}$$

7.14. Proposition. *Let $\mathcal{P} = \mathcal{R}(P) : \mathbb{C} \nrightarrow \mathbb{D}$ and $\mathcal{Q} = \mathcal{R}(Q) : \mathbb{D} \nrightarrow \mathbb{B}$ be suitable. Then the induced $\upsilon_* \mathcal{Q} : \mathbb{C} *_{\mathcal{F}} \mathbb{D} \nrightarrow \mathbb{B}$ is also suitable.*

Proof. Any dense family A in $\mathbb{D}$ gives a dense family $\{\upsilon(\alpha) \mid \alpha \in A\}$ in $\mathbb{C} * \mathbb{D}$ since P is suitable. Thus, if Q is suitable, then the family

$$\{\phi_m(\tau_g) \mid \tau \in \mathsf{Z}_{\mathbb{B}}(D),\, g : Y \to D,\, m \in Q(X, Y)\}$$

is dense in $\mathbb{C} * \mathbb{D}$ because it is dense in $\mathbb{D}$. ∎

8. Main results

We turn to an analysis of the isotropy of the categories $\mathbb{X}[\kappa]$ defined in § 6. We first study the problem of characterizing the isotropy group and rank of a category which arises as a sequential colimit of categories. Together with the results on collages (§ 7) we then have enough machinery to prove the desired results. As another application of these techniques we shall construct, for each κ, a small category $\mathbb{D}[\kappa]$ for which the sequence of subgroups (12) is strictly increasing up to rank κ.

8.1. Isotropy and colimits. We establish suitable and sufficient conditions on a sequential diagram of categories for computing the isotropy rank of the colimit in terms of the ranks of the approximations. Throughout, we consider an ordinal κ (regarded as a small category) and a cocontinuous functor

$$\mathbb{C}[-] : \kappa \to \mathfrak{Cat}\,. \tag{23}$$

We write $\iota = \iota_{\lambda' < \lambda} : \mathbb{C}[\lambda'] \to \mathbb{C}[\lambda]$ for the transition maps, and

$$\mathbb{C}[\kappa] = \mathrm{Colim}_{\lambda < \kappa} \mathbb{C}[\lambda]\,.$$

Our goal is to identify conditions on (23) that give us control over the isotropy of the colimit $\mathbb{C}[\kappa]$.

Suppose that the functor (23) has the property that each of the transition maps preserves isotropy. We then obtain an induced diagram

$$\begin{tikzcd}
\mathbb{C}[0] \arrow[r] \arrow[d] & \mathbb{C}[1] \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{C}[\lambda] \arrow[r] \arrow[d] & \mathbb{C}[\lambda+1] \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{C}[\kappa] \arrow[d, dashed, "K"] \\
\mathbb{C}[0]^{\{1\}} \arrow[r] & \mathbb{C}[1]^{\{1\}} \arrow[r] & \cdots \arrow[r] & \mathbb{C}[\lambda]^{\{1\}} \arrow[r] & \mathbb{C}[\lambda+1]^{\{1\}} \arrow[r] & \cdots \arrow[r] & \mathrm{Colim}_{\lambda}(\mathbb{C}[\lambda]^{\{1\}})
\end{tikzcd}$$

where in each row we form the colimit, and where the vertical K is induced by the colimiting property of $\mathbb{C}[\kappa]$. We would like to compare the functor K with the isotropy quotient of $\mathbb{C}[\kappa]$. To this end, we need to ensure that the maps $\mathbb{C}[\lambda] \to \mathbb{C}[\kappa]$ preserve isotropy as well. The following lemma provides a sufficient condition.

8.2. LEMMA. *Suppose that all of the transition maps $\mathbb{C}[\lambda] \to \mathbb{C}[\lambda']$ are stable* (§ 3). *Then the components of the colimiting cocone $\mathbb{C}[\lambda] \to \mathbb{C}[\kappa]$ preserve isotropy.*

PROOF. Assume stability of the transition functors $\mathbb{C}[\lambda] \to \mathbb{C}[\lambda']$, and consider an element of isotropy $\tau \in \mathsf{Z}_{\mathbb{C}[\lambda]}(C)$. We shall show that this induces a compatible element of isotropy $\sigma \in \mathsf{Z}_{\mathbb{C}[\kappa]}$. The component of σ at 1_C may be taken to be $[\tau_C]$, the image under $\mathbb{C}[\lambda] \to \mathbb{C}[\kappa]$ of τ_C. Now given $[f] : D \to C$ in $\mathbb{C}[\kappa]$, pick $\lambda' \geq \lambda$ for which $[f]$ is in the image of $\mathbb{C}[\lambda'] \to \mathbb{C}[\kappa]$. Since $\mathbb{C}[\lambda] \to \mathbb{C}[\lambda']$ is assumed to be stable, we may consider the unique $\tau' \in \mathsf{Z}_{\mathbb{C}[\lambda']}$ compatible with τ; this τ' has a component τ'_f at f. We can now set $\sigma_{[f]}$ to be $[\tau'_f]$. It is routine to check that this is well-defined, that σ is indeed an element of isotropy, and that it is compatible with τ. ∎

In particular, in the situation of the above lemma, we obtain, for each $\lambda < \kappa$, a factorization

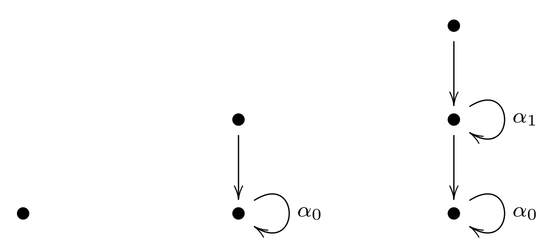
$$\begin{tikzcd} \mathbb{C}[\lambda] \arrow[r] \arrow[d] & \mathbb{C}[\kappa] \arrow[d] \\ C[\lambda]^{\{1\}} \arrow[r, dashed] & \mathbb{C}[\kappa]^{\{1\}} \end{tikzcd} \tag{24}$$

which in turn gives a unique factorization

$$\begin{tikzcd} \mathbb{C}[\kappa] \arrow[d, "K"'] \arrow[dr] & \\ \mathrm{Colim}_\lambda(\mathbb{C}[\lambda]^{\{1\}}) \arrow[r, dashed, "L"'] & \mathbb{C}[\kappa]^{\{1\}} \end{tikzcd} \tag{25}$$

of the isotropy quotient through K.

8.3. EXAMPLE. This example shows that even if all transition maps of (23) are stable, the comparison map L need not be an equivalence. Consider the following countable sequence of inclusions of categories $\mathbb{C}[n]$:



$$\mathbb{C}[0] \longrightarrow \mathbb{C}[1] \longrightarrow \mathbb{C}[2] \longrightarrow \cdots$$

The idea is that α_0 doesn't lift along the vertical map in $\mathbb{C}[1]$, but lifts to α_1 at the next stage. Similarly, α_1 doesn't lift in $\mathbb{C}[2]$, but does lift in the next stage. Consequently, none of the α_i lift along all maps, and all $\mathbb{C}[i]$ have trivial isotropy group. However, in the colimit $\mathbb{C}[\omega]$, α_0 is part of isotropy, so that the isotropy group of $\mathbb{C}[\omega]$ is non-trivial.

Example 8.3 works essentially because the transition maps are not full inclusions. This explains the assumptions in the following theorem.

8.4. THEOREM. *Suppose that a cocontinuous functor* (23) *has the property that its transition maps are full inclusions that are stable. Then the canonical comparison map L in* (25) *is an equivalence. Hence,* $\mathbb{C}[\kappa]^{\{1\}} \simeq \mathrm{Colim}_\lambda(\mathbb{C}[\lambda]^{\{1\}})$.

PROOF. We show that an automorphism $\alpha \in \mathsf{Aut}(X)$ in $\mathbb{C}[\kappa]$ is isotropy if and only if there is a stage $\lambda < \kappa$ where α is isotropy. Pick any λ for which α is in $\mathbb{C}[\lambda]$. The map $\iota : \mathbb{C}[\lambda] \to \mathbb{C}[\kappa]$ is full and faithful, and therefore by Proposition 3.12 gives a comparison homomorphism $\rho : \mathsf{Z}_{\mathbb{C}[\kappa]}(\iota X) \to \mathsf{Z}_{\mathbb{C}[\lambda]}(X)$. Thus, if α is isotropy, then applying ρ yields an element of isotropy in $\mathbb{C}[\lambda]$ whose underlying automorphism is α . This proves that the subgroup $\mathsf{I}_{\mathbb{C}[\kappa]}(X) \subseteq \mathsf{Aut}(X)$ in $\mathbb{C}[\kappa]$ is the union of the approximants $\mathsf{I}_{\mathbb{C}[\lambda]}(X)$. Hence, the isotropy congruence coincides with the congruence induced by K . ■

We would like to have a similar result for the iterated isotropy quotients. Using Theorem 5.10 we arrive at the following sufficient condition.

8.5. THEOREM. *Suppose that a cocontinuous functor* (23) *has the property that its transition maps are sieve inclusions* (§ 3.11)*. Write* $\mathbb{C}[\kappa] = \mathrm{Colim}_{\lambda<\kappa}\mathbb{C}[\lambda]$ *. Then:*

(i) *for each ordinal* θ *, the canonical map* $\mathrm{Colim}_{\lambda<\kappa}(\mathbb{C}[\lambda]^{\{\theta\}}) \to \mathbb{C}[\kappa]^{\{\theta\}}$ *is an isomorphism.*

(ii) $\mathsf{IR}(\mathbb{C}[\kappa]) = \bigvee_{\lambda<\kappa} \mathsf{IR}(\mathbb{C}[\lambda])$.

PROOF. It follows from Theorem 5.10 that each morphism $\mathbb{C}[\lambda]^{\{\theta\}} \to \mathbb{C}[\lambda']^{\{\theta\}}$ preserves isotropy, and is fully faithful. Using induction on θ (i) now follows from Theorem 8.4. For (ii) suppose first that we have a non-trivial automorphism α in $\mathbb{C}[\lambda]^{\{\theta\}}$ that is isotropy. Then the image of α under the transition map $\mathbb{C}[\lambda]^{\{\theta\}} \to \mathbb{C}[\kappa]^{\{\theta\}}$ is again non-trivial and isotropy. This shows that $\mathsf{IR}(\mathbb{C}[\kappa]) \geq \mathsf{IR}(\mathbb{C}[\lambda])$ for all $\lambda < \kappa$. Therefore, we have $\mathsf{IR}(\mathbb{C}[\kappa]) \geq \bigvee_{\lambda<\kappa} \mathsf{IR}(C[\lambda])$. If this inequality were strict, then there would be an ordinal $\theta = \bigvee_\lambda \mathsf{IR}(\mathbb{C}[\lambda])$ at which the isotropy quotient $\mathbb{C}[\kappa]^{\{\theta\}} \to \mathbb{C}[\kappa]^{\{\theta+1\}}$ is non-trivial. However, we have already seen that the isotropy congruence on the colimit is the union of those of the approximants, which is a contradiction. ■

We end the discussion of isotropy of colimits with a corollary concerning profunctors out of colimits, which we use in Prop. 8.8. Suppose that for each $\lambda < \kappa$ we are given an $\mathcal{A}$-system $P[\lambda]$ from $\mathbb{C}[\lambda]$ to $\mathbb{D}$. Using the cocone inclusions ι^κ_λ we may lift these (using the method described in Section 6.2) to $\mathcal{A}$-systems from $\mathbb{C}[\kappa]$ to $\mathbb{D}$, and then form their coproduct:

$$P[\kappa] =_{\mathrm{def}} \coprod_{\lambda<\kappa} (\iota^\kappa_\lambda)_*(P[\lambda]) \ .$$

8.6. Corollary. *Let $\mathbb{C}[-] : \kappa \to \mathfrak{Cat}$ be a sequence of categories satisfying the assumptions of Theorem 8.5. For each $\lambda < \kappa$, let $P[\lambda]$ be an $\mathcal{A}$-system from $\mathbb{C}[\lambda]$ to $\mathbb{D}$, inducing $P[\kappa]$ as described above. If*

$$\bigvee \{\lambda < \kappa \mid P[\lambda] \text{ is maximally obstructive}\} = \kappa ,$$

then $P[\kappa]$ is maximally obstructive as well.

Proof. The cofinality assumption says that removing those λ for which $P[\lambda]$ is not maximally obstructive does not affect the colimit. Hence, without loss of generality we may assume that all $P[\lambda]$ are maximally obstructive. Then for $\tau \in \mathsf{Z}_{\mathbb{D}}(D)$ with $\tau_D \neq 1$, we have

$$\begin{array}{rcll} \bigvee_{\substack{g:Y\to D \\ m\in P[\kappa](X,Y)}} \mathsf{IR}(\phi_m(\tau_g)) & = & \bigvee_{\lambda<\kappa} \bigvee_{\substack{g:Y\to D \\ m\in P[\lambda](X,Y)}} \mathsf{IR}(\phi_m(\tau_g)) & \text{def. of } P[\kappa] \\ & = & \bigvee_{\lambda<\kappa} \mathsf{IR}(\mathbb{C}[\lambda]) & P[\lambda] \text{ is max. obstr.} \\ & = & \mathsf{IR}(\mathbb{C}[\kappa]) & \text{by Thm. 8.5} \end{array}$$

■

8.7. Proof of main result. We now look at the specific case of the categories $\mathbb{X}[\kappa]$, as defined in Section 6.6.

8.8. Proposition. *For each ordinal κ:*

(i) $\mathsf{IR}(\mathbb{X}[\kappa]) = \kappa$*;*

(ii) $\mathbb{X}[\kappa]$ *is eventually rigid in the sense that all automorphisms have non-negative isotropy rank;*

(iii) *if κ is not a successor of a limit ordinal, then the profunctors $\mathcal{P}[\kappa] : \mathbb{X}[\kappa] \nrightarrow \mathbb{X}$ and $\mathcal{Q}[\kappa] : \mathbb{X}[\kappa] \nrightarrow \mathbb{X}[1]$ are maximally obstructive (equivalently, suitable).*

Proof. We argue by transfinite induction on κ. In each case, the claim about eventual rigidity is easily seen.
Base case. For $\kappa = 0, 1, 2$, the claims about isotropy ranks have been established in § 4. The fact that $\mathcal{P}[2] : \mathbb{X}[2] \nrightarrow \mathbb{X}$ and $\mathcal{Q}[2]$ are maximally obstructive is immediate by inspection.
Limit case. Suppose that κ is a limit ordinal. We have $\mathbb{X}[\kappa] = \mathrm{Colim}_{\lambda<\kappa}\mathbb{X}[\lambda]$: this has isotropy rank

$$\begin{array}{rcll} \mathsf{IR}(\mathbb{X}[\kappa]) & = & \mathsf{IR}(\mathrm{Colim}_{\lambda<\kappa}\mathbb{X}[\lambda]) & \\ & = & \bigvee_{\lambda<\kappa} \mathsf{IR}(\mathbb{X}[\lambda]) & \text{by Theorem 8.5} \\ & = & \bigvee_{\lambda<\kappa} \lambda & \text{induction hypothesis} \\ & = & \kappa . & \end{array}$$

Thus, it remains to consider the profunctors $\mathcal{P}[\kappa] : \mathbb{X}[\kappa] \nrightarrow \mathbb{X}[2]$ and $\mathcal{Q}[\kappa] : \mathbb{X}[\kappa] \nrightarrow \mathbb{X}[1]$. By the induction hypothesis, we know that all of the $\mathcal{P}[\lambda]$ and $\mathcal{Q}[\lambda]$ are maximally obstructive; since the subset $\{\lambda < \kappa \mid \lambda \text{ is not a successor of a limit ordinal}\}$ is cofinal, by Corollary 8.6 the induced $\mathcal{P}[\kappa]$ and $\mathcal{Q}[\kappa]$ are maximally obstructive as well.

Successor case, exceptions. Suppose the results hold for κ where κ is a limit ordinal. Consider $\mathbb{X}[\kappa+1]$, which is defined to be $\mathbb{X}[\kappa+1] = \mathbb{X}[\kappa] *_{\mathcal{Q}[\kappa]} \mathbb{X}[1]$ for the maximally obstructive $\mathcal{Q}[\kappa]$. Therefore, by Theorem 7.13 we have $\mathsf{IR}(\mathbb{X}[\kappa+1]) = \kappa + \mathsf{IR}(\mathbb{X}[1]) = \kappa+1$ as required.

Successor case, standard. If $\kappa = \lambda+1$ is a successor ordinal, then $\mathbb{X}[\kappa+1] = \mathbb{X}[\lambda] *_{\mathcal{P}[\lambda]} \mathbb{X}[2]$, so that $\mathsf{IR}(\mathbb{X}[\kappa+1]) = \lambda+2 = \kappa+1$, again by Theorem 7.13. By Proposition 7.14, it is immediate from their construction that the profunctors $\mathcal{P}[\kappa+1]$ and $\mathcal{Q}[\kappa+1]$ are maximally obstructive. ∎

8.9. Application. Given an ordinal κ, we construct a category $\mathbb{D}[\kappa]$ containing an object Y with the property that the chain of subgroups (12) of $\mathsf{Aut}(Y)$ induced by the higher isotropy groups is strict up to κ.

Let $F(\kappa)$ denote the free group on κ generators, considered as a category with a single object. Let t_λ denote the generator corresponding to $\lambda < \kappa$. We construct a suitable profunctor whose task it is to force $\mathsf{IR}(t_\lambda) = \lambda+1$. Choose for each $\lambda < \kappa$ a small category $\mathbb{C}_\lambda$ whose isotropy rank is $\lambda+1$. Next choose an automorphism $\tau_\lambda \in \mathsf{Aut}(C_\lambda)$ in $\mathbb{C}_\lambda$ with $\mathsf{IR}(\tau_\lambda) = \lambda+1$ (Lemma 5.13). Then define an $\mathcal{A}$-system A_λ from $\mathbb{C}_\lambda$ to $F(\kappa)$ with one generating heteromorphism and corresponding group homomorphism $F(\kappa) \to \mathsf{Aut}(C_\lambda)$, which sends the generator t_λ to τ_λ and all other generators to 1. Let $\mathbb{C} = \coprod_{\lambda<\kappa} \mathbb{C}_\lambda$, and write $i_\lambda : \mathbb{C}_\lambda \to \mathbb{C}$ for the induced $\mathcal{A}$-system from $\mathbb{C}$ to $F(\kappa)$ given by

$$A = \coprod_{\lambda<\kappa} (i_\lambda)_* A_\lambda \,.$$

Then it is easily seen that A is suitable, and that each t_λ has rank $\lambda+1$ as promised. Hence the collage $\mathbb{D}[\kappa]$ of A has the desired properties.

9. Topos-theoretic aspects

We apply the theory of isotropy for small categories that we have developed to isotropy for toposes [Funk et al., 2012]. After a brief and informal recapitulation of isotropy groups of toposes we prove that the isotropy quotient of a presheaf topos is equivalent to presheaves on the isotropy quotient of the underlying small category. We show how this extends to general sites. Finally, we prove that every Grothendieck topos has a well-defined isotropy rank.

9.1. The isotropy group of a topos. If X denotes an object of a topos $\mathcal{E}$, then an automorphism of the geometric morphism $\mathcal{E}/X \to \mathcal{E}$ associated with X may be regarded as a natural automorphism of the left adjoint $\Sigma_X : \mathcal{E}/X \to \mathcal{E}$. The set of these automorphisms forms a group $\mathcal{Z}(X)$, and by whiskering a functor

$$\mathcal{Z} : \mathcal{E}^{\mathrm{op}} \to \mathsf{Grp} \,.$$

It follows that $\mathcal{Z}$ is representable in the sense that there is an object Z of $\mathcal{E}$ and a natural bijection

$$\mathcal{E}(X, \mathsf{Z}) \cong \mathcal{Z}(X) \ .$$

It follows also that Z carries the structure of a group object internal to $\mathcal{E}$, which we call the isotropy group of $\mathcal{E}$. We say that a topos $\mathcal{E}$ is anisotropic if Z is trivial.

9.2. Proposition. *The isotropy group of a small category* $\mathbb{C}$ *as defined in Def. 2.2 and the isotropy group of the topos* $\mathcal{E} = \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ *as defined above coincide.*

Thus, we may equally regard $\mathsf{Z}_{\mathbb{C}}$ as a presheaf of groups on $\mathbb{C}$, or as a group internal to the topos $\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$.

The role of the isotropy group of $\mathcal{E}$ is clarified through its universal action; every object X of $\mathcal{E}$ carries a canonical action by the group Z:

$$\theta_X : X \times \mathsf{Z} \to X \ ; \ \ \theta_X(f, \tau) = f \cdot \tau = f\tau_Y \ . \tag{26}$$

The following diagram (left) explains the universal action.

$$\begin{tikzcd} & Y \arrow[r, "\tau_Y"] \arrow[dl, "(f{,}\tau)"'] \arrow[d, "f\cdot\tau"] & Y \arrow[dl, "f"] \\ X \times \mathsf{Z} \arrow[r, "\theta_X"] & X & \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[r, "\tau_C"] \arrow[dl, "(x{,}\tau)"'] \arrow[d, "x\cdot\tau"] & C \arrow[dl, "x"] \\ X \times \mathsf{Z} \arrow[r, "\theta_X"] & X & \end{tikzcd}$$

The same diagram above (right) interpreted for the topos of presheaves on $\mathbb{C}$ recovers the universal action discussed in § 2.11. Moreover, every morphism of $\mathcal{E}$ is equivariant with respect to the universal action, the action of Z on itself must be conjugation, and the isotropy group Z is terminal with these properties in a certain sense [Funk et al., 2012].

9.3. Isotropy quotient of a topos. We shall say that an object X of a topos $\mathcal{E}$ is *isotropically trivial* if the universal action θ_X is trivial, i.e., is the projection. The inclusion in $\mathcal{E}$ of the full subcategory $\mathcal{E}_\theta$ of isotropically trivial objects is the inverse image functor ψ^* of a connected, atomic geometric morphism

$$\psi : \mathcal{E} \to \mathcal{E}_\theta \ . \tag{27}$$

The left adjoint $\psi_! \dashv \psi^*$ is given by an "orbit space" construction, as in the following coequalizer diagram:

$$\begin{tikzcd} X \times Z \arrow[r, shift left, bend left, "\theta_X"] \arrow[r, shift right, bend right, "\pi_X"'] & X \arrow[r, two heads] & \psi_! X \end{tikzcd} \qquad \begin{tikzcd} \psi_*(X) \arrow[r] & X \arrow[r, shift left, bend left, "\widehat{\theta_X}"] \arrow[r, shift right, bend right, "\widehat{\pi_X}"'] & X^Z \end{tikzcd} \ .$$

The right adjoint $\psi^* \dashv \psi_*$ produces the largest subobject on which the action of Z is trivial, given by the above equalizer diagram of exponential transposes.

We refer to ψ (27) as the (first) isotropy quotient of $\mathcal{E}$. When we wish to consider the iterated quotients we favour the notation $\mathcal{E}^{\{1\}}$ for the first isotropy quotient. It follows that a topos $\mathcal{E}$ is anisotropic if and only if (27) is an equivalence.

9.4. Proposition. *The geometric morphism of presheaf toposes associated with the category isotropy quotient* $\psi : \mathbb{C} \to \mathbb{C}/\mathsf{I}$ *is equivalent to the isotropy quotient* (27) *for* $\mathcal{E} = \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ *.*

Proof. Assume that a presheaf P on $\mathbb{C}$ factors through the functor ψ^{op}, so that it is an object of $\mathsf{Set}^{\mathbb{C}/\mathsf{I}^{\mathrm{op}}}$. This means that for each $\alpha \in \mathsf{I}$, we have $P(\alpha) = 1$. Then the universal action of Z on P must be trivial because the universal action is determined entirely by the action on P of morphisms $\nu_C(\tau) = \tau_C$. Therefore, P is an object of $\mathcal{E}_\theta$. Conversely, if P is an object of $\mathcal{E}_\theta$, then $P(\tau_C) = 1$ for all $\tau \in \mathsf{Z}(C)$, so that P factors through ψ^{op}. ■

9.5. Remark. A consequence of Prop. 9.4 is that the isotropy quotient of a small category commutes with splitting idempotents in the sense that the isotropy quotient of the idempotent splitting of a small category is equivalent to the idempotent splitting of its isotropy quotient.

9.6. Extension to sites. It turns out that if a Grothendieck site $(\mathbb{C}, J)$ is subcanonical in the sense that the representable presheaves are sheaves, then the isotropy group of presheaves $\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ is a sheaf, and it is the isotropy group of the sheaf topos $Sh(\mathbb{C}, J)$ [Funk et al., 2012]. The following example illustrates the necessity of the assumption of subcanonicity.

9.7. Example. Let $\mathbb{C}$ be the category with two objects A, B, one non-trivial automorphism $\alpha \in \mathsf{Aut}(A)$, and $\mathbb{C}(B, A) = \{f, \alpha f\}$. Thus α is not isotropy, and $\mathbb{C}$ has trivial isotropy. However, when we consider the topology on $\mathbb{C}$ for which f is a cover, the resulting subtopos is simply the topos of G-sets for which $G = \mathsf{Aut}(A)$. The latter has non-trivial isotropy. Informally speaking, non-subcanonical topologies can remove the obstruction to an automorphism being isotropy.

For a given subcanonical site $(\mathbb{C}, J)$ consider the topology J/I on $\mathbb{C}/\mathsf{I}$ induced by the isotropy quotient $\psi : C \to \mathbb{C}/\mathsf{I}$. A sieve S on an object C of $\mathbb{C}/\mathsf{I}$ is covering if and only if it is the image of a covering sieve in $\mathbb{C}$. Because of the nature of the congruence, and the fact that $f \in S$ if and only if $f\alpha \in S$ when α is an isomorphism, this actually means that ψ creates covers. Indeed, once $f \in S$, then all g with $g \sim f$ must also be in S.

9.8. Proposition. *If a site* $(\mathbb{C}, J)$ *is subcanonical, then the geometric morphism associated with the site morphism* $\psi : (\mathbb{C}, J) \to (\mathbb{C}/\mathsf{I}, J/\mathsf{I})$ *is equivalent to the isotropy quotient* (27) *for* $\mathcal{E} = Sh(\mathbb{C}, J)$ *.*

Proof. We must show that a sheaf on $(\mathbb{C}/\mathsf{I}, J/\mathsf{I})$ is the same thing as a sheaf on $(\mathbb{C}, J)$ on which the isotropy group Z of $\mathbb{C}$ acts trivially. If F is a sheaf on $(\mathbb{C}/\mathsf{I}, J/\mathsf{I})$, then F is a presheaf on $\mathbb{C}/\mathsf{I}$, so Z acts trivially on it. And when regarded as a presheaf on $\mathbb{C}$, F is a J-sheaf because the quotient map ψ creates covers. The converse is similar. ■

As a matter of fact, in the above situation there is a commutative diagram of toposes

$$\begin{tikzcd} Sh(\mathbb{C}, J) \arrow[r] \arrow[d, tail] & Sh(\mathbb{C}/\mathsf{I}, J/\mathsf{I}) \arrow[d, tail] \\ \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}} \arrow[r] & \mathsf{Set}^{\mathbb{C}/\mathsf{I}^{\mathrm{op}}} \end{tikzcd}$$

where the horizontal maps are the isotropy quotients. The category

$$Sh(\mathbb{C}, J)_\theta \simeq Sh(\mathbb{C}/\mathsf{I}, J/\mathsf{I})$$

is simply the intersection of the two subcategories $Sh(\mathbb{C}, J)$ and $(\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}})_\theta$ of $\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$.

We also need the following consequence.

9.9. Corollary. *If a category $\mathbb{C}$ is anisotropic, then so is $Sh(\mathbb{C}, J)$ for any subcanonical topology J on $\mathbb{C}$.*

9.10. Rank of a topos. We consider the notion of isotropy rank for toposes. We find that it agrees with isotropy rank for small categories in the case of a presheaf topos.

For a Grothendieck topos $\mathcal{E}$, we define by transfinite induction the following sequence of toposes:

- $\mathcal{E}^{\{0\}} = \mathcal{E}$
- $\mathcal{E}^{\{\lambda+1\}} = \mathcal{E}^{\{\lambda\}}{}_\theta$, the isotropy quotient of $\mathcal{E}^{\{\lambda\}}$
- $\mathcal{E}^{\{\kappa\}} = \mathrm{Colim}_{\lambda<\kappa}\mathcal{E}^{\{\lambda\}}$, where this colimit is taken in the category of Grothendieck toposes and geometric morphisms.

Thus, we have the following sequence Grothendieck toposes and geometric morphisms.

$$\mathcal{E} \to \mathcal{E}^{\{1\}} \to \mathcal{E}^{\{2\}} \to \cdots \to \mathcal{E}^{\{\omega\}} \to \mathcal{E}^{\{\omega+1\}} \to \cdots . \tag{28}$$

9.11. Proposition. *For any ordinal κ, the composite*

$$\psi_\kappa : \mathcal{E} \to \mathcal{E}^{\{\kappa\}} \tag{29}$$

is connected, atomic.

Proof. A successor quotient $\mathcal{E}^{\{\lambda+1\}}$ is a full subcategory of $\mathcal{E}^{\{\lambda\}}$, which is closed under subquotients and contains the subobject classifier. In the case of a limit ordinal κ, the topos $\mathcal{E}^{\{\kappa\}}$ is as a full subcategory of $\mathcal{E}$ simply the intersection of the subcategories $\mathcal{E}^{\{\lambda\}}$ for $\lambda < \kappa$. Consequently, $\mathcal{E}^{\{\kappa\}}$ is also closed under subquotients and the subobject classifier in $\mathcal{E}$. ■

9.12. DEFINITION. *For a Grothendieck topos* $\mathcal{E}$ *, we let the* isotropy rank *of* $\mathcal{E}$ *be the least ordinal* κ *for which the isotropy quotient* $\mathcal{E}^{\{\kappa\}} \to \mathcal{E}^{\{\kappa+1\}}$ *is an equivalence. Let* $\mathsf{IR}(\mathcal{E})$ *denote the isotropy rank of* $\mathcal{E}$ *.*

Equivalently, $\mathsf{IR}(\mathcal{E})$ is the least ordinal κ such that $\mathcal{E}^{\{\kappa\}}$ in (28) is anisotropic.

In order to show that an arbitrary Grothendieck topos has an isotropy rank we find that Proposition 9.8 does not help us because as it happens the isotropy quotient of a subcanonical site may not be subcanonical. However, the following approach based on the notion of isotropy rank of an object solves the problem.

For any object A of $\mathcal{E}$, consider the sequence

$$A = A_0 \twoheadrightarrow A_1 \twoheadrightarrow A_2 \twoheadrightarrow \cdots \twoheadrightarrow A_\omega \twoheadrightarrow A_{\omega+1} \twoheadrightarrow \cdots \tag{30}$$

of objects in $\mathcal{E}$. In this sequence, A_κ denotes the orbit object ${\psi_\kappa}^*\psi_{\kappa!}(A)$ associated with ψ_κ (29). Every morphism in the sequence is a counit of an adjunction for a corresponding geometric morphism in the sequence (28): they are all epimorphisms. Because a Grothendieck topos is co-wellpowered there must be a least ordinal at which (30) stabilizes in the following sense.

9.13. DEFINITION. The isotropy rank *of an object* A *of a topos* $\mathcal{E}$ *is the least ordinal* κ *for which* $A_\kappa \to A_{\kappa+1}$ *is an isomorphism. Let* $\mathsf{IR}(A)$ *denote the isotropy rank of* A *.*

Equivalently, $\mathsf{IR}(A)$ is the least ordinal κ for which A_κ in (30) is isotropically trivial in $\mathcal{E}^{\{\kappa\}}$.

For the following we rely on the fact that if $\phi : \mathcal{E} \to \mathcal{F}$ is connected, atomic and X is a generator of $\mathcal{E}$, then $\phi_!(X)$ is a generator of $\mathcal{F}$ (Prop. 6.3, [Funk et al., 2012]).

9.14. THEOREM. *Every Grothendieck topos has an isotropy rank. Moreover, if* X *is a generator of* $\mathcal{E}$ *, then* $\mathsf{IR}(\mathcal{E}) = \mathsf{IR}(X)$ *.*

PROOF. For any ordinal λ , if X is a generator of $\mathcal{E}$, then X_λ is a generator of $\mathcal{E}^{\{\lambda\}}$, so that as a subcategory of $\mathcal{E}$, $\mathcal{E}^{\{\lambda\}}$ is generated from X_λ by taking all subquotients of copowers of X_λ . Therefore, because the sequence (30) of quotients of X stabilizes at some ordinal, so must the sequence (28) of quotients of $\mathcal{E}$. ∎

We may now prove that isotropy rank for toposes agrees with isotropy rank for small categories in the presheaf case.

9.15. PROPOSITION. *If* $\mathbb{C}$ *is a small category, then for any ordinal* κ *, we have*

$$\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}\{\kappa\}} \simeq \mathsf{Set}^{\mathbb{C}^{\{\kappa\}\mathrm{op}}} .$$

PROOF. This is vacuously true for $\kappa = 0$. It is proved for successor ordinals in Prop. 9.4. If κ is a limit ordinal, then we may use the fact that the free cocompletion $\mathbb{C} \mapsto \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ commutes with colimits. In other words, for any small diagram $F : \mathbb{D} \to \mathfrak{Cat}$ of small categories, we have

$$\mathrm{Colim}_{\mathbb{D}} \; (\mathsf{Set}^{-^{\mathrm{op}}} \circ F) \simeq \mathsf{Set}^{(\mathrm{Colim}_{\mathbb{D}} F)^{\mathrm{op}}} .$$

∎

9.16. Proposition. *For a small category* $\mathbb{C}$ *, we have* $\mathsf{IR}(\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}) = \mathsf{IR}(\mathbb{C})$ *.*

Proof. This is an immediate consequence of Proposition 9.15 and Corollary 5.6. ∎

9.17. Anisotropic objects and isotropy rank. We consider a few implications of the existence of the sequence of isotropy quotients of a topos. We recall the 'fundamental lemma' of isotropy theory [Funk & Hofstra, 2018], which states that for any object A of $\mathcal{E}$, the following two squares are canonically equivalent:

$$
\begin{tikzcd}
\mathcal{E}/A \arrow[r, "\psi/A"] \arrow[d] & \mathcal{E}^{\{1\}}/\psi_!(A) \arrow[d] \\
\mathcal{E} \arrow[r, "\psi"'] & \mathcal{E}^{\{1\}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}/A \arrow[r, "\psi'"] \arrow[d] & (\mathcal{E}/A)^{\{1\}} \arrow[d] \\
\mathcal{E} \arrow[r, "\psi"'] & \mathcal{E}^{\{1\}}
\end{tikzcd}
$$

where the left-hand square arises from factoring the composite $\mathcal{E}/A \to \mathcal{E} \to \mathcal{E}^{\{1\}}$ as a connected, atomic map followed by an étale map, and where the right-hand map arises from taking the isotropy quotient of $\mathcal{E}/A$. In words, the isotropy quotient of a slice over A is the slice of the quotient by the orbit space of A . Our first aim is to generalize this to higher quotients, and prove the following.

9.18. Theorem. *For a Grothendieck topos $\mathcal{E}$, an object A of $\mathcal{E}$ and an ordinal κ, we have*

$$(\mathcal{E}/A)^{\{\kappa\}} \simeq \mathcal{E}^{\{\kappa\}}/A_\kappa \ . \tag{31}$$

Proof. We argue by transfinite induction on κ . The successor case is taken care of by the fundamental lemma. We must therefore investigate the case of a limit ordinal κ . It suffices to prove that for any κ-indexed sequence of connected atomic toposes

$$\mathcal{E}_0 \xrightarrow{\psi_0} \mathcal{E}_1 \xrightarrow{\psi_1} \cdots \longrightarrow \mathcal{E}_\kappa = \mathrm{Colim}_{\lambda<\kappa}\mathcal{E}_\lambda$$

and induced sequence

$$A = A_0 \to A_1 \to \cdots \to A_\kappa = \mathrm{Colim}_{\lambda<\kappa}A_\lambda$$

(the colimit of the sequence (30)) the top sequence in

$$
\begin{tikzcd}
\mathcal{E}_0/A_0 \arrow[r, "\psi_0/A_0"] \arrow[d] & \mathcal{E}_1/A_1 \arrow[r, "\psi_1/A_1"] \arrow[d] & \cdots \arrow[r] & \mathcal{E}_\kappa/A_\kappa \arrow[d] \\
\mathcal{E}_0 \arrow[r, "\psi_0"] & \mathcal{E}_1 \arrow[r, "\psi_1"] & \cdots \arrow[r] & \mathcal{E}_\kappa = \mathrm{Colim}_{\lambda<\kappa}\mathcal{E}_\lambda
\end{tikzcd}
$$

is again a colimit. This is a routine argument which we leave to the reader. ∎

Let us say that an object A of a topos $\mathcal{E}$ is *anisotropic* if the slice topos $\mathcal{E}/A$ is anisotropic. It follows that A is anisotropic if and only if the canonical action θ_A (26) is free in the sense that the map $A \times \mathsf{Z} \to A \times A$, which we may write in element-style notation as $(a, z) \mapsto (az, a)$, is injective.

9.19. LEMMA. *If A is anisotropic object of a topos $\mathcal{E}$, then for any ordinal κ, A_κ is an anisotropic object of $\mathcal{E}^{\{\kappa\}}$.*

PROOF. If A is an anisotropic object of $\mathcal{E}$, then the anisotropic $\mathcal{E}/A$ is equivalent to $(\mathcal{E}/A)^{\{\kappa\}}$, whence to $\mathcal{E}^{\{\kappa\}}/A_\kappa$ by (31). Therefore, the latter topos is anisotropic, so that A_κ is an anisotropic object of $\mathcal{E}^{\{\kappa\}}$. ∎

9.20. LEMMA. *If a topos has a globally supported, anisotropic, isotropically trivial object, then the topos is anisotropic.*

PROOF. Suppose that A is such an object of a topos $\mathcal{E}$. By isotropically trivial, the isotropy group of $\mathcal{E}/A$ is the projection $A \times \mathsf{Z} \to A$. By anisotropic, $A \cong A \times \mathsf{Z}$ over A. By globally supported, we have $\mathsf{Z} \cong 1$. ∎

Compare the following result with Theorem 9.14.

9.21. PROPOSITION. *If A is a globally supported, anisotropic object of a topos $\mathcal{E}$, then $\mathsf{IR}(\mathcal{E}) = \mathsf{IR}(A)$.*

PROOF. It suffices to prove that $\mathsf{IR}(\mathcal{E}) \leq \mathsf{IR}(A)$. Suppose that κ is an ordinal such that A_κ is isotropically trivial in $\mathcal{E}^{\{\kappa\}}$. A_κ is also anisotropic $\mathcal{E}^{\{\kappa\}}$ (Lemma 9.19), and globally supported. By Lemma 9.20, $\mathcal{E}^{\{\kappa\}}$ is anisotropic. ∎

10. Conclusion and Further Remarks

The results in the present paper are a first step towards a systematic study of the notion of higher isotropy for small categories, establishing among other things the existence of small categories with arbitrarily high isotropy rank.

There are several possible avenues of investigation left open for future work. The first is somewhat technical, aiming to make the idea of isotropy as generalized conjugation more precise by means of a monad T on $\mathfrak{Cat}$ that freely turns a category $\mathbb{C}$ into one in which $\mathsf{Aut}(-)$ is functorial and isotropy is full. An algebra $T\mathbb{C} \to \mathbb{C}$ amounts to a homomorphic section $\mathsf{Aut}(C) \to \mathsf{Z}(C)$ of ν_C for each C.

In the present paper we have only presented those aspects of the functoriality of the isotropy group needed for the main results. It should be interesting to develop the theory of isotropy spans in more detail on the level of geometric morphisms.

The concept of isotropy group admits generalizations in several directions. One of these is due to Simon Henry [Henry, 2017], which he calls *the localic isotropy group.* This is a localic group internal to the topos, which sometimes carries more information than the discrete isotropy group discussed in our work. Henry shows that it is possible to form the isotropy quotient of a topos with respect to the localic isotropy group; interestingly, this quotient is then anisotropic, so that there is no higher isotropy in this sense.

Another interesting generalization is from groups to monoids (or even to Lawvere Theories). As the authors of the present paper recently learned, Peter Freyd's concept of core algebras [Freyd, 2007] is closely related to the isotropy group; in fact, the core of a Grothendieck topos could be called the isotropy monoid of the topos. We believe that the techniques and concepts offered in the present paper should have interesting interpretations for Freyd's original motivation, namely parametric polymorphism.

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Department of Mathematics and Computer Science, City University of New York,
Queensborough Community College, Bayside, New York, United States.
Department of Mathematics and Statistics, University of Ottawa
585 King Edward Ave., Ottawa, Ontario K1N 6N5 Canada.
Center for Computational Mathematics in Industry and Commerce
c/o Amanda Guderian, Applied Mathematics, MC
University of Waterloo, 200 University Avenue West,
Waterloo, Ontario, N2L 3G1 Canada.

Email: jfunk@qcc.cuny.edu
phofstra@uottawa.ca
sh9khan@uwaterloo.ca

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